Ramsey 4.2-Tangent and Normal-Q.21

Vaibhav Chhabra - AI20BTECH11022

August 30, 2021

Problem

Ramsey 4.2-Tangent and Normal-Q.21

Verify that the perpendicular bisector of the chord joining two points $\mathbf{x_1}$,$\mathbf{x_2}$ on the circle

$$\mathbf{x}^\top \mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} + c = 0 \tag{1}$$

passes through the centre.

Solution

Since $\mathbf{x_1}$ and $\mathbf{x_2}$ lie on the circle,

$$\mathbf{x_1}^\top \mathbf{x_1} + 2\begin{pmatrix} g & f \end{pmatrix} \mathbf{x_1} + c = 0 \tag{2}$$

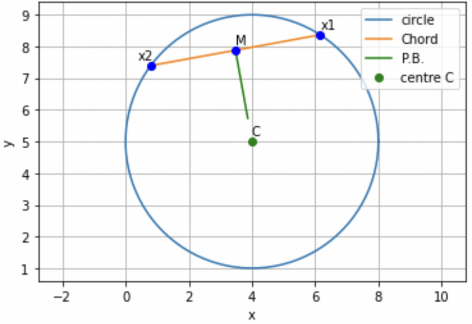
$$\mathbf{x_2}^\top \mathbf{x_2} + 2\begin{pmatrix} g & f \end{pmatrix} \mathbf{x_2} + c = 0 \tag{3}$$

Subtracting the two, we get,

$$\mathbf{x_2}^\top \mathbf{x_2} - \mathbf{x_1}^\top \mathbf{x_1} = -2\begin{pmatrix} g & f \end{pmatrix} (\mathbf{x_2} - \mathbf{x_1}) \tag{4}$$

The midpoint of the chord $\mathbf{x_1}\mathbf{x_2}$ is

$$\mathbf{M} = \frac{\mathbf{x_1} + \mathbf{x_2}}{2} \tag{5}$$



Since perpendicular bisector of $\mathbf{x_1x_2}$ is perpendicular to $\mathbf{x_1x_2}$ and passes through $\mathbf{M}$, any general point $\mathbf{P}$ on the perpendicular bisector can be given using the relation

$$(\mathbf{x_2}-\mathbf{x_1})^\top(\mathbf{P}-\mathbf{M})=0 \tag{6}$$

$$\implies (\mathbf{x_2}-\mathbf{x_1})^\top\mathbf{P}-\left(\mathbf{x_2}^\top-\mathbf{x_1}^\top\right)\left(\frac{\mathbf{x_1}+\mathbf{x_2}}{2}\right)=0 \tag{7}$$

$$\implies (\mathbf{x_2}-\mathbf{x_1})^\top\mathbf{P}=\frac{\mathbf{x_2}^\top\mathbf{x_2}-\mathbf{x_1}^\top\mathbf{x_1}+\mathbf{x_2}^\top\mathbf{x_1}-\mathbf{x_1}^\top\mathbf{x_2}}{2} \tag{8}$$

Since $\mathbf{x_1}^\top \mathbf{x_2} = \mathbf{x_2}^\top \mathbf{x_1}$, the equation reduces to

$$2\left(\mathbf{x_2} - \mathbf{x_1}\right)^\top \mathbf{P} = \mathbf{x_2}^\top \mathbf{x_2} - \mathbf{x_1}^\top \mathbf{x_1} \tag{9}$$

Using (4), we get

$$2\left(\mathbf{x_2} - \mathbf{x_1}\right)^\top \mathbf{P} = -2\begin{pmatrix} g & f \end{pmatrix}\left(\mathbf{x_2} - \mathbf{x_1}\right) \tag{10}$$

$$\implies \left(\mathbf{x_2} - \mathbf{x_1}\right)^\top \mathbf{P} = \left(\mathbf{x_2} - \mathbf{x_1}\right)^\top \begin{pmatrix} -g \\ -f \end{pmatrix} \tag{11}$$

Since the centre of the circle is $\mathbf{C} = \begin{pmatrix} -g \\ -f \end{pmatrix}$, we can clearly see that $\mathbf{P} = \mathbf{C}$ satisfies the equation of perpendicular bisector.
Hence, the perpendicular bisector of any chord of a circle passes through the centre of the circle.

Example

For example, consider the circle

$$\mathbf{x}^\top \mathbf{x} + 2\begin{pmatrix} -4 & -5 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{12}$$

and let $\mathbf{x_1} = \begin{pmatrix} 0 \\ 5 - 2\sqrt{5} \end{pmatrix}$ and $\mathbf{x_2} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$. We can clearly see in the figure that the perpendicular bisector of the chord $\mathbf{x_1 x_2}$ passes through the centre $\mathbf{C}$ of the circle.

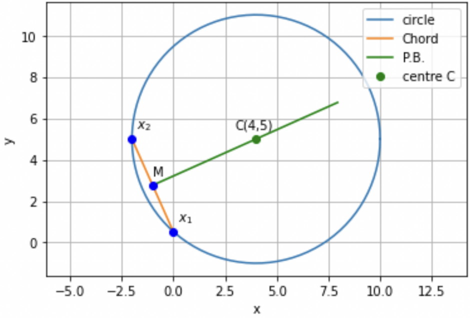


Figure: Example figure