Assignment 5

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Download all latex codes from

https://github.com/vaibhavchhabra25/AI1103-course/blob/main/Assignment-5/main.tex

1 Problem

(UGC/MATH 2018 (June set-a)-Q.106) Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with $E(X_i) = 0$ and $V(X_i) = 1$. Which of the following are true?

1) $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \to 0$ in probability
2) $\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i \to 0$ in probability
3) $\frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \to 0$ in probability
4) $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \to 1$ in probability

2 Solution

Theorem 2.1. *(Strong Law of Large Numbers) Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with finite $E(X_i)$. Then, as $n \to \infty$,*

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to E(X_i) \tag{2.0.1}$$

Given,

$$E(X_i) = 0 \tag{2.0.2}$$
$$V(X_i) = 1 \tag{2.0.3}$$

Also, we know that,

$$E(X_i^2) = V(X_i) + (E(X_i))^2 \tag{2.0.4}$$

Putting given values, we get,

$$E(X_i^2) = 1 + 0^2 = 1 \tag{2.0.5}$$

So $E(X_i^2)$ is finite. And as $\{X_i\}$ is a sequence of i.i.d. random variables, $\{X_i^2\}$ should also be a sequence of i.i.d. random variables. So, $\{X_i^2\}$ follows Strong Law of Large numbers.

Applying Strong Law of Large numbers for $\{X_i^2\}$, we get that as $n \to \infty$,

$$\frac{1}{n} \sum_{i=1}^{n} X_i^2 \to E(X_i^2) \tag{2.0.6}$$

Using (2.0.5),

$$\frac{1}{n} \sum_{i=1}^{n} X_i^2 \to 1 \tag{2.0.7}$$

with probability 1, or in probability.
So, option 1 is wrong and option 4 is correct.

Now, we define

$$Z_n = \frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i \tag{2.0.8}$$

Then,

$$E(Z_n) = \frac{1}{n^{3/4}} E\left(\sum_{i=1}^{n} X_i\right) \tag{2.0.9}$$

$$\implies E(Z_n) = \frac{1}{n^{3/4}} \left(E(X_1) + E(X_2) + \cdots + E(X_n)\right) \tag{2.0.10}$$

Using (2.0.2)

$$E(Z_n) = \frac{1}{n^{3/4}} (0) = 0 \tag{2.0.11}$$

And so, as $n \to \infty$,

$$E(Z_n) = 0 \tag{2.0.12}$$

Now,

$$V(Z_n) = V\left(\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i\right) \tag{2.0.13}$$

$$\implies V(Z_n) = \frac{1}{n^{3/2}} V\left(X_1 + X_2 + \cdots + X_n\right) \tag{2.0.14}$$

As $X_1, X_2, \cdots X_n$ are independent of each other,

$$V(Z_n) = \frac{1}{n^{3/2}} \left(V(X_1) + V(X_2) + \cdots + V(X_n)\right) \tag{2.0.15}$$

Using (2.0.3)

$$V(Z_n) = \frac{1}{n^{3/2}}(1 + 1 + \cdots + 1) = \frac{1}{n^{3/2}} \times n = \frac{1}{n^{1/2}} \tag{2.0.16}$$

So, as $n \to \infty$,

$$V(Z_n) \to 0 \tag{2.0.17}$$

So, by (2.0.12) and (2.0.17), as $n \to \infty$

$$Z_n \to N(0, 0) = 0 \tag{2.0.18}$$

Or as $n \to \infty$,

$$\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i \to 0 \tag{2.0.19}$$

with probability 1, or in probability.
So, option 2 is correct.

Now, let

$$Y_n = \frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \tag{2.0.20}$$

Then, similarly as Z_n,

$$E(Y_n) = 0 \tag{2.0.21}$$

And,

$$V(Y_n) = V\left(\frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} V(X_1 + X_2 + \cdots + X_n) \tag{2.0.22}$$

$$\implies V(Y_n) = \frac{1}{n}(V(X_1) + V(X_2) + \cdots + V(X_n)) \tag{2.0.23}$$

$$\implies V(Y_n) = \frac{1}{n} \times n = 1 \tag{2.0.24}$$

By (2.0.21) and (2.0.24), it is clear that as $n \to \infty$, Y_n converges to a distribution $N(0, 1)$.
So, option 3 is also wrong.

So the answer must be options 2,4.