NEW TOOLS FOR SMOOTHED ANALYSIS: Least Singular Value Bounds for Random Matrices with Dependent Entries

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STOC '24

PERTURBED MATRICES

Underlying base variables $\vec{x} = (x_1, \dots, x_n)$

$$M = \begin{bmatrix} P_{11}(\vec{x}) & P_{12}(\vec{x}) & \\ P_{21}(\vec{x}) & \ddots & \\ & & P_{m_1 m_2}(\vec{x}) \end{bmatrix}$$

matrix of polynomial entries in $\vec{x}$

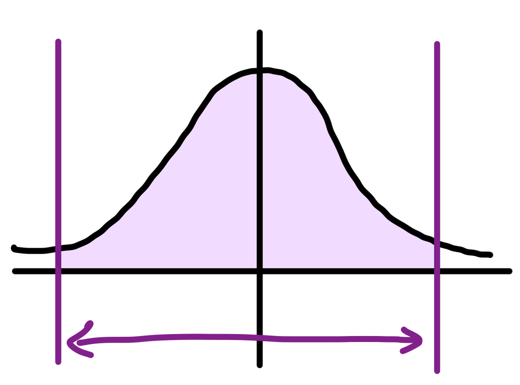
Draw values: $x_i = \underbrace{v_i}_{\text{arbitrary}} + \underbrace{\eta_i}_{\text{"perturbation"} \sim \mathcal{N}(0,\rho^2)}$

> WANT TO SHOW: M is robustly full-rank
>
> CHALLENGE: Entries of M are highly dependent!

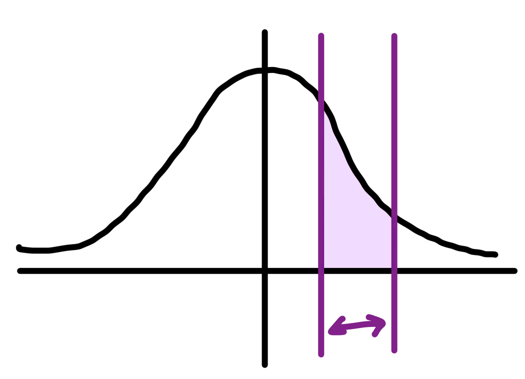
MATRIX ANTI-CONCENTRATION

Concentration | **Anti-concentration**

1D



"likely to fall in this big window"



"unlikely to fall in any fixed small window"

matrix

"action of M unlikely to be too far from 0 matrix"
$\Longleftrightarrow$
$\sigma_{max}(M) \le poly(n, \rho)$ with high probability

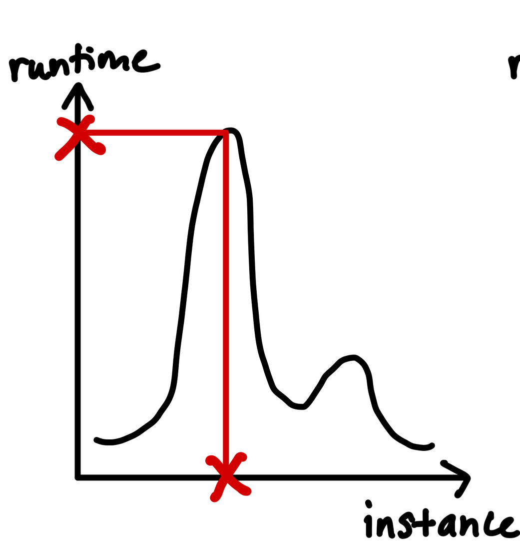
"action of M unlikely to be too close to 0 matrix"
$\Longleftrightarrow$
$\sigma_{min}(M) \ge poly(\frac{1}{n}, \rho)$ with high probability

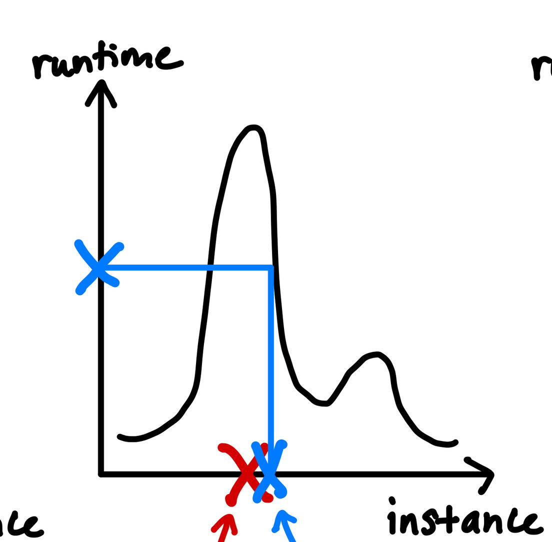
SMOOTHED ANALYSIS

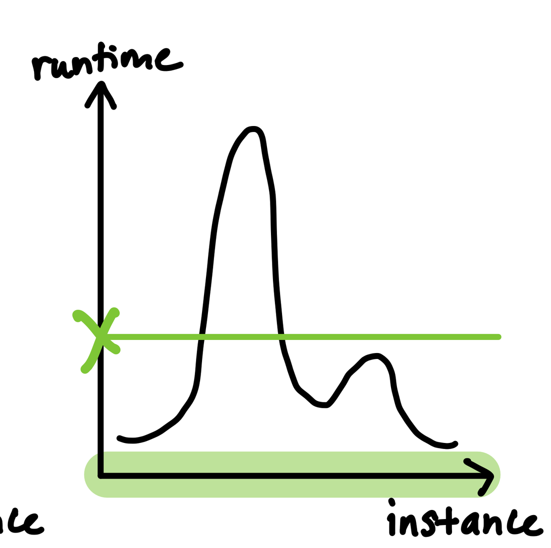
Worst-case analysis: analyze algorithm on adversarially chosen instance

Smoothed analysis: analyze algorithm on perturbed instance

Average-case analysis: analyze algorithm on instance drawn from fixed distribution







CONTRIBUTIONS

▷ 2 new approaches for proving anti-concentration/least singular value bounds for structured random matrices

1. New hierarchical net construction suited to anti-concentration
2. Inductive argument to analyze "smoothed modal contractions" of an arbitrary tensor

▷ New results and alternate proofs of known results

▷ Resolve conjectured smoothed analysis bounds for problems including

- ▷ Subspace clustering
  [Garg Kayal Saha, FOCS '20]
  [Chandra Garg Kayal Mittal Sinha, ITCS '24]
- ▷ Computing linear sections of varieties
  [Johnston Lovitz Vijayaraghavan, FOCS '23]
- ▷ Power-sum decompositions of polynomials
  [Bafna Hsieh Kothari Xu, FOCS '22]

SAMPLE THEOREM

The symmetric lift of a subspace V is

$$V^{\otimes d} = span\{a^{\otimes d} : a \in V\}$$

where $a^{\otimes d}$ is the outer product of a with itself d times, taken as a vector

Informal Theorem [BESV '24]

Let Φ be a projection of sufficiently high rank,
$\tilde{\mathcal{U}}$ be an $n \times m$ matrix (for $n > m$) perturbed by magnitude ρ,
and d be a constant,
then

$$\sigma_{min}(\Phi\, \tilde{\mathcal{U}}^{\otimes d}) \ge poly(\tfrac{1}{n}, \rho)$$

with high probability.

"$\tilde{\mathcal{U}}^{\otimes d}$ is unlikely to nontrivially intersect the kernel of Φ."

"Φ is likely invertible over the subspace $\tilde{\mathcal{U}}^{\otimes d}$."