APS360: Applied Fundamentals of Deep Learning

Week 2: Artificial Neural Networks

Content

- Neuron
- Activation Function
- Training an Artificial Neuron
- Loss Functions
- Gradient Descent
- Neural Network Architectures

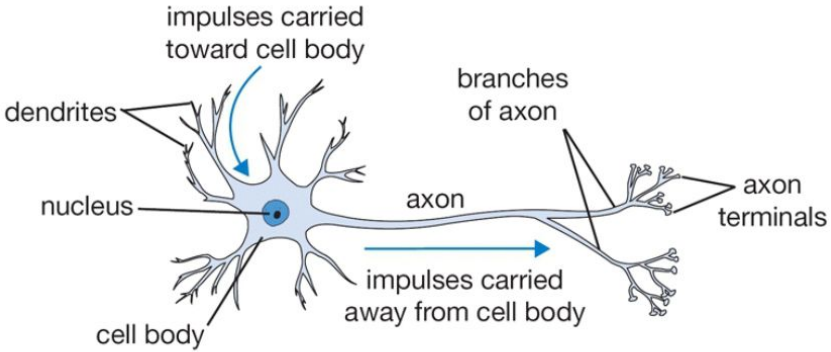
Neuron

Simplified Biological Neuron

Dendrites receive information from other neurons.

Cell body consolidates information from the dendrites.

Axon passes information to other neurons.



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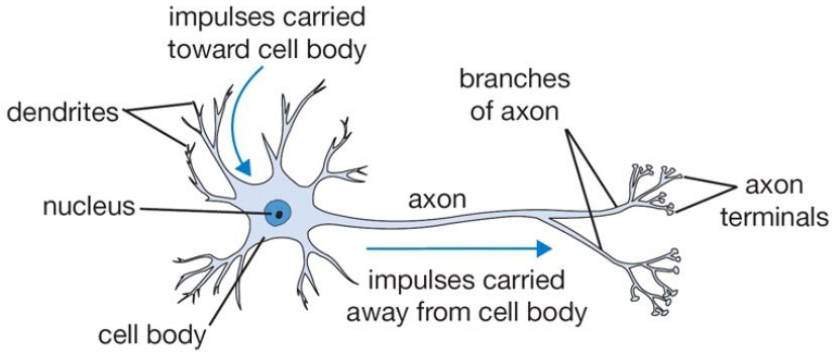
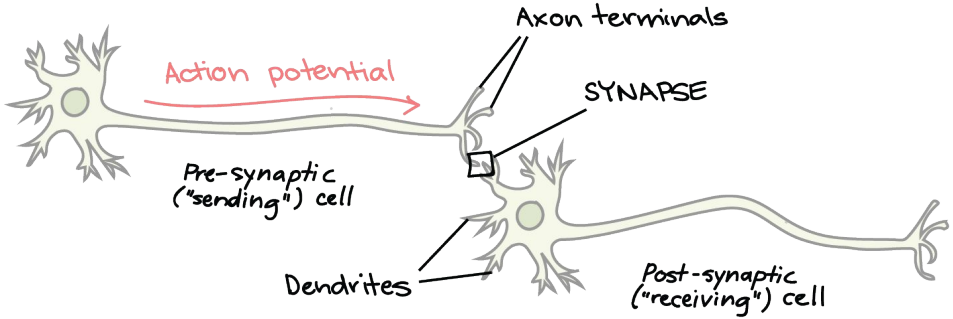
Synapse is the area where the axon of one neuron and the dendrite of another connect.





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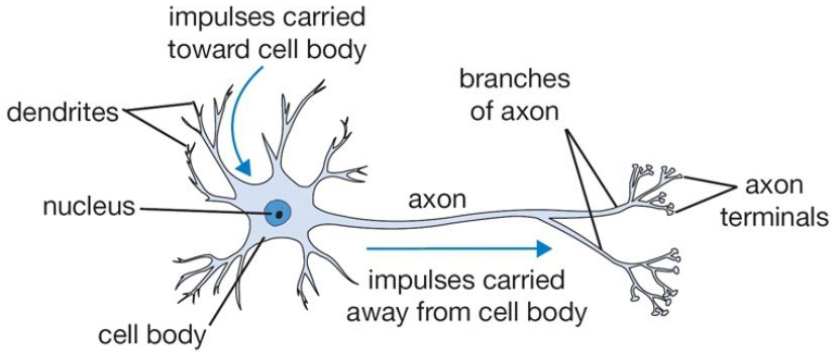
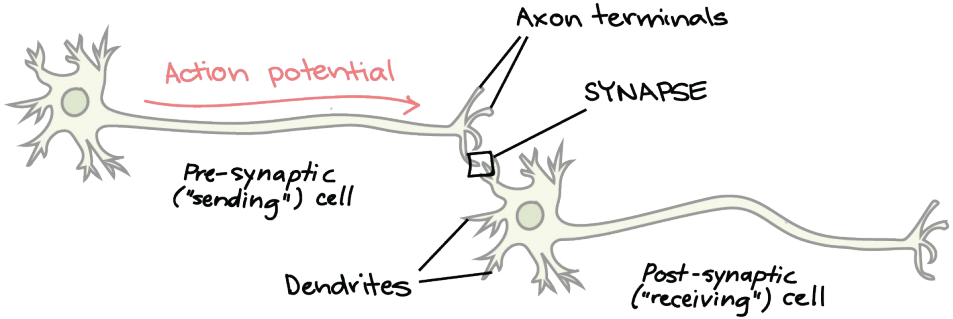




Neurons **fire** on stimuli like: edges, lines, angles, movements, familiar faces etc. regardless of scale, rotation and translation!

Artificial Neuron

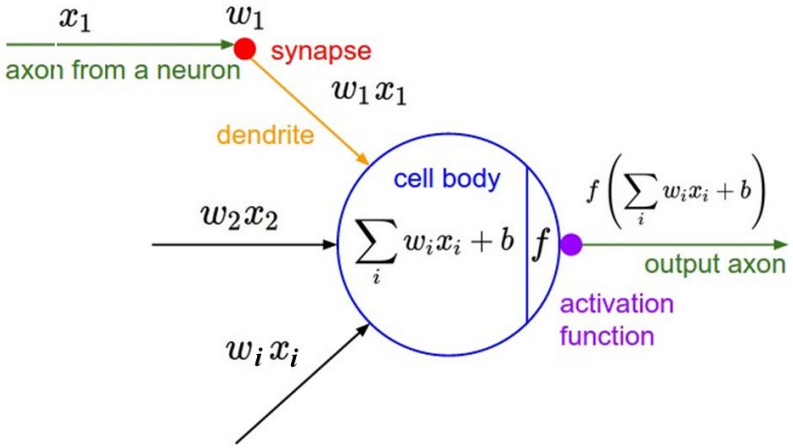
x_i is the **input** such as a pixel in an image

w_i is the **weight** for input x_i that we learn for this particular input

b is the **bias**, a weight we learn with no input

f is the **activation function** that determines how our output changes with the sum of all weight-input products

y is the **output** such as the class an image belongs to



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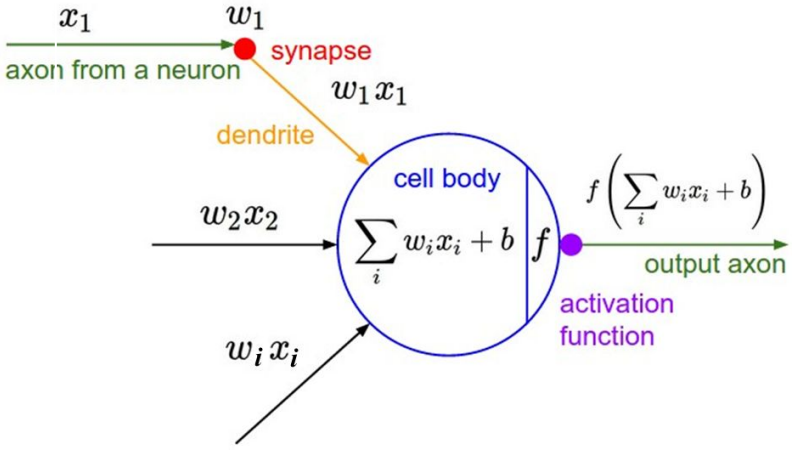
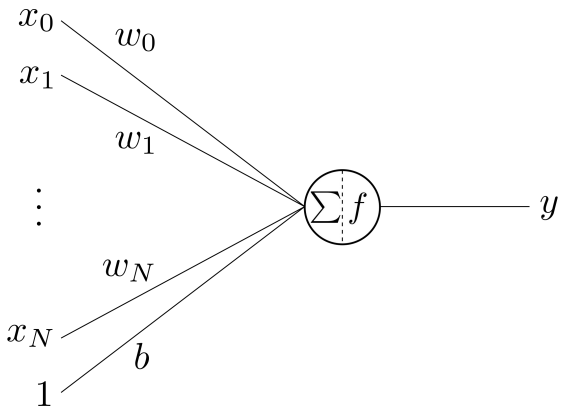
$$y = f(\mathbf{w}\cdot\mathbf{x}+b)$$

This equation looks vaguely familiar...



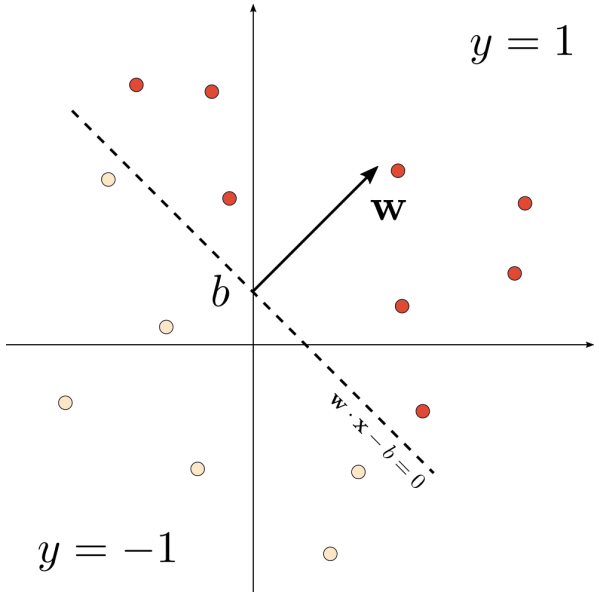


Activation Function

Linear Activation Function

Suppose the activation function is a simple linear function

$$y = f(\mathbf{w}\cdot\mathbf{x}+b) \longrightarrow y = \mathbf{w}\cdot\mathbf{x}+b$$



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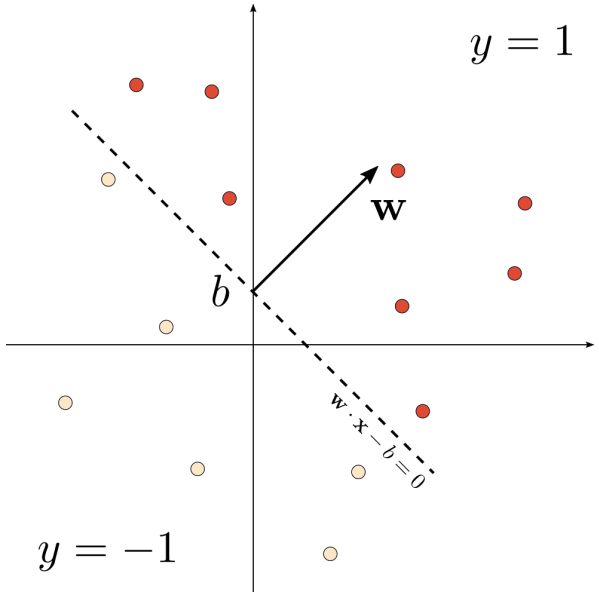
$$y = w_0x_0+w_1x_1+ b$$

Recall general equation of line: $Ax+By-C=0$



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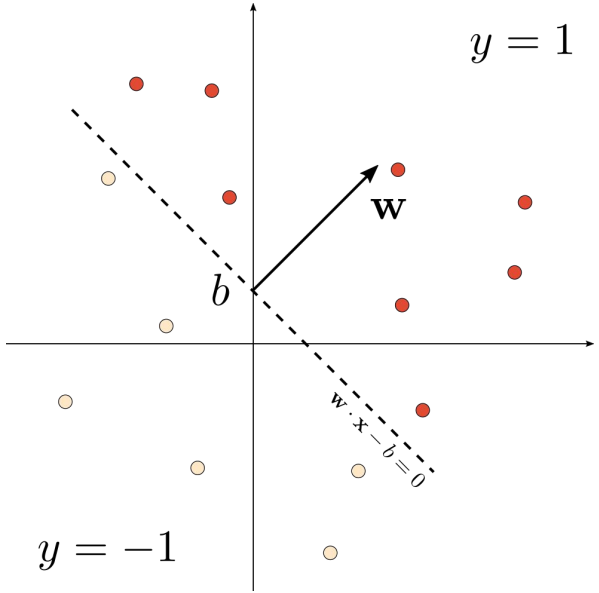
Recall general equation of line: $Ax+By-C=0$

The bias is related to the offset of the line from the origin

$y=\mathbf{w}\cdot\mathbf{x}+b$ is a generalized line for any dimension, known as a ***hyperplane***, splitting the n-dimensional input space into 2

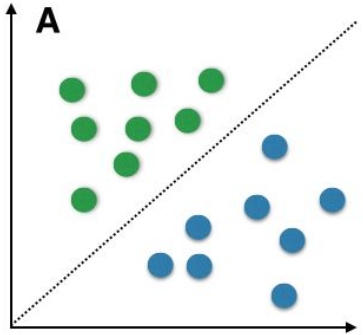


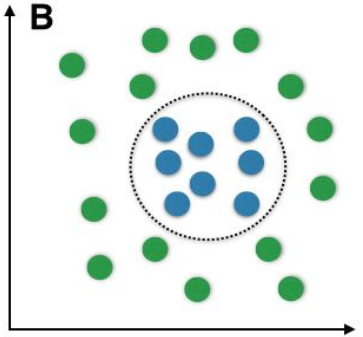
Neuron with a Linear Activation Function

What's wrong with a linear activation function?

- Most real datasets are not **linearly separable**, e.g. we can't find a line that separates classes well in a classification problem
- We can learn non-linear transformations of our data to help
- Multiple layers with non-linear transformations help
- **No advantage from multiple linear layers** (composite is a linear layer)

$$\underbrace{\mathbf{W}^{(1)}\mathbf{W}^{(2)}\mathbf{W}^{(3)}}\mathbf{x} = \mathbf{W}'\mathbf{x}$$





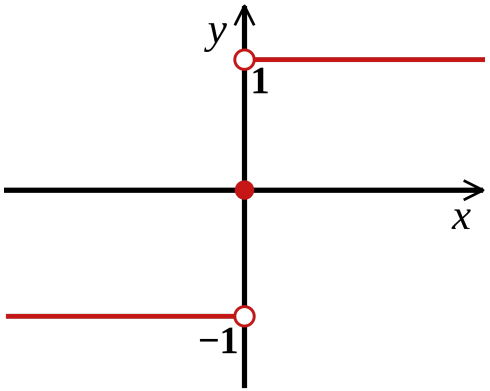
https://sebastianraschka.com/Articles/2014_naive_bayes_1.html

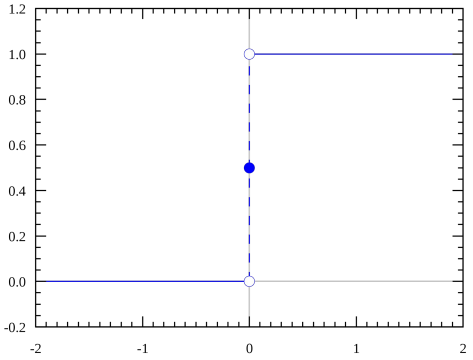
Early Activation Functions: Perceptrons

First artificial neurons (1943-70s) used a simple binary activation function based on which side of the hyperplane the input is:

$$f(x) = \text{sign}(x)$$ **Sign function**

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$ **Heaviside (unit) step function**





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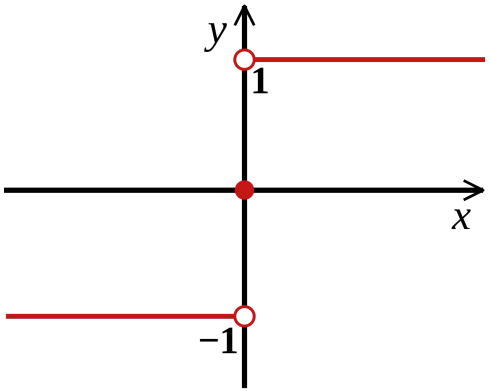

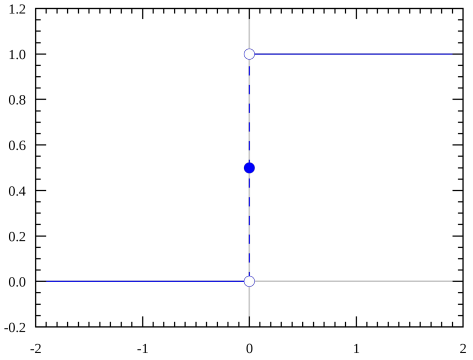

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These functions are not differentiable, continuous, or smooth

Sigmoid Activation Function

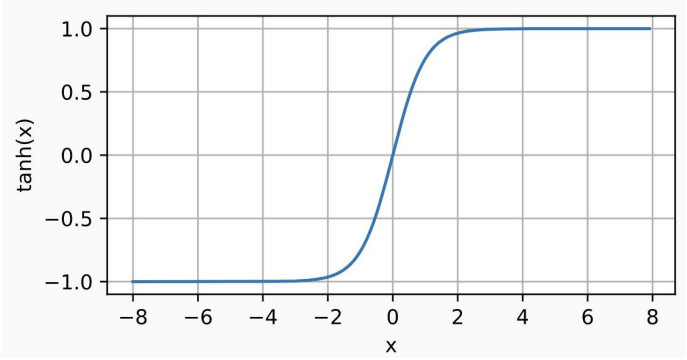
Sigmoid activation functions were the most common before 2012:

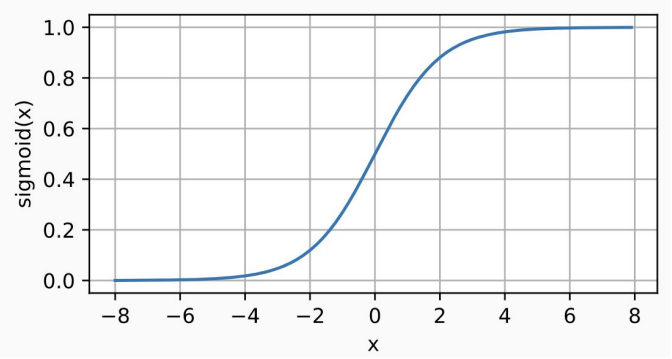
- Easily differentiable, smooth, continuous
- Range between [-1, 1] or [0, 1]

There are *many* sigmoid functions, the most common are:

$$f(x) = \tanh(x)$$ **Hyperbolic tangent**

$$f(x) = \frac{1}{1 + e^{-x}}$$ **Logistic function**





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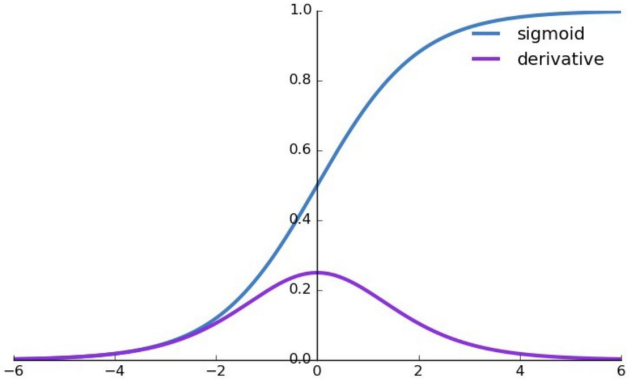
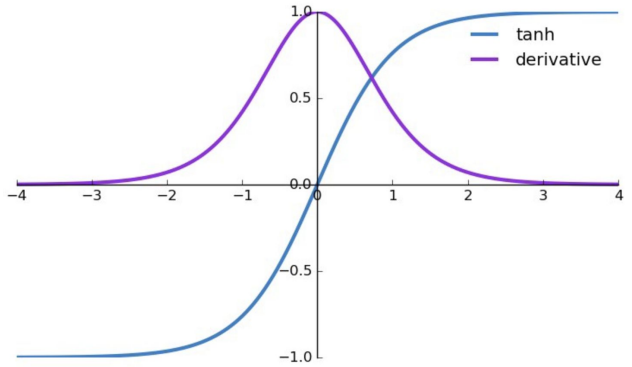
Saturated neurons "kill" the gradients

Gradients become vanishingly small very quickly away from x=0





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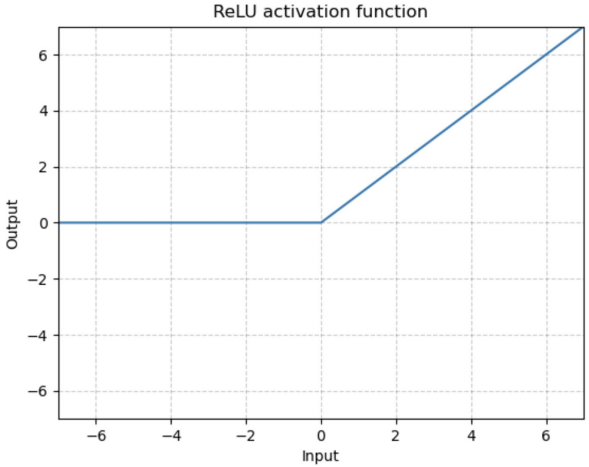
Modern deep learning typically uses the Rectified Linear Unit (ReLU) based activation functions:

$$\mathrm{ReLU}(x) = (x)^+ = \max(0, x)$$

ReLU



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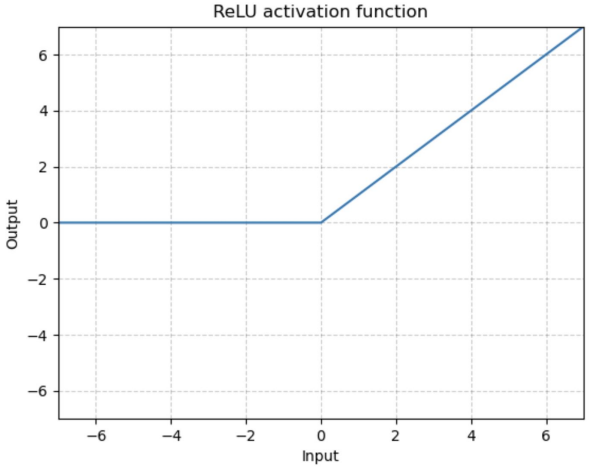
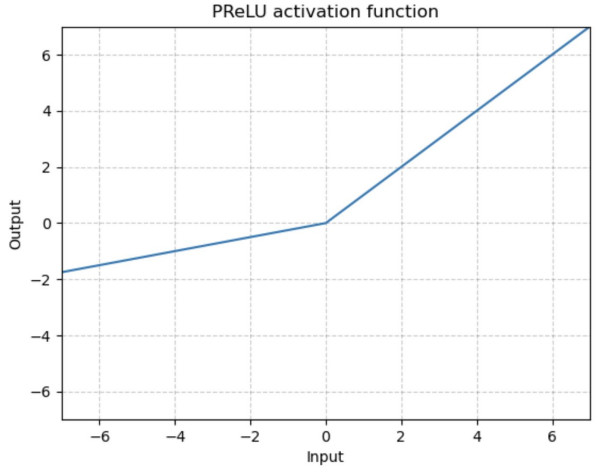
$$\text{LeakyRELU}(x) = \begin{cases} x, & \text{if } x \geq 0 \\ \text{negative\_slope} \times x, & \text{otherwise} \end{cases}$$ **Leaky ReLU**

$$\text{PReLU}(x) = \begin{cases} x, & \text{if } x \geq 0 \\ ax, & \text{otherwise} \end{cases}$$ **Parametric ReLU**

Very easy derivatives 0 or 1, use 0 at x=0





Continuous Approximations of ReLU

We can approximate ReLU activation by continuous functions:

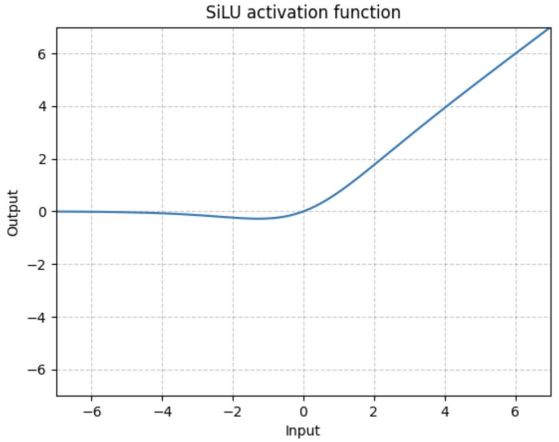
$$\text{SiLU}(x) = x.\sigma(x) = \frac{x}{1+e^{-x}}$$

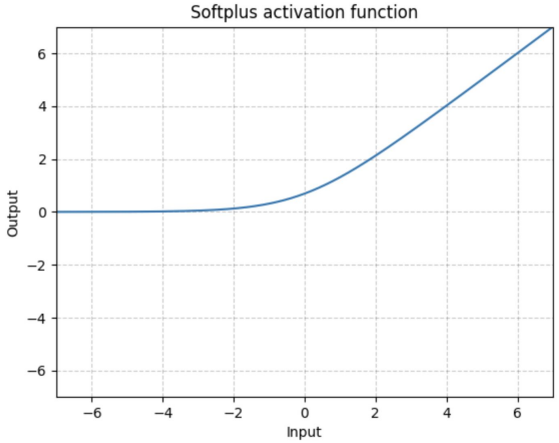
SiLU (Swish)

$$\text{SoftPlus}(x) = \frac{1}{\beta}\log\left(1+e^{\beta x}\right)$$

SoftPlus

Work on par or better than ReLU functions





Training Neural Networks

Training a Neuron

How do we **learn** the **weights** (and bias) of a neural network?

We can use our prediction error to decide how to change weights.

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input: $\mathbf{x}$, predicted output: y, ground truth label: t, Neuron M($\mathbf{w}$;$\mathbf{x}$)

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Used for both training and inference

↑

Forward pass

Backward pass

↓

Used only for training

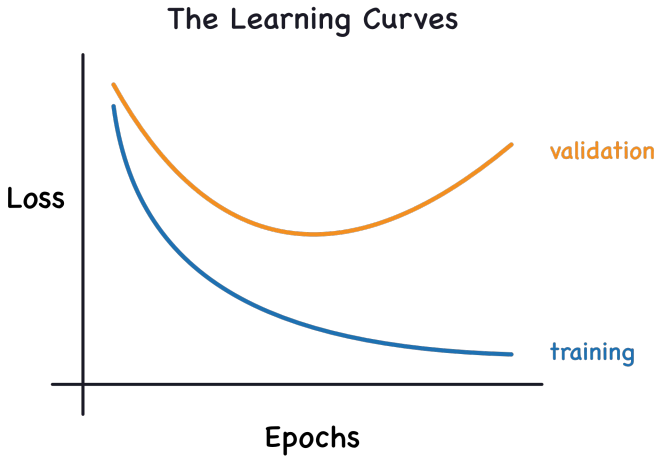
Loss Function

Loss function

A **loss function** computes how bad predictions are compared to the ground truth labels.

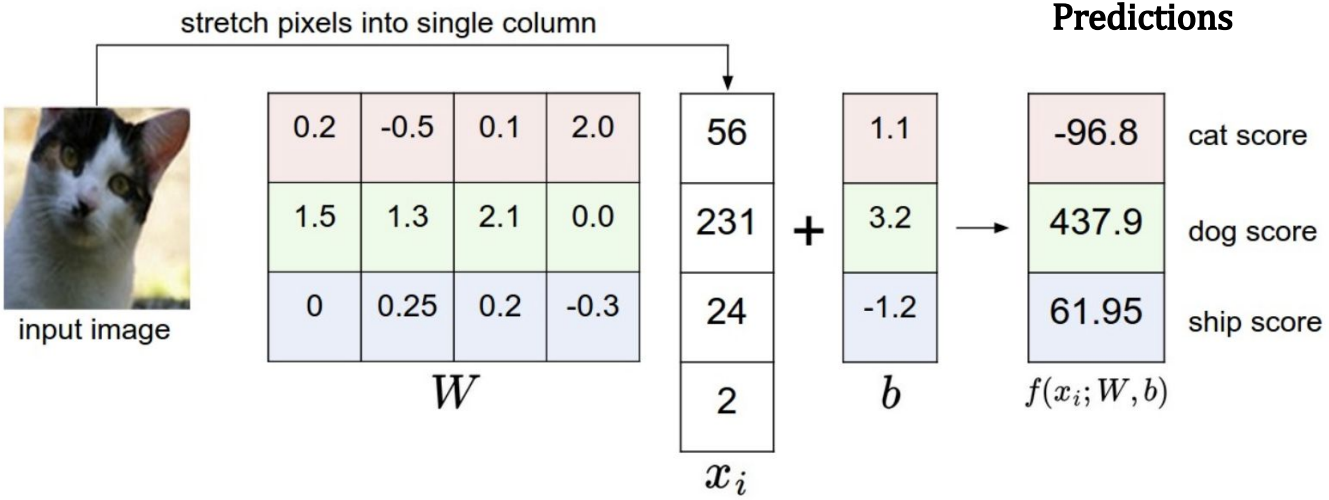
- **Large loss:** the network's prediction differs from the ground truth
- **Small loss:** the network's prediction matches the ground truth

We want to calculate the error over **all training samples** (average error)



Loss function

Suppose we want to train a linear neuron to differentiate images into three classes:



https://cs231n.github.io/linear-classify/

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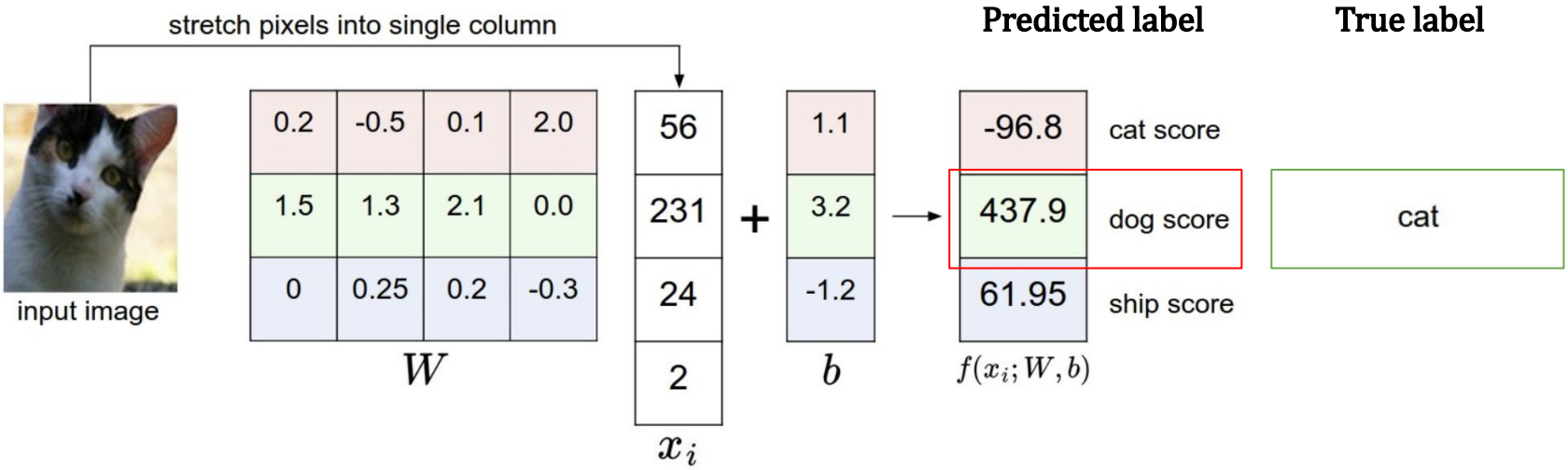

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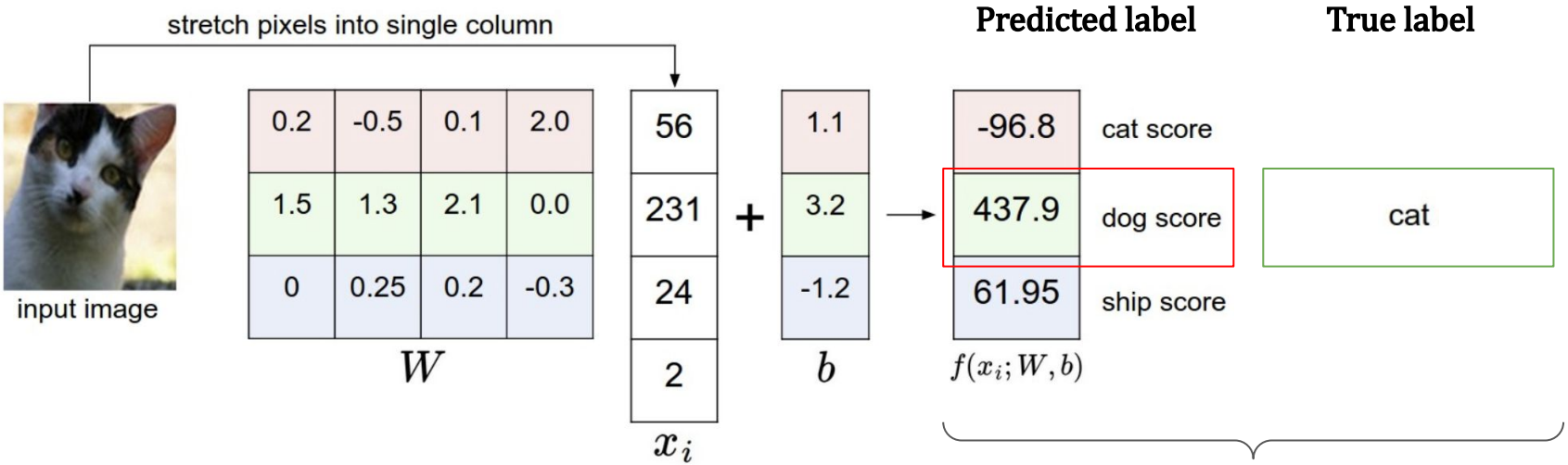
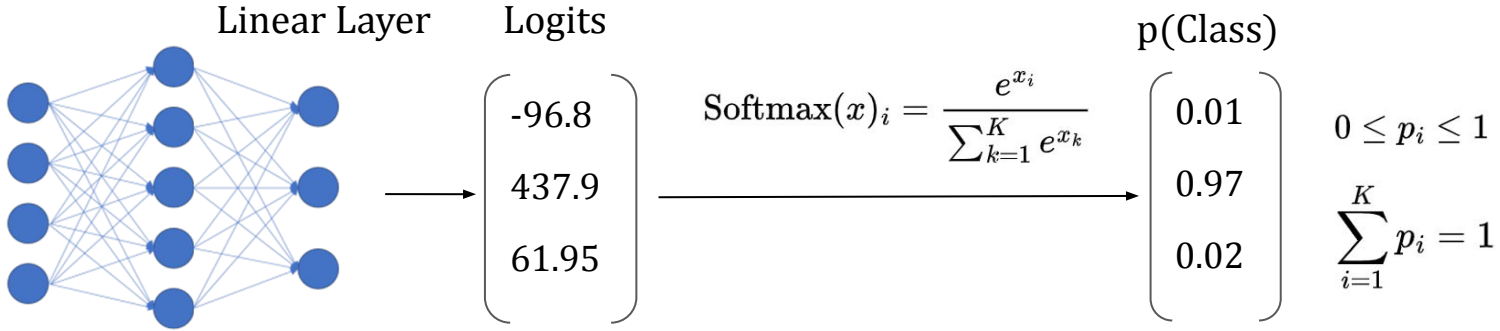


We cannot compare 437.9 to "cat" →
We need Canonical representation

https://cs231n.github.io/linear-classify/

Loss function

Softmax function → normalizes the **logits** into a categorical probability distribution over all possible classes.



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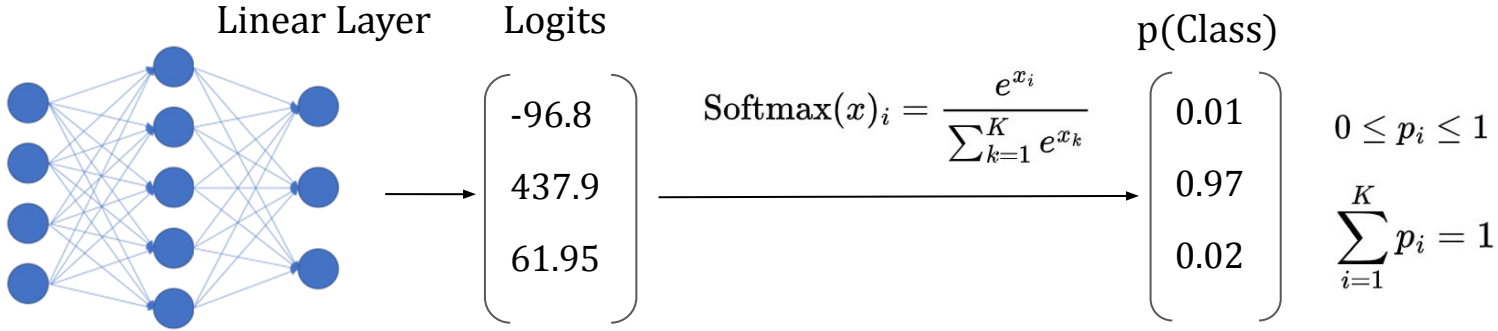
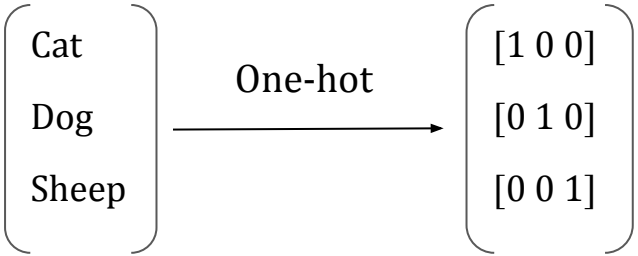


One-hot encoding → Maps categories to vector representation.



Loss function

Mean Squared Error (MSE) → Mostly used for regression problems.

$$\text{MSE} = \frac{1}{N}\sum_{n=1}^{N}(y_n - t_n)^2$$

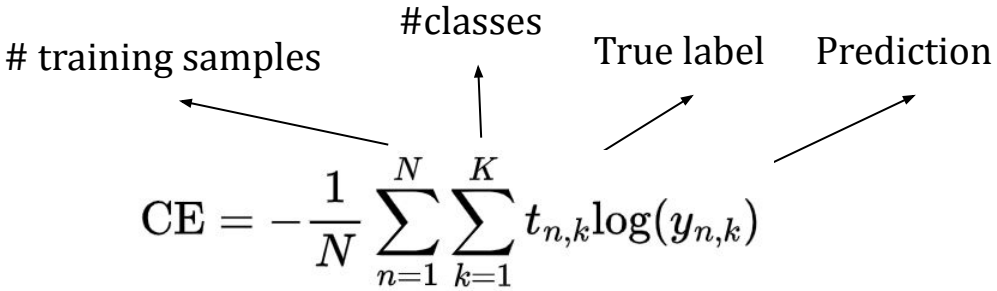
- N: Number of training samples
- y_n: Prediction
- t_n: True label

Predicted p(Class)	Ground truth
0.01	1.0
0.97	0.0
0.02	0.0

$$\text{MSE} = (0.01 - 1.0)^2 = 0.98$$

Loss function

Cross Entropy (CE) → Mostly used for classification problems.



Predicted p(Class)	Ground truth
0.01	1.0
0.97	0.0
0.02	0.0

$$\mathrm{CE} = -[1.0 \times \log_2(0.01) + 0.0 \times \log_2(0.97) + 0.0 \times \log_2(0.02)] = 6.64$$

Loss function

Cross Entropy (CE)

$$\mathrm{CE} = -\frac{1}{N}\sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k}\log(y_{n,k})$$

Binary Cross Entropy (BCE)

$$\mathrm{BCE} = -\frac{1}{N}\sum_{n=1}^{N}\left[t_n\log(y_n) + (1-t_n)\log(1-y_n)\right]$$

Forward-Pass with Error Calculations

```
import math

x = [[1.0, 0.1,-0.2],    # data
     [1.0,-0.1, 0.9],
     [1.0, 1.2, 0.1],
     [1.0, 1.1, 1.5]]
t = [0, 0, 0, 1]          # labels
w = [1, -1, 1]           # initial weights

def simple_ANN(x, w, t):
    total_e, e, y = 0, [], []
    for n in range(len(x)):
        v = 0
        for d in range(len(x[0])):
            v += x[n][d] * w[d]
        y.append(1/1+math.e**(-v))      # sigmoid
        e.append((y[n]-t[n])**2)        # MSE
     total_e = sum(e)/len(x)
     return (y, w, total_e)
```

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        for d in range(len(x[0])):
            v += x[n][d] * w[d]
        y.append(1/1+math.e**(-v))      # sigmoid
        e.append(-t[n]*math.log(y[n])-(1-t[n])*math.log(1-y[n]))  # BCE
     total_e = sum(e)/len(x)
     return (y, w, total_e)
```

Intermission
(5 to 10 min break)

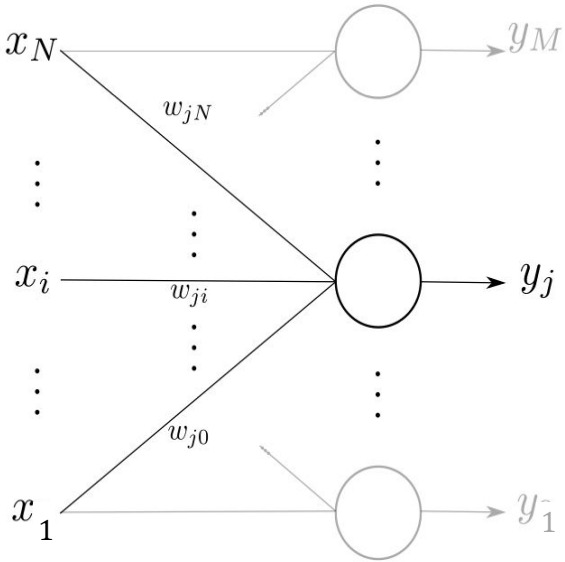
Gradient Descent

Neural Network Layer (Vector, Matrices, and Tensors)

A neural network *layer* with two neurons:

$$y_1 = f(\mathbf{w}_1 \cdot \mathbf{x} + b_1)$$
$$y_2 = f(\mathbf{w}_2 \cdot \mathbf{x} + b_2)$$

Can represent NN layer easier with a *weight matrix*, e.g. where each neuron's weight vector is a **row** of the weight matrix **W** and the input is a **column** vector **x**:



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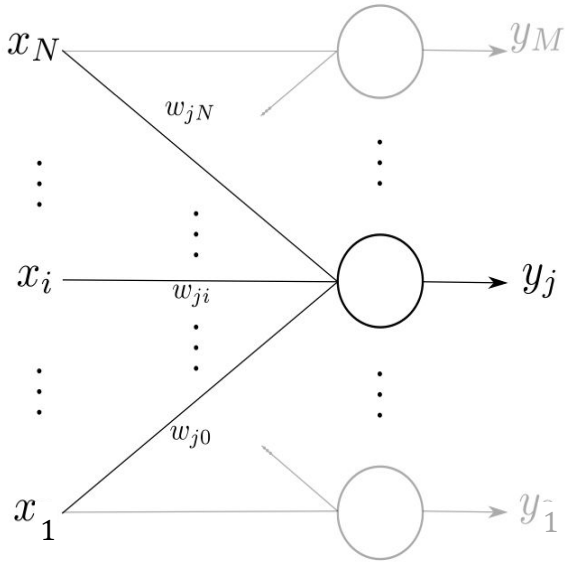
$$\mathbf{y} = f(\mathbf{W}\mathbf{x} + \mathbf{b})$$

This is important to know about because you will spend a lot of time debugging the dimensions of your tensors (where we add other dimensions such as batch size also)!



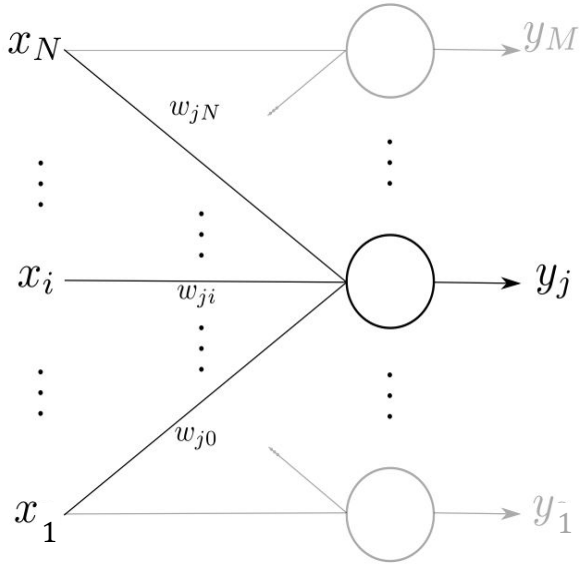
Neural Network Single Layer Training: Delta Rule

We want to know **how to change each of our neuron's weights** w_{ji} to reduce this error E

First we need to know how our error changes with each weight…

$$\frac{\partial E}{\partial w_{ji}}$$

This is relatively simple to calculate *adjacent to the output layer*



Neural Network Single Layer Training

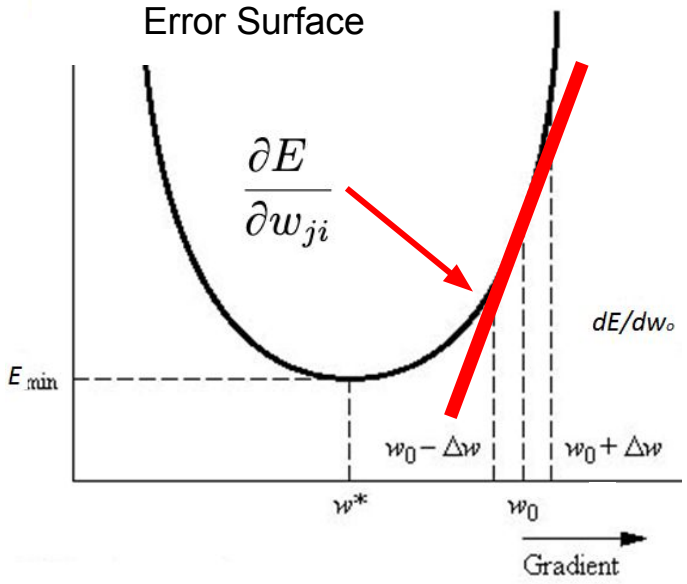
Vector of partial derivatives for all weights is the **gradient**

- Direction of the gradient is the direction in which the function increases most quickly
- Magnitude of the gradient is the rate of increase

Adjusting weights according to the slope (gradient) will guide us the minimum (or maximum) error

$$w_{ji}^{t+1} = w_{ji}^{t} - \gamma \frac{\partial E}{\partial w_{ji}} \qquad \Delta w_{ij} = \gamma \frac{\partial E}{\partial w_{ji}}$$

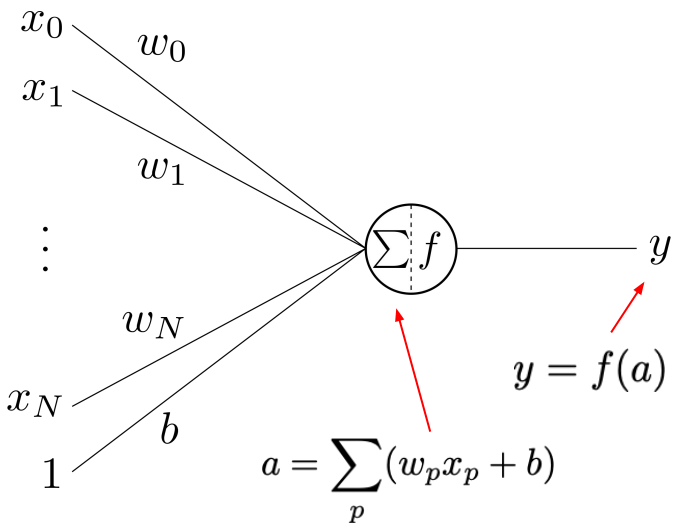
learning rate (step size)



Delta Rule for Single Weight/Training Sample

$$E = (y - t)^2 \qquad f(x) = \frac{1}{1 + e^{-x}}$$

$$\frac{dE}{dw_p} = \text{???}$$



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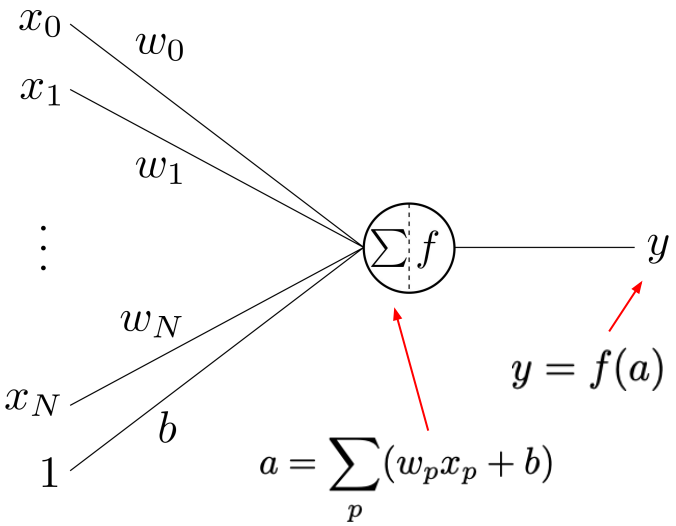

Chain rule!

$$\frac{dE}{dw_p} = \left(\frac{dE}{dy}\right)\left(\frac{dy}{da}\right)\left(\frac{da}{dw_p}\right)$$

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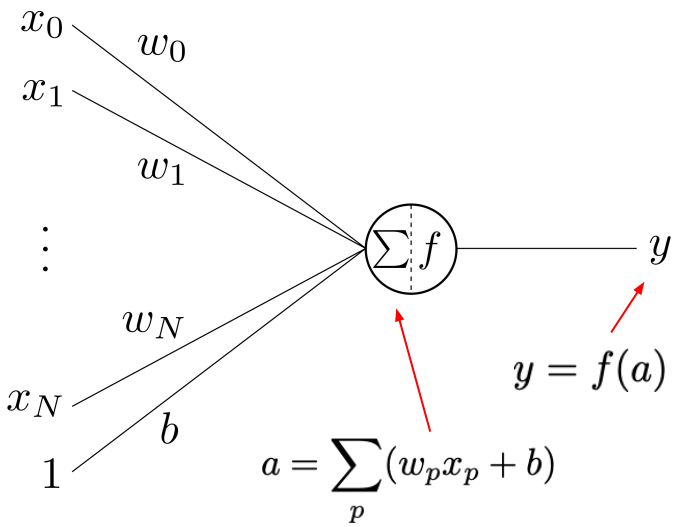

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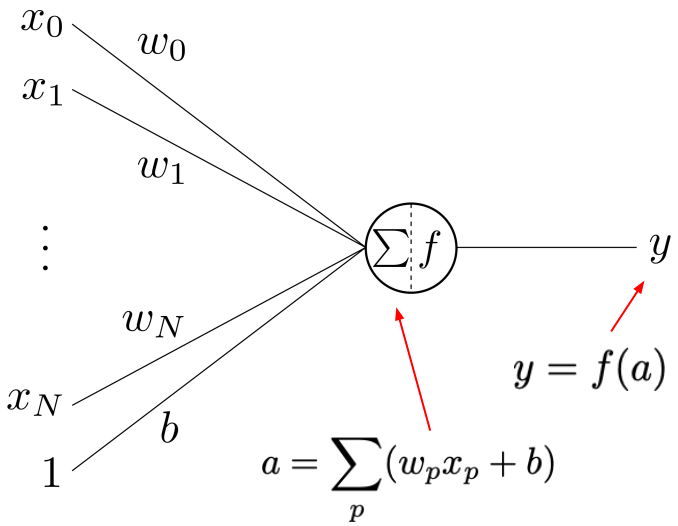

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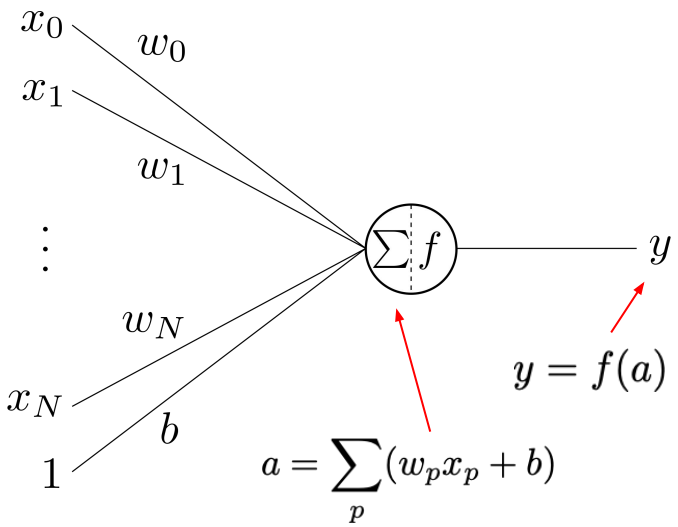

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$$\frac{da}{dw_p} = x_p$$

Delta Rule for Single Weight/Training Sample

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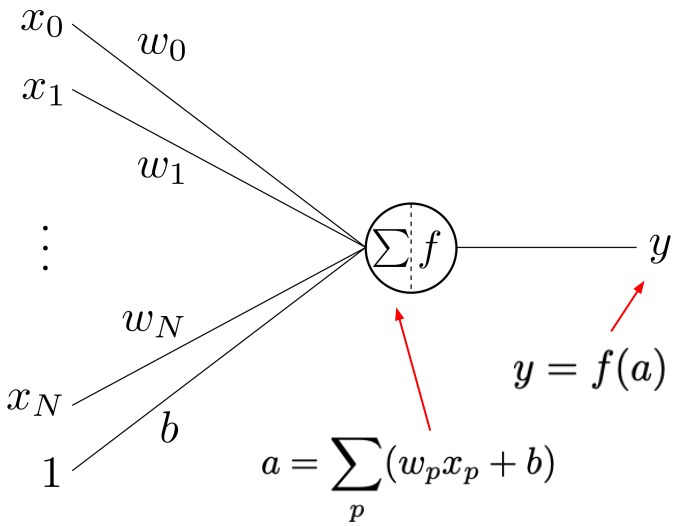

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$$\frac{da}{dw_p} = x_p$$

$$\frac{dE}{dw_p} = 2(x_p)((y - t)((1 - y)(y)))$$

Forward-Pass & Backward-Pass

```
def simple_ANN(x, w, t, iter, lr):
    total_e = 0
    for i in range(iter):
        e, y = [], []
        for n in range(len(x)):
            v = 0
            for d in range(len(x[0])):
                v += x[n][d] * w[d]
            y.append(1/1+math.e**(-v))      # sigmoid
            e.append((y[n]-t[n])**2)        # MSE

            # gradient descent to update weights
            for p in range(len(w)):
                d = 2*x[n][p]*(y[n]-t[n])*(1-y[n])*y[n]
                W[p] -= lr*d
     total_e = sum(e)/len(x)
     return (y, w, e)
```

Neural Network Architectures

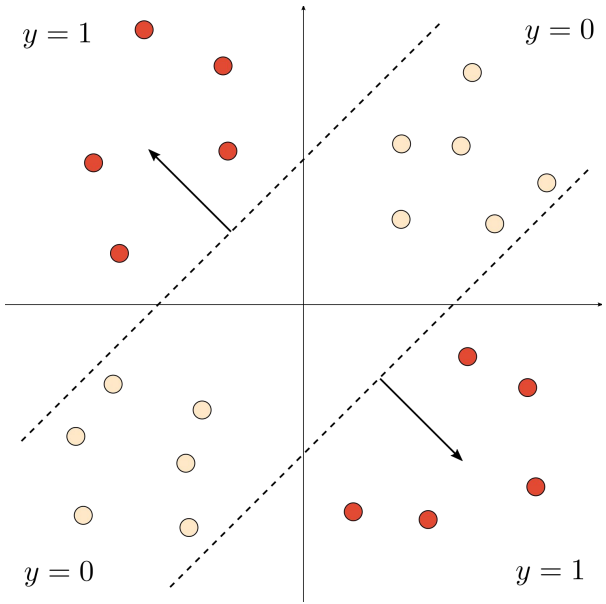
Multiple Layers are Important: XOR

Having a single decision boundary (a single NN layer) is not enough to solve many problems

The most famous such problem is the XOR function, which needs two decision boundaries to solve

We solve this by having **at least one hidden neural network layer** (i.e., two layers)

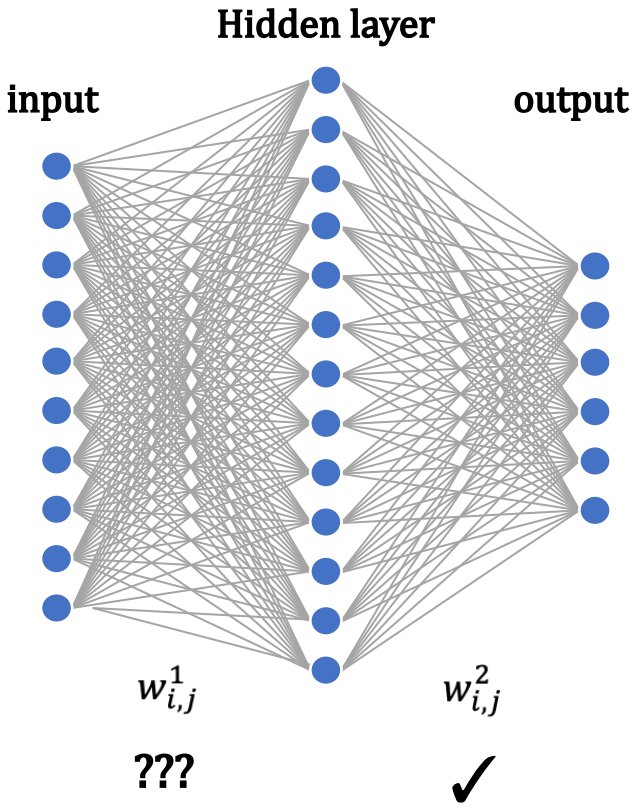
In fact in the limit of an infinitely-wide neural network with at least one hidden layer, NN is a **universal function approximator**



Backpropagation: Solving Credit Assignment Problem

Neural networks up until the 1970s were not very useful for two main reasons:

- Not clear how to train a NN of more than 1 layer → known as the **credit assignment problem**
- A neural network of only one layer cannot describe complex functions, two or more can represent any function (in theory with infinite width).



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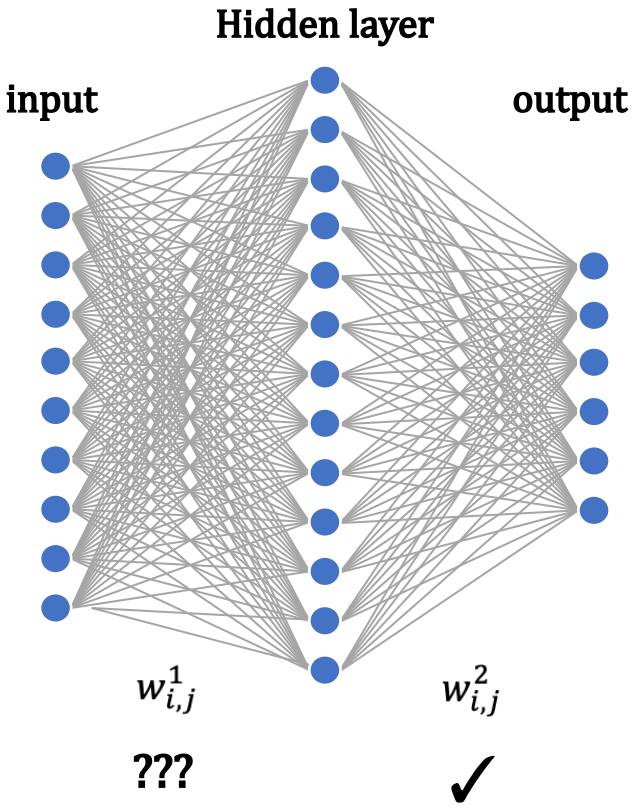
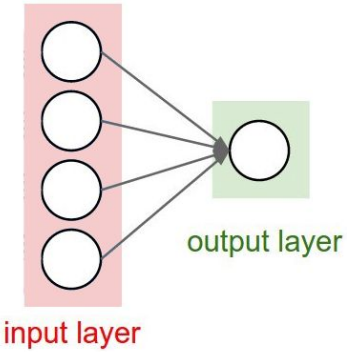
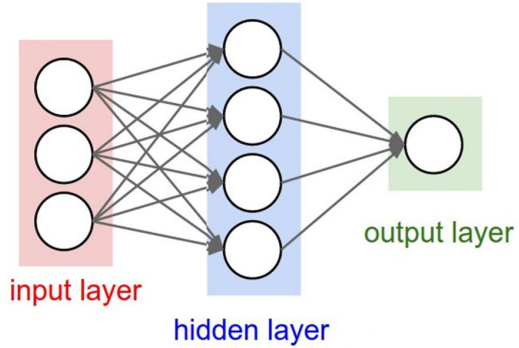
The credit assignment problem was solved by ***backpropagation***, a method that describes how to distribute errors to neurons **not adjacent to the output layer**

Solution: **Dynamic programing**

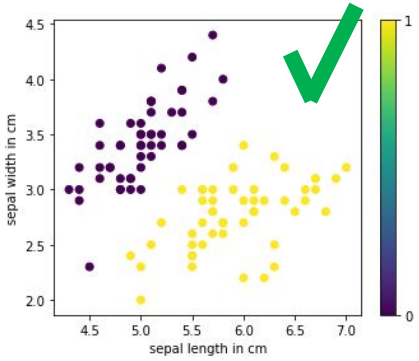


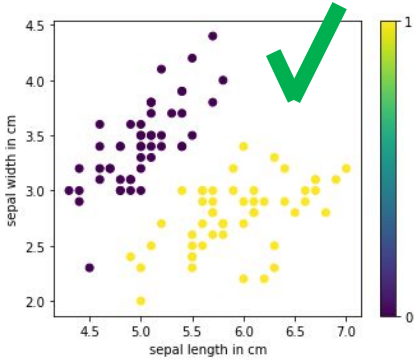
Multiple Layers with Non-Linearity



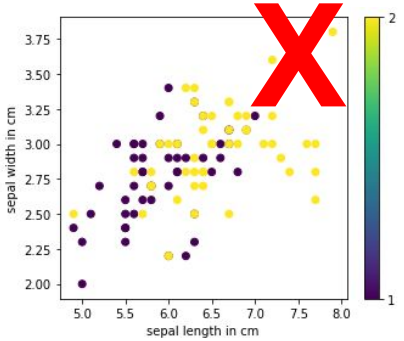


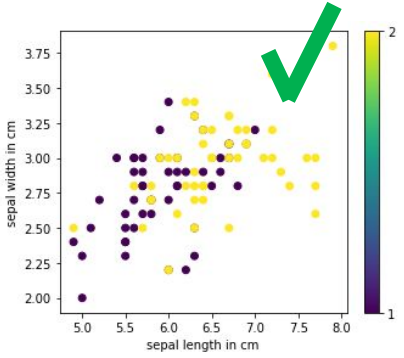
Linear





Non-Linear





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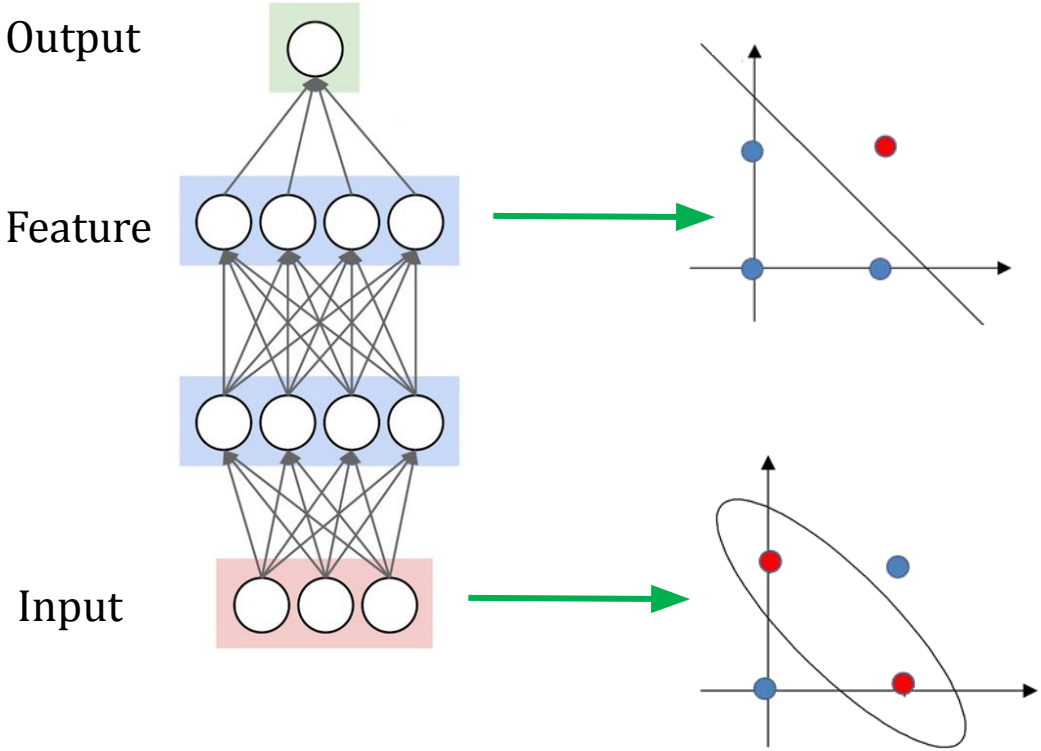
Multiple Layers with Non-Linearity

Neural Networks can be viewed as a way of **learning features** directly and end-to-end from raw input data

You can use the **activations** of the layer before the last layer as high-level features representing the input data

The goal being that the final layer is presented with a **linear separation**

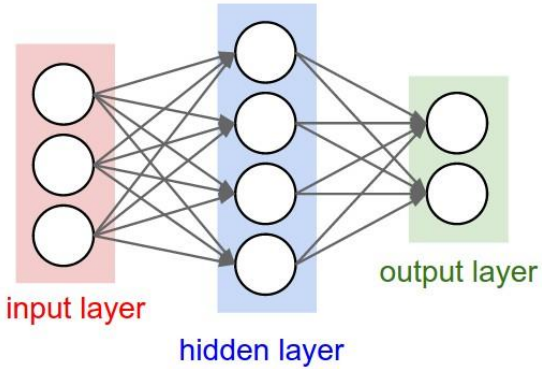


Neural Network Architecture

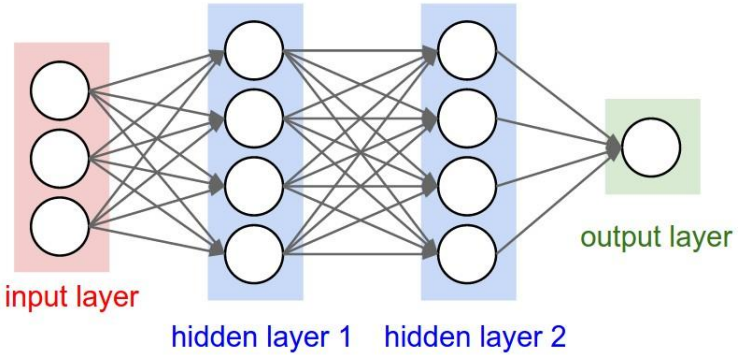
Feed-Forward Network: Information only flows forward *from one layer to a later layer*, from the input to the output.

Fully-Connected Network: Neurons between adjacent layers are fully connected.

Number of Layers: Number of hidden layers + output layer



2-layer neural network



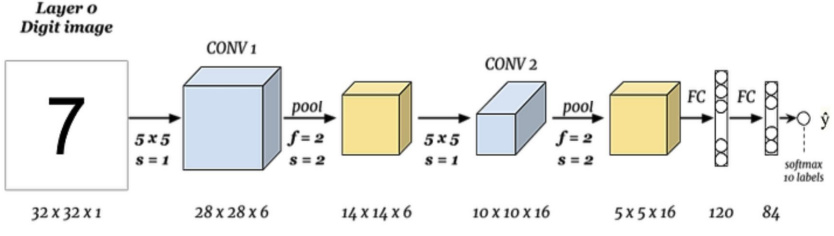
3-layer neural network

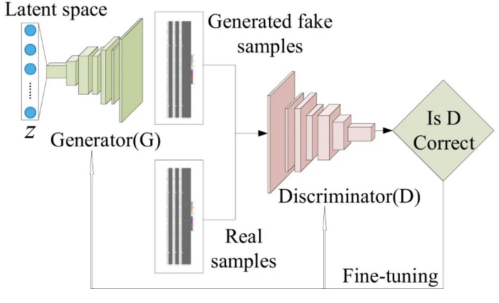
Neural Network Architecture

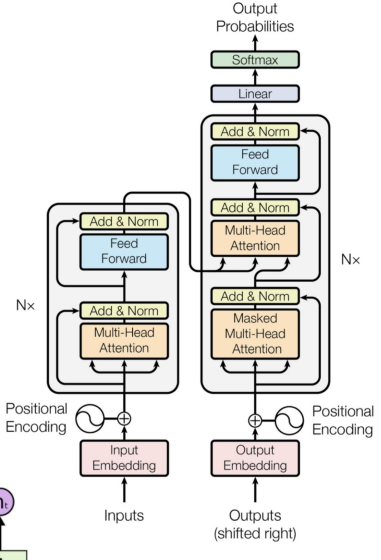
An architecture of a neural network describes the neurons and their connectivity.

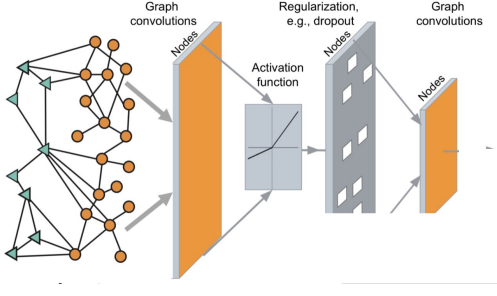
Architecture selection will greatly affect model performance.

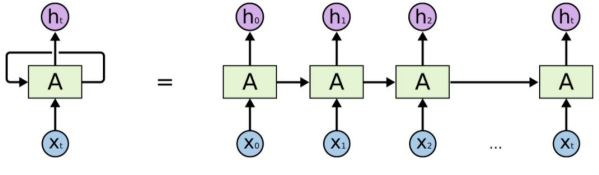
In future weeks we will introduce more neural network architecture.











Questions?