CSE 250A. Principles of AI

Probabilistic Reasoning and Decision-Making

Lecture 2 – Probability and Commonsense Reasoning

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Fall 2021

Motivation

- **Modeling of uncertainty**
  - Inherent randomness (e.g., radioactive decay)
  - Gross statistical dependencies of complex deterministic world (e.g., coin toss, free throw percentage, chance of rain)
- **Probability as guardian of commonsense reasoning**
- **Many empirical successes**
  - robotics
  - vision
  - speech
  - natural language processing
  - bioinformatics

Outline

1. Review of probability
2. Example of commonsense reasoning

Review of probability

What should be familiar:

- Discrete random variables
- Basic axioms of probability
- Marginalization, product rule, Bayes rule

What may be less familiar:

- Nuances of conditional dependence and independence
- How probabilities capture commonsense reasoning

Discrete random variables

- A *discrete random variable* X has a *domain of possible values* $\{x_1, x_2, \ldots, x_m\}$.
- Note the distinction: we capitalize the former, not the latter.
- Example: weather W with possible values
$$\{w_1 = \texttt{sunny}, w_2 = \texttt{rainy}\}$$
- The *unconditional probability* $P(X = x)$ denotes our degree of belief that $X = x$ in the absence of all other knowledge.

Basic axioms

1. **Probabilities are nonnegative:**

$$P(X = x) \;\geq\; 0$$

2. **Probabilities are normalized; they sum to one:**

$$\sum_i P(X = x_i) \;=\; 1$$

3. **Probabilities add for the union of mutually exclusive events:**

$$P(X = x_i \text{ or } X = x_j) \;=\; P(X = x_i) + P(X = x_j) \quad \text{if} \quad x_i \neq x_j$$

Conditional probabilities

The *conditional probability* $P(X = x_i | Y = y_j)$ denotes our degree of belief that $X = x_i$ given $Y = y_j$.

Test your understanding:

When is it true that $P(X = x_i | Y = y_j) = P(X = x_i)$?

(a) Never

(b) Always

(c) Sometimes

The answer is (c). It is sometimes true, depending on the events represented by X and Y and the values of x_i and y_j.

Ex: (marginally) dependent random variables

weather W	$\{w_1 = \texttt{sunny}, w_2 = \texttt{rainy}\}$
month M	$\{m_1 = \texttt{Jan}, m_2 = \texttt{Feb}, \ldots, m_{12} = \texttt{Dec}\}$

$$
\begin{aligned}
P(W = \texttt{sunny}) &= 0.9 \\
P(W = \texttt{sunny} \mid M = \texttt{Aug}) &= 0.97 \quad \textbf{higher} \\
P(W = \texttt{sunny} \mid M = \texttt{Jan}) &= 0.83 \quad \textbf{lower}
\end{aligned}
$$

Ex: (marginally) independent random variables

weather W	$\{w_1 = \texttt{sunny}, w_2 = \texttt{rainy}\}$
day of week D	$\{d_1 = \texttt{Sun}, d_2 = \texttt{Mon}, \ldots, d_7 = \texttt{Sat}\}$

$$
\begin{aligned}
P(W = \texttt{sunny}) &= 0.9 \\
P(W = \texttt{sunny} | D = \texttt{Sun}) &= 0.9 \\
P(W = \texttt{sunny} | D = \texttt{Mon}) &= 0.9 \\
&\vdots \\
P(W = \texttt{sunny} | D = \texttt{Sat}) &= 0.9
\end{aligned}
$$

Questions?

Ex: conditionally independent random variables

weather W	$\{w_1 = \texttt{sunny}, w_2 = \texttt{rainy}\}$
month M	$\{m_1 = \texttt{Jan}, \ldots, m_{12} = \texttt{Dec}\}$
sidewalk S	$\{s_1 = \texttt{dry}, s_2 = \texttt{wet}\}$

Note that S and M **are not marginally independent**:

$$P(S = \texttt{wet}) \; < \; P(S = \texttt{wet} | M = \texttt{Jan})$$

But S **is conditionally independent** of M given W:

$$\begin{aligned} P(S = \texttt{wet} | W = \texttt{sunny}) &= P(S = \texttt{wet} | W = \texttt{sunny}, M = \texttt{Jan}) \\ P(S = \texttt{wet} | W = \texttt{rainy}) &= P(S = \texttt{wet} | W = \texttt{rainy}, M = \texttt{Jan}) \end{aligned}$$

And these equalities hold for every value of the month M.

Ex: conditionally dependent random variables

We'll see an example of this at the end of today's lecture ...

More elementary properties

1. **Conditional probabilities are nonnegative:**

$$P(X\!=\!x_i|Y\!=\!y_j) \;\geq\; 0.$$

2. **Conditional probabilities are normalized:**

$$\sum_i P(X\!=\!x_i|Y\!=\!y_j) \;=\; 1.$$

Sum over i not j!

3. **Conditional probabilities add for the union of mutually exclusive events** $(x_i\!\neq\!x_j)$:

$$P\left(X\!\in\{x_i,x_j\}|Y\!=\!y_k\right) \;=\; P(X\!=\!x_i|Y\!=\!y_k)+P(X\!=\!x_j|Y\!=\!y_k).$$

Joint probabilities

The *joint probability* $P(X=x_i, Y=y_j)$ denotes our degree of belief that $X=x_i$ and $Y=y_j$.

Test yourself: which of the following is always true?

(a) $P(X=x_i \text{ or } Y=y_j) \;\leq\; P(X=x_i, Y=y_j)$
(b) $P(X=x_i \text{ or } Y=y_j) \;\geq\; P(X=x_i, Y=y_j)$
(c) $P(X=x_i \text{ or } Y=y_j) \;=\; P(X=x_i) + P(Y=y_j)$
(d) None of the above.

The answer is (b). The union of two events is never less probable than their intersection.

Assessing probabilities

Probabilities can measure our degrees of belief in many situations:

$$\text{e.g.,} \quad P(X=x_i, Y=y_j, Z=z_k | A=a_\ell, B=b_m, C=c_n)$$

Sometimes we need to elicit probabilities from domain experts.
Certain probabilities are easier to assess than others:

easier	$P(Z=z_k \mid X=x_i, Y=y_j)$	predicting a single outcome that is informed by other events
harder	$P(X=x_i, Y=y_j, Z=z_k)$	predicting multiple simultaneous outcomes without context

More rules

- **Product rule:**

$$
\begin{aligned}
P(X=x_i, Y=y_j) &= P(X=x_i)\,P(Y=y_j|X=x_i),\\
P(X=x_i, Y=y_j) &= P(Y=y_j)\,P(X=x_i|Y=y_j).
\end{aligned}
$$

- **Marginalization:**

$$
\begin{aligned}
P(X=x_i) &= \sum_{j} P(X=x_i, Y=y_j),\\
P(X=x_i, Y=y_j) &= \sum_{k} P(X=x_i, Y=y_j, Z=Z_k).
\end{aligned}
$$

Questions?

Shorthand notation

- **Implied universality:**

  When random variables are not assigned to values, assume the equation holds for all possible assignments.

$$P(X,Y) = P(X)\,P(Y|X) \quad \textbf{(product rule)}$$

$$P(X) = \sum_{y} P(X, Y{=}y) \quad \textbf{(marginalization)}$$

- **Implied assignment:**

$$P(x,y,z) = P(X{=}x, Y{=}y, Z{=}z)$$

Generalized product rule

- **Product rule for multiple variables:**

$$P(A, B, C, D, \ldots) \;=\; P(A)\, P(B|A)\, P(C|A, B)\, P(D|A, B, C)\, \ldots$$

- **Many orders are possible:**

$$P(A, B, C, D, \ldots) \;=\; P(D)\, P(C|D)\, P(B|C, D)\, P(A|B, C, D)\, \ldots$$

- **Which order to use in practice?**

  Use the order that expresses the joint probability in terms of probabilities that are already known or easier to compute.

Bayes rule

Recall the product rule:

$$\begin{aligned} P(X, Y) &= P(X)\,P(Y|X), \\ P(X, Y) &= P(Y)\,P(X|Y). \end{aligned}$$

Equating the right hand sides, we obtain:

$$P(X|Y) = \frac{P(Y|X)\,P(X)}{P(Y)} \qquad \boxed{\textbf{Bayes rule}}$$

Why so important?

Because it expresses $P(X|Y)$ in terms of $P(Y|X)$, which may be already known or easier to compute.

More rules

Basic rules:

$$P(X,Y) = P(X)\,P(Y|X) \qquad \textbf{(product rule)}$$

$$P(X) = \sum_y P(X, Y\!=\!y) \qquad \textbf{(marginalization)}$$

$$P(X|Y) = \frac{P(Y|X)\,P(X)}{P(Y)} \qquad \textbf{(Bayes rule)}$$

Also true, when conditioning on background evidence E:

$$P(X,Y|E) = P(X|E)\,P(Y|X,E)$$

$$P(X|E) = \sum_y P(X, Y\!=\!y|E)$$

$$P(X|Y,E) = \frac{P(Y|X,E)\,P(X|E)}{P(Y|E)}$$

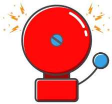
You will prove these results in HW 1.

Outline

1. **Review of probability**

2. **Example of commonsense reasoning**

Alarm example



- **Binary random variables**

$$\begin{array}{ll} B \in \{0,1\} & \text{Was there a burglary?} \\ E \in \{0,1\} & \text{Was there an earthquake?} \\ A \in \{0,1\} & \text{Was the alarm triggered?} \end{array}$$

- **Joint distribution**

$$P(B,E,A) \;=\; P(B)\,P(E|B)\,P(A|B,E)$$

Domain knowledge

- **Burglaries are rare events:**

$$P(B=1) = 0.001$$

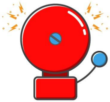
- **Earthquakes are rare events:**

$$P(E=1) = 0.002$$

- **Burglaries and earthquakes are (marginally) independent:**

$$P(E|B) \;=\; P(E)$$

More domain knowledge



How likely is the alarm to be triggered?

B	E	$P(A=1 \vert B, E)$
0	0	**0.001**
1	0	**0.94**
0	1	**0.29**
1	1	**0.95**

Complementary events

Probabilities of complementary events are easy to compute:

$$P(B\!=\!0) \;=\; 1 - P(B\!=\!1) \;=\; 0.999$$
$$P(E\!=\!0) \;=\; 1 - P(E\!=\!1) \;=\; 0.998$$

B	E	$P(A\!=\!1\vert B,E)$	$P(A\!=\!0\vert B,E)$
0	0	0.001	**0.999**
1	0	0.94	**0.06**
0	1	0.29	**0.71**
1	1	0.95	**0.05**

Questions?

Inference

Do the rules of probability capture commonsense reasoning?

Let's compare the following probabilities:

1. $P(B\!=\!1) \;=\; \mathbf{0.001}$
2. $P(B\!=\!1|A\!=\!1) \;=\; ?$
3. $P(B\!=\!1|A\!=\!1, E\!=\!1) \;=\; ?$

Test yourself: intuitively, what do you expect?

Inference — but how?

- **Here are the probabilities we want to compute:**

$$P(B=1|A=1)$$
$$P(B=1|A=1, E=1)$$

- **Here are the probabilities whose values we know:**

$$P(B)$$
$$P(E)$$
$$P(A|B, E)$$

- **Here are the tools at our disposal:**
  - marginal independence (of B and E)
  - product rule
  - marginalization
  - Bayes rule

  and their conditionalized versions

Computing $P(B=1|A=1)$

Where to start?

$$P(B=1|A=1) = \frac{P(A=1|B=1)\,P(B=1)}{P(A=1)}$$

$\boxed{\textbf{Bayes rule}}$

One of these terms we already know:

$$P(B=1) = 0.001$$

Let's compute the other terms:

- $P(A=1|B=1)$ in the numerator
- $P(A=1)$ in the denominator

Term in numerator

$$
\begin{aligned}
P(A=1|B=1) & \\
&= \sum_e P(A=1, E=e|B=1) \quad \boxed{\textbf{marginalization}} \\
&= \sum_e P(E=e|B=1)\, P(A=1|E=e, B=1) \quad \boxed{\textbf{product rule}} \\
&= \sum_e P(E=e)\, P(A=1|E=e, B=1) \quad \boxed{\textbf{independence}} \\
&= P(E=0)\, P(A=1|E=0, B=1) \; + \; P(E=1)\, P(A=1|E=1, B=1) \\
&= (0.998)\,(0.94) + (0.002)\,(0.95) \quad \boxed{\textbf{substitute and sum}} \\
&= \mathbf{0.94002}
\end{aligned}
$$

Term in denominator

$$P(A=1)$$

$$= \sum_{b,e} P(B=b, E=e, A=1) \quad \boxed{\textbf{marginalization}}$$

$$= \sum_{b,e} P(B=b)\, P(E=e|B=b)\, P(A=1|B=b, E=e) \quad \boxed{\textbf{product rule}}$$

$$= \sum_{b,e} P(B=b)\, P(E=e)\, P(A=1|B=b, E=e) \quad \boxed{\textbf{independence}}$$

$$\begin{aligned} = \; & P(B=0)\, P(E=0)\, P(A=1|B=0, E=0) \quad \boxed{\textbf{sum}} \\ & + P(B=0)\, P(E=1)\, P(A=1|B=0, E=1) \\ & + P(B=1)\, P(E=0)\, P(A=1|B=1, E=0) \\ & + P(B=1)\, P(E=1)\, P(A=1|B=1, E=1) \end{aligned}$$

$$= \mathbf{0.00252} \quad \boxed{\textbf{substitute}}$$

Computing $P(B=1|A=1)$

From Bayes rule:

$$\begin{aligned} P(B=1|A=1) &= \frac{P(A=1|B=1)\,P(B=1)}{P(A=1)} \\ &= \frac{(0.94002)\,(0.001)}{0.00252} \\ &= \mathbf{0.37} \end{aligned}$$

Comparing probabilities:

$$\begin{aligned} P(B=1) &= 0.001 \\ P(B=1|A=1) &= 0.37 \quad \textbf{(intuitively, much larger)} \\ P(B=1|A=1, {\color{red}E=1}) &= {\color{red}????} \end{aligned}$$

Questions?

Computing $P(B=1|A=1, E=1)$

$$P(B=1|A=1, E=1)$$

$$= \frac{P(A=1|B=1, E=1)\, P(B=1|E=1)}{P(A=1|E=1)} \qquad \boxed{\textbf{Bayes rule}}$$

$$= \frac{P(A=1|B=1, E=1)\, P(B=1)}{P(A=1|E=1)} \qquad \boxed{\textbf{independence}}$$

We already know both terms in the numerator:

$$\begin{aligned} P(A=1|B=1, E=1) &= 0.95 \\ P(B=1) &= 0.001 \end{aligned}$$

Term in denominator

Q: How to compute $P(A=1|E=1)$?
A: In exactly the same way we computed $P(A=1|B=1)$ ...

$$P(A=1|E=1)$$

$$= \sum_b P(A=1, B=b|E=1) \quad \boxed{\textbf{marginalization}}$$

$$= \sum_b P(B=b|E=1)\, P(A=1|B=b, E=1) \quad \boxed{\textbf{product rule}}$$

$$= \sum_b P(B=b)\, P(A=1|B=b, E=1) \quad \boxed{\textbf{independence}}$$

$$= \mathbf{0.29066} \quad \boxed{\textbf{sum and substitute}}$$

Computing $P(B=1|A=1)$

From Bayes rule:

$$
\begin{aligned}
P(B=1|A=1, E=1) &= \frac{P(A=1|B=1, E=1)\, P(B=1|E=1)}{P(A=1|E=1)} \\
&= \frac{(0.95)\,(0.001)}{0.29066} \\
&= \mathbf{0.0033}
\end{aligned}
$$

Comparing probabilities:

$$
\begin{aligned}
P(B=1) &= 0.001 & \\
P(B=1|A=1) &= 0.37 & (\uparrow) \\
P(B=1|A=1, E=1) &= 0.0033 & (\downarrow)
\end{aligned}
$$

Questions?

Example of commonsense reasoning

Comparing probabilities:

$$\left.\begin{array}{rcl} P(B=1) & = & 0.001 \\ P(B=1|A=1) & = & 0.37 \\ P(B=1|A=1, E=1) & = & 0.0033 \end{array}\right\}$$

This is an example of *non-monotonic* reasoning.

This pattern of reasoning is known as explaining away:

The earthquake *explains away* the alarm,
diminishing our belief in the burglary.

Ex: conditionally dependent random variables

- **B and E are marginally independent:**

$$\begin{aligned} P(B) &= P(B|E) \\ P(E) &= P(E|B) \\ P(B,E) &= P(B)\,P(E) \end{aligned}$$

- **But B and E are conditionally dependent given A:**

$$\begin{aligned} P(B|A) &\neq P(B|E,A) \\ P(E|A) &\neq P(E|B,A) \\ P(B,E|A) &\neq P(B|A)\,P(E|A) \end{aligned}$$

Next lecture — belief networks!
Also HW 1 is posted on Canvas.