Assignment 3

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Download all latex codes from

https://github.com/vaibhavchhabra25/AI1103-course/blob/main/Assignment-3/main.tex

1 Problem

(GATE 2001 (MA), Q. 2.24) Let (X, Y) be a two-dimensional random variable such that $E(X) = E(Y) = 3$, $Var(X) = Var(Y) = 1$ and $Cov(X, Y) = 1/2$. Then, $P(|X - Y| > 6)$ is

1) less than 1/6
2) equal to 1/2
3) equal to 1/3
4) greater than 1/2

2 Solution

Given,

$$E(X) = E(Y) = 3 \tag{2.0.1}$$

$$Var(X) = Var(Y) = 1 \tag{2.0.2}$$

$$Cov(X, Y) = 1/2 \tag{2.0.3}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \tag{2.0.4}$$

Substituting given values, we get,

$$1 = E(X^2) - 3^2 \tag{2.0.5}$$

So,

$$E(X^2) = 10 \tag{2.0.6}$$

Similarly for Y,

$$E(Y^2) = 10 \tag{2.0.7}$$

Also,

$$Cov(X, Y) = E(XY) - E(X)E(Y) \tag{2.0.8}$$

Substituting given values, we get,

$$1/2 = E(XY) - (3)(3) \tag{2.0.9}$$

So,

$$E(XY) = 19/2 \tag{2.0.10}$$

Let Z be a random variable defined as

$$Z = X - Y \tag{2.0.11}$$

Then using (2.0.1),

$$E(Z) = E(X - Y) = E(X) - E(Y) = 0 \tag{2.0.12}$$

Now, using (2.0.12)

$$Var(Z) = E(Z^2) - (E(Z))^2 = E(Z^2) \tag{2.0.13}$$

$$Var(Z) = E((X - Y)^2) \tag{2.0.14}$$

$$Var(Z) = E(X^2) + E(Y^2) - 2E(XY) \tag{2.0.15}$$

Using (2.0.6), (2.0.7) and (2.0.10),

$$Var(Z) = 10 + 10 - 2 \times 19/2 \tag{2.0.16}$$

$$Var(Z) = 1 \tag{2.0.17}$$

Theorem 2.1. *(Chebychev's Inequality) Let T be an arbitrary random variable, with finite mean E(T), then for all $a > 0$,*

$$\Pr(|T - E(T)| \geq a) \leq \frac{Var(T)}{a^2} \tag{2.0.18}$$

Proof. Let T be a random variable with probablility distribution function $f(T)$ and $a > 0$ be any real number. Then,

$$\Pr(|T - E(T)| \geq a) = \int_{-\infty}^{-E(T)-a} f(T)dT + \int_{E(T)+a}^{\infty} f(T)dT \tag{2.0.19}$$

$$\Pr(|T - E(T)| \geq a) = \int_{|T-E(T)|\geq a} f(T)dT \tag{2.0.20}$$

Now,

$$
\begin{aligned}
Var(T) &= \int_{-\infty}^{\infty} (T - E(T))^2 f(T) dT \\
&\geq \int_{|T-E(T)|\geq a} (T - E(T))^2 f(T) dT \\
&\geq a^2 \int_{|T-E(T)|\geq a} f(T) dT
\end{aligned} \tag{2.0.21}
$$

So, we finally get,

$$Var(T) \geq a^2 \int_{|T-E(T)|\geq a} f(T) dT \tag{2.0.22}$$

Using (2.0.20),

$$Var(T) \geq a^2 \Pr\left(|T - E(T)| \geq a\right) \tag{2.0.23}$$

Or,

$$\Pr\left(|T - E(T)| \geq a\right) \leq \frac{Var(T)}{a^2} \tag{2.0.24}$$

$\square$

Applying Chebychev's Inequality for Z with $a = 6$, we get,

$$\Pr\left(|Z - E(Z)| \geq 6\right) \leq \frac{Var(Z)}{6^2} \tag{2.0.25}$$

Using (2.0.12) and (2.0.17),

$$\Pr\left(|Z - 0| \geq 6\right) \leq \frac{1}{36} \tag{2.0.26}$$

As $Z = X - Y$,

$$\Pr\left(|X - Y| \geq 6\right) \leq \frac{1}{36} \tag{2.0.27}$$

So, option 1 is correct.