

This handout includes space for every question that requires a written response. Please feel free to use it to handwrite your solutions (legibly, please). If you choose to typeset your solutions, the `README.md` for this assignment includes instructions to regenerate this handout with your typeset L^AT_EX solutions.

1.a

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinite. For $K(x, z) = K_1(x, z) + K_2(x, z)$

Lets evaluate symmetric nature for K:

It is given in the problem set that K_1 and K_2 are valid kernels.

The kernel matrix corresponding to $K(x, z)$ is sum of two valid kernel matrices K_1 and K_2 .

So $K(x, z)$ is Symmetric

PSD:

$$z^T K z$$

$$= z^T (K_1 + K_2) z$$

$$= z^T K_1 z + z^T K_2 z$$

$$\geq 0 \text{ Note that, } z^T K_1 z \geq 0 \text{ and } z^T K_2 z \geq 0 \text{ as they are valid kernels}$$

Since $K(x, z) = K_1(x, z) + K_2(x, z)$ is Symmetric and PSD it is necessarily a kernel

1.b

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinite. For $K(x, z) = K_1(x, z) - K_2(x, z)$

Lets evaluate symmetric nature for K:

It is given in the problem set that K_1 and K_2 are valid kernels.

The kernel matrix corresponding to $K(x, z)$ is difference of two valid kernel matrices K_1 and K_2 .

So $K(x, z)$ is Symmetric

PSD:

$$z^T K z$$

$$= z^T (K_1 - K_2) z$$

$$= z^T K_1 z - z^T K_2 z$$

$$\geq 0 \text{ This is true only for cases when } z^T K_1 z > z^T K_2 z$$

Since $K(x, z) = K_1(x, z) - K_2(x, z)$ is Symmetric but not PSD it is not necessarily a kernel

1.c

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinite.
 For $K(x, z) = aK_1(x, z)$

Lets evaluate symmetric nature for K:

It is given in the problem set that K_1 is valid kernels.

It is also given in the problem set that a is scalar.

The kernel matrix corresponding to $K(x, z)$ is scaled by scalar a

So $K(x, z)$ is Symmetric

PSD:

$$z^T K z$$

$$= z^T (aK_1) z$$

$$= a z^T K_1 z$$

$$\geq 0 \text{ Because } a \text{ is positive real number } a \in R^+ \text{ and } z^T K_1 z \geq 0 \text{ is valid kernel}$$

Since $K(x, z) = aK_1(x, z)$ is Symmetric and PSD it is necessarily a kernel

1.d

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinite.
 For $K(x, z) = -aK_1(x, z)$

Lets evaluate symmetric nature for K:

It is given in the problem set that K_1 is valid kernel.

It is also given in the problem set that a is scalar.

The kernel matrix corresponding to $K(x, z)$ is scaled by scalar a

So $K(x, z)$ is Symmetric

PSD:

$$z^T K z$$

$$= z^T (-aK_1) z$$

$$= -a z^T K_1 z$$

Not a PSD because a is positive real number $a \in R^+$

Since $K(x, z) = -aK_1(x, z)$ is Symmetric but not PSD it is not necessarily a kernel.

1.e

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinite.
For $K(x, z) = K_1(x, z)K_2(x, z)$

Lets evaluate symmetric nature for K:

It is given in the problem set that K_1 and K_2 are valid kernels.

So $K(x, z)$ is Symmetric because of commutative property

PSD:

$$K(x, z) = K_1(x, z)K_2(x, z)$$

$$K(x, z) = \sum_i \phi_i^{(1)}(x)\phi_i^{(1)}(z) \sum_j \phi_j^{(2)}(x)\phi_j^{(2)}(z)$$

$$K(x, z) = \sum_i \sum_j \phi_i^{(1)}(x)\phi_i^{(1)}(z)\phi_j^{(2)}(x)\phi_j^{(2)}(z)$$

$$K(x, z) = \sum_i \sum_j [\phi_i^{(1)}(x)\phi_j^{(2)}(x)][\phi_i^{(1)}(z)\phi_j^{(2)}(z)]$$

$$\text{def } \psi(.) = \phi^{(1)}(.)\phi^{(2)}(.)$$

So we have

$$K(x, z) = \sum_i \sum_j \psi_{i,j}(x)\psi_{i,j}(z)$$

$$K(x, z) = \psi(x)^T \psi(z)$$

Now lets replace

$$z^T K z$$

$$= z^T \psi(x)^T \psi(z) z$$

$$= \geq 0 \text{ so a PSD}$$

Since $K(x, z) = K_1(x, z)K_2(x, z)$ is Symmetric and PSD it is necessarily a kernel.

1.f

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinite.
For $K(x, z) = f(x)f(z)$

Lets evaluate symmetric nature for K:

It is given in the problem set that f is real valued function

So $K(x, z)$ is Symmetric because scalar multiplication is commutative

PSD:

Let $f(x) = \psi(x)$ as in 1.e

So $K(x, z) = f(x)f(z)$ is PSD as per 1.e

Since $K(x, z) = f(x)f(z)$ is Symmetric and PSD it is necessarily a kernel.

1.g

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinite.
For $K(x, z) = K_3(\phi(x), \phi(z))$

Lets evaluate symmetric nature for K:

It is given in the problem set that K_3 is a Kernel

So $K(x, z)$ is Symmetric because by definition of K_3

PSD:

$$z^T K z$$

$$= z^T K_3 z$$

$$= \geq 0 \text{ so a PSD}$$

Since $K(x, z) = K_3(\phi(x), \phi(z))$ is Symmetric and PSD it is necessarily a kernel.

Also K_3 is valid kernel for any finite set $\{x^{(1)}, \dots, x^{(n)}\}$ This also includes $\{\phi(x^{(1)}), \dots, \phi(x^{(n)})\}$ so K is necessarily a kernel.

1.h

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinite.
 For $K(x, z) = p(K_1(x, z))$

Lets evaluate symmetric nature for K:

It is given in the problem set that p is a polynomial over x with postive coefficients

Let $p(x) = \sum_i a_i x^i$

So if we expand for $p(K_1)$ we will get terms from 1.a, 1.c, 1.e 1.f which are all symmetric

So $K(x, z)$ is symmetric

PSD:

$$z^T K z$$

$$= z^T (\sum_i a_i K_1^i) z$$

$$= \sum_i a_i z^T K_1^i z$$

Here a_i are postive coefficients and from 1.e we can conclude that $\sum_i a_i z^T K_1^i z$ will be ≥ 0

Hence a PSD

Since $K(x, z) = p(K_1(x, z))$ is Symmetric and PSD it is necessarily a kernel.

So any polynomial of kernel K_1 will also be a kernel