Problem 1

4. Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$
e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$

35 marks, 5 marks each

answer:

For each problem, we first write down the characteristic equation and find its roots. Using this we write down the general solution. We then plug in the initial conditions to obtain a system of linear equations. We solve these equations to determine the arbitrary constants in the general solution, and finally we write down the unique answer. □

(a) answer:

$r^2 - r - 6 = 0, \quad r_1 = -2, r_2 = 3$
$a_n = \alpha_1(-2)^n + \alpha_2 3^n$
$3 = \alpha_1 + \alpha_2$
$6 = -2\alpha_1 + 3\alpha_2$
$\alpha_1 = 3/5, \quad \alpha_2 = 12/5$
$a_n = (3/5)(-2)^n + (12/5)3^n$ □

(b)answer:

$r^2 - 7r + 10 = 0, \quad r_1 = 2, r_2 = 5$
$a_n = \alpha_1 2^n + \alpha_2 5^n$
$2 = \alpha_1 + \alpha_2$
$1 = 2\alpha_1 + 5\alpha_2$
$\alpha_1 = 3, \quad \alpha_2 = -1$
$a_n = 3(2)^n - 5^n$ □

(c)answer:

$r^2 - 6r + 8 = 0, \quad r_1 = 2, r_2 = 4$
$a_n = \alpha_1 2^n + \alpha_2 4^n$
$4 = \alpha_1 + \alpha_2$
$10 = 2\alpha_1 + 4\alpha_2$
$\alpha_1 = 3, \quad \alpha_2 = 1$
$a_n = 3(2)^n + 4^n$ □

(d)answer:

$r^2 - 2r + 1 = 0, \quad r = 1, 1$
$a_n = \alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$
$4 = \alpha_1$
$1 = \alpha_1 + \alpha_2$
$\alpha_1 = 4, \quad \alpha_2 = -3$
$a_n = 4 - 3n$ □

(e)answer:

$r^2 - 1 = 0, \quad r_1 = -1, r_2 = 1$
$a_n = \alpha_1(-1)^n + \alpha_2 1^n = \alpha_1(-1)^n + \alpha_2$
$5 = \alpha_1 + \alpha_2$
$-1 = -\alpha_1 + \alpha_2$
$\alpha_1 = 3, \quad \alpha_2 = 2$
$a_n = 3 \cdot (-1)^n + 2$ □

(f)answer:

$r^2 + 6r + 9 = 0, \quad r = -3, -3$
$a_n = \alpha_1(-3)^n + \alpha_2 n(-3)^n$
$3 = \alpha_1$
$-3 = (-3)\alpha_1 - 3\alpha_2$
$\alpha_1 = 3, \quad \alpha_2 = -2$
$a_n = 3(-3)^n - 2n(-3)^n = (3 - 2n)(-3)^n$ □

(g)answer:

$r^2 + 4r - 5 = 0. \quad r_1 = -5, r_2 = 1$
$a_n = \alpha_1(-5)^n + \alpha_2 1^n = \alpha_1(-5)^n + \alpha_2$
$2 = \alpha_1 + \alpha_2$
$8 = -5\alpha_1 + \alpha_2$

$\alpha_1 = -1, \quad \alpha_2 = 3$
$a_n = -(-5)^n + 3$ $\square$

Problem 2

12. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

7 marks

answer:

The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$. This factors as $(r-1)(r+1)(r-2) = 0$, so the roots are 1, -1, and 2. Therefore the general solution is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n$. Plugging in initial conditions gives $3 = \alpha_1 + \alpha_2 + \alpha_3$, $6 = \alpha_1 - \alpha_2 + 2\alpha_3$, and $0 = \alpha_1 + \alpha_2 + 4\alpha_3$. The solution to this system of equations is $\alpha_1 = 6, \alpha_2 = -2$, and $\alpha_3 = -1$. Therefore the answer is $a_n = 6 - 2(-1)^n - 2^n$. $\square$

Problem 3

4. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) a is taller than b.

b) a and b were born on the same day.

c) a has the same first name as b.

d) a and b have a common grandparent.

16 marks, 4 marks each

(a)answer:

Being taller than is not reflexive (I am not taller than myself), nor symmetric (I am taller than my daughter, but she is not taller than I). It is antisymmetric (vacuously, since we never have A taller than B, and B taller than A, even if $A = B$). It is clearly transitive. $\square$

(b)answer:

This is clearly reflexive, symmetric, and transitive (it is an equivalence relation). It is not antisymmetric, since twins, for example, are unequal people born on the same day. □

(c)answer:

This has exactly the same answers as part (b), since having the same first name is just like having the same birthday. □

(d)answer:

This is clearly reflexive and symmetric. It is not antisymmetric, since my cousin and I have a common grandparent, and I and my cousin have a common grandparent, but I am not equal to my cousin. This relation is not transitive. My cousin and I have a common grandparent; my cousin and her cousin on the other side of her family have a common grandparent. My cousin's cousin and I do not have a common grandparent. □

Problem 4

6. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x + y = 0$. **b)** $x = \pm y$.
c) $x - y$ is a rational number.
d) $x = 2y$. **e)** $xy \geq 0$.
f) $xy = 0$. **g)** $x = 1$.
h) $x = 1$ or $y = 1$.

32 marks, 4 marks each

(a)answer:

Since $1+1 \neq 0$, this relation is not reflexive. Since $x+y = y+x$, it follows that $x + y = 0$ if and only if $y + x = 0$, so the relation is symmetric. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. The relation is not transitive; for example, $(1, -1) \in R$ and $(-1, 1) \in R$, but $(1, 1) \notin R$. □

(b)answer:

Since $x = \pm x$(choosing the plus sign), the relation is reflexive. Since $x = \pm y$ if and only if $y = \pm x$, the relation is symmetric. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. The relation is transitive, essentially because the product of 1's and -1's is ± 1. □

(c)answer:

The relation is reflexive, since $x - x = 0$ is a rational number. The relation is symmetric, because if $x - y$ is rational, then so is $-(x - y) = y - x$. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, note that if $(x, y) \in R$ and $(y, z) \in R$, then $x - y$ and $y - z$ are rational numbers. Therefore their sum $x - z$ is rational, and that means that $(x, z) \in R$. □

(d)answer:

Since $1 \neq 2 \cdot 1$, this relation is not reflexive. It is not symmetric, since $(2, 1) \in R$, but $(1, 2) \notin R$. To see that it is antisymmetric, suppose that $x = 2y$ and $y = 2x$. Then $y = 4y$, from which it follows that $y = 0$ and hence $x = 0$. Thus the only time that (x, y) and (y, x) are both is R is when $x = y$ (and both are 0). This relation is clearly not transitive, since $(4, 2) \in R$ and $(2, 1) \in R$, but $(4, 1) \notin R$. □

(e)answer:

This relation is reflexive since squares are always nonnegative. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 3)$ and $(3, 2)$ are both in R. It is not transitive; for example, $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$. □

(f)answer:

This relation is not reflexive, since $(1, 1) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 0)$ and $(0, 2)$ are both in R. It is not transitive; for example, $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$. □

(g)answer:

This relation is not reflexive, since $(2,2) \notin R$. It is not symmetric, since $(1,2) \in R$ but $(2,1) \notin R$. It is antisymmetric, because if $(x,y) \in R$ and $(y,x) \in R$, then $x = 1$ and $y = 1$, so $x = y$. It is transitive, because if $(x,y) \in R$ and $(y,z) \in R$, then $x = 1$ (and $y = 1$, although that doesn't matter), so $(x,z) \in R$. □

(h)answer:

This relation is not reflexive, since $(2,2) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2,1)$ and $(1,2)$ are both in R. It is not transitive; for example, $(3,1) \in R$ and $(1,7) \in R$, but $(3,7) \notin R$. □

Problem 5

2. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a, b) \mid a$ and b are the same age$\}$
b) $\{(a, b) \mid a$ and b have the same parents$\}$
c) $\{(a, b) \mid a$ and b share a common parent$\}$
d) $\{(a, b) \mid a$ and b have met$\}$
e) $\{(a, b) \mid a$ and b speak a common language$\}$

10 marks, 2 marks each

(a)answer:

This is an equivalence relation. □

(b)answer:

This is an equivalence relation. □

(c)answer:

This is not an equivalence relation, since it need not be transitive. (We assume that biological parentage is at issue here, so it is possible for A to be the child of W and X, B to be the child of X and Y, and C to be the child of Y and Z. Then A is related to B, and B is related to C, but A is not related to C.) □

(d)answer:

This is not an equivalence relation since it is clearly not transitive. □

(e)answer:

Again, just as in part (c), this is not transitive. □