Lecture 1: Matrices, System of linear equations, Gauss Elimination Method

Matrix: A matrix is a rectangular array of numbers or symbols, arranged in rows and columns. We say a matrix has order $m \times n$ if there are m rows and n columns. We write a matrix of order $m \times n$ as follows:

$$A_{m\times n} = \begin{array}{cl} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{l} \\ \\ \\ \to i\text{th row} \\ \\ \\ \end{array} \\ \downarrow \\ j\text{th column} & \end{array}$$

The element a_{ij} is called the ij^{th} entry of $A_{m\times n}$.
In short, we write a matrix as $A = (a_{ij})$
where $i = 1, 2, \ldots, m$ & $j = 1, 2, \ldots, n$

Addition of Matrices: Let A and B be two matrices of same order $m \times n$. Let $A = (a_{ij})$, $B = (b_{ij})$ where $i = 1, 2, \ldots, m$ & $j = 1, 2, \ldots, n$ then $A + B$ is a matrix of order $m \times n$ given by

$$A + B = (a_{ij} + b_{ij})$$

Multiplication of Matrices Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$. The matrix AB can be defined if and only if $n = p$ i.e. number of columns of A is same as number of rows of B. The matrix AB is of order $m \times q$ and is defined as follows:

$$AB = (c_{ij})$$

where $$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

$$A = (a_{is}) \quad \& \quad B = (b_{tj})$$

$i = 1, 2, \ldots, m$
$j = 1, 2, \ldots, q$
$s = 1, 2, \ldots, n$
$t = 1, 2, \ldots, p$

Matrix theory has applications widely in every branches of scientific and engineering field. For example, matrices are useful in determining positions of a body in a space under rigid motions.

Let M be a rigid body in 3-dimensional space and (m_1, m_2, m_3) be coordinate of center of mass of M. Suppose M is rotated by an angle θ about z-axis. Then new coordinate of center of mass is

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

In Mathematics, matrix theory is used in solving system of linear equations, determining curvatures of surfaces, adjacency relation in Graph Theory etc.

System of Linear Equations

Consider m- linear equations with n-unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots \cdots \cdots \cdots$$

$$a_{m1} + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

This system of linear equations can be written as

$$AX = B$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Gaussian Elimination Method

Consider the equations:

$$2x + 3y = 1 \quad \cdots\cdots \quad (a)$$
$$x + 2y = 2 \quad \cdots\cdots \quad (b)$$

Perform the operation $(b) - \frac{1}{2} \times (a)$, the above equations become

$$2x + 3y = 1 \quad \cdots\cdots \quad (a_1)$$
$$0x + \frac{1}{2}y = 3/2 \quad \cdots\cdots \quad (b_1)$$

Thus $y = 3$.

Put $y = 3$ in (a_1). we get $x = -4$

In terms of language of matrices, we have done the following:

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 2 & 2 \end{array}\right) \xrightarrow{R_2' = R_2 - \frac{1}{2} R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1/2 & 3/2 \end{array}\right)$$

Augmented Matrix

Row operations.

Suppose we are given a system of linear equations $Ax = B$, where A is a square matrix of order n i.e A is of order $n \times n$.

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

The Augmented Matrix is $\left(A \mid B \right)$

Gauss-Elimination Method is to apply 'row operations' on the Augmented Matrix so that A becomes an upper triangular matrix i.e. after row operations A turns out to be →

$$\begin{pmatrix} \times & \times & \times & \cdots & \times \\ 0 & \times & \cdots & \cdots & \times \\ 0 & 0 & \times & \cdots & \times \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots 0 & \times \end{pmatrix}_{n \times n}$$

→ Upper triangular Matrix (all entries below the diagonal entries are zero i.e. $a_{ij} = 0 \ \forall \ i > j$)

Row Operations: There are 3 types of elementary row operations performed on a matrix

Type 1: Swap the positions of two rows.

Type 2: Multiply a row by a non-zero scalar.

Type 3: Add to one row a scalar multiple of another row.

Apply row operations on the Augmented matrix $(A|B)$ so that A becomes an upper triangular matrix.

Let $(A'|B')$ be the transformed augmented matrix after applying row operations on $(A|B)$, where A' is upper triangular.

The solutions for the system of linear equations $Ax = B$ is same as that of

$$A'x = B'$$

It is easy to find solutions of $A'x = B'$.

☆ A system of linear equations can have no solution, for example

$$2x + y = 1$$
$$6x + 3y = 2$$

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 6 & 3 & 2 \end{array}\right) \xrightarrow{R_2' = R_2 - 3R_1} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right)$$

$$2x + y = 1$$
$$0x + 0y = -1$$

clearly has no solutions

☆ A system of linear equations can have infinitely many solutions, ex. $2x + 3y = 1$ . .

Proposition: A system of linear equations $AX = B$ either has no solution or one solution or infinitely many solutions. (Here, A need not be square matrix)

Proof: Suppose there exists two solutions u & v of $AX = B$ such that $u \neq v$.

Then, $Au = B$ & $Av = B$

$$\Rightarrow A(u - v) = 0$$

$$\Rightarrow A(\alpha(u - v)) = 0 \quad \forall \alpha \in \mathbb{R} \text{ (set of real numbers)}$$

Thus, $A(u + \alpha(u - v)) = B$

$\Rightarrow u + \alpha(u - v)$ is also a solution of $AX = B$

This is true for all $\alpha \in \mathbb{R}$, hence there exists infinitely many solutions.