NOTES ON HOMOLOGY

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1. Definitions

(1) Let K : simplicial complex, p : integer dimension. A p-chain is a formal sum of p-simplices in K. Standard notation is $c = \sum a_i \sigma_i$ where a_i are the coefficients (either 0 or 1 that is, modulo 2 coefficients).

(2) Chains are added componentwise $\sum a_i \sigma_i + b_i \sigma_i = \sum (a_i + b_i) \sigma_i$ where $1 + 1 = 0$. Denote by $(C_p, +)$ the abelian group of p-chains with addition.

(3) For $p < 0$ and for $p > \dim K$, the group $C_p(K)$ is trivial.

(4) Define *boundary* : p-chain $\to$ $(p-1)$-chain as the sum of the $(p-1)$-dimensional faces, denoted by $\partial_p c$. Extend linearly so we have $\partial_p : C_p \to C_{p-1}$. This makes the boundary map a group homomorphism.

(5) The *Chain complex* is the sequence of chain groups connected by boundary maps —

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots$$

(6) A *p-cycle* is a p-chain with empty boundary that is, $\partial_p c = 0$. The p-cycles themselves form an abelian group denoted $Z_p = Z_p(K) = \ker \partial_p$. We have $Z_p \leq C_p$, $\forall p$ (subgroup).

(7) $Z_0 = \ker \partial_0 = C_0$.

(8) A *p-boundary* is a p-chain that is the boundary of a $(p+1)$-chain, $c = \partial d$ for $d \in C_{p+1}$. The p-boundaries also form an abelian group denoted $B_p = \operatorname{im} \partial_{p+1}$ which is also a subgroup of C_p.

(9) *Fundamental lemma of homology:* $\partial_p \partial_{p+1} d = 0$, for every p : integer and every $d \in C_{p+1}$.
Consequence: every p-boundary is also a p-cycle that is, $B_p \leq Z_p$ (subgroup).

(10) The p^{th} *homology group* is the defined as the abelian group $H_p = Z_p/B_p$. The p^{th} *Betti number* is the rank of this group $\beta_p = \text{rank } H_p$. (the rank of a group is the smallest cardinality of agenerating set of the group).
So elements of H_p are of the form $c + B_p$ where $c \in Z_p$. Each element is an equivalence class also called a *homology class*. Any two cycles in the same homology class are said to be *homologous*, denoted $c \sim c'$.

(11) $|C_p(K)| = 2^{\# \text{ of } p\text{-dimensional simplices in } K}$ and $C_p \cong (\mathbb{Z}_2^n, \text{XOR})$. Also have,

$$\text{ord } H_p = \text{ord } Z_p/\text{ord } B_p$$

and

$$\beta_p = \text{rank } H_p = \text{rank } Z_p - \text{rank} B_p$$

(12) *Euler-Poincaré formula:* The Euler characteristic of a simplicial complex is the alternating sum of the number of simplices in each dimension. Let the rank of C_p, Z_p and B_p be denoted by n_p, z_p and b_p respectively. We have

$$\begin{aligned}\chi &= \sum_{p\geq 0}(-1)^p n_p \\ &= \sum_{p\geq 0}(-1)^p (z_p + b_{p-1}) \\ &\quad \text{(by Rank-nullity by viewing groups as a } \mathbb{Z}_2\text{-vspace} \\ &= \sum_{p\geq 0}(-1)^p (z_p - b_p) \\ &= \sum_{p\geq 0}(-1)^p \beta_p\end{aligned}$$

(13) The p^{th}-*boundary matrix* represents the $(p-1)$-simplices as rows and the p-simplices as columns (assuming arbitrary by fixed ordering of simplices). The (i, j)-entry is $a_i^j = 1$ if the i^{th} $(p-1)$-simplex is a face of the j^{th} p-simplex and $a_i^j = 0$ otherwise, $\forall i \in \{1, \ldots, n_{p-1}\}$, $\forall j \in \{1, \ldots, n_p\}$.
Given a p-chain $c = \sum a_k\sigma_k$, the boundary $\partial_p c$ can be computed by matrix multiplication of $\partial_p = [a_i^j]$ with the column vector of coefficients (a_k)

(14) A collection of columns of ∂_p represents a p-chain and the sum of these columns gives its boundary. A collection of rows represents a $(p-1)$-chain and the sum of these rows gives its coboundary.

(15) The rows of ∂_p form a basis of the group C_{p-1} and the columns form a basis of the group C_p.

(16) *Smith normal form:* The matrix ∂_p (with entries in $\mathbb{Z}_2$) is in Smith normal form if the initial segment of the diagonal is 1 (doesn't need to be the entire diagonal, just a segment) and everything else is 0.

$$\begin{bmatrix} \mathbf{I}_{b_{p-1}\times(n_p-z_p)} & \mathbf{O}_{b_{p-1}\times z_p} \\ \mathbf{O}_{(n_{p-1}-b_{p-1})\times(n_p-z_p)} & \mathbf{O}_{(n_{p-1}-b_{p-1})\times z_p}\end{bmatrix}_{n_{p-1}\times n_p}$$

Here, the leftmost b_{p-1} columns represent p-chains whose nonzero boundaries generate the group B_{p-1}. The rightmost z_p columns represent p-cycles that generate Z_p (these are zero columns in ∂_p but will become non-zero rows in ∂_{p+1}).

(17) Once all the boundary matrices (for different p) are in Smith normal form, the Betti numbers can be calculated as differences between ranks —

$$\begin{aligned}\tilde{\beta}_p &= \text{rank} Z_p - \text{rank} B_p \\ &= z_p - b_p \\ &= (\#\text{ of zero columns in } N_p) - (\#\text{ of nonzero rows in } N_{p+1})\end{aligned}$$

where N_p is ∂_p in Smith normal form.

2. Examples

(1) $H_1(\text{Torus}) = \{B_1, x + B_1, y + B_1, x + y + B_1\}$ where B_1 is the 1-boundary group, and x and y are the one non-bounding 1-cycles that go once around the arm and the hole of the torus.
Therefore, $H_1(\text{Torus}) = \langle x + B_1, y + B_1\rangle$ and hence $\beta_1 = 2$. This situation can be represented as a square with vertices $\{0, x, y, x + y\}$.

(2) Closed ball is any triangulated topological space that is homeomorphic to $\mathbb{B}^k = \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$. $H_p(\text{Ball}) = 0,\ \forall p \geq 1$. But rank $H_0(\text{Ball}) = 1$ because $H_0 \cong \mathbb{Z}_2$.
If K is the set of faces of a single simplex of dimension k then rank $C_p = \binom{k+1}{p+1}$. For every $c \in Z_p$, one can find a $d \in B_p$ such that $\partial_p d = c$. Hence $Z_p = B_p$ and $H_p = 0$ for $p \geq 1$.

(3) If we build a tetrahedron one dimension at a time.
   (a) First there are only four vertices. Then $\partial_0 = [1\ 1\ 1\ 1]$. Its normal form N_0 has 1 nonzero row and 3 zero columns. Therefore $\tilde{\beta}_0 = 3$.
   (b) Now add the six vertices. Then $\partial_0 = [1\ 1\ 1\ 1]$ and $\partial_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$.
   ∂_1's normal form N_1 has 3 nonzero rows and 3 nonzero columns. $\tilde{\beta}_0 = 3(\text{from } N_0) - 3(\text{from } N_1) = 3 - 3 = 0$ and $\tilde{\beta}_1 = 3(\text{from } N_1)$.
   (c) At next step, we'll have $\tilde{\beta}_0 = 3 - 3 = 0$, $\tilde{\beta}_1 = 3 - 3 = 0$ and $\tilde{\beta}_2 = 1$.
   (d) At the final step we'll have $\tilde{\beta}_0 = 3-3 = 0$, $\tilde{\beta}_1 = 3-3 = 0$, $\tilde{\beta}_2 = 1-1 = 0$ and $\tilde{\beta}_3 = 0$.