Assignment 5

Vaibhav Chhabra
AI20BTECH11022

Download all latex codes from

https://github.com/vaibhavchhabra25/AI1103−course/blob/main/Assignment−5/main.tex

1 Problem

(UGC/MATH 2018 (June set-a)-Q.106) Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with $E(X_i) = 0$ and $V(X_i) = 1$. Which of the following are true?

1) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 0$ in probability
2) $\frac{1}{n^{3/4}}\sum_{i=1}^{n} X_i \to 0$ in probability
3) $\frac{1}{n^{1/2}}\sum_{i=1}^{n} X_i \to 0$ in probability
4) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 1$ in probability

2 Solution

Definition 1. *(Convergence in probability)* *Let $X1, X2, \dots$ be an infinite sequence of random variables, and let Y be another random variable. Then the sequence $\{X_n\}$ converges in probability to Y, if for all $\epsilon > 0$,*

$$\lim_{n\to\infty} \Pr\left(|X_n - Y| \geq \epsilon\right) = 0 \tag{2.0.1}$$

And we write as $n \to \infty$, $X_n \to Y$ in probability.

Theorem 2.1. *(Strong Law of Large Numbers)* *Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables, each having finite mean $E(X_i)$. Then for any $\epsilon > 0$,*

$$\lim_{n\to\infty} \Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - E(X_i)\right| \geq \epsilon\right) = 0 \tag{2.0.2}$$

Or, $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges in probability to $E(X_i)$.

Given,

$$E(X_i) = 0 \tag{2.0.3}$$
$$V(X_i) = 1 \tag{2.0.4}$$

Also, we know that,

$$E(X_i^2) = V(X_i) + (E(X_i))^2 \tag{2.0.5}$$

Putting given values, we get,

$$E(X_i^2) = 1 + 0^2 = 1 \tag{2.0.6}$$
$$\implies E(X_i^2) = 1 \tag{2.0.7}$$

So, $E(X_i^2)$ is finite.
As $\{X_i\}$ is sequence of i.i.d. random variables, it follows the following conditions $\forall x, x_i \in \mathbb{R}$:

1) $\Pr\left(X_1 \leq x\right) = \Pr\left(X_k \leq x\right) \ \ \forall k \in \{1, 2, 3 \cdots n\}$
2) $\Pr\left(X_1 \leq x_1, X_2 \leq x_2 \cdots X_n \leq x_n\right) =$ $\Pr\left(X_1 \leq x_1\right) \cdot \Pr\left(X_2 \leq x_2\right) \cdots \Pr\left(X_n \leq x_n\right)$

We can rewrite (1) as

$$\Pr\left(X_1^2 \leq x^2\right) = \Pr\left(X_k^2 \leq x^2\right) \tag{2.0.8}$$

Putting $x^2 = y$, we get,

$$\Pr\left(X_1^2 \leq y\right) = \Pr\left(X_k^2 \leq y\right) \tag{2.0.9}$$

Also, we can rewrite (2) as

$$\Pr\left(X_1^2 \leq x_1^2, X_2^2 \leq x_2^2 \cdots X_n^2 \leq x_n^2\right) = \Pr\left(X_1^2 \leq x_1^2\right) \cdot \Pr\left(X_2^2 \leq x_2^2\right) \cdots \Pr\left(X_n^2 \leq x_n^2\right) \tag{2.0.10}$$

Putting $x_i^2 = y_i$, we get,

$$\Pr\left(X_1^2 \leq y_1, X_2^2 \leq y_2 \cdots X_n^2 \leq y_n\right) = \Pr\left(X_1^2 \leq y_1\right) \cdot \Pr\left(X_2^2 \leq y_2\right) \cdots \Pr\left(X_n^2 \leq y_n\right) \tag{2.0.11}$$

By (2.0.9) and (2.0.11), $\{X_i^2\}$ must also be a sequence of i.i.d. random variables.
So, we can apply S.L.L.N. to this sequence.
Then, $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ converges in probability to $E(X_i^2)$.

Or, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 1$ in probability.
Thus, option 1 is wrong and option 4 is correct.

Now, we define

$$Y_n = \frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i \tag{2.0.12}$$

Then,

$$E(Y_n) = \frac{1}{n^{3/4}} E\left(\sum_{i=1}^{n} X_i\right) \tag{2.0.13}$$

$$\implies E(Y_n) = \frac{1}{n^{3/4}} \left(E(X_1) + E(X_2) + \cdots + E(X_n)\right) \tag{2.0.14}$$

Using (2.0.3)

$$E(Y_n) = \frac{1}{n^{3/4}} (0) = 0 \tag{2.0.15}$$

Now,

$$V(Y_n) = V\left(\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i\right) \tag{2.0.16}$$

$$\implies V(Y_n) = \frac{1}{n^{3/2}} V\left(X_1 + X_2 + \cdots + X_n\right) \tag{2.0.17}$$

As $X_1, X_2, \cdots X_n$ are independent of each other,

$$V(Y_n) = \frac{1}{n^{3/2}} \left(V(X_1) + V(X_2) + \cdots + V(X_n)\right) \tag{2.0.18}$$

Using (2.0.4)

$$V(Y_n) = \frac{1}{n^{3/2}} (1 + 1 + \cdots + 1) = \frac{1}{n^{3/2}} \times n = \frac{1}{n^{1/2}} \tag{2.0.19}$$

Now for any $\epsilon > 0$, consider the probability

$$\Pr\left(|Y_n - 0| \geq \epsilon\right) = \Pr\left(|Y_n - E(Y_n)| \geq \epsilon\right) \tag{2.0.20}$$

Applying Chebyschev's inequality here, we get,

$$\Pr\left(|Y_n - 0| \geq \epsilon\right) \leq \frac{V(Y_n)}{\epsilon^2} = \frac{1}{n^{1/2}\epsilon^2} \tag{2.0.21}$$

So,

$$\lim_{n\to\infty} \Pr\left(|Y_n - 0| \geq \epsilon\right) \leq \lim_{n\to\infty} \frac{1}{n^{1/2}\epsilon^2} = 0 \tag{2.0.22}$$

$$\implies \lim_{n\to\infty} \Pr\left(\left|\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i - 0\right| \geq \epsilon\right) = 0 \tag{2.0.23}$$

So, $\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i \to 0$ in probability.

Thus, option 2 is also correct.
So the answer must be options 2,4.