CSE 250A. Principles of AI

Probabilistic Reasoning and Decision-Making

Lecture 17 – Markov decision processes

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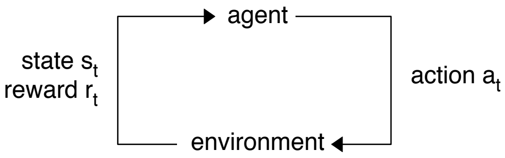
Outline

1. Review and example
2. Value functions
3. Algorithms

Reinforcement learning (RL)

- **Learning from experience in the world**



- **Formalization as Markov decision process**

$\mathcal{S}$	state space
$\mathcal{A}$	action space
$P(s'\|s,a)$	transition probabilities
$R(s)$	reward function

MDP $\{\mathcal{S}, \mathcal{A}, P(s'|s,a), R(s)\}$

Policies and their expected returns

- **Decision-making in MDPs**

  A **policy** $\pi : \mathcal{S} \rightarrow \mathcal{A}$ is a mapping of states to actions. There are combinatorially many policies:

  $$\#\text{ policies} \;=\; |\mathcal{A}|^{|\mathcal{S}|}$$

- **Experience under policy π**

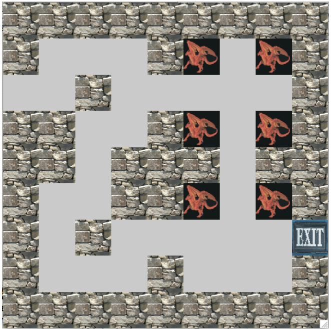
  $$\begin{array}{lccccccl} \text{state} & s_0 & \xrightarrow{\;\pi(s_0)\;} & s_1 & \xrightarrow{\;\pi(s_1)\;} & s_2 & \cdots \\ \text{reward} & r_0 & & r_1 & & r_2 & \cdots \end{array}$$

- **Expected returns**

  $$\mathrm{E}^{\pi}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) \,\middle|\, s_0 = s\right] \;=$$

  the expected value of the discounted infinite-horizon return, starting in state s at time $t=0$, and following policy π.

Example

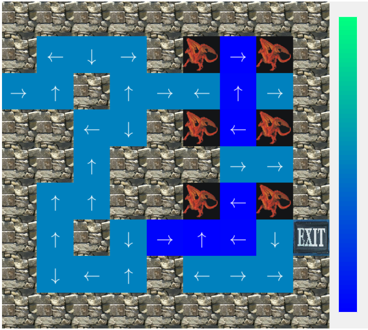


How to exit the maze with high probability?

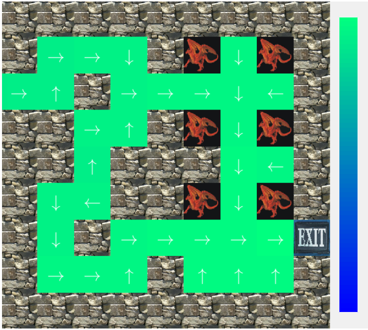
$\mathcal{S}$	location in maze
$\mathcal{A}$	$\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$
$P(s' \mid s, a)$	move *with some probability* in direction of arrow
$R(s)$	$+1$ (exit), -1 (dragon), 0 (otherwise)
γ	0.99 (close to one)

Example — no uncertainty

$\mathcal{S}$	location in maze
$\mathcal{A}$	$\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$
$P(s'\|s, a)$	move **deterministically** in direction of action
$R(s)$	+1 (exit), -1 (dragon), 0 (otherwise)
γ	0.99



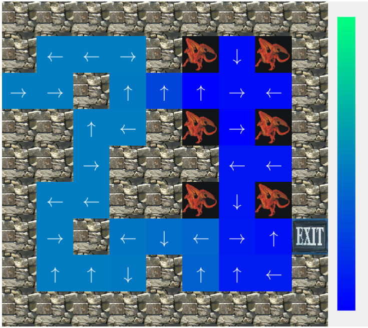
random policy



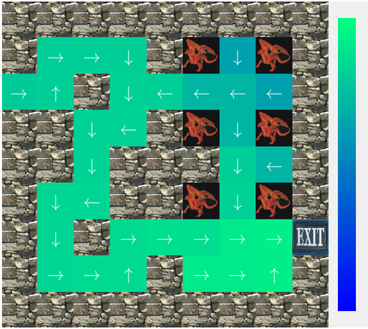
optimal policy

Example — low uncertainty

$\mathcal{S}$	location in maze
$\mathcal{A}$	$\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$
$P(s'\|s,a)$	move **80%** in direction of action
$R(s)$	+1 (exit), -1 (dragon), 0 (otherwise)
γ	0.99



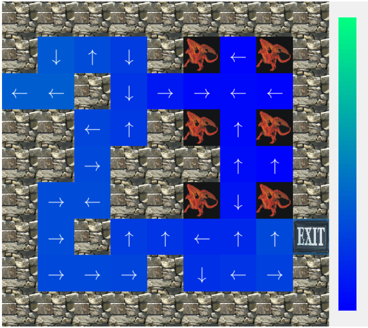
random policy



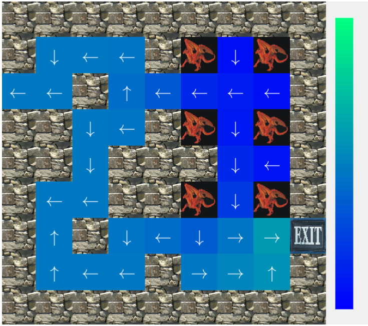
optimal policy

Example — high uncertainty

$\mathcal{S}$	location in maze
$\mathcal{A}$	$\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$
$P(s'\|s,a)$	move **20%** in direction of action
$R(s)$	+1 (exit), -1 (dragon), 0 (otherwise)
γ	0.99



random policy



optimal policy

Outline

1. Review and example
2. $\boxed{\text{Value functions}}$
3. Algorithms

State value function

$$V^\pi(s) \;=\; \mathrm{E}^\pi\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) \,\middle|\, s_0 = s\right]$$

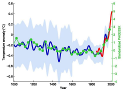
expected return,
starting in state s,
following policy π

- **Values versus rewards:**

  The reward $R(s)$ give **immediate** feedback to the agent.
  The value $V^\pi(s)$ computes the expected **long-term** return.

- **Types of behaviors:**

  Sacrifice now for long-term gain: $R(s) < 0,\ V^\pi(s) > 0$.
  Win now at the expense of later: $R(s) > 0,\ V^\pi(s) < 0$.



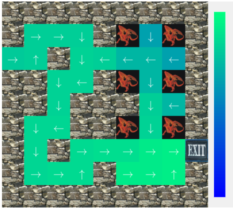
Properties of the state value function

- **Experience under policy π**

$$\begin{array}{lccccc} \text{state} & s_0 & \xrightarrow{\pi(s_0)} & s_1 & \xrightarrow{\pi(s_1)} & s_2 & \cdots \\ \text{reward} & r_0 & & r_1 & & r_2 & \cdots \end{array}$$

- **Adjacent states**

  States (s, s') can be visited in succession if $P(s'|s, \pi(s)) > 0$. The values $V^\pi(s)$ and $V^\pi(s')$ should be related, but how?



The **Bellman equation** tells us how.

Bellman equation

$$
\begin{aligned}
V^\pi(s) &= \mathrm{E}^\pi\left[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots \,\middle|\, s_0 = s\right] \\
&= R(s) \;+\; \gamma\, \mathrm{E}^\pi\left[R(s_1) + \gamma R(s_2) + \cdots \,\middle|\, s_0 = s\right] \\
&= R(s) \;+\; \gamma \sum_{s'} P(s'|s,\pi(s))\, \mathrm{E}^\pi\left[R(s_1) + \gamma R(s_2) + \cdots \,\middle|\, s_1 = s'\right] \\
&= R(s) \;+\; \gamma \sum_{s'} P(s'|s,\pi(s))\, V^\pi(s')
\end{aligned}
$$

The Bellman equation is the basis for much that will follow:

$$
\boxed{V^\pi(s) \;=\; R(s) \;+\; \gamma \sum_{s'} P(s'|s,\pi(s))\, V^\pi(s')}
$$

Action value function

$$Q^\pi(s, a) = \mathrm{E}^\pi\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) \,\middle|\, s_0 = s, a_0 = a\right]$$

expected return,
starting from state s,
taking action a,
then following policy π

- **Motivation**

  Useful to imagine how small changes affect expected outcomes. What if (just once) the agent acted differently in state s?

- **Analogous to the Bellman equation:**

$$Q^\pi(s, a) = R(s) + \gamma \sum_{s'} P(s'|s, a)\, V^\pi(s')$$

$$V^\pi(s) = R(s) + \gamma \sum_{s'} P(s'|s, \pi(s))\, V^\pi(s')$$

Optimality

- **Theorem**

  There exists at least one policy π^* (and perhaps many) such that $V^{\pi^*}(s) \geq V^{\pi}(s)$ for all policies π and states s of the MDP.

- **Proof**

  Later — by construction.

- **Notation**

$$
\begin{aligned}
V^*(s) &= V^{\pi^*}(s) \\
Q^*(s,a) &= Q^{\pi^*}(s,a)
\end{aligned}
$$

  These optimal value functions are **unique**.
  (All optimal policies share the same value functions.)

Relations at optimality

- **From the optimal action value function:**

$$\begin{aligned} V^*(s) &= \max_a \left[Q^*(s,a)\right] \\ \pi^*(s) &= \operatorname*{argmax}_a \left[Q^*(s,a)\right] \end{aligned}$$

- **From the optimal state value function:**

$$\begin{aligned} Q^*(s,a) &= R(s) + \gamma \sum_{s'} P(s'|s,a) V^*(s') \\ \pi^*(s) &= \operatorname*{argmax}_a \left[R(s) + \gamma \sum_{s'} P(s'|s,a) V^*(s')\right] \end{aligned}$$

- **Why are these relations useful?**

  Sometimes it can be easier to estimate $Q^*(s,a)$ or $V^*(s)$ (which are continuous) than to learn $\pi^*(s)$ (which is discrete).

Outline

1. **Review**
2. **Value functions**
3. **Algorithms**

Planning in MDPs

Given a complete model of the agent and its environment as a Markov decision process, namely

$$\text{MDP} \;=\; \{\mathcal{S}, \mathcal{A}, P(s'|s,a), R(s), \gamma\},$$

how can we *efficiently* compute (i.e., in time *polynomial in the number of states*) any of the following:

1. an optimal policy $\pi^*(s)$?
2. the optimal state value function $V^*(s)$?
3. the optimal action value function $Q^*(s,a)$?

This is the problem of **planning** in MDPs.

Three algorithms

Today we'll describe three basic algorithms.
Each of them solves a core problem in MDPs.

1. **Policy evaluation**

   How to compute $V^{\pi}(s)$ for some fixed policy π?

2. **Policy improvement**

   How to compute a policy π' such that $V^{\pi'}(s) \geq V^{\pi}(s)$?

3. **Policy iteration**

   How to compute an optimal policy $\pi^*(s)$?

Policy evaluation

For a fixed policy π, how to compute the state value function

$$V^{\pi}(s) \;=\; \mathrm{E}^{\pi}\left[\sum_{t=0}^{\infty} \gamma^{t} R(s_t) \,\middle|\, s_0 = s\right]?$$

From the Bellman equation:

$$V^{\pi}(s) \;=\; R(s) + \gamma \sum_{s'} P(s'|s,\pi(s))\, V^{\pi}(s').$$

This is a system of n linear equations for n unknowns;
the unknowns are $V^{\pi}(s)$ for $s \in \{1, 2, \ldots, n\}$ where $n = |\mathcal{S}|$.

Solving the linear system

- **From the Bellman equation:**

$$V^{\pi}(s) \;=\; R(s) + \gamma \sum_{s'} P(s'|s,\pi(s))\, V^{\pi}(s').$$

- **Rearranging terms:**

$$\begin{aligned} R(s) &= V^{\pi}(s) - \gamma \sum_{s'} P(s'|s,\pi(s))\, V^{\pi}(s') \\ &= \sum_{s'} \Big[\underbrace{I(s,s')}_{\text{identity matrix}} - \gamma P(s'|s,\pi(s)) \Big] V^{\pi}(s') \end{aligned}$$

- **In matrix-vector form:**

$$R \;=\; \Big[I - \gamma P^{\pi} \Big] V^{\pi}$$

$$\left[\begin{array}{c} \text{column vector of} \\ n \text{ known rewards} \end{array} \right] = \left[\begin{array}{c} n \times n \text{ matrix} \\ \text{(known)} \end{array} \right] \left[\begin{array}{c} \text{column vector of} \\ n \text{ unknown values} \end{array} \right]$$

Solving the linear system (con't)

- **Solution**

$$R = \left[I - \gamma P^\pi\right] V^\pi \quad \Longrightarrow \quad V^\pi = \underbrace{(I - \gamma P^\pi)^{-1}}_{\text{matrix inverse}} R$$

- **Complexity**

  It takes $O(n^2)$ operations to solve this system of equations.
  (For very large n, HW 9.5 develops an iterative solution.)

- **Example**

  Let $\mathcal{S} = \{1, 2\}$ and $P(s'|s, \pi(s)) = 0.5$ for all (s, s').

$$\begin{bmatrix} V^\pi(1) \\ V^\pi(2) \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \gamma \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \right)^{-1} \begin{bmatrix} R(1) \\ R(2) \end{bmatrix}.$$

Policy improvement

- **Problem statement**

  Given a policy π and its state value function $V^{\pi}(s)$, how to compute a policy π' such that

$$V^{\pi'}(s) \ \geq V^{\pi}(s) \quad \text{for all states } s?$$

- **Definition**

  Given the action value function $Q^{\pi}(s,a)$ for policy π, we define the **greedy policy** π' by

$$\pi'(s) \ = \ \operatorname*{argmax}_a \Big[Q^{\pi}(s,a) \Big].$$

  Why *greedy*? Because we change the action in state s to whatever appears to improve the expected return.

Greedy policies

- **In terms of the state value function:**

$$\begin{aligned}
\pi'(s) &= \operatorname*{argmax}_a \Big[Q^\pi(s,a) \Big] \\
&= \operatorname*{argmax}_a \Big[R(s) + \gamma \textstyle\sum_{s'} P(s'|s,a)\, V^\pi(s') \Big] \\
&= \operatorname*{argmax}_a \Big[\textstyle\sum_{s'} P(s'|s,a)\, V^\pi(s') \Big]
\end{aligned}$$

- **Test your understanding:**

$\pi'(s) = \pi(s)$ for some $s \in \mathcal{S}$? **not necessarily**

$\pi'(s) \neq \pi(s)$ for some $s \in \mathcal{S}$? **not necessarily**

$Q^\pi(s, \pi'(s)) \geq Q^\pi(s, \pi(s))$ for all $s \in \mathcal{S}$? **TRUE**

Policy improvement

- **Theorem**

  The greedy policy π' everywhere dominates the policy π from whose value functions it was derived:

$$V^{\pi'}(s) \;\geq V^{\pi}(s) \quad \text{for all states } s \in \mathcal{S}.$$

- **Proof:** *next lecture.*

- **Is this surprising?**

  Yes — greedy methods (in general) often have shortcomings.

  No — the MDP lends itself to this approach.

Policy iteration

How to compute π^*?

1. **Choose an initial policy $\pi : \mathcal{S} \to \mathcal{A}$.**
2. **Repeat until convergence:**

   Compute the action value function $Q^\pi(s, a)$.
   Compute the greedy policy $\pi'(s) = \operatorname{argmax}_a Q^\pi(s, a)$.
   Replace π by π'.

$$\pi_0 \xrightarrow{\text{evaluate}} \begin{matrix} V^{\pi_0}(s) \\ Q^{\pi_0}(s, a) \end{matrix} \xrightarrow{\text{improve}} \pi_1 \xrightarrow{\text{evaluate}} \begin{matrix} V^{\pi_1}(s) \\ Q^{\pi_1}(s, a) \end{matrix} \xrightarrow{\text{improve}} \cdots$$

Convergence of policy iteration

$$\pi_0 \xrightarrow{\text{evaluate}} \begin{matrix} V^{\pi_0}(s) \\ Q^{\pi_0}(s,a) \end{matrix} \xrightarrow{\text{improve}} \pi_1 \xrightarrow{\text{evaluate}} \cdots$$

- **Why must this converge?**

  Recall that $V^{\pi'}(s) \geq V^{\pi}(s)$ for all $s \in \mathcal{S}$.
  Thus we cannot cycle back to old policies.

  Also the number of policies is finite: $|\mathcal{A}|^{|\mathcal{S}|} < \infty$.
  Thus we cannot improve indefinitely.

- **Why does this converge to an optimal policy?**

  We will prove this in the next lecture.

Convergence of policy iteration

$$\pi_0 \xrightarrow{\text{evaluate}} \begin{matrix} V^{\pi_0}(s) \\ Q^{\pi_0}(s,a) \end{matrix} \xrightarrow{\text{improve}} \pi_1 \xrightarrow{\text{evaluate}} \cdots$$

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- **Why does this converge to an optimal policy?**

  We will prove this in the next lecture.
  Also there will be **DEMOS** ...