Assignment 3

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https://github.com/vaibhavchhabra25/AI1103−course/blob/main/Assignment−3/main.tex

1 Problem

(GATE 2001 (MA), Q. 2.24) Let (X, Y) be a two-dimensional random variable such that $E(X) = E(Y) = 1/2$, $Var(X) = Var(Y) = 1$ and $Cov(X, Y) = 1/2$.
Then, $P(|X - Y| > 6)$ is

1) less than 1/6
2) equal to 1/2
3) equal to 1/3
4) greater than 1/2

2 Solution

Given,

$$E(X) = E(Y) = 3 \tag{2.0.1}$$

$$Var(X) = Var(Y) = 1 \tag{2.0.2}$$

$$Cov(X, Y) = 1/2 \tag{2.0.3}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \tag{2.0.4}$$

Substituting given values, we get,

$$1 = E(X^2) - 3^2 \tag{2.0.5}$$

So,

$$E(X^2) = 10 \tag{2.0.6}$$

Similarly for Y,

$$E(Y^2) = 10 \tag{2.0.7}$$

Also,

$$Cov(X, Y) = E(XY) - E(X)E(Y) \tag{2.0.8}$$

Substituting given values, we get,

$$1/2 = E(XY) - (3)(3) \tag{2.0.9}$$

So,

$$E(XY) = 19/2 \tag{2.0.10}$$

Let Z be a random variable defined as

$$Z = X - Y \tag{2.0.11}$$

Then,

$$E(Z) = E(X - Y) = E(X) - E(Y) \tag{2.0.12}$$

Using (2.0.1),

$$E(Z) = 0 \tag{2.0.13}$$

Now, variance of Z,

$$Var(Z) = E(Z^2) - (E(Z))^2 \tag{2.0.14}$$

Using (2.0.13)

$$Var(Z) = E(Z^2) \tag{2.0.15}$$

$$Var(Z) = E((X - Y)^2) \tag{2.0.16}$$

$$Var(Z) = E(X^2 + Y^2 - 2XY) \tag{2.0.17}$$

$$Var(Z) = E(X^2) + E(Y^2) - 2E(XY) \tag{2.0.18}$$

Using (2.0.6), (2.0.7) and (2.0.10),

$$Var(Z) = 10 + 10 - 2 \times 19/2 \tag{2.0.19}$$

$$Var(Z) = 1 \tag{2.0.20}$$

Theorem 2.1. *(Chebychev's Inequality) Let T be an arbitrary random variable, with finite mean E(T), then for all* $a > 0$,

$$\Pr(|T - E(T)| \geq a) \leq \frac{Var(T)}{a^2} \tag{2.0.21}$$

Proof. Let A be a non-negative random variable and $a > 0$ be any real number. Define a new random variable B by

$$B = \begin{cases} a & A \geq a \\ 0 & A < a \end{cases} \tag{2.0.22}$$

Then clearly $B \leq A$ and by monotonicity,

$$E(B) \leq E(A) \tag{2.0.23}$$

$$E(B) = a \Pr(B = a) + 0 \Pr(B = 0) \tag{2.0.24}$$

$$E(B) = a \Pr(A \geq a) \tag{2.0.25}$$

By (2.0.23) and (2.0.25),

$$a \Pr(A \geq a) \leq E(A) \tag{2.0.26}$$

$$\Pr(A \geq a) \leq \frac{E(A)}{a} \tag{2.0.27}$$

Set $A = (T - E(T))^2$. Then,

$$\Pr(|T - E(T)| \geq a) = \Pr\left(|T - E(T)|^2 \geq a^2\right) \tag{2.0.28}$$

Using (2.0.27),

$$\Pr(|T - E(T)| \geq a) = \Pr\left(A \geq a^2\right) \leq \frac{E(A)}{a^2} \tag{2.0.29}$$

As $A = (T - E(T))^2$,

$$\Pr(|T - E(T)| \geq a) \leq \frac{E(T - E(T))^2}{a^2} \tag{2.0.30}$$

$$\Pr(|T - E(T)| \geq a) \leq \frac{Var(T)}{a^2} \tag{2.0.31}$$

□

Applying Chebychev's Inequality for Z with $a = 6$, we get,

$$\Pr(|Z - E(Z)| \geq 6) \leq \frac{Var(Z)}{6^2} \tag{2.0.32}$$

Using (2.0.13) and (2.0.20),

$$\Pr(|Z - 0| \geq 6) \leq \frac{1}{36} \tag{2.0.33}$$

As $Z = X - Y$,

$$\Pr(|X - Y| \geq 6) \leq \frac{1}{36} \tag{2.0.34}$$

So, option 1 is correct.