When to Learn: Conformal Scores as Online Update Criteria

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Key Elements

We introduce a lightweight and novel framework designed for adaptive model updates in streaming environments: **Conformal Online Learning (COL)**, inspired by conformal prediction but repurposed for learning. We demonstrate the effectiveness of COL through its application to online learning of Koopman linear embeddings for non-linear dynamical systems

Koopman Operator [1]

Instead of directly modeling the nonlinear dynamics $x_{t+1} = Tx_t$, Koopman operator theory considers the evolution of lifted states through a linear operator:

$$\Phi_\theta(x_{t+1}) \approx K\Phi_\theta(x_t).$$

This approach enables the use of linear spectral analysis tools.

Conformal Online Learning of Koopman embeddings

The goal is to incrementally Φ_θ and K from sequentially observed dynamics. At each time step t, the parameters (θ_t, K_t) are updated by minimizing the multi-step prediction loss within $\mathcal{D}_t = \{x_{t-w}, \dots, x_t\}$. The score function at time step t is defined as

$$s_t(x_t, (\theta_t, K_t)) := \sum_{\tau=1}^{w} \|\Phi_{\theta_t}(x_t) - K_t^\tau \Phi_{\theta_t}(x_{t-\tau})\|^2$$

to evaluate the consistency of (θ_t, K_t).

Prediction Score Set

Given a newly observed state x_t and a conformity threshold $q_t > 0$, the *prediction score set* at time t is defined as

$$S_t = s_t(x_t, \mathrm{Param}_t),$$

where $\mathrm{Param}_t = \{(\theta, K) \text{ such that } s = s_t(x_t, (\theta, K)) \le q_t\}$. This set contains all prediction scores attainable at x_t by Koopman models that satisfy the current calibration constraint.

Dynamic threshold by Conformal PI Control [2]

Rather than fixing the quantile threshold q_t in advance, the method updates it online in response to conformity violations. After observing whether $s_t \in S_t$, a binary error signal $e_t = \mathbf{1}\{s_t \notin S_t\}$ is computed, and the threshold q_t is adjusted via:

$$q_{t+1} = q_t + \underbrace{\gamma(e_t - \alpha)}_{\text{Proportional term (P)}} + \underbrace{r_t\left(\sum_{i=1}^{t}(e_i - \alpha)\right)}_{\text{Integral term (I)}}$$

Dynamic Regret Theorem

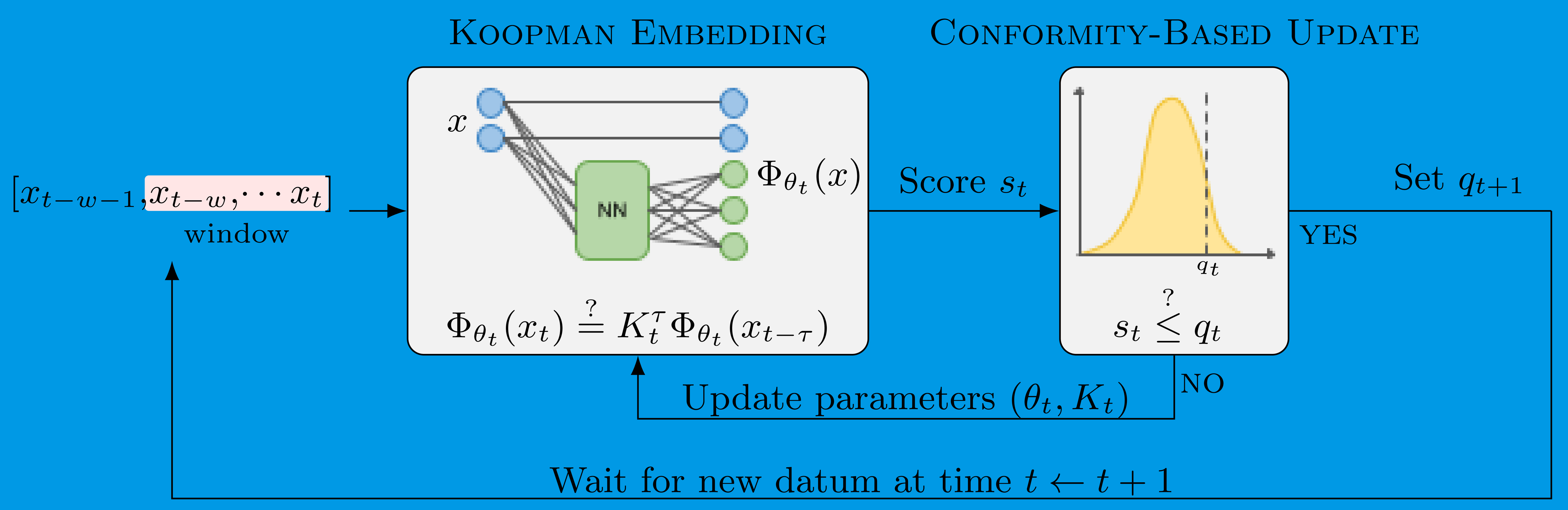
Let (θ_t, K_t) be the parameters produced by COL and let $(\theta_t^*, K_t^*) \in \operatorname{argmin}_{(\theta,K)} \mathcal{L}_t(\theta, K)$ denote any time-dependent optimal model minimizing the loss at step t. Further assume:

- (A1) Each $\mathcal{L}_t$ is L-smooth with $\|\nabla \mathcal{L}_t(\theta, K)\| \le B$;
- (A2) Bounded total variation and squared variation for oracle path: $V_T := \sum_{t=1}^{T} \|(\theta_{t+1}^*, K_{t+1}^*) - (\theta_t^*, K_t^*)\| < \infty$, $S_T := \sum_{t=1}^{T} \|(\theta_{t+1}^*, K_{t+1}^*) - (\theta_t^*, K_t^*)\|^2 < \infty$;
- (A3) $\sum_{t=1}^{T} q_t \le \mathcal{O}(\alpha h(T))$ for some sublinear, nonnegative, nondecreasing function h;

Then the dynamic regret satisfies:

$$\sum_{t=1}^{T} [\mathcal{L}_t(\theta_t, K_t) - \mathcal{L}_t(\theta_t^*, K_t^*)] \le \mathcal{O}(\alpha h(T) + V_T + S_T).$$

Shifting from producing conformal prediction sets to **assessing the model's consistency** on new observations, model updates are triggered only when its conformal score $s_t(\text{model}_t)$ is greater than a dynamic threshold q_t, and continue until $s_t(\text{model}_t) \le q_t$.



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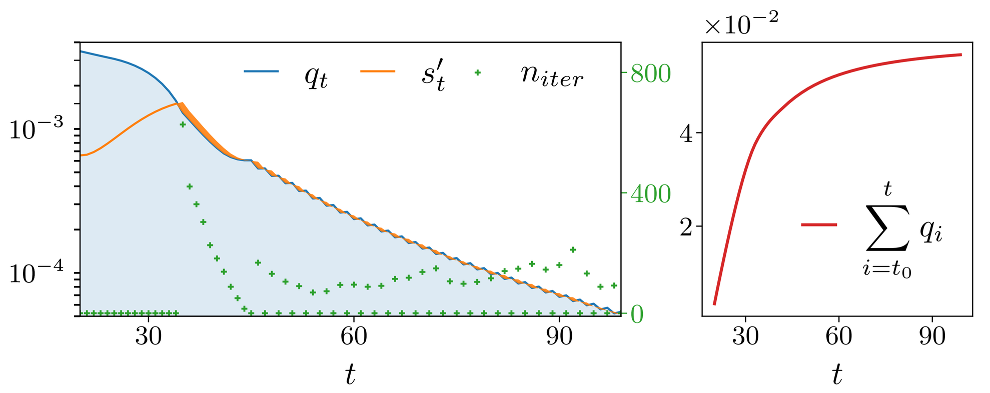
Compare to baselines

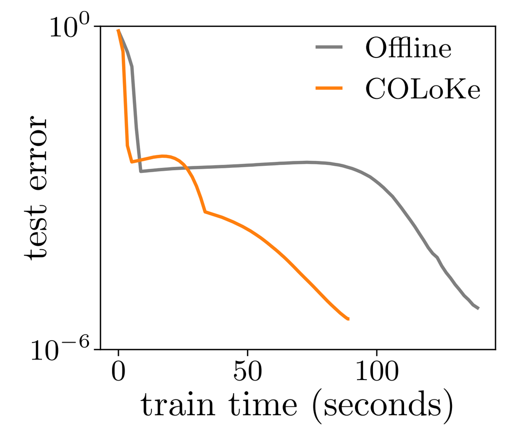
Models are trained using the training trajectories while computing the online prediction error. After the training, models are evaluated on the held-out trajectories to compute generalization error. We report averages across five splits, along with the standard deviation of the averages.

	ODMD	OEDMD	OnlineAE	OLoKe	COLoKe
Single attractor	$1.1 \cdot 10^{-3}$ $(\pm 3.6 \cdot 10^{-5})$ $4.6 \cdot 10^{-5}$ $(\pm 7.3 \cdot 10^{-7})$	$2.5 \cdot 10^{-2}$ $(\pm 2.8 \cdot 10^{-4})$ $1.5 \cdot 10^{-2}$ $(\pm 4.7 \cdot 10^{-4})$	$1.0 \cdot 10^{-2}$ $(\pm 7.7 \cdot 10^{-4})$ $7.4 \cdot 10^{-5}$ $(\pm 2.8 \cdot 10^{-5})$	$2.1 \cdot 10^{-6}$ $(\pm 6.6 \cdot 10^{-7})$ $7.5 \cdot 10^{-6}$ $(\pm 2.5 \cdot 10^{-6})$	$\mathbf{2.4 \cdot 10^{-7}}$ $(\pm \mathbf{3.6 \cdot 10^{-8}})$ $\mathbf{7.6 \cdot 10^{-7}}$ $(\pm \mathbf{9.6 \cdot 10^{-8}})$
Duffing oscillator	$2.5 \cdot 10^{-4}$ $(\pm 7.8 \cdot 10^{-6})$ $1.9 \cdot 10^{-4}$ $(\pm 1.5 \cdot 10^{-6})$	$6.8 \cdot 10^{-3}$ $(\pm 4.5 \cdot 10^{-3})$ $3.8 \cdot 10^{-3}$ $(\pm 3.3 \cdot 10^{-4})$	$8.7 \cdot 10^{-3}$ $(\pm 2.5 \cdot 10^{-3})$ $2.0 \cdot 10^{-3}$ $(\pm 6.2 \cdot 10^{-4})$	$5.5 \cdot 10^{-5}$ $(\pm 1.0 \cdot 10^{-5})$ $2.3 \cdot 10^{-4}$ $(\pm 4.0 \cdot 10^{-5})$	$\mathbf{3.1 \cdot 10^{-6}}$ $(\pm \mathbf{2.3 \cdot 10^{-7}})$ $\mathbf{7.3 \cdot 10^{-5}}$ $(\pm \mathbf{1.9 \cdot 10^{-5}})$
VdP oscillator	$2.1 \cdot 10^{-3}$ $(\pm 3.6 \cdot 10^{-5})$ $1.1 \cdot 10^{-3}$ $(\pm 4.7 \cdot 10^{-6})$	$2.1 \cdot 10^{-3}$ $(\pm 3.2 \cdot 10^{-5})$ $1.1 \cdot 10^{-3}$ $(\pm 7.8 \cdot 10^{-6})$	$1.7 \cdot 10^{-2}$ $(\pm 3.0 \cdot 10^{-3})$ $3.8 \cdot 10^{-3}$ $(\pm 1.0 \cdot 10^{-3})$	$6.6 \cdot 10^{-4}$ $(\pm 1.5 \cdot 10^{-4})$ $9.2 \cdot 10^{-4}$ $(\pm 3.0 \cdot 10^{-4})$	$\mathbf{3.8 \cdot 10^{-4}}$ $(\pm \mathbf{1.2 \cdot 10^{-5}})$ $\mathbf{6.0 \cdot 10^{-4}}$ $(\pm \mathbf{1.4 \cdot 10^{-4}})$
Lorenz system	$2.7 \cdot 10^{-1}$ $(\pm 1.3 \cdot 10^{-3})$ $1.0 \cdot 10^{-1}$ $(\pm 5.8 \cdot 10^{-4})$	$5.5 \cdot 10^{-1}$ $(\pm 2.2 \cdot 10^{-2})$ $2.7 \cdot 10^{-1}$ $(\pm 3.1 \cdot 10^{-2})$	$5.9 \cdot 10^{-1}$ $(\pm 8.4 \cdot 10^{-2})$ $3.8 \cdot 10^{-2}$ $(\pm 2.6 \cdot 10^{-3})$	$7.6 \cdot 10^{-3}$ $(\pm 1.8 \cdot 10^{-4})$ $4.7 \cdot 10^{-3}$ $(\pm 3.0 \cdot 10^{-4})$	$\mathbf{6.5 \cdot 10^{-3}}$ $(\pm \mathbf{1.0 \cdot 10^{-4}})$ $\mathbf{3.3 \cdot 10^{-3}}$ $(\pm \mathbf{1.1 \cdot 10^{-4}})$
ETD $\star$	$\mathbf{1.4 \cdot 10^{-1}}$ $(\pm \mathbf{2.9 \cdot 10^{-1}})$ $1.2 \cdot 10^{-1}$ $(\pm 2.9 \cdot 10^{-1})$	$2.7 \cdot 10^{-1}$ $(\pm 3.0 \cdot 10^{-1})$ $1.5 \cdot 10^{-1}$ $(\pm 2.6 \cdot 10^{-1})$	$2.1 \cdot 10^{-1}$ $(\pm 3.4 \cdot 10^{-1})$ $7.9 \cdot 10^{-2}$ $(\pm 7.3 \cdot 10^{-2})$	$2.1 \cdot 10^{-1}$ $(\pm 3.4 \cdot 10^{-2})$ $9.7 \cdot 10^{-2}$ $(\pm 8.5 \cdot 10^{-2})$	$2.1 \cdot 10^{-1}$ $(\pm 8.6 \cdot 10{-}2)$ $\mathbf{7.3 \cdot 10^{-2}}$ $(\pm \mathbf{6.3 \cdot 10^{-2}})$

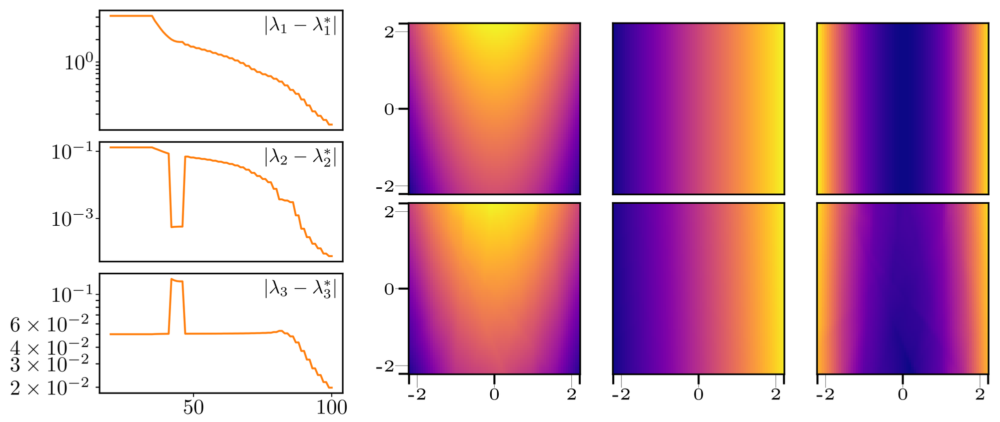
Adaptive update behavior



Learning efficiency



Convergence of eigen-properties



Acknowledgements

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References

1. S. L. Brunton, M. Budišić, E. Kaiser, and J. N. Kutz, "Modern Koopman Theory for Dynamical Systems," *SIAM Review*, vol. 64, no. 2, pp. 229–340, 2022.
2. A. Angelopoulos, E. Candès, and R. J. Tibshirani, "Conformal PID Control for Time Series Prediction," *Advances in Neural Information Processing Systems*, vol. 36, pp. 23047–23074, 2023.