CSE 250A FINAL EXAM
FALL QUARTER 2020

OUT: `Sat Dec 12, 3:00 PM PST (Canvas)`
DUE: `Sun Dec 13, 3:00 PM PST (Gradescope)`

Question	**Points**
Academic integrity statement	*
1. d-separation	10
2. Polytree inference	10
3. Naïve Bayes vs logistic regression	10
4. Nonnegative random variables	5
5. EM algorithm	10
6. Inference in HMMs	8

Question	**Points**
7. Most likely hidden states	7
8. Gaussian random variables	6
9. Policy improvement	7
10. The simplest MDP	3
11. Noisy parity model	9
12. Gamer rating engine	15
Total:	**100**

The exam is expected to take one full afternoon, but you may work as long as needed until the deadline. **Be sure to sign and submit the statement of academic integrity with your completed exam.** *Exams without signed statements will not be graded.*

You are <u>**not**</u> required to typeset your solutions. We do expect your writing to be legible and your final answers clearly indicated. Also, please allow sufficient time to upload your solutions.

You are allowed to check your answers with programs in Matlab, Mathematica, Maple, NumPy, etc. But this should not be necessary, and be aware that these programs may not produce the intermediate steps needed to receive credit.

The later parts of problems often do not depend on the results of earlier ones; *therefore it is a good strategy to attempt all parts of every problem.*

An asterisk may indicate the need for a somewhat more laborious calculation. If you do not see the solution quickly, you may wish to return to these parts later.

If something is unclear, state the assumptions that seem most natural to you and proceed under those assumptions. Out of fairness, we will not be answering questions about the technical content of the exam on Piazza or by email.

Academic Integrity Statement

This take-home, open-book exam represents work that I have completed on my own. In particular, during the twenty-four hour period of the exam, I hereby affirm[1] the following:

(i) I have not communicated with other students in the class about these problems.

(ii) I have not consulted other knowledgeable researchers or acquaintances.

(iii) I have not contracted for help on any parts of the exam over the Internet.

I understand that any of the above actions, if proven, will result in a failing grade on the exam. In addition, I understand the following:

(i) I am obligated to report any violations of academic integrity by others that come to my attention during the exam.

(ii) All other students in the course (past and present) are under the same obligation.

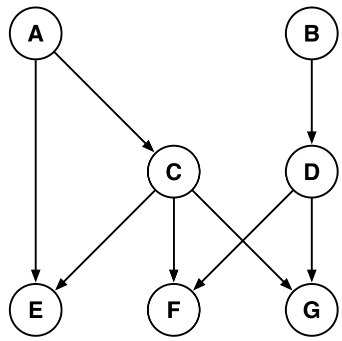
__________________________ __________________________

Signature **Date**

[1] **If you do not have a printer, you may simply copy, sign, and date the following statement with your solutions:** *I affirm the statement of academic integrity on page 2 of the exam.*

1. d-separation (10 pts)

Consider the following statements of marginal or conditional independence in the belief network shown below. For each statement, indicate whether it is true or false, and then *justify your answer* as shown in the provided examples. *Note that credit will only be awarded to answers that are supported by further explanation.*



Examples:

(i) $P(F|A,C) \stackrel{?}{=} P(F|C)$

True. There are four paths from node A to node F. They are
$A \rightarrow C \rightarrow F$,
$A \rightarrow E \leftarrow C \rightarrow F$,
$A \rightarrow C \rightarrow G \leftarrow D \rightarrow F$,
$A \rightarrow E \leftarrow C \rightarrow G \leftarrow D \rightarrow F$,
and each of these paths is blocked by node C. (Some of these paths are also blocked by other nodes, but the answer is already complete as written.)

(ii) $P(C,D|G) \stackrel{?}{=} P(C|G)\,P(D|G)$

False. The path $C \rightarrow G \leftarrow D$ is not blocked.

Problems:

(a) $P(B|F) \stackrel{?}{=} P(B|E,F)$

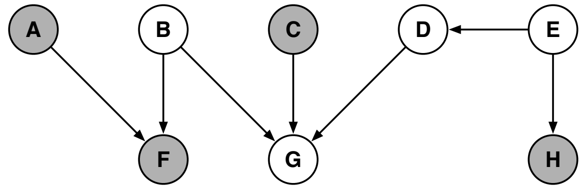
(b) $P(B,E) \stackrel{?}{=} P(B)\,P(E)$

(c) $P(A|B,F) \stackrel{?}{=} P(A|F)$

(d) $P(A,B|E) \stackrel{?}{=} P(A|E)\,P(B|E)$

(e) $P(C|A,E,F,G) \stackrel{?}{=} P(C|A,B,E,F,G)$

2. Polytree inference (10 pts)



For the belief network shown above, consider how to *efficiently* compute the conditional probability $P(G|A, C, F, H)$. This can be done in four consecutive steps in which some later steps rely on the results from earlier ones.

Complete the procedure below for this inference by showing how to compute the following probabilities. Make each computation as efficient as possible, and *briefly justify each step* in your solutions for full credit. Your answers should be expressed in terms of the CPTs of the belief network and (as needed) the results of previous steps.

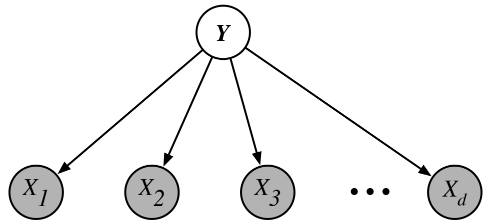
(a) Compute $P(E|H)$.
(2 pts)

(b) Compute $P(D|H)$.
(2 pts)

(c) Compute $P(B|A, F)$.
(3 pts)

(d) Compute $P(G|A, C, F, H)$.
(3 pts)

3. Naïve Bayes versus logistic regression (10 pts)



(a) Naïve Bayes model (2 pts)

Consider the belief network of discrete random variables shown above. Show how to compute the conditional probability

$$P(y|x_1, x_2, \ldots, x_d)$$

in terms of the belief network's probability tables for $P(y)$ and $P(x_i|y)$. Justify your steps to receive full credit.

(b) Log-odds (2 pts)

Consider the special case where all the variables in this belief network are binary-valued. For this special case, compute the log-odds

$$\log \frac{P(Y\!=\!1|x_1, x_2, \ldots, x_d)}{P(Y\!=\!0|x_1, x_2, \ldots, x_d)}$$

in terms of the belief network's probability tables. Simplify your answer as much as possible; this will be helpful for the next parts of the problem.

(c*) Linear decision boundary (3 pts)

Show that the log-odds from part (b) is a linear function of the values of $x_1, x_2, \ldots, x_n$. In particular, show that it can be written in the form:

$$\log \frac{P(Y\!=\!1|x_1, x_2, \ldots, x_d)}{P(Y\!=\!0|x_1, x_2, \ldots, x_d)} \;=\; a_0 + \sum_{i=1}^{d} a_i x_i$$

for appropriately chosen values of $a_0, a_1, \ldots, a_d$. Your solution should express these values in terms of the belief network's probability tables.

Hint: since each x_i is equal to either 0 or 1, it may be a useful notation to write

$$\log \frac{P(x_i|Y\!=\!1)}{P(x_i|Y\!=\!0)} \;=\; x_i \log \frac{P(X_i\!=\!1|Y\!=\!1)}{P(X_i\!=\!1|Y\!=\!0)} \;+\; (1\!-\!x_i) \log \frac{P(X_i\!=\!0|Y\!=\!1)}{P(X_i\!=\!0|Y\!=\!0)}.$$

(d) Logistic regression (2 pts)

Consider whether it is possible to express your result for $P(Y=1|x_1, x_2, \ldots, x_d)$ in the form of a logistic regression. In particular, are there parameters $w_0, w_1, \ldots, w_d$ such that

$$P(Y=1|x_1, x_2, \ldots, x_d) \;=\; \sigma\left(w_0 + \sum_{i=1}^{d} w_i x_i\right),$$

where $\sigma(z) = (1 + e^{-z})^{-1}$ is the sigmoid function? If yes, show how to choose these parameters so that this is true. If not, explain why not.

Hint: What is the inverse of the sigmoid function?

4. Nonnegative random variables (5 pts)

Let $\mu > 0$. In this problem you will derive some elementary but useful properties of the exponential distribution

$$P(z) \;=\; \frac{1}{\mu}\exp\left(-\frac{z}{\mu}\right)$$

for a continuous, **nonegative** random variable with mean μ. (You will be exploiting these properties to solve the last problem on the exam.)

(a) Log-likelihood (1 pt)

Let $\{z_1, z_2, \ldots, z_T\}$ be an *i.i.d.* data set of nonnegative values. Assuming each value was drawn from an exponential distribution, compute the log-likelihood

$$\mathcal{L}(\mu) \;=\; \sum_{t=1}^{T} \log P(z_t)$$

in terms of the distribution's parameter μ.

(b) Maximum likelihood estimation (1 pt)

Show that the maximum likelihood estimate for μ is given by the sample mean of the data.

(c) Cumulative distribution (1 pt)

Calculate the *cumulative distribution*, given by

$$P(Z\!<\!a) \;=\; \int_0^a dz\, P(z),$$

when Z is exponentially distributed with mean μ.

(d) Comparison (2 pts)

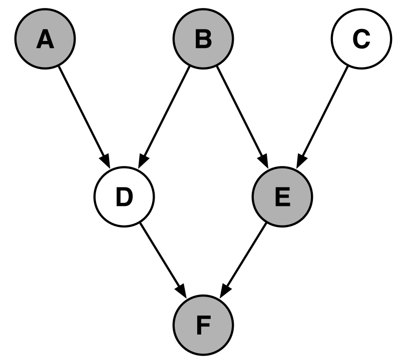
Suppose that Z_1 and Z_2 are independent, exponentially distributed random variables with means μ_1 and μ_2, respectively. Show that

$$P(Z_1\!>\!Z_2) \;=\; \frac{\mu_1}{\mu_1+\mu_2},$$

where the left side denotes the probability that Z_1 exceeds Z_2 in value.

Hint: note that $P(Z_1 > Z_2) = \int_0^\infty da\, P(Z_1 = a)\, P(Z_2 < a)$, and use your result from the previous part of this problem.

5. EM algorithm (10 pts)



Consider the belief network shown at the right, with observed nodes $\{A, B, E, F\}$ and hidden nodes $\{C, D\}$. On the problems below, *simplify your answers as much as possible, and briefly justify your steps to receive full credit.*

(a) Hidden node C (2 pts)

Show how to compute the posterior probability $P(C|A, B, E, F)$ in terms of the conditional probability tables (CPTs) of the belief network.

(b) Hidden node D (2 pts)

Show how to compute the posterior probability $P(D|A, B, E, F)$ in terms of the CPTs of the belief network.

(c) Both hidden nodes (1 pt)

Show how to compute the posterior probability $P(C, D|A, B, E, F)$ in terms of your previous results.

(d) **Log-likelihood** (2 pts)

Consider a data set of T partially labeled examples $\{a_t, b_t, e_t, f_t\}_{t=1}^{T}$ over the observed nodes of the network. The log-likelihood of the data set is given by:

$$\mathcal{L} = \sum_t \log P(A\!=\!a_t, B\!=\!b_t, E\!=\!e_t, F\!=\!f_t)$$

Compute this expression in terms of the CPTs of the belief network.

(e) EM algorithm (3 pts)

Consider the EM updates for the CPTs of this belief network that maximize the log-likelihood in part (d). Provide these updates for the following:

(i) $P(C\!=\!c)$

(ii) $P(D\!=\!d|A\!=\!a, B\!=\!b)$

(iii) $P(E\!=\!e|B\!=\!b, C\!=\!c)$

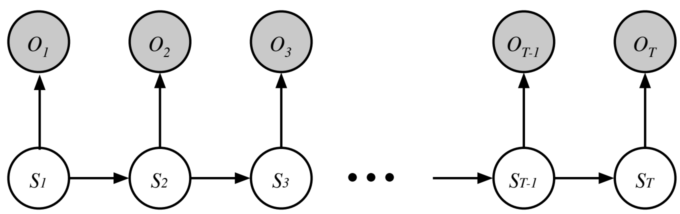
For this part of the problem, it is *not* necessary to justify your steps; it is only necessary to state the correct updates.

6. Inference in HMMs (8 pts)

Consider a discrete HMM with the belief network shown below. As usual, let $s_t \in \{1, 2, \ldots, n\}$ and $o_t \in \{1, 2, \ldots, m\}$ denote, respectively, the hidden state and observation at time t; also, let

$$\begin{aligned} \pi_i &= P(S_1 = i), \\ a_{ij} &= P(S_{t+1} = j | S_t = i), \\ b_{ik} &= P(O_t = k | S_t = i), \end{aligned}$$

denote the initial distribution over hidden states, the transition matrix, and the emission matrix. In your answers you may also use $b_i(k)$ to denote the matrix element b_{ik}.



(a) **Inference** (4 pts)

Let a^t denotes the tth power (via matrix multiplication) of the transition matrix, and assume that a^0 denotes the identity matrix. Prove by induction or otherwise that

$$P(S_t = i) = \sum_{k=1}^{n} \pi_k (a^{t-1})_{ki}.$$

(b) **More inference** (4 pts)

The forward-backward algorithm in discrete HMMs computes the probabilities

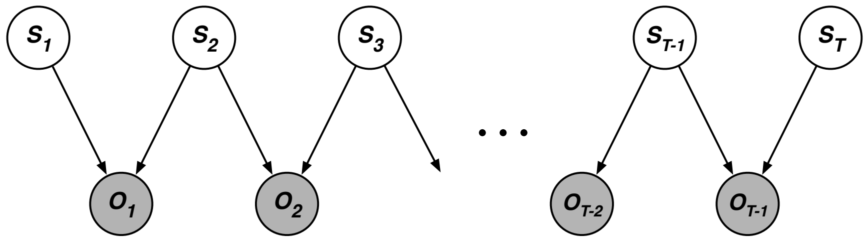
$$\begin{aligned} \alpha_{it} &= P(o_1, o_2, \ldots, o_t, S_t = i), \\ \beta_{it} &= P(o_{t+1}, o_{t+2}, \ldots, o_T | S_t = i). \end{aligned}$$

In terms of these probabilities (which you may assume to be given) and the parameters (a_{ij}, b_{ik}, π_i) of the HMM, show how to compute the conditional probability

$$P(o_1, o_2, \ldots, o_T | S_t = i, S_{t+1} = j)$$

for times $1 \le t < T$. (Note that this is *not* the same conditional probability computed in lecture.) Show your work for full credit, justifying each step in your derivation, and simplifying your answer as much as possible. *Hint:* your result from part (a) may be useful.

7. Most likely hidden states (7 pts)



Consider the belief network shown above, where $s_t \in \{1, 2, \ldots, n\}$ and $o_t \in \{1, 2, \ldots, m\}$ denote, respectively, the hidden state and observation at time t. In this problem you will derive an efficient algorithm for computing the most likely sequence of hidden states

$$\{s_1^*, s_2^*, \ldots, s_T^*\} = \underset{s_1, s_2, \ldots, s_t}{\operatorname{argmax}} P(s_1, s_2, \ldots, s_t | o_1, o_2, \ldots, o_{t-1}).$$

Note that this is *not* a hidden Markov model: in particular, every observation node has two parents, every hidden node has none, and the joint distribution is given by

$$P(S_1, S_2, \ldots, S_T, O_1, O_2, \ldots, O_{T-1}) = \prod_{t=1}^{T} P(S_t) \prod_{t=1}^{T-1} P(O_t | S_t, S_{t+1}).$$

(a) Forward pass (4 pts)

Given a sequence of $T-1$ observations, the most likely hidden states are most easily found by computing the T-column matrix with elements

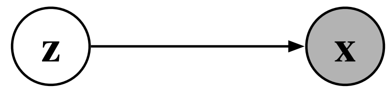
$$\ell_{it}^* = \max_{s_1, s_2, \ldots, s_{t-1}} \log P(s_1, s_2, \ldots, s_{t-1}, S_t = i, o_1, o_2, \ldots, o_{t-1}).$$

Note that the log probability for ℓ_{it}^* in this network only includes observations up to time t–1. Give an efficient algorithm to compute these matrix elements for $1 \leq i \leq n$ and $1 \leq t \leq T$.

(b) Backward pass (3 pts)

Show how to efficiently derive the sequence of most likely hidden states from the results of the forward pass in part (a).

8. Gaussian random variables (6 pts)



$\mathbf{z} \sim \mathcal{N}(\mathbf{0}, I)$ $\qquad\qquad$ $\mathbf{x} \sim \mathcal{N}(\mathbf{\Lambda z}, I)$

Consider the belief network of multivariate Gaussian random variables shown above, where $\mathbf{z} \in \Re^d$ is hidden and $\mathbf{x} \in \Re^D$ is observed. In this network,

$$\begin{aligned} P(\mathbf{z}) &= \frac{1}{(2\pi)^{d/2}} \exp\left\{-\tfrac{1}{2}\,\mathbf{z}^\top\mathbf{z}\right\}, \\ P(\mathbf{x}|\mathbf{z}) &= \frac{1}{(2\pi)^{D/2}} \exp\left\{-\tfrac{1}{2}\,(\mathbf{x}-\mathbf{\Lambda z})^\top(\mathbf{x}-\mathbf{\Lambda z})\right\}. \end{aligned}$$

Here, the parameter $\mathbf{\Lambda}$ is a $D \times d$ matrix. Note that both these multivariate Gaussian distributions have an identity covariance matrix.

(a) Posterior mode (2 pts)

Let $\mathbf{z}^* = \arg\max_{\mathbf{z}} P(\mathbf{z}|\mathbf{x})$ denote the mode of the posterior distribution (i.e., the location of its global maximum). Prove that we may also compute this mode from

$$\mathbf{z}^* = \underset{\mathbf{z}}{\operatorname{argmax}} \, \log\left[P(\mathbf{z})\,P(\mathbf{x}|\mathbf{z})\right].$$

(b*) Maximization (3 pts)

Compute $\mathbf{z}^*$ by explicitly maximizing $\log\left[P(\mathbf{z})\,P(\mathbf{x}|\mathbf{z})\right]$ as suggested in part (a). Your answer should express $\mathbf{z}^*$ in terms of the observed value of $\mathbf{x}$ and the matrix parameter $\mathbf{\Lambda}$.

(c) Posterior mean (1 pt)

Consider the following assertion:

The posterior mean $\mathrm{E}[\mathbf{z}|\mathbf{x}]$*, defined by the multidimensional integral*

$$\mathrm{E}[\mathbf{z}|\mathbf{x}] = \int_{\mathbf{z}\in\Re^d} \mathbf{dz}\, P(\mathbf{z}|\mathbf{x})\,\mathbf{z},$$

is equal to the posterior mode $\mathbf{z}^* = \arg\max_{\mathbf{z}} P(\mathbf{z}|\mathbf{x})$ *that was obtained (much more simply) by differentiation in part (b).*

Is the above statement **true** or **false**? If true, explain by appealing to the properties of normal distributions; if false, provide a counterexample. (You are not meant to evaluate the integral.)

9. Policy improvement (7 pts)

Consider the Markov decision process (MDP) with two states $s \in \{0,1\}$, two actions $a \in \{\downarrow, \uparrow\}$, discount factor $\gamma = \frac{2}{3}$, and the reward function and transition matrices as shown below:

s	$R(s)$
0	-4
1	8

s	s'	$P(s'\|s, a{=}\downarrow)$
0	0	$\frac{3}{4}$
0	1	$\frac{1}{4}$
1	0	$\frac{1}{4}$
1	1	$\frac{3}{4}$

s	s'	$P(s'\|s, a{=}\uparrow)$
0	0	$\frac{1}{2}$
0	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$
1	1	$\frac{1}{2}$

(a) State value function (4 pts)

Consider the policy π that chooses the action $a = \downarrow$ in each state, and compute the state value function $V^{\pi}(s)$ for $s \in \{0,1\}$. Show your work for full credit.

Hint: the elements of the state value function are integers.

(b) **Action value function** (2 pts)

Compute the action value function $Q^{\pi}(s,a)$ for this same policy, where $s \in \{0,1\}$ and $a \in \{\downarrow, \uparrow\}$. Show your work for full credit.

(c) **Greedy policy** (1 pt)

Compute the greedy policy π' with respect to these value functions. Your final answer should clearly specify $\pi'(s) \in \{\downarrow, \uparrow\}$ for $s \in \{0,1\}$. Show your work for full credit.

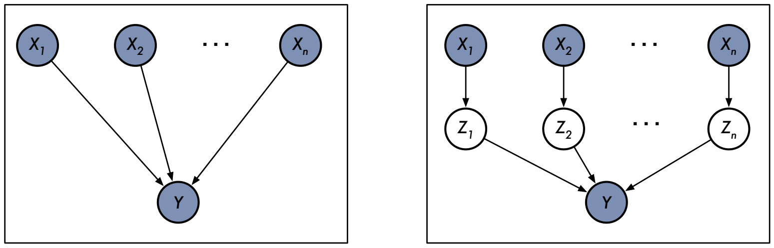
10. The simplest MDP (3 pts)

Consider a Markov decision process (MDP) in which the reward function is constant: i.e., there is a scalar reward r such that

$$R(s) \;=\; r$$

for every state s in the state space. In such an MDP, prove that every policy is optimal.

11. Noisy parity model (9 pts)



(a) Noisy parity (2 pts)

Consider the belief network of binary (0/1) random variables shown *on the left*. Furthermore, suppose that the conditional probability table (CPT) at node Y takes the form

$$P(Y\!=\!1|x_1, x_2, \ldots, x_n) \;=\; \frac{1}{2}\left[1 - \prod_{i=1}^{n}(1-2p_i)^{x_i}\right],$$

where $p_i \in [0,1]$ are parameters in the unit interval. Show that when all the parameters p_i are equal to one, the node Y deterministically computes the *parity* of its parent's bit vector:

$$P(Y\!=\!1|x_1, x_2, \ldots, x_n) \;=\; \begin{cases} 1 & \text{if } \sum_i x_i \text{ is odd,} \\ 0 & \text{if } \sum_i x_i \text{ is even.} \end{cases}$$

(b) Noisy copy (1 pt)

Consider the belief network of binary (0/1) random variables shown *on the right*. Suppose thast each hidden variable Z_i is a noisy copy of its parent X_i in the following sense:

$$\begin{aligned} P(Z_i\!=\!0|X_i\!=\!0) &= 1, \\ P(Z_i\!=\!1|X_i\!=\!1) &= p_i \quad \text{where } p_i \in [0,1]. \end{aligned}$$

Use these probabilities to derive the following simple identity (which may be useful for the next part of this problem); in particular, for $x_i \in \{0, 1\}$, show that

$$P(Z_i\!=\!0|X_i\!=\!x_i) - P(Z_i\!=\!1|X_i\!=\!x_i) \;=\; (1-2p_i)^{x_i},$$

where in this context it is understood that any real number (including zero) raised to the zeroth power is equal to one.

(c*) Latent variable model (6 pts)

In the belief network *on the right*, suppose that the node Y deterministically computes the parity of its parents:

$$P(Y\!=\!1|z_1, z_2, \ldots, z_n) \;=\; \frac{1}{2}\left[1 - \prod_{i=1}^{n}(-1)^{z_i}\right].$$

Show in this case that the belief network on the right yields the same conditional probability $P(Y = 1|x_1, x_2, \ldots, x_n)$ as given in part (a).

12. Gamer rating engine (15 pts)

A gaming company pits n users on the Internet against each other in a popular two-player game. The game involves a combination of skill and chance, so that in each game the more skilled of the two players is likely but not guaranteed to win.

Over time the company records the wins and losses of these games in an $n \times n$ matrix with elements G_{ij}. Specifically, the element G_{ij} records the number of games in which user i beat user j. Note that $G_{ii} = 0$, because users cannot play against themselves, and also that in general $G_{ij} \neq G_{ji}$, because in each matchup the more skilled player is more likely to win than lose.

From this matrix, the company wants to assign each user a rating so that it can suggest new matches between players of comparable skill. Let $r_i \in \Re$ denote this rating for the ith user, and let $\sigma(z) = (1 + e^{-z})^{-1}$ denote the sigmoid function. In a game between players with ratings r_i and r_j, the company hypothesizes that the player with rating r_i should win with probability $\sigma(r_i - r_j)$. The log-likelihood of their data in this model is therefore given by

$$\mathcal{L} \;=\; \sum_{i,j=1}^{n} G_{ij} \log \sigma(r_i - r_j),$$

and the best model of this form is found by choosing the user ratings to maximize this expression. This is where they turn to you for help.

(a) Gradient ascent (3 pts)

The simplest approach to maximize the log-likelihood is gradient ascent. In this case, the user ratings are adapted by

$$r_k \leftarrow\; r_k + \eta \left(\frac{\partial \mathcal{L}}{\partial r_k}\right)$$

for some small learning rate $\eta > 0$. Compute the partial derivative

$$\frac{\partial \mathcal{L}}{\partial r_k}$$

that appears in this learning rule, and simplify your final expression as much as possible. Be sure to show your work for full credit.

(b) Self-check (2 pts)

Intuitively, if the user ratings have been properly estimated, we might expect that the number of expected wins for each user is equal to the number of observed wins. More precisely, for the kth user, we might expect that

$$\sum_{j=1}^{n} (G_{kj} + G_{jk})\sigma(r_k - r_j) \;=\; \sum_{j=1}^{n} G_{kj},$$

where the left and right sides compute, respectively, the number of expected and observed wins. Using your answer from part (a), show that the elements of the gradient vanish when the above condition is satisfied.

(c) Nonnegative ratings (1 pt)

Unfortunately, the marketing department of the company objects to this model; it refuses on principle to assign any user a zero or negative skill rating. (Too demoralizing, they say.)

To this end, let $\mu_i = e^{r_i}$ be a new (strictly positive) rating for each user with original (real-valued) rating r_i. Consider how the probability of a win is expressed in terms of these of new ratings. In particular, show that

$$\sigma(r_i - r_j) \;=\; \frac{\mu_i}{\mu_i + \mu_j}.$$

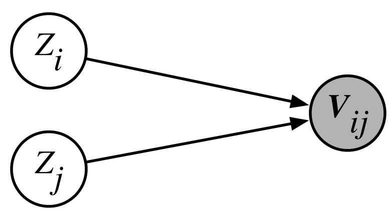
Note how this expression matches the form of the probability you computed in problem 4(d). We can thus interpret these transformed ratings as parameters of a latent variable model. In this model, each game between the ith and jth user is simulated by the following experiment:

- For the ith user, sample an exponentially distributed random variable Z_i with mean μ_i.
- For the jth user, sample an exponentially distributed random variable Z_j with mean μ_j.
- Award victory to the user with the larger sampled variable.

This latent variable model is depicted by the belief network shown below, where the binary random variable V_{ij} indicates a victory by player i (for $V_{ij}\!=\!1$) or player j (for $V_{ij}\!=\!0$).

$$P(Z_i\!=\!z) \;=\; \tfrac{1}{\mu_i}\exp(-z/\mu_i)$$

$$P(Z_j\!=\!z) \;=\; \tfrac{1}{\mu_j}\exp(-z/\mu_j)$$



$$P(V_{ij}\!=\!1|Z_i, Z_j) \;=\; \begin{cases} 1 & \text{if } Z_i > Z_j \\ 0 & \text{otherwise} \end{cases}$$

Not surprisingly, the EM algorithm for this model takes a familiar form. The E-step computes posterior means, such as $\mathrm{E}[Z_i|V_{ij}]$, in terms of the current user ratings, and the M-step uses these expected values to derive new user ratings. The rest of this problem guides you through these steps.

(See next page.)

(d) M-step (3 pts)

The EM updates in this model are derived as usual from an auxiliary function. In this case, the auxiliary function is given by

$$Q(\boldsymbol{\mu}, \boldsymbol{\mu}_{\mathrm{old}}) \;=\; -\sum_{i,j=1}^{n} \left\{ G_{ij}\left(\log \mu_i \;+\; \frac{\mathrm{E}[Z_i|V_{ij}\!=\!1]}{\mu_i}\right) \;+\; G_{ji}\left(\log \mu_i + \frac{\mathrm{E}[Z_i|V_{ij}\!=\!0]}{\mu_i}\right)\right\},$$

where $\boldsymbol{\mu}$ is shorthand for $(\mu_1, \mu_2, \ldots, \mu_n)$, and where the posterior means on the right side are computed in terms of the current user ratings $\boldsymbol{\mu}_{\mathrm{old}}$. From the above expression (which you are not asked to prove), derive the EM updates by maximizing the auxiliary function with respect to its first argument:

$$\boldsymbol{\mu}_{\mathrm{new}} \;=\; \operatorname*{argmax}_{\boldsymbol{\mu}} Q(\boldsymbol{\mu}, \boldsymbol{\mu}_{\mathrm{old}}).$$

Your answer should show how to re-estimate the rating μ_i for the ith user in terms of the counts G_{ij} and G_{ji} as well as the posterior means $\mathrm{E}[Z_i|V_{ij}\!=\!1]$ and $\mathrm{E}[Z_i|V_{ij}\!=\!0]$ (which you will compute next).

(e) Posterior mean (when the player wins) (3 pts)

To complete the EM algorithm, you must compute the posterior means that appear in your answer to part (d). For the latent variable model on the previous page, it can be shown that

$$\mathrm{E}[Z_i|V_{ij}\!=\!1] \;=\; \frac{1}{P(Z_i\!>\!Z_j)} \int_0^{\infty}\! dz\; P(Z_i\!=\!z)\, P(Z_j\!<\!z)\, z.$$

You do not need to prove[2] this result, but assuming it to be true, use it to calculate $\mathrm{E}[Z_i|V_{ij}\!=\!1]$ in terms of the current user ratings.

Hint: substitute your result from 4(d) for the term $P(Z_i > Z_j)$, your result from 4(c) for the term $P(Z_j\!<\!z)$, and the exponential distribution for $P(Z_i\!=\!z)$. Then perform the integral.

(See next page.)

[2]It follows from Bayes rule and marginalization—nothing magical or mysterious—but you already have enough to do on this exam.

(f) Posterior mean (when the player loses) (3 pts)

Here's some good news for the last problem: you can compute the other posterior mean $\mathrm{E}[Z_i|V_{ij}\!=\!0]$ without evaluating yet another integral.

To do this, you only need a result that follows from the most basic rules of probability. Note that by definition, the prior and posterior means of Z_i are given by

$$\begin{aligned}
\mathrm{E}[Z_i] &= \int_0^\infty dz\, z\, P(Z_i\!=\!z), \\
\mathrm{E}[Z_i|V_{ij}\!=\!0] &= \int_0^\infty dz\, z\, P(Z_i\!=\!z|V_{ij}\!=\!0), \\
\mathrm{E}[Z_i|V_{ij}\!=\!1] &= \int_0^\infty dz\, z\, P(Z_i\!=\!z|V_{ij}\!=\!1).
\end{aligned}$$

It turns out that two of these expected values, along with the marginal probabilities $P(V_{ij}\!=\!0)$ and $P(V_{ij}\!=\!1)$, are sufficient to determine the third. To prove this fact, show that

$$\mathrm{E}[Z_i] \;=\; P(V_{ij}\!=\!0)\, \mathrm{E}[Z_i|V_{ij}\!=\!0] \;+\; P(V_{ij}\!=\!1)\, \mathrm{E}[Z_i|V_{ij}\!=\!1].$$

Hint: Write out the expected values on the right side as integrals, then use the product rule.

The line across the page is to make clear that no more work is required! But for completeness, here is how this identity yields the other posterior mean. Rearranging the identity:

$$\mathrm{E}[Z_i|V_{ij}\!=\!0] \;=\; \frac{\mathrm{E}[Z_i] \;-\; P(V_{ij}\!=\!1)\, \mathrm{E}[Z_i|V_{ij}\!=\!1]}{P(V_{ij}\!=\!0)}.$$

Now simply note that in the course of this exam, you have already computed every term on the right side. In particular,

$$\begin{aligned}
\mathrm{E}[Z_i] &= \mu_i, \\
P(V_{ij}\!=\!1) &= P(Z_i \!>\! Z_j), \\
P(V_{ij}\!=\!0) &= 1 - P(V_{ij}\!=\!1),
\end{aligned}$$

and $\mathrm{E}[Z_i|V_{ij} = 1]$ was computed in part (e). Again, for emphasis, you do **not** need to substitute these previous results into the identity; **after proving the identity, you are done**.

Notes from the instructors

- **From Lawrence:**

  Congratulations on your hard work in CSE 250A. We covered a lot of material during the quarter, often at a brisk pace. Nevertheless I hope that you enjoyed the course. One of my favorite moments in this class is to congratulate students when they hand in their final exam. Unfortunately that is not possible this quarter, so this brief note will have to suffice.

  I will post solutions on Canvas once the TAs and I verify that all the exams have been submitted. It will also be possible to inspect your exams after they are graded. In the meantime, good luck on your remaining exams, and enjoy the winter break.

- **From Aditi:**

  I hope you found this course helpful, and that some/all of you want to explore the concepts introduced in this course further! Happy holidays, and stay safe! :)

- **From Jennifer:**

  Congratulations on completing CSE250A! You worked extremely hard and learned a lot this quarter! Please take a moment to reflect on that and appreciate the incredible journey you have been on this quarter. You are certainly better off for it. I hope you enjoyed the course as much as I did, and I look forward to seeing you around campus. Please don't hesitate to reach out if you ever want to chat about more ML topics or anything GradWIC related. Thank you for being such wonderful, engaged, and curious students :)

- **From Jessica:**

  Congratulations on completing CSE 250A, especially in these weird times! I'm sure you enjoyed it and did wonderfully on the exam as well. Good luck for the rest of your exams and happy holidays!

- **From Udayan:**

  Congratulations on completing CSE250A. Hope you all had a great time. This was my first time being a TA, and I enjoyed it a lot. If you don't remember me, I am the TA who liked giving blunt answers on Piazza XD. Remember the fundamentals you learned in this course, they are very useful. We tried our best to keep the course as authentic and close to previous offerings as possible, despite being online. Hoping to meet a lot of you in person someday. Wishing all of you best of luck in your future at UC San Diego.

- **From Xinghan:**

  Time flies, and you are all about to complete CSE 250A! Each one of you deserves a round of applause for doing all the hard work, and I believe for sure that you have something to take away from this course. I would also like to thank you for your encouragement, kind notes, and support for my OH and discussion sessions, which make me a better TA. Best of luck on all your exams, and have a nice winter break!