1)

a) True

b) False

c) False.

d) True.

e) ~~True~~ False

f) ~~False~~ True

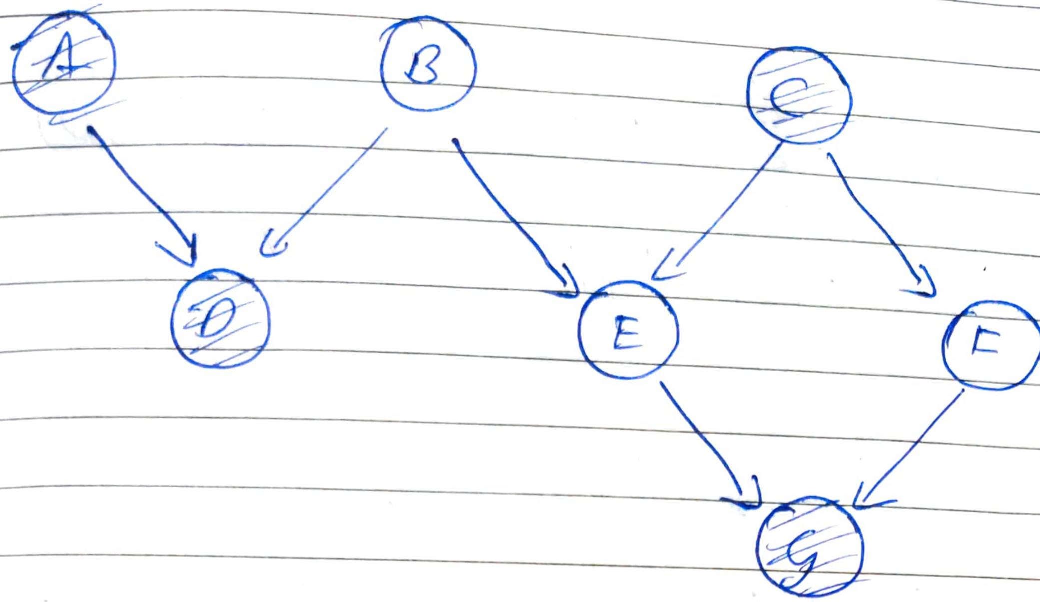
g) True

h) False

i) True

j) False

2/



a)

$$P(B|A,D) = \frac{P(A,B|D)}{P(A|D)} \qquad \boxed{\text{P.R.}}$$

$$= \frac{P(D|A,B)\,P(A,B)}{P(D)\,P(A|D)} \qquad \boxed{\text{Bayes}}$$

$$= \frac{P(D|A,B)\,\cancel{P(A)}\,P(B)}{\cancel{P(D)}\,\frac{P(D|A)\,\cancel{P(A)}}{\cancel{P(D)}}} \qquad \boxed{\text{C.I. \& Bayes}}$$

$$= \frac{P(D|A,B)\,P(B)}{\sum_b P(B,D|A)} \qquad \boxed{\text{marg}}$$

$$= \frac{P(D|A,B)\,P(B)}{\sum_b P(D|A,B)\,P(B|A)} \qquad \boxed{\text{P.R.}}$$

$$= \frac{P(D|A,B)\,P(B)}{\sum_b P(D|A,B=b)\,P(B=b)} \qquad \boxed{\text{C.I.}}$$

b) $P(E|A,C,D) = \sum_{b} P(B,E|A,C,D)$ $\boxed{\text{marg}}$

$= \sum_{b} P(E|A,B,C,D)\,P(B|A,C,D)$ $\boxed{\text{P.R.}}$

$= \sum_{b} P(E|B,C)\,P(B|A,D)$ $\boxed{\text{C.I.}}$

c) $P(G|A,C,D,F)$

$\Rightarrow \sum_{e} P(E=e, G|A,C,D,F)$ $\boxed{\text{marg}}$

$\Rightarrow \sum_{e} P(G|A,C,D,E,F)\,P(E|A,C,D,F)$ $\boxed{\text{P.R.}}$

$\Rightarrow \sum_{e} P(G|E,F)\,P(E|A,C,D)$ $\boxed{\text{C.I.}}$

d) $P(F|A,C,D,G)$

$$\Rightarrow \frac{P(G|A,C,D,F)\,P(F|A,C,D)}{P(G|A,C,D)} \quad \boxed{\text{Bayes}}$$

$$\Rightarrow \frac{P(G|A,C,D,F)\,P(F|C)}{\sum_{f} P(G,F|A,C,D)} \quad \boxed{\text{marg \& C.I.}}$$

$$\Rightarrow \frac{P(G|A,C,D,F)\,P(F|C)}{\sum_{f} P(G|A,C,D,F)\,P(F|C)} \quad \boxed{\text{P.R. \& C.I}}$$

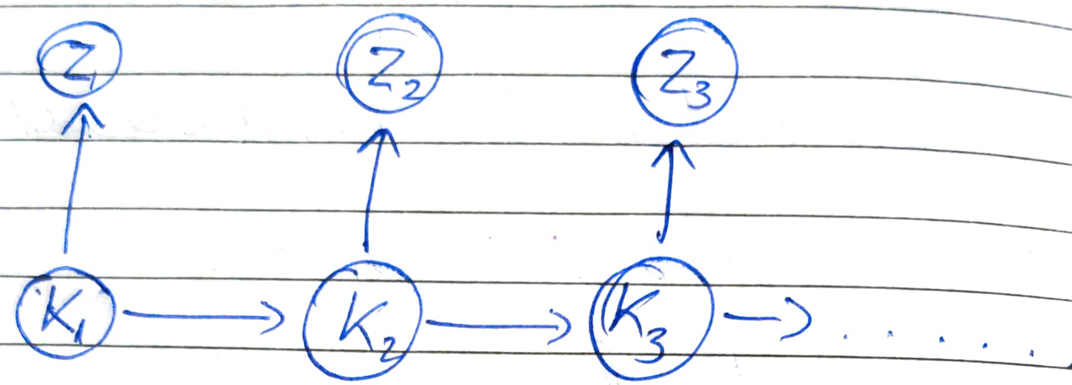
3

a) No

- Definition of a polytree states that a polytree is a singly connected belief network, i.e., between any two nodes there is at most one path. Alternatively, a polytree is a belief network without any loops.

- Justification :- There are loops present in this belief network. Thus, there are more than one paths ~~are~~ that exist between two nodes in this network.
Thus this belief network is NOT a polytree

b) The X_i & Y_i nodes can be clustered in the given belief network to obtain a HMM.



Where, K_i node has been created after clustering X_i & Y_i nodes

- Complexity of the forward algo. is $O(N^2T)$ where N is the no. of states of hidden states.

For the combined variable K, we have n^2 states $\Rightarrow$ total complexity is $\boxed{O(n^4T)}$

4

a) No

There can be loops in the network.

b)

$P(y_1, y_2 \ldots y_n) \cancel{= P(}$

$\Rightarrow \sum_{x_1, x_2 \ldots x_m} P(y_1, y_2 \ldots y_n | x_1, x_2 \ldots$

$\Rightarrow \sum_{x_1, x_2 \ldots x_m} P(y_1, y_2 \ldots y_n, x_1, x_2 \ldots x_m)$ $\boxed{\text{marg}}$

$\Rightarrow \sum_{x_1, x_2 \ldots x_m} P(y_1, y_2 \ldots y_n | x_1, x_2 \ldots x_m) P(x_1, x_2 \ldots x_m)$ $\boxed{\text{P.R.}}$

$\Rightarrow \sum_{x_1, x_2 \ldots x_m} \prod_{i=1}^{n} P(y_i | x_1, \ldots x_m) \prod_{j=1}^{m} P(x_j)$ $\boxed{\text{C.I.}}$

c) $O(n * 2^m)$

Each x_i needs to iterate over 2 states $\{0,1\}$ and all the resultant sums need to be iterated over by 2 states of X_{i+1} $\Rightarrow 2^m$

For each calculation of $\vec{x}$ states, n values of $\vec{y}$ will be calculated

$\Rightarrow$ Total complexity $O(n2^m)$

d) $P(x_1, x_2 \ldots x_m) = \prod_{i=1}^{m} \mu_i^{x_i}(1-\mu_i)^{1-x_i}$

$$\Rightarrow \sum_{x_1 x_2 \ldots x_m} \prod_{i=1}^{m} \mu_i^{x_i}(1-\mu_i)^{1-x_i}.$$

$$\Rightarrow \sum_{x_1, x_2 \ldots x_m \setminus x_i} \left(\sum_{x_i=\{0,1\}} \mu_i^{x_i}(1-\mu_i)^{1-x_i}\right) \prod_{j=1}^{m\setminus i} \mu_j^{x_j}(1-\mu_j)^{1-x_j}$$

$$\Rightarrow \sum_{x_1, x_2 \ldots x_m \setminus x_i} \left(\mu_i^0(1-\mu_i)^1 + \mu_i^1(1-\mu_i)^0\right) \prod_{j=1}^{m\setminus i} \mu_j^{x_j}(1-\mu_j)^{1-x_j}$$

$$\Rightarrow \sum_{x_1, x_2 \ldots x_m \setminus x_i} (1.) \prod_{j=1}^{m\setminus i} \mu_j^{x_j}(1-\mu_j)^{1-x_j}$$

$\Rightarrow$ Repeating this for all m ~~sets~~ binary variables of x

$$\Rightarrow \underbrace{1 \cdot 1 \cdot 1 \cdots\cdots 1}_{m}$$

$\Rightarrow 1$

c. From (b)

$$P(y_1, y_2 \dots y_n) =$$

$$\Rightarrow \sum_{x_1, x_2 \dots x_m} \prod_{j=1}^{n} P(y_j / x_1, x_2 \dots x_m) \prod_{i=1}^{m} P(x_i)$$

$$\Rightarrow P(y_1 = 0, y_2 = 0 \dots y_n = 0) =$$

$$\Rightarrow \sum_{x_1, x_2 \dots x_m} \prod_{j=1}^{n} \prod_{i=1}^{m} (1 - p_{ij})^{x_i} \prod_{i=1}^{m} \mu_i^{x_i} (1 - \mu_i)^{1-x_i}$$

$$\left[\because P(x_i = 1) = \mu_i \right]$$

f) From (e)

$$P(Y_1 = 0, Y_2 = 0 \ldots Y_n = 0)$$

$$\Rightarrow \sum_{x_1, x_2 \ldots x_m} \prod_{j=1}^{n} \prod_{i=1}^{m} (1-P_{ij})^{x_i} \prod_{j=1}^{m} \mu_i^{x_i} (1-\mu_i)^{1-x_i}$$

$$\Rightarrow \sum_{x_1, x_2 \ldots x_m} \prod_{j=1}^{n} \prod_{i=1}^{m} (1-P_{ij})^{x_i} \mu_i^{x_i} (1-\mu_i)^{1-x_i}$$

$$\Rightarrow \sum_{x_1, x_2 \ldots x_m \setminus x_i} \left(\sum_{x_i = \{0,1\}} \prod_{j=1}^{n} (1-P_{ij})^{x_i} \mu_i^{x_i} (1-\mu_i)^{1-x_i} \right)$$

$$* \left(\prod_{k=1}^{m \setminus i} (1-P_{kj})^{x_k} \mu_k^{x_k} (1-\mu_k)^{1-x_k} \right)$$

$$\Rightarrow \sum_{x_1, x_2 \ldots x_m \setminus x_i} \left(\prod_{j=1}^{n} ((1-P_{ij})\mu_i + (1-\mu_i)) \right)$$

$$* \prod_{k=1}^{m \setminus i} (1-P_{kj})^{x_k} \mu_k^{x_k} (1-\mu_k)^{1-x_k}$$

$$\Rightarrow \sum_{x_1, x_2 \ldots x_m \setminus x_i} \prod_{j=1}^{n} (1-\mu_i P_{ij}) \prod_{k=1}^{m \setminus i} (1-P_{kj})^{x_k} \mu_k^{x_k} (1-\mu_k)^{1-x_k}$$

$\Rightarrow$ Repeating $m-1$ times $\Rightarrow$

$$\Rightarrow \boxed{\prod_{j=1}^{n} \prod_{i=1}^{m} (1 - \mu_i P_{ij})}$$

Thus, the complexity is $O(mn)$

5

a) $P(Y=1|\vec{x}) = \gamma\sigma(\vec{\omega}.\vec{x})$

$$\Rightarrow P(y_t|\vec{x}_t) = P(y_t=1|\vec{x}_t)^{y_t} P(y_t=0|\vec{x}_t)^{(1-y_t)}$$

$$\Rightarrow \log P(y_t|\vec{x}_t) = \log P(y_t=1|\vec{x}_t)^{y_t} + \log P(y_t=0|\vec{x}_t)^{(1-y_t)}$$

$$\Rightarrow \log P(y_t|\vec{x}_t) = y_t \log \gamma\sigma(\vec{\omega}.\vec{x}) + (1-y_t)\log(1-\gamma\sigma(\vec{\omega}.\vec{x}))$$

$$\Rightarrow \boxed{\sum_{t=1}^{T} \log P(y_t|\vec{x}_t) = \sum_{t=1}^{T} y_t \log \gamma\sigma(\vec{\omega}.\vec{x}) + (1-y_t)\log(1-\gamma\sigma(\vec{\omega}.\vec{x}))}$$

b) From (a)

$$\mathcal{L}(\vec{\omega}, \gamma) = \sum_{t=1}^{T} y_t \log \gamma \sigma(\vec{\omega} \cdot \vec{x})$$

$$+ (1-y_t) \log(1 - \gamma \sigma(\vec{\omega} \cdot \vec{x}))$$

$$\Rightarrow \frac{\partial \mathcal{L}(\vec{\omega}, \gamma)}{\partial \vec{\omega}} =$$

$$\Rightarrow \sum_{t=1}^{T} y_t \frac{\gamma \sigma'(\vec{\omega} \cdot \vec{x}_t) \vec{x}_t}{\gamma \sigma(\vec{\omega} \cdot \vec{x}_t)}$$

$$- (1-y_t) \frac{\gamma \sigma'(\vec{\omega} \cdot \vec{x}_t) \vec{x}_t}{(1 - \gamma \sigma(\vec{\omega} \cdot \vec{x}_t))}$$

$$\Rightarrow \sum_{t=1}^{T} y_t \sigma(-\vec{\omega} \cdot \vec{x}_t) \vec{x}_t$$

$$- (1-y_t) \frac{\gamma \sigma(\vec{\omega} \cdot \vec{x}_t) \sigma(-\vec{\omega} \cdot \vec{x}_t) \vec{x}}{1 - \gamma \sigma(\vec{\omega} \cdot \vec{x}_t)}$$

$$\Rightarrow \sum_{t=1}^{T} \vec{x}_t \sigma(-\vec{\omega} \cdot \vec{x}_t) \left(y_t - \frac{(1-y_t) \gamma \sigma(\vec{\omega} \cdot \vec{x}_t)}{1 - \gamma \sigma(\vec{\omega} \cdot \vec{x}_t)} \right)$$

$$\Rightarrow \sum_{t=1}^{T} \vec{x}_t \sigma(-\vec{\omega} \cdot \vec{x}_t) \frac{(y_t - \gamma \sigma(\vec{\omega} \cdot \vec{x}_t))}{(1 - \gamma \sigma(\vec{\omega} \cdot \vec{x}_t))}$$

$$\Rightarrow \boxed{\sum_{t=1}^{T} \sigma(-\vec{\omega}\cdot\vec{x}_t)\left(\frac{y-p_t}{1-p_t}\right)\vec{x}_t} \quad \text{(iii)}$$

c) $$P(Y=1|U,Z) = I(U,1)\,I(Z,1) \quad ..(i)$$

$$\left[\because P(Y=1|U,Z)=1 \text{ if } U=Z=1\right]$$

$$\cancel{\Rightarrow P(Y=1|U,Z)=}$$

$$\Rightarrow P(Y=1|\vec{x}) = \sum_{U} P(Y=1,U|\vec{x}) \quad \boxed{\text{marg}}$$

$$\Rightarrow \sum_{U} P(Y=1|U,\vec{x})\,P(U|\vec{x}) \quad \boxed{\text{P.R.}}$$

$$\Rightarrow \sum_{U,Z} P(Y=1,Z|U)\,P(U|\vec{x}) \quad \boxed{\text{marg \& C.I.}}$$

$$\Rightarrow \sum_{U,Z} P(Y=1|U,Z)\,P(Z|U)\,P(U|\vec{x})$$

$$\boxed{\text{P.R.}}$$

$$\Rightarrow \sum_{U,Z} P(Y=1|U,Z)\,P(Z)\,P(U|\vec{x}) \quad \boxed{\text{C.I.}}$$

$\Rightarrow \quad P(Z=1)\, P(U=1|\vec{x})$

[$\because P(Y=1|U,Z) = 0$ for all other U, Z combinat^n]

$\Rightarrow \quad \gamma\, \sigma(\vec{\omega}\cdot\vec{x})$

$$\Rightarrow \boxed{P(Y=1|\vec{x}) = \gamma\,\sigma(\vec{\omega}\cdot\vec{x})}$$

d) $P(Z=1|\vec{x}, Y=1) \Rightarrow$

$$\Rightarrow \frac{P(Y|Z,\vec{x})\,P(Z|\vec{x})}{P(Y|\vec{x})} \qquad \boxed{\text{bayes}}$$

$$\Rightarrow \frac{\sum_U P(Y,U|Z,\vec{x})\,P(Z)}{P(Y|\vec{x})} \qquad \boxed{\text{marg \& C.I.}}$$

$$\Rightarrow \frac{\sum_U P(Y|U,Z)\,P(U|\vec{x},Z)\,P(Z)}{P(Y|\vec{x})} \qquad \boxed{\text{P.R.}}$$

$$\Rightarrow \sum_{U} \frac{P(Y|U,Z)P(U|\vec{x})P(Z)}{P(Y|\vec{x})} \quad \boxed{C.I.}$$

$$\Rightarrow P(Z=1|\vec{x};Y=1) =$$

$$\sum_{U} \frac{P(Y=1|U,Z=1)P(U|\vec{x})P(Z=1)}{P(Y=1|\vec{x})}$$

$$\Rightarrow \frac{P(Y=1|U=1,Z=1)P(U=1|\vec{x})P(Z=1)}{P(Y=1|\vec{x})}$$

$$\left[\because P(Y=1|U,Z) = I(U,1)\, I(Z,1)\right]$$

$$\Rightarrow 1 \qquad \left[\because P(Y=1|\vec{x}) = P(U=1|\vec{x})P(Z=1)\right]$$

$$\Rightarrow P(Z=1|\vec{x},Y=1) = 1$$

e) From (d)

$$P(Z|\vec{x},Y) = \sum_{U} \frac{P(Y|U,Z)P(U|\vec{x})P(Z=1)}{P(Y|\vec{x})}$$

$$\Rightarrow P(Z=1 \mid \vec{x}, Y=0) =$$

$$\frac{\sum_{U} P(Y=0 \mid U, Z=1) P(U \mid \vec{x}) P(Z=1)}{P(Y=0 \mid \vec{x})}$$

$$\Rightarrow \frac{P(Y=0 \mid U=0, Z=1) P(U=0 \mid \vec{x}) P(Z=1)}{1 - P(Y=1 \mid \vec{x})}$$

$$\left[\because P(Y=0 \mid U=1, Z=1) = 0 \right]$$

$$\Rightarrow \boxed{\frac{\left(1-\sigma(\vec{w}\cdot\vec{x})\right)\gamma}{1-\gamma\sigma(\vec{w}\cdot\vec{x})}}$$

f) Since, Z is a root node,

$$P(Z=1) \leftarrow \sum_{t} \frac{P(Z=1, \pi(Z) \mid V_t = v_t)}{T}$$

$$\Rightarrow P(Z=1) \leftarrow \frac{1}{T} \sum_{t} P(Z=1 \mid \vec{x}_t, Y=y_t)$$

$$\Rightarrow \gamma \leftarrow \frac{1}{T} \sum_{t} P(Z=1 \mid \vec{x}_t, Y=y_t)$$

6/

a) $P(B|A,C,D)$

$$\Rightarrow \frac{P(D|A,B,C)P(B|A,C)}{P(D|A,C)} \quad \boxed{\text{bayes}}$$

$$\Rightarrow \frac{P(D|B,C)\,P(B|A)}{\sum_b P(B,D|A,C)} \quad \boxed{\text{C.I. \& marg}}$$

$$\Rightarrow \frac{P(D|B,C)P(B|A)}{\sum_b P(D|A,B,C)\,P(B|A,C)} \quad \boxed{\text{P.R.}}$$

$$\Rightarrow \frac{P(D|B,C)\,P(B|A)}{\sum_b P(D|B,C)\,P(B|A)} \quad \boxed{\text{C.I.}}$$

b) $$\mathcal{L} = \sum_t \log P(D|A,C)$$

$$= \sum_t \log \sum_b P(B,D|A,C) \quad \boxed{\text{marg.}}$$

$$= \sum_t \log \sum_b P(D|A,B,C)P(B|A,C) \quad \boxed{\text{P.R.}}$$

$$= \sum_t \log \sum_b P(D|B,C)P(B|A) \quad \boxed{\text{C.I.}}$$

$$= \sum_t \log \sum_b P(D=d_t|B,C=c_t)P(B|A=a_t)$$

c) • $$P(B=b \mid A=a) \leftarrow \frac{\sum_t P(B=b, A=a \mid A=a_t, C=c_t, D=d_t)}{\sum_t P(A=a \mid A=a_t, C=c_t, D=d_t)}$$

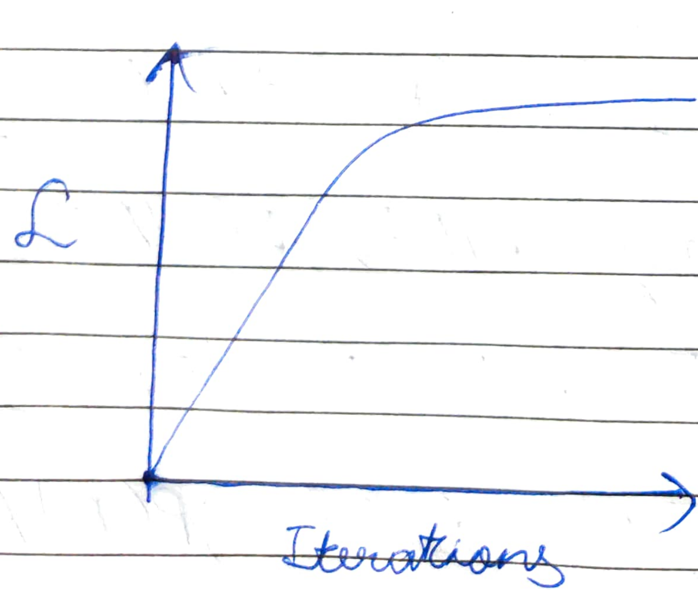
$$\leftarrow \sum_t \frac{I(a,a_t)\, P(b \mid a_t, c_t, d_t)}{\sum_t I(a,a_t)}$$

• $$P(D=d \mid B=b, C=c)$$

$$\leftarrow \frac{\sum_t P(d, b, c \mid a_t, c_t, d_t)}{\sum_t P(b, c \mid a_t, c_t, d_t)}$$

$$\leftarrow \sum_t \frac{I(c,c_t)\, I(d,d_t)\, P(b \mid a_t, c_t, d_t)}{\sum_t I(c,c_t)\, P(b \mid a_t, c_t, d_t)}$$

d)



7

a) $P(S_{t+1}=j \mid S_t=i, O_1, O_2 \dots O_T)$

$$\Rightarrow \frac{P(S_t=i, S_{t+1}=j, O_1, \dots O_T)}{P(S_t=i, O_1, O_2 \dots O_T)} \quad \boxed{P.R.}$$

$$\Rightarrow \frac{P(S_t=i, O_1, \dots, O_t) P(S_{t+1}=j \mid S_t=i) P(O_{t+1} \mid S_{t+1}) * P(O_{t+2}, \dots O_T \mid S_{t+1}=j)}{P(S_t=i, O_1, \dots, O_t) P(O_{t+1}, \dots O_T \mid S_t=i)}$$

$\boxed{C.I.}$

$$\Rightarrow \frac{\alpha_{it}\, a_{ij}\, b_j(O_{t+1})\, \beta_{j,t+1}}{\alpha_{it}\beta_{it}}$$

$$\Rightarrow \frac{a_{ij}\, b_j(O_{t+1})\, \beta_{j,t+1}}{\beta_{it}}$$

b) Base Case:-

$$[U^{(1)}]_{ij} = P(S_2 = j \mid S_1 = i, O_1, O_2 \ldots O_T)$$

From (a) $\Rightarrow$ Base case is true

Assumption:-

let;

$$[U^{(1)} U^{(2)} \ldots U^{(t-1)}]_{ij}$$
$$= P(S_t = j \mid S_1 = i, O_1, O_2 \ldots O_T)$$

$\Rightarrow$ For $(t+1)$ case, we have,

$$P(S_{t+1} = j \mid S_1 = i, O_1, O_2 \ldots O_T)$$

$$\Rightarrow \sum_k P(S_{t+1} = j, S_t = k \mid S_1 = i, O_1, O_2 \ldots O_T) \quad \boxed{\text{marg.}}$$

$$\Rightarrow \sum_k P(S_{t+1} = j \mid S_t = k, O_1, O_2 \ldots O_T, S_1 = i)$$
$$* P(S_t = k \mid S_1 = i, O_1, O_2 \ldots O_T) \quad \boxed{\text{P.R.}}$$

$$\Rightarrow \sum_k P(S_{t+1}=j \mid S_t=k, O_1, O_2 \ldots O_T) \quad \boxed{C.I.}$$

$$* \; P(S_t=k \mid S_1=i, O_1, O_2 \ldots O_T)$$

$$\Rightarrow \sum_k U^{(t)}_{kj} \left[U^{(1)} U^{(2)} \ldots\ldots U^{(t-1)}\right]_{ik}$$

$$\Rightarrow \left[U^{(1)} U^{(2)} \ldots\ldots U^{(t)}\right]_{ij}$$

Hence, for $t = T$, we have.

$$\boxed{\Rightarrow P(S_T=j \mid S_1=i, O_1, O_2 \ldots O_T) = \left[U^{(1)} U^{(2)} \ldots\ldots U^{(T-1)}\right]_{ij}}$$

Hence proved

8

a) $$r^*_{i,T-1} = \max_j \left[\log P(S_T = j, O_T \mid S_{T-1} = i) \right]$$

$$= \max_j \left[\log P(S_T = j \mid S_{T-1} = i) P(O_T \mid S_T = j, S_{T-1} = i) \right]$$

$\boxed{P.R.}$

$$= \max_j \left[\log P(S_T = j \mid S_{T-1} = i) P(O_T \mid S_T = j) \right]$$

$\boxed{C.I.}$

$$= \max_j \left[\log a_{ij} b_j(O_T) \right]$$

$$= \max_j \left[\log a_{ij} + \log b_j(O_T) \right]$$

b) $$r^*_{it} = \max_{S_{t+2}, \ldots S_T} \max_j \left[\log P(S_{t+1} = j, S_{t+2} \ldots S_T, O_{t+1} \ldots O_T \mid S_t = i) \right]$$

$$\Rightarrow \max_{S_{t+2}, \ldots S_T} \max_j \left[\log \left(P(S_{t+2}, \ldots S_T, O_{t+1}, \ldots O_T \mid S_{t+1} = j, S_t = i) * P(S_{t+1} = j \mid S_t = i) \right) \right]$$

$\boxed{P.R.}$

O_{t+1}

$$\Rightarrow \max_{S_{t+2},\dots S_T} \max_j \Big[\log\Big(P(S_{t+2},\dots S_T, O_{t+2},\dots O_T \mid S_{t+1}=j, O_{t+1})$$
$$* P(S_{t+1}=j \mid S_t=i) P(O_{t+1} \mid S_{t+1}=j)\Big)\Big]$$

$\boxed{\text{C. I. \& P. R.}}$

$$\Rightarrow \max_j \Big[\max_{S_{t+2}\dots S_T} \Big[\log P(S_{t+2},\dots S_T, O_{t+2},\dots O_T \mid S_{t+1}=j)\Big]$$
$$+ \log P(S_{t+1}=j \mid S_t=i)$$
$$+ \log P(O_{t+1} \mid S_{t+1}=j)\Big]$$

$$\Rightarrow \max_j \Big[r^*_{j\,t+1} + \log a_{ij} + \log b_j(O_{t+1})\Big]$$

c) $r^*_{jT} = 0 \quad \forall j$

$$\Rightarrow r^*_{i\,T-1} = \max_j \Big[r^*_{jT} + \log a_{ij} + \log b_j(O_T)\Big]$$

from(b)

$$\Rightarrow r^*_{i\,T-1} = \max_j \Big[\log a_{ij} + \log b_j(O_T)\Big]$$

d)

$$S_1^* = \underset{i}{\operatorname{argmax}} \max_{S_2, \dots S_T} \left[\log P(S_1 = i, S_2 \dots S_T, O_1 \dots O_T) \right]$$

$$\Rightarrow \underset{i}{\operatorname{argmax}} \max_{S_2, \dots S_T} \left[\log \left(P(S_2 \dots S_T, O_1, \dots O_T \mid S_1 = i) \cdot P(S_1 = i) \right) \right] \quad \boxed{P.R.}$$

$$\Rightarrow \underset{i}{\operatorname{argmax}} \max_{S_2, \dots S_T} \left[\log \left(P(S_2, \dots S_T, O_2, \dots O_T \mid S_1 = i, O_1) \cdot P(S_1 = i) P(O_1 \mid S_1 = i) \right) \right] \quad \boxed{P.R.}$$

$$\Rightarrow \underset{i}{\operatorname{argmax}} \max_{S_2, \dots S_T} \left[\log \left(P(S_2, \dots S_T, O_2, \dots O_T \mid S_1 = i) \cdot \pi_i \, b_i(O_1) \right) \right] \quad \boxed{C.I.}$$

$$\Rightarrow \underset{i}{\operatorname{argmax}} \left[r_{i1}^* + \log \pi_i + \log b_i(O_1) \right]$$

e) $S_1^* = \underset{i}{\operatorname{argmax}} \max_{S_2 \dots S_T} \left[\log P(S_1 = i, S_2 \dots S_T, O_1 \dots O_T) \right]$

$\Rightarrow S_2^* = \underset{j}{\operatorname{argmax}} \left[\underset{i}{\operatorname{argmax}} \max_{S_3 \dots S_T} \left[\log P(S_1 = i, S_2 = j, \dots S_T, O_1 \dots O_T) \right] \right]$

$\Rightarrow$ from (d)

$\Rightarrow S_2^* = \underset{j}{\operatorname{argmax}} \left[\underset{i}{\operatorname{argmax}} \max_{S_3 \dots S_T} \left[\log P(S_2 = j, \dots S_T, O_2 \dots O_T \mid S_1 = i) + \log \pi_i b_i(O_1) \right] \right]$

$\Rightarrow S_2^* = \underset{j}{\operatorname{argmax}} \left[\underset{i}{\operatorname{argmax}} \max_{S_3 \dots S_T} \left[\log P(S_3 \dots S_T, O_3 \dots O_T \mid S_2 = j) + \log P(S_2 = j \mid S_1 = i) + \log P(O_2 \mid S_2 = j) + \log \pi_i b_i(O_1) \right] \right]$

$$\Rightarrow S_2^* = \underset{j}{\operatorname{argmax}} \left[r^*_{j2} + \log a_{ij} + \log b_j(O_2) \right]$$

where, $\cancel{S_1}$ $i = S_1^*$

$\Rightarrow$ For S_t^* we have;

$$\boxed{\begin{aligned} & i = S_{t-1}^* \\ & S_t^* = \underset{j}{\operatorname{argmax}} \left[r^*_{jt} + \log a_{ij} + \log b_j(O_t) \right] \end{aligned}}$$

9)

a) For policy evaluation, we'll use bellman equation.

$$V^{\pi}(S) = R(S) + \gamma \sum_{S'} P(S' | S, \pi(S)) V^{\pi}(S')$$

- $$V^{\pi}(2) = R(2) + \gamma \sum_{S'} P(S' | 2, \uparrow) V^{\pi}(S')$$

$$\Rightarrow V^{\pi}(2) = R(2) + \frac{3}{4}\left(P(1|2,\uparrow) V^{\pi}(1) + P(2|2,\uparrow) V^{\pi}(2) + P(3|2,\uparrow) V^{\pi}(3) \right)$$

$$\Rightarrow V^{\pi}(2) = -3 + \frac{\cancel{3}}{4}\left(0 + \frac{1}{\cancel{3}} V^{\pi}(2) + \frac{2}{\cancel{3}} V^{\pi}(3) \right)$$

$$\Rightarrow V^{\pi}(2) = -3 + \frac{V^{\pi}(2)}{4} + \frac{V^{\pi}(3)}{2}$$

$$\Rightarrow \frac{3}{4} V^{\pi}(2) = \frac{V^{\pi}(3)}{2} - 3$$

$$\Rightarrow V^{\pi}(2) = \frac{4}{3}\left(\frac{V^{\pi}(3)}{2} - 3 \right) \quad \ldots (i)$$

- $V^{\pi}(3) = R(3) + \gamma \sum_{s'} P(s'|3, \uparrow)\, V^{\pi}(s')$

$\Rightarrow V^{\pi}(3) = R(3) + \frac{3}{4}\Big(P(1|3,\uparrow)V^{\pi}(1) + P(2|3,\uparrow)V^{\pi}(2) + P(3|3,\uparrow)V^{\pi}(3)\Big)$

$\Rightarrow V^{\pi}(3) = 6 + \frac{\cancel{3}}{4}\left(\frac{2}{\cancel{3}} \times 24 + 0 + \frac{1}{\cancel{3}} \times V^{\pi}(3)\right)$

$\Rightarrow V^{\pi}(3) = 6 + 12 + \frac{V^{\pi}(3)}{4}$

$\Rightarrow \frac{\cancel{3}\, V^{\pi}(3)}{4} = \cancel{18}\,6$

$\Rightarrow V^{\pi}(3) = 24 \quad \ldots (ii)$

$\Rightarrow$ Substituting (ii) in (i)

$V^{\pi}(2) = 12$

S	$\pi(S)$	$V^{\pi}(S)$
1	$\uparrow$	24
2	$\uparrow$	12
3	$\uparrow$	24

b) $$\pi'(s) = \underset{a}{\operatorname{argmax}} \left[\sum_{s'} P(s'|s,a) V^{\pi}(s') \right]$$

• ~~Evaluating for $s=1$~~

~~$a = \uparrow$~~

$$\pi'(s) = \underset{a}{\operatorname{argmax}} \left[Q^{\pi}(s,a) \right]$$

• $$Q^{\pi}(1,\uparrow) = R(1) + \gamma \sum_{s'} P(s'|1,\uparrow) V^{\pi}(s')$$

$$= 24 \qquad \text{[Given in table]}$$

• $$Q^{\pi}(1,\downarrow) = R(1) + \gamma \sum_{s'} P(s'|1,\downarrow) V^{\pi}(s')$$

$$= 12 + \frac{\cancel{3}}{4}\left(\frac{1}{\cancel{3}} \times 24 + 0 + \frac{2}{\cancel{3}} \times 16\right)$$

$$= 12 + 6 + 8$$

$$= 26$$

$\Rightarrow \pi'(1) = \downarrow$

- $$Q^{\pi}(2,\uparrow) = R(2) + \gamma \sum_{s'} P(s'|2,\uparrow)\, V^{\pi}(s')$$

$$= -3 + \frac{\cancel{3}}{4}\left(0 + \frac{1}{\cancel{3}} \times 12 + \frac{2}{\cancel{3}} \times 16\right)$$

$$= -3 + 3 + 8$$

$$= 8$$

- $Q^{\pi}(2,\downarrow) = 12$ [Given in table]

$\Rightarrow \pi'(2) = \downarrow$

i)

S	$\pi(S)$	$V^{\pi}(S)$	$\pi'(S)$
1	↑	24	↓
2	↓	12	↓
3	↓	16	↑

10

a) Yes.

Since, any linear combination of a random variable, which is normally distributed, is also normally distributed.

b) (i) $\sum_{t=1}^{\infty} \alpha_t = \infty$

(ii) $\sum_{t=1}^{\infty} \alpha_t^2 < \infty$

c) No.

(ii) Condition is violated.

d)

Base Case :-

$$\mu_1 = \mu_0 + \alpha(x_1 - \mu_0)$$

$$\mu_1 = \alpha x_1 \qquad (i)$$

$$\mu_1 = \alpha \sum_{t=1}^{T} (1-\alpha)^{T-t} x_t$$

$$= \alpha \sum_{t=1}^{1} (1-\alpha)^{1-t} x_t$$

$$= \alpha x_1$$

$\Rightarrow$ Base case is true.

Assumption :-

Let;

$$\mu_T = \alpha \sum_{t=1}^{T} (1-\alpha)^{T-t} x_t$$

be true,

Then, to find μ_{T+1}

$$\Rightarrow \mu_{T+1} = \mu_T + \alpha(x_{T+1} - \mu_T)$$

$$= \mu_T(1-\alpha) + \alpha x_{T+1}$$

$$= \alpha\left[\sum_{t=1}^{T}(1-\alpha)^{T-t}x_t(1-\alpha) + x_{T+1}\right]$$

$$= \alpha\left[\sum_{t=1}^{T}(1-\alpha)^{T+1-t}x_t + x_{T+1}\right]$$

$$\boxed{\mu_{T+1} = \alpha\sum_{t=1}^{T+1}(1-\alpha)^{T+1-t}x_t}$$

Hence, proved

e)

$$\mu_T = \alpha\sum_{t=1}^{T}(1-\alpha)^{T-t}x_t$$

$$\Rightarrow E[\mu_T] = E\left[\alpha\sum_{t=1}^{T}(1-\alpha)^{T-t}x_t\right].$$

$$= \alpha\sum_{t=1}^{T}(1-\alpha)^{T-t}E[x_t]$$

$$\Rightarrow \lim_{T\to\infty} E[\mu_T] = \lim_{T\to\infty} \alpha \sum_{t=1}^{T} (1-\alpha)^{T-t} E[x_t]$$

$\because$ Expectation of a sample drawn from a distribution will be equal to the Expectation of the distribution

$$\Rightarrow \lim_{T\to\infty} E[\mu_T] = \lim_{T\to\infty} \alpha \sum_{t=1}^{T} (1-\alpha)^{T-t} E[X]$$

$$= \lim_{T\to\infty} \alpha \sum_{k=0}^{T} (1-\alpha)^{k} E[X]$$

$$= \alpha \times \frac{1}{(1-(1-\alpha))} \times E[X]$$

$$\boxed{\lim_{T\to\infty} E[\mu_T] = E[X]}$$

f)

$$\mu_T = \alpha \sum_{t=1}^{T} (1-\alpha)^{T-t} x_t$$

$$Var(\mu_T) = Var\left(\alpha \sum_{t=1}^{T} (1-\alpha)^{T-t} x_t\right)$$

$$\Rightarrow \underbrace{E\left[(\mu_T - E[\mu_T])^2\right]}_{\text{given}} = \underbrace{\alpha^2 \sum_{t=1}^{T} (1-\alpha)^{2(T-t)} Var(x_t)}_{\text{property of variance}}$$

$$\Rightarrow \lim_{T\to\infty} E\left[(\mu_T - E[\mu_T])^2\right] =$$

$$\lim_{T\to\infty} \alpha^2 \sum_{t=1}^{T} ((1-\alpha)^2)^{T-t} \sigma^2$$

∵ Variance of a sample drawn from a distribution will be equal to the variance of random variable representing the distribution

$$\Rightarrow \lim_{T\to\infty} E\left[(\mu_T - E[\mu_T])^2\right] =$$

$$\lim_{T\to\infty} \alpha^2 \sum_{k=0}^{T} ((1-\alpha)^2)^k \sigma^2$$

$$\Rightarrow \lim_{T \to \infty} E\left[\left(\mu_T - E[\mu_T]\right)^2\right]$$

$$= \frac{\alpha^2}{\left(1-(1-\alpha)^2\right)} \sigma^2$$

$$= \frac{\alpha^2}{2\alpha - \alpha^2} \sigma^2$$

$$= \left(\frac{\alpha}{2-\alpha}\right) \sigma^2$$

11

I affirm the statement of academic integrity on page 2 of the exam

Vaibhav

12/6/2021