(GATE 2001 (MA), Q. 2.24)

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Question

Let (X, Y) be a two-dimensional random variable such that $E(X) = E(Y) = 3$, $Var(X) = Var(Y) = 1$ and $Cov(X, Y) = 1/2$. Then, $P(|X - Y| > 6)$ is

1. less than $1/6$
2. equal to $1/2$
3. equal to $1/3$
4. greater than $1/2$

Solution

Given,

$$E(X) = E(Y) = 3 \tag{1}$$

$$Var(X) = Var(Y) = 1 \tag{2}$$

$$Cov(X, Y) = 1/2 \tag{3}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \tag{4}$$

Substituting given values, we get,

$$1 = E(X^2) - 3^2 \tag{5}$$

So,

$$E(X^2) = 10 \tag{6}$$

Similarly for Y,

$$E(Y^2) = 10 \tag{7}$$

Solution

Also,

$$Cov(X, Y) = E(XY) - E(X)E(Y) \tag{8}$$

Substituting given values, we get,

$$1/2 = E(XY) - (3)(3) \tag{9}$$

So,

$$E(XY) = 19/2 \tag{10}$$

Let Z be a random variable defined as

$$Z = X - Y \tag{11}$$

Then using (1),

$$E(Z) = E(X - Y) = E(X) - E(Y) = 0 \tag{12}$$

Solution

Now, using (12)

$$Var(Z) = E(Z^2) - (E(Z))^2 = E(Z^2) \tag{13}$$

$$Var(Z) = E((X - Y)^2) \tag{14}$$

$$Var(Z) = E(X^2) + E(Y^2) - 2E(XY) \tag{15}$$

Using (6), (7) and (10),

$$Var(Z) = 10 + 10 - 2 \times 19/2 \tag{16}$$

$$Var(Z) = 1 \tag{17}$$

Chebychev's Inequality

Theorem

(Chebychev's Inequality) Let T be an arbitrary random variable, with finite mean E(T), then for all $a > 0$,

$$\Pr\left(|T - E(T)| \geq a\right) \leq \frac{Var(T)}{a^2} \tag{18}$$

Proof:

Let T be a random variable with probablility distribution function $f(T)$ and $a > 0$ be any real number. Then,

$$\Pr\left(|T - E(T)| \geq a\right) = \int_{-\infty}^{-E(T)-a} f(T)dT + \int_{E(T)+a}^{\infty} f(T)dT \tag{19}$$

$$\Pr\left(|T - E(T)| \geq a\right) = \int_{|T-E(T)|\geq a} f(T)dT \tag{20}$$

Now,

$$Var(T) = \int_{-\infty}^{\infty} (T - E(T))^2 f(T) dT$$
$$\geq \int_{|T-E(T)|\geq a} (T - E(T))^2 f(T) dT$$
$$\geq a^2 \int_{|T-E(T)|\geq a} f(T) dT \tag{21}$$

So, we finally get,

$$Var(T) \geq a^2 \int_{|T-E(T)|\geq a} f(T) dT \tag{22}$$

Using (20),

$$Var(T) \geq a^2 \Pr(|T - E(T)| \geq a) \tag{23}$$

Or,

$$\Pr(|T - E(T)| \geq a) \leq \frac{Var(T)}{a^2} \tag{24}$$

Solution Contd.

Applying Chebychev's Inequality for Z with $a = 6$, we get,

$$\Pr(|Z - E(Z)| \geq 6) \leq \frac{Var(Z)}{6^2} \tag{25}$$

Using (12) and (17),

$$\Pr(|Z - 0| \geq 6) \leq \frac{1}{36} \tag{26}$$

As $Z = X - Y$,

$$\Pr(|X - Y| \geq 6) \leq \frac{1}{36} \tag{27}$$

So, option 1 is correct.