Linear Models for Regression

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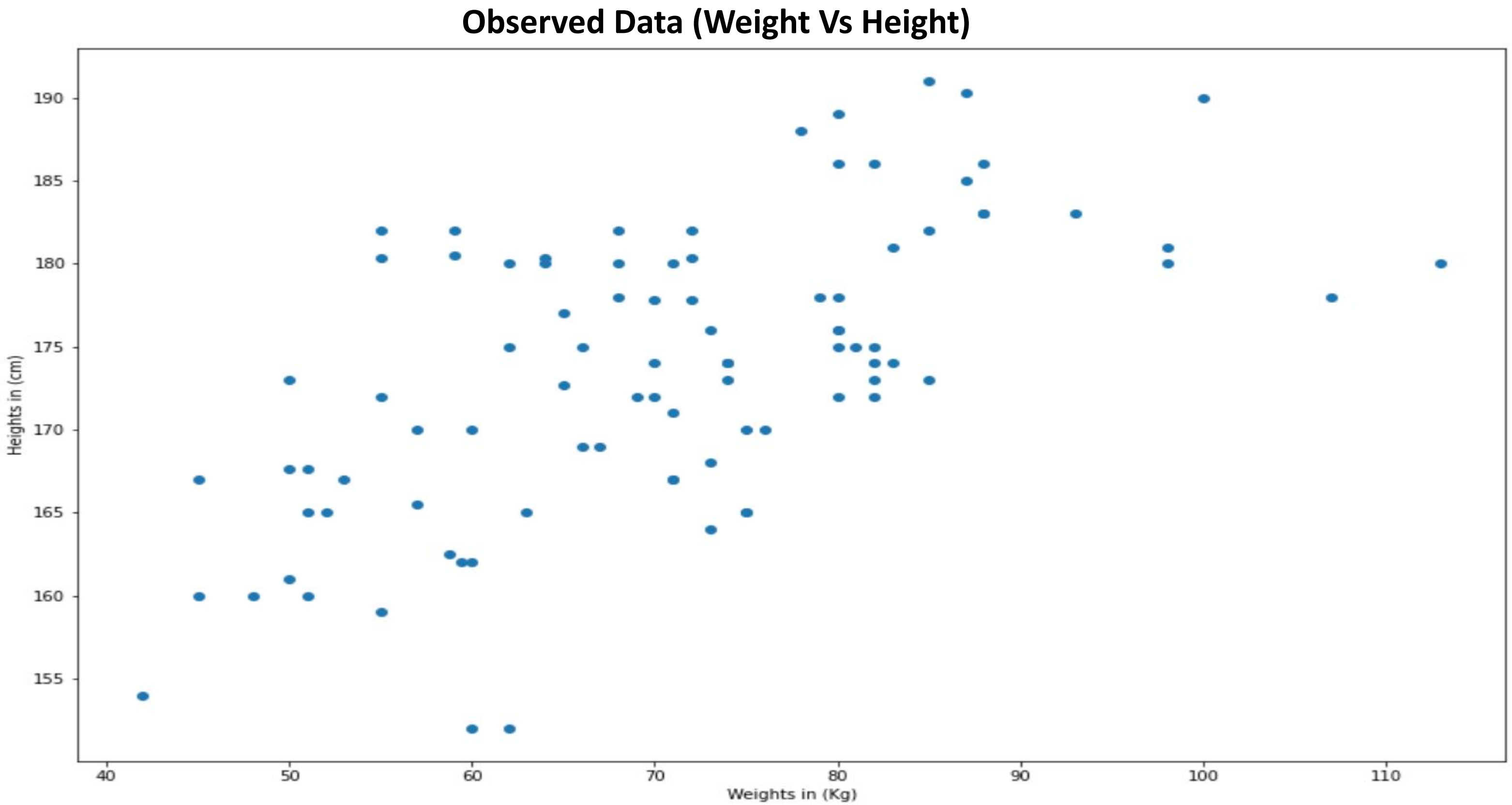
The Problem



Observed Data (Weight Vs Height)
Heights in (cm)
Weights in (Kg)
190
185
180
175
170
165
160
155
40
50
60
70
80
90
100
110

Motivation

- Height and weight are random.

- Can we accurately predict what will be someone's height given her weight?
  - Difficult to estimate from "a priori" models
  - But, we have lots of data from which to build a model

- Assumptions:
  - Linear correspondence (relationship) exists between height and weight.
    - Usually true
  - Observation noise follows a normal distribution.
    - Also usually true!

Formal Problem Settings

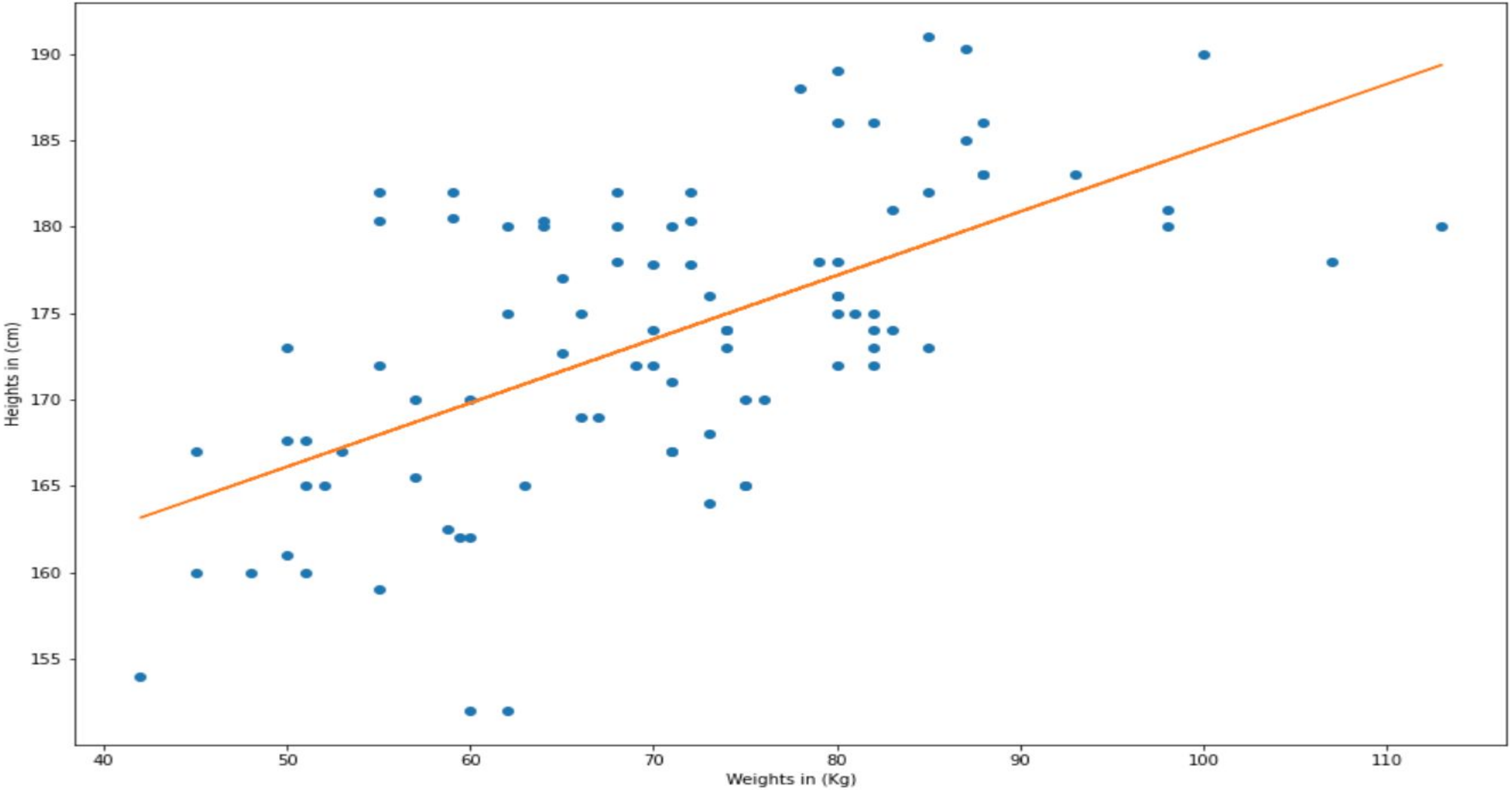
- **Input: D** = $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$
- $\mathbf{x_i} \in \mathbf{X}, \mathbf{y_i} \in \mathbf{Y}$ where X, Y ε R [real number]

- Hypothesis $y_i = w^T x_i + \varepsilon$
  - Linear Correspondence -> $w^T x_i$
  - Normal Distribution of noise -> $\varepsilon \sim N(0, \sigma^2)$

- $y_i \sim N(w^T x_i, \sigma^2)$

$$P(y_i|\vec{x}_i; w) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{w^T x_i - yi}{2\sigma}\right)^2}$$

A simple model



How to get W?

- **Maximum Likelihood Estimation (MLE)** gives us the solution which maximises the likelihood.
  - Find w that maximizes the probability of the data D
    - $argmax\ P(D|w)$

- **Maximum A Posterior (MAP)** gives us the solution which maximises the posterior probability.
  - Find w that is most likely given the data D.
    - $P(w|D) = P(D|w) * P(w)/P(D)$
    - Assumes the availability of the prior $P(w) \sim N(0, \sigma_0^2)$
      - E.g. in case of transfer learning as initial weights w_0

How to get W: Maximum Likelihood Estimation (MLE)

- MLE gives us the solution which maximises the Likelihood

$$\underset{w}{\operatorname{argmax}} \prod_{i=1}^{n} P(y_i|\vec{x}_i; w) = \underset{w}{\operatorname{argmax}} \sum_{i=1}^{n} log P(y_i|\vec{x}_i; w)$$

- Both will provide the same maxima. Now putting the value of the normal distribution probability:

$$= \underset{w}{\operatorname{argmax}} \sum_{i=1}^{n} log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}(w^T x_i - y_i)^2$$

- The log term is independent of w, *hence we can get rid of it*

Finding model parameters [Optimization]

- Simplifying it further:

$$= \underset{w}{\operatorname{argmax}} -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(w^T x_i - y_i)^2$$

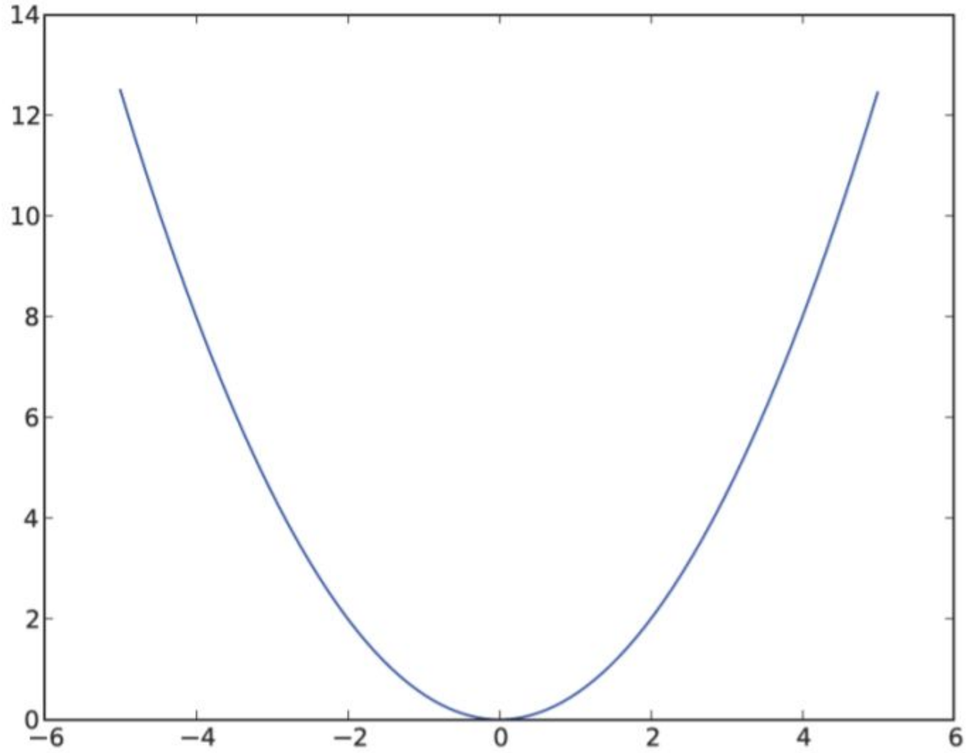
- $1/2\sigma^2$ is again a constant. Further, negative of maximization is same as minimization, hence:

$$= \underset{w}{\operatorname{argmin}} \sum_{i=1}^{n}(w^T x_i - y_i)^2$$

- This is the L_2 loss. To add interpretability to the above, we need to take an average:

$$J(w) = \underset{w}{\operatorname{argmin}} \frac{1}{n}\sum_{i=1}^{n}(w^T x_i - y_i)^2$$

Loss functions: Squared Loss (L_2 Loss)



$$l(\hat{y}, y_i) = (\hat{y} - y_i)^2 = (w^T x_i - y_i)^2$$

Analytical Solution

- In the multivariate case:

$$W = (X'X)^{-1}X'Y$$

- Exercise 1: Find the the solution for unidimensional case:

Hint:

$$J(w) = \sum_{i=1}^{n} (w_0 + w_1 x_i - y_i)^2$$

$$\frac{\partial}{\partial w_0} = ?$$

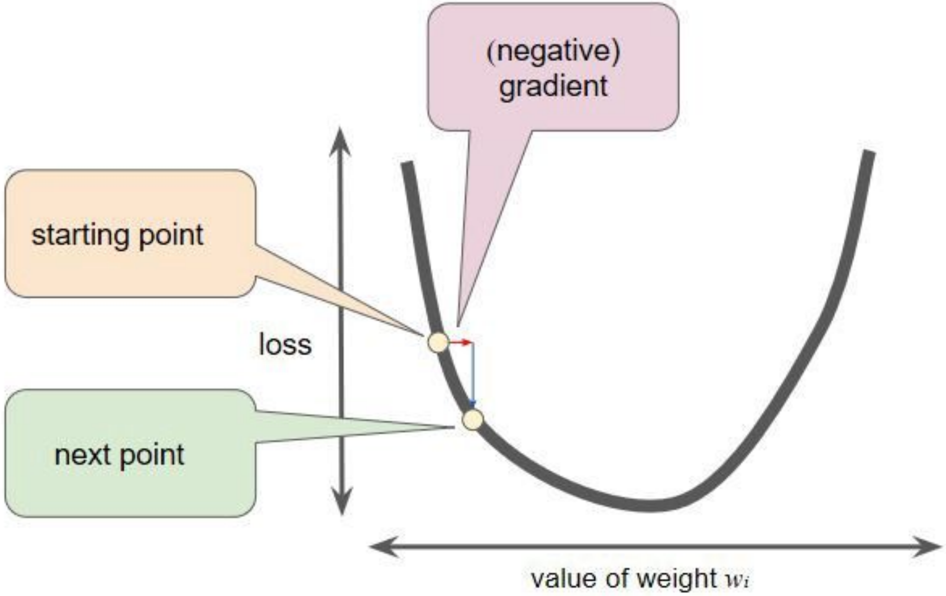
$$\frac{\partial}{\partial w_1} = ?$$

- For a multivariate case, the computational complexity is very high!
  - Hence, an iterative solution such as gradient descent is preferred.

Gradient descent

$$w_i = w_i - \eta \frac{\partial J(w)}{\partial w_i}$$

- Repeat until "convergence"



Gradient descent: Demo

- https://lukaszkujawa.github.io/gradient-descent.html

How to get W: Maximum A Posterior Estimation (MAP)

- Study the Maximum A Posteriori (MAP) solution for Linear Regression