This handout includes space for every question that requires a written response. Please feel free to use it to handwrite your solutions (legibly, please). If you choose to typeset your solutions, the `README.md` for this assignment includes instructions to regenerate this handout with your typeset LaTeX solutions.

1.a

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinte.
For $K(x, z) = K_1(x, z) + K_2(x, z)$

Lets evalute symmetric nature for K:
It is given in the problem set that K_1 and K_2 are valid kernels.
The kernel matrix corresponding to $K(x, z)$ is sum of two valid kernel matrices K_1 and K_2.

So $K(x, z)$ is Symmetric

PSD:
$z^T K z$
$= z^T (K_1 + K_2) z$
$= z^T K_1 z + z^T K_2 z$
≥ 0 Note that, $z^T K_1 z \geq 0$ and $z^T K_2 z \geq 0$ as they are valid kernels

Since $K(x, z) = K_1(x, z) + K_2(x, z)$ is Symmetric and PSD it is necessarily a kernel

1.b

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinte.
For $K(x,z) = K_1(x,z) - K_2(x,z)$

Lets evalute symmetric nature for K:
It is given in the problem set that K_1 and K_2 are valid kernels.
The kernel matrix corresponding to $K(x,z)$ is difference of two valid kernel matrices K_1 and K_2.

So $K(x,z)$ is Symmetric

PSD:
$z^T K z$
$= z^T (K_1 - K_2) z$
$= z^T K_1 z - z^T K_2 z$
≥ 0 This is true only for cases when $z^T K_1 z > z^T K_2 z$

Since $K(x,z) = K_1(x,z) - K_2(x,z)$ is Symmetric but not PSD it is not necessarily a kernel

1.c

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinte.
For $K(x,z) = aK_1(x,z)$

Lets evalute symmetric nature for K:
It is given in the problem set that K_1 is valid kernels.
It is also given in the problem set that a is scalar.
The kernel matrix corresponding to $K(x,z)$ is scaled by scalar a

So $K(x,z)$ is Symmetric

PSD:
$z^T K z$
$= z^T (aK_1) z$
$= a z^T K_1 z$
≥ 0 Because a is positive real number $a \in R^+$ and $z^T K_1 z \geq 0$ is valid kernel

Since $K(x,z) = aK_1(x,z)$ is Symmetric and PSD it is necessarily a kernel

1.d

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinte.
For $K(x, z) = -aK_1(x, z)$

Lets evalute symmetric nature for K:
It is given in the problem set that K_1 is valid kernel.
It is also given in the problem set that a is scalar.
The kernel matrix corresponding to $K(x, z)$ is scaled by scalar a

So $K(x, z)$ is Symmetric

PSD:
$z^T K z$
$= z^T(-aK_1)z$
$= -az^T K_1 z$
Not a PSD because a is positive real number $a \in R^+$

Since $K(x, z) = -aK_1(x, z)$ is Symmetric but not PSD it is not necessarily a kernel.

1.e

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinte.
For $K(x,z) = K_1(x,z)K_2(x,z)$

Lets evalute symmetric nature for K:
It is given in the problem set that K_1 and K_2 are valid kernels.
So $K(x,z)$ is Symmetric because of commutative property

PSD:
$K(x,z) = K_1(x,z)K_2(x,z)$
$K(x,z) = \sum_i \phi_i^{(1)}(x)\phi_i^{(1)}(z) \sum_j \phi_j^{(2)}(x)\phi_j^{(2)}(z)$
$K(x,z) = \sum_i \sum_j \phi_i^{(1)}(x)\phi_i^{(1)}(z)\phi_j^{(2)}(x)\phi_j^{(2)}(z)$
$K(x,z) = \sum_i \sum_j [\phi_i^{(1)}(x)\phi_j^{(2)}(x)][\phi_i^{(1)}(z)\phi_j^{(2)}(z)]$
def $\psi(.) = \phi^{(1)}(.)\phi^{(2)}(.)$
So we have

$K(x,z) = \sum_i \sum_j \psi_{i,j}(x)\psi_{i,j}(z)$
$K(x,z) = \psi(x)^T\psi(z)$

Now lets replace
$z^T K z$
$= z^t\psi(x)^T\psi(z)z$
$=\geq 0$ so a PSD
Since $K(x,z) = K_1(x,z)K_2(x,z)$ is Symmetric and PSD it is necessarily a kernel.

1.f

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinte.
For $K(x, z) = f(x)f(z)$

Lets evalute symmetric nature for K:
It is given in the problem set that f is real valued function
So $K(x, z)$ is Symmetric because scalar multiplication is commutiative

PSD:
Let f(x) = $\psi(x)$ as in 1.e
So $K(x, z) = f(x)f(z)$ is PSD as per 1.e

Since $K(x, z) = f(x)f(z)$ is Symmetric and PSD it is necessarily a kernel.

1.g

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinte.
For $K(x,z) = K_3(\phi(x), \phi(z))$

Lets evalute symmetric nature for K:
It is given in the problem set that K_3 is a Kernel
So $K(x,z)$ is Symmetric because by definition of K_3

PSD:
$z^T K z$
$= z^T K_3 z$
$=\geq 0$ so a PSD

Since $K(x,z) = K3(\phi(x), \phi(z))$ is Symmetric and PSD it is necessarily a kernel.
Also K_3 is valid kernel for any finite set $\{x^{(1)}, ..., x^{(n)}\}$ This also includes $\{\phi(x^{(1)}), ..., \phi(x^{(n)})\}$ so K is necessarily a kernel.

1.h

As given in the problem set any kernel to be necessarily a kernel it should be symmetric and positive semidefinte.
For $K(x, z) = p(K1(x, z))$

Lets evalute symmetric nature for K:
It is given in the problem set that p is a polynomial over x with postive coefficients
Let p(x) = $\sum_i a_i x^i$
So if we expand for $p(K1)$ we will get terms from 1.a, 1.c, 1.e 1.f which are all symmetric
So $K(x, z)$ is symmetric

PSD:
$z^T K z$
$= z^T (\sum_i a_i K_1^i) z$
$= \sum_i a_i z^T K_1^i z$
Here a_i are postive coefficients and from 1.e we can conclude that $\sum_i a_i z^T K_1^i z$ will be ≥ 0
Hence a PSD
Since $K(x, z) = p(K1(x, z))$ is Symmetric and PSD it is necessarily a kernel.
So any polynomial of kernel K_1 will also be a kernel