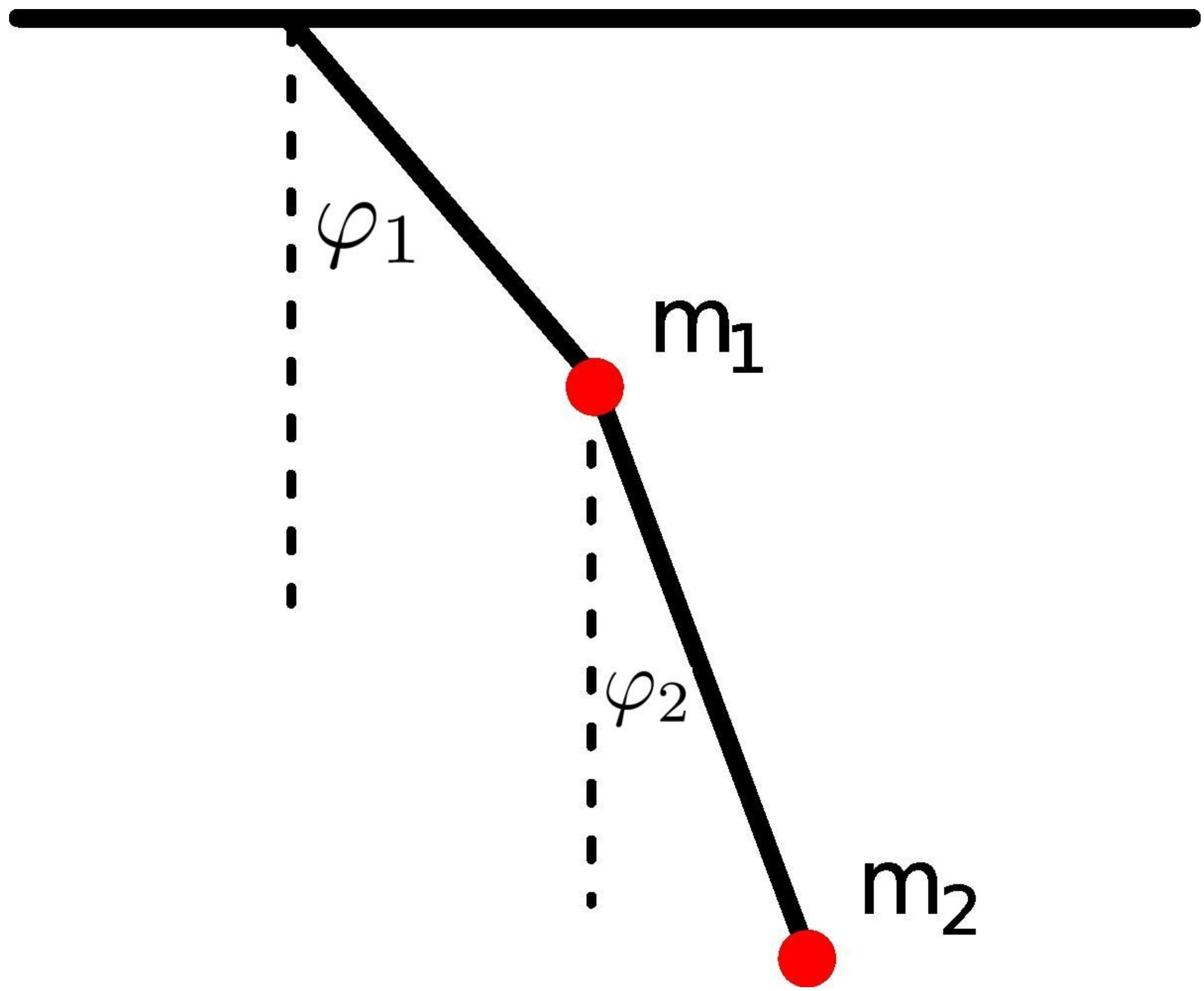
Assignment-8

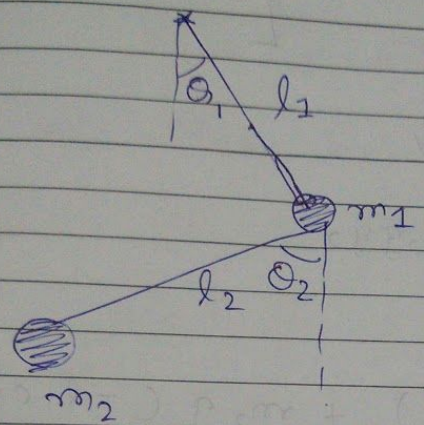
Problem statement
A double pendulum consists of one pendulum attached to another. Double pendula are an example of a simple physical system which can exhibit chaotic behavior. Consider a double bob pendulum with masses m1 and m2 attached by rigid massless wires of lengths L1 and L2. Further, let the angles the two wires make with the vertical be denoted theta1 and theta2. Finally, let gravity be given by *g*.
Derive lagrangian equations of motion,and using that simulate this problem. Show chaotic behaviour using plots, their phase space diagrams.

Assumptions:
We neglect air drag here.
There is no friction between hinge and rigid massless wire.
Gravity (g) remains constant.
Rigid massless wires are attached with point mass objects.



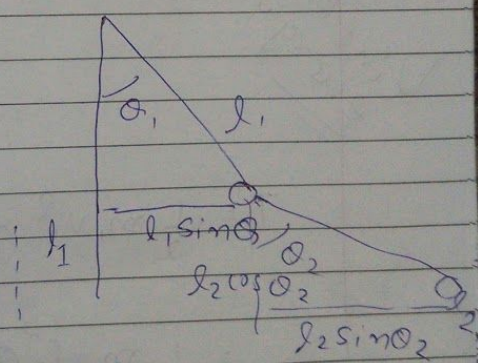
Analytical Solution:

Double Pendulum :-



movement of m_1 in $x \rightarrow x_1$
movement of m_1 in $y \rightarrow y_1$
movement of m_2 in $x \rightarrow x_2$
movement of m_2 in $y \rightarrow y_2$

$x_1 = l_1 \sin\theta_1$
$y_1 = -l_1 \cos\theta_1$



$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$

$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$

potential energy $\Rightarrow$ of pendula

$V = m_1 g y_1 + m_2 g y_2$

Lagrangian is L

$$L = T - V$$

Potential energy

$$V = m_1 g y_1 + m_2 g y_2$$

$$= m_1 g(-l_1 \cos\theta_1) + m_2 g(-l_1 \cos\theta_1 - l_2 \cos\theta_2)$$

$$= (-l_1 \cos\theta_1)(m_1 + m_2) g + m_2 g(-l_2 \cos\theta_2)$$

Kinetic energy

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [\quad - \quad]$$

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$x_1 = l_1 \sin\theta_1$

$\Rightarrow v_{1x} = l_1 \cos\theta_1 \, \dot{\theta}_1$

$y_1 = -l_1 \cos\theta_1$

$\Rightarrow v_{1y} = +l_1 \sin\theta_1 \, \dot{\theta}_1$

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$

$\Rightarrow \cancel{v_2 = l_1 \sin\theta_1}$

$v_{2x} = l_1 \cos\theta_1 \, \dot{\theta}_1 + l_2 (-\cos\theta_2)\, \dot{\theta}_2$

$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$

$v_{y2} = +l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2$

putting in eqn 1

$$T = K = \frac{1}{2} m_1 \left[l_1^2 \cos^2\theta_1 + l_2^2 \sin^2\theta_2 \right] \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1 \cos\theta_1 \dot{\theta}_1 - l_2 \cos\theta_2 \dot{\theta}_2 \right]^2 + \frac{1}{2} m_2 \left[+ l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2 \right]^2 \quad (1)$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_2^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right].$$

using Lagrangian - Euler Method

$L = T - V$

$= \frac{1}{2}(m_1+m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2$

$+ \cancel{\frac{1}{2}} m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos(\theta_1 - \theta_2))$

$+ (m_1+m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$

for θ_1

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \quad \text{--- (2)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

$$= (m_2 + m_1) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \quad \text{--- (3)}$$

$$\frac{\partial L}{\partial \theta_1} = -l_1 m_1 g \sin\theta_1 - l_1 g m_2 \sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \quad \text{--- (4)}$$

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for θ_2

$$\frac{\partial L}{\partial \dot\theta_2} = m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1 \cos(\theta_1 - \theta_2) \quad — (5)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_1 \sin(\theta_1 - \theta_2)(\dot\theta_1 - \dot\theta_2) \quad — (6)$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - l_2 m_2 g \sin\theta_2 \quad — (7)$$

using (2) and (3)

$$(m_1 + m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot\theta_2 \sin(\theta_1 - \theta_2) + l_1 g (m_1 + m_2) \sin\theta_1 = 0 \quad — (8)$$

using (5) and (6)

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m_2 (l_1)(l_2) \dot\theta_1^2 \sin(\theta_1 - \theta_2) + l_2 m_2 g \sin\theta_2 = 0 \quad — (9)$$

eq^n (8) and (9) are main eq^n that can be ~~made~~ transform into matlab code by simple tweeking.

after solving (8) and (9) we get :->

$$\cancel{\ddot{\theta}_1 = -m_2 l_2 \dot{\theta}_2^2 \sin}$$

$$\ddot{\theta}_1 = \frac{\left(-m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) - g(m_1+m_2)\sin(\theta_1)\right)(m_2 l_2) - [m_2 l_2 \cos(\theta_1-\theta_2)][m_2 l_1 (\dot{\theta}_2)^2 \sin(\theta_1 - \; - m_2 g \sin(\theta_2)]}{(m_1+m_2) l_1 (m_2)(l_2) - m_2 l_1 \cos(\theta_1-\theta_2) \times m_2 l_2 \cos(\theta_1-\theta_2)}$$

(a) $\dot{\theta}_1 = \dfrac{d\theta_1}{dt}$

$\dot{\theta}_2 = \dfrac{d\theta_2}{dt}$

Code:

```
clear;
close all;
m1 = 2;
m2 = 1;
L1 = 1;
L2 = 2;
g=9.8;
time=20;
%options = odeset('RelTol' , 1.0e-6);
[t,y] = ode45(@rhs , [0 time] , [1.57; 0.0 ; 3.14 ; 0.0] );
V = -(m1+m2)*g*L1*cos(y(:,1))-m2*g*L2*cos(y(:,3));

x1=y(:,1);    % theta 1
x2=y(:,3);    % theta 2
v1=y(:,2);    % theta dot 1
v2=y(:,4);    % theta dot 2

len=size(x1);
T=zeros(len);
for i=1 : len
   T(i)= 0.5*m1*L1*L1*v1(i)^2
+0.5*m2*(L1*L1*v1(i)^2+L2*L2*v2(i)^2+2*L1*L2*v1(i)*v2(i)*cos(x1(i)-x2(i))
);
end
```

```
E=T+V;
%plot(t,E);

%plotting the graphs:::::

%position of pendulum1 in x
px1=L1*sin(x1);

%position of pendulum1 in y
py1=-L1 * cos(x1);

%position of pendulum2 in x
px2=L1*sin(x1) +L2*sin(x2);

%position of pendulum2 in y
py2=-L1*cos(x1) -L2*cos(x2);

%velocity of pendulum1 in x
vx1=L1*cos(x1).*v1;

%velocity of pendulum1 in y
vy1=L1 * sin(x1) .* v1 ;

%velocity of pendulum2 in x
vx2=L1*cos(x1).* v1 -L2*cos(x2) .* v2;

%velocity of pendulum2 in y
vy2=L1*sin(x1).*v1 + L2*sin(x2).*v2;

% graph of position of pendulum 1
figure
plot( px1 , py1 ) ;
title('position of pendulum 1');
```

```
xlabel( 'abcissa');
ylabel( 'ordinate');

% graph of position of pendulum 2
figure
plot( px2 , py2 ) ;
title('position of pendulum 2 ');
xlabel( 'abcissa');
ylabel( 'ordinate');

% graph of velocity of pendulum 1
figure
plot( vx1 , vy1) ;
title('graph of velocity for 1st pendulum');
xlabel( 'velocity in x ');
ylabel( 'velocity in y');

% graph of velocity of pendulum 2
figure
plot( vx2 , vy2) ;
title('graph of velocity for 2nd pendulum');
xlabel( 'velocity in x ');
ylabel( 'velocity in y');


%phasespace diagrams for pendulum 1
figure
plot(x1,v1)
title('phasespace for 1st pendulum');
xlabel( 'angular position');
ylabel( 'angular velocity');

%phasespace diagrams for pendulum 2
```

```
figure
plot(x2,v2)
title('phasespace for 2nd pendulum');
xlabel( 'angular position');
ylabel( 'angular velocity');


% phase space diagrams for pendulum 1
figure
quiver (px1,px2,vy1,vy2)
title('quiver for 1st pendulum');
xlabel( 'position');
ylabel( 'velocity');
figure
plot( t , x1 ) ;



%angular displacement for pendulum 1
figure
plot( t , x1 ) ;
title('angular displacement for pendulum 1');
xlabel( 'time');
ylabel( 'angular displacement');

%angular displacement for pendulum 2
figure
plot( t , x2 ) ;
title('angular displacement for pendulum 2');
xlabel( 'time');
ylabel( 'angular displacement');
```

```
%angular velocity for pendulum 1
figure
plot( t , v1 ) ;
title('angular velocity for pendulum 1');
xlabel( 'time');
ylabel( 'angular velocity');

%angular velocity for pendulum 2
figure
plot( t , v2 ) ;
title('angular velocity for pendulum 2');
xlabel( 'time');
ylabel( 'angular velocity');


% Kinetic energy of pendula system
figure
plot( t , T ) ;
title('Kinetic energy of pendula system');
xlabel( 'time');
ylabel( 'Kinetic energy');


% Potential energy of pendula system
figure
plot( t , V ) ;
title('Potential energy of pendula system');
xlabel( 'time');
ylabel( 'Potential energy');

E=zeros(size(t),1);
E=E+20;
% total energy of pendula system
```

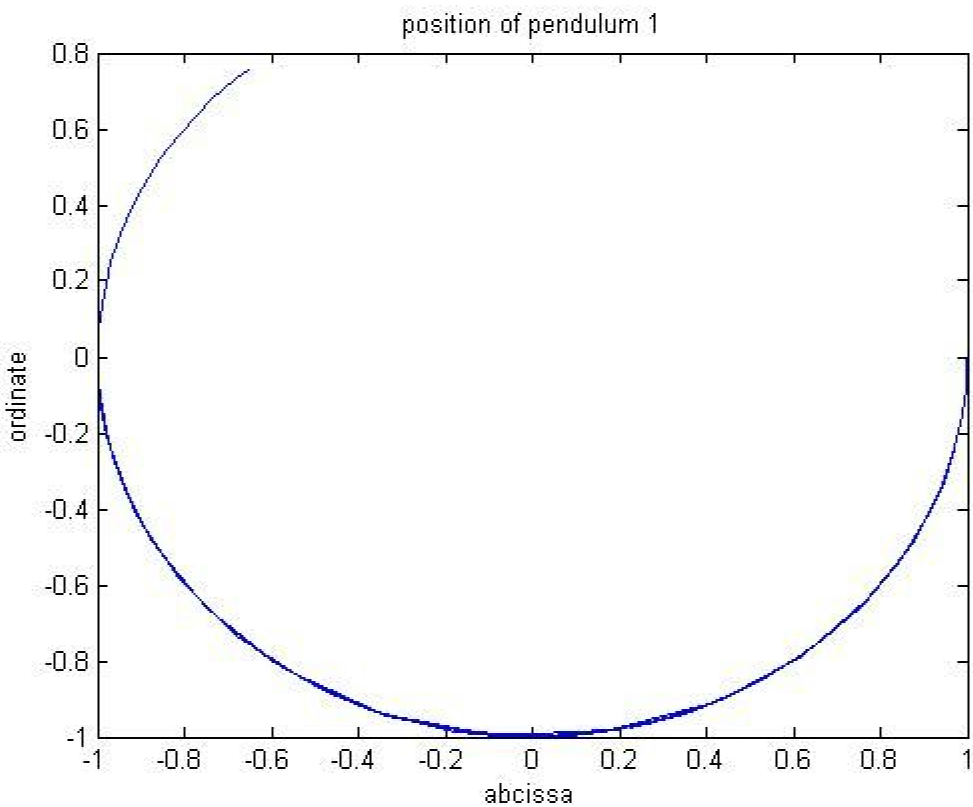
```
figure
plot( t , E ) ;
title('total energy of pendula system');
xlabel( 'time');
ylabel( 'Total energy');


% all energy in one plot
figure
plot(t,T,'r')
hold on
plot(t,V,'b')
hold on
plot(t,E,'k')
title('All energy in one plot');
xlabel( 'time');
ylabel( 'energy in jule');
ylim([-100 100])


min ( E )
max ( E )
( max ( E)-min ( E ) ) / max ( E )
```

RHS:

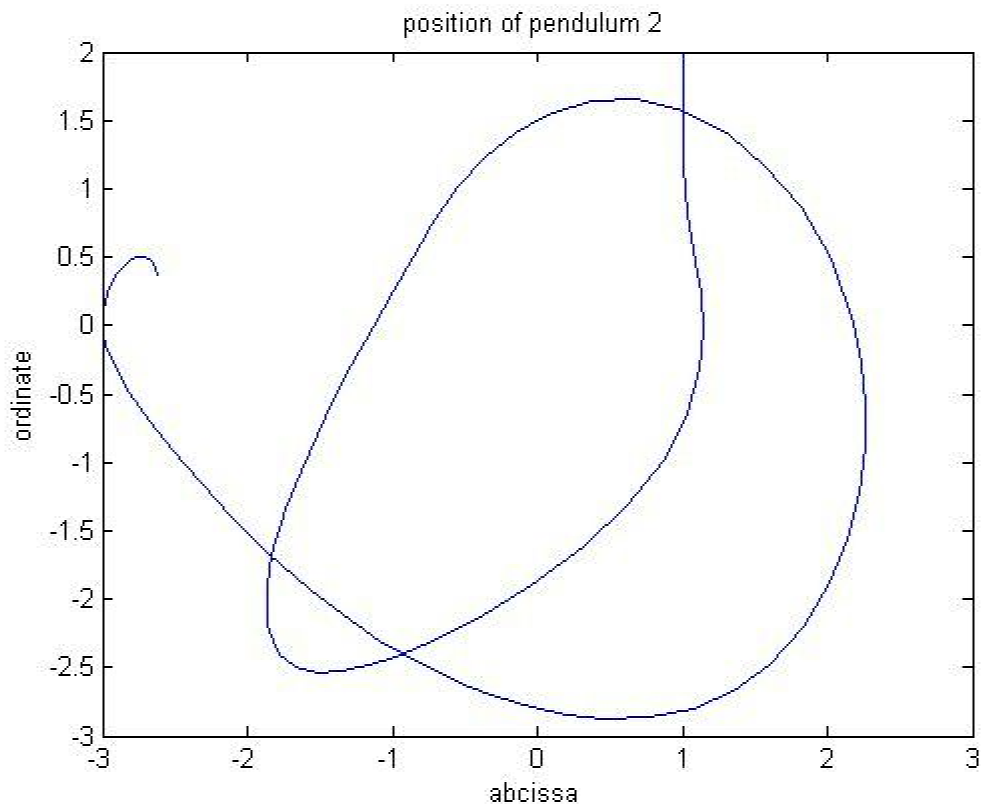
```
function [ F ] =rhs( t , y )

F = zeros( 4 , 1 ) ;

m1 = 2 ;
m2 = 1 ;
```

```
L1 = 1 ;
L2 = 2 ;
g = 9.8 ;



F( 1 ) = y ( 2 ) ;
F ( 3 ) = y ( 4 ) ;
F ( 2 ) = ( (-m2*L2*y ( 4 )*y ( 4 )*sin( y ( 1 )-y ( 3 ) )-g*( m1+m2 )*sin( y ( 1 )
))*(m2*L2 )-(m2*L2*cos( y ( 1 )-y ( 3 ) ))*( m2*L1*y ( 2 )*y ( 2 )*sin( y ( 1 )-y
( 3 ) )-m2*g*sin( y ( 3 ) )) ) / ( (( m1+m2 )*L1)*(m2*L2 )-(m2*L1*cos( y ( 1 )-y
( 3 ) ) )*(m2*L2*cos( y ( 1 )-y ( 3 ) )) ) ;
F ( 4 ) = ( (( m1+m2 )*L1)*( m2*L1*y ( 2 )*y ( 2 )*sin( y ( 1 )-y ( 3 )
)-m2*g*sin( y ( 3 ) ))-(m2*L1*cos( y ( 1 )-y ( 3 ) ) )*(-m2*L2*y ( 4 )*y ( 4 )*sin(
y ( 1 )-y ( 3 ) )-g*( m1+m2 )*sin( y ( 1 ) )) ) / ( (( m1+m2 )*L1)*(m2*L2
)-(m2*L1*cos( y ( 1 )-y ( 3 ) ) )*(m2*L2*cos( y ( 1 )-y ( 3 ) )) ) ;


end
```

Graphs with explanation:

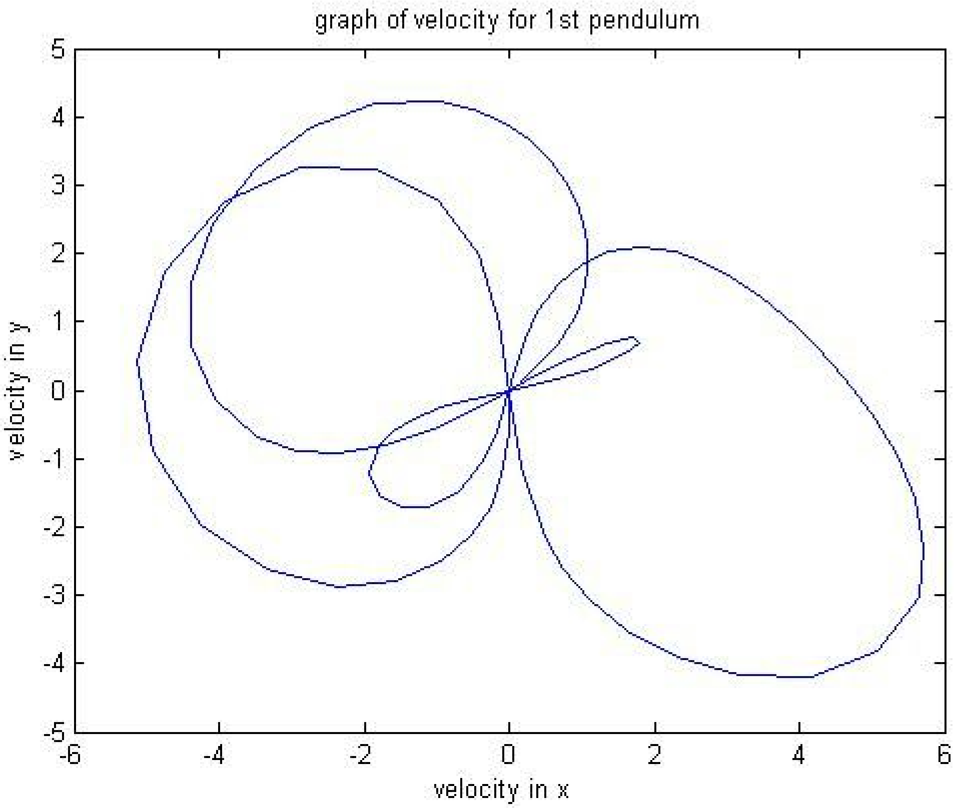


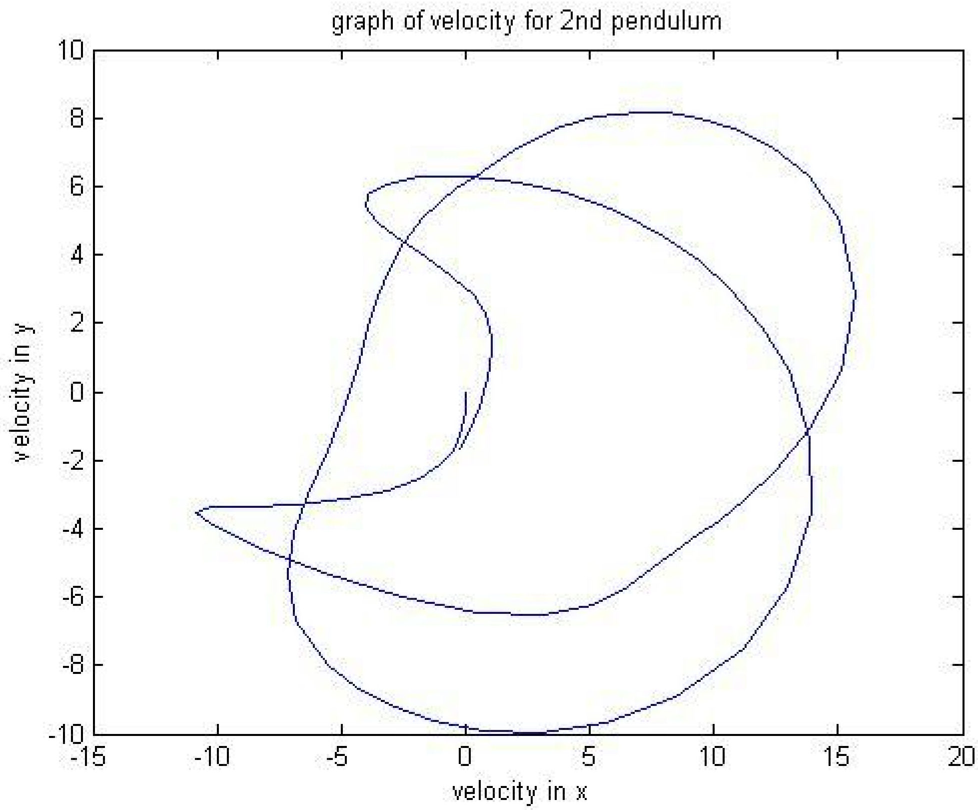
The above graph shows the trajectory of 1st pendulum. Here trajectory is circular as we consider rigid massless wire, as wire cannot change its shape, it can only move circular. Here, its motion is not restricted to 180 degree, if mass of other bob connected to it is more, i.e. it gives more force, the first can move above 180(go over the hinge).



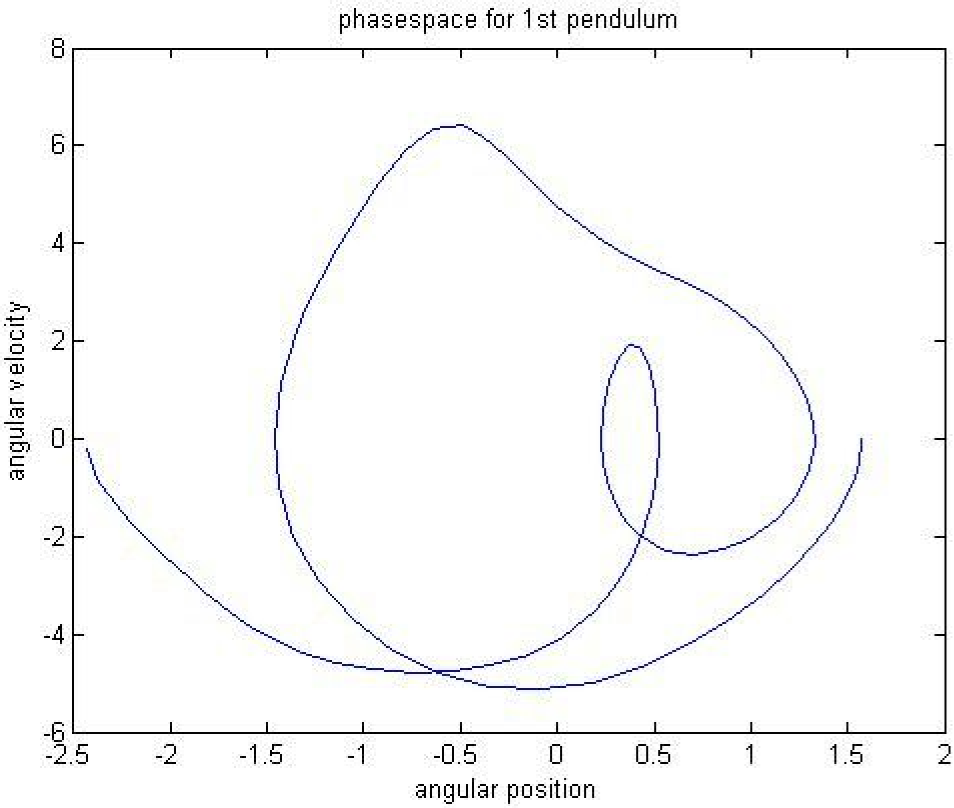
Here, initial position of the second bob with mass m2 starts at (1,2).
This seems a very random graph as the hinge of this pendulum changes continuously, so we cannot find any regularity or known behaviour in this graph. This is due to combined effect of both pendulums.

Below is given the velocity of each pendulum in x and y direction.

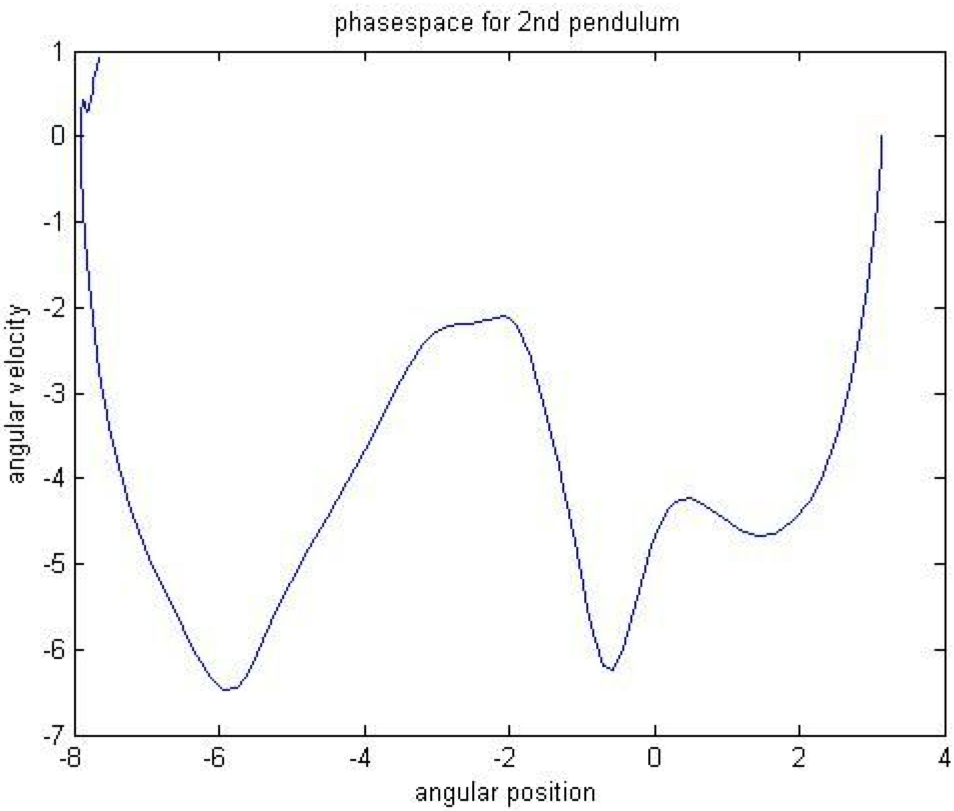




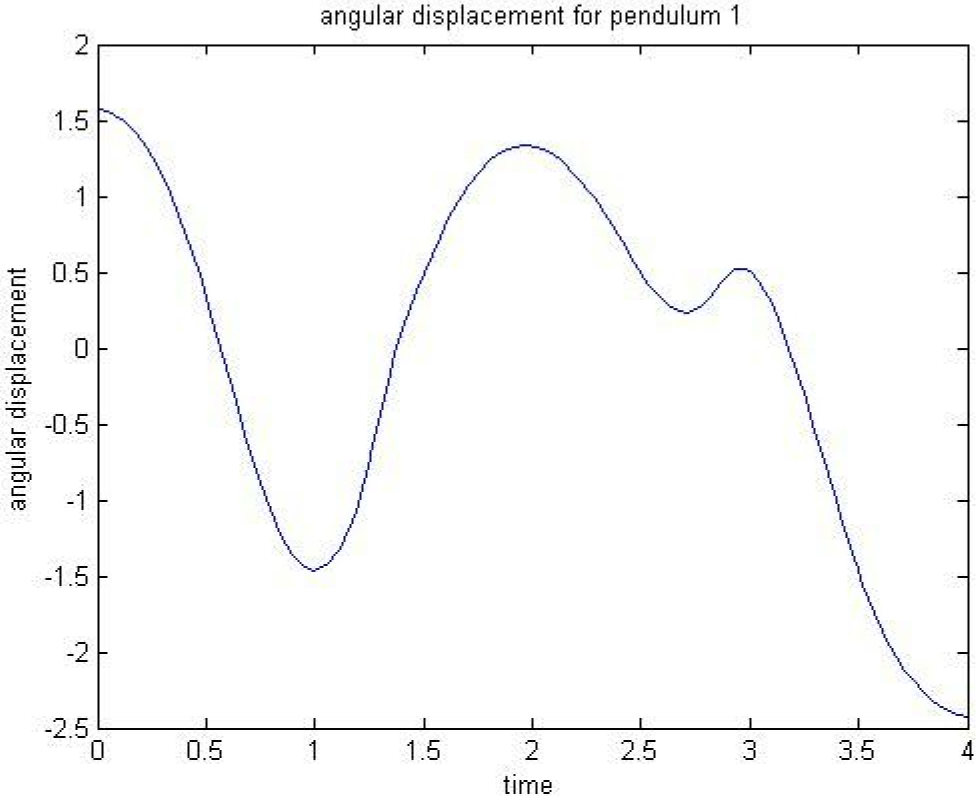
The velocity of the second pendulum is very chaotic because whenever there is a change of position in pendulum 1 , it gives jerk to pendulum 2 . which is highly Impulsive and Impulsive force are very hard to calculate. And change in momentum is equal to Impulse*dt integration.

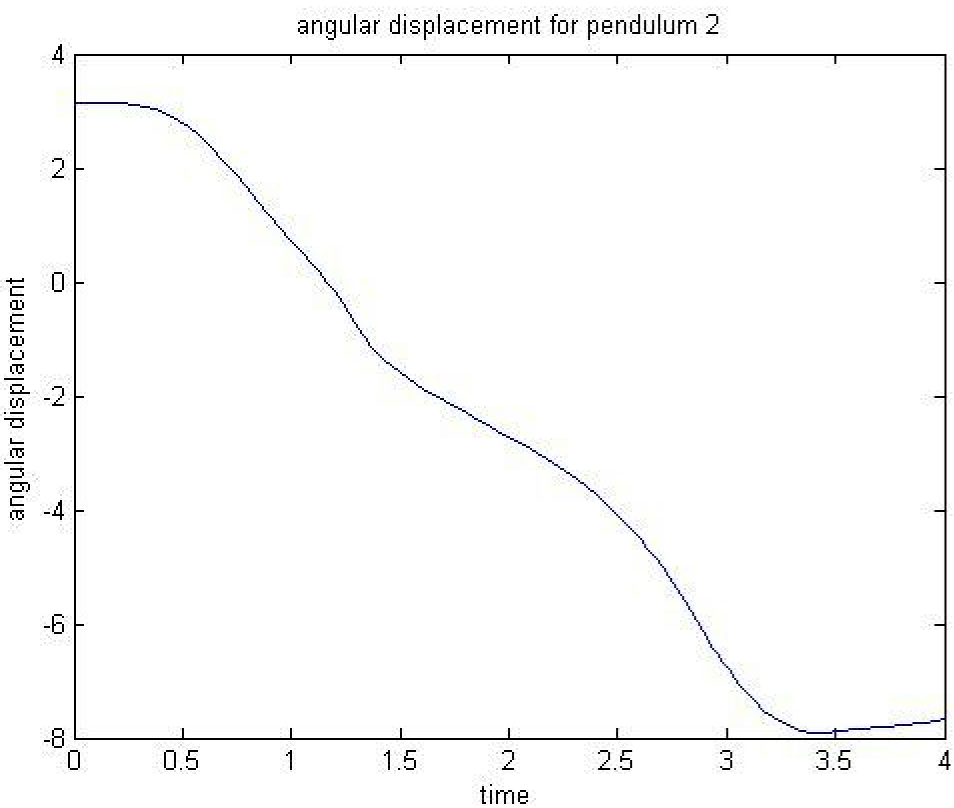


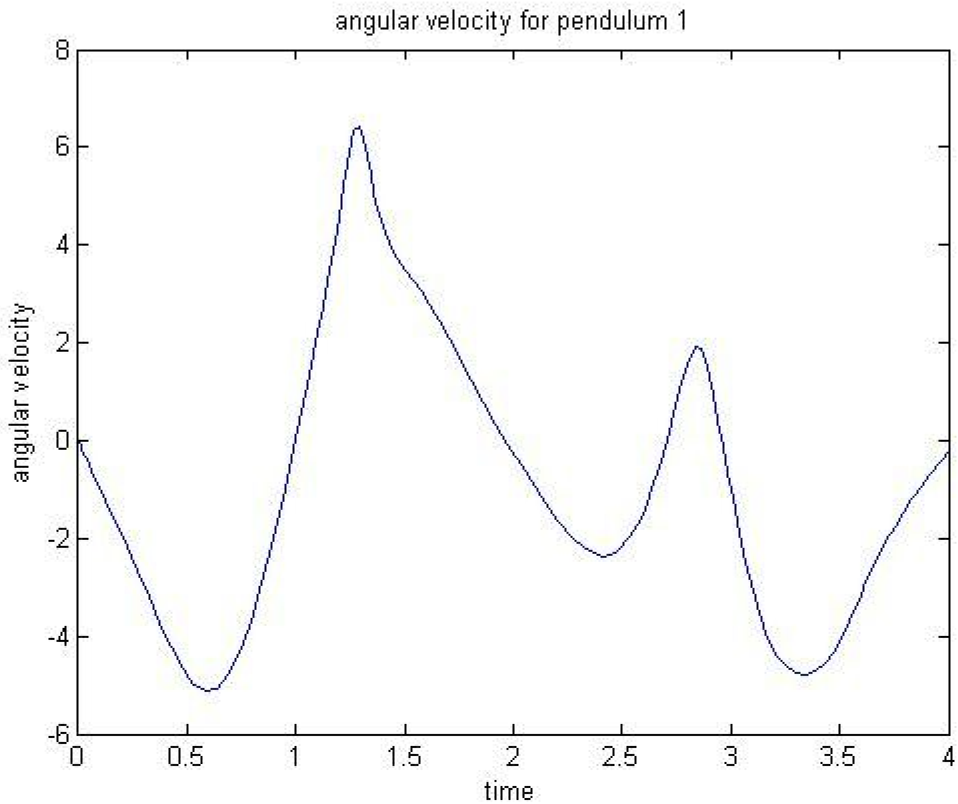
Here mass of pendulum is more than mass of pendulum 2. So It is more near to ideal behaviour.

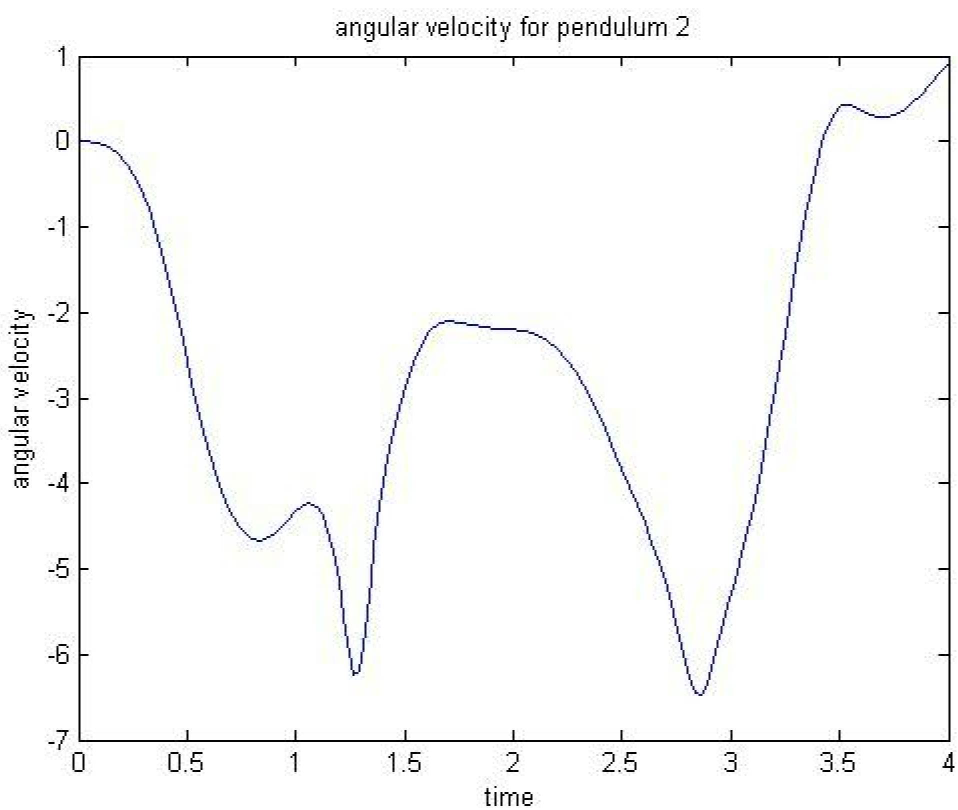


Here pendulum 2's hinge is not fixed so it's motion is very chaotic. We can't explain it's phasespace. Though its behaviour is very different from ideal behaviour.

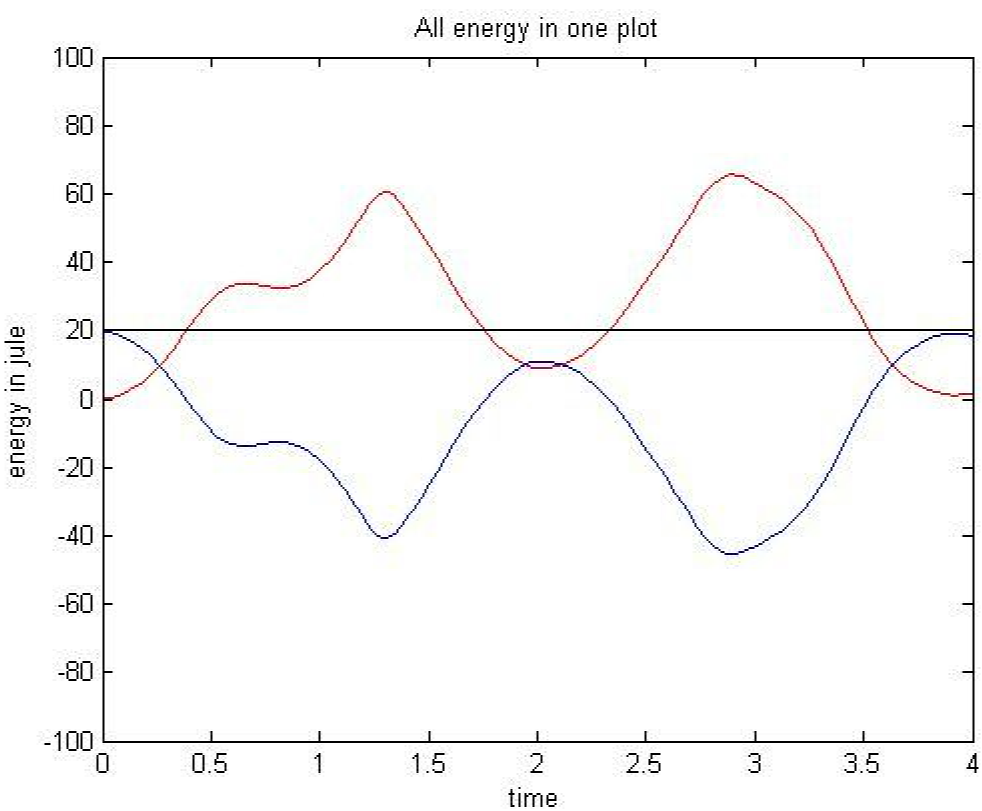

angular displacement for pendulum 1
angular displacement
time
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.5
1
1.5
2
2.5
3
3.5
4


angular displacement for pendulum 2
angular displacement
time
4
2
0
-2
-4
-6
-8
0
0.5
1
1.5
2
2.5
3
3.5
4


angular velocity for pendulum 1
angular velocity
time
8
6
4
2
0
-2
-4
-6
0
0.5
1
1.5
2
2.5
3
3.5
4


angular velocity for pendulum 2
angular velocity
time
1
0
-1
-2
-3
-4
-5
-6
-7
0
0.5
1
1.5
2
2.5
3
3.5
4

Energy plot:

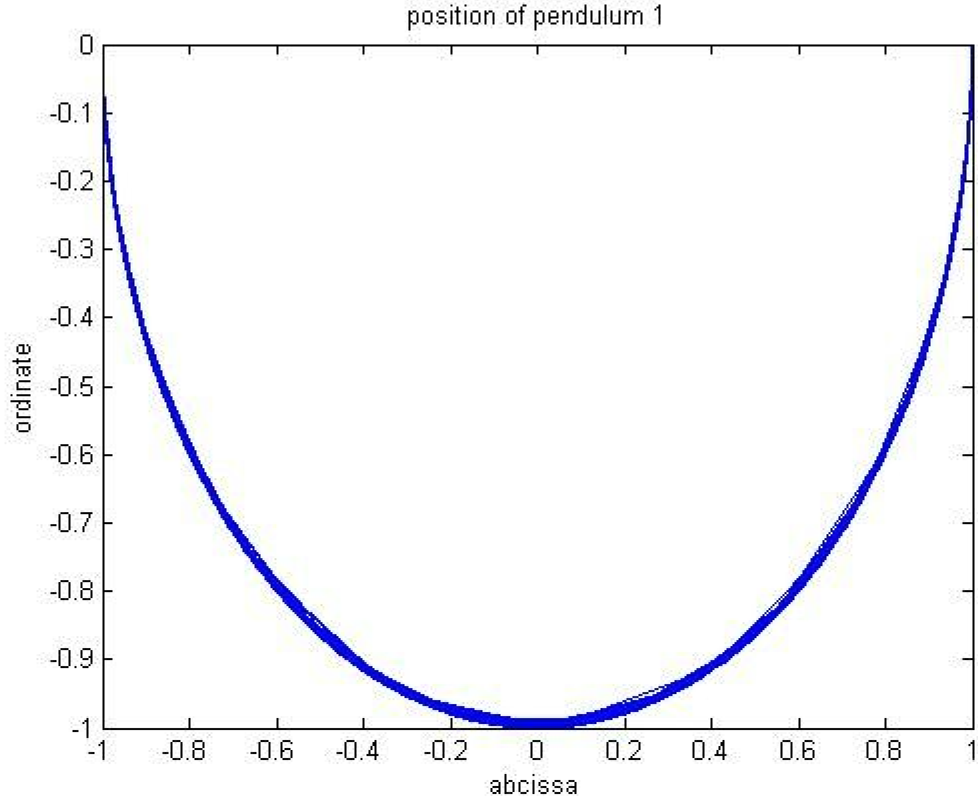


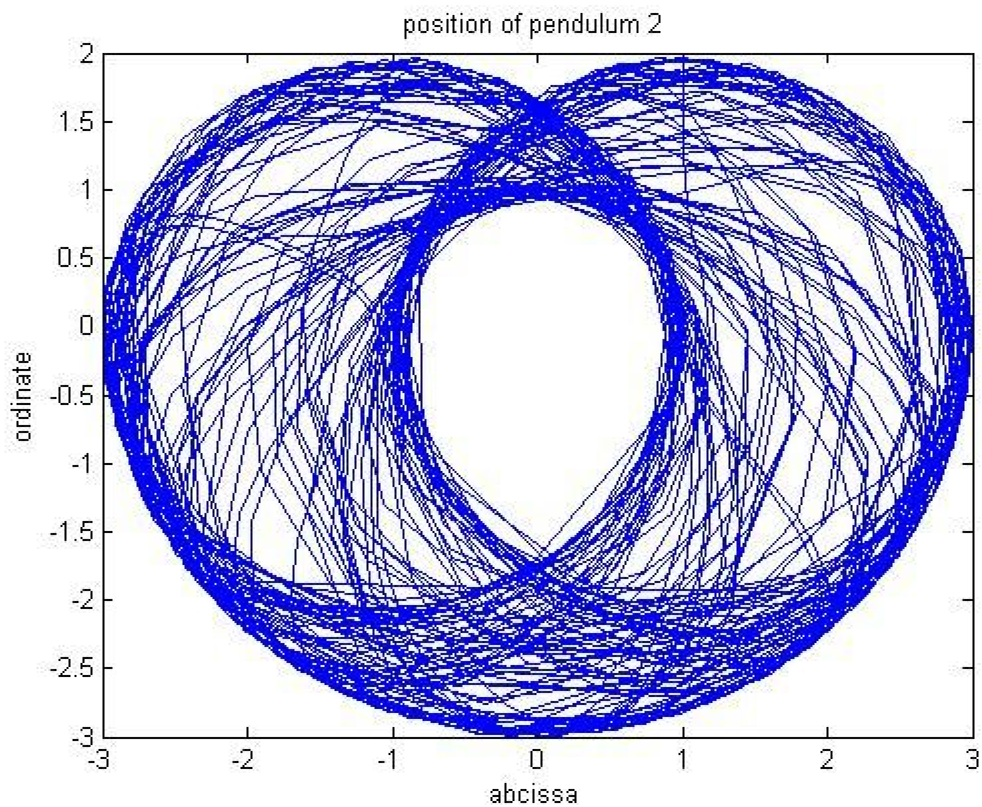
All energy in one plot
energy in jule
time
100
80
60
40
20
0
-20
-40
-60
-80
-100
0
0.5
1
1.5
2
2.5
3
3.5
4

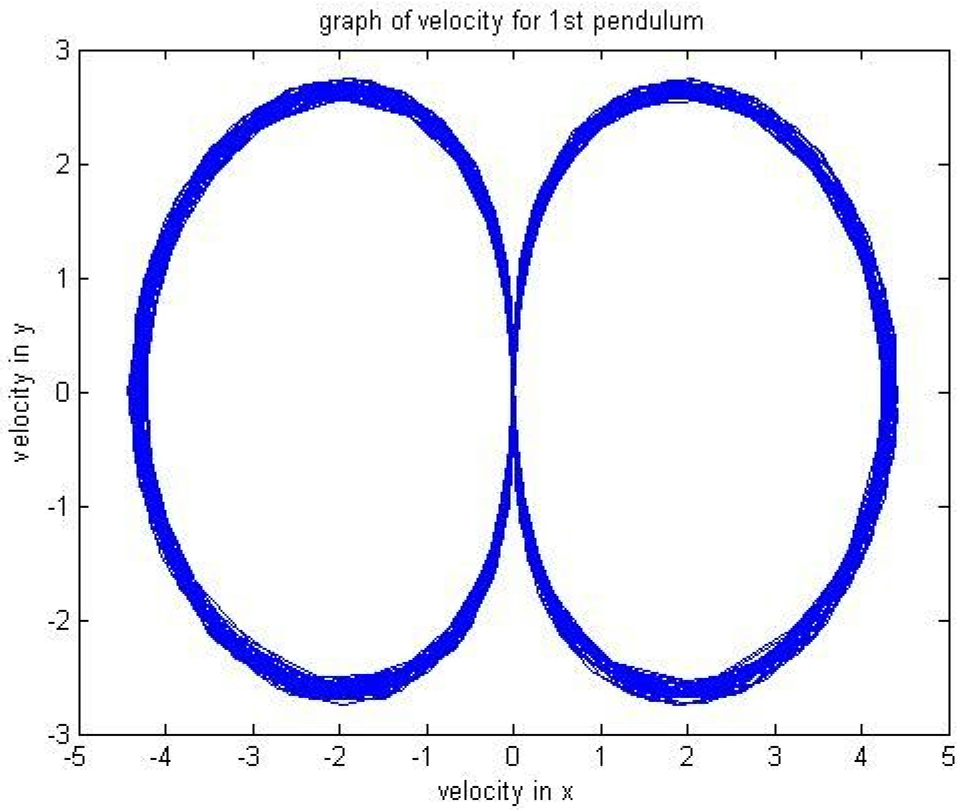
This graph shows energy conservation in this problem.
Red: Kinetic Energy
Blue: Potential Energy
The straight line shows the total energy of system.
As it is straight line,(constant value) we can say that energy is conserved.

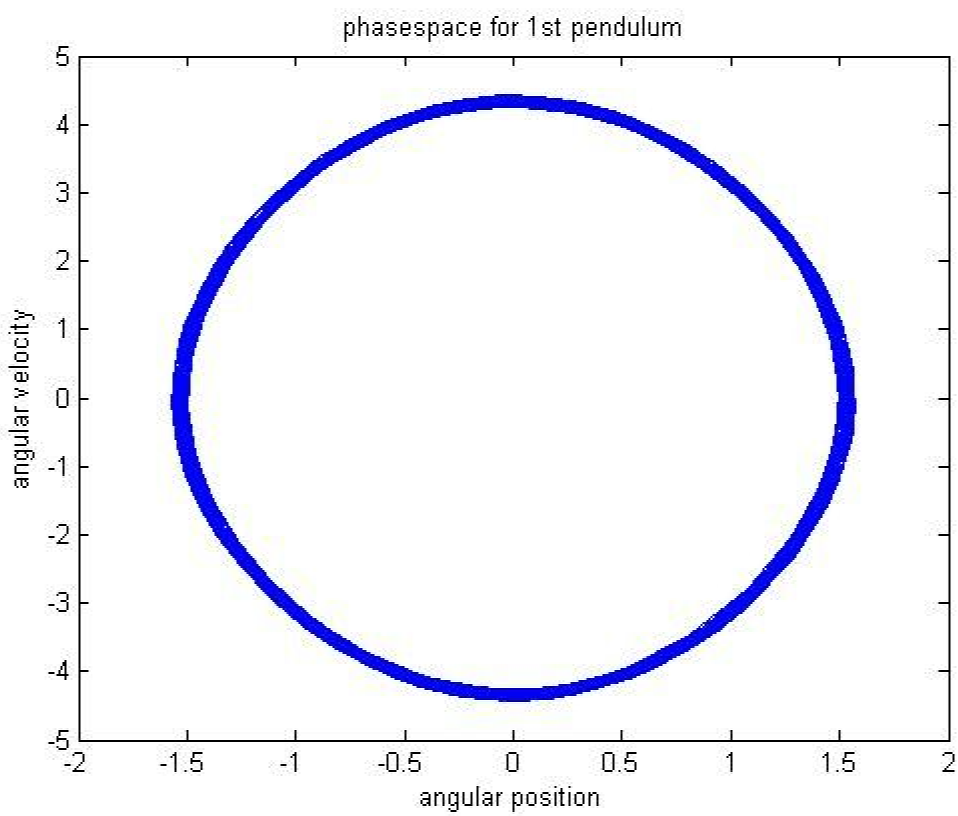
Graph: If mass of the second pendulum is so small in comparison with mass of first pendulum m1>>m2

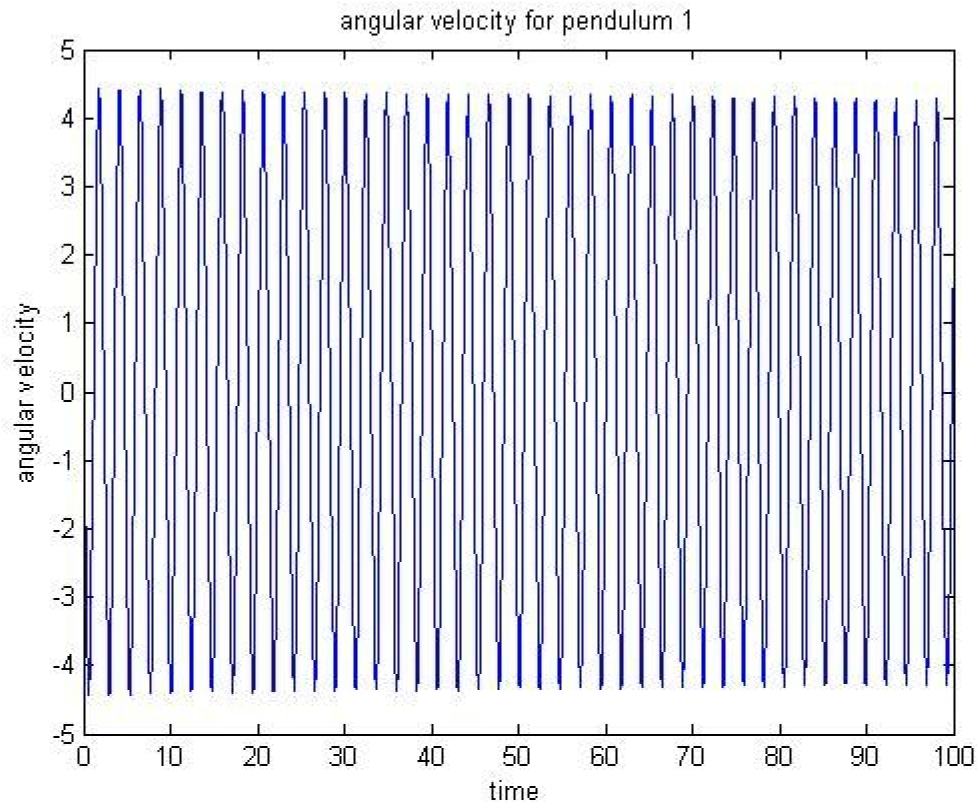
The behaviour of the mass m1 is very near to ideal behaviour because it is moving like there is no mass m2 whatsoever .

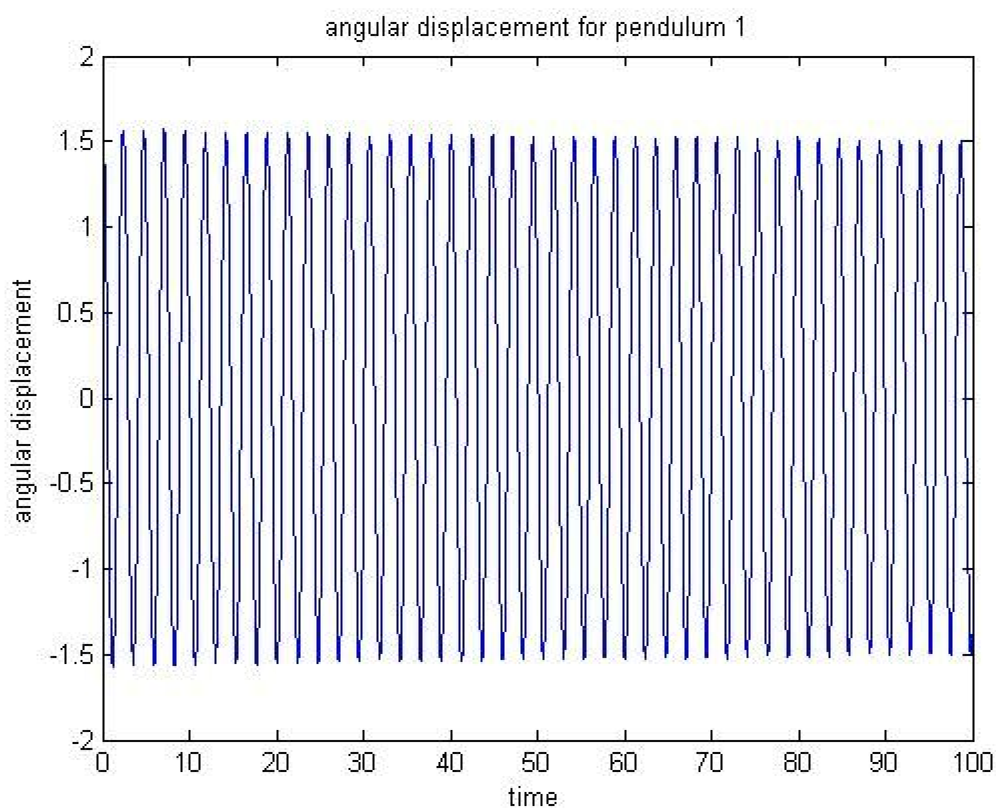

position of pendulum 1
ordinate
abcissa
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


position of pendulum 2
ordinate
abcissa
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3
-2
-1
0
1
2
3

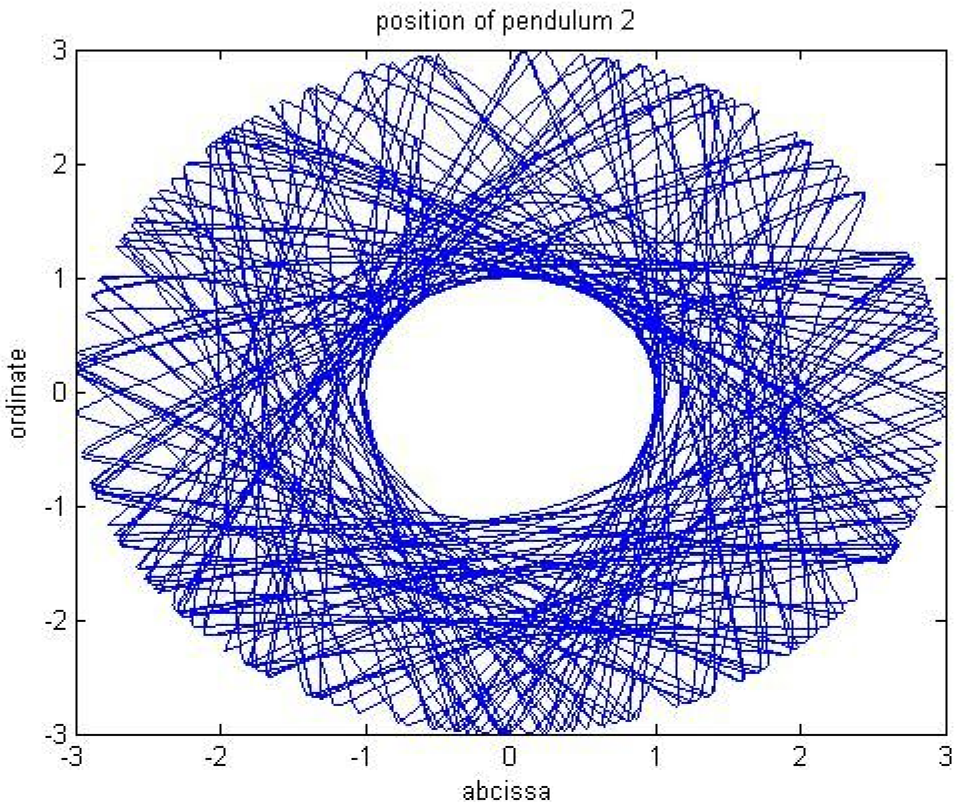

graph of velocity for 1st pendulum
velocity in y
velocity in x
3
2
1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5

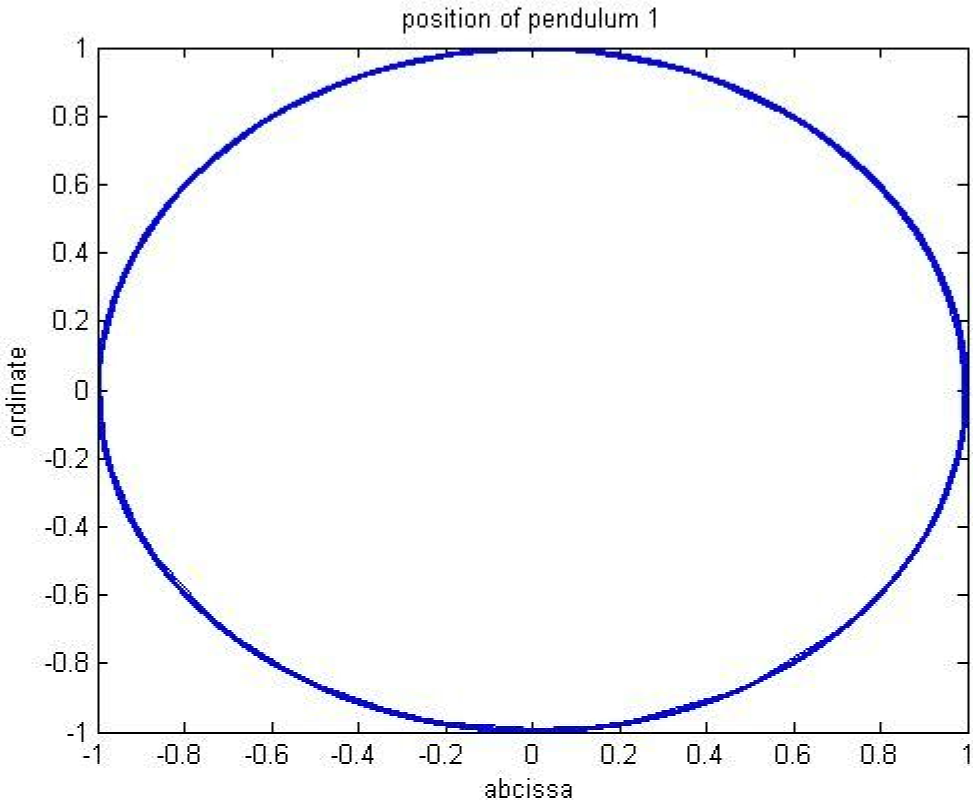

phasespace for 1st pendulum
angular velocity
angular position
5
4
3
2
1
0
-1
-2
-3
-4
-5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


angular velocity for pendulum 1
angular velocity
time
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
10
20
30
40
50
60
70
80
90
100


angular displacement for pendulum 1
angular displacement
time
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
10
20
30
40
50
60
70
80
90
100

Graph: If mass of the second pendulum is very large in comparison with mass of first pendulum m2>>m1




position of pendulum 1
ordinate
abcissa
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1