2.6 SPECIAL CASES OF LINEAR PROGRAMMING

The special cases of the solution of the linear programming problem can be categorized as shown below.

(a) Infeasible solution
(b) Unbounded solution
(c) Unbounded solution space with finite solution
(d) Alternate optimum solution
(e) Degenerate solution.

In this section, the special cases of the linear programming problems are explained with suitable illustrations:

Example 2.22 **(Infeasible solution)** Solve the following LP problem graphically:

$$\text{Maximize } Z = 10X_1 + 3X_2$$

subject to

$$2X_1 + 3X_2 \le 18$$
$$6X_1 + 5X_2 \ge 60$$
$$X_1 \text{ and } X_2 \ge 0$$

Solution The introduction of the non-negative constraints ($X_1 \ge 0$ and $X_2 \ge 0$) of the given problem will eliminate the second, third and fourth quadrants of the X_1X_2 plane, as shown in Figure 2.4. Compute the coordinates to plot equations relating to different constraints on the X_1X_2 plane, as shown below.

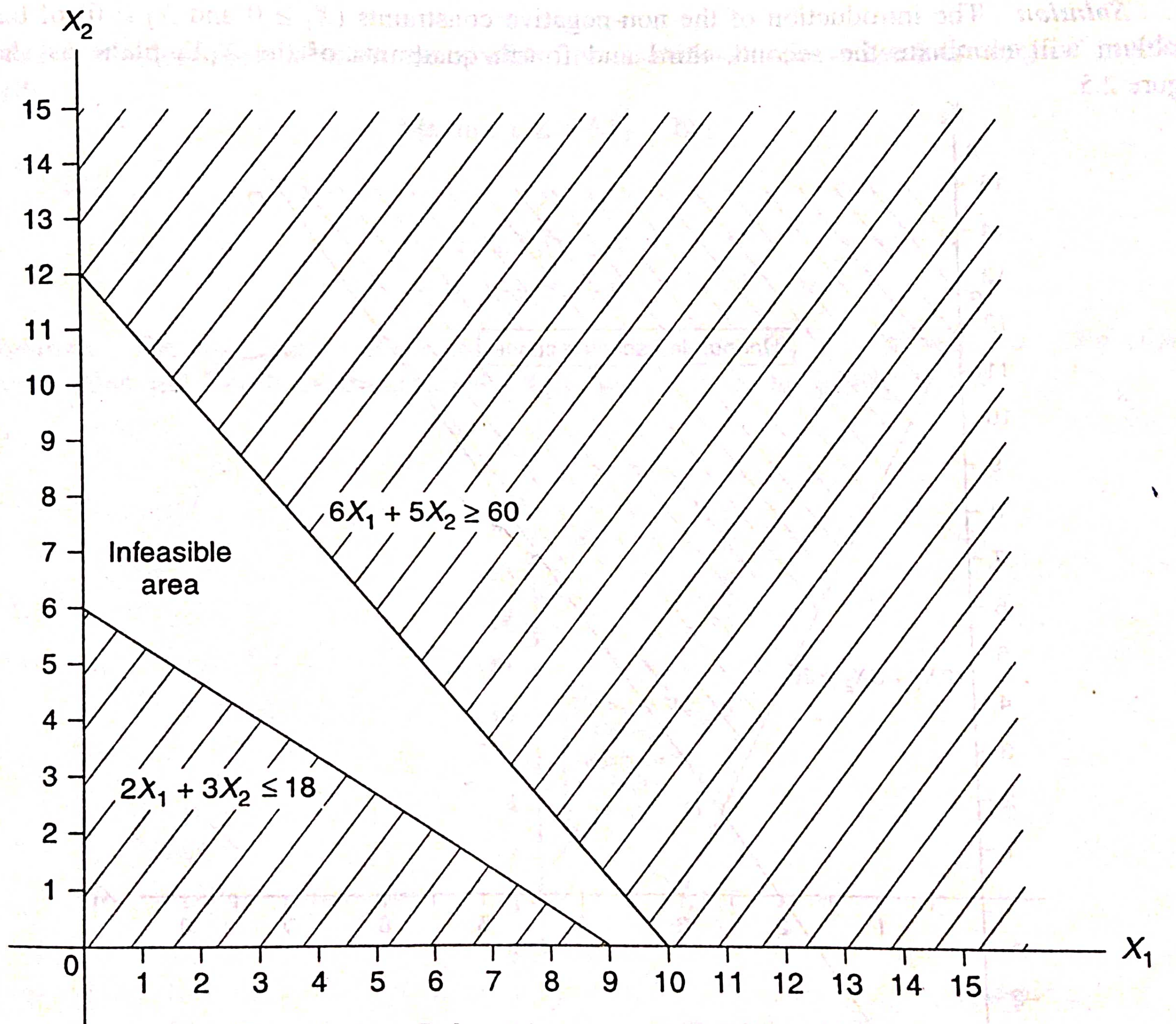

Figure 2.4 Infeasible region for Example 2.22.

From the first constraint in equation form

$$2X_1 + 3X_2 = 18$$

we get $X_2 = 6$, when $X_1 = 0$; and $X_1 = 9$, when $X_2 = 0$. Also, from the second constraint in equation form

$$6X_1 + 5X_2 = 60$$

we have $X_2 = 12$, when $X_1 = 0$; and $X_1 = 10$, when $X_2 = 0$.

Now, plot the constraints 1 and 2 as shown in Figure 2.4. The feasible side of each constraint is shaded. From Figure 2.4, it is clear that there is no common intersecting area of the shaded regions. This means that the problem has infeasible solution.

Example 2.23 **(Unbounded solution)** Solve the following LP problem using graphical method.

$$\text{Maximize } Z = 12X_1 + 25X_2$$

subject to

$$12X_1 + 3X_2 \geq 36$$
$$15X_1 - 5X_2 \leq 30$$
$$X_1 \text{ and } X_2 \geq 0$$

Solution The introduction of the non-negative constraints ($X_1 \geq 0$ and $X_2 \geq 0$) of the given problem will eliminate the second, third and fourth quadrants of the X_1X_2 plane as shown in Figure 2.5.

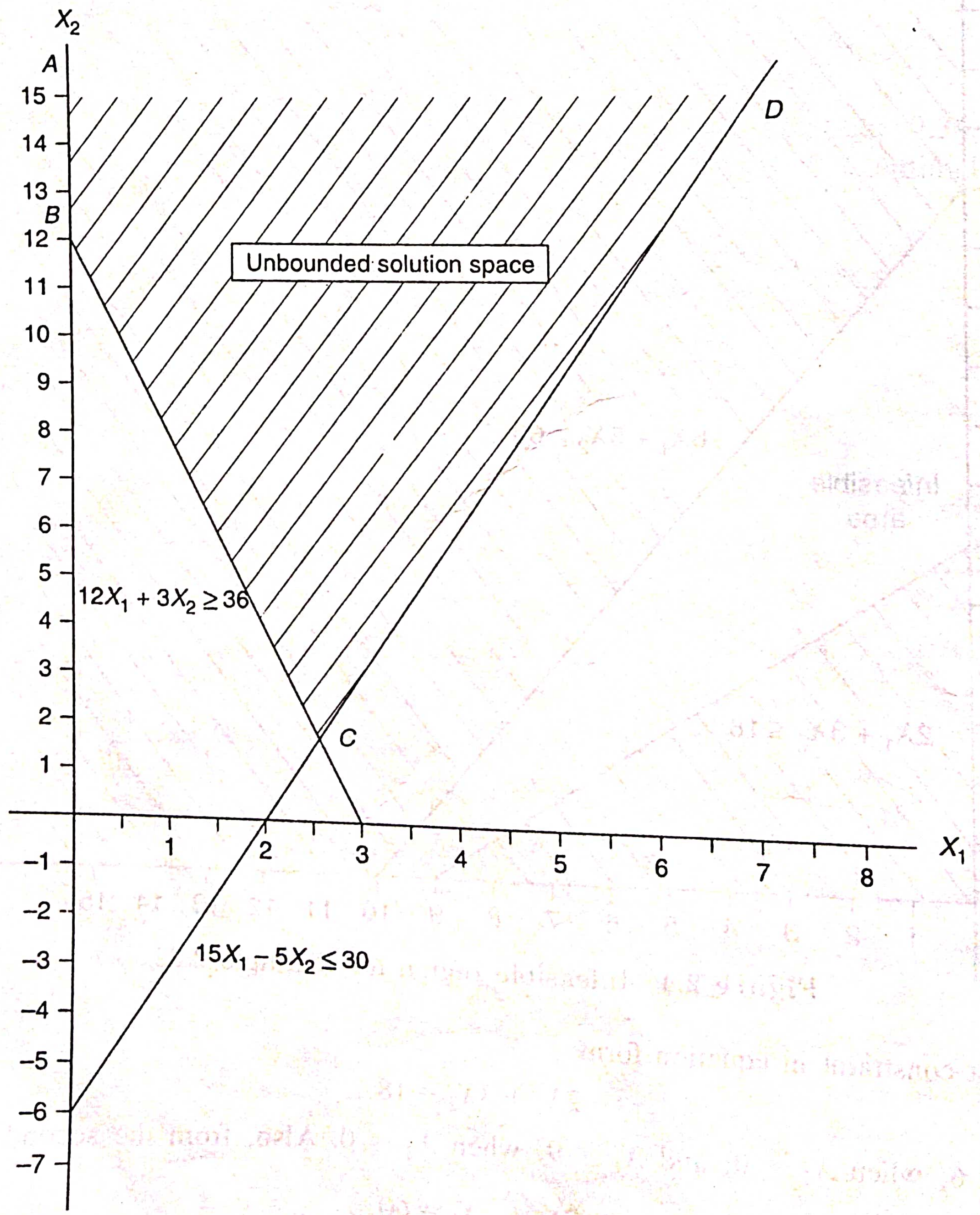

Figure 2.5 Unbounded solution space for Example 2.23.

Compute the coordinates to plot the equations relating to different constraints on the X_1X_2 plane as shown below. From the first constraint in equation form

$$12X_1 + 3X_2 = 36$$

we get $X_2 = 12$, when $X_1 = 0$; and $X_1 = 3$, when $X_2 = 0$. Also, from the second constraint in equation form

$$15X_1 - 5X_2 = 30$$

we have $X_2 = -6$, when $X_1 = 0$; and $X_1 = 2$, when $X_2 = 0$.

Now, plot constraints 1 and 2 as shown in Figure 2.5. In this figure, the solution space is denoted by A, B, C and D. One of the sides of this shaded region is not closed. This indicates the unbounded nature of the solution of the given problem. The objective function can be increased to infinity.

Example 2.24 **(Unbounded solution space with finite solution)** Solve the following LP problem graphically.

$$\text{Maximize } Z = 5X_1 - 2X_2$$

subject to

$$X_1 \le 2$$

$$-X_1 + 2X_2 \ge 4$$

$$X_1 \text{ and } X_2 \ge 0$$

Solution The introduction of the non-negative constraints ($X_1 \ge 0$ and $X_2 \ge 0$) will eliminate the second, third and fourth quadrants of the X_1X_2 plane as shown in Figure 2.6.

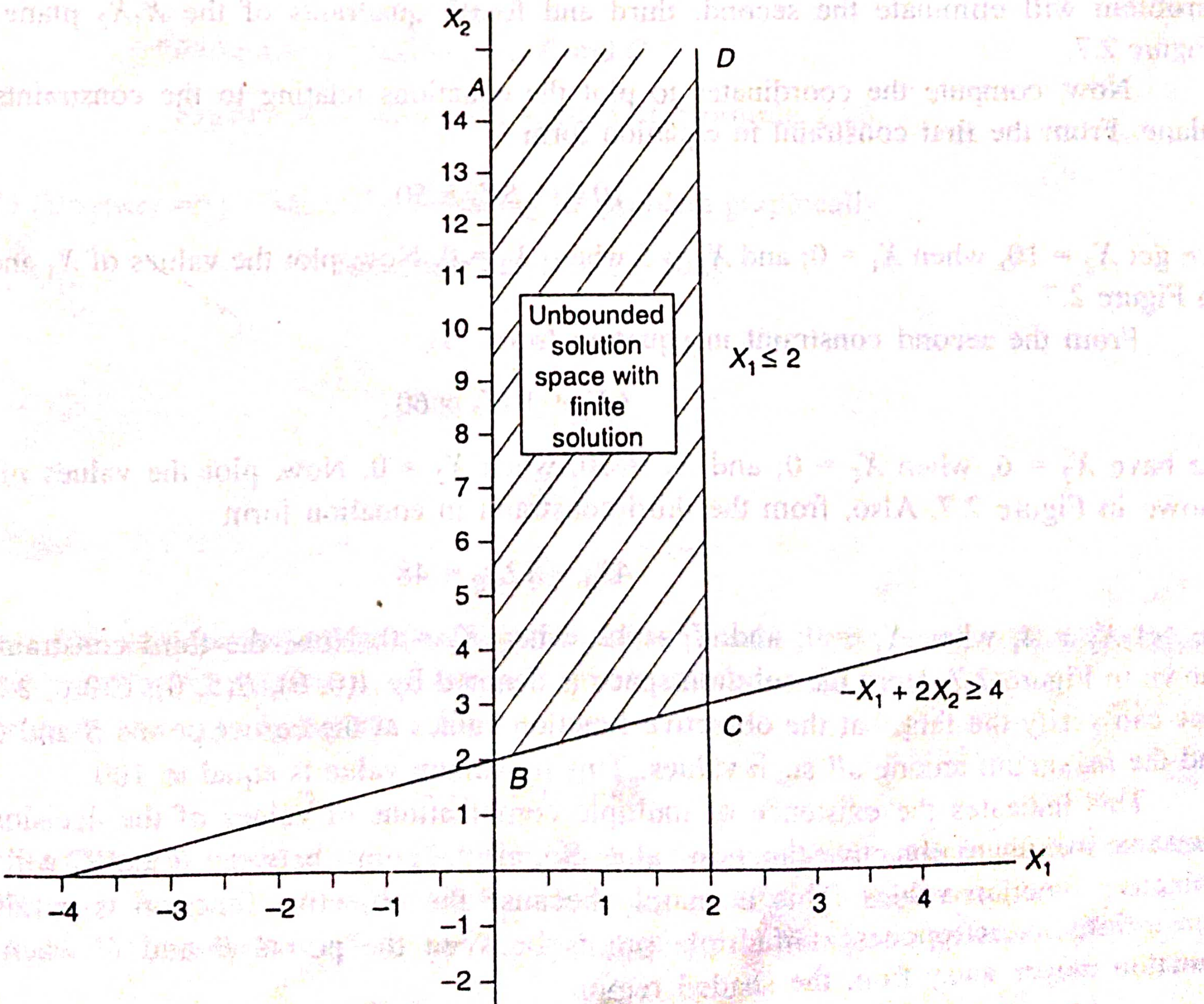

Figure 2.6 Unbounded solution space with finite solution of Example 2.24.

Now, from the first constraint, we get $X_1 = 2$, and from the second constraint in equation form

$$-X_1 + 2X_2 = 4$$

we get $X_2 = 2$, when $X_1 = 0$; and $X_1 = -4$, when $X_2 = 0$. We plot the constraints 1 and 2 as shown in Figure 2.6. In Figure 2.6, the solution space $ABCD$ is unbounded because the value of the variable X_2 is unlimited. Since the coefficient of the variable X_2 in the objective function is negative, any increase in X_2 will decrease the objective function value. But the value of the variable X_1 is limited. So, the solution space has a feasible and optimal solution at C. The corresponding results are:

$$X_1 = 2, X_2 = 3 \quad \text{and} \quad Z(\text{optimum}) = 4$$

Example 2.25 **(Alternate optima/multiple optimum solution)** Solve the following LP problem using graphical method:

$$\text{Maximize } Z = 20X_1 + 10X_2$$

subject to

$$10X_1 + 5X_2 \leq 50$$
$$6X_1 + 10X_2 \leq 60$$
$$4X_1 + 12X_2 \leq 48$$
$$X_1 \text{ and } X_2 \geq 0$$

Solution The introduction of the non-negative constraints ($X_1 \geq 0$ and $X_2 \geq 0$) of the given problem will eliminate the second, third and fourth quadrants of the X_1X_2 plane as shown in Figure 2.7.

Now, compute the coordinates to plot the equations relating to the constraints on the X_1X_2 plane. From the first constraint in equation form

$$10X_1 + 5X_2 = 50$$

we get $X_2 = 10$, when $X_1 = 0$; and $X_1 = 5$ when $X_2 = 0$. Now, plot the values of X_1 and X_2 as shown in Figure 2.7.

From the second constraint in equation form

$$6X_1 + 10X_2 = 60$$

we have $X_2 = 6$, when $X_1 = 0$; and $X_1 = 10$, when $X_2 = 0$. Now, plot the values of X_1 and X_2 as shown in Figure 2.7. Also, from the third constraint in equation form

$$4X_1 + 12X_2 = 48$$

we get $X_2 = 4$, when $X_1 = 0$; and $X_1 = 12$, when $X_2 = 0$. Now, the third constraint is plotted as shown in Figure 2.7. Here the solution space is denoted by $A(0, 0)$, $B(5, 0)$, $C(3.6, 2.8)$ and $D(0, 4)$. One can verify the fact that the objective function values at the corner points B and C are the same and the maximum among all such values. This maximum value is equal to 100.

This indicates the existence of multiple combinations of values of the decision variables for the same maximum objective function value. So, all the points between B and C will have the same objective function values. This is mainly because the objective function is parallel to the first constraint. So, it touches at multiple points between the points B and C when the objective function moves away from the shaded region.

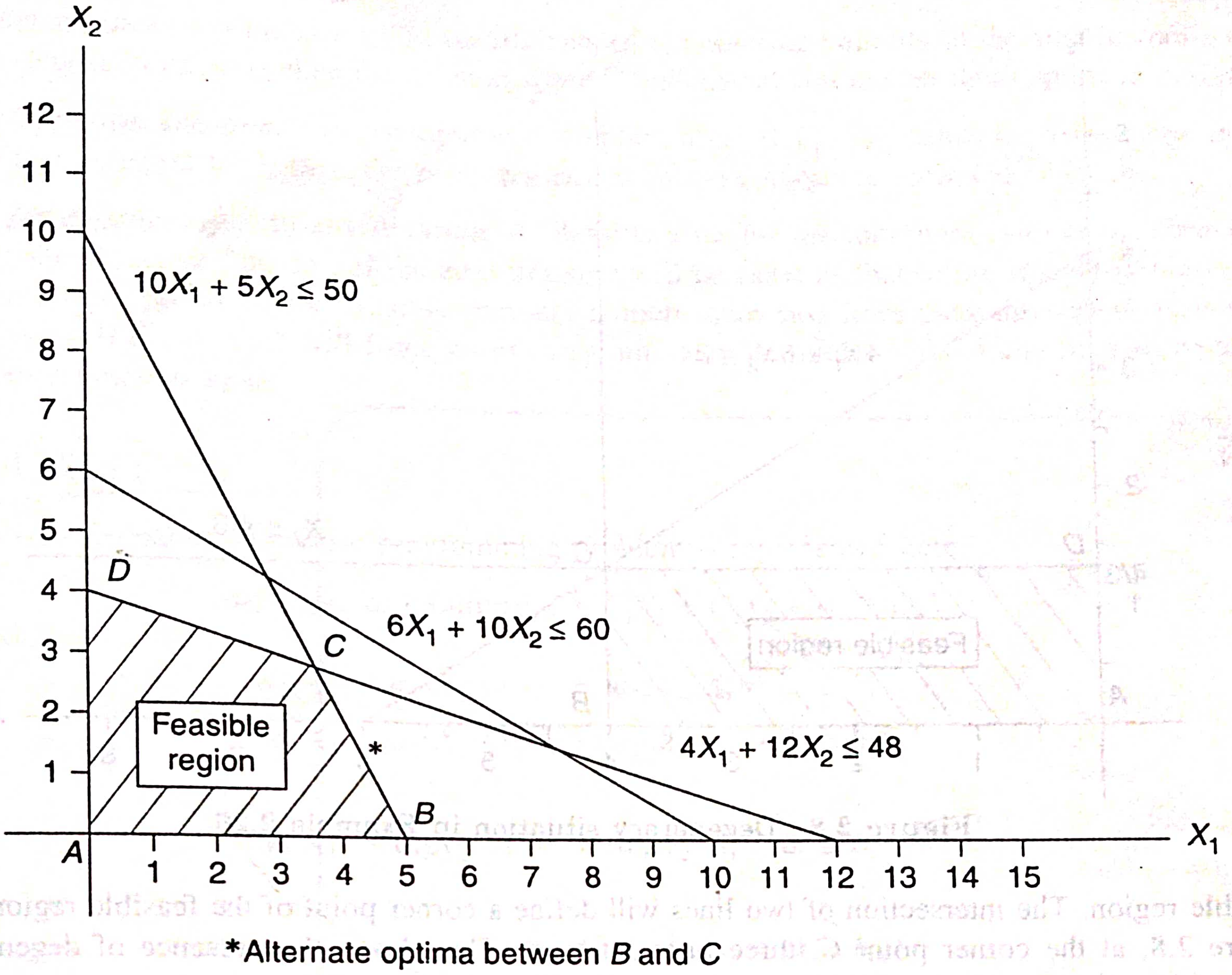

Figure 2.7 Alternate optima of Example 2.25.

***Example 2.26* (Degeneracy)** Solve the following LP problem graphically.

$$\text{Maximize } Z = 100X_1 + 50X_2$$

subject to

$$4X_1 + 6X_2 \le 24$$

$$X_1 \le 4$$

$$X_2 \le \frac{4}{3}$$

$$X_1, X_2 \ge 0$$

Solution The introduction of the non-negative constraints ($X_1 \ge 0$ and $X_2 \ge 0$) will eliminate the second, third and fourth quadrants of the X_1X_2 plane, as shown in Figure 2.8.

From the first constraint in equation form

$$4X_1 + 6X_2 = 24$$

we have $X_2 = 4$, when $X_1 = 0$; and $X_1 = 6$, when $X_2 = 0$. Now, plot the constraint 1. The second constraint in equation form $X_1 = 4$ helps plot the constraint 2. Also the third constraint in equation form $X_2 = 4/3$, plots the constraint 3 as shown in Figure 2.8. The closed polygon $ABCD$ is the

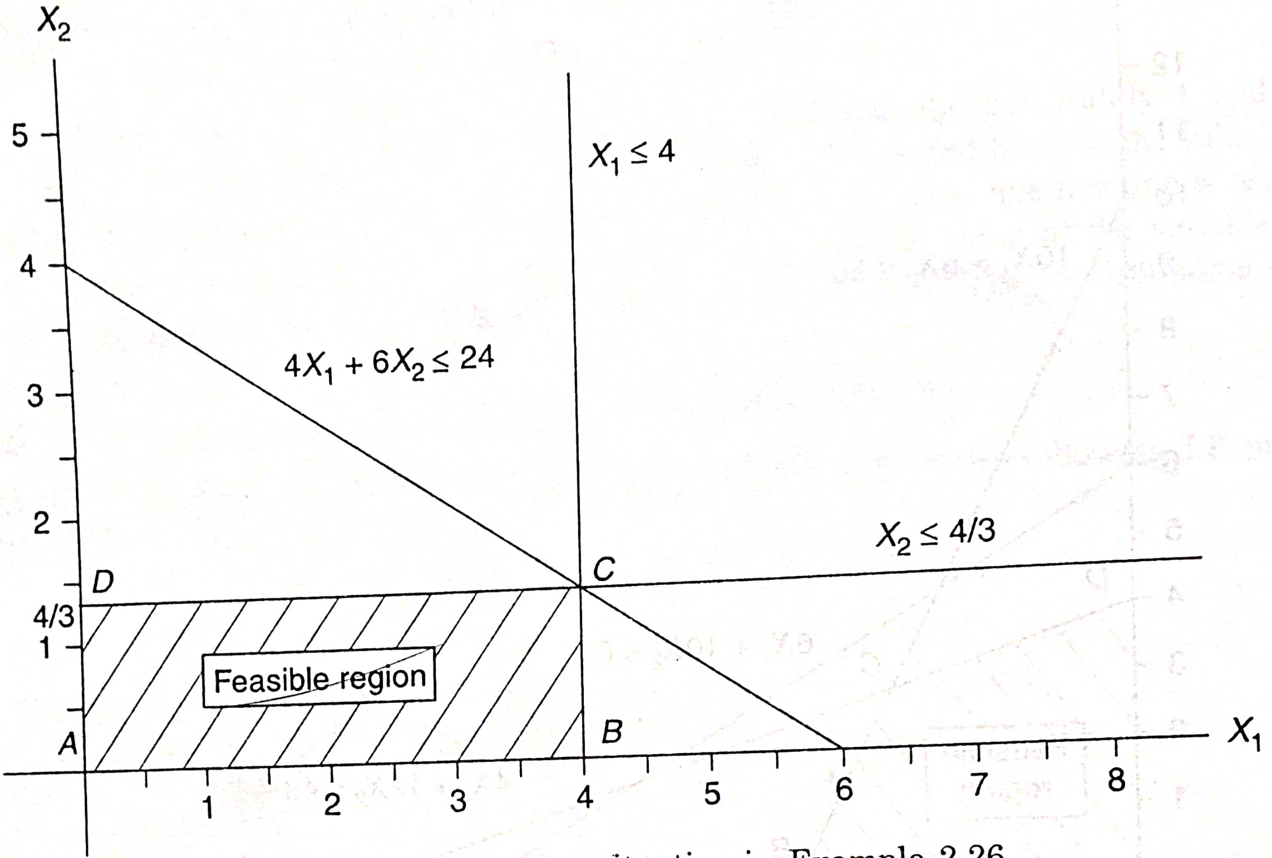

Figure 2.8 Degeneracy situation in Example 2.26.

feasible region. The intersection of two lines will define a corner point of the feasible region. But in Figure 2.8, at the corner point C, three lines intersect. This shows the presence of degeneracy in the problem.

The objective function value at each of the corner points of the closed polygon is computed by substituting its coordinates in the objective function. The coordinates of the corner point C are (4, 4/3). The values of Z for the different points are:

$$Z(A) = 0$$

$$Z(B) = 400$$

$$Z(C) = \frac{1400}{3} = 466.67$$

$$Z(D) = 66.67$$

The Z value is maximum for the corner point C. Hence, the corresponding solution is presented as follows:

$$X_1^* = 4, \qquad X_2^* = \frac{4}{3} = 1.33, \qquad Z(\text{optimum}) = 466.67$$