CSE 250A. Principles of AI

Probabilistic Reasoning and Decision-Making

Lecture 14 – Hidden Markov models

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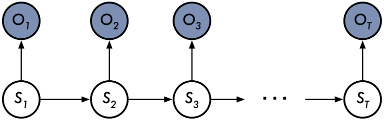
Outline

Hidden Markov models

- **Belief network**



observations $o_t \in \{1, 2, \ldots, m\}$

states $s_t \in \{1, 2, \ldots, n\}$

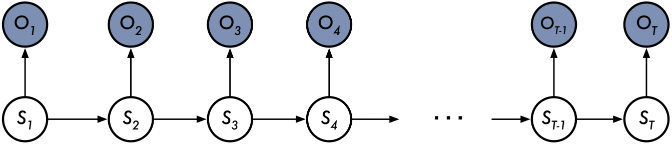
- **Parameters**

$$\begin{aligned} a_{ij} &= P(S_{t+1}=j|S_t=i) & \textbf{transition matrix} \\ b_{ik} &= P(O_t=k|S_t=i) & \textbf{emission matrix} \\ \pi_i &= P(S_1=i) & \textbf{initial state distribution} \end{aligned}$$

- **Notation**

Sometimes we'll write $b_i(k) = b_{ik}$ to avoid double subscripts.

Key computations in HMMs



Inference

1. How to compute the likelihood $P(o_1, o_2, \ldots, o_T)$?
2. How to compute the most likely state sequence $\text{argmax}_{\vec{s}} P(\vec{s}|\vec{o})$?
3. How to update beliefs by computing $P(s_t|o_1, o_2, \ldots, o_t)$?

Learning

How to estimate parameters $\{\pi_i, a_{ij}, b_{ik}\}$ that maximize the log-likelihood of observed sequences?

HW 7.1 Decoding a hidden message

- **What you are given**

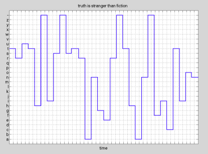
  HMM parameters $\{a_{ij}, b_{ik}, \pi_i\}$ where $n = 27$ and $m = 2$
  Binary observations $\{o_1, o_2, \ldots, o_T\}$ where $T \approx 10^5$

- **What you will compute**

  Most likely sequence of hidden states

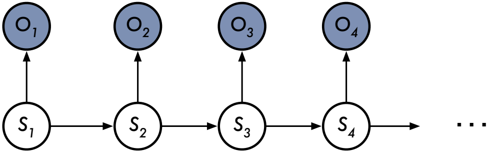
  $$\arg\max_{s_1, s_2, \ldots, s_T} P(s_1, s_2, \ldots, s_T | o_1, o_2 \ldots, o_T)$$

- **What you will plot**

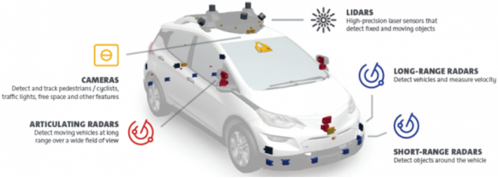


Convert states 1–27 to letters A–Z,⟨SPC⟩.
Ignore repeated states to reveal a message:
Truth is stranger than fiction.

HW 7.4 Belief updating



How to compute $P(s_t | o_1, o_2, \ldots, o_t)$?
Important for real-time monitoring of dynamical systems.



Outline

1. Review & HW
2. **Forward algorithm**
3. Viterbi algorithm
4. Backward algorithm

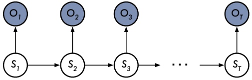
Computing the likelihood $P(o_1, o_2, \ldots, o_T)$

- **One strategy that is not quite right**

  Compute $P(o_1)$.
  Compute $P(o_1, o_2)$.
  Compute $P(o_1, o_2, o_3)$.
  Etc.

- **Why this is appealing**

  We expect an iterative approach of this kind in a BN with a chain-like DAG.



- **Why this doesn't work**

  We need to build up a slightly different quantity.
  This is what the **forward algorithm** does.

Computing $P(o_1, o_2, \ldots, o_t, S_t = i)$

- **Definition**

  For a particular sequence of observations $\{o_1, o_2, \ldots, o_T\}$, define the matrix with elements:

$$\alpha_{it} = P(o_1, o_2, \ldots, o_t, S_t = i) \qquad n \text{ rows} \left\{ \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,T-1} & \alpha_{1T} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,T-1} & \alpha_{2T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{n,T-1} & \alpha_{nT} \end{bmatrix} \right.$$

- **First column ($t = 1$)**

$$\begin{aligned} \alpha_{i1} &= P(o_1, S_1 = i) \\ &= P(S_1 = i)\, P(o_1 | S_1 = i) && \boxed{\textbf{product rule}} \\ &= \pi_i\, b_i(o_1) && \boxed{\textbf{CPTs}} \end{aligned}$$

Computing $\alpha_{it} = P(o_1, o_2, \ldots, o_t, S_t = i)$

- **Next columns ($t > 1$)**

$$\begin{aligned}
\alpha_{j,t+1} &= P(o_1, o_2, \ldots, o_{t+1}, S_{t+1} = j) \\
&= \sum_{i=1}^{n} P(o_1, o_2, \ldots, o_{t+1}, S_t = i, S_{t+1} = j) \quad \boxed{\textbf{marginalization}} \\
&= \sum_{i=1}^{n} \Big[P(o_1, o_2, \ldots, o_t, S_t = i) \cdot \\
&\qquad P(S_{t+1} = j | o_1, o_2, \ldots, o_t, S_t = i) \cdot \\
&\qquad P(o_{t+1} | o_1, o_2, \ldots, o_t, S_t = i, S_{t+1} = j) \Big] \quad \boxed{\textbf{product rule}} \\
&= \sum_{i=1}^{n} \Big[P(o_1, o_2, \ldots, o_t, S_t = i)\, P(S_{t+1} = j | S_t = i)\, P(o_{t+1} | S_{t+1} = j) \Big] \quad \boxed{\textbf{CI}} \\
&= \sum_{i=1}^{n} \alpha_{it}\, a_{ij}\, b_j(o_{t+1}) \quad \boxed{\textbf{CPTs}}
\end{aligned}$$

Forward algorithm

The forward algorithm fills in the matrix of α_{it} elements one column at a time:

$$\boxed{\begin{aligned} \alpha_{i1} &= \pi_i\, b_i(o_1) \\ \alpha_{j,t+1} &= \sum_{i=1}^{n} \alpha_{it}\, a_{ij}\, b_j(o_{t+1}) \end{aligned}}$$

$$n \text{ rows} \left\{ \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,T-1} & \alpha_{1T} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,T-1} & \alpha_{2T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{n,T-1} & \alpha_{nT} \end{bmatrix} \right.$$

Warning: for long sequences, beware of numerical underflow ...

Computing the likelihood $P(o_1, o_2, \ldots, o_T)$

Sum!

$$n \text{ rows} \left\{ \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,T-1} & \alpha_{1T} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,T-1} & \alpha_{2T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{n,T-1} & \alpha_{nT} \end{bmatrix} \right.$$

$$\begin{aligned} & P(o_1, o_2, \ldots, o_T) \\ = & \sum_{i=1}^{n} P(o_1, o_2, \ldots, o_T, s_T = i) \qquad \boxed{\textbf{marginalization}} \\ = & \sum_{i=1}^{n} \alpha_{iT} \qquad \boxed{\textbf{sum of last column}} \end{aligned}$$

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1. **Review & HW**
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3. **Viterbi algorithm**
4. **Backward algorithm**

The most likely sequence of hidden states

$$
\begin{aligned}
\{s_1^*, s_2^*, \ldots, s_T^*\} \\
&= \operatorname{argmax}_{s_1, s_2, \ldots, s_T} P(s_1, s_2, \ldots, s_T | o_1, o_2, \ldots, o_T) \\
&= \operatorname{argmax}_{s_1, s_2, \ldots, s_T} \left[\frac{P(s_1, s_2, \ldots, s_T, o_1, o_2, \ldots, o_T)}{P(o_1, o_2, \ldots, o_T)} \right] && \boxed{\textbf{product rule}} \\
&= \operatorname{argmax}_{s_1, s_2, \ldots, s_T} P(s_1, s_2, \ldots, s_T, o_1, o_2, \ldots, o_T) && \boxed{\textbf{no states in denominator}} \\
&= \operatorname{argmax}_{s_1, s_2, \ldots, s_T} \log P(s_1, s_2, \ldots, s_T, o_1, o_2, \ldots, o_T) && \boxed{\textbf{log is monotonic}}
\end{aligned}
$$

Q: Why is this progress?

A: Because **logs** of **joint probabilities** are easy to compute.

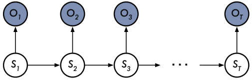
Computing the most likely state sequence

- **One strategy that is not quite right**

  Compute $\text{argmax}_{s_1} \log P(s_1, o_1)$.
  Compute $\text{argmax}_{s_1,s_2} \log P(s_1, s_2, o_1, o_2)$.
  Compute $\text{argmax}_{s_1,s_2,s_3} \log P(s_1, s_2, s_3, o_1, o_2, o_3)$.
  Etc.

- **Why this is appealing**

  We expect an iterative approach of this kind in a BN with a chain-like DAG.



- **Why this doesn't work**

  We need to build up a slightly different quantity.
  This is what the **Viterbi algorithm** does.

The matrix ℓ^*

- **Definition**

  For a particular sequence of observations $\{o_1, o_2, \ldots, o_T\}$, we define the following matrix:

$$\ell^*_{it} \;=\; \max_{s_1, s_2, \ldots, s_{t-1}} \log P(s_1, s_2, \ldots, s_{t-1}, S_t = i, o_1, o_2, \ldots, o_t)$$

$$n \text{ rows} \left\{ \begin{bmatrix} \ell^*_{11} & \ell^*_{12} & \cdots & \ell^*_{1,T-1} & \ell^*_{1T} \\ \ell^*_{21} & \ell^*_{22} & \cdots & \ell^*_{2,T-1} & \ell^*_{2T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell^*_{n1} & \ell^*_{n2} & \cdots & \ell^*_{n,T-1} & \ell^*_{nT} \end{bmatrix} \right.$$

- **Intuition**

ℓ^*_{it}	log-probability of the *t-step path* of hidden states $s_1, s_2, \ldots, s_t$ that best explains the observations $o_1, o_2, \ldots, o_t$ and ends at state $S_t = i$ at time t

Computing the matrix ℓ^*

$$\boxed{\ell^*_{it} = \max_{s_1,s_2,\ldots,s_{t-1}} \log P(s_1, s_2, \ldots, s_{t-1}, S_t = i, o_1, o_2, \ldots, o_t)}$$

$$n \text{ rows} \left\{ \begin{bmatrix} \ell^*_{11} & \ell^*_{12} & \cdots & \ell^*_{1,T-1} & \ell^*_{1T} \\ \ell^*_{21} & \ell^*_{22} & \cdots & \ell^*_{2,T-1} & \ell^*_{2T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell^*_{n1} & \ell^*_{n2} & \cdots & \ell^*_{n,T-1} & \ell^*_{nT} \end{bmatrix} \right.$$

- **First column ($t = 1$)**

$$\begin{aligned} \ell^*_{i1} &= \log P(S_1 = i, o_1) \\ &= \log \Big[P(S_1 = i)\, P(o_1 | S_1 = i) \Big] && \boxed{\textbf{product rule}} \\ &= \log \pi_i + \log b_i(o_1) && \boxed{\textbf{CPTs}} \end{aligned}$$

Computing the matrix ℓ^*

- **Next columns ($t > 1$)**

$$
\begin{aligned}
\ell^*_{j,t+1} &= \max_{s_1,\ldots,s_t} \log P(s_1,\ldots,s_t, S_{t+1}=j, o_1,\ldots,o_{t+1}) \\
&= \max_i \max_{s_1,\ldots,s_{t-1}} \log P(s_1,\ldots,s_{t-1}, S_t=i, S_{t+1}=j, o_1,\ldots,o_{t+1}) \\
&= \max_i \max_{s_1,\ldots,s_{t-1}} \log \Big[P(s_1,\ldots,s_{t-1}, S_t=i, o_1,\ldots,o_t) \cdot \quad \boxed{\textbf{product rule}} \\
&\qquad P(S_{t+1}=j | s_1,\ldots,s_{t-1}, S_t=i, o_1,\ldots,o_t) \cdot \\
&\qquad P(o_{t+1} | s_1,\ldots,s_{t-1}, S_t=i, S_{t+1}=j, o_1,\ldots,o_t) \Big] \\
&= \max_i \max_{s_1,\ldots,s_{t-1}} \log \Big[P(s_1,\ldots,s_{t-1}, S_t=i, o_1,\ldots,o_t) \cdot \\
&\qquad P(S_{t+1}=j | S_t=i) \cdot P(o_{t+1} | S_{t+1}=j) \Big] \quad \boxed{\textbf{CI}} \\
&= \max_i \max_{s_1,\ldots,s_{t-1}} \Big[\log P(s_1,\ldots,s_{t-1}, S_t=i, o_1,\ldots,o_t) + \log a_{ij} + \log b_j(o_{t+1}) \Big] \\
&= \max_i \Big[\ell^*_{it} + \log a_{ij} \Big] + \log b_j(o_{t+1})
\end{aligned}
$$

Summary

We have shown how to efficiently compute the matrix ℓ^* one column at a time, from left to right:

$$\boxed{\begin{aligned} \ell^*_{i1} &= \log \pi_i + \log b_i(o_1) \\ \ell^*_{j,t+1} &= \max_i \left[\ell^*_{it} + \log a_{ij} \right] + \log b_j(o_{t+1}) \end{aligned}}$$

$$n \text{ rows} \left\{ \begin{bmatrix} \ell^*_{11} & \ell^*_{12} & \cdots & \ell^*_{1,T-1} & \ell^*_{1T} \\ \ell^*_{21} & \ell^*_{22} & \cdots & \ell^*_{2,T-1} & \ell^*_{2T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell^*_{n1} & \ell^*_{n2} & \cdots & \ell^*_{n,T-1} & \ell^*_{nT} \end{bmatrix} \right.$$

But how do we derive $\{s_1^*, s_2^*, \ldots, s_T^*\}$ from this matrix?

Computing $\{s_1^*, s_2^*, \ldots, s_T^*\}$

- **Form one more matrix:**

$$\Phi_{t+1}(j) \;=\; \arg\max_i \left[\ell_{it}^* + \log a_{ij}\right]$$

Intuitively, given the observations $o_1, o_2, \ldots, o_{t+1}$, record the most likely state at time t given that $S_{t+1} = j$.

- **Compute the most likely states by backtracking:**

$$s_T^* \;=\; \arg\max_i \left[\ell_{iT}^*\right]$$

for $t = T-1$ **to 1**

$$s_t^* \;=\; \Phi_{t+1}(s_{t+1}^*)$$

end

Max!

$$\begin{bmatrix} \ell_{11}^* & \ell_{12}^* & \cdots & \ell_{1,T-1}^* & \ell_{1T}^* \\ \ell_{21}^* & \ell_{22}^* & \cdots & \ell_{2,T-1}^* & \ell_{2T}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{n1}^* & \ell_{n2}^* & \cdots & \ell_{n,T-1}^* & \ell_{nT}^* \end{bmatrix}$$

Summary of Viterbi algorithm

- **Fill ℓ^* matrix from left to right:**

$$\boxed{t=1} \qquad \ell^*_{i1} = \log \pi_i + \log b_i(o_1)$$

$$\boxed{t>1} \qquad \ell^*_{j,t+1} = \max_i \left[\ell^*_{it} + \log a_{ij} \right] + \log b_j(o_{t+1})$$

- **Backtrack through ℓ^* from right to left:**

$$\boxed{t=T} \qquad s^*_T = \text{argmax}_i \left[\ell^*_{iT} \right]$$

$$\boxed{t<T} \qquad s^*_t = \text{argmax}_i \left[\ell^*_{it} + \log a_{is^*_{t+1}} \right]$$

- **Where you've seen this before:**

  This algorithm is an instance of **dynamic programming**.
  Sometimes $\{s^*_1, s^*_2, \ldots, s^*_T\}$ is called the **Viterbi path**.

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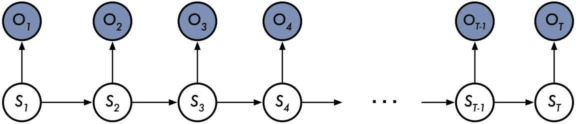
Learning in HMMs



Given: one or more sequences of observations $\{o_1, o_2, \ldots, o_T\}$.
For simplicity, we'll assume just one.

Goal: estimate $\{\pi_i, a_{ij}, b_{ik}\}$ to maximize $P(o_1, o_2, \ldots, o_T)$, the likelihood of the observed data.

Assume: the cardinality n of the hidden state space is fixed.
$s_t \in \{1, 2, \ldots, n\}$

EM algorithm for HMMs

- **CPTs to re-estimate:**

$$\begin{aligned} \pi_i &= P(S_1 = i) \\ a_{ij} &= P(S_{t+1} = j | S_t = i) \\ b_{ik} &= P(O_t = k | S_t = i) \end{aligned}$$

- **How EM works in general:**

  To re-estimate $P(X_i = x | \mathrm{pa}_i = \pi)$ in the M-step,
  we must compute $P(X_i = x, \mathrm{pa}_i = \pi | V)$ in the E-step.

- **E-step in HMMs must compute:**

$$\begin{aligned} P(S_1 = i | o_1, o_2, \ldots, o_T) \\ P(S_{t+1} = j, S_t = i | o_1, o_2, \ldots, o_T) \\ P(O_t = k, S_t = i | o_1, o_2, \ldots, o_T) \end{aligned}$$

How to efficiently compute these posteriors?

We need one more matrix ...

$$\begin{aligned}\textbf{Analogous to } \alpha_{it} &= P(o_1, o_2, \ldots, o_t, S_t = i),\\ \textbf{define } \beta_{it} &= P(o_{t+1}, o_{t+2}, \ldots, o_T | S_t = i).\end{aligned}$$

$$n \text{ rows} \left\{ \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1,T-1} & \beta_{1T} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2,T-1} & \beta_{2T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{n,T-1} & \beta_{nT} \end{bmatrix} \right.$$

Understand the **differences** between these matrices:

- α_{it} predicts observations up to and including time t.
- β_{it} predicts observations from **time** $t+1$ to time T.

Computing $\beta_{it} = P(o_{t+1}, o_{t+2}, \ldots, o_T | S_t = i)$

- **Last column ($t = T$)**

$$\beta_{iT} \;=\; P(\underline{\quad}\,|S_T = i)$$

What does this mean?

Note: β_{it} computes the probability of the future given $S_t = i$.

But we don't see *any* observations beyond time T.
Put another way, the future after time T is unspecified.

What is the probability of some unspecified future occurring?
By definition, we set:

$$\boxed{\beta_{iT} \;=\; 1 \quad \textbf{for all} \quad i \in \{1, 2, \ldots, n\}}$$

Computing $\beta_{it} = P(o_{t+1}, o_{t+2}, \ldots, o_T | S_t = i)$

- **Previous columns ($t < T$)**

$$
\begin{aligned}
\beta_{it} &= P(o_{t+1}, o_{t+2}, \ldots, o_T | S_t = i) \\
&= \sum_{j=1}^{n} P(S_{t+1} = j, o_{t+1}, o_{t+2}, \ldots, o_T | S_t = i) \qquad \boxed{\textbf{marginalization}} \\
&= \sum_{j=1}^{n} \Big[P(S_{t+1} = j | S_t = i) \cdot \\
&\qquad P(o_{t+1} | S_t = i, S_{t+1} = j) \cdot \\
&\qquad P(o_{t+2}, \ldots, o_T | S_t = i, S_{t+1} = j, o_{t+1}) \Big] \qquad \boxed{\textbf{product rule}} \\
&= \sum_{j=1}^{n} \Big[P(S_{t+1} = j | S_t = i)\, P(o_{t+1} | S_{t+1} = j)\, P(o_{t+2}, \ldots, o_T | S_{t+1} = j) \Big] \qquad \boxed{\textbf{CI}} \\
&= \sum_{j=1}^{n} a_{ij}\, b_j(o_{t+1})\, \beta_{j,t+1} \qquad \boxed{\textbf{CPTs}}
\end{aligned}
$$

Backward algorithm

The backward algorithm fills in the matrix of β_{it} elements one column at a time:

$$
\boxed{
\begin{aligned}
\beta_{iT} &= 1 \quad \textbf{for} \quad i \in \{1, 2, \ldots, n\} \\
\beta_{it} &= \sum_{j=1}^{n} a_{ij}\, b_j(o_{t+1})\, \beta_{j,t+1}
\end{aligned}
}
$$

$$
n \text{ rows} \left\{ \begin{bmatrix}
\beta_{11} & \beta_{12} & \cdots & \beta_{1,T-1} & 1 \\
\beta_{21} & \beta_{22} & \cdots & \beta_{2,T-1} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{n1} & \beta_{n2} & \cdots & \beta_{n,T-1} & 1
\end{bmatrix} \right.
$$

Next lecture: EM with the forward-backward algorithm.