Ans-1

a. $x_{k+1} = x_k - \frac{f(x)}{d}$ where $d \to$ constant

~~for go~~

let $\phi(x) = x - \frac{f(x)}{d}$ and for convergence

$|\phi'(x)| < 1 \;\forall\; x \in [a,b]$

$-1 < \phi'(x) < 1$

now we have assumed $\phi(x) = x - \frac{f(x)}{d}$

$\Rightarrow \phi'(x) = 1 - \frac{f'(x)}{d}$

$\Rightarrow -1 < 1 - \frac{f'(x)}{d} < 1$

$\Rightarrow \frac{f'(x)}{d} < 2$

$\Rightarrow d > \frac{f'(x)}{2}$

hence condition for d is

→ $d > \frac{f'(x)}{2}$ for it to be convergent.

→ d should be of the same sign as $f'(x)$

→ $f'(x) \neq 0$.

ii. For calculation of general rate of convergence let x^* be such $f(x^*) = 0$ and let x_k be the calculated value by newton's method

$x_k = x^* + \varepsilon_k$ where ε is the error value and is considered very small.

we know that $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

replacing x_k by $x^* + \varepsilon_k$ we get

$$x_{k+1} = x^* + \varepsilon_k - \frac{f(x^* + \varepsilon_k)}{f'(x^* + \varepsilon_k)}$$

we can say $x^* + \varepsilon_{k+1} = x_{k+1}$ [~~as~~ similar to what we did for x_k]

$$\cancel{x^*} + \varepsilon_{k+1} = \cancel{x^*} + \varepsilon_k - \frac{f(x^* + \varepsilon_k)}{f'(x^* + \varepsilon_k)}$$

$$\varepsilon_{k+1} = \varepsilon_k - \frac{f(x^* + \varepsilon_k)}{f'(x^* + \varepsilon_k)}$$

using talyor expansion for $f(x^* + \varepsilon_k)$ & $f'(x^* + \varepsilon_k)$

$$\varepsilon_{k+1} = \varepsilon_k - \left\{ \frac{f(x^*) + \varepsilon_k f'(x^*) + \frac{\varepsilon_k}{2!} f''(x^*) + \cdots}{f'(x^* + \varepsilon_k)} \right\}$$

we have replaced $f'(x^* + \varepsilon_k)$ by d

neglecting

$$\varepsilon_{k+1} = \varepsilon_k - \left\{ \frac{\overset{0}{\cancel{f(x^*)}} + \varepsilon_k f'(x^*) + \cancel{\frac{\varepsilon_k}{2!} f''(x^*) + \cdots}}{d} \right.$$

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\varepsilon_k f'(x^*)}{d}$$

$$\varepsilon_{k+1} = \varepsilon_k \left\{ 1 - \frac{f'(x^*)}{d} \right\} \quad \text{—— (1)}$$

let $1 - \frac{f'(x^*)}{d} = P$

$$\varepsilon_{k+1} = P\varepsilon_k$$

comparing with $\varepsilon_{k+1} = A\varepsilon_k^t$ where $t \to$ rate of convergence we get

$$t = 1$$

Hence rate of convergence is 1

iii From the above part we obtained $\varepsilon_{k+1} = \varepsilon_k \left\{ 1 - \frac{f'(x^*)}{d} \right\}$

From the above we can see that the quadratic convergence is not possible as we are getting only ε_k and not ε_k^2

Hence there won't be any quadratic convergence.

b. Given to us $x^2-1=0$; $(x-1)^4=0$ & $x-\cos x=0$

Function	convergence
$x^2-1=0$	1.999 ∼ 2
$(x-1)^4=0$	1.000... ∼ 1
$x-\cos x=0$	1.9988 ∼ 2

→ value approximately equal to calculated value (for $x^2-1=0$)

→ value approximately equal to calculated value (for $x-\cos x=0$)

for the function $(x-1)^4$ we will try to reason the convergence rate

so actual root $= x^* = 1$ and the computed root is 1.002857

we can clearly see that the value of error is high that is $\varepsilon_k = 0.002857$ hence we cannot omit the higher values in the talyor expansion of $f(x^*+\varepsilon_k)$ & $f'(x^*+\varepsilon_k)$ and the value obtained which is multiplied by ε_k^c would be greater (effectively value A) and hence c which is the convergence rate drops to 1.

Ans-2 It is observed that the values ~~closer~~ which converge to an answer give lower value of residual and hence lower value of error. while the error and residual is high for those values which do not converge.

Ans-3

a. Given: Chebyshev's polynomial follow the relation

$$f_0(t) = 1$$
$$f_1(t) = t$$
$$f_{n+1}(t) = 2tf_n(t) - f_{n-1}(t)$$

Required to show: $f_n(t) = \cos(n \arccos(t))$ is a Chebyshev's polynomial

Proof: let $f_n(t) = \cos(n \arccos(t))$

$\rightarrow$ $f_0(t) = \cos(\cancel{n}\arccos(t))$

$= \cos 0 = 1$

hence $f_0(t) = 1$ — (I)

$\rightarrow$ $f_1(t) = \cos(\arccos(t))$

$= t$

hence $f_1(t) = t$ — (II)

$\rightarrow$ $f_{n+1}(t) = \cos\{(n+1)\cos^{-1}(t)\}$

let $\cos^{-1}(t) = A$

$f_{n+1}(t) = \cos\{(n+1)A\}$

$= \cos(nA + A)$

$= \cos nA \cos A - \sin nA \sin A$

$f_{n+1}(t) = \cos nA \cos A - \sin nA \sin A$ — (III)

$f_{n-1}(t) = \cos\{(n-1)\cos^{-1} t\}$

let $\cos^{-1} t = A$

$\Rightarrow \cos\{(n-1)A\}$

$= \cos(nA - A)$

$\cos(nA - A) = \cos(nA)\cos(A) + \sin(nA)\sin A$

$\cos(nA - A) = \cos(nA)\cos(A) + \sin(nA)\sin A$

$\Rightarrow f_{n-1}(t) = \cos(nA)\cos(A) + \sin(nA)\sin(A)$ — (IV)

adding (III) & (IV) we get

$f_{n+1}(t) + f_{(n-1)}(t) = 2\cos(nA)\cos(A)$

replacing A by $\cos^{-1} t$

$\because A = \cos^{-1} t$ by assumption

$f_{n+1}(t) + f_{n-1}(t) = 2\cos(n\cos^{-1} t)\cos(\cos^{-1} t)$

$f_{n+1}(t) + f_{n-1}(t) = 2t\cos(n\cos^{-1} t) \rightarrow f_n(t)$

$f_{n+1}(t) + f_{n-1}(t) = 2t f_n(t)$.

$f_{n+1}(t) = 2t f_n(t) - f_{n-1}(t)$ — (V)

from (I) (II) & (V) we can say $f_n(t) = \cos(n \arccos(t))$ holds chebyshev's three term recurrence

Hence proved.

Given: $\cos((n+1)\cos^{-1}(t))$ and we need to try to show it is chebyshev's three term recurrence

$\rightarrow f_0(t) = \cos(\cos^{-1}(t)) = t$

$\because f_0(t) \neq 1$

Hence it doesn't holds chebyshev's three term recurrence.

b. Given: $f_n(t) = \cos(n \cos^{-1}(t))$

To prove: $f_n(t)$ is a polynomial

Proof: $f_n(t) = \cos(n \cos^{-1} t)$

let $A = \cos^{-1} t$

$f_n(t) = \cos(nA)$

we know that $\cos\theta + i\sin\theta = e^{i\theta}$ and $\cos n\theta + i\sin n\theta = e^{ni\theta}$

$\cos n\theta + i\sin n\theta = (e^{i\theta})^n$

$\Rightarrow \cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$\Rightarrow \cos n\theta + i\sin n\theta = {}^nC_0 \cos^n\theta + {}^nC_1 \cos^{n-1}\theta . i\sin\theta -$
${}^nC_2 \cos^{n-2}\theta \sin^2\theta \text{ - - - - - - - - } +$
${}^nC_n (i\sin\theta)^n$

$\cos n\theta = {}^nC_0 \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \text{ - - - - - - - } +$

$= {}^nC_0 \cos^n\theta - {}^nC_2 \cos^{n-2}\theta (1 - \cos^2\theta) + \text{ - - - - - - - } +$

$= {}^nC_0 \cos^n\theta - {}^nC_2 \cos^{n-2}\theta + {}^nC_2 \cos^n\theta + \text{ - - - - - }$

now replacing θ by $A = \cos^{-1} t$

$\cos nA = {}^nC_0 (\cos \cos^{-1} t)^n - {}^nC_2 (\cos \cos^{-1} t)^{n-2} + {}^nC_2 (\cos \cos^{-1}$
- - - - - - -

$\cos n \cos^{-1} t = {}^nC_0 t^n - {}^nC_2 t^{n-2} + {}^nC_2 t^n \text{ - - - - - - - - - }$

$f_n(t) = {}^nC_0 t^n - {}^nC_2 t^{n-2} + {}^nC_2 t^n \text{ - - - - - - - - }$

hence we can clearly see that $A_n(t)$ is a polynomial.

Hence proved

Ans-4 Given: formula for divide difference is

$$f[t_1, t_2 \cdots t_k] := \frac{f[t_2, t_3 \cdots t_k] - f[t_1, t_2 \cdots t_{k-1}]}{t_k - t_1}$$

To prove: This formula is correct

Proof: Let $p(t)$ be newton interpolation polynomial for arbitary j^{th} basis

$$p(t) = \alpha_1 + \alpha_2 (t - t_1) + \alpha_2 (t - t_p) + \cdots + \alpha_n (t - t_n) + \cdots (t - t_{n-1})$$

Base case: for $k = 1$

for $k = 1$

$p(t) = \alpha_1$

$p(t_1) = f(t_1) = f[t_1]$ $\quad [\because p$ interpolates f in $t_1]$

hence base case holds valid

Induction hypothesis: The $f[t_1, t_2 \cdots t_{n-1}] := \dfrac{f[t_2 \cdots t_{n-1}] - f[t_1 \cdots t_{n-2}]}{t_{n-1} - t_1}$

holds true.

Induction step: We need to show that the formula holds true for $k = m - 1$

Let $a_i = f[t_1, t_2 \ldots\ldots t_i]$ $\forall\, i \in 1, 2 \ldots\ldots n-1$

let $p_n(t) = \alpha_1 + \alpha_2(t-t_1) + \text{----} \quad \alpha_n\{(t-t_1)(t-t_2)\ldots(t-t_{n-1})\}$

we know the values of $\alpha_1, \alpha_2 \ldots\ldots \alpha_{n-1}$
[Induction hypothesis]

hence we can use this to ~~calculate and~~ prove the identity

$f(t_n) = p(t_n)$ [interpolation condition]

$$f(t_n) = \alpha_1 + \alpha_2(t_n - t_1) + \text{-----} + \alpha_n\{(t_n-t_1)(t_n-t_2)\ldots(t_n-t_{n-1})\}$$

$$\frac{f(t_n) - \alpha_1}{t_n - t_1} = \alpha_2 + \alpha_3(t_n - t_2) + \text{------} + \alpha_n\{(t_n-t_2)\ldots(t_n - t_{n-1})\}$$

$$\frac{f(t_n) - f(t_1)}{t_n - t_1} = \alpha_2 + \alpha_3(t_n - t_2) + \text{-----} + \alpha_n\{(t_n-t_2)\ldots(t_n-t_{n-1})\}$$

$$f[t_1, t_n] = \alpha_2 + \alpha_3(t_n - t_2) + \text{--------} + \alpha_n\{(t_n-t_2)\ldots(t_n-t_{n-1})\}$$

$$\frac{f[t_1, t_n] - \alpha_2}{t_n - t_2} = \alpha_3 + \alpha_4(t_n - t_3) + \text{-----} + \alpha_n\{(t_n-t_3)\ldots(t_n-t_{n-1})\}$$

$$f[t_1, t_2, t_n] = \alpha_3 + \alpha_4(t_n - t_3) + \text{--------} \alpha_n\{(t_n-t_3)\ldots(t_n-t_{n-1})\}$$

repeating the above steps $n-1$ time
we get

$$f[t_1, t_2 \cdots t_{n-2}\, t_n] = \alpha_{n-1} + \alpha_n (t_n - t_{n-1})$$

$$\frac{f[t_1, t_2 \cdots, t_n] - \alpha_{n-1}}{t_n - t_{n-1}} = \alpha_n$$

$$\boxed{f[t_1, t_2 \cdots t_n] = \alpha_n}$$

hence from above we can say the statement holds for $k = n$.

Hence formula for divided difference is correct

In general

$$f[t_1, t_2 \cdots t_j] = \alpha_j$$

Hence proved.

(b) Given points $(-1, 1)$; $(0, 0)$; $(1, 1)$

To find: interpolating polynomial with monomial, lagrange and newton basis.

(a) monomial basis

let $p_{n-1}(t) = \alpha_1 + \alpha_2 t + \cdots + \alpha_n t^{n-1}$

$$Ax = \begin{bmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ 1 & t_2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & & \\ 1 & t_n & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

given data point $(-1,1)$ $(0,0)$ $(1,1)$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$x_1 - x_2 + x_3 = 1$ — (I)

$x_1 = 0$ — (II)

$x_1 + x_2 + x_3 = 1$ — (III)

adding (I) & (III) we get

$2x_1 + 2x_3 = 2$

$\because x_1 = 0$ [from (II)]

$2x_3 = 2$

$x_3 = 1$

substituting $x_1 = 0$ & $x_3 = 1$ in (I) we get

$x_2 = 0$

$(x_1, x_2, x_3) = (0, 0, 1)$

$sol^n = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

interpolating polynomial for monomial basis $= t^2 = p_2(t)$ — (A)

b. Lagrange basis

for lagrange basis function is

$$l_j(t) = \prod_{k=1, k\neq j}^{n} (t - t_k) \Big/ \prod_{k=1\, k\neq j}^{n} (t_j - t_k)$$

interpolating polynomial

$$p_{n-1}(t) = y_1 l_1(t) + y_2 l_2(t) + \text{------} + y_n l_n(t)$$

$$p_2(t) = y_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + y_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} +$$

$$y_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$

$$p_2(t) = 1\left\{\frac{(t-0)(t-1)}{-1(-2)}\right\} + 0\left\{\frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}\right\}$$

$$+ \left\{\frac{(t+1)(t-0)}{2}\right\}$$

$$= \frac{(t-1)t}{2} + \frac{(t+1)t}{2}$$

$$= \frac{t}{2}\left\{(t-1)+(t+1)\right\}$$

$$= \frac{t}{\cancel{2}} \times \cancel{2}t = t^2$$

$$p_2(t) = t^2 \text{ — } B$$

c. Newton basis function

$$\pi_j(t) = \prod_{k=1}^{j-1} (t - t_k) \; ; \; j = 1, 2 \ldots n$$

given points $(-1, 1)$; $(0, 0)$; $(1, 1)$

interpolating polynomial

$$\phi_{n-1}(t) = \alpha_1 + \alpha_2 (t - t_1) + \alpha_3 (t - t_1)(t - t_2) \ldots$$
$$+ \alpha_n (t - t_1)(t - t_2) \ldots (t - t_{n-1})$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & t_2 - t_1 & 0 \\ 1 & t_3 - t_1 & (t_3 - t_1)(t_3 - t_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

substituting values we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$x_1 = 1$ — (I)

$x_1 + x_2 = 0$ — (II)

$x_1 + 2x_2 + 2x_3 = 1$ — (III)

from (I) & (II) we get $x_2 = -1$

from (I) (II) & (III) we get

$2x_3 = 2$

$x_3 = 1$

sol^n vector $= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

hence the interpolating polynomial is

$$p_2(t) = 1 - 1(t+1) + 1(t+1)t$$

$$= \cancel{1} - \cancel{t} - \cancel{1} + t^2 + \cancel{t}$$

$$\boxed{p_2(t) = t^2} \longrightarrow C$$

d. from A B & C we can clearly see the interpolating polynomial is same for all basis ie $\boxed{p_2(t) = t^2}$