Assignment 6

Vaibhav Chhabra
AI20BTECH11022

Download all latex codes from

https://github.com/vaibhavchhabra25/AI1103-course/blob/main/Assignment-6/main.tex

1 Problem

(CSIR UGC NET EXAM June 2013 - Q.75)
Let X be a non-negative integer valued random variable with probability mass function $f(x)$ satisfying $(x+1)f(x+1) = (\alpha+\beta x)f(x)$, $x = 0, 1, 2, ...; \beta \neq 1$. You may assume that $E(X)$ and $Var(X)$ exist. Then which of the following statements are true?

1) $E(X) = \dfrac{\alpha}{1-\beta}$
2) $E(X) = \dfrac{\alpha^2}{(1-\beta)(1+\alpha)}$
3) $Var(X) = \dfrac{\alpha^2}{(1-\beta)^2}$
4) $Var(X) = \dfrac{\alpha}{(1-\beta)^2}$

2 Solution

For a discrete random variable X with P.D.F. $f(x)$ and which can take values from a set $\mathbb{S}$,

$$E(X) = \sum_{x\in\mathbb{S}} xf(x) \tag{2.0.1}$$

And,

$$E(X^2) = \sum_{x\in\mathbb{S}} x^2 f(x) \tag{2.0.2}$$

Also, as $f(x)$ is the P.D.F.,

$$\sum_{x\in\mathbb{S}} f(x) = 1 \tag{2.0.3}$$

Given, for $x \in \mathbb{S} = \{0, 1, 2, ...n\}$,

$$(x+1)f(x+1) = (\alpha+\beta x)f(x) \tag{2.0.4}$$

Summing both sides for $x \in \mathbb{S}$ we get,

$$\sum_{x=0}^{n}(x+1)f(x+1) = \sum_{x=0}^{n}(\alpha+\beta x)f(x) \tag{2.0.5}$$

Replacing $x+1$ with x in L.H.S. we get,

$$\sum_{x=1}^{n+1} xf(x) = \sum_{x=0}^{n}(\alpha+\beta x)f(x) \tag{2.0.6}$$

Rewriting LHS, we get,

$$\sum_{x=0}^{n} xf(x) + (n+1)f(n+1) = \sum_{x=0}^{n}(\alpha+\beta x)f(x) \tag{2.0.7}$$

But as $x \in \{0, 1, 2...n\}$, $f(n+1) = 0$. So the equation becomes

$$\sum_{x=0}^{n} xf(x) = \alpha\sum_{x=0}^{n} f(x) + \beta\sum_{x=0}^{n} xf(x) \tag{2.0.8}$$

Using (2.0.1) and (2.0.3), we get,

$$E(X) = \alpha(1) + \beta E(X) \tag{2.0.9}$$

So,

$$E(X) = \frac{\alpha}{1-\beta} \tag{2.0.10}$$

Now in (2.0.4), multiplying both sides by $(x+1)$, we get,

$$(x+1)^2 f(x+1) = (\alpha+\beta x)(x+1)f(x) \tag{2.0.11}$$

Summing both sides for $x \in \mathbb{S}$ we get,

$$\sum_{x=0}^{n}(x+1)^2 f(x+1) = \sum_{x=0}^{n}(\alpha+\beta x)(x+1)f(x) \tag{2.0.12}$$

Replacing $x+1$ with x in L.H.S. we get,

$$\sum_{x=1}^{n+1} x^2 f(x) = \sum_{x=0}^{n}(\beta x^2 f(x) + (\alpha+\beta)xf(x) + \alpha f(x)) \tag{2.0.13}$$

Rewriting LHS similarly as before, we get,

$$\sum_{x=0}^{n} x^2 f(x) = \beta \sum_{x=0}^{n} x^2 f(x) + \\ (\alpha+\beta)\sum_{x=0}^{n} x f(x) + \alpha \sum_{x=0}^{n} f(x) \tag{2.0.14}$$

Using (2.0.1), (2.0.2) and (2.0.3), we get,

$$E(X^2) = \beta E(X^2) + (\alpha+\beta)E(X) + \alpha(1) \tag{2.0.15}$$

Using (2.0.10)

$$E(X^2)(1-\beta) = \frac{\alpha(\alpha+\beta)}{1-\beta} + \alpha \tag{2.0.16}$$

So,

$$E(X^2) = \frac{\alpha^2+\alpha}{(1-\beta)^2} \tag{2.0.17}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \tag{2.0.18}$$

Using (2.0.10) and (2.0.17),

$$Var(X) = \frac{\alpha^2+\alpha}{(1-\beta)^2} - \frac{\alpha^2}{(1-\beta)^2} \tag{2.0.19}$$

So,

$$Var(X) = \frac{\alpha}{(1-\beta)^2} \tag{2.0.20}$$

So, options 1 and 4 are correct.