New Tools for Smoothed Analysis:

Least Singular Value Bounds for Random Matrices with Dependent Entries

Aditya Bhaskara
University of Utah

Eric Evert
Northwestern University
→ University of Florida

Vaidehi Srinivas
Northwestern University

Aravindan Vijayaraghavan
Northwestern University

PERTURBED MATRICES

Underlying base variables: $\vec{x} = (x_1, \dots, x_n)$

$$M = \begin{array}{cc} & \xleftrightarrow{\qquad m_2 \qquad} \\ m_1 \updownarrow & \begin{bmatrix} p_{11}(\vec{x}) & p_{21}(\vec{x}) & & \\ p_{12}(\vec{x}) & & & \\ & & \ddots & \\ & & & p_{m_1 m_2}(\vec{x}) \end{bmatrix} \end{array}$$

matrix of polynomial entries in $\vec{x}$

"perturbation"

Draw values for each variable: $x_i = v_i + \eta_i \leftarrow \mathcal{N}(0, \rho^2)$

arbitrary

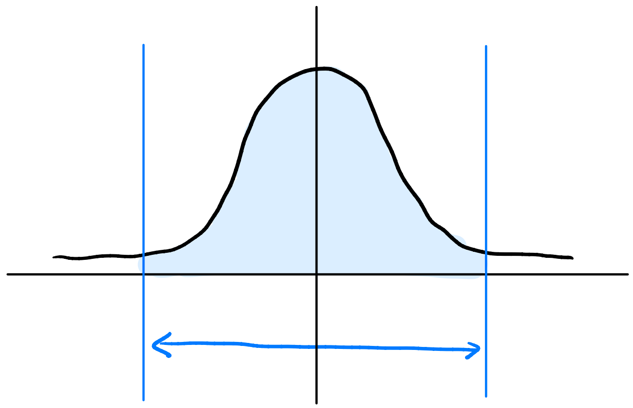
WANT: M is robustly full-rank $\iff \sigma_{min}(M) \geq \mathrm{poly}\left(\frac{1}{n}, \rho\right)$

CHALLENGE: entries of M are highly dependent!

MATRIX ANTI-CONCENTRATION

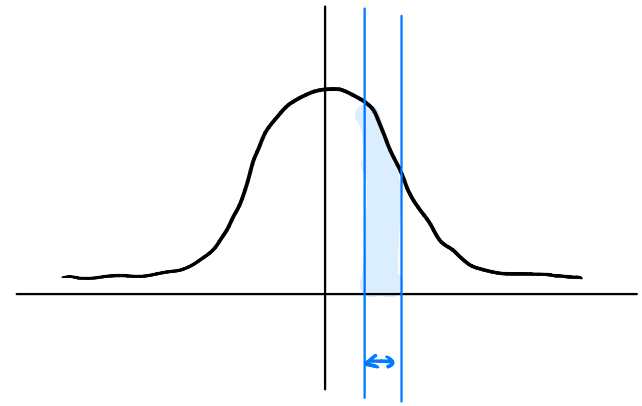
1 dimension:

Concentration/Large deviation:



"likely to fall in this window"

Anti-concentration:



"unlikely to fall in <u>any</u> <u>fixed</u> small window"

Matrices:

Concentration/Large deviation: Action of M unlikely to be too far from the 0 matrix.

$$\Longleftrightarrow$$

$$\sigma_{max}(M) \leq \mathrm{poly}(n, \rho)$$

Anti-concentration: Action of M unlikely to be close to any fixed matrix M'

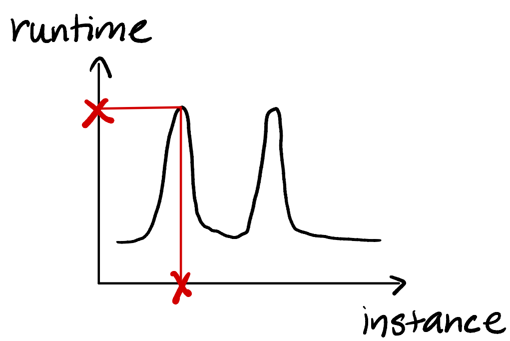
$$\Longleftrightarrow$$

$$\sigma_{min}\left(M - M'\right) \geq \mathrm{poly}\left(\frac{1}{n}, \rho\right)$$

SMOOTHED ANALYSIS

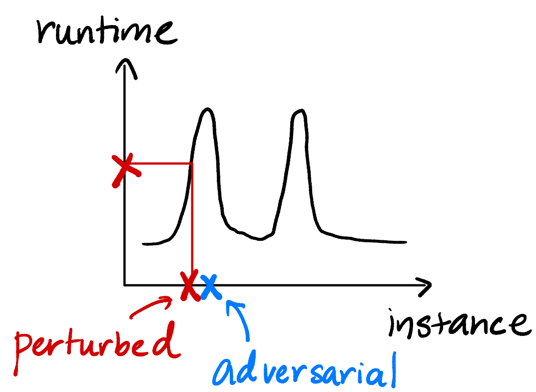
worst-case analysis

analyze algorithm on adversarially chosen instance



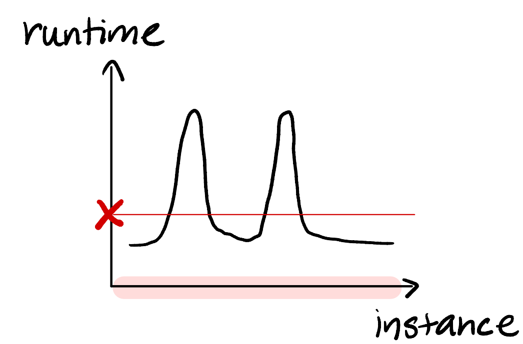
smoothed analysis

analyze algorithm on perturbed instance



average-case analysis

analyze algorithm on instance drawn from fixed distribution



▷ Matrix anti-concentration helps us argue that "in a small ball around <u>any</u> instance, most instances are well-behaved"

APPLICATION 1: SUBSPACE CLUSTERING

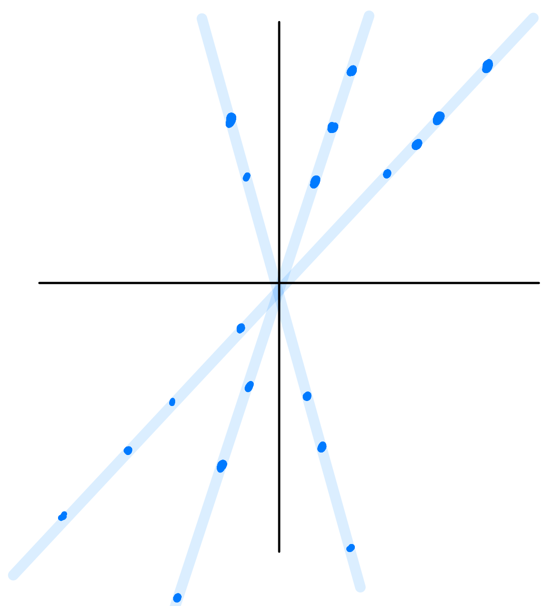
Collection of s hidden subspaces (perturbed) $W_1, \dots, W_s$ $\qquad W_i \in \mathbb{R}^{n \times t}$

See points $A = A_1 \cup \cdots \cup A_s \subseteq \mathbb{R}^n$ $\qquad A_i \subseteq W_i$

"sufficiently perturbed points"

↳ smoothed analysis

GOAL: Recover $W_1, \dots, W_s$ from A.



ALGORITHM: Robust Vector Space Decomposition (RVSD)

[Garg Kayal Saha '20, Chandra Garg Kayal Mittal Saha '24]

Works when subspaces are mutually independent

⚠ immediately fails if $s \cdot t > n$!

FIX: Consider $\{ a^{\otimes d} \mid a \in A \}$ instead.

For $a \in W_i$, $\quad a^{\otimes d} \in W_i^{\otimes d} \in \mathbb{R}^{\approx n^d \times t^d}$

Sanity check: choose d so $s \cdot t^d \leq n^d$

RVSD works as long as $W_i^{\otimes d}$ "look random"

PSEUDO RANDOM SUBSPACES

What does it mean for $w_i^{\otimes d}$ to "look random"?

NEW MATRIX PRODUCT UNLOCKED!

Symmetric Lift

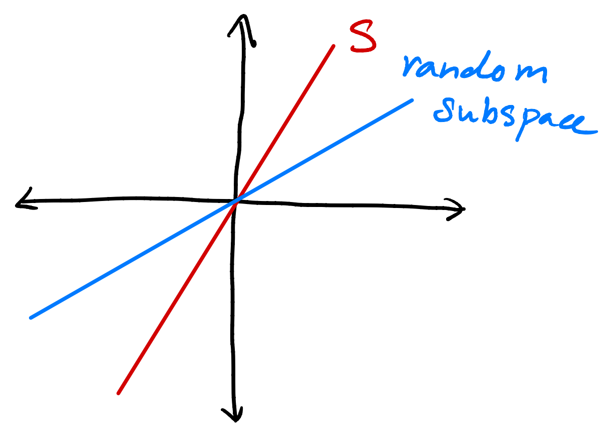
$$\boxed{V^{\otimes d} = \left\{ \mathrm{Sym}\left(v_{j_1} \otimes \cdots \otimes v_{j_d}\right) \;\middle|\; j_1 \le \cdots \le j_d \right\}}$$

- If $\tilde{V}$ is m-dimensional subspace of $\mathbb{R}^n$,

  $\tilde{V}^{\otimes d}$ is an $\binom{m+d+1}{d} \approx m^d$ dimensional subspace of $\mathbb{R}^{\binom{n+d+1}{d} \approx n^d}$

  the symmetric subspace

- Fix an arbitrary low-dimensional subspace S.



- A random low-dimensional subspace is unlikely to intersect S. $\approx m^d \times n^d$ degrees of randomness

WANT: a perturbed structured low-dimensional subspace is unlikely to intersect S. $\approx mn$ degrees of randomness

SAMPLE THEOREM 1 : SYMMETRIC LIFTS

Theorem [Bhaskara Evert S. Vijayaraghavan '24]

Let Φ be a projection matrix of rank $\delta \binom{n+d-1}{d}^{\approx n^d}$ for $\delta > 0$,

$U \in \mathbb{R}^{n \times m}$ is an arbitrary matrix, $\tilde{U} = U + \left(\mathcal{N}(0, \rho^2)\right)^{n \times m}$,

then for $m < cn$ for constant c

$$\sigma_{\min}\left(\Phi \, \tilde{U}^{\otimes d}\right) \geq \mathrm{poly}\left(\rho, \tfrac{1}{n}\right)$$

with probability $1 - \exp(-\Omega(n))$.

" $\tilde{U}^{\otimes d}$ is unlikely to intersect the kernel of Φ. "

" Φ is likely invertible over $\tilde{U}^{\otimes d}$. "

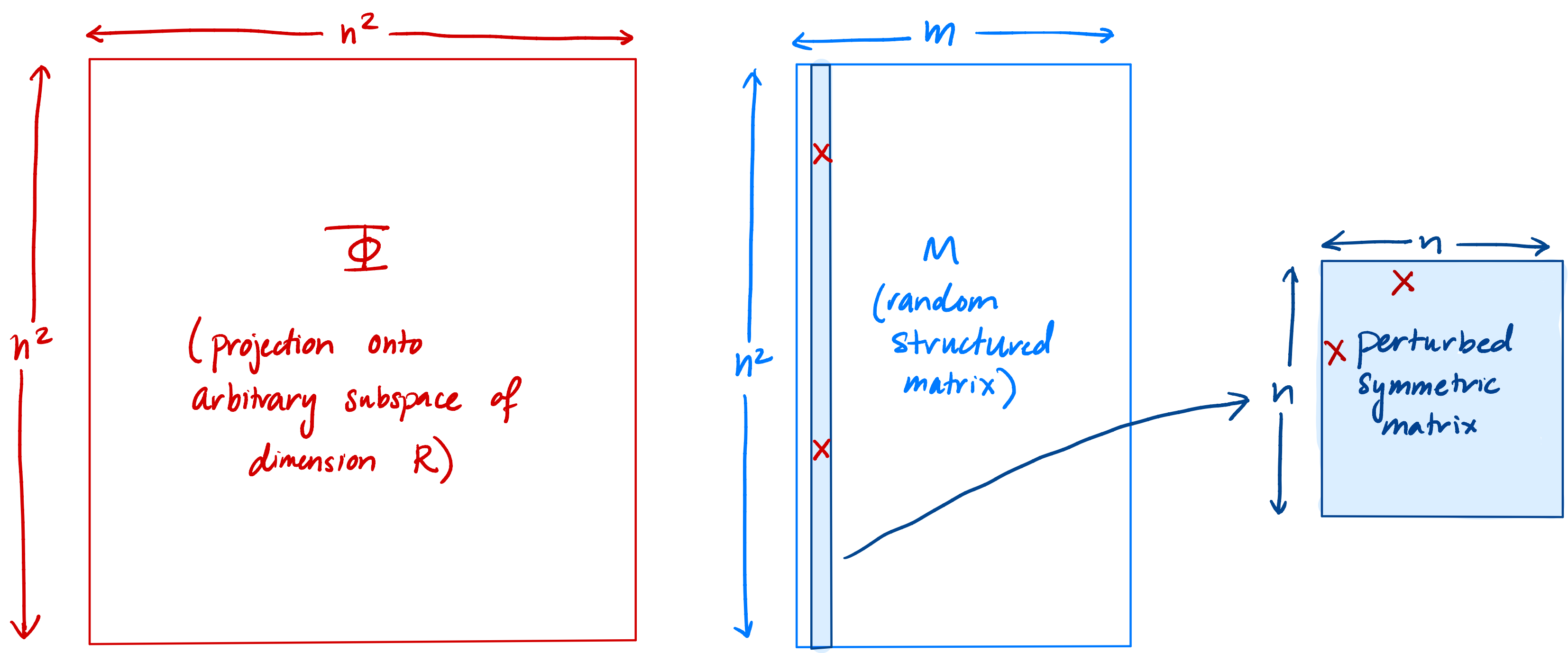
▷ Previous work focuses on matrices where columns have independent randomness

[Bhaskara Charikar Moitra Vijayaraghavan '14], [Anari Daskalakis Maass Papadimitriou Saberi Vempala '18], [Bhaskara Chen Perreault Vijayaraghavan '19]

POTENTIAL PITFALL: LOW-DIMENSIONAL STRUCTURE

WANT: ΦM is robustly rank m

STRUCTURE: M has perturbed symmetric columns



R is δn^2 for constant $\delta > 0$.

PROBLEM: Symmetric matrices form a subspace

Φ could project onto the anti-symmetric space

HIGHER-ORDER KRONECKER PRODUCTS

WANT: ΦM is robustly rank m^d

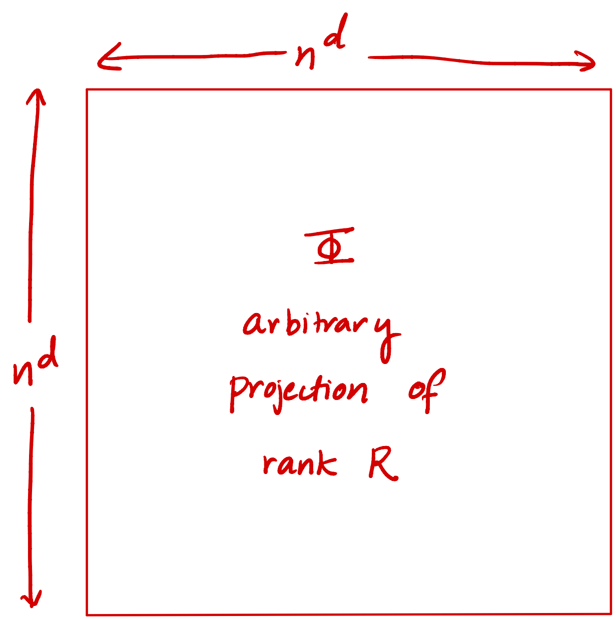
STRUCTURE: M is a Kronecker product of perturbed matrices

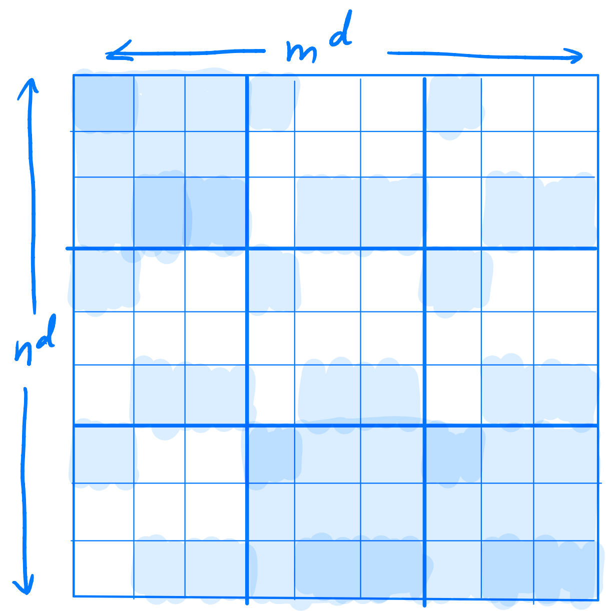
decoupling step

$\tilde{U}^{(1)} \otimes \tilde{U}^{(2)} \otimes \cdots \otimes \tilde{U}^{(d)}$

$\tilde{U}^{(i)} \in \mathbb{R}^{n \times m}$

$m < c_d \cdot n$





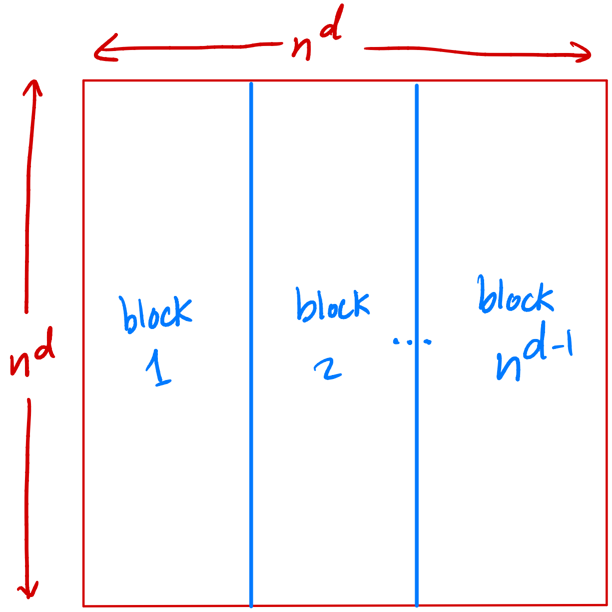
dnm "degrees of randomness"

Kronecker product

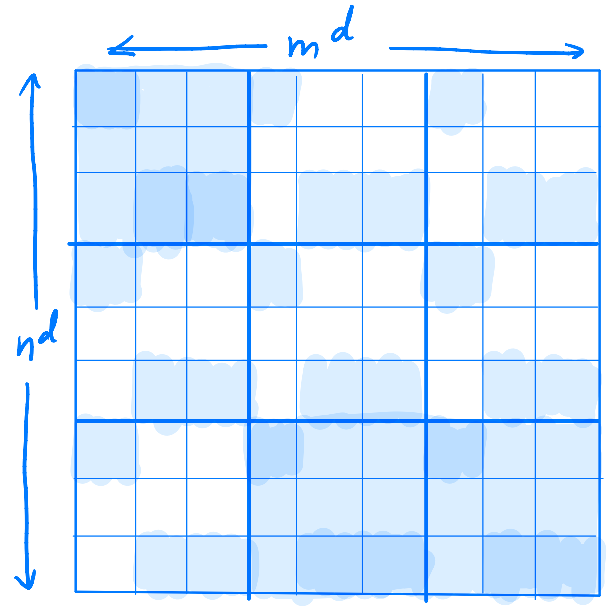
$$\left\{ \tilde{u}^{(1)}_{i_1} \otimes \cdots \otimes \tilde{u}^{(d)}_{i_d} \;\middle|\; i_1, \ldots, i_d \in [n] \right\}$$

NEED TO ARGUE: Structure of M cannot line up badly with structure of Φ.

KEY STEP



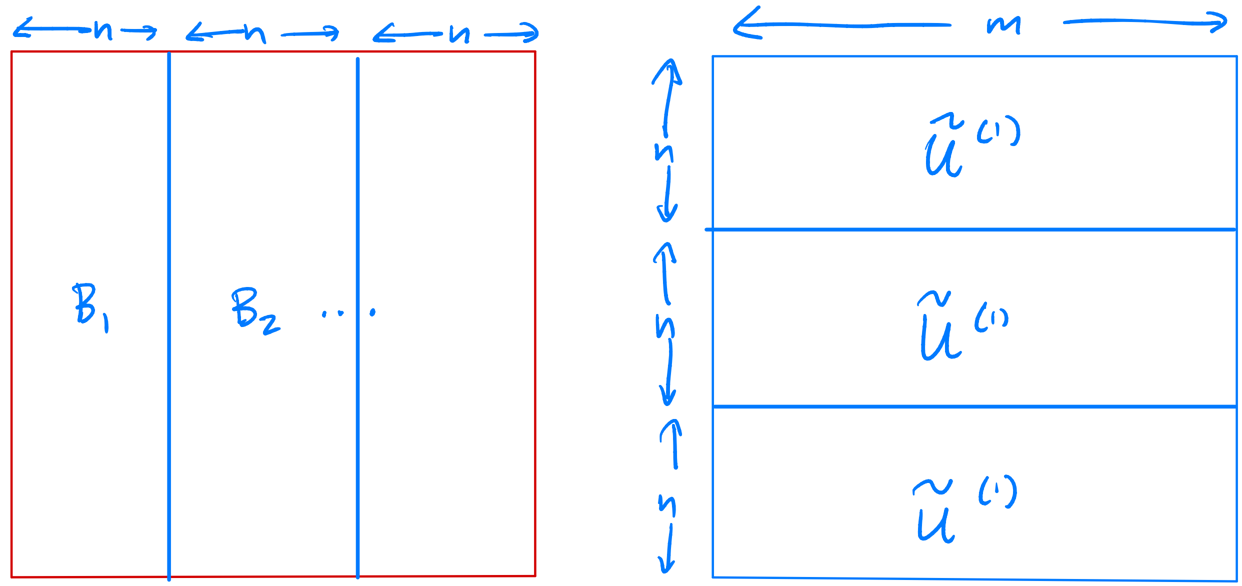
Φ: arbitrary projection of rank R



perturbed matrix with Kronecker structure

Apply induction on blocks with $\tilde{u}^{(2)} \otimes \cdots \otimes \tilde{u}^{(d)}$

KEY STEP



Φ: arbitrary projection of rank R

perturbed matrix with Kronecker Structure

- Prove structural property of high-rank Φ:

  Φ has many blocks that are "sufficiently independent" of each other

  "good blocks"

- Product of good blocks with $\tilde{u}^{(1)}$ display sufficient anti-concentration

SAMPLE THEOREM 2: KHATRI-RAO PRODUCTS

NEW MATRIX PRODUCT UNLOCKED!

Khatri-Rao Product

$$V^{\odot d} = \{ v_i^{\otimes d} \mid i \in [n] \}$$

Theorem [Bhaskara Evert S. Vijayaraghavan '24]

Let $U \in \mathbb{R}^{n \times m}$ be an arbitrary matrix, $\tilde{U} = U + (\mathcal{N}(0, \rho^2))^{n \times m}$

then for $m < cdn$ for constant c, we have

$$\sigma_{\min}\left(\tilde{U}^{\odot d}\right) \geq \mathrm{poly}\left(\rho, \frac{1}{n}\right)$$

with probability $1 - \exp(-\Omega(n))$.

- Alternate proof of results from [Bhaskara Charikar Moitra Vijayaraghavan '14], [Anari Daskalakis Maass Papadimitriou Saberi Vempala '18]
- Proof extends to more general matrices of low-degree polynomials

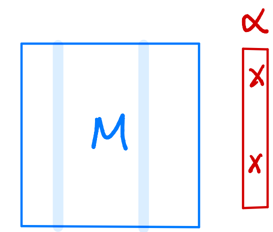
TECHNIQUE 2: HIERARCHICAL NETS

HAVE: ρ-perturbed matrix M

WANT: $\|M\alpha\| \geq \mathrm{poly}\left(\frac{1}{n}, \rho\right)$ for all unit vectors α

ATTEMPTS

① ε-net over all vectors α



☹ If α is sparse, $M\alpha$ has "insufficient randomness" for union bound

OBSERVATION: Matrices of interest anti-concentrated over dense vectors

"Combination amplifies anticoncentration" (CAA) property:

For any α with k entries of magnitude $\geq \delta$,
$\|M\alpha\| \geq \Omega(\delta)$, with probability $1 - \exp(-\omega(k))$.

② Hierarchy of nets based on sparsity of α

▷ Related to previous techniques such as [Rudelson Vershynin '08], but we require a more fine-grained notion of sparsity

APPLICATION 1: SUBSPACE CLUSTERING

Collection of s hidden subspaces $W_1, \dots, W_s$ $\quad W_i \in \mathbb{R}^{n \times t}$

See points $A = A_1 \cup \cdots \cup A_s \subseteq \mathbb{R}^n$ $\quad A_i \subseteq W_i$

"sufficiently perturbed points"

↳ smoothed analysis

GOAL: Recover $W_1, \dots, W_s$ from A.

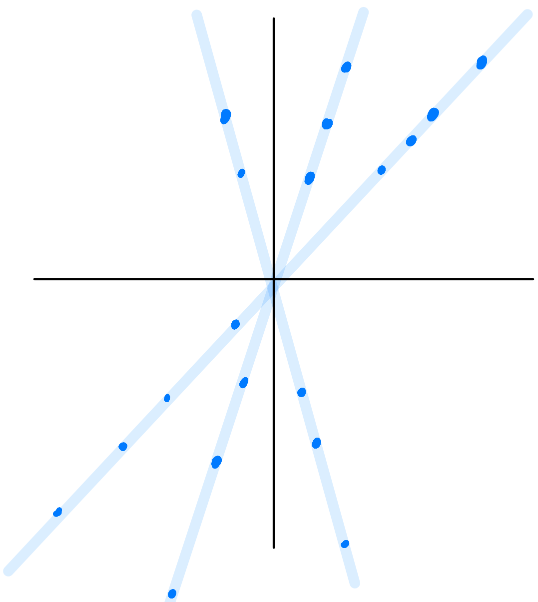
ALGORITHM: Robust Vector Space Decomposition (RVSD)

[Garg Kayal Saha '20, Chandra Garg Kayal Mittal Sinha '24]

Works when subspaces are mutually independent

⚠ immediately fails if $s \cdot t > n$!

▷ Conjectured smoothed analysis guarantee



NEW! [Bhaskara Evert S. Vijayaraghavan '24]

The algorithm of [CGKMS '24] for subspace clustering admits a smoothed analysis guarantee.

APPLICATION 2 : RANK-1 MATRIX IN A SUBSPACE

GIVEN: ρ-perturbed subspace $\tilde{\mathcal{U}}$ of matrices

WANT: certify every rank-1 matrix is "far" from $\tilde{\mathcal{U}}$

ALGORITHM: [Johnston Lovitz Vijayaraghavan '23] (Guarantee for generic instances)

▷ rank-1 matrices form a low-degree algebraic variety $\mathcal{X}$

↳ set of common zeroes of collection of polynomials

$\tilde{\mathcal{U}}$ is far from rank-1 matrices when:

$$\sigma_{min}\left(\Phi^d_{\mathcal{X}} \quad \tilde{\mathcal{U}}^{\otimes d} \right) \geq poly\left(\frac{1}{n}, \rho\right).$$

↳ based on Hilbert's projective nullstellensatz certificates for variety $\mathcal{X}$

NEW! [Bhaskara Evert S. Vijayaraghavan '24]

Algorithm of [JLV '23] for certifying distance between subspace and variety admits a smoothed analysis guarantee.

APPLICATION 3: POWER-SUM DECOMPOSITIONS OF POLYNOMIALS

GIVEN $p(x) = \sum_{t \in [m]}^{T} a_t(x)^3 + e(x)$

$a_t(x)$ are perturbed homogeneous deg. 2

WANT: recover $\{ a_t(x) \mid t \in [m] \}$.

ALGORITHM: [Bafna Hsieh Kothari Xu '22]: (Guarantee for random instances)

- If we had all intermediate "FOIL" terms of $\sum_t a_t(x)^3$, can recover $a_t(x)$ using tensor decomposition

  ⚠ combining like terms "symmetrizes" the input $x_1 x_2 x_3 = x_2 x_3 x_1$

- Use structure of polynomial to infer subspace of $\{a_t(x)\}$

  $\Rightarrow a_t(x)^3$ live in Kronecker space of $\{a_t(x)\}$, symmetrization invertible!

NEW! [Bhaskara Evert S. Vijayaraghavan '24]

Algorithm of [BHKX '22] for decomposition of sums of cubes of quadratic polynomials admits a smoothed analysis guarantee.

CONCLUSION

- We make progress on understanding anti-concentration properties of perturbed matrices with dependent columns
- Results imply smoothed analysis guarantees for a variety of algorithms
- Need to prove projection Φ is invertible over perturbed structured M? We may be able to help!

THANKS!

vaidehi@u.northwestern.edu

arxiv.org/abs/2405.01517