# Empirical Risk Minimization

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# Learning Process

Three components

1. Generator of random vectors *x*, i.i.d. from a fixed but <u>unknown</u> distribution $P(x)$

2. A supervisor (oracle/astrologer) which returns an output vector *y*, for every input vector *x*, as per the conditional distribution $P(y/x)$, also fixed but <u>unknown</u>.

3. A learning machine capable of implementing a set of functions

$$f(x, w), w \epsilon W$$

# Learning Process

- The learning problem is to choose from the given set of functions the one which best approximates the supervisor's response.

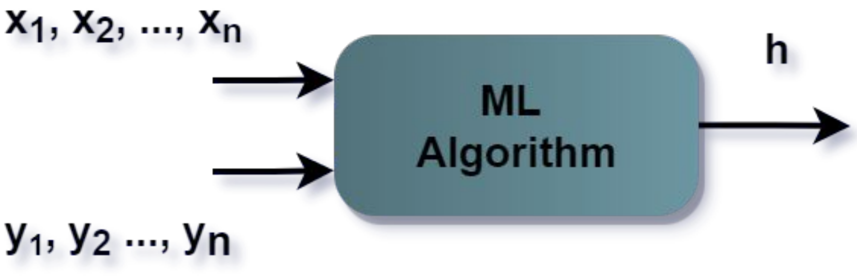
- The selection is based on training samples

$$(x_i, y_i); i = 1, 2, 3 \ldots l$$

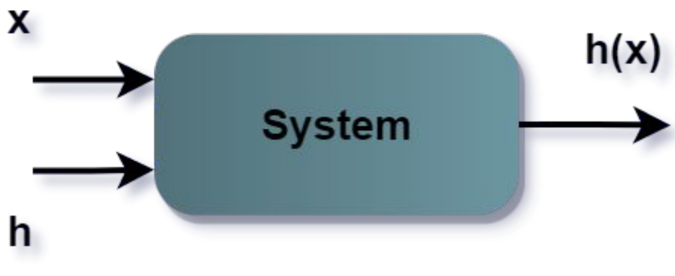
# Problem Setting

- Dataset is a set of possible instances **D** = $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$
- $\mathbf{x_i} \in \mathbf{X}$**:** Each sample is a vector with $\mathbf{R^d}$ drawn from distribution ***P(x)***
- $\mathbf{y_i} \in \mathbf{Y}$ :
  - Classification: Each label is a single integer value out of two [binary] or more classes [multiclass]
  - Regression: Y = R [real number]
- Unknown target function **f : X -> Y** as distribution ***P(y/x)***
- Set of function hypotheses H={ h | h : X->Y }
- **Input:** Training examples $\{<x_i,y_i>\}$
- **Output:** Hypothesis h $\in$ H that best approximates target function f

# Training vs Testing




Training Phase




Testing/Deployment Phase

# Sample Data

| | Person | height(in feet) | weight(in lbs) | foot size(in inches) |
|---|---|---|---|---|
| 0 | male | 6.00 | 180 | 12 |
| 1 | male | 5.92 | 190 | 11 |
| 2 | male | 5.58 | 170 | 12 |
| 3 | male | 5.92 | 165 | 10 |
| 4 | female | 5.00 | 100 | 6 |
| 5 | female | 5.50 | 150 | 8 |

- X = <height, weight, foot size>
- Y = <male, female> | <0, 1> | <-1, +1>
- A sample instance $(x_1, y_1)$ = (<6.00, 180, 12>, male)
- Dimensionality d in $X \in R^d = 3$
- Unknown target function f: height, weight, foot size -> male/female

# How to choose h?

- Randomly:
  - Advantage: Really fast
  - Disadvantage: Terrible performance

- Scan entire **H** and pick the best:
  - Advantage: Great performance
  - Disadvantage: Terribly slow

- Intelligent Way: Learn it using the performance metric
  - Loss functions help to evaluate the performance of the $\mathbf{h_i}$**'s** $\in$ **H** to identify the **optimal h.**

# Loss Functions

- To choose the best function, it makes sense to minimize a loss (or cost or discrepancy) between the response of the supervisor and the learning machine, given an input (x, y)

$$\mathcal{L}(y, f(x))$$

- We want to minimize the loss over all samples

$$L(h) = \frac{1}{n}\sum_{i=1}^{n} l(f(x_i), y_i)$$

- Loss functions are always non-negative.
  - Hence, minimum possible loss is which means we are not making any mistakes.

# Loss Functions - Classification

- **0-1 Loss:** Binary Classification with equal weights on misclassification

$$L(h) = \frac{1}{n}\sum_{i=1}^{n} l_{01}(f(x_i), y_i)$$

$$l_{01}(f(x_i), y_i) = \begin{cases} 0, if \quad f(x_i) = y_i \\ 1, if \quad f(x_i) \neq y_i \end{cases}$$

| **ID** | **Model Prediction** **$f(x_i)$** | **Ground Truth** **$y_i$** | **Loss** **$l_{01}(f(x_i, y_i))$** |
|---|---|---|---|
| Mango 1 | Good | Good | 0 |
| Mango 2 | Good | Bad | 1 |
| Mango 3 | Bad | Good | 1 |
| Mango 4 | Bad | Bad | 0 |
| **Total** | | | **0.5** |

# Loss Functions - $h_1$ or $h_2$

- The aim of the function is to select a hypothesis with the lowest loss.

| ID | Model Prediction $[h_1]$ $f(x_i)$ | Model Prediction $[h_2]$ $f(x_i)$ | Ground Truth $y_i$ | Loss $l_{01}(f(x_i, y_i))$ | Loss $l_{01}(f(x_i, y_i))$ |
|---|---|---|---|---|---|
| M1 | Good | Good | Good | 0 | 0 |
| M2 | Good | Bad | Bad | 1 | 0 |
| M 3 | Bad | Good | Good | 1 | 0 |
| M4 | Bad | Bad | Bad | 0 | 0 |
| **Total Loss** | | | | **0.5** | **0** |

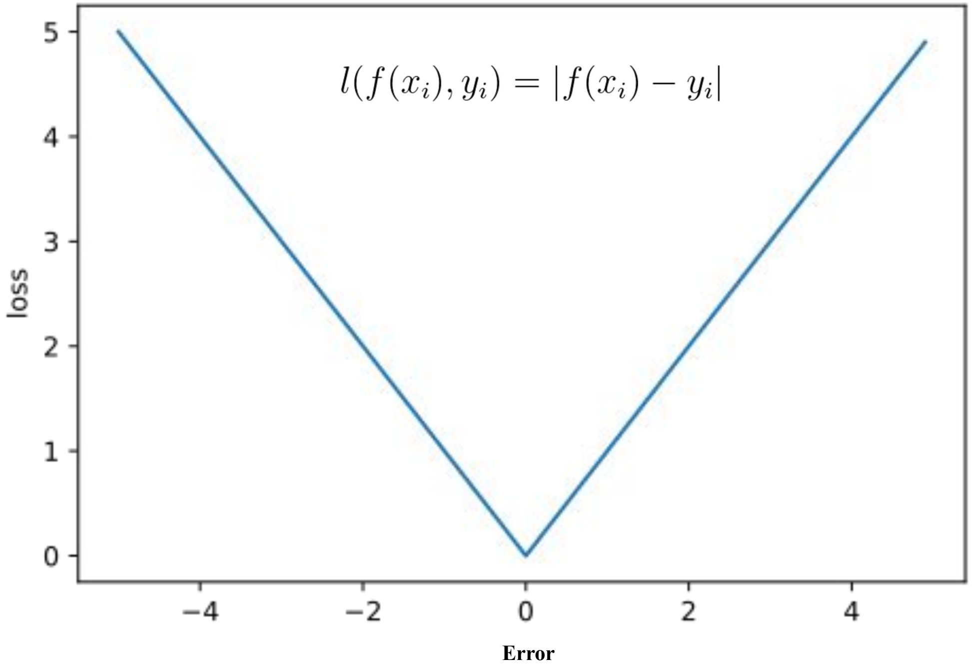
# Loss Functions - Unequal Weights

- Classification with unequal weights on misclassification
  - Minimize a 0-$10^7$-500 loss

$$l(f(x_i), y_i) = \begin{cases} 0, if & f(x_i) = y_i \\ 500, if & f(x_i) = 1, y_i = 0 \\ 10^7, if & f(x_i) = 0, y_i = 1 \end{cases}$$

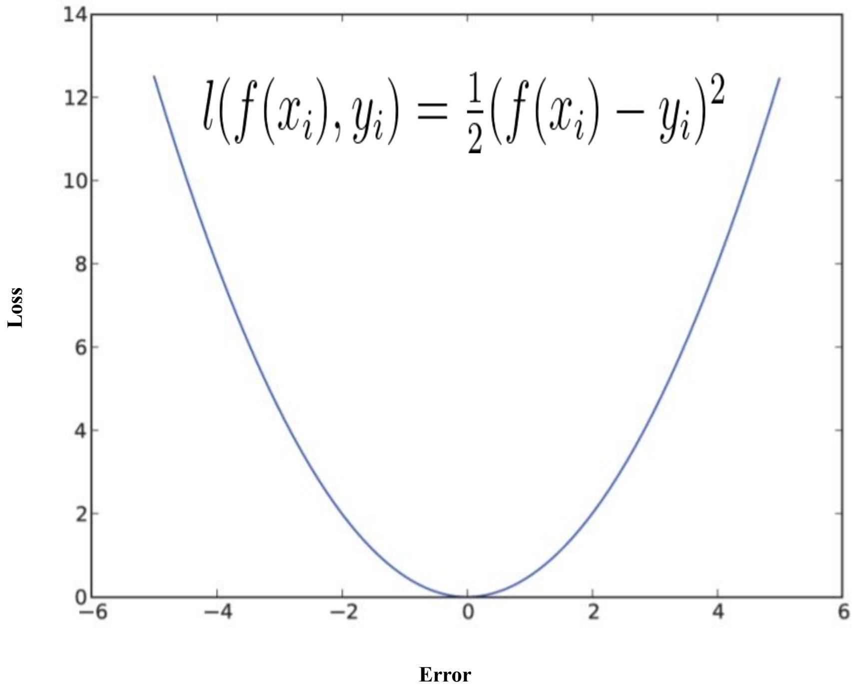
# Loss functions: Regression - $L_1$ Loss

$$l(f(x_i), y_i) = |f(x_i) - y_i|$$



# Loss functions: Regression - $L_2$ Loss

$$l(f(x_i), y_i) = \frac{1}{2}(f(x_i) - y_i)^2$$



# Mean Absolute Error vs. Mean Square Error

- Mean Absolute Error (MAE)

$$L(h) = MAE = \frac{1}{n}\sum_{i=1}^{n} |f(x_i) - y_i|$$

- Mean Square Error (MSE)

$$L(h) = MSE = \frac{1}{n}\sum_{i=1}^{n} (f(x_i) - y_i)^2$$

**MAE = 1, RMSE = 1.50**

| ID | Error | \|Error\| | Error² |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 |
| 3 | 2 | 2 | 4 |
| 4 | -0.5 | 0.5 | 0.25 |
| 5 | 1.5 | 1.5 | 2.25 |
| Total | | 5 | 7.50 |

# MSE vs. MAE (L2 loss vs L1 loss)

- Robust to Outliers? $L_1$ or $L_2$
  - An individual's height being too tall or small.
- Differentiability?
  - **What:** Continuous and smooth function over a region.
  - **Why Do We Care:** To get the rate of increase/decrease defined at all points.
  - **Buy Why!:** It allows to find the minima/maxima and hence the optimal model.

# Generalization

- What about performance on unknown or new data samples?
- Our true goal is to know and minimize the loss of the *unknown test samples* drawn from same distribution *P*

$$h^* = \underset{f(x))}{\operatorname{argmin}} \frac{1}{m-n} \sum_{i=n+1}^{m} l(f(x_i), y_i)$$

- In other words, we want to know and minimize the *Expected Loss l(f(x, y))* and hence the *Risk* associated with function *f*.

$$R(f) = \int \int p(x_i, y_i) l(f(x_i), y_i) dx dy$$

# Expected Loss

- Usually we don't know the test points and their labels in advance!!!
  - We do not know the $p(x_i, y_i)$
  - Hence, we do not know the *expected loss* or the $R(f)$

- ***Our goal:*** *Expected loss should be closer to the actual loss*

- The law of large numbers (LLN) states that if the amount of exposure to losses increases, then the predicted loss will be closer to the actual loss.
  - Example: Insurance in Real Life

# Empirical Risk Minimization Principle

- So. by LLN the *statistical risk* associated with function *f* becomes equal to the *empirical risk*

$$\frac{1}{m-n}\sum_{i=n+1}^{m} l(f(x_i), y_i) \overset{m\to\infty}{\to} R_{L,P}(f)$$

- Picking the function *f (via hypothesis h)* that minimizes the *empirical risk* is known as *empirical risk minimization.*

$$h^* = f^* = \underset{f(x)}{\mathrm{argmin}} R_{L,P}(f)$$

- **Our hope:**

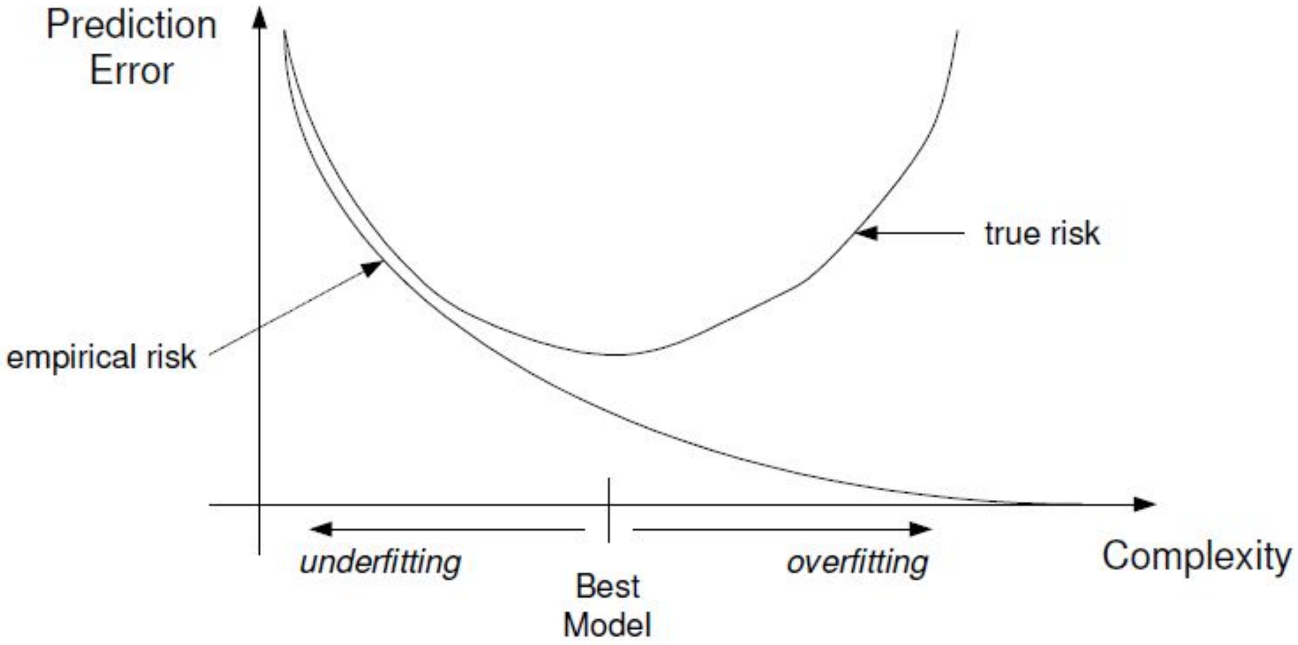
$$\underset{f(x)\epsilon F}{\mathrm{argmin}} R_{L,P}(f) \approx \underset{f(x)\epsilon F}{\mathrm{argmin}} R_{L,P}^{true}(f)$$

# Empirical Risk Minimization Principle

Empirical risk minimization depends on following:

1. **How much data we have:** For any given function $f$, as we get more and more data, we can expect that $R(f) \rightarrow R_{true}(f)$
2. **The true distribution p**: Depending on how "complex" the true distribution is, more or less data may be necessary to get a good approximation of it.
3. **The loss function L**: If the loss function is very "weird"– giving extremely high loss in certain unlikely situations, this can lead to trouble.
4. **The class of functions F**: Roughly speaking, if the size of F is "large", and the functions in F are "complex", this worsens the approximation, all else being equal.

# Effect of Function Complexity




- At fixed number of samples, overly complicated models -> overfitting.
- Empirical risk is no longer a good indicator of true risk.

# References

- Principles of Risk Minimization for Learning Theory