Simpler Approximations for the Network Steiner Tree Problem

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SCS Senior Thesis

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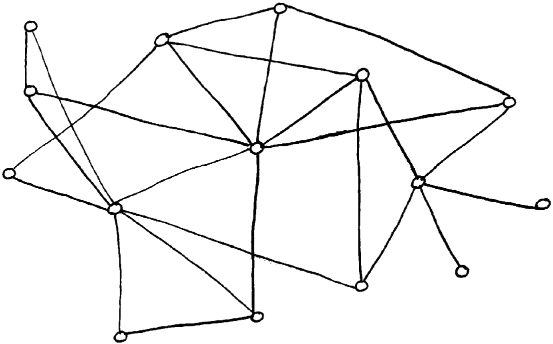
Overview

- Introduce the Steiner Tree Problem
- Describe previous results
- Introduce Submodular Function Maximization
- Show how we used this to simplify previous approximation algorithms

The Network Steiner Tree Problem

- Fix a graph $G_{\text{full}} = (V_{\text{full}}, E_{\text{full}})$



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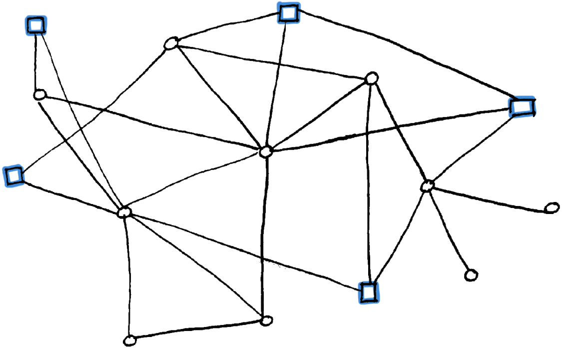

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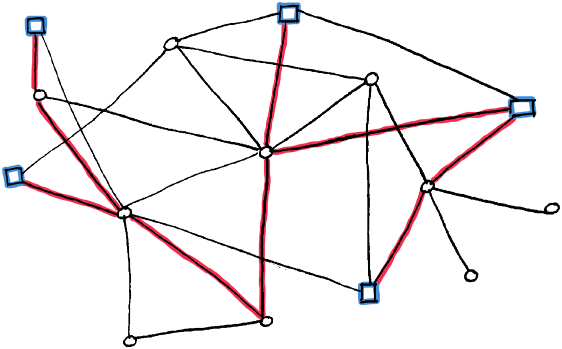

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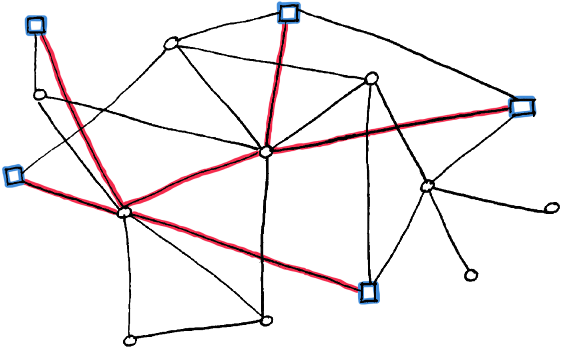

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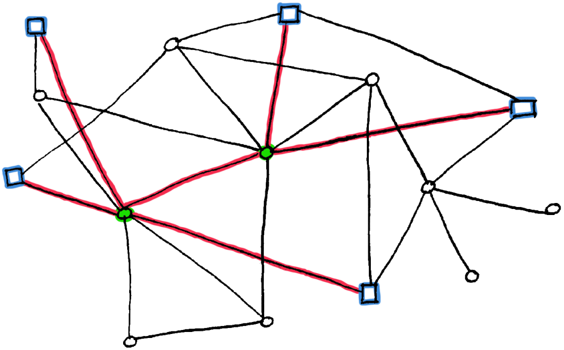
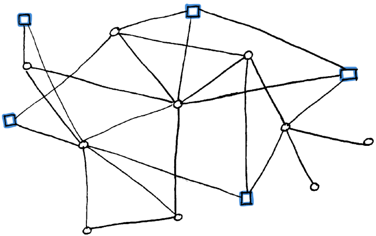
- Non-terminals in a Steiner Tree are **Steiner nodes** (green)

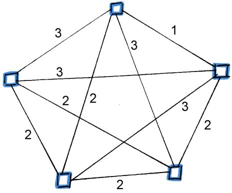


The Shortest Path Graph

In general, this is NP-hard, so we look for polynomial-time approximations.

We can associate $G_{\text{full}} = (V_{\text{full}}, E_{\text{full}})$ with $G = (V, E)$, the shortest path graph on just the terminals.





MST of Shortest Path Graph is a 2-Approximation

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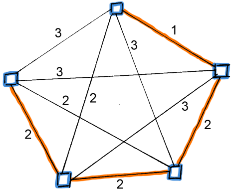

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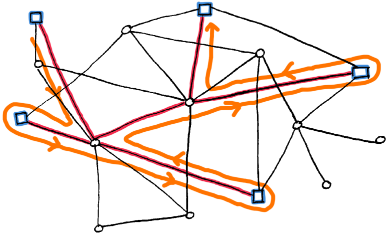

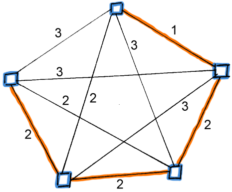

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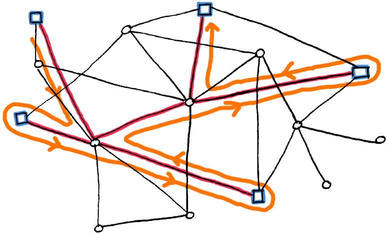
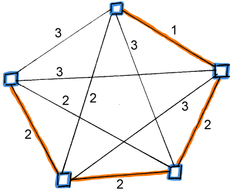
- Denote the MST of G as $\text{MST}(G)$. Denote the cost of $\text{MST}(G)$ as $\text{mst}(G)$.





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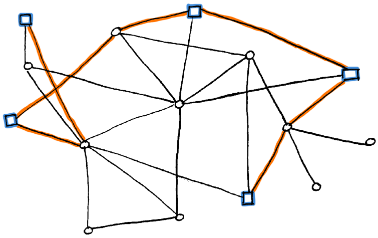
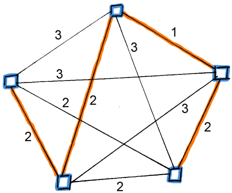
Where do we lose when we use the MST?





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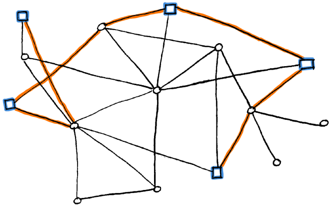
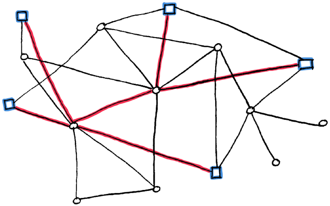
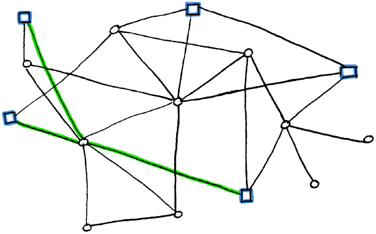
We don't reuse the Steiner nodes effectively, so we redo work.

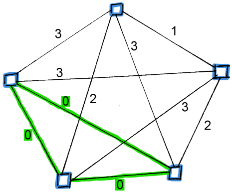




Contracting Components

- We contract a component K by sinking the cost for K, and setting the connections within K to 0 in G.





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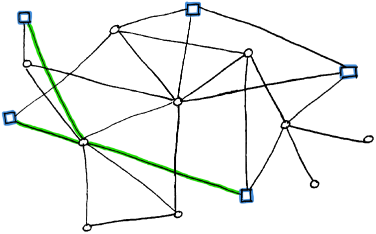
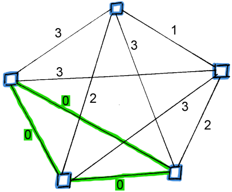
- We denote this $G[K]$.





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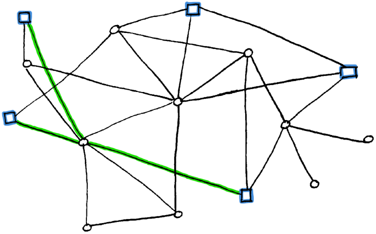
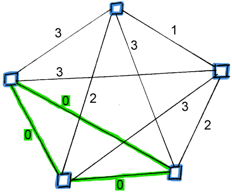
- Define: $\text{save}_G(K) = \text{mst}(G) - \text{mst}(G[K])$.





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In the analysis, Zelikovsky proves that:

- A greedy choice of triples does well compared to the optimal choice of triples.
- There is a set of triples that we can contract that approximates the best Steiner Tree well.

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Key Observation [Cha18]

The save_G function that we are optimizing is **submodular**.

Submodular Functions

For a universe U, a function f over subsets of U

$$f : \mathcal{P}(U) \to \mathbb{R}$$

is **submodular**, if for any $A \subseteq B \subseteq U$, and new element $e \notin B$

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The **marginal value** of adding e to a subset is larger.

An Example of a Submodular Function

- Task: build cell towers

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- Universe (U): potential tower locations
- Value (f): sum of data connection speeds to each house

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Consider the marginal value of e (blue) to set A (solid towers).

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Consider the marginal value of e (blue) to set B (solid towers), a superset of A.

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- Question: Given a budget, what is the highest value set we can build?
- This is an example of a **knapsack constraint**.

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- Small edit allows us to increase cost of S, in exchange for increased value.

Our Results

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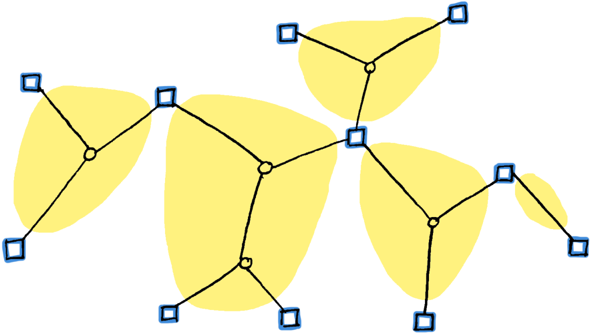

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Improving this idea for [Zel93]:

- Tuning cost v. benefit in submodular maximization gives us an approximation factor

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Based on modified contraction idea from Robins and Zelikovsky [RZ05]:

- Approximation factor:

$$1 + \frac{\ln 3}{2} + \epsilon \approx 1.693.$$

- This matches their bound.

Takeaways

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- In the future we could look at whether this approach can simplify other approximations for this problem, based on linear programming and other techniques.

References

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