CSE 250A. Principles of AI

Probabilistic Reasoning and Decision-Making

Lecture 4 – CPTs, d-separation, inference

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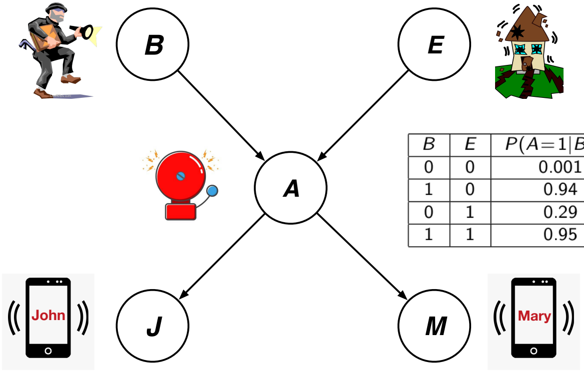
Outline

Alarm example

$P(B=1) = 0.001$

$P(E=1) = 0.002$



B	E	$P(A=1 \mid B, E)$
0	0	0.001
1	0	0.94
0	1	0.29
1	1	0.95

A	$P(J=1 \mid A)$
0	0.05
1	0.9

A	$P(M=1 \mid A)$
0	0.01
1	0.7

Belief networks

A **belief network** (BN) is a directed acyclic graph (DAG) in which:

1. Nodes represent random variables.
2. Edges represent dependencies.
3. Conditional probability tables (CPTs) describe how each node depends on its parents.

$$\boxed{\textbf{BN} = \textbf{DAG} + \textbf{CPTs}}$$

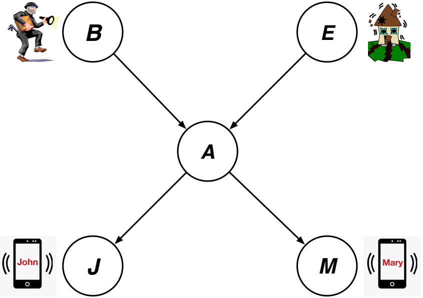
Marginal and conditional independence in DAGs

- **Missing edges encode assumptions of independence:**

$$P(X_i|X_1,\ldots,X_{i-1}) \;=\; P(X_i|\mathrm{pa}(X_i))$$

where $\mathrm{pa}(X_i) \subseteq \{X_1,\ldots,X_{i-1}\}$ denotes the **parents** of node X_i.

- **Alarm example:**



$$\begin{aligned} P(E) &= P(E|B) \\ P(J|A) &= P(J|A,B,E) \\ P(M|A) &= P(M|A,B,E,J) \end{aligned}$$

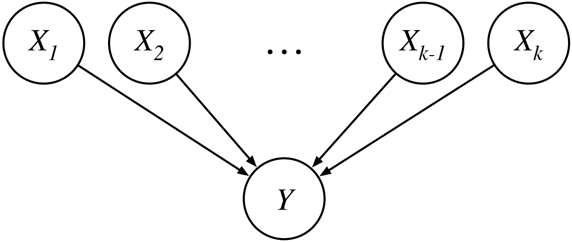
These are true no matter what CPTs are attached to the nodes in the DAG.

Questions?

Outline

1. Review
2. $\boxed{\text{Conditional probability tables (CPTs)}}$
3. d-separation and examples
4. Inference

Representing CPTs



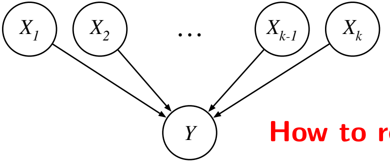
- **How to represent $P(Y|X_1, X_2, \ldots, X_k)$?**

- **Simplest case:**

  Suppose $X_i \in \{0, 1\}$, $Y \in \{0, 1\}$ are **binary** random variables.

  How to represent $P(Y=1|X_1, X_2, \ldots, X_k)$?

Types of CPTs

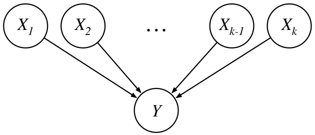


How to represent $P(Y=1|X_1, X_2, \ldots, X_k)$**?**

Some possibilities:

1. Tabular
2. Logical / Deterministic
3. Noisy-OR
4. Sigmoid

1. Tabular CPT



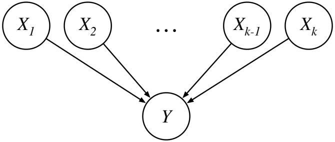
X_1	X_2	$\cdots$	X_k	$P(Y=1\|X_1, X_2, \ldots, X_k)$
0	0	$\cdots$	0	0.1
1	0	$\cdots$	0	0.6
0	1	$\cdots$	0	0.3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	$\cdots$	1	0.2

A lookup table can exhaustively enumerate a conditional probability for every configuration of parents.

Pro Able to model arbitrarily complicated dependence.

Con A table with 2^k rows is too unwieldy for large k.

2. Logical / Deterministic CPT



CPTs can also mimic the behavior of logical circuits.

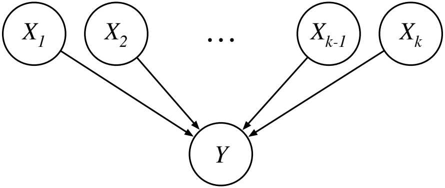
AND gate $$P(Y=1|X_1, X_2, \ldots, X_k) = \prod_{i=1}^{k} X_i$$

OR gate $$P(Y=0|X_1, X_2, \ldots, X_k) = \prod_{i=1}^{k} (1 - X_i)$$

Pro Compact representation for large k.

Con No model of uncertainty.

3. Noisy-OR CPT



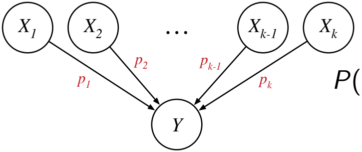
Use k numbers $p_i \in [0, 1]$ to parameterize all 2^k entries in the CPT:

$$P(Y=0|X_1, X_2, \ldots, X_k) = \prod_{i=1}^{k} (1 - p_i)^{X_i}$$

$$P(Y=1|X_1, X_2, \ldots, X_k) = 1 - \prod_{i=1}^{k} (1 - p_i)^{X_i}$$

But why is this called Noisy-OR?

Noisy-OR CPT (con't)



$$P(Y=1|X_1,X_2,\ldots,X_k) \;=\; 1-\prod_{i=1}^{k}(1-p_i)^{X_i}$$

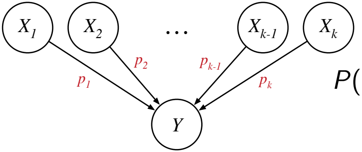
- **When all parents are equal to zero:**

$$P(Y=1|X_1=0,X_2=0,\ldots,X_k=0) \;=\; 1-\prod_{i=1}^{k}(1-p_i)^0 \;=\; 1-\prod_{i=1}^{k}(1) \;=\; 0$$

- **When exactly one parent X_j is equal to one:**

$$\begin{aligned} P(Y=1|X_1=0,\ldots,X_{j-1}=0,X_j=1,X_{j+1}=0,\ldots,X_k=0) \\ &= 1-(1-p_1)^0\cdots(1-p_{j-1})^0(1-p_j)^1(1-p_{j+1})^0\cdots(1-p_k)^0 \\ &= 1-(1-p_j) \\ &= p_j \end{aligned}$$

Noisy-OR CPT (con't)



$$P(Y=1|X_1, X_2, \ldots, X_k) \;=\; 1 - \prod_{i=1}^{k}(1-p_i)^{X_i}$$

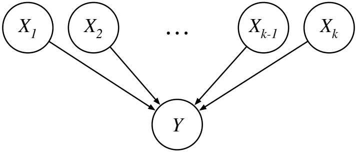
- **Modeling uncertainty**
  Intuitively, $p_i \in [0,1]$ is the probability that $X_i = 1$ **by itself** triggers $Y = 1$.

- **Logical OR as special case**
  We recover a logical OR gate by taking the limit $p_i \to 1$ for all parents $i = 1, 2, \ldots, k$.

- **Canonical application**
  The parents $\{X_i\}_{i=1}^{k}$ are diseases, and the child Y is a symptom. The more diseases, the more likely is the symptom.

Questions?

4. Sigmoid CPT

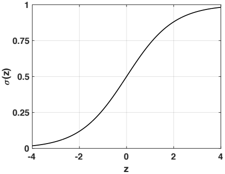


Use k real numbers $\theta_i \in \Re$ to parameterize all 2^k entries in the CPT:

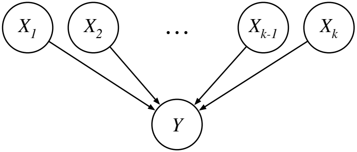
$$P(Y=1|X_1, X_2, \ldots, X_k) \;=\; \sigma\left(\sum_{i=1}^{k} \theta_i X_i\right)$$

The function on the right hand side is called the **sigmoid** function:

$$\sigma(z) \;=\; \frac{1}{1+e^{-z}}$$



4. Sigmoid CPT (con't)



$$P(Y=1|X_1, X_2, \ldots, X_k) = \sigma\left(\sum_{i=1}^{k} \theta_i X_i\right)$$

Other uses of sigmoid functions:

- Activation function in neural nets
- Inverse of the link function for logistic regression

Properties:

- If $\theta_i > 0$, then $X_i = 1$ favors $Y = 1$.
- If $\theta_i < 0$, then $X_i = 1$ inhibits $Y = 1$.
- These effects can mix in a sigmoid CPT (unlike noisy-OR).

Questions?

Outline

1. **Review**
2. **Conditional probability tables (CPTs)**
3. **d-separation and examples**
4. **Inference**

Conditional independence

- **What we've already seen**

  A node X_i is conditionally independent of its non-parent ancestors given its parents:

  $$P(X_i|X_1, X_2, \ldots, X_{i-1}) \;=\; P(X_i|\mathrm{pa}(X_i))$$

- **What we can ask more generally**

  Let X, Y, and E refer to disjoint *sets* of nodes in a BN.
  When is X conditionally independent of Y given E?

  $$\text{When is} \left\{ \begin{array}{rcl} P(X|E,Y) & = & P(X|E) \\ P(Y|E,X) & = & P(Y|E) \\ P(X,Y|E) & = & P(X|E)\,P(Y|E) \end{array} \right\} \quad ?$$

- **Above is special case**

  $$X = \{X_i\}, \quad E = \mathrm{pa}(X_i) \quad Y = \{X_1, X_2, \ldots, X_{i-1}\} - \mathrm{pa}(X_i)$$

Overview

- **Key idea:**

  We can relate conditional independence of random variables in a BN to properties of its DAG.

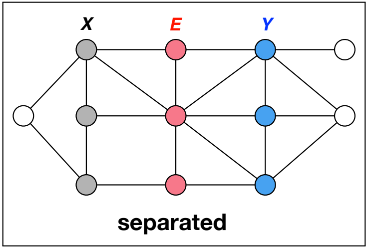
- **What's new here?**

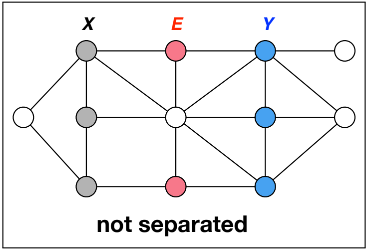
  We must generalize a highly intuitive and natural property of undirected graphs to a more subtle property of DAGs.

Separation in undirected graphs

Let X, Y, and E be disjoint sets of nodes in an **undirected** graph.

Definition: X and Y are said to be *separated* by E if every path from a node in X to a node in Y contains one or more nodes in E.





Questions?

d-separation in DAGs

d-separation = direction-dependent separation

- **Motivation**

  How is conditional independence in a BN encoded by the structure of its DAG?

- **Theorem**

  $P(X, Y|E) = P(X|E)\,P(Y|E)$ if and only if every *path* from a node in X to a node in Y is *blocked* by E.

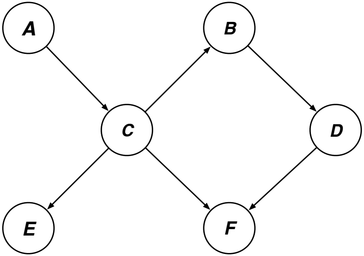
What counts as a path, and when is it blocked?

Paths in DAGs

- **Definition**

  A **path** is any sequence of nodes connected by edges (*regardless of their directionalities*); it is also assumed that no nodes repeat.

- **Examples**



Two paths from A to D:

(1) $A \to C \to B \to D$

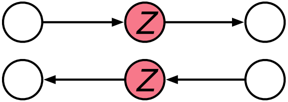
(2) $A \to C \to F \leftarrow D$

Blocked paths

- **Definition**

  A path π is **blocked** by a set of nodes E if there exists a node $Z \in \pi$ for which one of the three following conditions hold.

(1) $Z \in E$

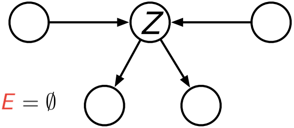


edges **align**

(2) $Z \in E$

edges **diverge**

(3) $Z \notin E$



edges **converge**

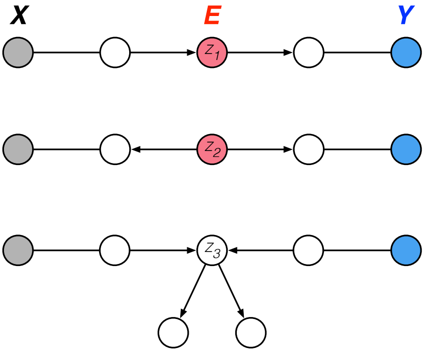
descendants$(Z) \cap E = \emptyset$

d-separation

- **Theorem**

  $P(X, Y|E) = P(X|E)\,P(Y|E)$ if and only if every *path* from a node in X to a node in Y is *blocked* by E.

- **Intuition**



$Z_1 \in E$ is an intervening event in a causal chain

$Z_2 \in E$ is a common explanation or cause

$Z_3 \notin E, \mathrm{desc}(Z_3) \cap E = \emptyset$ is an unobserved common effect

Questions?

d-separation

- **Theorem**

  $P(X, Y|E) = P(X|E)\,P(Y|E)$ if and only if every *path* from a node in X to a node in Y is *blocked* by E.

- **Proof** (not given)

  The proof of the theorem is non-trivial.
  You are **not** responsible for its proof.

- **How useful is the theorem? Very!**

  There are efficient algorithms to test d-separation in large BNs.
  You should become skilled at these tests in simple BNs.

Alarm example

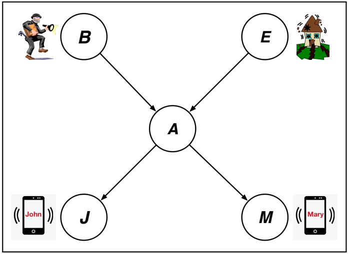
TRUE or FALSE?

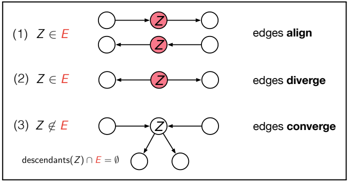
1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.
   There is one path $B \rightarrow A \rightarrow M$.
   Node A satisfies condition (1).
   The statement is **true**.

2. $P(J, M|A) \stackrel{?}{=} P(J|A)\, P(M|A)$

   The evidence is $\{A\}$.
   There is one path $J \leftarrow A \rightarrow M$.
   Node A satisfies condition (2).
   The statement is **true**.





Alarm example (con't)

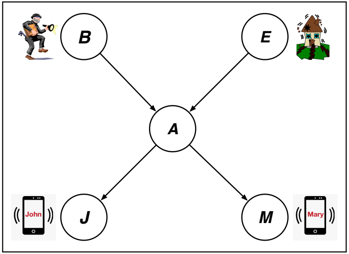
TRUE or FALSE?

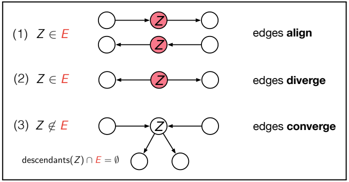
3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Node A satisfies condition (3).
   The statement is **true**.

4. $P(B|M) \stackrel{?}{=} P(B|M, E)$

   The evidence is $\{M\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Note that $M \in \mathrm{desc}(A)$.
   The statement is **false**.





Loopy example

TRUE or FALSE?

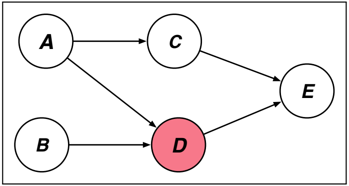
5. $P(B|D,E) \stackrel{?}{=} P(B|D)$

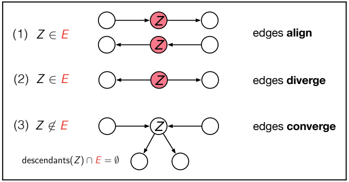
   The evidence is $\{D\}$.
   There are two paths from B to E.

   Path $B \to D \to E$
   is blocked by node D,
   satisfying condition (1).

   Path $B \to D \leftarrow A \to C \to E$
   is not blocked by any node.

   The statement is **false**.

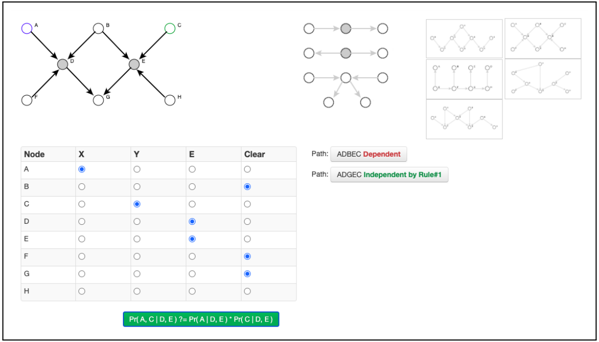




Questions?

More examples

d-separation true/false demo:



`https://tinyurl.com/d-sep-demo`

Outline

1. Review
2. Conditional probability tables (CPTs)
3. d-separation and examples
4. $\boxed{\text{Inference}}$

Inference

- **Problem**

  Given a set E of evidence nodes, and a set Q of query nodes, how to compute the posterior distribution $P(Q|E)$?

- **Special cases of the above**

diagnostic reasoning	(from effects to causes)	$P(B=1\|M=1)$
causal reasoning	(from causes to effects)	$P(M=1\|B=1)$
explaining away	(competing causes)	$P(B=1\|A=1, E=1)$
mixed reasoning	(about past and future)	$P(B=1, M=1\|A=1)$

- **Next lecture**

  When can inference be done efficiently
  (i.e., polynomial time in the size of the belief network)?