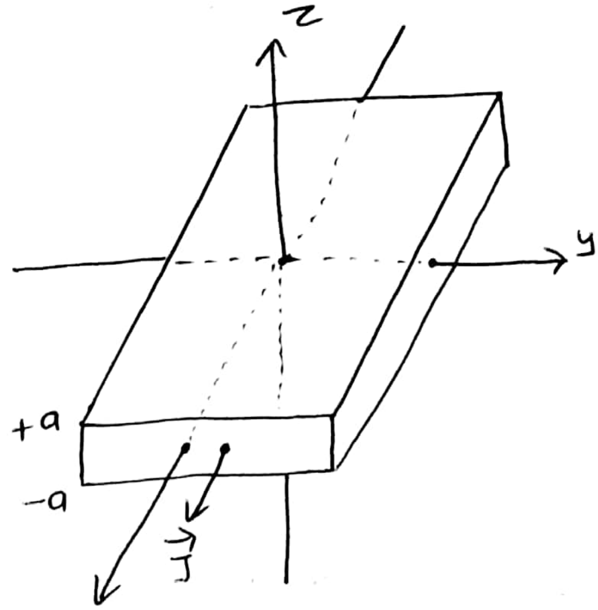
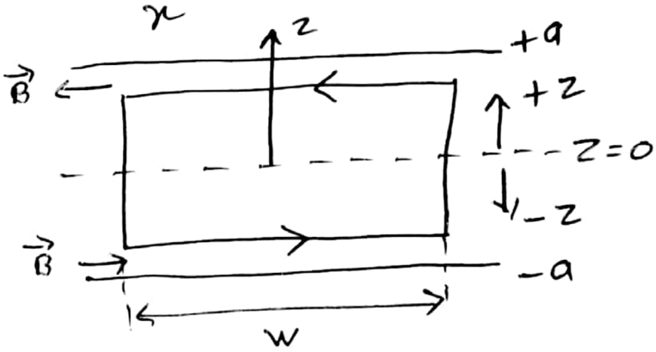
10/10/2017

# Assignment # 9

**# 9.1** Using the right hand rule, magnetic field for $z>0$ will be in $-\hat{y}$ direction and for $z<0$ in the $+\hat{y}$ direction.




Take an Amperian loop of width $w$ and height $2z$ and apply Ampere's law:



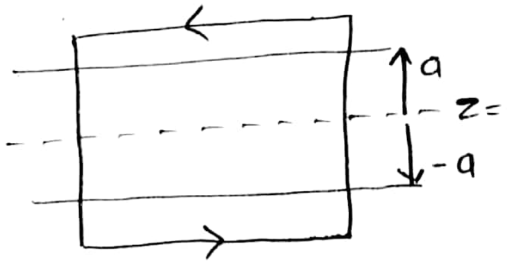

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\Rightarrow |\vec{B}|.2w = \mu_0 J.w.2z$$

$$\Rightarrow |\vec{B}| = \mu_0 z J \qquad \text{for } |z|<a$$

So, $\vec{B} = -\mu_0 z J \hat{y}$ for $|z|<a$

Similarly for $|z|\geqslant a$, we get




$$\vec{B} = -\mu_0 a J \hat{y} \qquad \text{for } |z|\geqslant a$$

So,

$$\vec{B} = \begin{cases} -\mu_0 z J \hat{y} & |z|<a \\ \mp \mu_0 a J \hat{y} & |z|\geqslant a \end{cases} \qquad \begin{pmatrix} -\text{ sign for } z\geqslant 0 \\ +\text{ sign for } z<0 \end{pmatrix}$$

Vector potential $\vec{A}$: $\vec{\nabla}\times\vec{A} = \vec{B}$

So,

$$\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = \begin{cases} -\mu_0 z J & |z|<a \\ \mp \mu_0 a z & |z|\geqslant a \end{cases} \qquad \begin{pmatrix} -\text{ve sign for } z\geqslant 0 \\ +\text{ve sign for } z<0 \end{pmatrix}$$

If we take $A_z = 0$, then:

$$A_x = \begin{cases} -\dfrac{\mu_0 J}{2} z^2 & |z|<a \\ \mp \mu_0 J a z + c & |z|\geqslant a \end{cases}$$

Continuity of potential at $z = \pm a$ gives:

$$-\frac{\mu_0 J a^2}{2} = -\mu_0 J a^2 + c \Rightarrow c = \frac{\mu_0 J a^2}{2}$$

So, $$\vec{A}_1 = -\frac{\mu_0 J}{2} z^2 \hat{i} \qquad (|z| < a)$$

$$= \left(\frac{\mu_0 J a^2}{2} \mp \mu_0 J a z\right)\hat{i} \qquad (|z| \geq a) \text{ (with proper sign)}$$

Alternatively, we have.

$$\frac{\partial A_x}{\partial z} = -\frac{\mu_0 z J}{2} \qquad (|z| < a)$$

$$= \mp \frac{\mu_0 a J}{2} \qquad (|z| > a) \text{ (with proper sign)}$$

and $$\frac{\partial A_z}{\partial x} = +\frac{\mu_0 z J}{2} \qquad (|z| < a)$$

$$= \pm \frac{\mu_0 a J}{2} \qquad (|z| > a) \text{ (with proper sign)}$$

This gives for $z > 0$,

$$A_x = -\frac{\mu_0 J z^2}{4} \qquad (z < a)$$

$$= -\frac{\mu_0 J a}{2} z + c_1 \qquad (z > a)$$

with $c_1 = \frac{\mu_0 J a^2}{4}$ (using boundary condition)

& $$A_z = \frac{\mu_0 z J}{2} x \qquad z < a$$

$$= \pm \frac{\mu_0 a J}{2} x + c_2 \qquad z > a$$

with $c_2 = 0$ (using boundary condition)

So, $$\vec{A_2} = -\frac{\mu_0 J z^2}{4}\hat{i} + \frac{\mu_0 J z x}{2}\hat{k} \qquad (z<a)$$

$$= \left(\frac{\mu_0 J a^2}{4} - \frac{\mu_0 J a z}{2}\right)\hat{i} + \frac{\mu_0 J a x}{2}\hat{k} \qquad (z>a)$$

So, $$\vec{A_2} - \vec{A_1} = \frac{\mu_0 J z^2}{4}\hat{i} + \frac{\mu_0 J z x}{2}\hat{k} \qquad (z<a)$$

$$= \left(-\frac{\mu_0 J a^2}{4} + \frac{\mu_0 J a z}{2}\right)\hat{i} + \frac{\mu_0 J a x}{2}\hat{k} \qquad (z>a)$$

This difference can be shown as the gradient of a function $f(x,y)$ such that, (for $z>0$)

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\mu_0 J z^2}{4} \\ \frac{\partial f}{\partial z} &= \frac{\mu_0 J z x}{2} \end{aligned}\right] \quad (z<a)$$

&

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= -\frac{\mu_0 J a^2}{4} + \frac{\mu_0 J a z}{2} \\ \frac{\partial f}{\partial z} &= \frac{\mu_0 a J x}{2} \end{aligned}\right] \quad (z>a)$$

so, $$f(x,y) = \frac{\mu_0 J z^2 x}{4} \qquad (z<a)$$

$$= -\frac{\mu_0 J a^2}{4}x + \frac{\mu_0 J a z x}{2} \qquad (z>a)$$

#9.2 To solve this problem, firstly we will calculate the vector potential for a spherical shell of radius (R) carrying a uniform surface charge density ($\sigma$), spinning at angular velocity $\vec{\omega}$.

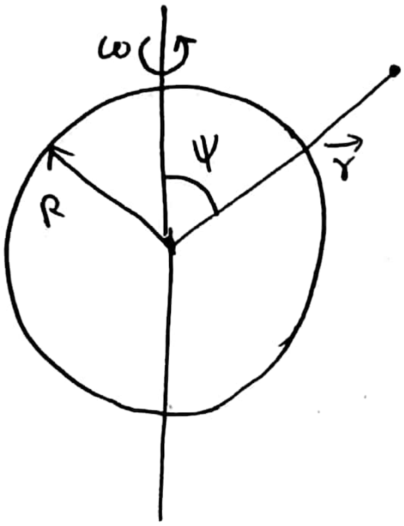



Fig (1)

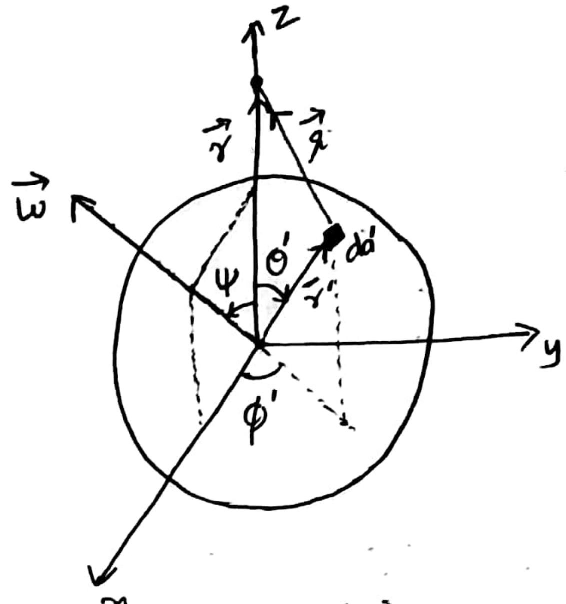



Fig (2)

Most common would be to set the polar axis along $\vec{\omega}$ (Fig 1) but calculations will be easier if we put $\vec{r}$ along $\hat{z}$ and $\vec{\omega}$ is tilted at angle $\psi$ (Fig 2) and $\vec{\omega}$ lies in the $xz$ plane.

Vector potential for surface current:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')}{\mathfrak{r}}\, da'$$

$\vec{K} = \sigma \vec{v}$ &

$\mathfrak{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$

$da' = R^2 \sin\theta'\, d\theta'\, d\phi'$

velocity of point $\vec{r}'$ is given by $\vec{\omega} \times \vec{r}'$ so,

$$\vec{v} = \vec{\omega}\times\vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\Big[-(\cos\psi\sin\theta'\sin\phi')\,\hat{x} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{y} + (\sin\psi\sin\theta'\sin\phi')\,\hat{z}\Big]$$

Each term which involve $\sin\phi'$ or $\cos\phi'$ will vanish because $\int_0^{2\pi} \sin\phi' d\phi' = \int_0^{2\pi} \cos\phi' d\phi' = 0$

So,
$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \int_0^{\pi} \frac{\cos\theta' \sin\theta' d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \hat{y}$$

substituting $u = \cos\theta'$, integral becomes
$$\int_{-1}^{+1} \frac{u\,du}{\sqrt{R^2 + r^2 - 2Rru}} = -\frac{(R^2 + r^2 + Rru)}{3R^2 r^2} \sqrt{R^2 + r^2 - 2Rru}\,\Big|_{-1}^{+1}$$

$$= -\frac{1}{3R^2 r^2}\left[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r)\right]$$

If point $\vec{r}$ lies inside the sphere, $R > r$, and the expression reduces to $\left(\frac{2r}{3R^2}\right)$, and if $\vec{r}$ lies outside the sphere, then $R < r$, it reduces to $\left(\frac{2R}{3r^2}\right)$. Also, $(\vec{\omega} \times \vec{r}) = -\omega r \sin\psi\,\hat{y}$. So,

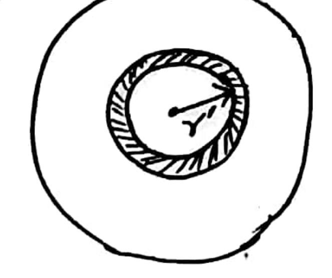
$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 R \sigma}{3} (\vec{\omega} \times \vec{r}) & r < R \\ \frac{\mu_0 R^4 \sigma}{3 r^3} (\vec{\omega} \times \vec{r}) & r > R \end{cases}$$

Let's revert back to natural coordinates as Fig (1) where $\vec{\omega}$ is in $\hat{z}$ direction,

$$\vec{A}(r, \theta, \phi) = \begin{cases} \frac{\mu_0 R \omega \sigma}{3} r \sin\theta\,\hat{\phi} & r \leq R \\ \frac{\mu_0 R^4 \omega \sigma}{3} \frac{\sin\theta}{r^2}\,\hat{\phi} & r \geq R. \end{cases}$$

Now we will calculate magnetic field due to uniformly charged sphere.

A uniformly charged sphere can be divided into shells, with each shell carrying total charge $= 4\pi r'^2 dr' \rho$




So, surface charge density on the surface of a shell of radius $r'$ & thickness $dr'$

$$\sigma(r') = \frac{4\pi r'^2 dr' \rho}{4\pi r'^2} = \rho dr'$$

This shell produces following vector potential

$$d\vec{A}(r,\theta,\phi) = \begin{cases} \frac{\mu_0 r' \omega \rho dr'}{3} r\sin\theta\,\hat{\phi} & r \leq r' \\ \frac{\mu_0 r'^4 \omega \rho dr'}{3} \frac{\sin\theta}{r^2}\hat{\phi} & r \geq r' \end{cases}$$

So, total vector potential due to entire sphere is

$$\vec{A}(r,\theta,\phi) = \frac{\mu_0 \omega \rho}{3}\frac{\sin\theta}{r^2}\int_0^r r'^4 dr'\,\hat{\phi} + \frac{\mu_0 \omega \rho}{3} r\sin\theta \int_r^R r' dr'\,\hat{\phi}$$

$$= \frac{\mu_0 \omega \rho \sin\theta}{3}\left[\frac{R^2 r}{2} - \frac{r^3}{2} + \frac{r^3}{5}\right]\hat{\phi}$$

$$\boxed{\vec{A}(r,\theta,\phi) = \frac{\mu_0 \omega \rho}{3}\left(\frac{R^2 r}{2} - \frac{3r^3}{10}\right)\sin\theta\,\hat{\phi}}$$

So, $\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta}$

$$= \frac{\mu_0\omega\rho}{3}\frac{1}{r\sin\theta}\left(\frac{R^2 r}{2} - \frac{3r^3}{10}\right)2\sin\theta\cos\theta\,\hat{r} - \frac{\mu_0\omega\rho}{3}\frac{1}{r}\left(R^2 r - \frac{6}{5}r^3\right)\sin\theta\,\hat{\theta}$$

$$= \frac{\mu_0\omega\rho}{3}\left[\left(R^2 - \frac{3}{5}r^2\right)\cos\theta\,\hat{r} - \left(R^2 - \frac{6r^2}{5}\right)\sin\theta\,\hat{\theta}\right]$$

$$= \frac{\mu_0\omega Q}{\frac{4}{3}\pi R^3 . 3}\left[\left(R^2 - \frac{3}{5}r^2\right)\cos\theta\,\hat{r} - \left(R^2 - \frac{6r^2}{5}\right)\sin\theta\,\hat{\theta}\right]$$

$$\Rightarrow \vec{B} = \frac{\mu_0\omega Q}{4\pi R^3}\left[\left(R^2 - \frac{3}{5}r^2\right)\cos\theta\,\hat{r} - \left(R^2 - \frac{6r^2}{5}\right)\sin\theta\,\hat{\theta}\right]$$

CHECK: $\vec{\nabla} \cdot \vec{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, B_\theta)$

$$= \frac{\mu_0 \omega Q}{4\pi R^3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(R^2 r^2 - \frac{3r^4}{5}\right)\cos\theta - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left\{\left(R^2 - \frac{6r^2}{5}\right)\sin^2\theta\right\}\right]$$

$$= 0.$$

&

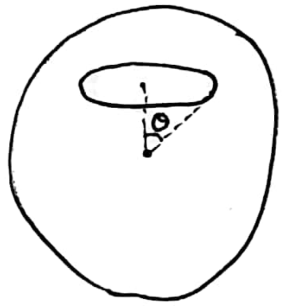
$$\vec{\nabla}\times\vec{B} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{\phi}$$

$$= \frac{\mu_0 \omega Q}{4\pi R^3 r}\left[-\left(R^2 - \frac{18r^2}{5}\right)\sin\theta + \left(R^2 - \frac{3}{5}r^2\right)\sin\theta\right]\hat{\phi}$$

$$\vec{\nabla}\times\vec{B} = \frac{3\mu_0 \omega Q r}{4\pi R^3}\sin\theta\,\hat{\phi}$$

R.H.S. should be equal to $\mu_0 \vec{J}$. So let's check if it matches.

$$\vec{J}(r,\theta,\phi) = \frac{\text{current due to the ring shown}}{\text{area}}\,\hat{\phi}$$



$$= \frac{\frac{\omega}{2\pi}\cdot 2\pi r^2 dr \sin\theta\, d\theta \cdot \rho}{r\,dr\,d\theta}$$

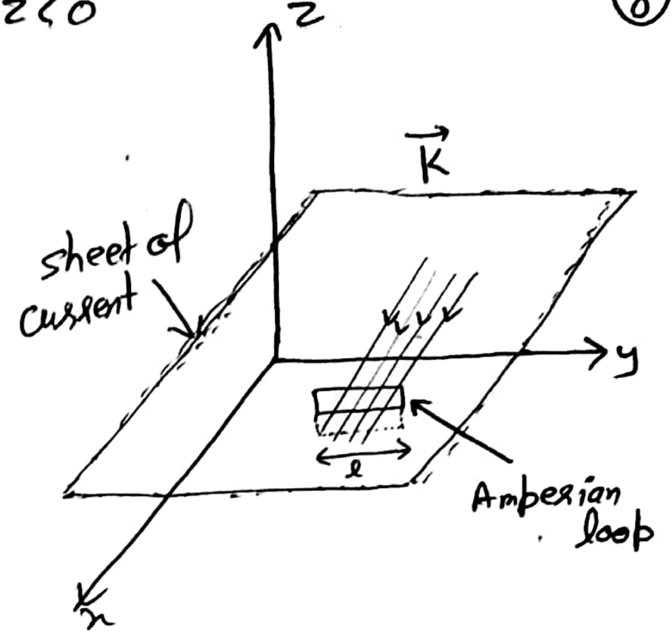
$$= \omega r \rho \sin\theta\,\hat{\phi}$$

$$= \frac{3\omega r Q \sin\theta}{4\pi R^3}\,\hat{\phi}$$

So, $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

$\vec{B}$ is correct as it correctly satisfies both divergence and curl equations.

#9.3 The field will be in $\hat{y}$ for $z<0$ and $-\hat{y}$ for $z>0$




So,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$= B\cdot 2l = \mu_0 K\cdot l$$

$$\Rightarrow \vec{B} = \mp \frac{\mu_0 K}{2}\hat{y} \quad \begin{matrix} - \text{ for } z>0 \\ + \text{ for } z<0 \end{matrix}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

for $B_y$, $\Rightarrow B_y = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$

(i) If $A_x = 0 \Rightarrow A_z = +\frac{\mu_0 K x}{2} \quad z>0$

$= -\frac{\mu_0 K x}{2} \quad z<0$

(ii) If $A_z = 0 \Rightarrow A_x = -\frac{\mu_0 K z}{2} \quad z>0$

$= +\frac{\mu_0 K z}{2} \quad z<0$

Calculating $\vec{A}$ from direct integration:

$$\vec{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}\,d\tau'$$

Now, $\vec{J}(\bar{r}') = K\,\delta(z')\,\hat{x}$

Because $\int \vec{J}(\bar{r}')dz' = \int K\,\delta(z')dz' = K$

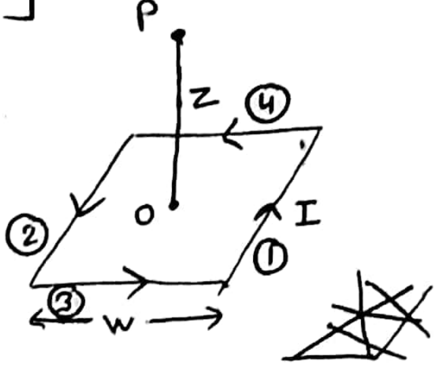
So, $\vec{A}(\bar{r}) = \frac{\mu_0 K \hat{x}}{4\pi}\int \frac{dx'dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$

Now, from electrostatics, uniformly charged sheet in $x,y$ plane, the electrostatic potential,

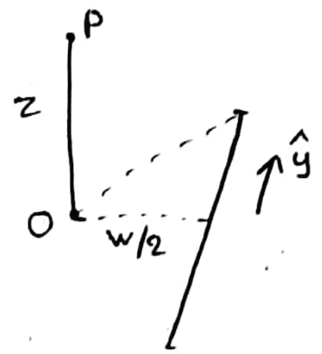
$$\phi(\bar{r}) = \frac{\sigma}{4\pi\epsilon_0}\int \frac{dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}} = \mp\frac{\sigma z}{2\epsilon_0} \quad \begin{matrix} - \text{ for } z>0 \\ + \text{ for } z<0 \end{matrix}$$

So, $$\int \frac{dx'dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}} = \mp 2\pi z$$

So, $$\vec{A}(\bar{r}) = \mp \frac{\mu_0 K z}{2}\hat{x} \qquad \begin{bmatrix} - \text{ for } z>0 \\ + \text{ for } z<0 \end{bmatrix}$$




**# 9.4** The problem requires calculation of field $\vec{B}$ due to a finite size square loop. Field due to each wire of the loop will be the same and after addition only vertical component servies.




$$\vec{B}_1 = \frac{\mu_0 I}{4\pi}\int \frac{d\bar{l}'\times(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{dy'\hat{j}\times(z\hat{k}-\frac{w}{2}\hat{i}-y'\hat{j})}{\left(\frac{w^2}{4}+y'^2+z^2\right)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{dy'(z\hat{i}+\frac{w}{2}\hat{k})}{\left(\frac{w^2}{4}+y'^2+z^2\right)^{3/2}}$$

From ~~the other side of~~ the wire on the other side, we get

$$\vec{B}_2 = \frac{\mu_0 I}{4\pi}\int \frac{-dy'\hat{j}\times(z\hat{k}+\frac{w}{2}\hat{i}-y'\hat{j})}{\left(\frac{w^2}{4}+z^2+y'^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\int \frac{dy'(-z\hat{i}+\frac{w}{2}\hat{k})}{\left(\frac{w^2}{4}+z^2+y'^2\right)^{3/2}}$$

So, $$\vec{B}_1+\vec{B}_2 = \vec{B}_{1+2} = \frac{\mu_0 I W}{4\pi}\int_{-W/2}^{W/2} \frac{dy'}{\left(\frac{W^2}{4}+z^2+y'^2\right)^{3/2}}$$

substitute $y' = \sqrt{\frac{w^2}{4}+z^2}\tan\theta \Rightarrow dy' = \sqrt{\frac{w^2}{4}+z^2}\sec^2\theta\, d\theta$

limit from $\theta = -\tan^{-1}\left[\frac{w}{2\sqrt{\frac{w^2}{4}+z^2}}\right]$ to

$\theta = +\tan^{-1}\left[\frac{w}{2\sqrt{\frac{w^2}{4}+z^2}}\right]$

$$\Rightarrow \vec{B}_{1+2} = \frac{\mu_0 I W}{4\pi} \frac{1}{\left(\frac{W^2}{4}+z^2\right)} \cdot \int_{-\tan^{-1}\left[\frac{W}{2\sqrt{\frac{W^2}{4}+z^2}}\right]}^{+\tan^{-1}\left[\frac{W}{2\sqrt{\frac{W^2}{4}+z^2}}\right]} d\theta$$

$$\vec{B}_{1+2} = \frac{\mu_0 I W}{4\pi} \frac{1}{\left(\frac{W^2}{4}+z^2\right)} 2\tan^{-1}\left[\frac{W}{2\sqrt{\frac{W^2}{4}+z^2}}\right] \hat{z}$$

So, total field for all 4 wires of the loop,

$$\vec{B}(z) = \frac{\mu_0 I W}{4\pi} \frac{1}{\left(\frac{W^2}{4}+z^2\right)} \cdot 4\tan^{-1}\left[\frac{W}{2\sqrt{\frac{W^2}{4}+z^2}}\right]$$

$$\Rightarrow \boxed{\vec{B}(z) = \frac{\mu_0 I W}{\pi} \frac{4}{(W^2+4z^2)} \tan^{-1}\left(\frac{W}{\sqrt{W^2+4z^2}}\right)}$$

For $z \gg W$, we get

$$\vec{B} \simeq \frac{\mu_0 I W}{\pi} \frac{4}{4z^2} \cdot \frac{W}{2z} = \frac{\mu_0}{4\pi}\left(\frac{2 I W^2}{z^3}\right)$$

Magnetic field due to magnetic dipole,

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{m}\cdot\hat{z})\hat{z} - \vec{m}}{z^3}$$

for $\vec{m} = I W^2 \hat{z}$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{2 I W^2}{z^3} \hat{z} \quad \Rightarrow \text{same as above.}$$