Lecture 2 : Echelon form, Elementary Matrices & Row Reduction.

Consider system of linear equations

$$AX = B \quad \text{where}$$

$A = (a_{ij})$ is a matrix of order $m \times n$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ & $B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

To solve it, by applying row operations we will redued the augmented matrix $(A|B)$ to 'row reduced echelon form'.

Row Echelon Form : A matrix is said to be in Row Echelon Form if

* all non-zero rows (rows with atleast one non-zero element) are above any rows of all zeroes. (i.e. all zero rows will be placed in the bottom of the matrix)

* the leading coeffecient (i.e. the first non-zero element from left of a non-zero row) of a non-zero row is always strictly to the right of the leading coeffocient of the row above it.

Example:

$$\begin{pmatrix} 2 & 5 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}$$ or

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For further simplification, we will apply row operations so that the augmented matrix is of 'reduced row echelon form'

Reduced Row Echelon Form (RREF):

A matrix is in RREF if

* it is in row echelon form
* every leading coeffecient is 1 (one) and is the only non-zero entry in its column.

Example:

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Consider the system of linear equations:

$$2x + 3y + 4z = 5$$
$$3x + 4y + 5z = 6$$
$$x + 2y + 3z = 4$$

The Augmented Matrix is

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \end{array}\right) \xrightarrow{R_{13}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \end{array}\right)$$

$$\xrightarrow{R_{23}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{array}\right)$$

$$\xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{(-1)R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1-2R_2}$$

Row Reduced Echelon Form $\rightarrow$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Identity Matrix: It is a square matrix all whose diagonal entries are 1 (one) and rest of the entries are zeroes. An identity matrix of order n is denoted by I_n.

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}$$

Elementary Matrices: Let I_n be identity matrix of order n.

There are 3 types of elementary matrices:

(i) $E_i(c)$ is the matrix obtained from I_n by multiplying i-th row by non-zero real number c.

(ii) E_{ij} is obtained from I_n by interchanging i-th and j-th row.

(iii) $E_{ij}(c)$ is obtained from I_n by adding i-th row with c multiple of j-th row, where $c \in \mathbb{R} \setminus \{0\}$.

Effect of Elementary Matrices on Multiplication

Let $A = (a_{ij})$ be a matrix of order $m \times n$.

(i) Pre-multiplication : Let E be an elementary matrix of order m.

★ If $E = E_i(c)$, then $EA = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ ca_{i1} & \cdots & ca_{in} \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

is the matrix obtained from A by multiplying ith row by $c \in \mathbb{R} \setminus \{0\}$.

★ If $E = E_{ij}$, then EA is the matrix obtained from A by swapping ith row of A with jth row.

★ If $E = E_{ij}(c)$, then EA from A by adding ith row of A to c multiple of jth row.

Observe that pre-multiplication by elementary matrices is same as performing row operations.

(ii) Post-multiplication: Let E be an elementary matrix of order n. Then AE is one of the followings:

(a) multiplying jth column of A by $c\ (\neq 0)$

(b) interchanging ith column of A with jth column.

(c) adding ith column to c multiple of jth column.

Theorem: Let A be a matrix of order $m \times n$. Then by applying row and column operations on A i.e. (pre-multiplying and post-multiplying by a sequence of elementary matrices) A can be reduced to the form

$$\begin{pmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{pmatrix}$$ where 'O' is zero matrix,

where $0 \leq r \leq \min\{m, n\}$

Proof: If there exists an entry $a_{ij} \neq 0$ of A, then by interchanging ith row with 1st row and jth column with 1st column, we will get a matrix with entry in $(1,1)$ position to be a_{ij}. Multiply the 1st row by $1/a_{ij}$. Then by applying row and column operations of 3rd type on this matrix to get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & \vdots & & & \\ \vdots & & & & \\ 0 & b_{m1} & \cdots & \cdots & b_{mn} \end{pmatrix}$$

Continue this process on the matrix (b_{ij}). $\square$

Similarly, we can prove that a square matrix can be reduced to RREF by applying elementary operations.

Gauss - Jordan Method of Reduction :

Theorem: Let A be a matrix of any order. Then by applying elementary row operations on A, it can be reduced to RREF.

Proof:- We will prove it by applying induction on number of columns of A.

Let n be number of columns of A.

$n=1$: Suppose A has m rows then

$$A = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$$

if $\exists\ a_{r1} \neq 0$, we apply row operation R_{r1} & then row operations $R_1\left(\frac{1}{a_{r1}}\right)$ & of 3rd type to get

A of the form $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (which is RREF)

Thus, A can be reduced to $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

The statement is true for $n=1$.

Let the statement be true for all matrices with number of columns $\leq n-1$.

Let A be a matrix of order $m \times n$.

Case i) Suppose in the 1st column $\exists\ a_{r1} \neq 0$

Perform R_{r1} & then $R_1\left(\frac{1}{a_{r1}}\right)$ & then row operations of type 3 to get A of the form

$$\begin{pmatrix} 1 & b_{11} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & & \cdots & \\ 0 & b_{m2} & & b_{mn} \end{pmatrix}$$

Let $B = \begin{pmatrix} b_{22} & \cdots & b_{2n} \\ - & - & - \\ b_{m2} & \cdots & b_{mn} \end{pmatrix}$. By induction hypothesis, B can be reduced to RREF B' by row operations. Applying these row operations on $\begin{pmatrix} 1 & b_{11} & \cdots & b_{1n} \\ 0 & & B & \\ 0 & & & \end{pmatrix}$

we get $\begin{pmatrix} 1 & b_{11} & \cdots & b_{1n} \\ 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{pmatrix}$

For $i>1$, if b_{ij} is leading coefficient then by a row operation of type 3, we can make $b_{1j} = 0$.

Thus A is transformed to RREF

Case ii) 1st column of A is $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

def $B = \begin{pmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m2} & \cdots & a_{mn} \end{pmatrix}$, then

$$A = \begin{pmatrix} 0 & B \end{pmatrix}$$

B has $(n-1)$ columns, by induction hypothesis B can be reduced to RREF. Thus, A can be reduced to RREF.

FACT: RREF of matrix is unique i.e. a matrix cannot be transformed to two different RREF