Lower Bounds and Separations for Torus Polynomials

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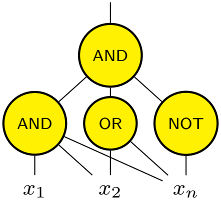

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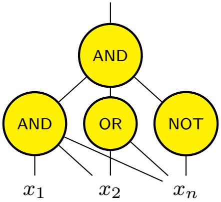

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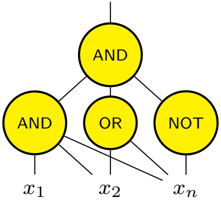

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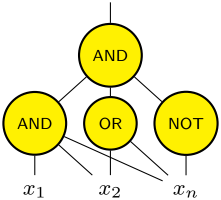

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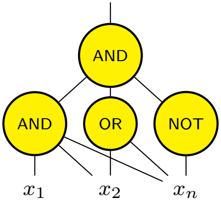

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AC^0: Constant-depth polynomial size with AND, OR, NOT gates.

The Landscape of Circuit Complexity

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$$\begin{tikzcd} \mathsf{NP} & \mathsf{AC}^0[p] \text{ (allow } \mathsf{MOD}_p\text{)} \\ \mathsf{P} \arrow[u] \arrow[r, dashed, no head, "\oplus \text{ (Håstad `86)}"'] & \mathsf{AC}^0 \arrow[u] \end{tikzcd}$$

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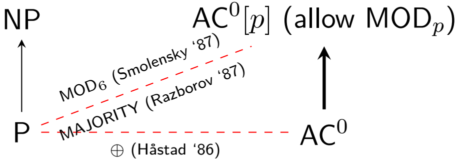

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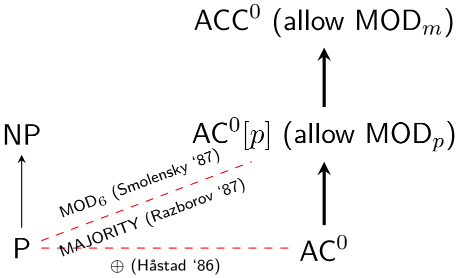

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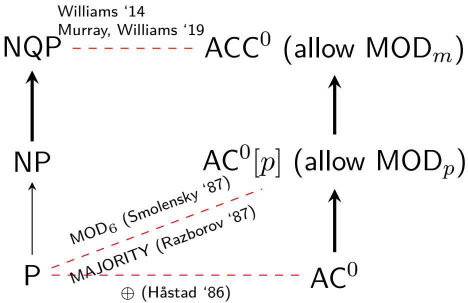

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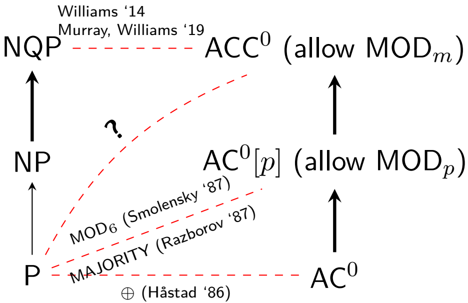

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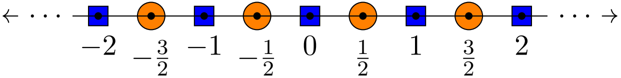

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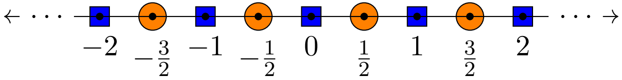

Trivial upper bound: degree n for any f.

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- Lower bound iff programs are infeasible iff duals are feasible.

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- Allows for incremental progress.

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- New nullspace vectors supported on a single Hamming layer.
  - Asymmetric construction, unlike previously known.

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  - Use lattice theory.

Thank you!