Graphical Structure of Unsatisfiable Boolean Formulae

PhD thesis defence

Vaibhav Karve
Department of Mathematics
University of Illinois at Urbana-Champaign

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Outline

Introduction

Definitions

2GraphSAT

Local Rewriting in Graphs

Reduction rules

Computational results

Conclusion

Key results

2GraphSAT

- a new graph decision problem
- invariant under graph homeomorphism (topological minoring)
- a complete (and finite) list of minimal obstructions (simple graphs)

3GraphSAT

- graph rewrite/reduction rules that preserve satisfiability
- systematic computer-aided search on looped-multi-hypergraphs
- an incomplete list of minimal obstructions
- a Python package called `graphsat`

Local graph rewriting theorem

$$G[v] \;=\; \mathrm{sphere}(G;v);\mathrm{star}(G;v)[v] \;=\; \bigcup_{\substack{g_i\ h_i\ :\ \mathrm{Graph} \\ g_i,h_i\ =\ \mathrm{link}(G,v)}} \mathrm{sphere}(G;v);(g_i \vee h_i)$$

CNFs

Definition (CNF)

A boolean formula is in *Conjunctive Normal Form* if it is a conjunction of disjunctions of variables and their negations.

For example,

$$(x_1 \text{ or } x_2 \text{ or } \text{ not } x_3) \text{ and } (x_3 \text{ or } x_4) \text{ and } (\text{ not } x_4)$$

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$$

- Variable $\equiv$ an element of a countably infinite set, i.e. $x_1, x_2, \ldots$
- Literal $\equiv$ a variable or its negation, i.e. $x_1, \neg x_1, x_2, \neg x_2, \ldots$
- Clause $\equiv$ a nonempty set of literals
- CNF $\equiv$ a nonempty set of clauses

$$\{\{x_1, x_2, \neg x_3\}, \{x_3, x_4\}, \{\neg x_4\}\}$$

SAT

Definition (Boolean Satisfiability)

A CNF is *satisfiable* if there exists a satisfying truth-assignment. Otherwise, it is *unsatisfiable*.

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For example,

- $[x_1 := \top;\ x_3 := \top;\ x_4 := \top]$ does not satisfy $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$
- $[x_1 := \top;\ x_3 := \top;\ x_4 := \bot]$ satisfies $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$
- Therefore, $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$ is satisfiable.

If we insist on constructing an unsatisfiable CNF —

- $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$ is satisfiable because $[x_1 := \top;\ x_3 := \top;\ x_4 := \bot]$

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If we insist on constructing an unsatisfiable CNF —

- $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$ is satisfiable because $[x_1 := \top;\ x_3 := \top;\ x_4 := \bot]$
- $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$ is satisfiable because $[x_1 := \bot;\ x_2 := \bot;\ x_3 := \top;\ x_4 := \bot]$

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A CNF is *satisfiable* if there exists a satisfying truth-assignment. Otherwise, it is *unsatisfiable*.

For example,

- $[x_1 := \top;\ x_3 := \top;\ x_4 := \top]$ does not satisfy $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$
- $[x_1 := \top;\ x_3 := \top;\ x_4 := \bot]$ satisfies $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$
- Therefore, $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$ is satisfiable.

If we insist on constructing an unsatisfiable CNF —

- $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$ is satisfiable because $[x_1 := \top;\ x_3 := \top;\ x_4 := \bot]$
- $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$ is satisfiable because $[x_1 := \bot;\ x_2 := \bot;\ x_3 := \top;\ x_4 := \bot]$
- $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_2 \vee \neg x_3 \vee x_4)$ is unsatisfiable.

SAT

History

- Stephen Cook (1971) & Leonid Levin (1973)
- First NP-complete problem

Definition (SAT decision problem)

- Instance: A boolean formula in conjunctive normal form.
- Question: Is the given formula satisfiable?
- Certificate: If yes, then the certificate is a truth assignment. If no, then there is no certificate.

It is easier to verify that a CNF is satisfiable, than to prove that it is unsatisfiable.

SAT using a computer

To verify that $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4)$ is satisfiable —

```
from cnf import cnf, Cnf
from sat import cnf_pysat_satcheck as sat

x : Cnf = cnf([[1, 2, -3], [3, 4], [-4]])
print(sat(x))
```

```
True
```

To prove that $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_2 \vee \neg x_3 \vee x_4)$ is unsatisfiable —

```
x2 : Cnf = cnf([[1, 2, -3], [3, 4], [-4], [-1, -3, 4], [-2, -3, 4]])
print(sat(x2))
```

```
False
```

Graphs

- Vertex $\equiv$ an element of a countably infinite set, i.e. points/nodes.
- Edge $\equiv$ a nonempty set of vertices, i.e. loops, simple connections, hyperedges.
- Graph $\equiv$ a nonempty set of edges, i.e. a looped-multi-hypergraph.

Examples

Graphs | CNFs

Each graph "supports" a set of CNFs. Each CNF supported on a graph has the same underlying structure.

$$(v_1 v_2 v_3);\ (v_3 v_4);\ (v_4) \quad \longleftrightarrow \quad \begin{array}{rl} 1. & (x_1 \vee x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_4) \\ 2. & (x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (x_4) \\ 3. & (x_1 \vee \neg x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_4) \\ & \vdots \\ 64. & (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_3 \vee \neg x_4) \wedge (\neg x_4) \end{array}$$

Definition (Satisfiability of graphs)

If every CNF supported on a graph is satisfiable, then the graph itself is *satisfiable*. If any CNF supported on a graph is unsatisfiable, then the graph is *unsatisfiable*.

$$G \text{ is sat} \iff \forall\, (x : \text{Cnf}) \in G;\ x \text{ is sat}$$

SAT | GraphSAT

Definition (Satisfiability of graphs)

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SAT decision problem

- Instance: A CNF.
- Question: Is the given formula satisfiable?
- Certificate: If yes, then the certificate is a truth assignment.
  If not, then there is no certificate.

It is easier to verify that a CNF is satisfiable, than to prove that it is unsatisfiable.

GraphSAT decision problem

- Instance: A looped-multi-hypergraph.
- Question: Is the given graph satisfiable?
- Certificate: If not, then the certificate is an unsatisfiable CNF supported on the graph.
  If yes, then there is no certificate.

It is easier to verify that a graph is unsatisfiable, than to prove that it is satisfiable.

SAT | GraphSAT

2SAT	$(x_1 \vee \neg x_2) \wedge (x_2 \vee x_3)$	P
3SAT	$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \neg x_4)$	NP-complete

2GraphSAT	$(v_1 v_2)_; (v_2 v_3)$	P
3GraphSAT	$(v_1 v_2 v_3)_; (v_2 v_3 v_4)$	??

2GraphSAT is in complexity class P

This is work that was presented in the Prelim exam.

Theorem (Polynomial time algorithm for 2GraphSAT)

Let G be a looped-multi-graph with n vertices. There exists an algorithm that can decide whether G is satisfiable in $\mathcal{O}(n)$ steps.

Proof sketch.

1. G is sat $\iff$ G does not contain four specific graphs as a topological minor.
   We will sketch this on the next slide.

2. Searching for fixed topological minors is an $\mathcal{O}(n)$ problem.
   »» N. Robertson and P.D. Seymour. "Graph Minors XIII. The Disjoint Paths Problem". In: *Journal of Combinatorial Theory, Series B* 63.1 (1995), pp. 65–110. ISSN: 0095-8956

3. Since we search for finitely many graphs, the algorithm still runs in $\mathcal{O}(n)$ steps.

$\square$

GraphSAT is homeomorphism-invariant

De nition (Topological minors)

The following operations are allowed

1. deletion of vertices
2. deletion of edges
3. smoothing of degree=2 vertices: $X; 12; 23\ 7! \quad X; 13$ where X has no edges incident on 1.

If we only allow operation 3., we obtain homeomorphic graphs.

GraphSAT is homeomorphism-invariant

We only need to consider graphs up to homeomorphisms!

Example (Satisfiability of graph families)

1. Tree graphs are satisfiable.
2. Cycle graphs are satisfiable.
3. A graph is satisfiable $\iff$ each of its connected components is satisfiable.
4. Graph containing an unsatisfiable subgraph is unsatisfiable.

Analogy for understanding local graph rewriting

CNFs can be evaluated at a variable

x : CNF $= (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_4)$
x_1 : Variable

$$\begin{aligned} x[x_1] &= (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_4)[x_1] \\ &= (x_1 \vee \neg x_2)[x_1] \wedge (x_2 \vee x_3)[x_1] \wedge (\neg x_4)[x_1] \\ &= (\top \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_4) \\ &= (x_2 \vee x_3) \wedge (\neg x_4) \end{aligned}$$

Evaluating at a variable and its negation keeps CNF-satisfiability unchanged!

Analogy for understanding local graph rewriting

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Evaluating at a variable and its negation keeps CNF-satisfiability unchanged!

	Physics analogy	Math analogy
CNF	state of particle	a function on boolean variables
Graph	superposition of many states	a family of functions
CNF evaluated at a variable	observing a collapsed state	function evaluated at a point
Graph evaluated at a vertex	sum over all states/paths/histories	??

Local graph rewriting

Theorem (Local graph rewriting)

Let G be a graph and let v be a vertex of G. Evaluating G at v yields the following set of CNFs:

$$G[v] = sphere(G; v); star(G; v)[v] = \bigcup_{\substack{g_i\ h_i\ :\ Graph \\ g_i, h_i\ =\ link(G, v)}} sphere(G; v); (g_i \vee h_i)$$

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$$G[v] = sphere(G; v); star(G; v)[v] = \bigcup_{\substack{g_i\ h_i\ :\ Graph \\ g_i, h_i\ =\ link(G, v)}} sphere(G; v); (g_i \vee h_i)$$

$$\begin{aligned}
(345; 567; 123; 124)[1] &= 345; 567; (23 \vee 24) \\
&\sim 345; 567; (\top\ \cup\ 234) \\
&= 345; 567; \top\ \cup\ 345; 567; 234 \\
&= 345; 567\ \cup\ 345; 567; 234 \\
&\sim 345; 567; 234
\end{aligned}$$

Local rewriting examples

```python
from mhgraph import mhgraph, MHGraph, vertex, Vertex
from graph_rewrite import local_rewrite

G : MHGraph = mhgraph([[1, 2], [1, 3], [1, 4], [1, 5]])
v : Vertex = vertex(1)

local_rewrite(G, v, True)
```

	(S),(3, 4),(3, 5),(2, 5),(2, 4)	(S),(2, 5),(2, 3),(2, 4)	(S),(3, 4),(3, 5),(2, 3)	(S),(4, 5),(3, 4),(2, 5),(2, 3)	(S),(2, 5),(3, 5),(4, 5)
0	(-S)(3,4)(3,-5)(-2,4)(-2,-5)	(-S)(2,3)(2,-5)(2,-4)	(-S)(3,4)(3,5)(-2,3)	(S)(-2,3)(3,-4)(-4,-5)(-2,-5)	(S)(4,5)(3,5)(2,5)
1	(-S)(-3,5)(-2,5)(-2,-4)(-3,-4)	(S)(2,5)(2,3)(2,-4)	(S)(3,5)(3,-4)(-2,3)	(-S)(3,4)(-2,3)(4,-5)(-2,-5)	(S)(2,-5)(3,-5)(-4,-5)
2	(S)(2,4)(2,5)(-3,4)(-3,5)	(S)(2,-5)(2,-3)(2,-4)	(-S)(3,5)(3,-4)(-2,3)	(S)(3,4)(2,3)(4,5)(2,5)	(-S)(4,5)(2,5)(-3,5)
3	(-S)(-3,4)(-3,5)(-2,5)(-2,4)	(S)(-2,5)(-2,4)(-2,-3)	(S)(3,4)(3,5)(-2,3)	(-S)(-2,3)(3,-4)(-4,-5)(-2,-5)	(-S)(4,5)(-3,5)(-2,5)
4	(-S)(3,5)(3,-4)(-2,5)(-2,-4)	(S)(-2,4)(-2,-5)(-2,-3)	(-S)(3,-5)(3,-4)(-2,3)	(S)(-4,5)(-2,5)(-3,-4)(-2,-3)	(-S)(3,5)(-4,5)(-2,5)
5	(S)(3,4)(2,4)(2,-5)(3,-5)	(-S)(2,-5)(2,-3)(2,-4)	(-S)(3,4)(2,3)(3,5)	(-S)(-3,4)(2,-3)(4,-5)(2,-5)	(-S)(2,-5)(-3,-5)(-4,-5)
6	(-S)(3,5)(2,5)(3,-4)(2,-4)	(-S)(2,5)(2,3)(2,-4)	(S)(2,3)(3,5)(3,-4)	(S)(2,5)(-4,5)(2,-3)(-3,-4)	(-S)(4,-5)(-3,-5)(-2,-5)
7	(-S)(3,-5)(3,-4)(-2,-4)(-2,-5)	(S)(2,3)(2,-5)(2,-4)	(-S)(-3,4)(-3,5)(-2,-3)	(S)(-4,5)(-2,3)(3,-4)(-2,5)	(-S)(2,5)(3,5)(-4,5)
8	(S)(3,4)(2,4)(3,5)(2,5)	(-S)(2,4)(2,-5)(2,-3)	(S)(3,4)(2,3)(3,5)	(-S)(3,4)(2,3)(2,-5)(4,-5)	(S)(-4,-5)(-3,-5)(-2,-5)
9	(-S)(3,4)(3,5)(-2,5)(-2,4)	(S)(-2,3)(-2,-5)(-2,-4)	(-S)(2,3)(3,5)(3,-4)	(-S)(-4,-5)(-3,-4)(-2,-5)(-2,-3)	(S)(2,-5)(-3,-5)(-4,-5)
10	(S)(-3,-5)(-2,-4)(-3,-4)(-2,-5)	(-S)(2,5)(2,-3)(2,-4)	(S)(3,4)(2,3)(3,-5)	(-S)(2,3)(2,-5)(3,-4)(-4,-5)	(S)(3,5)(-4,5)(-2,5)
11	(S)(-3,4)(-2,4)(-3,-5)(-2,-5)	(-S)(-2,3)(-2,-5)(-2,-4)	(-S)(2,3)(3,-5)(3,-4)	(S)(3,4)(2,3)(2,-5)(4,-5)	(S)(4,-5)(-3,-5)(-2,-5)
12	(-S)(2,5)(2,-4)(-3,5)(-3,-4)	(S)(-2,3)(-2,5)(-2,-4)	(-S)(-3,4)(2,-3)(-3,5)	(S)(2,3)(2,5)(-4,5)(3,-4)	(S)(3,-5)(-4,-5)(-2,-5)
13	(-S)(2,4)(-3,4)(2,-5)(-3,-5)	(S)(2,4)(2,5)(2,3)	(-S)(2,-3)(-3,-5)(-3,-4)	(S)(4,5)(2,5)(-3,4)(2,-3)	(-S)(2,-5)(4,-5)(-3,-5)
14	(-S)(3,4)(2,4)(2,-5)(3,-5)	(-S)(-2,5)(-2,-3)(-2,-4)	(S)(-3,4)(-3,-5)(-2,-3)	(S)(-3,4)(2,-3)(4,-5)(2,-5)	(S)(2,5)(-4,5)(-3,5)
15	(S)(-3,4)(-3,5)(-2,5)(-2,4)	(-S)(-2,-5)(-2,-3)(-2,-4)	(S)(-3,5)(-3,-4)(-2,-3)	(S)(4,5)(-3,4)(-2,5)(-2,-3)	(S)(4,5)(2,5)(-3,5)
16	(S)(3,5)(3,-4)(-2,5)(-2,-4)	(-S)(2,4)(2,5)(2,-3)	(-S)(-3,-5)(-3,-4)(-2,-3)	(-S)(-3,4)(4,-5)(-2,-5)(-2,-3)	(S)(4,5)(3,5)(-2,5)
17	(S)(-3,5)(-2,5)(-2,-4)(-3,-4)	(S)(2,4)(2,3)(2,-5)	(S)(-3,4)(-3,5)(-2,-3)	(S)(2,-5)(2,-3)(-4,-5)(-3,-4)	(-S)(2,-5)(3,-5)(4,-5)
18	(S)(3,4)(3,-5)(-2,4)(-2,-5)	(-S)(-2,4)(-2,-5)(-2,-3)	(S)(-3,-5)(-3,-4)(-2,-3)	(-S)(3,4)(2,3)(4,5)(2,5)	(-S)(2,-5)(3,-5)(-4,-5)
19	(-S)(2,4)(2,5)(-3,4)(-3,5)	(-S)(-2,5)(-2,4)(-2,-3)	(-S)(2,-3)(-3,5)(-3,-4)	(S)(3,4)(-2,3)(4,-5)(-2,-5)	(-S)(4,5)(2,5)(3,5)
20	(S)(2,4)(-3,4)(2,-5)(-3,-5)	(S)(-2,3)(-2,4)(-2,-5)	(-S)(-3,4)(2,-3)(-3,-5)	(-S)(-4,5)(-2,3)(3,-4)(-2,5)	(S)(2,-5)(3,-5)(4,-5)
21	(S)(2,5)(2,-4)(-3,5)(-3,-4)	(S)(-2,3)(-2,5)(-2,4)	(-S)(-3,5)(-3,-4)(-2,-3)	(-S)(2,-5)(2,-3)(-4,-5)(-3,-4)	(-S)(4,5)(3,5)(-2,5)
22	(S)(2,-4)(2,-5)(3,-5)(3,-4)	(-S)(2,4)(2,3)(2,-5)	(-S)(-3,4)(-3,-5)(-2,-3)	(S)(-3,4)(4,-5)(-2,-5)(-2,-3)	(S)(3,-5)(4,-5)(-2,-5)

Aside: graph disjunction

Let G_1 and G_2 be graphs. The disjunction of G_1 and G_2 is defined to be the following set of CNFs –

$$G_1 \vee G_2 = \{x_1 \vee x_2 \mid (x_1 : \text{CNF}) \in G_1;\ (x_2 : \text{CNF}) \in G_2\}$$

Table 3: Graph disjunctions where size of h_1 + size of h_2 is 2.

h_1		h_2		$h_1 \vee h_2$
a	$\vee$	a	$=$	$\top \cup a$
a	$\vee$	b	$=$	ab
a	$\vee$	bc	$=$	abc
a	$\vee$	ab	$=$	$\top \cup ab$
ab	$\vee$	cd	$=$	$abcd$
ab	$\vee$	ac	$=$	$\top \cup abc$
ab	$\vee$	ab	$=$	$\top \cup ab$

Table 4: Graph disjunctions where size of h_1 + size of h_2 is 3.

h_1		h_2		$h_1 \vee h_2$
a	$\vee$	a^2	$=$	a
a	$\vee$	b^2	$=$	a
a	$\vee$	$(ab)^2$	$=$	$\top \cup a \cup ab$
a	$\vee$	(b, ab)	$=$	$a \cup ab$
ab	$\vee$	c^2	$=$	ab
ab	$\vee$	(a, c)	$=$	$ab \cup abc$
ab	$\vee$	a^2	$=$	ab
ab	$\vee$	(a, b)	$=$	$\top \cup ab$
ab	$\vee$	(c, ac)	$=$	$ab \cup abc$
ab	$\vee$	(c, ab)	$=$	$ab \cup abc$
ab	$\vee$	(a, cd)	$=$	$ab \cup abcd$
ab	$\vee$	(a, ac)	$=$	$\top \cup ab \cup abc$
ab	$\vee$	(a, bc)	$=$	$\top \cup ab \cup abc$
ab	$\vee$	(a, ab)	$=$	$\top \cup ab$
ab	$\vee$	(ab, cd)	$=$	$ab \cup abcd$
ab	$\vee$	(ab, ac)	$=$	$\top \cup ab \cup abc$
ab	$\vee$	$(ac)^2$	$=$	$\top \cup ab \cup abc$
ab	$\vee$	(ac, bc)	$=$	$\top \cup ab \cup abc$
ab	$\vee$	$(ab)^2$	$=$	$\top \cup ab$

Table 5: Subset-superset pairs for graph disjunctions where size of h_1 + size of h_2 is at most 3.

Subset		h_1		h_2		Superset
		a	$\vee$	(b, c)	$\subset$	(ab, ac)
ab	$\subset$	a	$\vee$	(a, b)	$\subset$	$ab \cup (a, ab)$
		a	$\vee$	(b, cd)	$\subset$	(ab, acd)
abc	$\subset$	a	$\vee$	(a, bc)	$\subset$	$abc \cup (a, abc)$
ab	$\subset$	a	$\vee$	(b, ac)	$\subset$	$ab \cup (ab, ac)$
		a	$\vee$	(b, bc)	$\subset$	(ab, abc)
$\top \cup a \cup ab$	$\subset$	a	$\vee$	(a, ab)	$\subset$	$\top \cup a \cup ab \cup (a, ab)$
		a	$\vee$	(bc, de)	$\subset$	(abc, ade)
		a	$\vee$	(bc, bd)	$\subset$	(abc, abd)
		a	$\vee$	$(bc)^2$	$\subset$	$(abc)^2$
acd	$\subset$	a	$\vee$	(ab, cd)	$\subset$	$acd \cup (ab, acd)$
$ab \cup abc$	$\subset$	a	$\vee$	(ab, bc)	$\subset$	$ab \cup abc \cup (ab, abc)$
$\top \cup ab \cup ac$	$\subset$	a	$\vee$	(ab, ac)	$\subset$	$\top \cup ab \cup ac \cup (ab, ac)$
		ab	$\vee$	(c, d)	$\subset$	(abc, abd)
		ab	$\vee$	(c, de)	$\subset$	$(abc, abde)$
abc	$\subset$	ab	$\vee$	(c, cd)	$\subset$	$abc \cup (abc, abcd)$
abc	$\subset$	ab	$\vee$	(c, ad)	$\subset$	$abc \cup (abc, abd)$
		ab	$\vee$	(cd, ef)	$\subset$	$(abcd, abef)$
		ab	$\vee$	(cd, ce)	$\subset$	$(abcd, abce)$
		ab	$\vee$	$(cd)^2$	$\subset$	$(abcd)^2$
$abcd$	$\subset$	ab	$\vee$	(cd, ac)	$\subset$	$abcd \cup (abc, abcd)$
$\top \cup abc \cup abd$	$\subset$	ab	$\vee$	(ac, ad)	$\subset$	$\top \cup abc \cup abd \cup (abc, abd)$
$\top \cup abc \cup abd$	$\subset$	ab	$\vee$	(ac, bd)	$\subset$	$\top \cup abc \cup abd \cup (abc, abd)$
$abcd$	$\subset$	ab	$\vee$	(ac, cd)	$\subset$	$abcd \cup (abc, abcd)$

GraphSAT strategies

Given a looped-multi-hypergraph, we wish to ascertain its satisfiability status.

Strategy: Bruteforce

For a given graph G, check very CNF x_i supported on G. For every x_i, check every truth-assignment.

$$G \text{ is sat} \iff \forall x_i \in G;\ x_i \text{ is sat.}$$

G	CNF count
v_1v_2	4
$v_1v_2v_3$	8
$(v_1v_2v_3);\ (v_1v_3v_4);\ (v_1v_2v_4)$	512
$(v_1v_2v_3);\ (v_1v_3v_4);\ (v_1v_2v_4);\ (v_3v_4v_5);\ (v_3v_5v_6);\ (v_4v_5v_6)$	262,144

GraphSAT strategies

Strategy: Apply reduction rules

Idea: simplify the graph before subjecting it to the bruteforce strategy.

For example, assume that we are given the following reduction rule:

$$X ; (v_1 v_2 v_3) ; (v_1 v_2 v_4) \quad \longmapsto \quad X ; (v_2 v_3 v_4)$$

where X is any graph not having edges incident on the vertex v_1.

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Proof sketch of reduction rule.

$$G[v] \;=\; \text{sphere}(G, v), \text{star}(G, v)[v] \;=\; \bigcup_{\substack{g_i\ h_i\ :\ \text{Graph} \\ g_i, h_i\ =\ \text{link}(G, v)}} \text{sphere}(G, v), (g_i \vee h_i)$$

$$\begin{aligned}
(X, 123, 124)[1] &= X, (23 \vee 24) \\
&\sim X, (\top \cup 234) \\
&= X \cup X, 234 \\
&\sim X, 234
\end{aligned}$$

Graph reduction rules

Graph reduction rules

Minimal unsatisfiable looped-multi-hypergraphs

1. Start with all looped-multi-hypergraphs sorted smallest to largest.
2. Pick one and apply all known reduction rules to it.
3. Sat-check the reduced core of the graph.
4. If satisfiable, then pick the next one.
5. If unsatisfiable, then add to list of "minimal criminals".

Number of connected graphs with less than 7 vertices	143
Number of minimal unsatisfiable irreducible simple graphs	4
Number of connected graphs with less than 6 vertices	10080
Number of minimal unsatisfiable irreducible L-M-H-graphs	202
Number of connected graphs with less than 7 vertices	48,364,386
Number of minimal unsatisfiable irreducible L-M-H-graphs	??

A selection of unsatisfiable looped-multi-hypergraphs

Thickening of edges

One of the reduction rules is $X; (v_1v_2v_3); (v_1v_2v_4); (v_2v_3v_4) \longmapsto X; (v_3v_4)$,
where X is a graph having no edges incident on the vertices v_1 and/or v_2.
Applying this rule in reverse yields an easy way of creating (un)satisfiable hypergraphs.

(Un)satisfiable triangulations

1. Tetrahedron is satisfiable.

(Un)satisfiable triangulations

1. Tetrahedron is satisfiable.
2. Triangulation of a triangular prism –

```python
import mhgraph as mhg
import graph_rewrite as grw

prism1: mhg.MHGraph
prism1 = mhg.mhgraph([[1,2,3], [1,2,5], [1,3,4], [1,4,5], [2,3,6], [2,5,6], [3,4,6], [4,5,6]])
grw.decompose(prism1)

prism2: mhg.MHGraph
prism2 = mhg.mhgraph([[1,2,3], [1,2,5], [1,3,6], [1,4,5], [1,4,6], [2,3,6], [2,5,6], [4,5,6]])
grw.decompose(prism2)
```

Output:

```
(1, 2, 3),(1, 2, 5),(1, 3, 4),(1, 4, 5),(2, 3, 6),(2, 5, 6),(3, 4, 6),(4, 5, 6) is SAT
(1, 2, 3),(1, 2, 5),(1, 3, 6),(1, 4, 5),(1, 4, 6),(2, 3, 6),(2, 5, 6),(4, 5, 6) is SAT
```

(Un)satisfiable triangulations

3. Triangulation of a Möbius strip

```
import mhgraph as mhg
import graph_rewrite as grw

mobius_strip: mhg.MHGraph
mobius_strip = mhg.mhgraph([[1, 2, 4], [1, 4, 6], [2, 3, 5], [2, 4, 5], [3, 4, 6], [3, 5, 6]])
grw.decompose(mobius_strip)
```

Output:

```
(1, 2, 4),(1, 4, 6),(2, 3, 5),(2, 4, 5),(3, 4, 6),(3, 5, 6) is SAT
```

(Un)satisfiable triangulations

4. Triangulation of a Real projective plane ($\mathbb{RP}^2$)

```
import mhgraph as mhg
import graph_rewrite as grw

rp2: mhg.MHGraph
rp2 = mhg.mhgraph([[1, 2, 3], [3, 2, 6], [4, 6, 1], [4, 1, 2], [5, 2, 6],
                   [5, 6, 1], [1, 5, 3], [3, 6, 4], [4, 2, 5], [5, 3, 4]])
grw.decompose(rp2)
```

Output:

```
(1, 2, 3),(3, 2, 6),(4, 6, 1),(4, 1, 2),(5, 2, 6),(5, 6, 1),(1, 5, 3),(3, 6, 4),(4, 2, 5),(5, 3, 4) is SAT
```

(Un)satisfiable triangulations

5. Triangulation of a Klein bottle

Figure: The "242 triangulation" of a Klein bottle and its unsatisfiable subgraph.

6. Triangulation of a Torus

Figure: Minimal triangulation of a torus and its unsatisfiable subgraph.

Satisfiable infinite graphs

Recall,

- Vertex $\equiv$ an element of a countably infinite set, i.e. points/nodes.
- Edge $\equiv$ a nonempty set of vertices, i.e. loops, simple connections, hyperedges.
- Graph $\equiv$ a nonempty set of edges, i.e. a looped-multi-hypergraph.

Satisfiable infinite graphs

Key results

1. Set of all satisfiable simple graphs
   = Graphs forbidding $\{K_4,$ butterfly graph, bow-tie graph, $K_{1,1,3}\}$ as topological minors.

2. Set of all satisfiable looped-multi-graphs
   = Language of 2GraphSAT decision problem
   = Graphs forbidding $\{K_4,$ double-loop graph, dumbbell graph, $ab^4\}$ as topological minors.

3. There is a P-time algorithm for 2GraphSAT.

4. A list of graph reduction rules that preserve graph satisfiability.

5. The graph local rewriting theorem –

$$g[v] \;=\; \text{sphere}(g,v), \text{star}(g,v)[v] \;=\; \bigcup_{\substack{g_i\ h_i\ :\ \text{Graph} \\ g_i, h_i\ =\ \text{link}(g,v)}} \text{sphere}(g,v), (g_i \vee h_i)$$

6. An incomplete list of known unsatisfiable looped-multi-hypergraphs.
7. A Python package called `graphsat`.
   - : vaibhavkarve/graphsat
   - Supports Python version 3.9+, released under the GNU-GPL-v3.0 open license.
   - Functional-style, test-driven, literate programming, static type-checked.
   - Algorithms for SAT, GraphSAT, local rewriting, reduction of graphs under known rules, and searching for minimal unsatisfiable hypergraphs.

Conjectures and future work

1. We showed that the complexity class of 2GraphSAT is P. The complexity class for 3GraphSAT is not known. Moreover, the effect of local graph rewriting on 3GraphSAT is not known. Does local rewriting make the problem easier, or does it leave the complexity unchanged?

2. We have an incomplete list of unsatisfiable looped-multi-hypergraph.
   - **Conjecture 1**: The number of essential sat-invariant graph reduction rules is finite.
   - **Conjecture 2**: Each sat-invariant reduction rule is searchable in polynomial time.
   - **Conjecture 3**: The number of minimal unsatisfiable graphs under these reduction rules is also finite.

   If conjectures 1, 2, and 3 hold then we can conclude that 3GraphSAT is in P. This would give us an easy P-time heuristic check for 3SAT, simplifying some 3SAT cases. However, this will not affect the complexity class of 3SAT.

3. All unsatisfiable infinite graphs known so far have a finite unsatisfiable subgraph. Is there an unsatisfiable infinite graph whose every finite subgraph is satisfiable?
   Conjecture: No.

Conjectures and future work

4. Let $G_{a,b}$ denote the complete a-uniform hypergraph on b vertices. We can also think of this as the $(a-1)$-skeleton of a $(b-1)$-simplex.
   For example, we know that $G_{2,4} = K_4$ is unsatisfiable, while $G_{2,3} = C_3$ is satisfiable.

	b = 1	b = 2	b = 3	b = 4	b = 5	b = 6	b = 7	···
a = 1	sat	sat	sat	sat	sat	sat	sat	
a = 2	-	sat	sat	**unsat**	unsat	unsat	unsat	
a = 3	-	-	sat	sat	**unsat**	unsat	unsat	
a = 4	-	-	-	sat	sat	**sat**	??	
a = 5	-	-	-	-	sat	sat	??	
a = 6	-	-	-	-	-	sat	sat	
a = 7	-	-	-	-	-	-	sat	
···	-	-	-	-	-	-	-	

5. The generalized rule for n triangular hyperedges meeting at a common free vertex is not known. We know the reduction rule only for $n = 3$.

$$X; 123; 124; 134 \longmapsto X; 23 \cup X; 24 \cup X; 34$$

6. We have not explored random instances of GraphSAT, or graph analogues of random SAT.

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References

[1] Niklas Eén and Niklas Sörensson. "An Extensible SAT-solver". In: *Theory and Applications of Satisfiability Testing*. Ed. by Enrico Giunchiglia and Armando Tacchella. Berlin, Heidelberg: Springer Berlin Heidelberg, 2004, pp. 502–518. isbn: 978-3-540-24605-3.

[2] Alexey Ignatiev, Antonio Morgado, and Joao Marques-Silva. "PySAT: A Python Toolkit for Prototyping with SAT Oracles". In: *SAT*. 2018, pp. 428–437. doi: 10.1007/978-3-319-94144-8_26. url: https://doi.org/10.1007/978-3-319-94144-8_26.

[3] Vaibhav Karve and Anil N. Hirani. *GitHub: vaibhavkarve/graphsat*. Version v0.1.0. Apr. 2021. doi: 10.5281/zenodo.4662169. url: github.com/vaibhavkarve/graphsat.

[4] Vaibhav Karve and Anil N. Hirani. "The complete set of minimal simple graphs that support unsatisfiable 2-CNFs". In: *Discrete Applied Mathematics* 283 (2020), pp. 123–132. issn: 0166-218X. doi: 10.1016/j.dam.2019.12.017.

[5] N. Robertson and P.D. Seymour. "Graph Minors XIII. The Disjoint Paths Problem". In: *Journal of Combinatorial Theory, Series B* 63.1 (1995), pp. 65–110. issn: 0095-8956.

[6] Neil Robertson and P.D. Seymour. "Graph minors I. Excluding a forest". In: *Journal of Combinatorial Theory, Series B* 35.1 (1983), pp. 39–61. issn: 0095-8956.

[7] Neil Robertson and P.D. Seymour. "Graph Minors XX. Wagner's conjecture". In: *Journal of Combinatorial Theory, Series B* 92.2 (2004). Special Issue Dedicated to Professor W.T. Tutte, pp. 325–357. issn: 0095-8956.

URLs

General notes on GraphSAT	vaibhavkarve.github.io/satisfiability
A copy of the thesis	vaibhavkarve.github.io/satisfiability/thesis.pdf
A copy of these slides	vaibhavkarve.github.io/satisfiability/slides.pdf
Code repository	github.com/vaibhavkarve/graphsat