CS310-001: Discrete Computational Structures

Chapter 2. Basic Structures: Sets, Functions, Sequences, Sums and Matrices

Instructor: Boting Yang

Department of Computer Science
University of Regina

Outline

1. Sequences and Summations (2.4)

2. Cardinality of Sets (2.5)

Sequences

- A **sequence** is a function from a subset of the set of the integers (usually either the set $\{0, 1, 2, ...\}$ or the set $\{1, 2, 3, ...\}$) to a set S.
- A **geometric progression** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

  where the initial term a and the common ratio r are real numbers.
- An **arithmetic progression** is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

  where the initial term a and the common difference d are real numbers.

Sequences

- **Inferring a Rule:** Given some initial terms $a_0, a_1, \ldots, a_k$ of a sequence, construct a rule that is consistent with those initial terms.
- A **recursion** for a_n is a function whose arguments are earlier terms in the sequence.
- A **closed form** for a_n is a formula whose argument is the subscript n.

Example 1

- Find a closed form formula to produce a sequence with the first 10 terms $2, 5, 8, 11, 14, 17, 20, 23, 26, 29$.

Example 2

- How can we produce the terms of a sequence if the first 12 terms are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$?

Summations

- **Example 1.** $1 + 2 + 3 + \cdots + n$

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- **Example 2.** $1^2 + 2^2 + 3^2 + \cdots + n^2$

Summation Examples

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- Find a Σ-notation for $1-2+3-4+5-6+7-8+9-10$.

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- Find a Σ-notation for $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$.
- Find a Σ-notation for $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10$.
- Fill in the blanks: $\sum_{k=1}^{7} 2^k = \sum_{k=[\quad]}^{10} 2^{[\quad]}$.

Arithmetic progression

Theorem. If a and d are real numbers, then

$$\sum_{i=0}^{n}(a+id) = (n+1)a + \frac{n(n+1)d}{2}$$

Geometric progression

Theorem. If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1}, & \text{if } r \neq 1 \\ (n+1)a, & \text{if } r = 1 \end{cases}$$

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Example. Let x be a real number with $|x| < 1$. Find $\sum_{k=0}^{\infty} x^k$.

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Definition. A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$, and we write $|S| = \aleph_0$.

A set that is not countable is called *uncountable*.

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- **Hilbert's Grand Hotel.** How can we accommodate a new guest arriving at the fully occupied Grand Hotel without removing any of the current guests?

Cardinality of Sets

- **Theorem.** The set of positive rational numbers is countable.

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- **Theorem.** The set of positive rational numbers is countable.

- **Corollary.** The set of all rational numbers is countable.

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Schroder-Bernstein Theorem. If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

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Example Show that $|(0,1)| = |(0,1]|$.

Diagonalization argument

Theorem. The set of real numbers is uncountable.

Proof idea. Cantor diagonalization argument (1879).

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Applications. To prove the Halting Problem, to show that some languages are not Turing-recognizable, to show some functions are not computable, etc..

Uncomputable functions

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Theorem. There are functions from the set of positive integers to $\{0, 1\}$ that are uncomputable.