ME 221: Fluid Mechanics II

Instructor

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Section1: Viscous Flows

Lecture 1: Recap of Governing Equations

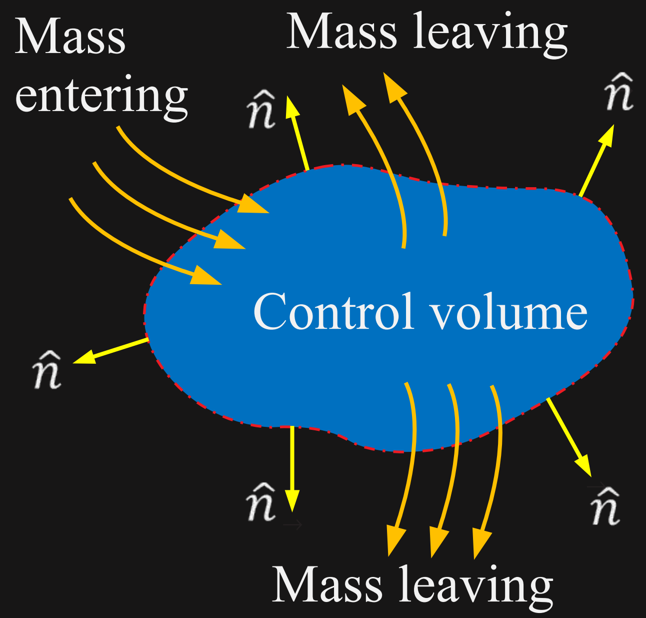
Reynolds Transport theorem

- Let B represent any **extensive property** (such as mass, energy, or momentum).
- Let $\beta = dB/dm$ represent the corresponding **intensive property** (the amount of B per unit mass in any small element of the fluid).
- Rate of change of the property B of the system = Rate of change of B of the control volume + Rate at which B is exiting the control volume by mass crossing the control surface.
- The final form of RTT for an arbitrary fixed CV is

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\left(\int_{CV} \rho\beta\, d\boldsymbol{v}\right) + \int_{CS} \rho\beta\vec{V}\cdot\hat{n}\, dA$$

- The RTT for a Moving and/or deforming control volume will be

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\left(\int_{CV} \rho\beta\, d\boldsymbol{v}\right) + \int_{CS} \rho\beta\vec{V}_r\cdot\hat{n}\, dA$$



Conservation of Mass

- **Conservation of Mass**: The system is a fixed quantity of mass. Thus the mass of a system is conserved (it does not change with time).

$$m_{sys} = \text{constant} \Rightarrow \frac{dm_{sys}}{dt} = 0$$

$$\frac{d}{dt}\left(\int_{CV} \rho \, d\boldsymbol{V}\right) + \int_{CS} \rho \vec{V}_r \cdot \hat{n} \, dA = 0$$

- **Incompressible Flow**: Density variations are negligible i.e. $\rho = \text{constant}$.

$$\int_{CS} \vec{V} \cdot \hat{n} \, dA = 0$$

- If the control volume has only a number of one-dimensional inlets and outlets and the flow is steady then

$$\sum_i (\rho_i V_i A_i)_{out} = \sum_i (\rho_i V_i A_i)_{in}$$

Conservation of Linear Momentum

- **Conservation of linear momentum**: Newton's second law states that the net external force $(\sum\vec{F})$ acting on a system is equal to rate of change of its linear momentum $(\vec{P})$.

$$\sum\vec{F} = \frac{d\vec{P}}{dt} = m\frac{d\vec{V}}{dt} = m\vec{a}$$

- To obtain the law of conservation of linear moment for a control volume we choose property B to be the linear momentum of the fluid, $B = m\vec{V}$ and $\beta = \frac{dB}{dm} = \vec{V}$.

$$\frac{d}{dt}(m\vec{V})_{sys} = \sum\vec{F} = \frac{d}{dt}\left(\int_{CV}\rho\vec{V}\,d\boldsymbol{v}\right) + \int_{CS}\rho\vec{V}(\vec{V}_r\cdot\hat{n})\,dA$$

- The net external force acting on a CV (CV and system are identical at time t) = Rate of change of momentum of the fluid inside CV+ net flux of momentum leaving through the CS.
- The entire equation is a vector relation; the equation thus has three components.

Noninertial Reference Frame

- The control volume formulation of linear momentum in noninertial coordinates merely adds force terms by integrating the added relative acceleration over each differential mass in the control volume.

$$\sum \vec{F} - \int_{CV} \rho \vec{a}_{rel} \, d\mathcal{V} = \frac{d}{dt}\left(\int_{CV} \rho \vec{V} \, d\mathcal{V} \right) + \int_{CS} \rho \vec{V} (\vec{V}_r \cdot \hat{n}) \, dA$$

$$\vec{a}_{rel} = \frac{d^2 \vec{R}}{dt^2} + \frac{\mathrm{d}\vec{\Omega}}{\mathrm{dt}} \times \vec{r} + 2\vec{\Omega} \times \vec{V} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

- $\frac{d^2 \vec{R}}{dt^2}$ is the acceleration of the noninertial origin of coordinates xyz.
- $\frac{d\vec{\Omega}}{dt} \times \vec{r}$ is the angular acceleration effect.
- $2\vec{\Omega} \times \vec{V}$ is the Coriolis acceleration.
- $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ is the centripetal acceleration, directed from the particle normal to the axis of rotation with magnitude $\Omega^2 L$, where L is the normal distance to the axis.

Conservation of Angular Momentum

- If O is the point about which moments are desired, the angular momentum about O is given by

$$\vec{H}_o = \int_{syst} (\vec{r} \times \vec{V})\, dm$$

- The amount of angular momentum per unit mass will be

$$\beta = \frac{d\vec{H}_o}{dm} = \vec{r} \times \vec{V}$$

- The most general case of the angular momentum theorem is for a deformable control volume associated with a non-inertial coordinate system.

$$\sum \left(\vec{r} \times \vec{F}\right)_o - \int_{CV} (\vec{r} \times \vec{a}_{rel})\, dm = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho\, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r.\hat{n})\, dA$$

The Bernoulli Equation

- The Bernoulli equation for steady frictionless incompressible flow along a streamline.

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 = \text{const}$$

- The Bernoulli equation is a momentum-based force relation and was derived using the following restrictive assumptions:

1. Steady flow: a common situation.
2. Incompressible flow: appropriate if the flow Mach number is less than 0.3. This restriction can be removed allowing for compressibility.
3. Frictionless flow: restrictive—solid walls and mixing introduce friction effects.
4. Flow along a single streamline: different streamlines may have different "Bernoulli constants"

$$w_0 = \frac{p}{\rho} + \frac{V^2}{2} + gz$$

Differential form

- Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Cartesian: $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical: $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

- Momentum Equation

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right) = \rho\vec{g} - \nabla p + \nabla \cdot \tau_{ij}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

Strain rate tensor

- We can mathematically combine linear strain rate and shear strain rate into one symmetric second-order tensor called the **strain rate tensor**

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}$$

- Linear strain rate

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \epsilon_{yy} = \frac{\partial v}{\partial y}, \epsilon_{zz} = \frac{\partial w}{\partial z}$$

- Shear strain rate is

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

- The **angular velocity vector** and is expressed in Cartesian coordinates as

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z) = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\jmath} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

Newtonian Fluid: Navier-Stokes Equations

- For a Newtonian fluid, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity.

$$\tau_{ii} = 2\mu\epsilon_{ii} - \frac{2}{3}\mu\nabla\cdot\vec{V}$$
$$\tau_{ij} = 2\mu\epsilon_{ij} \text{ for } i \neq j$$

- For incompressible flow, divergence of velocity is zero i.e. $\nabla\cdot\vec{V} = 0$ then

$$\tau_{ij} = 2\mu\epsilon_{ij}$$
$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}, \tau_{yy} = 2\mu\frac{\partial v}{\partial y}, \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$
$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

- The x component of the divergence term $\nabla\cdot\tau_{ij}$

$$\nabla\cdot\tau_{ij}\Big|_x = \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = \mu\nabla^2 u$$

Newtonian Fluid: Navier-Stokes Equations

- Then the linear momentum equation becomes:

$$\rho\left(\frac{\partial\vec{V}}{\partial t}+u\frac{\partial\vec{V}}{\partial x}+v\frac{\partial\vec{V}}{\partial y}+w\frac{\partial\vec{V}}{\partial z}\right)=\rho\vec{g}-\nabla p+\mu\nabla^2\vec{V}$$

- These are the incompressible flow Navier-Stokes equations.
- In Cartesian coordinates

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

- They are second-order nonlinear partial differential equations.

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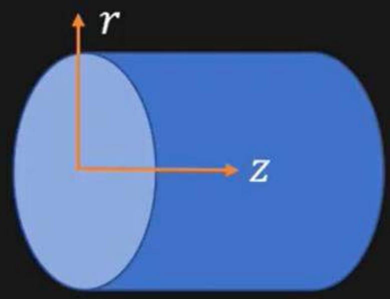
Lecture 2: Analytical solutions

Flow In a Circular Duct

- The most useful exact solution of the Navier-Stokes equation is for incompressible flow in a straight circular pipe of radius R.
- Here fully developed region will be used.
- Fully developed region is far enough from the entrance that the flow is purely axial.
- It means $u_z \neq 0$ while $u_r, u_\theta = 0$
- Here we are neglecting the gravity and also assuming that axial symmetry is present $\frac{\partial}{\partial \theta} = 0$
- In cylindrical co-ordinate



$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\Rightarrow \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\Rightarrow \nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \frac{\partial}{\partial z}(u_z)$$

Flow In a Circular Duct

$$\vec{V}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

- The flow proceeds straight down the pipe without radial motion.
- From continuity equation

$$\frac{\partial}{\partial z}(u_z) = 0 \Rightarrow u_z = u_z(r)$$

- The r momentum equation in the cylindrical co-ordinate

$$\frac{\partial u_r}{\partial t} + (\vec{V}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

- The θ momentum equation in the cylindrical co-ordinate

$$\frac{\partial u_\theta}{\partial t} + (\vec{V}\cdot\nabla)u_\theta + \frac{1}{r}u_r u_\theta = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)$$

Flow In a Circular Duct

- From r momentum equation

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(0) = 0$$

$$\Rightarrow \frac{\partial p}{\partial r} = 0 \quad \Rightarrow p = p(z)$$

- From z momentum equation

$$\frac{dp}{dz} = \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \text{constant}$$

- Now, solve the above equation

$$\Rightarrow \frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{\mathrm{r}}{\mu}\frac{dp}{dz}$$

$$\Rightarrow r\frac{du_z}{dr} = \frac{r^2}{2\mu}\frac{dp}{dz} + c_1$$

Flow In a Circular Duct

$$\Rightarrow \frac{du_z}{dr} = \frac{r}{2\mu}\frac{dp}{dz} + \frac{c_1}{r}$$

$$\Rightarrow u_z = \frac{r^2}{4\mu}\frac{dp}{dz} + c_1 \ln r + c_2$$

- Boundary conditions $u(z) = 0$ at $r = R$
- at $r = 0 \Rightarrow u(z)$ should be finite
- Therefore $c_1 = 0$

$$\Rightarrow 0 = \frac{R^2}{4\mu}\frac{dp}{dx} + c_2$$

$$u_z = -\frac{1}{4\mu}\frac{dp}{dz}(R^2 - r^2)$$

- The velocity profile is a paraboloid with a maximum at the centerline.

$$\Rightarrow u_{max} = u_z(r = 0) \Rightarrow -\frac{dp}{dz}\frac{R^2}{4\mu}$$

Flow In a Circular Duct

$$\Rightarrow \; u_z(r) = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

- Shear stress

$$\tau_{rz} = \mu \epsilon_{rz} = \mu\left(\cancel{\frac{\partial u_r}{\partial z}} + \frac{\partial u_z}{\partial r}\right)$$

$$\Rightarrow \tau_{rz} = \frac{r}{2}\frac{dp}{dz}$$

- Average velocity

$$u_{avg} = \frac{1}{\pi R^2}\int_0^R u_{max}\left[1 - \frac{r^2}{R^2}\right] 2\pi r \, dr$$

$$\Rightarrow u_{avg} = \frac{2u_{max}}{R^4}\int_0^R (R^2 r - r^3) dr$$

$$\Rightarrow u_{avg} = \frac{2u_{max}}{R^4}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R$$

Flow In a Circular Duct

$$u_{avg} = \frac{2u_{max}}{R^4}\frac{R^4}{4}$$

$$u_{avg} = \frac{1}{2}\mathrm{u_{max}}$$

- Volume flow rate $\dot{Q}$

$$Q = u_{avg}\pi R^2$$

$$Q = -\frac{dp}{dz}\frac{\pi R^4}{8\mu}$$

- Since dp/dz = constant $\Rightarrow -\frac{dp}{dz} = \frac{\Delta p}{L}$ where L is pipe length and Δp is the pressure drop.

$$Q = \frac{\pi R^4}{8\mu L}\Delta p = \frac{\pi D^4}{128\mu L}\Delta p$$

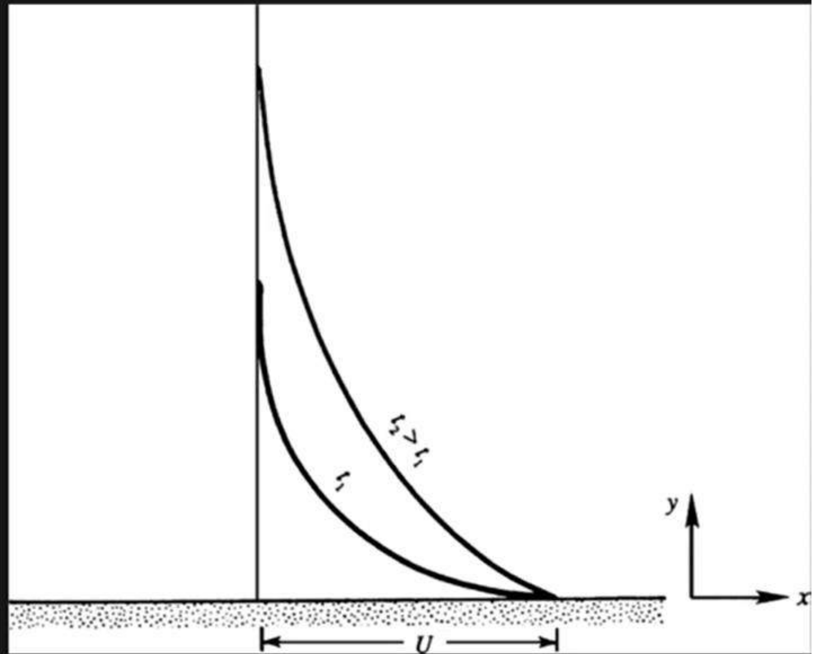
- Where D is the pipe diameter.

Stokes' first Problem

- The flow due to impulsive motion of a flat plate parallel to itself.
- The plate was at rest for $t \leq 0$. It starts moving along x-axis with velocity U.
- The plate is infinite along x and z directions. The fluid extends to infinity in y direction.
- Note that the problem is transient in nature.
- Based on these assumptions we can say $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$
- Then from continuity equation we get

$$\frac{\partial v}{\partial y} = 0$$
$$\Rightarrow v = f(t)$$

- Considering B.C. $v(t, y = 0) = 0$ leads to $v = 0$ everywhere.



Stokes' first Problem

- Consider the y-momentum equation

$$\rho \frac{D\cancel{v}}{\cancel{D}t} = -\frac{\partial p}{\partial y} + \mu \nabla^2 \cancel{v}$$

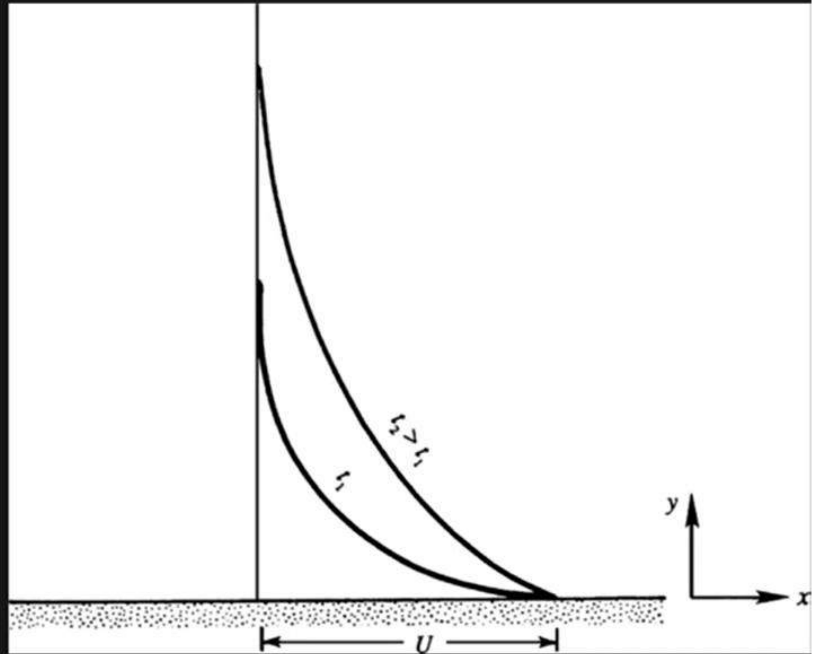
$$\Rightarrow \frac{\partial p}{\partial y} = 0$$

$$\Rightarrow p = p(t)$$

- Very far from the plate, the pressure is going to remain same. Therefore, we can conclude that $p =$ constant.
- Next, consider the x-momentum equation

$$\rho \left[\frac{\partial u}{\partial t} + u \cancel{\frac{\partial u}{\partial x}} + \cancel{v} \frac{\partial u}{\partial y}\right] = -\cancel{\frac{\partial p}{\partial x}} + \mu \left[\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\Rightarrow \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$



Stokes' first Problem

- Initial and boundary conditions are

$$u(0, y) = 0$$

$$u(t, y = 0) = U$$

$$u(t, y \to \infty) = 0$$

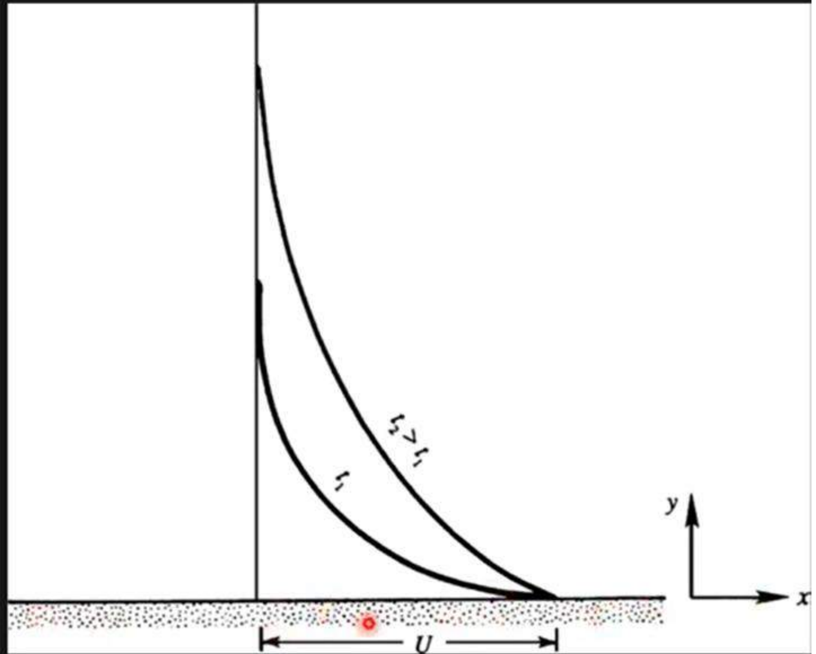
- We can convert the previous partial differential equation into an ODE by dimensional consideration. We can say

$$u = f(t, y, U, \nu)$$

- Using Pi theorem or Ipsen method, the above relation can be non-dimensionalized to

$$\frac{u}{U} = f\left(\frac{y}{\sqrt{\nu t}}\right)$$

- We say $\frac{u}{U} = F(\eta)$ where $\eta = \frac{y}{2\sqrt{\nu t}}$. It is called similarity variable.



Stokes' first Problem

- Then using chain rule

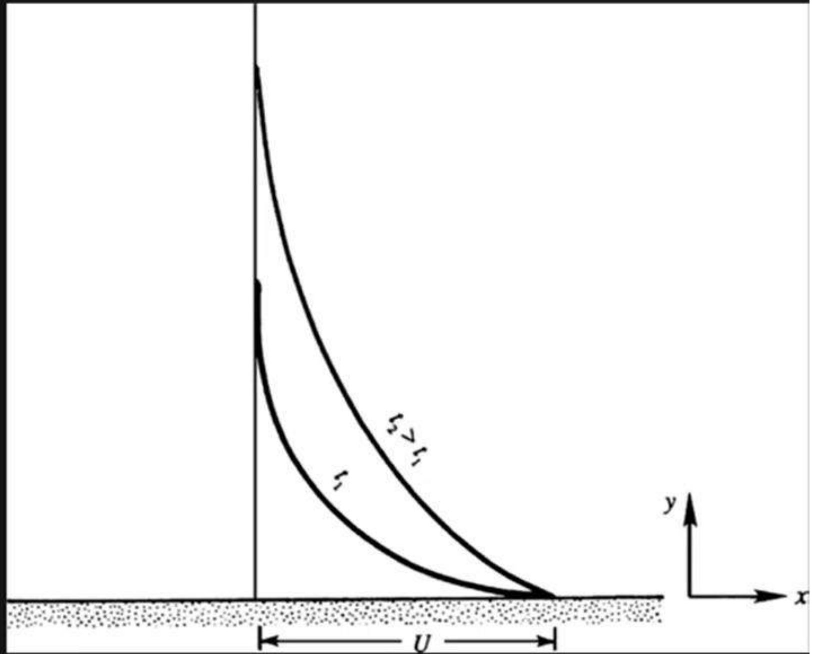
$$\frac{\partial u}{\partial t} = \frac{du}{d\eta}\frac{\partial \eta}{\partial t} = U\frac{dF}{d\eta}\left(-\frac{1}{2}\frac{y}{\sqrt{\nu}}\frac{1}{t\sqrt{t}}\right) = -\frac{U\eta}{2t}\frac{dF}{d\eta}$$

$$\frac{\partial u}{\partial y} = \frac{du}{d\eta}\frac{\partial \eta}{\partial y} = U\frac{dF}{d\eta}\left(\frac{1}{2\sqrt{\nu t}}\right) = \frac{\mathrm{U}}{2\sqrt{\nu t}}\frac{\mathrm{dF}}{\mathrm{d}\eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{d}{d\eta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial \eta}{\partial y} = \frac{U}{4\nu t}\frac{d^2 F}{d\eta^2}$$

- The PDE transforms to

$$-\frac{U\eta}{2t}\frac{dF}{d\eta} = \nu\frac{U}{4\nu t}\frac{d^2F}{d\eta^2} \Rightarrow F'' = -2\eta F'$$



Stokes' first Problem

- Initial and boundary conditions transforms to

$$u(0,y) = 0 \Rightarrow F(\eta \to \infty) = 0$$

$$u(t, y = 0) = U \Rightarrow F(0) = 1$$

$$u(t, y \to \infty) = 0 \Rightarrow F(\eta \to \infty) = 0$$

- Solving the ODE

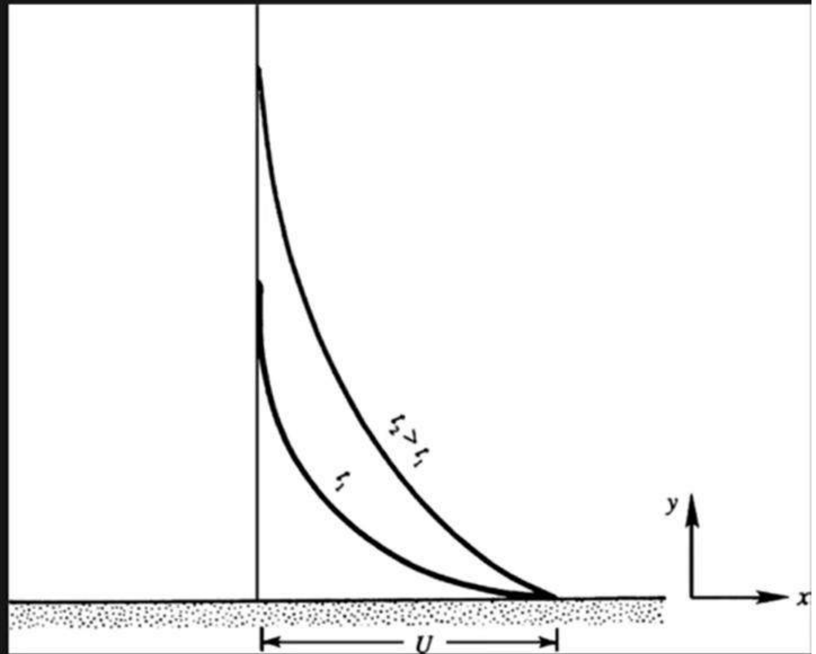
$$\frac{dF'}{d\eta} = -2\eta F'$$

$$\Rightarrow \ln F' = -\eta^2 + \ln A$$

$$\Rightarrow \frac{dF}{d\eta} = Ae^{-\eta^2}$$

$$\Rightarrow \int_0^\eta dF = A\int_0^\eta e^{-\eta^2}\, d\eta$$

$$\Rightarrow F(\eta) - F(0) = A\int_0^\eta e^{-\eta^2}\, d\eta$$



Stokes' first Problem

$$\Rightarrow F(\eta) = 1 + A\int_0^{\eta} e^{-\eta^2}\, d\eta$$

- $F(\eta \to \infty) = 0$

$$1 + A\int_0^{\infty} e^{-\eta^2}\, d\eta = 0$$

$$\Rightarrow 1 + A\frac{\sqrt{\pi}}{2} = 0$$

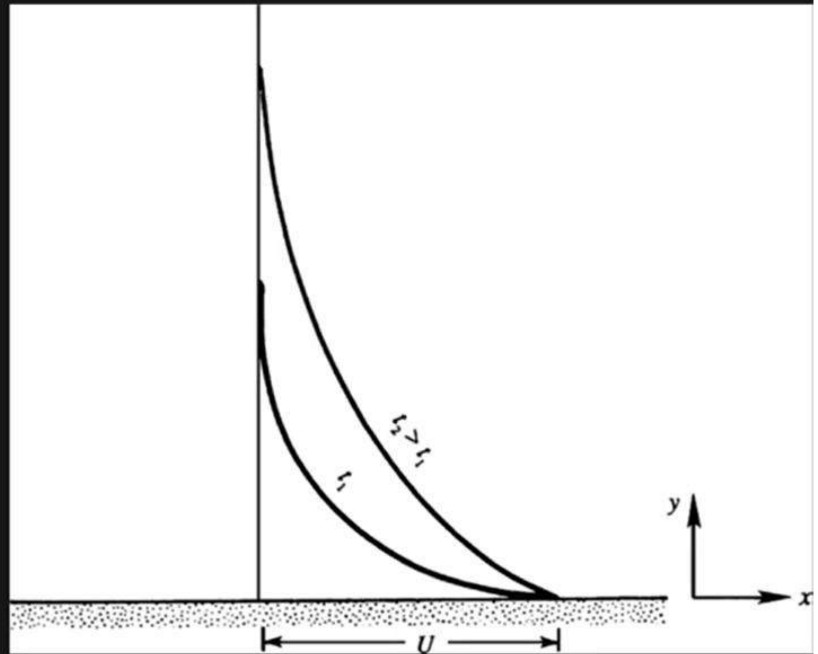
$$\Rightarrow A = -\frac{2}{\sqrt{\pi}}$$

- Therefore the non-dimensional velocity profile is

$$F(\eta) = 1 - \text{erf}(\eta)$$

- Where

$$\text{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\, d\eta$$



Stokes' first Problem

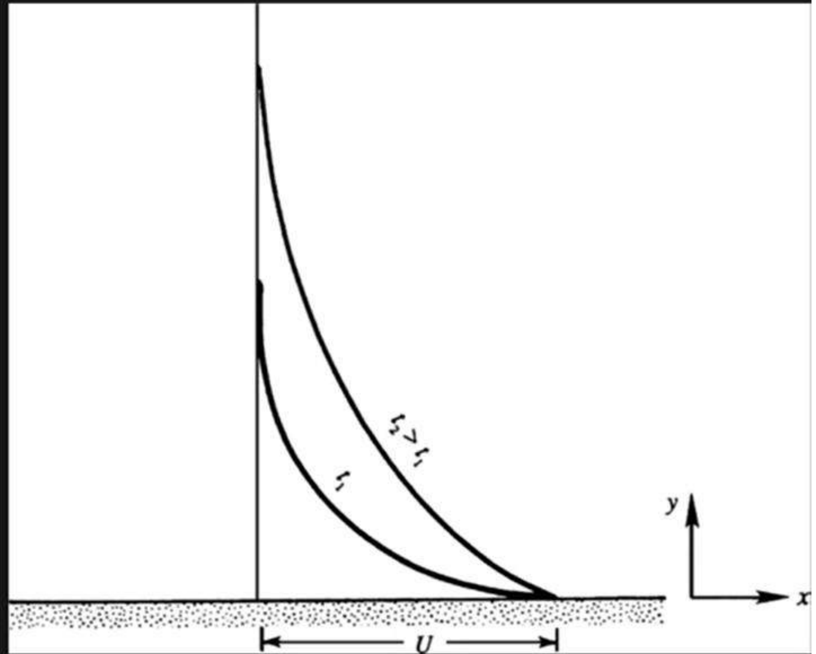
- We arbitrarily define the thickness of the diffusion layer (δ) as the distance from the plate at which u is 1% of U.

$$\frac{u(y=\delta)}{U} = 0.01$$

- Using numerical integration

$$\Rightarrow \eta = 1.82$$

$$\Rightarrow \delta = 3.64\sqrt{\nu t}$$



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Section1: Viscous Flows

Lecture 3: Energy equation

Conservation of Energy

- **Conservation of energy**: If δQ amount of heat is added to the system and δW amount of work is done by the system then the increase in the energy of system is

$$dE = \delta Q - \delta W$$
$$\Rightarrow \frac{dE}{dt} = \dot{Q} - \dot{W}$$

- This is also called the first law of thermodynamics.
- On putting $B = E$ and $\beta = e = \hat{u} + \frac{1}{2}V^2 + gz$ in RTT for a fixed CV we get

$$\frac{d}{dt}(E)_{sys} = \dot{Q} - \dot{W} = \frac{d}{dt}\left(\int_{CV} \rho e \, d\boldsymbol{\mathcal{V}}\right) + \int_{CS} \rho e(\vec{V} \cdot \hat{n}) \, dA$$

- Enthalpy $\hat{h} = \hat{u} + \frac{p}{\rho}$, therefore

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{d}{dt}\left(\int_{CV} \rho\left(\hat{u} + \frac{1}{2}V^2 + gz\right) d\boldsymbol{\mathcal{V}}\right) + \int_{CS} \rho\left(\hat{h} + \frac{1}{2}V^2 + gz\right)(\vec{V} \cdot \hat{n}) \, dA$$

Conservation of Energy: differential form

- The energy equation for the infinitesimal control volume

$$\dot{Q} - \dot{W} = \frac{\partial}{\partial t}(\rho e)\, dx\, dy\, dz + \nabla \cdot \left(\rho e \vec{V}\right) dx\, dy\, dz$$

- Let us assume heat transfer due to conduction only. Recall the Fourier's law of heat conduction, heat flux $q = q_x\hat{\imath} + q_y\hat{\jmath} + q_z\hat{k} = -k\nabla T$.

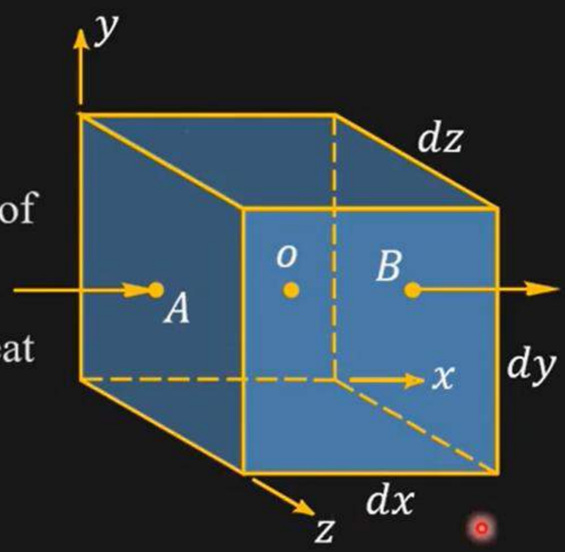
- Let q_0 be the heat flux at the center of the element then the net rate of heat entering from the faces having normal in x-direction is

$$\dot{Q}_x = \left[q_{0x} - \frac{\partial}{\partial x}(q_{0x})\frac{dx}{2}\right] dy\, dz\ - \left[q_{0x} + \frac{\partial}{\partial x}(q_{0x})\frac{dx}{2}\right] dy\, dz$$

$$\Rightarrow \dot{Q}_x = -\frac{\partial}{\partial x}(q_{0x})\, dx\, dy\, dz$$

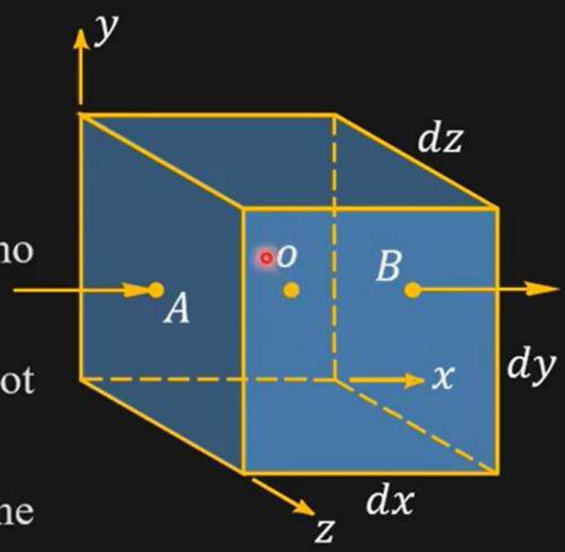
- The total rate of heat entering the element is

$$\dot{Q} = -\left[\frac{\partial}{\partial x}(q_{0x}) + \frac{\partial}{\partial y}\left(q_{0y}\right) + \frac{\partial}{\partial z}(q_{0z})\right] dx\, dy\, dz$$



Conservation of Energy: differential form

$$\Rightarrow \dot{Q} = -\nabla \cdot q \; dx \; dy \; dz$$
$$\Rightarrow \dot{Q} = \nabla \cdot (k\nabla T) \; dx \; dy \; dz$$



- The work is divided into three components $\dot{W} = \dot{W}_p + \dot{W}_s + \dot{W}_v$.
- In the limit of infinitesimal element, the shaft work will be zero since there is no such infinitesimal shaft.
- Let us calculate the pressure work. The rate of work done is equal to the dot product of force and velocity.
- For a small surface of area dA and unit normal $\hat{n}$, the rate of work done by the pressure force is

$$d\dot{W}_p = (-p \; dA \; \hat{n}) \cdot \vec{V}$$

- The total rate of work done by the CV is

$$\dot{W}_p = \int_{CS} -d\dot{W}_p$$

Conservation of Energy: differential form

$$\Rightarrow \dot{W}_p = \int_{CS} p\vec{V}.\hat{n}\, dA$$

$$\Rightarrow \dot{W}_p = \int_{CV} \nabla \cdot (p\vec{V})\, dV$$

- In the limit of a infinitesimal element

$$\dot{W}_p = \nabla \cdot (p\vec{V})\, dx\, dy\, dz$$

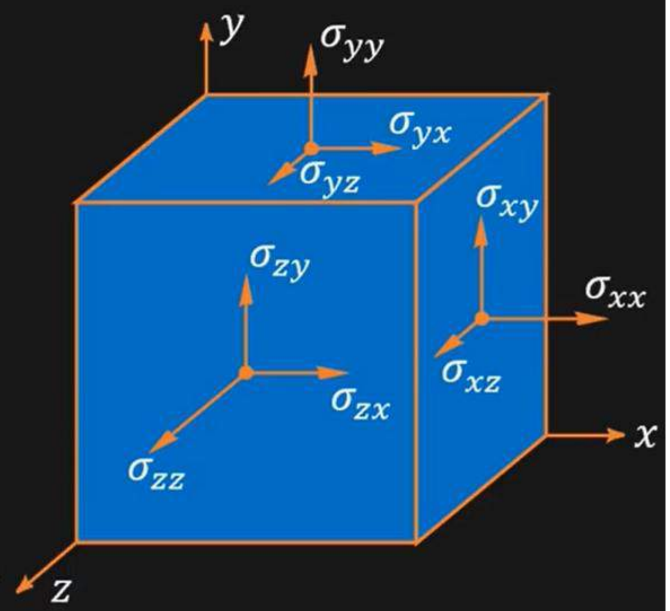
- Similarly, the rate of work done by viscous stresses will be

$$\dot{W}_v = -\nabla \cdot (\tau_{ij} \cdot \vec{V})\, dx\, dy\, dz$$

- You can verify this by considering the net rate of work done by viscous stresses on faces having normal along x, y and z-directions separately. Note that

$$\tau_{ij} \cdot \vec{V}\Big|_x = u\tau_{xx} + v\tau_{xy} + w\tau_{xz}$$

$$\tau_{ij} \cdot \vec{V}\Big|_y = u\tau_{yx} + v\tau_{yy} + w\tau_{yz}$$



Conservation of Energy: differential form

- Putting everything back in to the energy equation

$$\nabla \cdot (k\nabla T)\, dx\, dy\, dz - \nabla \cdot \left(p\vec{V}\right) dx\, dy\, dz + \nabla \cdot \left(\tau_{ij} \cdot \vec{V}\right) dx\, dy\, dz = \frac{\partial}{\partial t}(\rho e)\, dx\, dy\, dz + \nabla \cdot \left(\rho e\vec{V}\right) dx\, dy\, dz$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho e) + \nabla \cdot \left(\rho e\vec{V}\right) = \nabla \cdot (k\nabla T) + \nabla \cdot \left(\tau_{ij} \cdot \vec{V}\right) - \nabla \cdot \left(p\vec{V}\right)$$

- Let us apply chain rule of differentiation on the LHS

$$\frac{\partial}{\partial t}(\rho e) + \nabla \cdot \left(\rho e\vec{V}\right) = \rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t} + \mathrm{e}\nabla \cdot \left(\rho\vec{V}\right) + \left(\rho\vec{V}\right) \cdot \nabla e$$

$$= \rho\left[\frac{\partial e}{\partial t} + \vec{V} \cdot \nabla e\right] + e\left[\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho\vec{V}\right)\right]$$

$$= \rho\frac{De}{Dt}$$

- The energy equation becomes

$$\rho\frac{De}{Dt} = \nabla \cdot (k\nabla T) + \nabla \cdot \left(\tau_{ij} \cdot \vec{V}\right) - \vec{V} \cdot \nabla p - p\nabla \cdot \vec{V}$$

Conservation of Energy: differential form

- Taking total derivative of energy $e = \hat{u} + \frac{1}{2}V^2 + gz$

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \frac{D}{Dt}\left(\frac{1}{2}V^2\right) + \frac{D}{Dt}(gz)$$

$$= \frac{D\hat{u}}{Dt} + \frac{D}{Dt}\left(\frac{1}{2}V^2\right) + \frac{\partial}{\partial t}(gz) + \vec{V}\cdot\nabla(gz)$$

- Since $gz \neq f(t)$

$$\Rightarrow \frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \frac{D}{Dt}\left(\frac{1}{2}V^2\right) + wg$$

- The energy equation becomes

$$\rho\left[\frac{D\hat{u}}{Dt} + \frac{D}{Dt}\left(\frac{1}{2}V^2\right)\right] = \nabla\cdot(k\nabla T) + \nabla\cdot\left(\tau_{ij}\cdot\vec{V}\right) - \vec{V}\cdot\nabla p - p\nabla\cdot\vec{V} - \rho wg$$

- Recall the momentum equation

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p + \nabla\cdot\tau_{ij}$$

Conservation of Energy: differential form

- Taking dot product of the momentum equation with velocity and using $\vec{g} = -g\hat{k}$

$$\rho\vec{V}\cdot\frac{D\vec{V}}{Dt} = -\rho g w - \vec{V}\cdot\nabla p + \vec{V}\cdot(\nabla\cdot\tau_{ij})$$

$$\Rightarrow \rho\frac{D}{Dt}\left(\frac{1}{2}V^2\right) = -\rho g w - \vec{V}\cdot\nabla p + \vec{V}\cdot(\nabla\cdot\tau_{ij})$$

- Subtracting the above equation from the energy equation gives

$$\rho\frac{D\hat{u}}{Dt} = \nabla\cdot(k\nabla T) + \nabla\cdot(\tau_{ij}\cdot\vec{V}) - \vec{V}\cdot(\nabla\cdot\tau_{ij}) - p\nabla\cdot\vec{V}$$

$$\Rightarrow \rho\frac{D\hat{u}}{Dt} + p\nabla\cdot\vec{V} = \nabla\cdot(k\nabla T) + \Phi$$

- Where $\Phi = \nabla\cdot(\tau_{ij}\cdot\vec{V}) - \vec{V}\cdot(\nabla\cdot\tau_{ij})$ is called viscous dissipation term.
- Consider the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$$

Conservation of Energy: differential form

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

- Next we take the total derivative of $\hat{h} = \hat{u} + p/\rho$

$$\frac{D\hat{h}}{Dt} = \frac{D\hat{u}}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

$$= \frac{D\hat{u}}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\left(-\rho \nabla \cdot \vec{V}\right)$$

$$\Rightarrow \rho \frac{D\hat{h}}{Dt} = \rho \frac{D\hat{u}}{Dt} + \frac{Dp}{Dt} + p \nabla \cdot \vec{V}$$

$$\Rightarrow \rho \frac{D\hat{u}}{Dt} + p \nabla \cdot \vec{V} = \rho \frac{D\hat{h}}{Dt} - \frac{Dp}{Dt}$$

- Using the above relation for energy equation we get

Conservation of Energy: differential form

$$\rho \frac{D\hat{h}}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k\nabla T) + \Phi$$

- For perfect gas $dh = C_p \, dT$

$$\rho C_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k\nabla T) + \Phi$$

- Dp/Dt and Φ are small for low speed flows. Then the energy equation reduced to

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot (k\nabla T)$$

- If $k =$ constant then

$$\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T = \frac{k}{\rho C_p} \nabla^2 T$$

- For incompressible flows

$$\Phi = \mu \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 \right]$$

ME 221: Fluid Mechanics II

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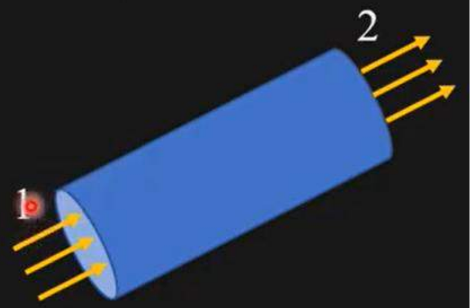
Section1: Viscous Flows

Lecture 4: Losses in piping system

Head loss and friction factor

- Consider the steady and incompressible flow through a pipe. Applying energy equation for a CV formed by the pipe walls and section 1 & 2

$$\dot{Q} - \dot{W}_s - \dot{W}_p - \dot{W}_v = \int_{CS} \rho e(\vec{V} \cdot \hat{n})\, dA$$



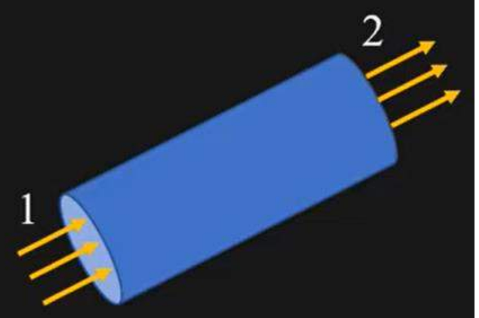
- We assume there is no shaft inside the pipe $\Rightarrow \dot{W}_s = 0$.
- Since $\dot{W} = \vec{F} \cdot \vec{V}$, $\dot{W}_v = 0$ at pipe walls due to the no-slip boundary condition.
- Normal stress is zero at inlet (section 1) due the fully developed flow condition. Only shear stress may be present which is normal to the inlet velocity. Therefore, viscous work is zero.
- Same can be said about the outlet (section 2).
- Therefore, $\dot{W}_v = 0$ for the CV.
- Pressure work

$$\dot{W}_p = \int_{CS} p\vec{V} \cdot \hat{n}\, dA$$

Head loss and friction factor

- Putting back in energy equation

$$\dot{Q} = \int_{CS} \rho\left(e + \frac{p}{\rho}\right)(\vec{V}\cdot\hat{n})\, dA$$



- The surface integral will be zero along pipe walls due to no-slip BC.
- At inlet and outlet, the fluid velocity is along the pip axis. Hence

$$\dot{Q} = \int_2 \rho\left(e + \frac{p}{\rho}\right)V\, dA - \int_1 \rho\left(e + \frac{p}{\rho}\right)V\, dA$$

- Using $e = \hat{u} + \frac{1}{2}V^2 + gz$, the integral can be written as

$$I_1 = \int_1 \rho\left(e + \frac{p}{\rho}\right)V\, dA$$

$$= \rho\int_1 \hat{u}V\, dA + \frac{1}{2}\rho\int_1 V^3\, dA + \rho\int_1 gzV\, dA + \int_1 pV\, dA$$

$$\approx \rho\hat{u}_1\int_1 V\, dA + \frac{1}{2}\rho\int_1 V^3\, dA + \rho g z_1\int_1 V\, dA + \int_1 pV\, dA$$

Head loss and friction factor

- Consider the definition of average velocity

$$\overline{V} = \frac{1}{A}\int_A V\,dA$$

- Also, pressure does not vary along the cross-section 1 & 2. Then

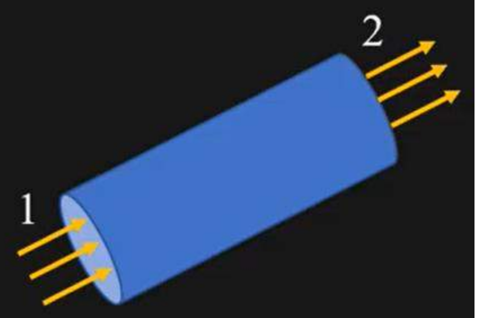
$$I_1 = \rho\hat{u}_1 A_1\overline{V}_1 + \frac{1}{2}\rho\int_1 V^3\,dA + \rho g z_1 A_1\overline{V}_1 + p_1 A_1\overline{V}_1$$

$$\Rightarrow I_1 = \dot{m}\hat{u}_1 + \frac{1}{2}\rho\int_1 V^3\,dA + \dot{m}gz_1 + \dot{m}\frac{p_1}{\rho}$$

- Where $\dot{m} = \rho A_1\overline{V}_1 = \rho A_2\overline{V}_2$ is the mass flow rate.
- Recall the definition of kinetic energy correction factor

$$\alpha = \frac{1}{A}\int_A \left(V/\overline{V}\right)^3 dA$$

$$\Rightarrow \frac{1}{2}\rho\int_A V^3\,dA = \frac{1}{2}\alpha\rho\overline{V}^3 A = \frac{1}{2}\alpha\dot{m}\overline{V}^2$$

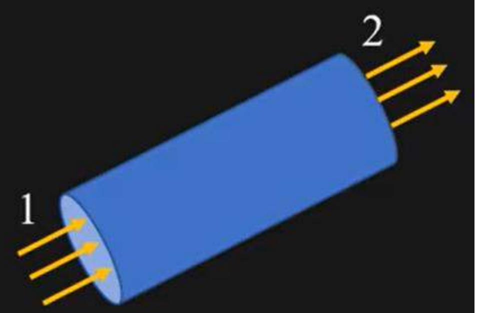


Head loss and friction factor

- Coming back to the integral

$$I_1 = \dot{m}\hat{u}_1 + \frac{1}{2}\rho\int_1 V^3\,dA + \dot{m}gz_1 + \dot{m}\frac{p_1}{\rho}$$

$$\Rightarrow I_1 = \dot{m}\hat{u}_1 + \frac{1}{2}\alpha\dot{m}\overline{V}_1^2 + \dot{m}gz_1 + \dot{m}\frac{p_1}{\rho}$$



- Coming back to the energy equation

$$\dot{Q} = I_2 - I_1$$

$$\Rightarrow \dot{Q} = \dot{m}(\hat{u}_2 - \hat{u}_1) + \frac{1}{2}\dot{m}\left(\alpha_2\overline{V}_2^2 - \alpha_1\overline{V}_1^2\right) + \dot{m}g(z_2 - z_1) + \frac{\dot{m}}{\rho}(p_2 - p_1)$$

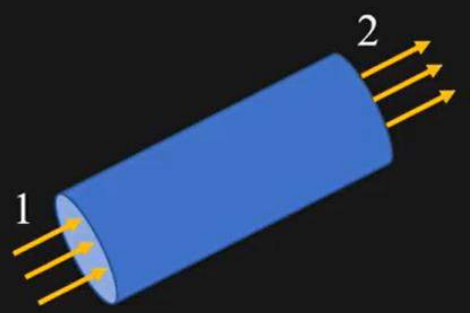
$$\Rightarrow \left(\frac{p_1}{\rho} + \frac{1}{2}\alpha_1\overline{V}_1^2 + gz_1\right) - \left(\frac{p_2}{\rho} + \frac{1}{2}\alpha_2\overline{V}_2^2 + gz_2\right) = \left(\hat{u}_2 - \hat{u}_1 - \frac{\dot{Q}}{\dot{m}}\right)$$

- The term $\frac{p}{\rho} + \frac{1}{2}\alpha\overline{V}^2 + gz$ represents mechanical energy per unit mass.
- Decrease in the mechanical energy = increase in the internal energy + heat transferred to the surrounding.

Head loss and friction factor

- Dividing the energy equation by g

$$\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\overline{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\overline{V}_2^2}{2g} + z_2\right) = h_f$$



- Where h_f is called the head loss.
- If the pipe has uniform cross-section then $\alpha_1 = \alpha_2$ and $\overline{V}_1 = \overline{V}_2$.
- If we also assume the pipe to be horizonal then

$$h_f = \frac{p_1 - p_2}{\rho g} = \frac{\Delta p}{\rho g}$$

- In general, head loss is the sum of major and minor losses.
- Minor loses are due to
  1. Pipe entrance or exit.
  2. Sudden expansion or contraction.
  3. Bends, elbows, tees, and other fittings.
  4. Valves, open or partially closed.
  5. Gradual expansions or contractions.

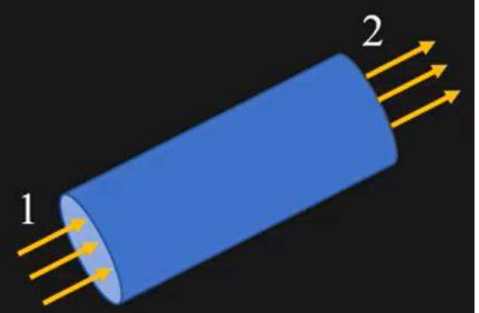
Head loss and friction factor

- Major head loss accounts for the losses due to friction effects.
- For laminar pipe (diameter D and length L) flow we saw

$$\Delta p = \frac{128\mu L}{\pi D^4} Q$$

$$\Rightarrow \Delta p = \frac{128\mu L}{\pi D^4} \times \pi \left(\frac{D}{2}\right)^2 \overline{V} = \frac{32\mu L}{D^2} \overline{V}$$

$$\Rightarrow \Delta p = 32 \frac{L}{D} \frac{\rho \overline{V}^2}{Re_D}$$



- Where $Re_D = \frac{\rho \overline{V} D}{\mu}$. Then the major head loss for laminar pipe flow is

$$h_f = \frac{\Delta p}{\rho g} = \frac{32}{Re_D} \frac{L}{D} \frac{\overline{V}^2}{g}$$

- There is no analytical expression for turbulent flows. We switch to dimensional analysis.

Head loss and friction factor

- The pressure drop is function of pipe diameter (D), length (L), surface roughness (e), average fluid velocity ($\overline{V}$), fluid density (ρ) and viscosity (μ).

$$\Delta p = F\left(D, L, e, \overline{V}, \rho, \mu\right)$$

- Use Ipsen of Pi method to obtain the non-dimensional form of the above equation

$$\frac{\Delta p}{\rho \overline{V}^2} = F\left(\frac{L}{D}, \frac{e}{D}, \frac{1}{Re}\right)$$

- From experiments, the pressure drop is found to be proportional to the pipe length

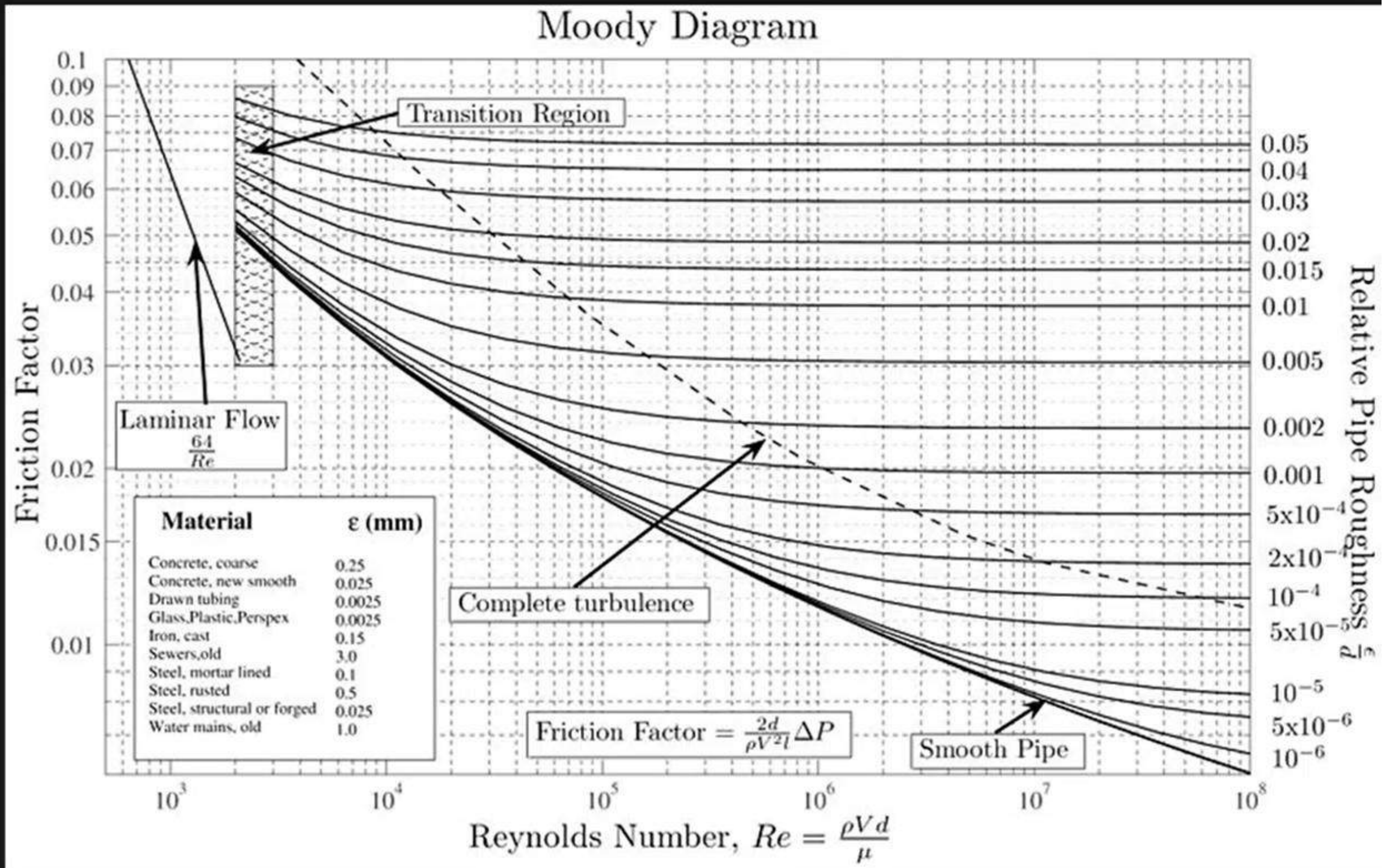
$$\frac{\Delta p}{\rho \overline{V}^2} = \frac{L}{D} f_1\left(\frac{e}{D}, \frac{1}{Re}\right)$$

$$\Rightarrow h_f = \frac{L}{D}\frac{\overline{V}^2}{g} f_1\left(\frac{e}{D}, \frac{1}{Re}\right)$$

$$\Rightarrow h_f = f\frac{L}{D}\frac{\overline{V}^2}{2g}$$

Head loss and friction factor

- f is called friction factor. It depends on Reynolds number and surface roughness.
- For laminar pipe flow
$$f = 64/Re_D$$
- For turbulent flows, following implicit relation exists
$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{e}{3.7D} + \frac{2.51}{Re_D\sqrt{f}}\right)$$
- The friction factor is plotted in the Moody chart for various roughness values.



ME 221: Fluid Mechanics II

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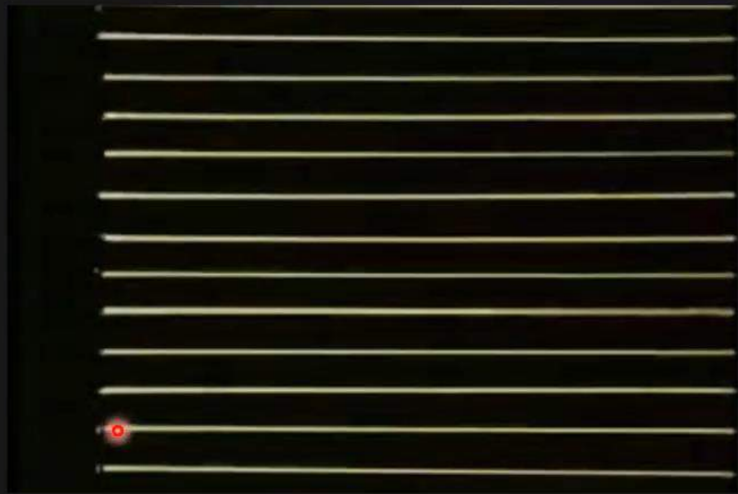
Section2: Boundary Layer Theory

Lecture 5: Introduction

Free stream flow

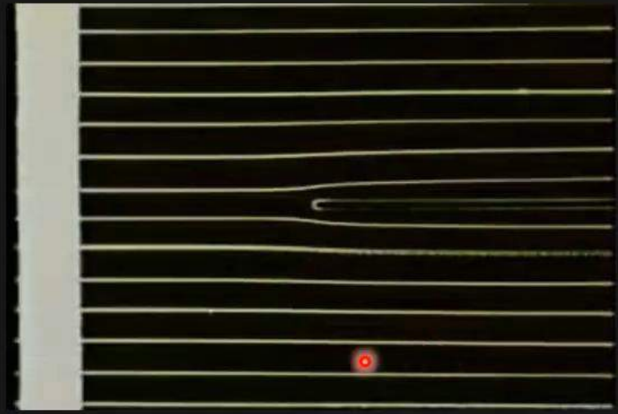
- When DC current passes through a thin wire submerged in water, hydrogen bubbles forms which allow us visualize the flow.
- The animation shows free stream flow of water.
- A wire has been insulated at regular intervals to produced lines of bubbles by passing a continuous DC current through the wire.
- Another wire is fully exposed to water and creates a sheet of bubbles when a pulse of current is passed.
- The streamlines formed by the bubbles are also streamlines since the flow is steady.
- The flow is irrotational and inviscid.



http://web.mit.edu/hml/ncfmf.html

Flow over a plate (Lab Frame)

- Now a thin stationary plate is placed in the free stream.
- Due to the no-slip boundary condition (fluid viscosity) the fluid in the contact with the plate remains stationary.
- The effect of the plate on the flow is felt in its vicinity in the form of retardation of the flow.
- This results in deflection of the streamlines.
- Far from the plate, the flow is still inviscid and irrotational. It can be described using the potential flow theory.
- The region near the plate where viscous effects are significant (fluid velocity is different from its free stream value) is called the boundary layer (BL).
- Inside BL, the viscous effect are significant while outside BL viscous forces are absent.



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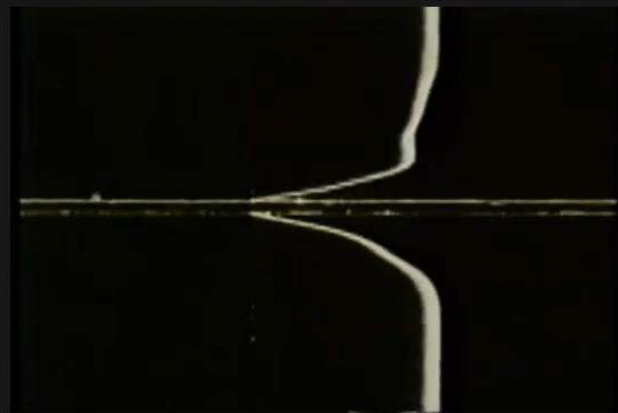
Flow over a plate (Moving Frame)

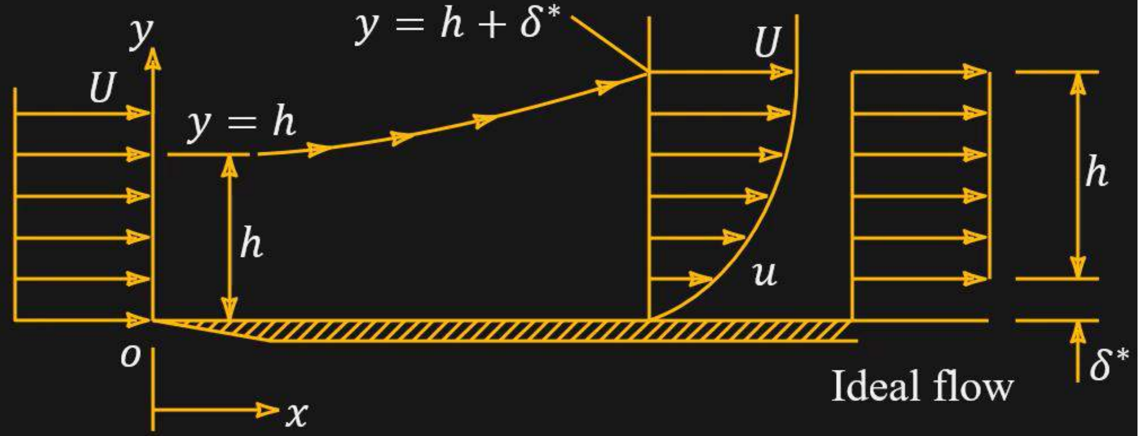
- The camera is moving with the flow in the first animation.
- The thickness of the BL can be defined arbitrarily as the distance from the plate where the fluid streamwise velocity is 99% of free stream velocity (U).
- If x and y represent the directions along and transverse to the plate, respectively, then BL region is where $u(x, y) < 0.99U$.
- The thickness of the BL increases as we move downstream along the plate.
- The BL region and its separation is responsible for non-zero lift and drag forces on the submerged bodies (D'Alembert's paradox).
- The second animation compares laminar BL (bottom) with the turbulent BL (top).



http://web.mit.edu/hml/ncfmf.html

Displacement Thickness

- It is the distance by which the plate would have to be displaced outwards in a hypothetical frictionless flow so as to maintain the same mass flux as in the actual flow.
- Consider the flow over a flat plate shown in figure. The mass flow rate per unit width between $y = 0$ and $y = h$ at any x is



$$\dot{m}_a = \int_0^h \rho u \, dy$$

- If the flow was inviscid everywhere then the flow velocity will be U at all the places and the plate needs to be displaced by a distance d in order to have the same mass flow rate as for the actual flow

$$\dot{m}_i = \rho U(h - d) = \dot{m}_a$$

$$\rho U\left(\int_0^h dy - d\right) = \int_0^h \rho u \, dy$$

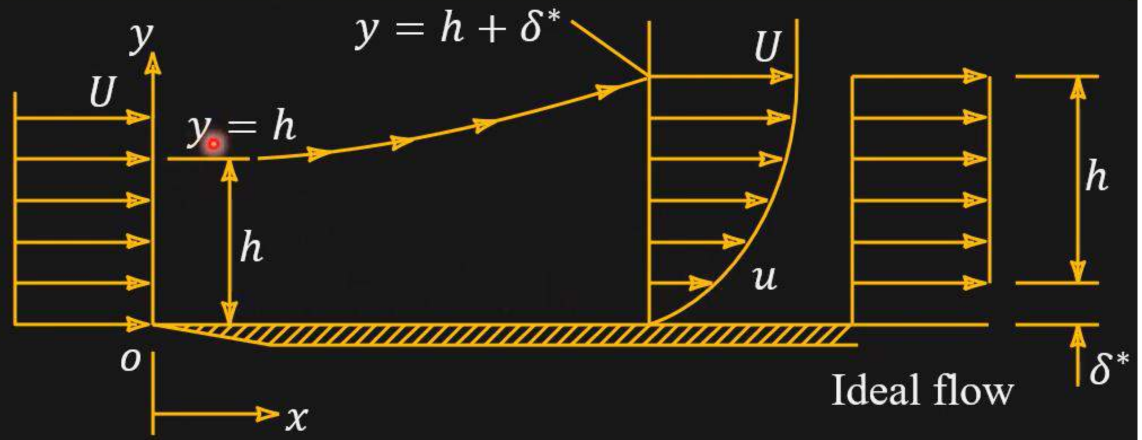
Displacement Thickness

$$d = \int_0^h \left(1 - \frac{u}{U}\right) dy$$

- In the limit $h \to \infty$, we get the displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

- Note that δ^* is function of x.



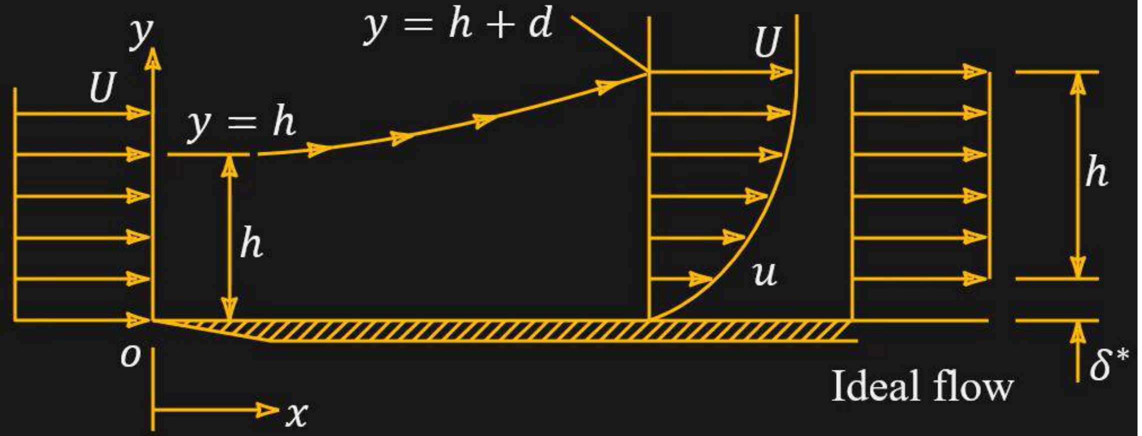
- Let the boundary layer thickness is δ then the integral in the definition of δ^* is very small for $y > \delta$ since $1 - u/U < 0.01$. Therefore, from engineering point of view

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

- There is also a physical interpretation of displacement thickness.

Physical significance of δ^*

- Let us consider a streamline that is located at height $y = h$ at the tip of the plate $(x = 0)$.
- The mass flow rate per unit width between the streamline and the plate at $x = 0$ is
$$\dot{m}_0 = \rho U h$$
- Recall that the mass flow rate in a stream tube (formed by the streamline and the plate in the present case) remains same.



- As we move along the plate, the streamwise flow speed reduces and therefore the streamline has to deflect upward to conserve the mass flow rate.
- Let the vertical displacement of the streamline at a position x is $d(x)$. The mass flow rate per unit width between the streamline and the plate at location x is
$$\dot{m}_x = \int_0^{h+d} \rho u \, dy$$

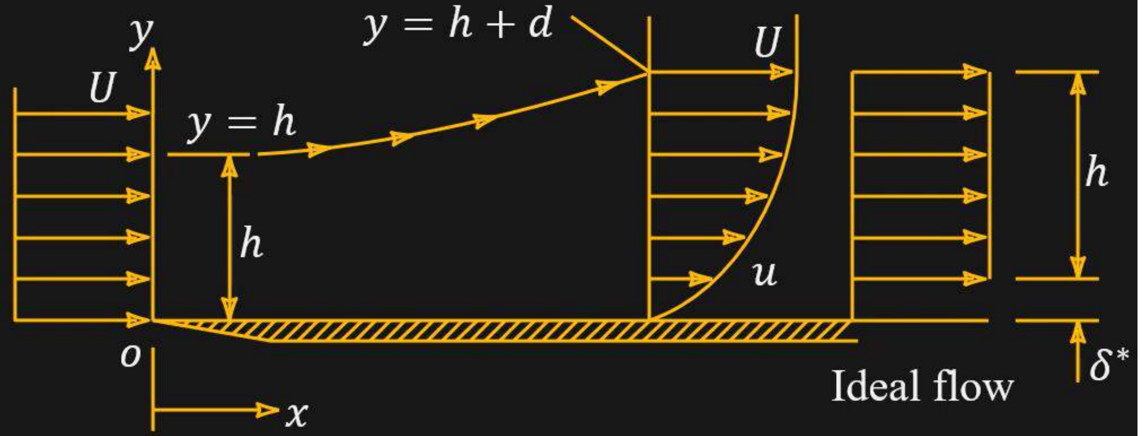
Physical significance of δ^*

$$\dot{m}_0 = \dot{m}_x$$

$$\rho U h = \int_0^{h+d} \rho u \, dy$$

$$\Rightarrow U h = \int_0^{h} u \, dy + \int_h^{h+d} u \, dy$$

$$\Rightarrow \int_0^{h} (U - u) \, dy = \int_h^{h+d} u \, dy$$



- Assuming $h > \delta(x)$ i.e. the streamline is outside the BL. Then, $u \approx U$ for $y \geq h > \delta$ therefore

$$\int_h^{h+d} u \, dy \approx \int_h^{h+d} U \, dy = Ud$$

- And the above equation becomes

$$\int_0^{h} (U - u) \, dy = Ud$$

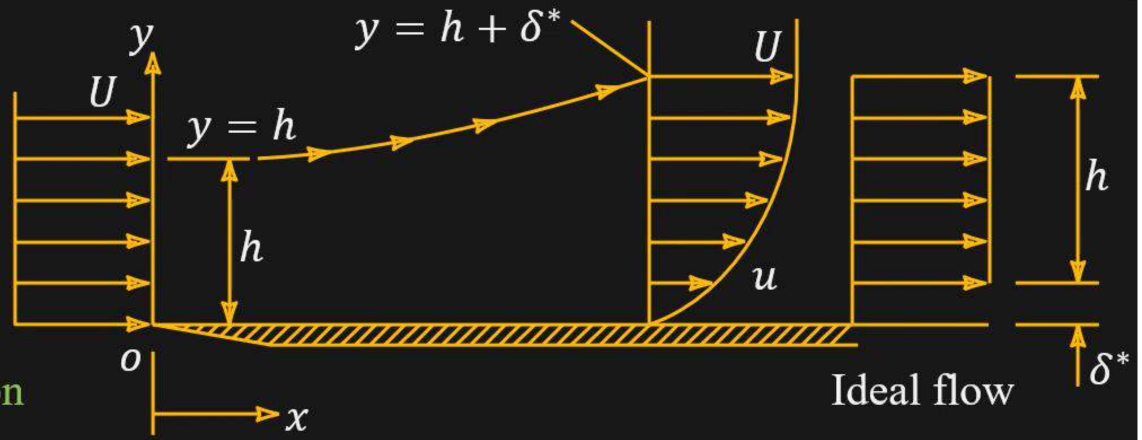
Physical significance of δ^*

$$\Rightarrow d = \int_0^h \left(1 - \frac{u}{U}\right) dy$$

- since $h > \delta\ (x)$, we can say

$$d = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

- Displacement thickness is equal to the deflection of the streamlines that are outside BL.



- The mass flow rate if the flow was inviscid everywhere $\dot{m}_i = \rho U \delta$
- The actual mass flux is $\dot{m}_a = \int_0^\delta \rho u\, dy$
- Loss of mass flow rate due to BL is

$$\dot{m}_i - \dot{m}_a = \int_0^\delta \rho(U - u) dy = \rho U \delta^*$$

ME 221: Fluid Mechanics II

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Section2: Boundary Layer Theory

Lecture 6: momentum thickness

Momentum Thickness

- The plate was displaced by the amount of displacement thickness in order to equate the mass flow rates in actual and ideal flow cases.
- However, the momentum flux can still be unbalanced.
- Recall that the loss of mass flux in actual flow is equal to $\rho U \delta^*$ if the plate is not displaced in the ideal flow.
- Similarly, the momentum thickness (θ) is defined such that the loss of momentum flux in actual flow is equal to $\rho U^2 \theta$ when the plate has been displacement by δ^* in the ideal flow.
- Since the flow outside the BL is same as (very close to) the ideal flow, we need to consider the momentum flux inside the BL to account for the momentum loss
- Therefore, by definition

$$\rho U^2 \theta = \int_{\delta^*}^{\delta} \rho U^2 \, dy - \int_0^{\delta} \rho u^2 \, dy$$

Momentum Thickness

$$\Rightarrow U^2\theta = \int_0^{\delta} U^2\, dy - U^2\delta^* - \int_0^{\delta} u^2\, dy$$

$$\Rightarrow U^2\theta = \int_0^{\delta} (U^2 - u^2)\, dy - U^2 \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

$$\Rightarrow \theta = \int_0^{\delta} \left(1 - \frac{u^2}{U^2}\right) dy - \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

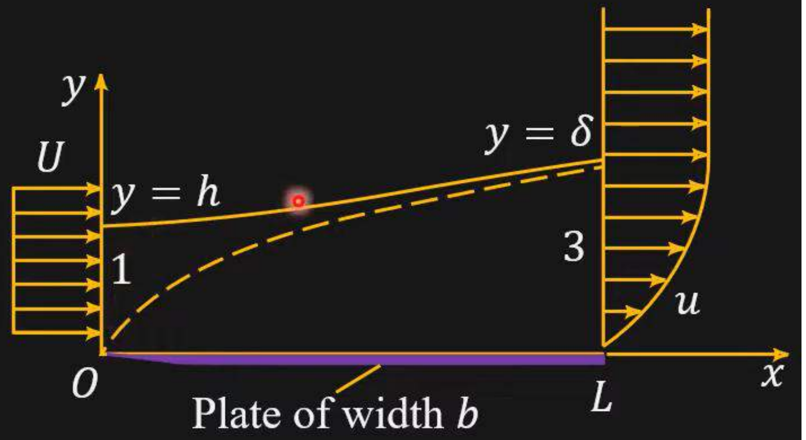
$$\Rightarrow \theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

- Once again, the exact definition will be

$$\Rightarrow \theta = \int_0^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Physical significance of θ

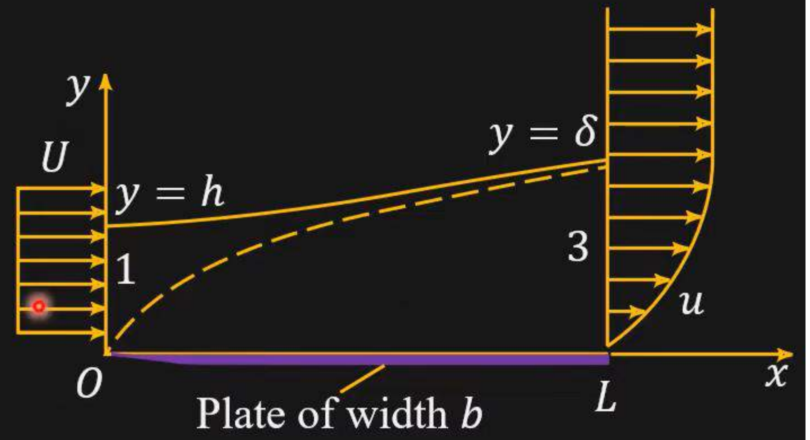
- The momentum thickness is a measure of drag force acting on the plate.
- We select the four-sided region from O to h to δ to L and back to the origin O as our control volume.
- Let us apply conservation of momentum to CV.
- If We had chosen to cut across horizontally from left to right along the height $y = h$ then we would have cut through the shear layer and exposed unknown shear stresses.
- Instead we follow the streamline passing through $(x, y) = (O, h)$, which is outside the shear layer and also has no mass flow across it.
- We also assume that the pressure is uniform everywhere so that there is no net pressure force on the CV.



Physical significance of θ

- The four control volume sides are thus:
- From (0,0) to $(0, h)$; a one dimensional inlet, $\vec{V}.\vec{n} = -U$
- From $(0, h)$ to (L, δ); streamline, no shear, $\vec{V}.\vec{n} = 0$
- From (L, δ) to $(L, 0)$; a two-dimensional outlet, $\vec{V}.\vec{n} = +u$
- From $(L, 0)$ to $(0,0)$; a streamline just above the plate surface $\vec{V}.\vec{n} = 0$. Shear forces summing to the drag force $-Di$ acting from the plate on to the retarded fluid
- The pressure is uniform, and there is no net pressure force. Since the flow is assumed incompressible and steady, then no unsteady terms and flows only across section $0h$ and $L\delta$:

$$\sum F_x = -D = \rho \int_1 u(0, y)(\vec{V}.\vec{n})\, dA + \rho \int_3 u\,(L, y)(\vec{V}.\vec{n})\, dA$$



Physical significance of θ

$$= \rho \int_0^h U(-U) b\, dy + \rho \int_0^\delta u(L,y)[u(L,y)] b\, dy$$

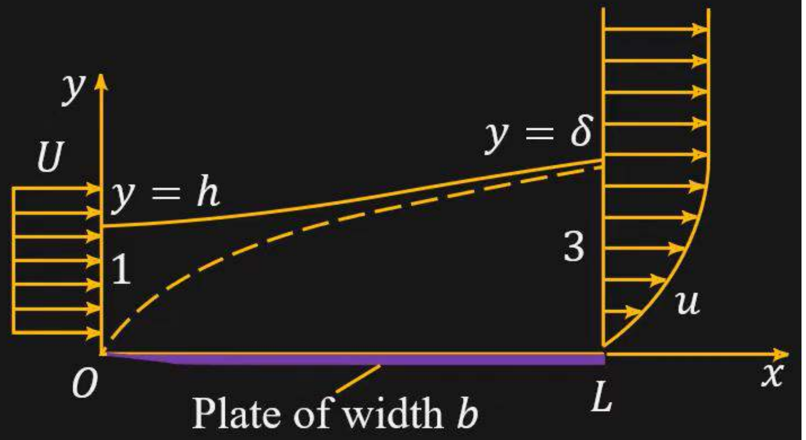
- Evaluating the first integral an rearranging it

$$D = \rho U^2 bh - \rho b \int_0^\delta u^2\, dy \Big|_{x=L}$$

- This could be considered the answer to the problem, but it is not useful because the height h is not known with respect to the shear layer thickness δ. This is found by applying the mass conservation, since the control volume forms a streamtube.

$$\rho \int_{CS} (\vec{V}.\vec{n})\ dA = 0$$

$$\Rightarrow \rho \int_0^h (-U) b\, dy + \rho \int_0^\delta ub\ dy \Big|_{x=L} = 0$$



Physical significance of θ

$$Uh = \int_0^{\delta} u \, dy \Big|_{x=L}$$

- Substituting the value of h in the expression of drag.

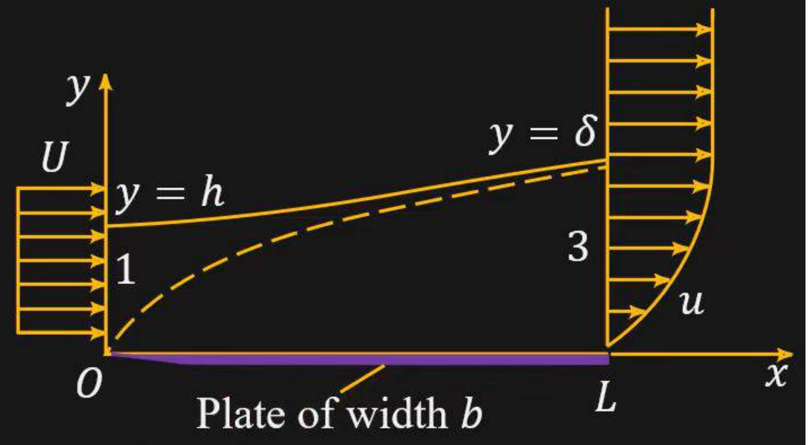
$$D = \rho U b \int_0^{\delta} u \, dy \Big|_{x=L} - \rho b \int_0^{\delta} u^2 \, dy \Big|_{x=L}$$

$$D = \rho b \int_0^{\delta} u(U-u) \, dy \Big|_{x=L}$$

$$\Rightarrow D = \rho U^2 b \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \Big|_{x=L}$$

$$\Rightarrow D = \rho U^2 b \theta$$

- Therefore the drag force on the plate can be found using the momentum thickness.



Order of Boundary Layer Thickness

- Recall the Stokes' first problem- a impulsively started plate in a stagnant fluid.
- We had defined the diffusion layer thickness as the distance from the plate where the fluid velocity is 1% of plate velocity.

$$u(y=\delta) = 0.01U$$

- From the analytical solution we found that $\delta \sim \sqrt{\nu t}$ where $\nu = \mu/\rho$.
- In other words, the effect of the moving plate is felt at distance of δ after time t.
- The diffusion layer thickness is similar to the boundary layer thickness in the sense that viscous effects are confined within these boundaries.
- In case of boundary layer, free stream fluid particles interacts with the plate for time $t \sim L/U$ when they reach a distance L from the tip of the plate. Therefore, the BL thickness is

$$\delta \sim \sqrt{\frac{\mu}{\rho}\frac{L}{U}}$$

Order of Boundary Layer Thickness

$$\Rightarrow \delta \sim \sqrt{\frac{\mu}{\rho U L} L^2}$$

$$\Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}$$

- We can see that BL thickness is inversely related to the Reynolds number.
- The above approximation can also be derived by considering the balance of inertia and viscous forces inside BL.
- At we move away from the plate, the viscous forces tends towards zero from a maximum value while the inertia forces increases from zero towards a maximum value.
- Inside boundary layer - Viscous force ~ inertia force.

$$\mu \frac{\partial^2 u}{\partial y^2} \sim \rho u \frac{\partial u}{\partial x}$$

Order of Boundary Layer Thickness

- Inside BL: $u \sim U$, $y \sim \delta$, $x \sim L$

$$\Rightarrow \mu\left(\frac{U}{\delta^2}\right) \sim \rho U \frac{U}{L}$$

$$\Rightarrow \left(\frac{L}{\delta}\right)^2 \sim \frac{\rho U L}{\mu}$$

$$\Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}$$

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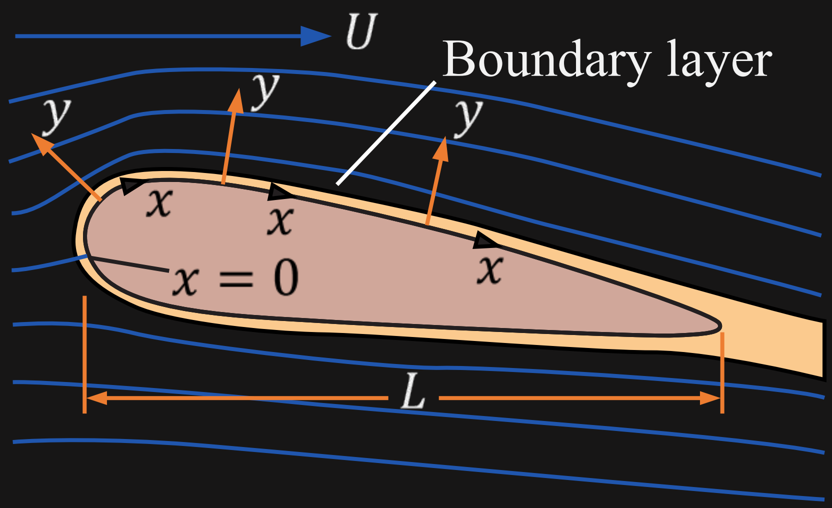
Section2: Boundary Layer Theory

Lecture 7: Boundary layer equation

Boundary layer equation

- Consider external flow over a body as shown.
- The coordinate system is chosen so that x/y-axis are tangent/normal the surface of the body everywhere.
- We make following assumptions:

1. 2D, steady and incompressible flow.
2. The curvature of the body is small so that the xy coordinate system can be approximated as a Cartesian coordinate system.
3. The boundary layer thickness is small.



- The last assumption can be justified in case of high Reynolds number flows. Recall that the magnitude of boundary layer thickness is

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}$$

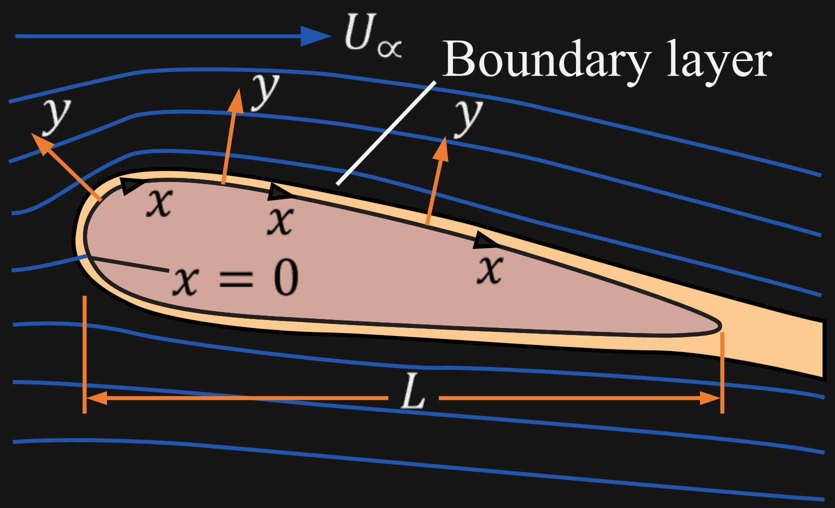
- For $Re_L \gg 1$ cases we have $\delta \ll L$

Boundary layer equation

- The Navier-Stokes equations will be simplified based on the assumptions.
- Let us first consider the non-dimensional continuity equation

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

- Where $U = u/U_\infty$ and $V = v/U_\infty$ are the non-dimensional velocity components along x and y-directions.
- $X = x/L$ and $Y = y/L$ are the non-dimensional coordinates.
- U_∞ and L are the characteristic (reference) velocity and length for the flow.
- Within boundary layer, the magnitude of stream velocity is in the order of free-stream velocity magnitude $\Rightarrow u \sim U_\infty$.
- Therefore, $U \sim O(1)$ where big O notation represents order of magnitude.

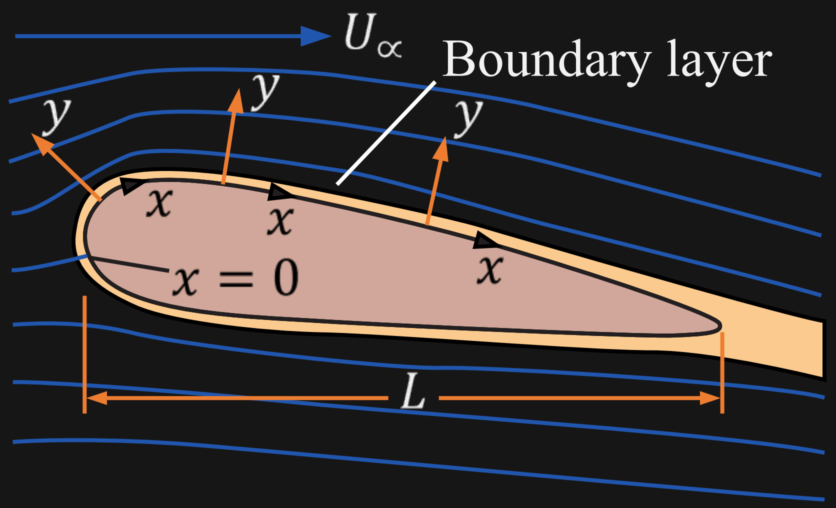


Boundary layer equation

- Within boundary layer, the stream wise coordinate is in the order of size of the body $\Rightarrow x \sim L$.
- Therefore, $X \sim O(1)$.
- Within boundary layer, the cross-stream coordinate is in the order of boundary layer thickness $\Rightarrow y \sim \delta$.
- Therefore, $Y \sim O(\epsilon)$ where $\epsilon = \delta/L \ll 1$.
- Putting back into the continuity equation

$$\frac{O(1)}{O(1)} + \frac{V}{O(\epsilon)} = 0$$

- From above equation we see that, for continuity equation to be satisfied, the order of magnitude of V should be small $\Rightarrow V \sim O(\epsilon)$ where $\epsilon^2 = 1/Re$.
- Therefore, $v \ll u$ inside boundary layer.



Boundary layer equation

- Next we consider the non-dimensional y-momentum equation

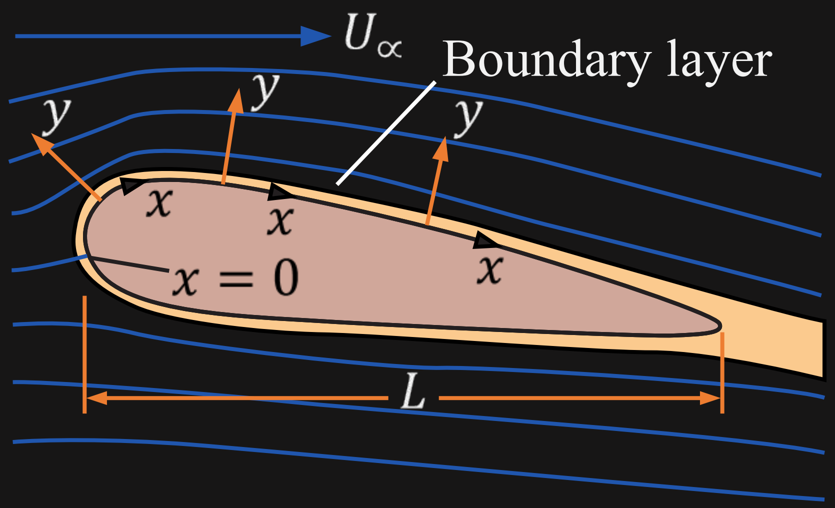
$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+\frac{1}{Re}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)$$

- Putting the order of magnitudes

$$O(1)\frac{O(\epsilon)}{O(1)}+O(\epsilon)\frac{O(\epsilon)}{O(\epsilon)}=-\frac{\partial P}{\partial Y}+O(\epsilon^2)\left(\frac{O(\epsilon)}{O(1)}+\frac{O(\epsilon)}{O(\epsilon^2)}\right)$$

$$\Rightarrow O(\epsilon)+O(\epsilon)=-\frac{\partial P}{\partial Y}+O(\epsilon^3)+O(\epsilon)$$

$$\Rightarrow \frac{\partial P}{\partial Y}=O(\epsilon)$$



- The above equation implies that there is negligible pressure variation in y-direction inside the boundary layer.
- Therefore, $p=p(x)$ inside the boundary layer.

Boundary layer equation

- Lastly, we consider the non-dimensional x-momentum equation

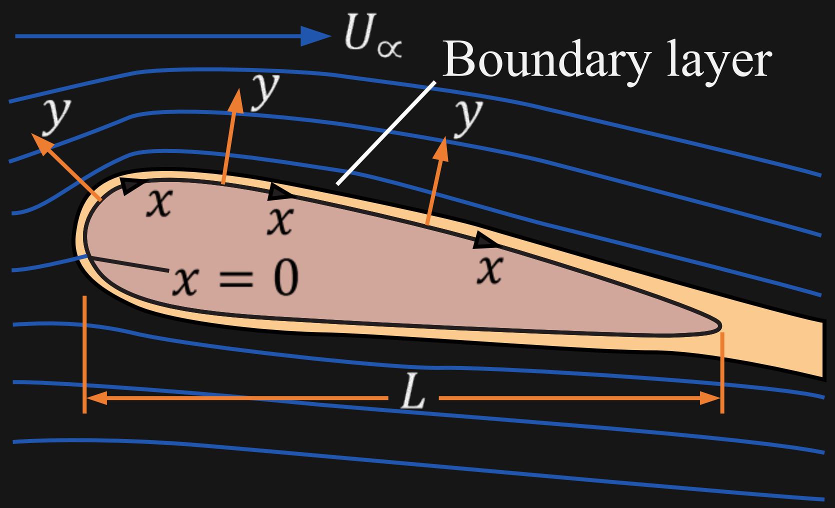
$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right)$$

- Putting the order of magnitudes

$$O(1)\frac{O(1)}{O(1)} + O(\epsilon)\frac{O(1)}{O(\epsilon)} = -\frac{\partial P}{\partial X} + \epsilon^2\left(\frac{O(1)}{O(1)} + \frac{O(1)}{O(\epsilon^2)}\right)$$

$$\Rightarrow O(1) + O(1) = -\frac{\partial P}{\partial X} + O(\epsilon^2) + O(1)$$

- The above equation implies that the term $\partial^2 U/\partial X^2$ can be neglected due to having smaller magnitude as compared to rest of the terms.
- Secondly, the pressure gradient in x-direction is significant.
- Since $\partial P/\partial Y$ is negligible inside BL, the value of pressure remains same as we move along a line $x =$ constant from inside BL to outside BL.



Boundary layer equation

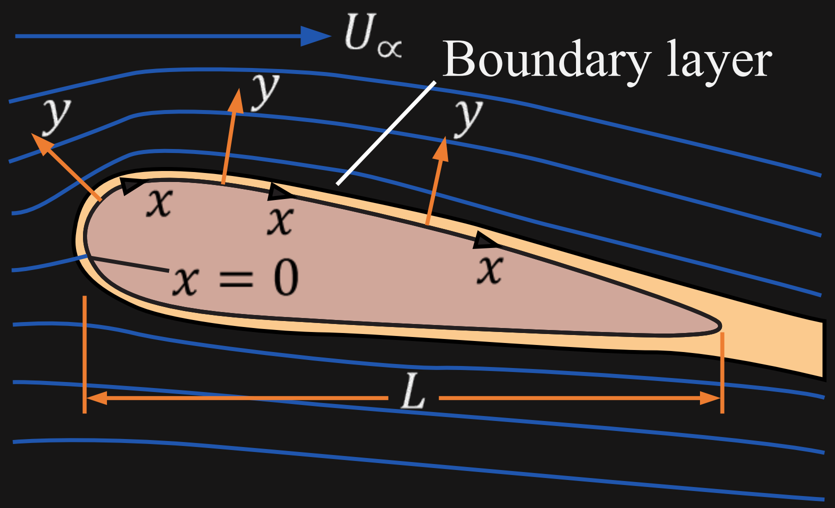
- Therefore, the pressure and thereby the term $\partial P/\partial X$ inside the BL can be obtained from obtained from outside BL.
- Outside BL, the flow is inviscid and the Bernoulli's equation is applicable (neglect gravity).

$$p + \frac{1}{2}\rho u_e^2 = \text{constant}$$

- u_e is the flow velocity in the inviscid region just outside the BL.
- Note that u_e can be different from U_∞ since the free stream could accelerate near the body (refer to potential flow around cylinder).
- Differentiating the Bernoulli's equation with respect to x

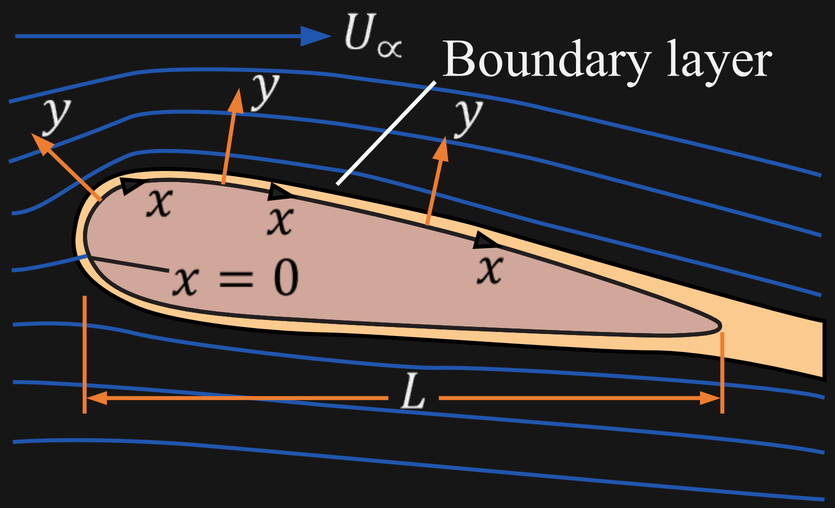
$$\frac{dp}{dx} + \rho u_e \frac{du_e}{dx} = 0$$

- We can get the non-dimensional form of the above equation by multiplying by $L/\rho U_\infty^2$.



Boundary layer equation

- $$\frac{1}{\rho U_\infty^2}\frac{dp}{d(x/L)} + \frac{u_e}{U_\infty}\frac{d(u_e/U_\infty)}{d(x/L)} = 0$$
$$\Rightarrow \frac{dP}{dX} + U_e\frac{dU_e}{dX} = 0$$
- where $P = p/\rho U_\infty^2$ and $U_e = u_e/U_\infty$.
- Finally, the boundary layer equations are
$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$
$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{dP}{dX} + \frac{1}{Re}\frac{\partial^2 U}{\partial Y^2}$$
$$\frac{dP}{dX} + U_e\frac{dU_e}{dX} = 0$$
- Ludwig Prandtl in 1904 is credited for these equations. BL equations are mathematically parabolic and are solved by beginning at the leading edge and marching downstream.



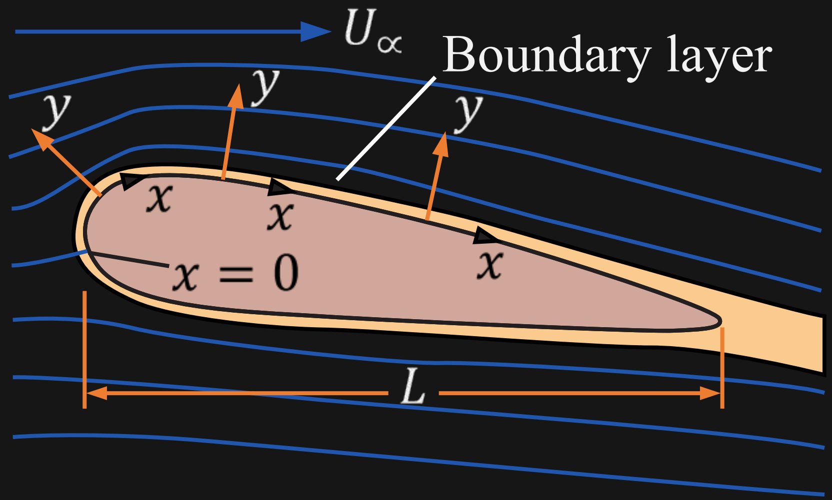
Boundary layer equation

- The dimensional form of the boundary layer equations is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{1}{\rho}\frac{dp}{dx}+u_e\frac{du_e}{dx}=0$$



- These equations are subjected to the following boundary conditions
- No slip: $u(x,0)=v(x,0)=0$
- Merger with the inviscid flow: $u(x,y\rightarrow\infty)=u_e$

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Section2: Boundary Layer Theory

Lecture 8: Blasius boundary layer

Flow over a flat plate

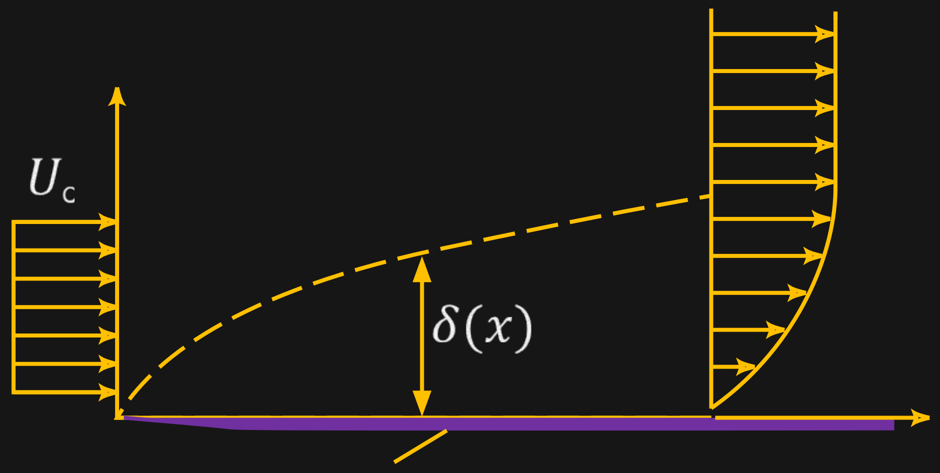
- Consider the flow over a flat thin plate.
- The plate is semi-infinite i.e. it extends from $x = 0$ to ∞.
- The flow is assumed to be 2D, steady and laminar.
- Since the plate and BL thickness are negligible, the inviscid flow is unaffected by the presence of the plate $\Rightarrow u_e = U_\infty$.
- Hence, no pressure gradient is imposed by the inviscid region

$$\frac{dp}{dx} = -\rho u_e \frac{du_e}{dx} = 0$$

- Hence, the flow in the flat plate boundary layer is governed by the following simplified boundary layer equations

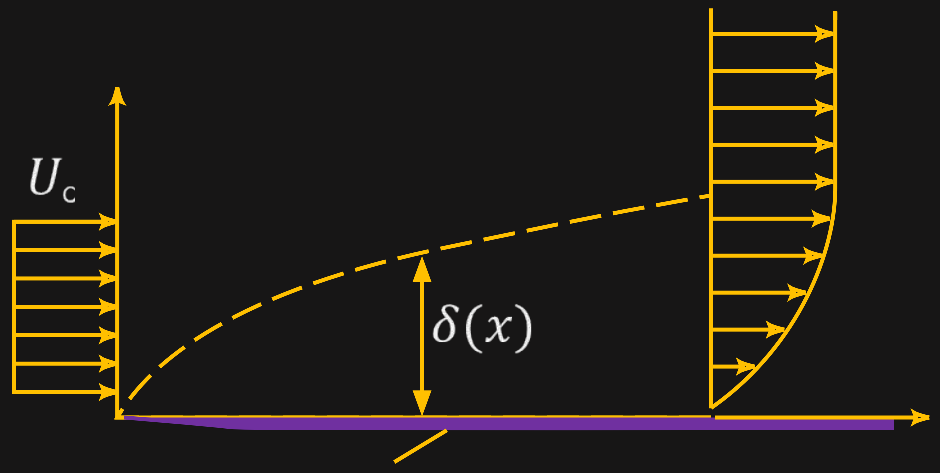
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$



Flow over a flat plate

- Subjected to the following boundary conditions
- No slip: $u(x,0) = v(x,0) = 0$.
- Initial profile: $u(0,y) = U_\infty$.
- Merger with the inviscid flow: $u(x, y \to \infty) = U_\infty$.
- Notice that there is no characteristic length scale for this flow.
- A similarity solution exists whenever a problem lacks a characteristic length or time scale.
- This is similar to what we had seen in the Stokes' first problem.
- The flat plate boundary layer velocity profile is self similar i.e. velocity profiles at different x locations are similar to each other.
- The non-dimensional velocity profile u/U_∞ is same at all x locations.



Flow over a flat plate

- Within boundary layer

$$\frac{u}{U_\infty} = F\left(\frac{y}{\delta}\right)$$

- We know that $\delta(x) \sim \frac{x}{\sqrt{Re}} = x\sqrt{\frac{\nu}{U_\infty x}} = \sqrt{\frac{\nu x}{U_\infty}}$

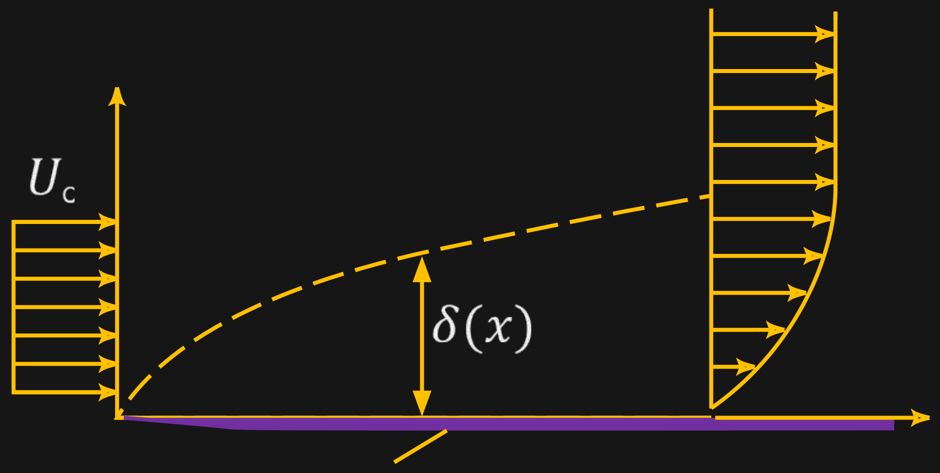
- Therefore, we can write

$$\frac{u}{U_\infty} = F(\eta) \text{ where } \eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

- Recall that the Continuity equation can automatically be satisfied by using stream function formulation

$$u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right) \Rightarrow \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0$$



Flow over a flat plate

- Let us find out the expression of ψ.

$$u \equiv U_\infty F(\eta) = \frac{\partial \psi}{\partial y}$$
$$\Rightarrow \psi = U_\infty \int F(\eta)\, dy$$

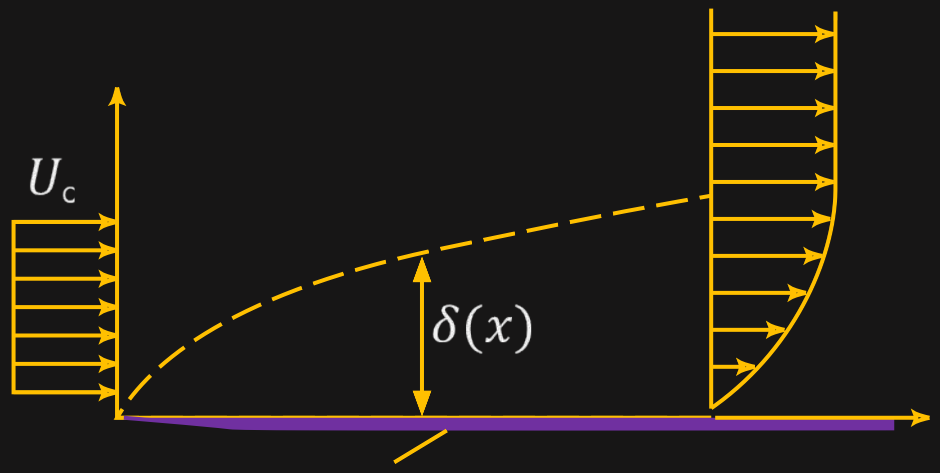
- Using $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$ we get

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \partial y = \sqrt{\frac{\nu x}{U_\infty}}\,\partial \eta$$

- Using this into the previous integral equation

$$\psi = U_\infty \sqrt{\frac{\nu x}{U_\infty}} \int F(\eta)\, d\eta$$
$$\psi = \sqrt{U_\infty \nu x}\, f(\eta)$$

- Where $f(\eta) = \int F(\eta)\, d\eta$.



Flow over a flat plate

- Now we can obtain the y component of fluid velocity

$$v = -\frac{\partial \psi}{\partial x}$$

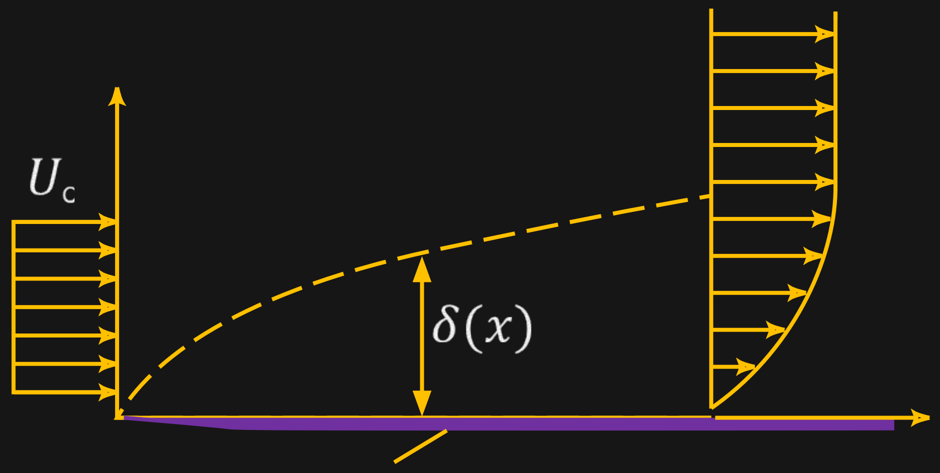
$$\Rightarrow v = -\frac{\partial}{\partial x}\left(\sqrt{U_\infty \nu x}\, f(\eta)\right)$$

- Using chain rule

$$v = -\left[\sqrt{U_\infty \nu}\,\frac{1}{2\sqrt{x}} f(\eta) + \sqrt{U_\infty \nu x}\,\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}\right]$$

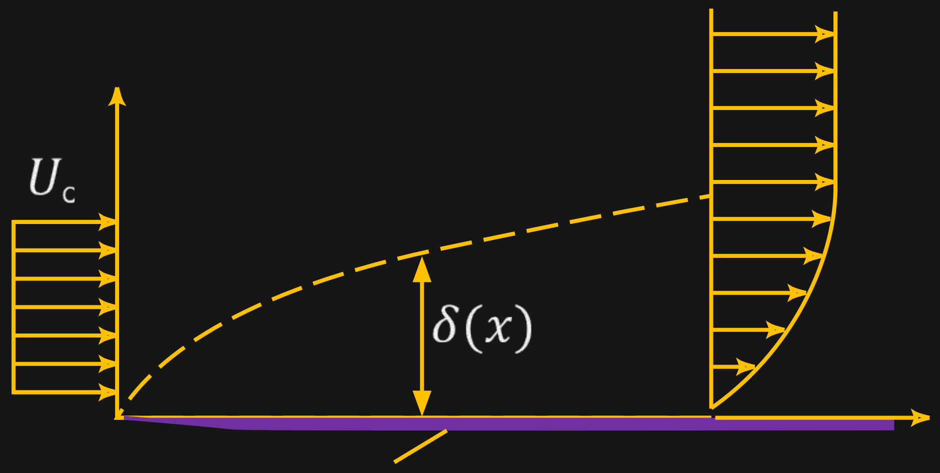
$$\Rightarrow v = -\sqrt{\frac{U_\infty \nu}{x}}\frac{f(\eta)}{2} - \sqrt{U_\infty \nu x}\, f'(\eta)\, y\sqrt{\frac{U_\infty}{\nu}}\left(-\frac{1}{2}\right)\frac{1}{x\sqrt{x}}$$

$$\Rightarrow v = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}\left[y\sqrt{\frac{U_\infty}{\nu x}} f'(\eta) - f(\eta)\right]$$



Flow over a flat plate

- $$\Rightarrow v = \frac{1}{2}\left[\eta f'(\eta) - f(\eta)\right]\sqrt{\frac{U_\infty \nu}{x}}$$
- Next, we express $\partial u/\partial x$ in terms of η.
$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[U_\infty F(\eta)\right] = U_\infty F'(\eta)\frac{\partial \eta}{\partial x}$$
- Note that $F'(\eta) = f(\eta)$ therefore
$$\frac{\partial u}{\partial x} = U_\infty f''(\eta) y \sqrt{\frac{U_\infty}{\nu}}\left(-\frac{1}{2x\sqrt{x}}\right)$$
$$= \frac{U_\infty}{2x} f''(\eta) y \sqrt{\frac{U_\infty}{\nu x}}$$
$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{U_\infty}{2}\frac{\eta}{x} f''(\eta)$$



Flow over a flat plate

- The next term in the momentum equation is $\partial u/\partial y$.

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial x}[U_\infty F(\eta)]=U_\infty F'(\eta)\frac{\partial \eta}{\partial y}$$

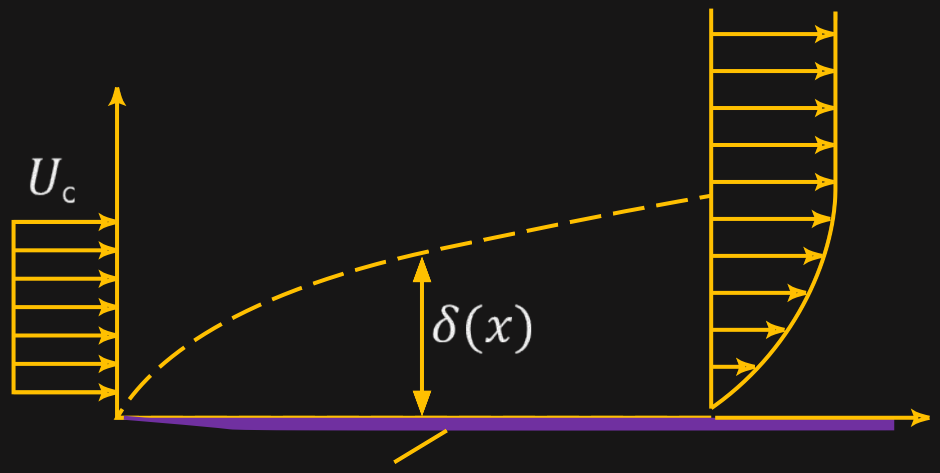
$$\Rightarrow \frac{\partial u}{\partial y}=U_\infty\sqrt{\frac{U_\infty}{\nu x}}f''(\eta)$$

- The last unknow terms is $\partial^2 u/\partial y^2$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(U_\infty\sqrt{\frac{U_\infty}{\nu x}}f''(\eta)\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2}=U_\infty\sqrt{\frac{U_\infty}{\nu x}}f'''(\eta)\frac{\partial \eta}{\partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2}=\frac{U_\infty^2}{\nu x}\,f'''\,(\eta)$$



Flow over a flat plate

- Replacing every term into the momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow U_\infty f'(\eta)\left[-\frac{U_\infty}{2}\frac{\eta}{x}f''(\eta)\right]+\frac{1}{2}\left[\eta f'(\eta)-f(\eta)\right]\sqrt{\frac{U_\infty \nu}{x}}\left[U_\infty\sqrt{\frac{U_\infty}{\nu x}}f''(\eta)\right]=\nu\frac{U_\infty^2}{\nu x}f'''(\eta)$$

$$\Rightarrow \frac{U_\infty^2}{x}\left[-\frac{1}{2}\eta f'f''+\frac{1}{2}(\eta f'f''-ff'')\right]=\frac{U_\infty^2}{x}f'''$$

$$\Rightarrow f'''+\frac{1}{2}ff''=0$$

- This is the Blasius equation for flow over a flat plate.
- The equation is a third order nonlinear ODE.
- No analytical solution is available. It needs to be solved using some numerical technique.
- Still simpler than the original BL equations.

Flow over a flat plate

-
- The boundary conditions also need to be transformed using $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$.
- No slip: $u(x,0) = v(x,0) = 0$

$$\frac{u}{U_\infty}(\eta = 0) = f'(0) = 0$$
$$\frac{v}{U_\infty}(\eta = 0) = f(0) = 0$$

- Initial profile: $u(0,y) = U_\infty$.

$$\frac{u}{U_\infty}(\eta \to \infty) = f'(\eta \to \infty) = 1$$

- Merger with the inviscid flow: $u(x, y \to \infty) = U_\infty$.

$$\frac{u}{U_\infty}(\eta \to \infty) = f'(\eta \to \infty) = 1$$

ME 221: Fluid Mechanics II

Instructor

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Section2: Boundary Layer Theory

Lecture 9: Solution of Blasius equation

Flow over a flat plate

- This is the Blasius equation for flow over a flat plate.

$$f''' + \frac{1}{2} f f'' = 0$$

- The equation is a third order nonlinear ODE.
- The boundary conditions are

$$f(0) = f'(0) = 0 \text{ and } f'(\eta \to \infty) = 1$$

- Let us assume the following series solution

$$f(\eta) = A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \cdots$$

- Then

$$f'(\eta) = A_1 + A_2\eta + \frac{A_3}{2!}\eta^2 + \frac{A_4}{3!}\eta^3 + \cdots$$

$$f''(\eta) = A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \cdots$$

Flow over a flat plate

- $$f'''(\eta) = A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \cdots$$
- From $f(0) = 0 \Rightarrow A_0 = 0$
- From $f'(0) = 0 \Rightarrow A_1 = 0$
- From the governing equation

$$f'''(0) = -\frac{1}{2}f(0)f''(0) = 0$$
$$\Rightarrow A_3 = 0$$

- Putting the series forms into the ODE

$$2\left[A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \cdots\right] + \left[\frac{A_2}{2!}\eta^2 + \frac{A_4}{4!}\eta^4 + \frac{A_5}{5!}\eta^5 + \cdots\right]\left[A_2 + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \cdots\right] = 0$$

$$\Rightarrow 2A_4\eta + \left[2\frac{A_5}{2!} + \frac{A_2^2}{2!}\right]\eta^2 + 2\frac{A_6}{3!}\eta^3 + \left[2\frac{A_7}{4!} + A_2\frac{A_4}{4!} + \frac{A_2}{2!}\frac{A_4}{2!}\right]\eta^4 + \left[2\frac{A_8}{5!} + A_2\frac{A_5}{5!} + \frac{A_2}{2!}\frac{A_5}{3!}\right]\eta^5 + \cdots = 0$$

Flow over a flat plate

- The last equation is satisfied for all η when the coefficients of different powers of η are equal to zero.

$$2A_4 = 0 \Rightarrow A_4 = 0$$

$$2\frac{A_5}{2!} + \frac{A_2^2}{2!} = 0 \Rightarrow A_5 = -\frac{A_2^2}{2}$$

$$2\frac{A_6}{3!} = 0 \Rightarrow A_6 = 0$$

$$2\frac{A_7}{4!} + A_2\frac{A_4}{4!} + \frac{A_2}{2!}\frac{A_4}{2!} = 0 \Rightarrow A_7 = 0$$

$$2\frac{A_8}{5!} + A_2\frac{A_5}{5!} + \frac{A_2}{2!}\frac{A_5}{3!} = 0 \Rightarrow A_8 = \frac{11}{4}A_2^3$$

- Substituting the values of different coefficients

$$f(\eta) = \frac{A_2}{2!}\eta^2 - \frac{A_2^2}{2}\frac{\eta^5}{5!} + \frac{11}{4}A_2^3\frac{\eta^8}{8!} - \frac{375}{8}A_2^4\frac{\eta^{11}}{11!} + \cdots$$

$$\Rightarrow f(\eta) = A_2^{1/3}\left[\frac{1}{2!}\left(A_2^{1/3}\eta\right)^2 - \frac{1}{2}\frac{1}{5!}\left(A_2^{1/3}\eta\right)^5 + \frac{11}{4}\frac{1}{8!}\left(A_2^{1/3}\eta\right)^8 - \frac{375}{8}\frac{1}{11!}\left(A_2^{1/3}\eta\right)^{11} + \cdots\right]$$

Flow over a flat plate

- which can also be written as

$$f(\eta) = A_2^{1/3} F\left(A_2^{1/3}\eta\right)$$

- The coefficient A_2 is still unknown which can be determined from the last boundary condition

$$\lim_{\eta\to\infty} f'(\eta) = 1$$

$$\lim_{\eta\to\infty} \left[A_2^{2/3} F'\left(A_2^{1/3}\eta\right)\right] = 1$$

$$A_2 = \left[\frac{1}{\lim\limits_{\eta\to\infty} F'\left(A_2^{1/3}\eta\right)}\right]^{3/2}$$

- The value of A_2 can be found numerically. Iteratively

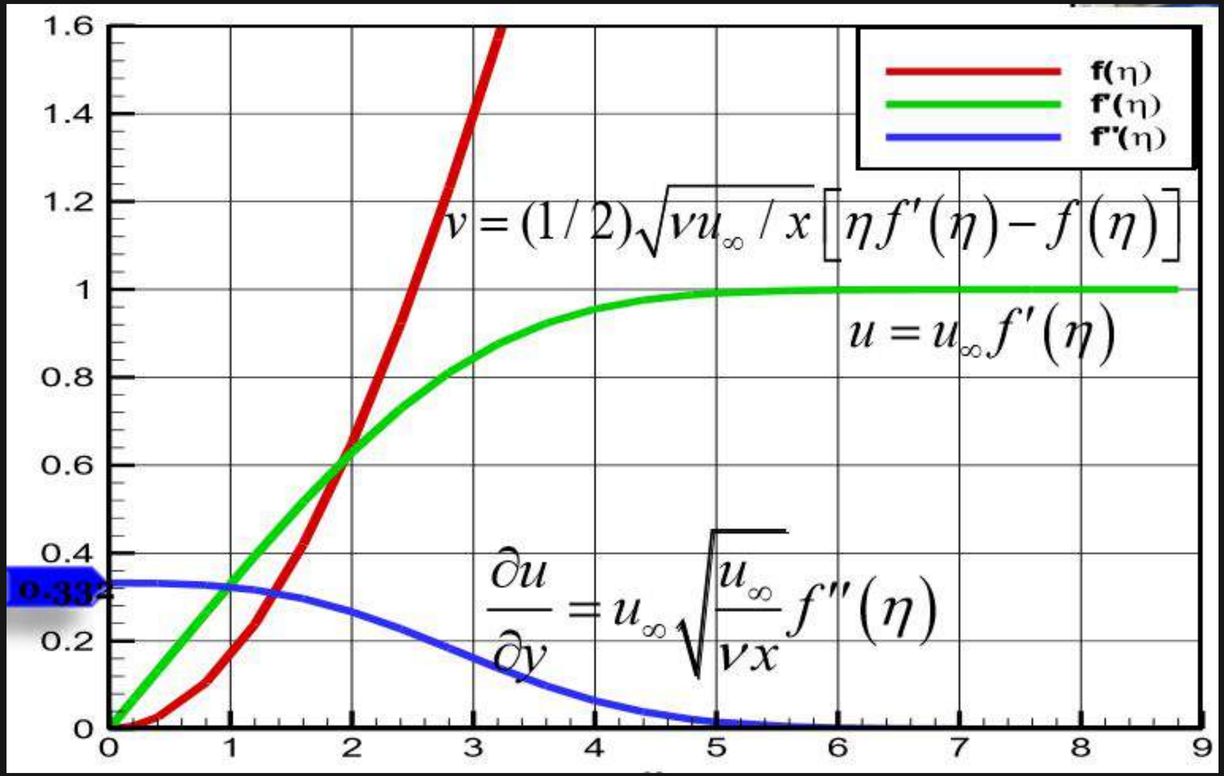
$$A_{2,k} = \left[\frac{1}{\lim\limits_{\eta\to\infty} F'\left(A_{2,k-1}^{1/3}\eta\right)}\right]^{3/2}$$

- Where k is the iteration number.

Flow over a flat plate

- We start the iteration by assuming a value of $A_{2,0}$ and then use the last equation to find $A_{2,1}$.
- Then $A_{2,1}$ is used to find $A_{2,2}$ and so on.
- We can stop when the value of A_2 between two successive iteration does not change by more than a predefined small value.
- The numerical solution gives $A_2 = f''(0) = 0.33206$.
- The velocity profile $\frac{u}{U_\infty} = f'(\eta)$ has been plotted in the figure.
- $f'(\eta) \approx 0.99$ for $\eta = 4.9$.

$$\Rightarrow \delta\sqrt{\frac{U_\infty}{\nu x}} = 4.9$$



Flow over a flat plate

- $$\Rightarrow \delta = 4.9\sqrt{\frac{\nu x}{U_\infty}} = 4.9x\sqrt{\frac{\nu}{U_\infty x}}$$
$$\Rightarrow \frac{\delta}{x} = \frac{4.9}{\sqrt{Re_x}}$$

- Where $Re_x = U_\infty x/\nu$.
- Wall shear stress

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu U_\infty f''(0)\frac{\partial \eta}{\partial y}$$
$$\Rightarrow \tau_w = \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} f''(0) = \frac{\mu U_\infty^2}{\nu}\sqrt{\frac{\nu}{U_\infty x}} f''(0)$$
$$\Rightarrow \tau_w(x) = \frac{\rho U_\infty^2}{\sqrt{Re_x}} f''(0) = 0.332\frac{\rho U_\infty^2}{\sqrt{Re_x}}$$

Flow over a flat plate

- Skin friction coefficient

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$$

$$\Rightarrow C_f = \frac{0.664}{\sqrt{Re_x}}$$

- Total drag force per unit width on length L

$$D = \int_0^L \tau_w \times (dx \times 1)$$

$$\Rightarrow D = 0.332\rho U_\infty^2 \int_0^L \frac{dx}{\sqrt{Re_x}} = 0.332\rho U_\infty^2 \sqrt{\frac{\nu}{U_\infty}} \int_0^L \frac{dx}{\sqrt{x}}$$

$$\Rightarrow D = 0.332\rho U_\infty^2 \sqrt{\frac{\nu}{U_\infty}}\, 2\sqrt{x}\Big|_0^L = 0.664\rho U_\infty^2 \sqrt{\frac{\nu}{U_\infty}}\, \sqrt{L}$$

Flow over a flat plate

- $$\Rightarrow D = \frac{0.664}{\sqrt{Re_L}} \rho U_\infty^2 L$$
- Drag coefficient

$$C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 L}$$

$$\Rightarrow C_D = \frac{1.328}{\sqrt{Re_L}}$$

- Note that

$$C_D = \frac{1}{L}\int_0^L C_f \, dx$$

- Displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy$$

Flow over a flat plate

-

$$\Rightarrow \delta^* = \int_0^\infty [1 - f'(n)] \sqrt{\frac{\nu x}{U_\infty}} dy$$

$$\Rightarrow \delta^* = 1.721 \sqrt{\frac{\nu_x}{U_\infty}}$$

$$\Rightarrow \frac{\delta^*}{x} = \frac{1.721}{\sqrt{Re_x}}$$

$$\Rightarrow \delta \approx 2.85\delta^*$$

- Recall that

$$D(x) = \rho U_\infty^2 \, \theta(x)$$

$$\Rightarrow \frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

$$\Rightarrow \delta \approx 7.38\theta$$

ME 221: Fluid Mechanics II

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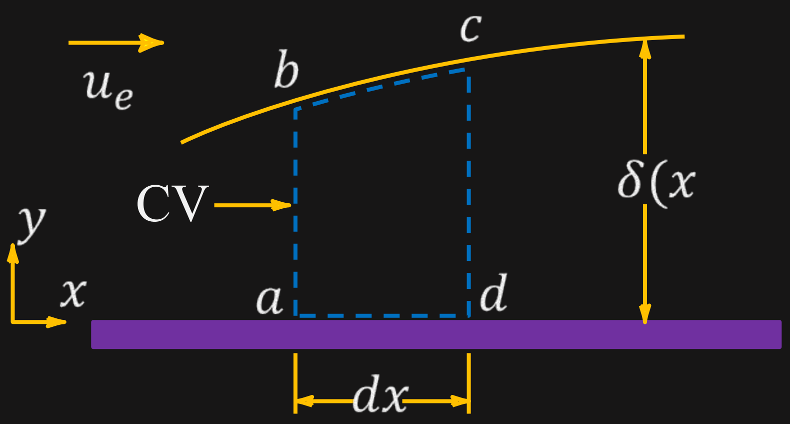
Section2: Boundary Layer Theory

Lecture 10: Momentum integral equation

Von-Karman momentum integral equation



- Consider the boundary layer region over a surface as shown.
- The flow is assumed to be incompressible, steady and 2D.
- We consider a differential control volume *abcd* as show in the figure.
- If $\dot{m}_{ab}$, $\dot{m}_{bc}$ and $\dot{m}_{cd}$ are mass flow rate leaving surfaces ab, bc and cd then from mass conservation for a steady flow

$$\dot{m}_{ab} + \dot{m}_{bc} + \dot{m}_{cd} = 0$$

- If $u(y)$ and δ are the velocity profile and boundary layer thickness at section ab then

$$\dot{m}_{ab} = -b \int_0^{\delta} \rho u \, dy = -\dot{m}_x$$

- Where b is the width (into the paper) of the control volume.
- For surface cd

$$\dot{m}_{cd} = \dot{m}_{x+dx}$$

Von-Karman momentum integral equation

- From Taylor series

$$\dot{m}_{cd} = \dot{m}_x + \frac{\partial \dot{m}_x}{\partial x}\, dx$$

- Then the mass flux from surface bc is (using mass conservation)

$$\dot{m}_{bc} = -(\dot{m}_{ab} + \dot{m}_{cd})$$

$$\Rightarrow \dot{m}_{bc} = -\left(-\dot{m}_x + \dot{m}_x + \frac{\partial \dot{m}_x}{\partial x}\, dx\right)$$

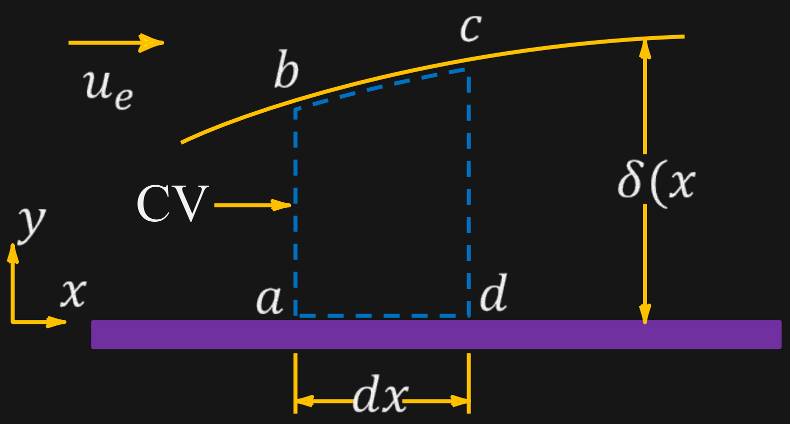
$$\Rightarrow \dot{m}_{bc} = -\frac{\partial \dot{m}_x}{\partial x}\, dx$$

$$\Rightarrow \dot{m}_{bc} = -b\frac{\partial}{\partial x}\left\{\int_0^{\delta} \rho u\, dy\right\} dx$$

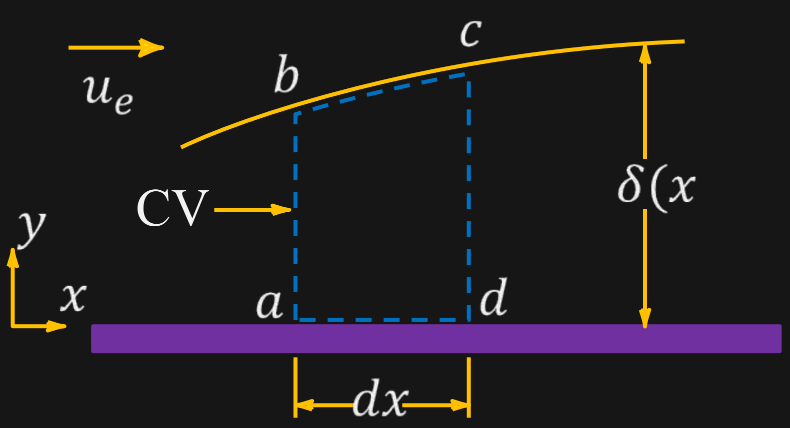
- Next we apply steady state x momentum equation to the CV.

$$F_s = \dot{M}_{ab} + \dot{M}_{bc} + \dot{M}_{cd}$$

- F_s and $\dot{M}$ represent x component of surface forces and momentum flux.



Von-Karman momentum integral equation



- Gravity and other body forces are assumed to be zero.
- The x component of momentum flux through surface ab is

$$\dot{M}_{ab} = -b\int_0^\delta \rho u^2\, dy = -\dot{M}_x$$

- For surface cd

$$\dot{M}_{cd} = \dot{M}_{x+dx} = \dot{M}_x + \frac{\partial \dot{M}_x}{\partial x}\, dx$$

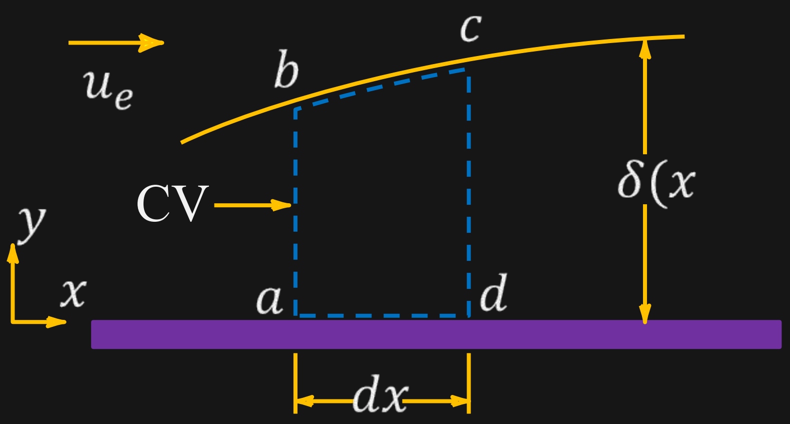
- Since surface bc is the edge of the boundary layer, the x component of velocity is u_e everywhere on the surface.
- The x component of momentum flux through surface bc is

$$\dot{M}_{bc} = \int_0^\delta \rho u_e (\vec{V}\cdot\hat{n})\, b\, dy$$

$$\Rightarrow \dot{M}_{bc} = u_e\int_0^\delta \rho(\vec{V}\cdot\hat{n}) b\, dy = u_e \dot{m}_{bc}$$

Von-Karman momentum integral equation

- $$\Rightarrow \dot{M}_{bc} = -u_e b \frac{\partial}{\partial x}\left\{\int_0^\delta \rho u \, dy\right\} dx$$



- Then the x momentum equation is

$$F_s = -\dot{M}_x + \dot{M}_x + \frac{\partial \dot{M}_x}{\partial x}\, dx - u_e b \frac{\partial}{\partial x}\left\{\int_0^\delta \rho u \, dy\right\} dx$$

$$\Rightarrow F_s = b\left[\frac{\partial}{\partial x}\left\{\int_0^\delta \rho u^2 \, dy\right\} dx - u_e \frac{\partial}{\partial x}\left\{\int_0^\delta \rho u \, dy\right\} dx\right]$$

- The shear force at surface bc is negligible since it is the edge of the BL.
- If τ_w is the shear stress at location x then shear force on surface ad is

$$F_{ad} \approx -\tau_w b \, dx$$

- A second order approximation will be

$$F_{ad} \approx -\left(\tau_w + \frac{1}{2}\frac{\partial \tau_w}{\partial x} dx\right) b \, dx$$

Von-Karman momentum integral equation

- Since pressure does not vary with y inside the BL, the pressure force on surface ab is

$$F_{ab} = pb\delta$$

- Similarly, pressure force on surface cd is

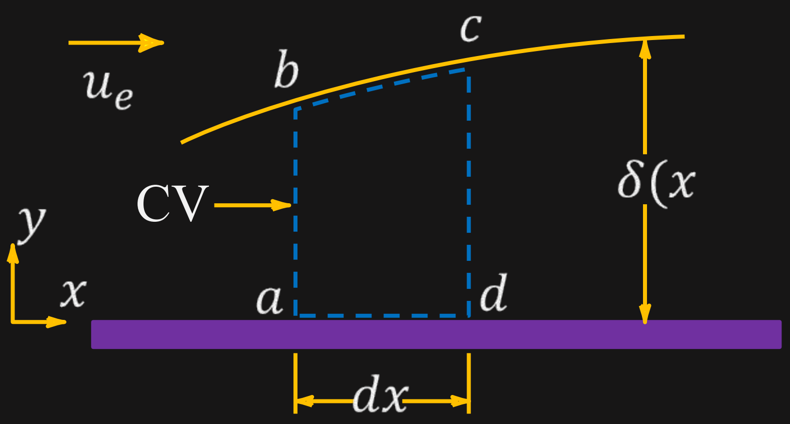
$$F_{cd} = -\left(p + \frac{dp}{dx}\,dx\right) b(\delta + d\delta)$$

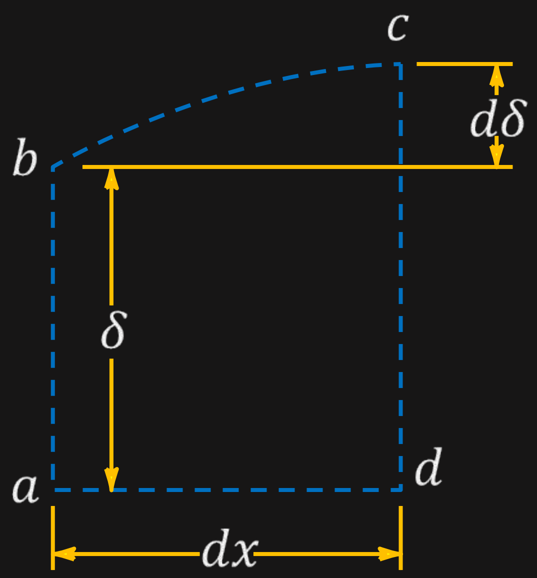
- The x component of the pressure force on surface bc will be equal to the pressure force on the projected area

$$F_{bc} = \left(p + \frac{1}{2}\frac{dp}{dx}\,dx\right) b\,d\delta$$

- The net surface force in x direction is

$$F_s = F_{ab} + F_{bc} + F_{cd} + F_{ad}$$





Von-Karman momentum integral equation

- $$\Rightarrow F_s = \cancel{pb\delta} + \left(\cancel{p} + \frac{1}{2}\frac{dp}{dx}\,dx\right) b\,d\delta - \left(\cancel{p} + \frac{dp}{dx}\,dx\right) b(\delta + d\delta) - \left(\tau_w + \frac{1}{2}\frac{\partial \tau_w}{\partial x}\,dx\right) b\,dx$$
$$\Rightarrow F_s = b\left(-\frac{1}{2}\frac{dp}{dx}\,dx\,d\delta - \frac{dp}{dx}\delta\,dx - \tau_w\,dx - \frac{1}{2}\frac{\partial \tau_w}{\partial x}\,dx^2\right)$$
- Since the CV is infinitesimal
$$F_s \approx b\left(-\frac{dp}{dx}\delta\,dx - \tau_w\,dx\right)$$
- Coming back to the x momentum equation
$$F_s = b\left[\frac{\partial}{\partial x}\left\{\int_0^\delta \rho u^2\,dy\right\}dx - u_e\frac{\partial}{\partial x}\left\{\int_0^\delta \rho u\,dy\right\}dx\right]$$
$$-\frac{dp}{dx}\delta - \tau_w = \frac{\partial}{\partial x}\left\{\int_0^\delta \rho u^2\,dy\right\} - u_e\frac{\partial}{\partial x}\left\{\int_0^\delta \rho u\,dy\right\}$$
- The pressure gradient can be obtained using the Bernoulli equation and $\delta = \int_0^\delta dy$ then

Von-Karman momentum integral equation

- $$\Rightarrow \rho u_e \frac{du_e}{dx}\int_0^\delta dy - \tau_w = \frac{\partial}{\partial x}\left\{\int_0^\delta \rho u^2\, dy\right\} - u_e \frac{\partial}{\partial x}\left\{\int_0^\delta \rho u\, dy\right\}$$
$$\Rightarrow \tau_w = -\frac{\partial}{\partial x}\left\{\int_0^\delta \rho u^2\, dy\right\} + u_e \frac{\partial}{\partial x}\left\{\int_0^\delta \rho u\, dy\right\} + \frac{du_e}{dx}\int_0^\delta \rho u_e\, dy$$
$$\Rightarrow \tau_w = -\frac{\partial}{\partial x}\left\{\int_0^\delta \rho u^2\, dy\right\} + \frac{\partial}{\partial x}\left\{u_e \int_0^\delta \rho u\, dy\right\} - \frac{du_e}{dx}\int_0^\delta \rho u\, dy + \frac{du_e}{dx}\int_0^\delta \rho u_e\, dy$$
$$\Rightarrow \tau_w = \frac{\partial}{\partial x}\left\{\int_0^\delta \rho u(u_e - u)\, dy\right\} + \frac{du_e}{dx}\int_0^\delta \rho(u_e - u)\, dy$$
$$\Rightarrow \tau_w = \frac{\partial}{\partial x}\left\{u_e^2 \int_0^\delta \rho \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy\right\} + \frac{du_e}{dx} u_e \int_0^\delta \rho \left(1 - \frac{u}{u_e}\right) dy$$
$$\Rightarrow \frac{\tau_w}{\rho} = \frac{d}{dx}(u_e^2 \theta) + u_e \frac{du_e}{dx}\delta^*$$

- This is the momentum integral equation. The boundary layer thickness can be estimated by assuming a velocity profile.

Second approach

- The momentum integral equation can also be derived by integrating the boundary layer equations.
- Consider the momentum equation for boundary layer

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_e\frac{du_e}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

- Integrate the above equation from 0 to δ

$$\int_0^{\delta}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)dy=\int_0^{\delta}u_e\frac{du_e}{dx}\,dy+\int_0^{\delta}\nu\frac{\partial^2 u}{\partial y^2}\,dy$$

$$=\int_0^{\delta}u_e\frac{du_e}{dx}\,dy+\frac{1}{\rho}\,\mu\frac{\partial u}{\partial y}\Big|_0^{\delta}$$

$$=\int_0^{\delta}u_e\frac{du_e}{dx}\,dy+\frac{1}{\rho}\,(0-\tau_w)$$

$$\Rightarrow\int_0^{\delta}\left(u\frac{\partial u}{\partial x}-u_e\frac{du_e}{dx}+v\frac{\partial u}{\partial y}\right)dy=-\frac{\tau_w}{\rho}$$

Second approach

- Next, we integrate the continuity equation.

$$\int_0^\delta \frac{\partial u}{\partial x}\,dy + \int_0^\delta \frac{\partial v}{\partial y}\,dy = 0$$

$$\int_0^\delta \frac{\partial u}{\partial x}\,dy + v - 0 = 0$$

$$v_e = -\int_0^\delta \frac{\partial u}{\partial x}\,dy$$

- Using integration by parts for the second advection term of the momentum equation

$$\int_0^\delta v\frac{\partial u}{\partial y}\,dy = v\int \frac{\partial u}{\partial y}\,dy\,\Big|_0^\delta - \int_0^\delta \frac{\partial v}{\partial y}\left(\int \frac{\partial u}{\partial y}\,dy\right)dy$$

$$= vu\,\Big|_0^\delta - \int_0^\delta u\frac{\partial v}{\partial y}dy$$

$$= (v_e u_e - 0) - \int_0^\delta u\left(-\frac{\partial u}{\partial x}\right)dy$$

Second approach

- $$\Rightarrow \int_0^\delta v \frac{\partial u}{\partial y}\, dy = -u_e \int_0^\delta \frac{\partial u}{\partial x}\, dy + \int_0^\delta u \frac{\partial u}{\partial x} dy$$
- Coming back to the integral form of momentum equation

$$\int_0^\delta \left(u \frac{\partial u}{\partial x} - u_e \frac{du_e}{dx} + v \frac{\partial u}{\partial y} \right) dy = -\frac{\tau_w}{\rho}$$

$$\Rightarrow \int_0^\delta \left(u \frac{\partial u}{\partial x} - u_e \frac{du_e}{dx} - u_e \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} \right) dy = -\frac{\tau_w}{\rho}$$

- Add and subtract the term $u\, du_e/dx$

$$\int_0^\delta \left(u \frac{du_e}{dx} - u_e \frac{du_e}{dx} \right) dy + \int_0^\delta \left(u \frac{\partial u}{\partial x} - u \frac{du_e}{dx} - u_e \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} \right) dy = -\frac{\tau_w}{\rho}$$

$$\Rightarrow \frac{du_e}{dx} \int_0^\delta (u - u_e) dy + \int_0^\delta \left[u \left(\frac{\partial u}{\partial x} - \frac{du_e}{dx} \right) + \frac{\partial u}{\partial x} (u - u_e) \right] dy = -\frac{\tau_w}{\rho}$$

Second approach

- $$\Rightarrow u_e \frac{du_e}{dx}\int_0^{\delta}\left(1-\frac{u}{u_e}\right)dy + \int_0^{\delta}\left[u\left(\frac{du_e}{dx}-\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial x}(u_e-u)\right]dy = \frac{\tau_w}{\rho}$$
$$\Rightarrow u_e \frac{du_e}{dx}\delta^* + \int_0^{\delta}\frac{\partial}{\partial x}[u(u_e-u)]dy = \frac{\tau_w}{\rho}$$
$$\Rightarrow u_e \frac{du_e}{dx}\delta^* + \frac{d}{dx}\int_0^{\delta}u_e^2\left[\frac{u}{u_e}\left(1-\frac{u}{u_e}\right)\right]dy = \frac{\tau_w}{\rho}$$
$$\Rightarrow u_e \frac{du_e}{dx}\delta^* + \frac{d}{dx}\left[u_e^2\int_0^{\delta}\frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy\right] = \frac{\tau_w}{\rho}$$
$$\frac{\tau_w}{\rho} = \frac{d}{dx}(u_e^2\theta) + u_e\frac{du_e}{dx}\delta^*$$

ME 221: Fluid Mechanics II

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Section2: Boundary Layer Theory

Lecture 11: Approximation method

Karman-Pohlhausen Approximate Method

- The momentum integral equation is

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(u_e^2\theta) + u_e \frac{du_e}{dx}\delta^*$$

- for flow over flat plate u_e = constant. Then the equation simplifies to

$$\frac{\tau_w}{\rho} = u_e^2 \frac{d\theta}{dx}$$

- Let us assume a linear velocity profile for the flat plate boundary layer

$$\frac{u}{U_\infty} = f'(\eta) = a_0 + a_1\eta$$

- Where $\eta = y/\delta$.
- The coefficients can be determined using boundary conditions
- No slip BC: $u(y = 0) = 0 \rightarrow u(\eta = 0) = 0 \rightarrow a_0 = 0$
- Merger with free stream: $u(y = \delta) = U_\infty \rightarrow u(\eta = 1) = U_\infty \rightarrow a_1 = 1$

Karman-Pohlhausen Approximate Method

- Therefore, the linear velocity profile for the flat plate boundary layer is

$$\frac{u}{U_\infty} = f'(\eta) = \eta$$

- Momentum thickness

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

$$\Rightarrow \theta = \delta \int_0^1 \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) d\eta$$

- Using the linear velocity profile

$$\theta = \delta \int_0^1 \eta(1-\eta)\, d\eta$$

$$\Rightarrow \theta = \delta \left[\frac{\eta^2}{2} - \frac{\eta^3}{3}\right]_0^1 = \frac{\delta}{6}$$

Karman-Pohlhausen Approximate Method

- We also need wall shear stress

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu}{\delta}\frac{\partial u}{\partial \eta}\bigg|_{\eta=0}$$

- Using linear velocity profile $u = U_\infty \eta$

$$\tau_w = \frac{\mu}{\delta} U_\infty$$

- Coming back to the momentum integral equation

$$\frac{\tau_w}{\rho} = u_e^2 \frac{d\theta}{dx} \Rightarrow \frac{1}{\rho}\frac{\mu}{\delta} U_\infty = U_\infty^2 \frac{1}{6}\frac{d\delta}{dx}$$

$$\int_0^\delta \delta \, d\delta = \frac{6\nu}{U_\infty}\int_0^x dx \Rightarrow \frac{1}{2}\delta^2 = \frac{6\nu}{U_\infty} x = \frac{6x^2}{Re_x}$$

$$\frac{\delta}{x} = \frac{\sqrt{12}}{Re_x} \approx \frac{3.46}{Re_x}$$

- Linear profile underpredicts the boundary layer thickness.

Karman-Pohlhausen Approximate Method

- Let us assume a quadratic velocity profile for the flat plate boundary layer
$$\frac{u}{U_\infty} = f'(\eta) = a_0 + a_1\eta + a_2\eta^2$$
- Where $\eta = y/\delta$.
- The coefficients can be determined using boundary conditions
- No slip BC: $u(y = 0) = 0 \rightarrow u(\eta = 0) = 0 \rightarrow a_0 = 0$
- Merger with free stream: $u(y = \delta) = U_\infty \rightarrow u(\eta = 1) = U_\infty \rightarrow a_1 + a_2 = 1$
- Zero shear: $\frac{\partial u}{\partial y}(y = \delta) = 0 \rightarrow \frac{\partial u}{\partial \eta}(\eta = 1) = 0 \rightarrow a_1 + 2a_2 = 0$
- On solving the above linear algebraic equation we get $a_1 = 2$ and $a_2 = -1$.
- Therefore, the quadratic velocity profile for the flat plate boundary layer is
$$\frac{u}{U_\infty} = f'(\eta) = 2\eta - \eta^2$$

Karman-Pohlhausen Approximate Method

- Momentum thickness

$$\theta = \delta \int_0^1 \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) d\eta$$

- Using the quadratic velocity profile

$$\theta = \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta$$

$$\Rightarrow \theta = \delta \int_0^1 (2\eta - \eta^2 - 4\eta^2 + 2\eta^3 + 2\eta^3 - \eta^4)\, d\eta$$

$$\Rightarrow \theta = \delta \left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right]_0^1 = \frac{2}{15}\delta$$

- Wall shear stress

$$\tau_w = \frac{\mu}{\delta}\frac{\partial u}{\partial \eta}\bigg|_{\eta=0} = 2\frac{\mu}{\delta}U_\infty$$

Karman-Pohlhausen Approximate Method

- Coming back to the momentum integral equation

$$\frac{\tau_w}{\rho} = u_e^2 \frac{d\theta}{dx}$$

$$\Rightarrow 2\frac{1}{\rho}\frac{\mu}{\delta}U_\infty = U_\infty^2 \frac{2}{15}\frac{d\delta}{dx}$$

$$\int_0^\delta \delta \, d\delta = \frac{15\nu}{U_\infty}\int_0^x dx$$

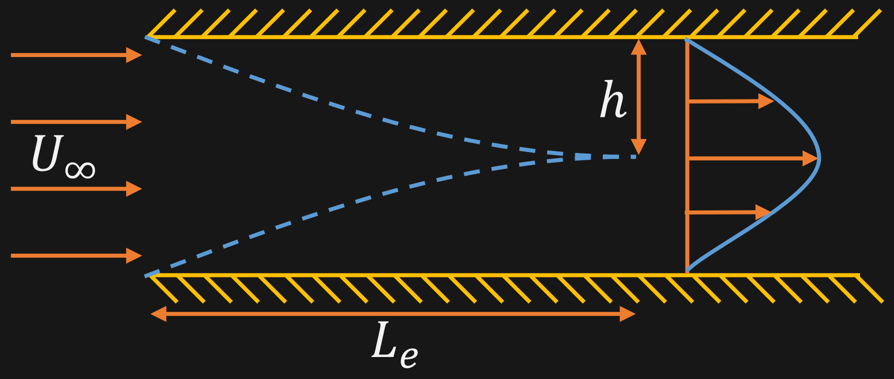
$$\Rightarrow \frac{1}{2}\delta^2 = \frac{15\nu}{U_\infty}x = \frac{15x^2}{Re_x}$$

$$\frac{\delta}{x} = \frac{\sqrt{30}}{Re_x} \approx \frac{5.48}{Re_x}$$

- Quadratic profile overpredicts the boundary layer thickness.

Entrance/Development length

- Consider a channel parallel plate channel of height $2h$.
- A Uniform streams of velocity U_∞ enters the channel.
- Boundary layer will develop at both the walls and will grow in size as we move along the channel length.
- The flow in the central region will remain inviscid until the two boundary layers merges.
- The area available for the inviscid central region will decrease due to growing boundary layer thickness.
- The flow becomes fully developed downstream to the point where the two boundary layers merge.
- Let us approximate the development length (L_e) as the distance at which the boundary layers merge.



Entrance/Development length

- Let us also assume that the boundary layer growth can approximated as the flat plate boundary layer

$$\frac{\delta}{x} = \frac{4.9}{\sqrt{Re_x}}$$

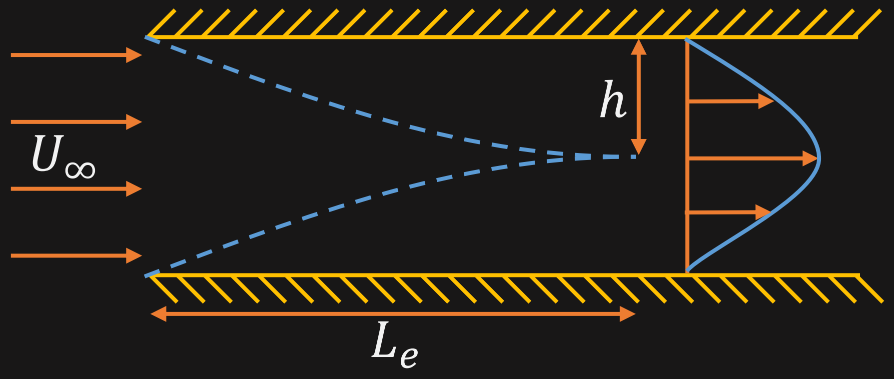
- At $x = L_e$ we have $\delta = h$ where $2h$ is the channel height. Therefore,

$$\frac{h}{L_e} = \frac{4.9}{\sqrt{Re_L}} = 4.9\sqrt{\frac{\nu}{u_e L_e}}$$

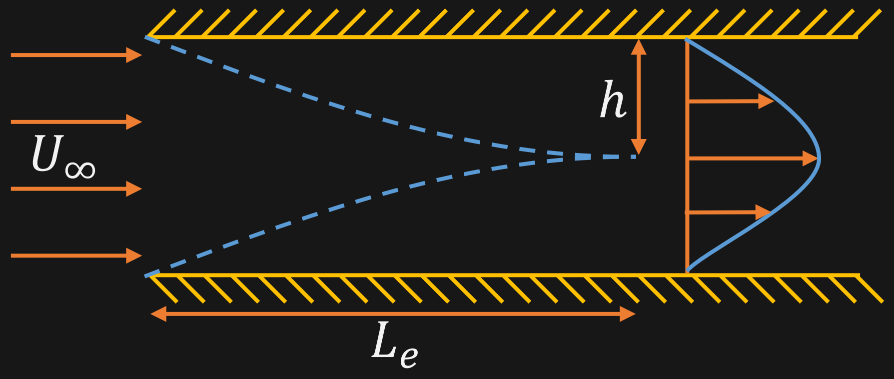
$$\Rightarrow \frac{h}{L_e} = \frac{4.9}{\sqrt{Re}}\sqrt{\frac{U_\infty 2h}{u_e L_e}}$$

$$\Rightarrow \frac{h}{L_e} = \frac{2(4.9)^2}{Re}\frac{U_\infty}{u_e}$$

- where $Re = U_\infty 2h/\nu$.



Entrance/Development length



- Using mass conservation

$$U_\infty \times 2h = u_e(2h - 2\delta^*)$$

- Where $\delta^* \approx 0.35\delta = 0.35h$

$$\Rightarrow U_\infty \times h = u_e(h - 0.35h)$$
$$\Rightarrow U_\infty = 0.65u_e$$
$$\Rightarrow u_e = 1.54U_\infty$$

- Coming back to the development length estimate

$$\frac{L_e}{h} = \frac{u_e}{U_\infty}\frac{1}{2(4.9)^2}Re = \frac{1.54}{2(4.9)^2}Re$$
$$\Rightarrow \frac{L_e}{h} \approx 0.03Re$$

- For laminar pipe flow $L_e/D = 0.06Re_d$ where $Re_d = U_\infty D/\nu$.
- The value of Re_d for laminar to turbulent transition in a pipe is 2300.

ME 221: Fluid Mechanics II

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Section2: Boundary Layer Theory

Lecture 12: BL in pressure gradient

Pressure gradient effect on BL

- Consider the boundary layer momentum equation at wall

$$\rho\left(\cancel{u}\frac{\partial u}{\partial x}+\cancel{v}\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\mu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \mu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}=\frac{dp}{dx}=-u_e\frac{du_e}{dx}$$

Adverse pressure gradient	Favorable pressure gradient	zero pressure gradient

Pressure gradient effect on BL

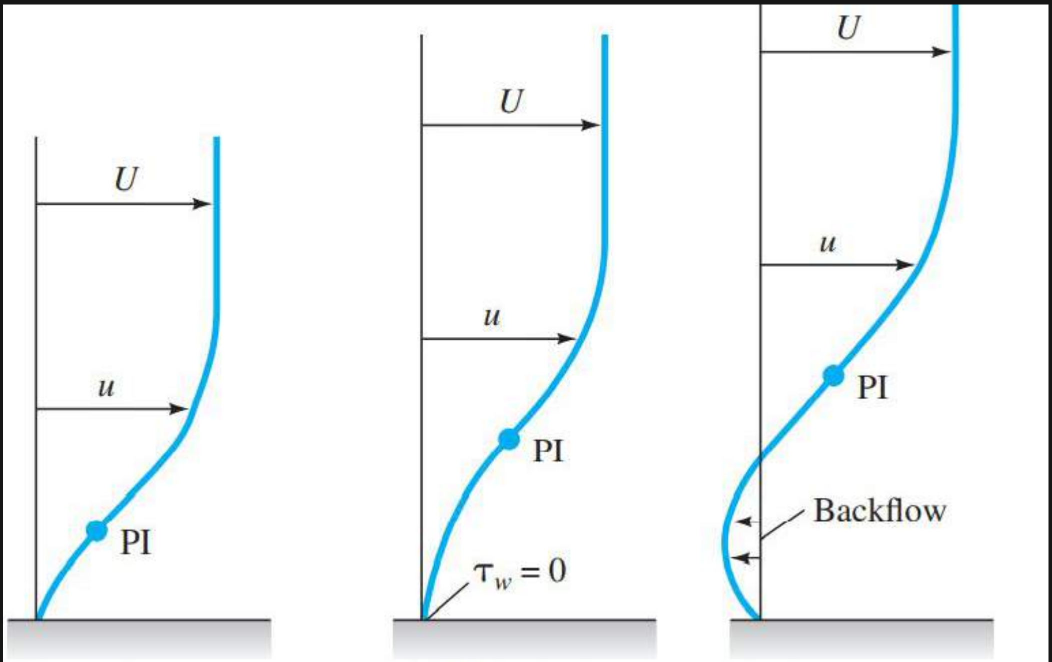
- What can we say about $\frac{\partial^2 u}{\partial y^2}|_{y=\delta}$?
- Shear stress at edge of the boundary layer, $\mu \frac{\partial u}{\partial y}|_{y=\delta}$, goes to zero from a positive value within the boundary layer. Therefore,

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=\delta} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) < 0$$

- For adverse pressure gradient case, $\frac{\partial^2 u}{\partial y^2}|_{y=0} > 0$ and $\frac{\partial^2 u}{\partial y^2}|_{y=\delta} < 0$.
- From intermediate value theorem, $\frac{\partial^2 u}{\partial y^2}$ should be zero somewhere within the boundary layer.
- It follows that the second derivative must pass through zero somewhere in between, at a point of inflection.
- Any boundary layer profile in an adverse gradient must exhibit a characteristic S shape.

Pressure gradient effect on BL

- Typical boundary layer velocity profiles are shown for adverse pressure gradient case.
- For small magnitude of adverse pressure gradient, the point of inflection (PI) is close the wall but flow is not separated.
- As the magnitude of adverse pressure gradient is increased, the wall shear stress becomes zero and the flow separates.
- Further increase in adverse pressure gradient causes reversed flow near the wall.
- Onset of flow separation happens where wall shear stress is zero.



(c) Weak adverse gradient:

$\frac{dU}{dx} < 0$

$\frac{dp}{dx} > 0$

No separation, PI in the flow

(d) Critical adverse gradient:

Zero slope at the wall:

Separation

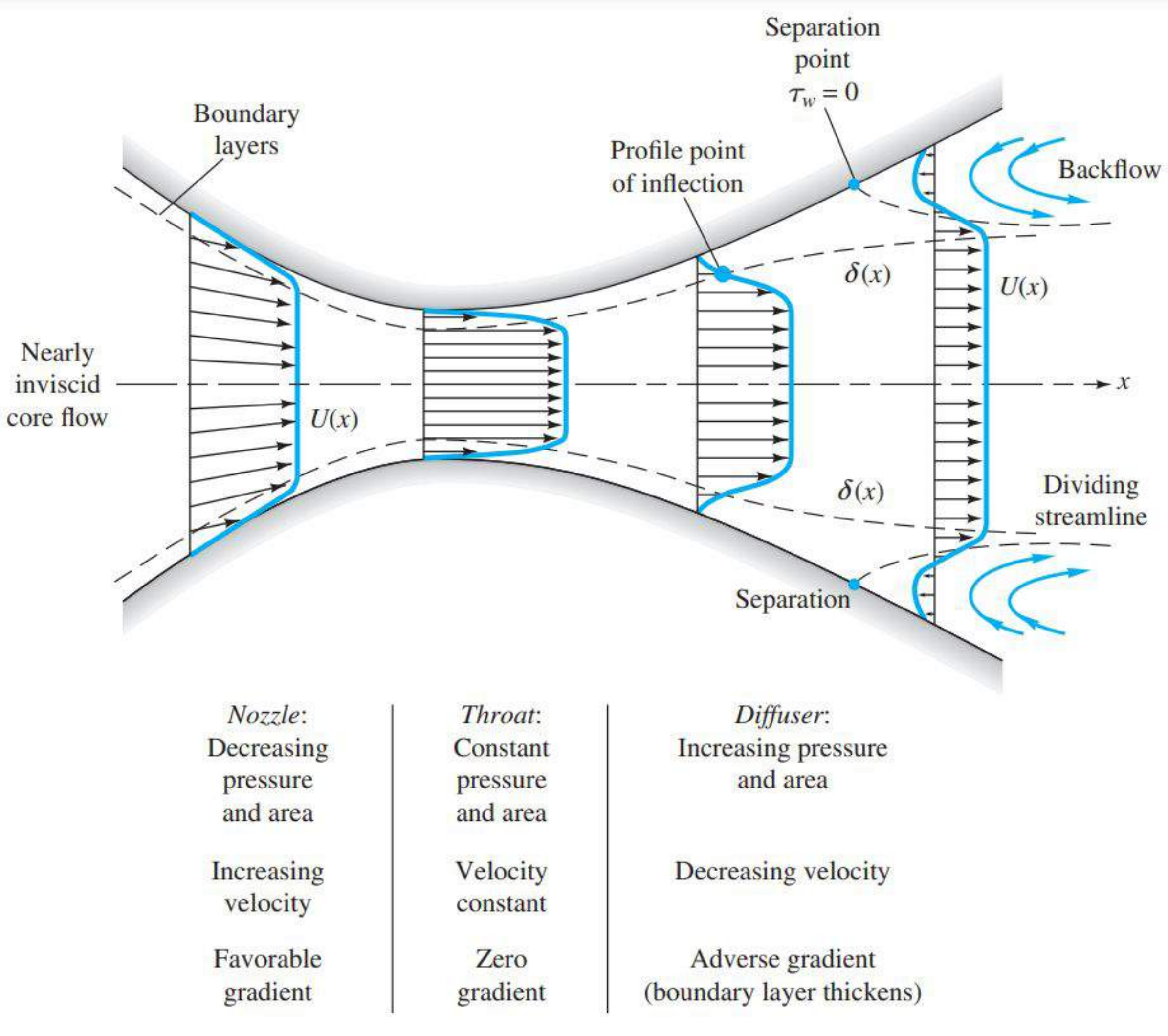
(e) Excessive adverse gradient:

Backflow at the wall:

Separated flow region

Pressure gradient effect on BL

- Consider the flow in a duct consisting of a nozzle, throat, and diffuser.
- The nozzle flow is a favorable gradient and never separates, nor does the throat flow where the pressure gradient is approximately zero.
- The expanding-area diffuser produces low velocity and increasing pressure, an adverse gradient.
- If the diffuser angle is too large, the adverse gradient is excessive, and the boundary layer will separate at one or both walls, with backflow.

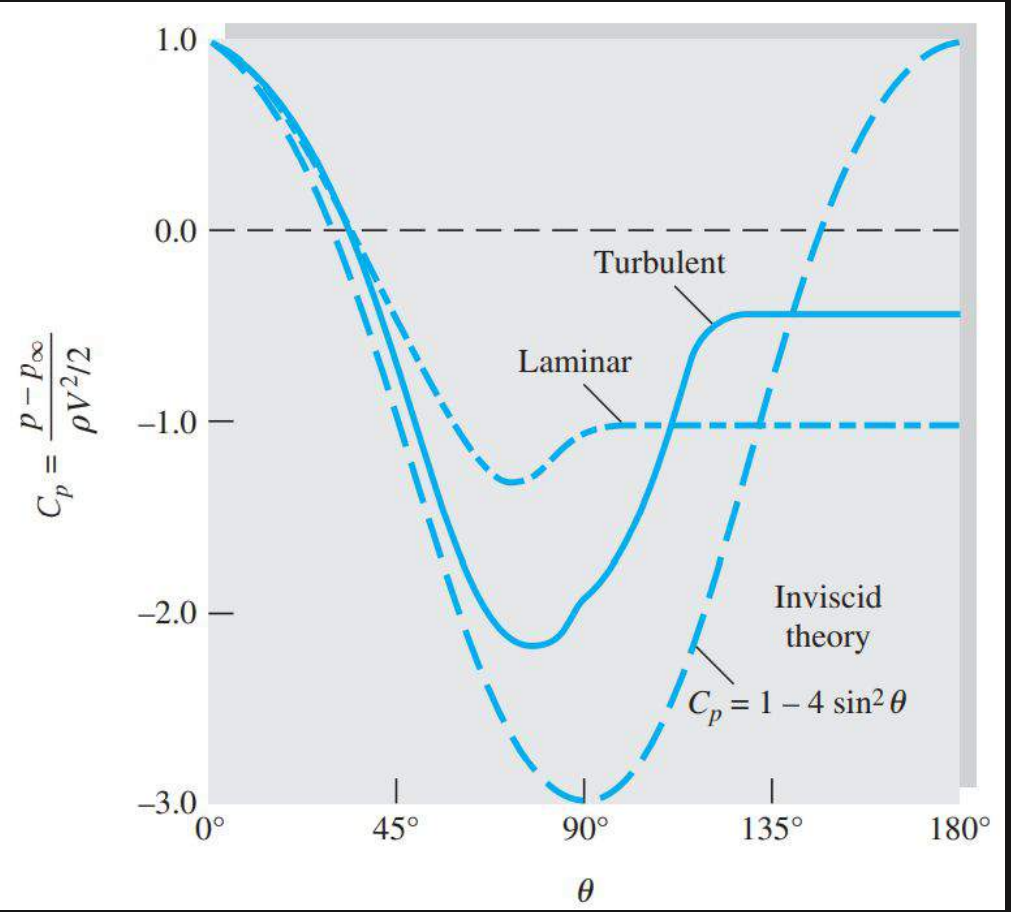


Flow around cylinder

- From potential flow theory

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta$$

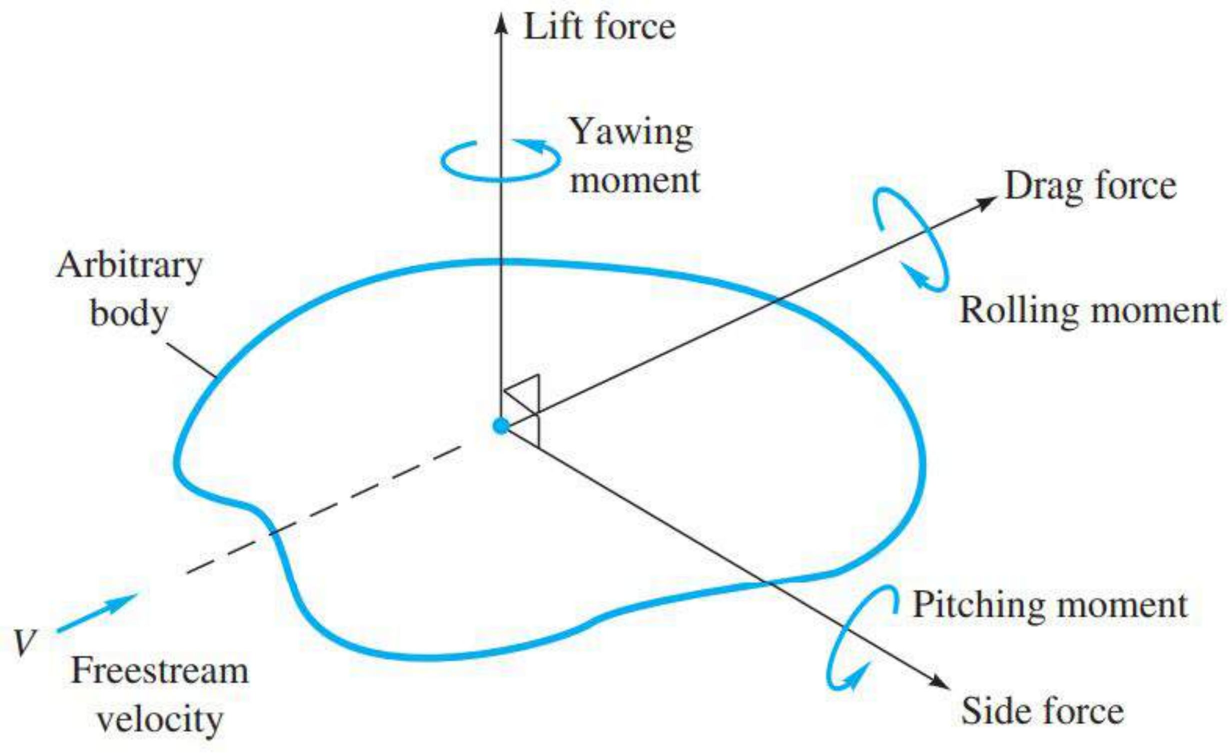
- The actual flow around cylinder is separated and results in low pressure region behind the cylinder.
- This results in a net force on the cylinder (contrary to zero force predicted by the potential flow theory).
- The location of flow separation depends on the Reynolds number.
- For cylinder, laminar flow has larger drag force as compared to turbulent flow.



Drag of Immersed Bodies

- Any body of any shape when immersed in a fluid stream will experience forces and moments from the flow.
- The force on the body along the flow direction is called drag, and the moment about that axis the rolling moment.
- Force perpendicular to the drag is called lift. The moment about the lift axis is called yaw.
- The third component is the side force, and about this axis is the pitching moment.
- Drag coefficients are defined by using a characteristic area A

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A}$$



Drag of Immersed Bodies

- The area A is usually one of three types:

1. Frontal area, the body as seen from the stream; suitable for thick, stubby bodies, such as spheres, cylinders, cars, trucks, missiles, projectiles, and torpedoes.
2. Planform area, the body area as seen from above; suitable for wide, flat bodies such as wings and hydrofoils.
3. Wetted area, customary for surface ships and barges.

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Section3: Compressible flows

Lecture 13: Equation of state

Compressible flow

- A flow where density changes are significant.
- Governing equations for the compressible flow are:
- Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

- Momentum Equation

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right] = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{V}$$

- Energy

$$\rho \frac{D\hat{u}}{Dt} + p(\nabla \cdot \vec{V}) = \nabla.(k\nabla T) + \Phi$$

- These are valid for Newtonian fluids and any kind of flow (incompressible, compressible).

Compressible flow

- For incompressible flow $\rho = const.$
- Therefore, $\vec{V}$, p, and T are three unknown and we have three equations.
- Note that energy equation is decoupled from continuity and momentum equations for incompressible flows.
- For compressible flow ρ, $\vec{V}$, p, and T are four unknown and we have three equations.
- The additional equation comes form the equation of state (Thermodynamics)
- Mach number

$$M_a = \frac{V}{a}$$

- Where, V is fluid velocity, a is sound speed in fluid.
- $M_a < 0.3$ incompressible flow (negligible density change).
- $0.3 < M_a < 0.8$ subsonic flow (significant density change, no shock).

Compressible flow

- $0.8 < M_a < 1.2$ transonic flow (both subsonic and shock region)
- $1.2 < M_a < 3.0$ supersonic flow (only shock region, no subsonic region)
- $3.0 < M_a$ hypersonic flow (strong shocks)
- These ranges are subjective.

Equation of state (thermodynamics)

- We can use the idea gas equation

$$p = \rho RT$$

- Where gas constant $R = R_u/M$.
- R_u is the Universal gas constant. $R_u = 8.314$ kJ/kmol-K.
- M is the Molecular mass of gas. For air $M = 28.97$ Kg/kmol and $R = 0.287$ kJ/Kg-K.
- Assumptions for the ideal gas equation: Gas molecules take up zero volume and do not interact with each other.
- For low pressure and/or high temperature, many gases follows ideal gas equation.
- For air $p < 30$ atm at $T = 25°C$ or $T > 130°C$ for $p = 1$ atm.

Thermodynamics

- In general, the internal energy of a simple substance may be expressed as a function of any two independent properties, for example

$$u = u(v, T)$$

- Where $v = 1/\rho$ is the specific volume. Then

$$du = \left(\frac{\partial u}{\partial T}\right)_v dT + \left(\frac{\partial u}{\partial v}\right)_T dv$$

- The specific heat at constant volume is defined as

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v$$

- Therefore

$$du = c_v dT + \left(\frac{\partial u}{\partial v}\right)_T dv$$

Thermodynamics

- For ideal gas, the internal energy u is a function of temperature only

$$u = u(T)$$
$$\Rightarrow du = \left(\frac{\partial u}{\partial T}\right)_v dT = C_v dT$$

- Where C_v is the specific heat at constant volume.
- The enthalpy of any substance is defined as

$$h = u + \frac{p}{\rho}$$

- For an ideal gas

$$h = u + RT$$

- Therefore, h is only function of temperature for ideal gas $h = h(T)$.

$$\Rightarrow dh = \left(\frac{\partial h}{\partial T}\right)_p dT$$
$$\Rightarrow dh = C_p dT$$

Thermodynamics

- Where C_p is specific heat at constant pressure

$$h = u + RT$$
$$\Rightarrow dh = du + RdT$$
$$\Rightarrow C_p - C_v = R$$

- Ratio of specific heat $k = C_p/C_v$. For air $k \approx 1.4$.
- Then we can express the specific heats as

$$C_p = \frac{kR}{k-1}$$
$$C_v = \frac{R}{k-1}$$

- Entropy is defined by the equation

$$dS = \left(\frac{\delta Q}{T}\right)_{rev}$$

Thermodynamics

- δQ is path dependent and thus not an exact differential.
- The inequality of Clausius, deduced from the second law, states that

$$\oint \frac{\delta Q}{T} \leq 0$$

- From second law of thermodynamics

$$dS \geq \frac{\delta Q}{T}$$

- Where the equality holds for the reversible process.

$$dS = \frac{\delta Q}{T}$$

- For an adiabatic process $\delta Q = 0 \Rightarrow dS \geq 0$
- For an adiabatic reversible process $dS = 0 \Rightarrow$ isentropic process.

Thermodynamics

- The first law of thermodynamics

$$\delta Q = dU + \delta W$$

- Where $\delta Q = TdS$ and $\delta w = pdV$

$$\Rightarrow TdS = dU + pdV$$

$$\Rightarrow T\frac{dS}{m} = d\left(\frac{U}{m}\right) + p\left(d\frac{V}{m}\right)$$

$$\Rightarrow Tds = du + pdv$$

$$\Rightarrow ds = \frac{du}{T} + \frac{p}{T}dv = C_v\frac{dT}{T} + R\frac{dv}{v}$$

- Where $v =$ specific volume $= 1/\rho$

$$u = h - pv$$

$$du = dh - pdv - v\,dp$$

- Which gives

$$Tds = dh - vdp$$

Thermodynamics

- And

$$ds = \frac{dh}{T} - v\frac{dp}{T} = C_p\frac{dT}{T} - R\frac{dp}{P}$$

- On integrating we will get (here assume C_v and C_p are constant

$$s_2 - s_1 = C_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right)$$

$$s_2 - s_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

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Section3: Compressible flows

Lecture 14: Speed of sound

Thermodynamics

- Entropy change for an ideal gas

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right)$$

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

- From ideal gas equation $pv = RT$, we can write $T_2/T_1 = p_2 v_2/p_1 v_1$. Then

$$s_2 - s_1 = c_v \ln\left(\frac{p_2 v_2}{p_1 v_1}\right) + R\ln\left(\frac{v_2}{v_1}\right)$$

$$s_2 - s_1 = c_v \ln\left(\frac{p_2}{p_1}\right) + c_v \ln\left(\frac{v_2}{v_1}\right) + R\ln\left(\frac{v_2}{v_1}\right)$$

- Using $c_p - c_v = R$ we get

$$s_2 - s_1 = c_v \ln\left(\frac{p_2}{p_1}\right) + c_p \ln\left(\frac{v_2}{v_1}\right)$$

Thermodynamics

- For isentropic process $s_1 = s_2$. From first relation

$$\Rightarrow C_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right) = 0$$

$$\Rightarrow \left(\frac{T_2}{T_1}\right)\left(\frac{v_2}{v_1}\right)^{R/C_v} = 1$$

$$\Rightarrow \left(\frac{T_2}{T_1}\right)\left(\frac{v_2}{v_1}\right)^{k-1} = 1$$

$$\Rightarrow T_1 v_1^{k-1} = T_2 v_2^{k-1}$$

$$\Rightarrow T v^{k-1} = \text{constant}$$

- Second relation

$$c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = 0$$

$$\left(\frac{T_2}{T_1}\right)\left(\frac{p_2}{p_1}\right)^{-R/c_p} = 1$$

Thermodynamics

- $$\Rightarrow \left(\frac{T_2}{T_1}\right)\left(\frac{p_2}{p_1}\right)^{-(k-1)/k} = 1$$
  $$\Rightarrow Tp^{1-k/k} = \text{constant}$$
- Third relation
  $$\Rightarrow c_v \ln\left(\frac{p_2}{p_1}\right) + c_p \ln\left(\frac{v_2}{v_1}\right) = 0$$
  $$\Rightarrow \left(\frac{p_2}{p_1}\right)\left(\frac{v_2}{v_1}\right)^{c_p/c_v} = 1$$
  $$\Rightarrow \left(\frac{p_2}{p_1}\right)\left(\frac{v_2}{v_1}\right)^{k} = 1$$
  $$\Rightarrow pv^k = \text{constant}$$
- These three relations are valid for an ideal gas undergoing isentropic process.

Thermodynamics

- For constant volume process $v_2 = v_1$ then

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right)$$

$$\Rightarrow T_2 = T_1 \exp\left(\frac{s_2 - s_1}{c_v}\right) \Rightarrow T = T_0 \exp\left(\frac{s - s_0}{c_v}\right)$$

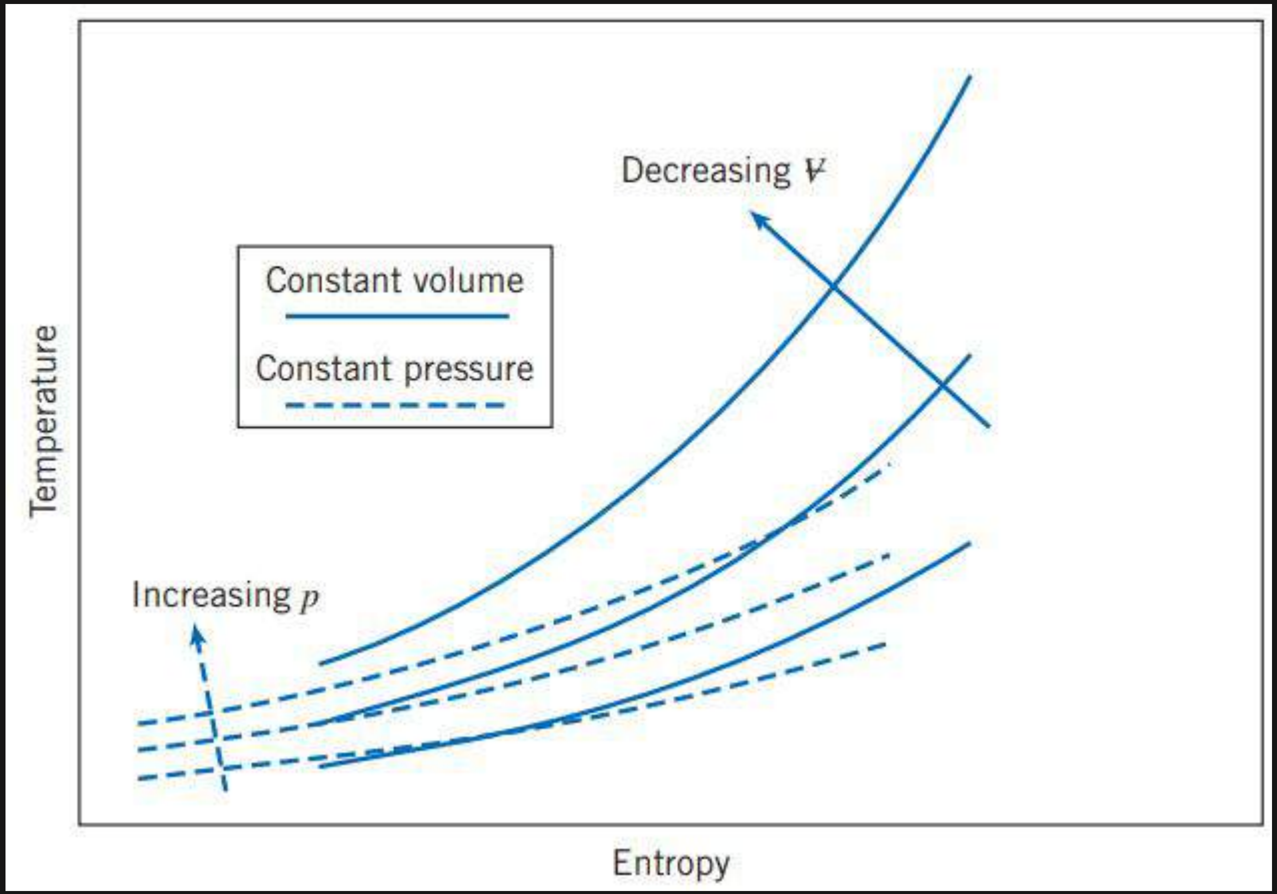
$$\Rightarrow \left.\frac{dT}{ds}\right|_v = \frac{T}{C_v}$$

- For constant pressure process $p_2 = p_1$ then

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right)$$

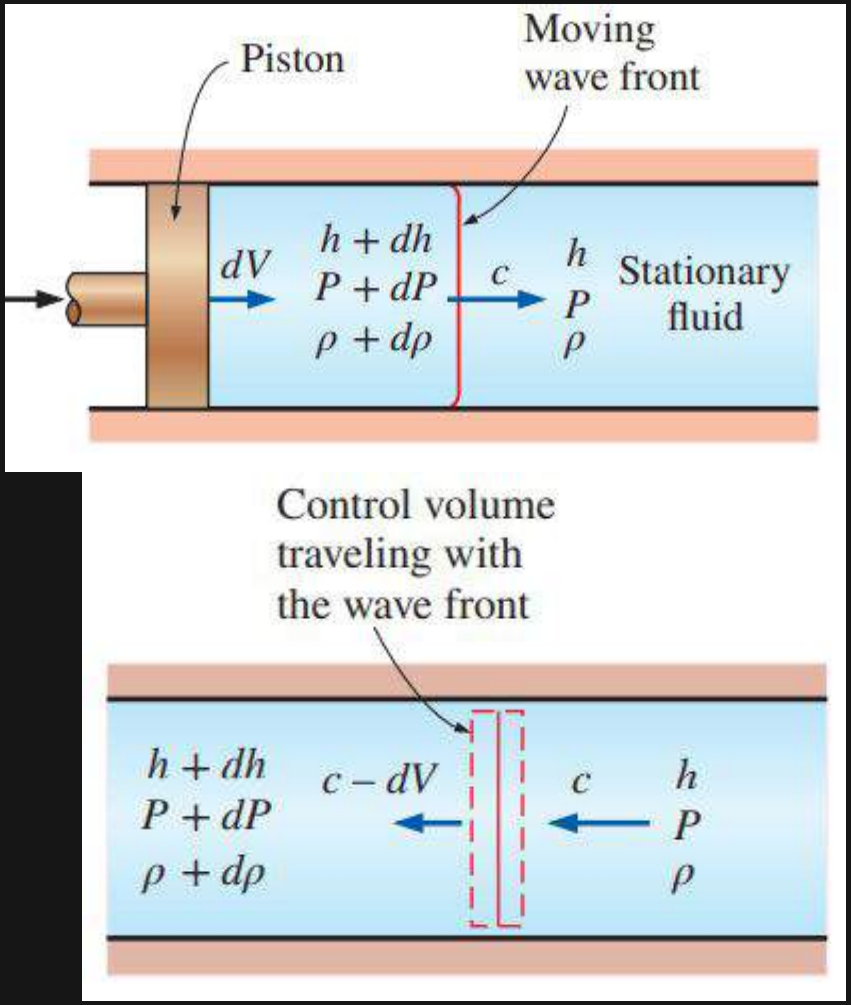
$$\Rightarrow T_2 = T_1 \exp\left(\frac{s_2 - s_1}{c_p}\right)$$

$$\Rightarrow \left.\frac{dT}{ds}\right|_p = \frac{T}{C_p}$$



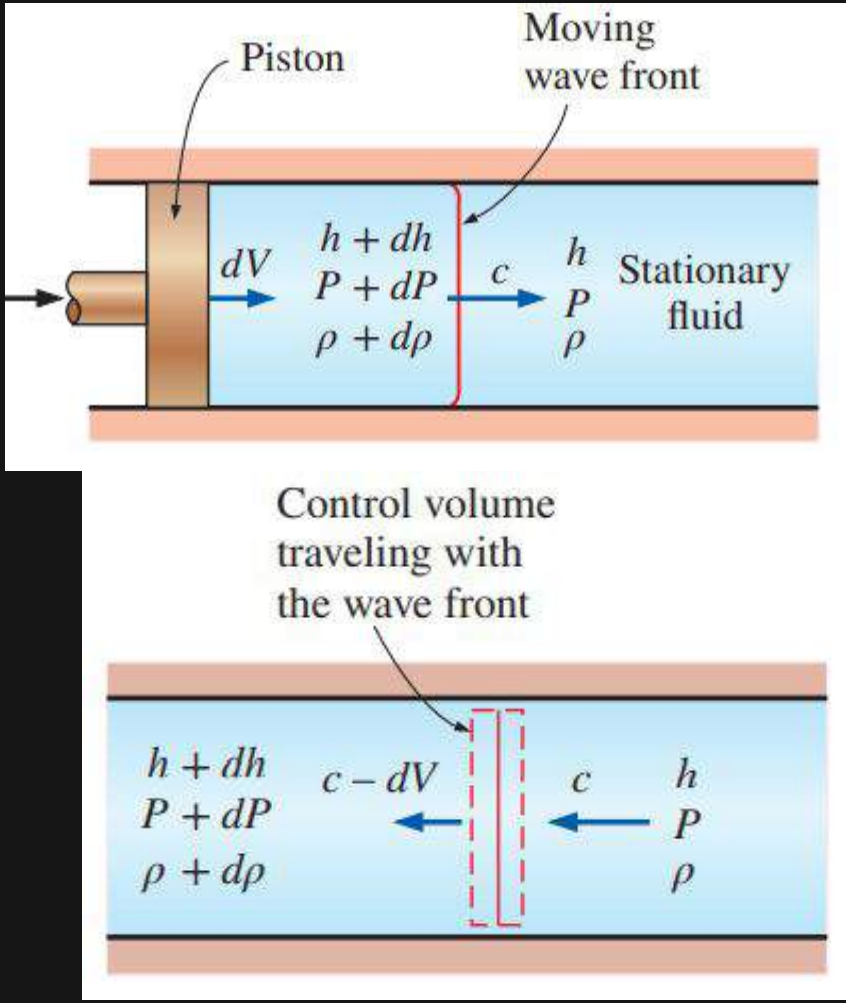
Speed of Sound

- It is the rate of propagation of a pressure pulse of infinitesimal strength through still fluid.
- It is a thermodynamic property of a fluid.
- In figure shown, the piston is moved with a velocity dV at some instant. Earlier, the fluid was stationary everywhere.
- An incompressible fluid will start moving immediately (to conserve mass).
- It will take sometime for compressible fluid to respond. A wave will travel from left to right.
- Fluid left to the wave front experiences an incremental change in its thermodynamics properties. The fluid on right is yet to change from its original state.
- Let us consider a CV around the wave front moving with it. In the moving reference frame



Speed of Sound

- $$\dot{m}_{in} = \dot{m}_{out}$$
$$\Rightarrow A\,\rho c = A\,(\rho + d\rho)(c - dV)$$
$$\Rightarrow \rho c = \rho c - (\rho + d\rho)dV + c\,d\rho$$
$$\Rightarrow dV = c\frac{d\rho}{\rho + d\rho}$$
- For infinitesimal pulse $d\rho \ll \rho$
$$dV = c\frac{d\rho}{\rho}$$
- Momentum balance
$$\sum \vec{F} = \dot{m}(V_{out} - V_{in})$$
$$\Rightarrow (p + \mathrm{d}p)A - pA = \rho Ac[-(c - dV) - (-c)]$$
$$\Rightarrow dp = \rho c\,dV$$



Speed of Sound

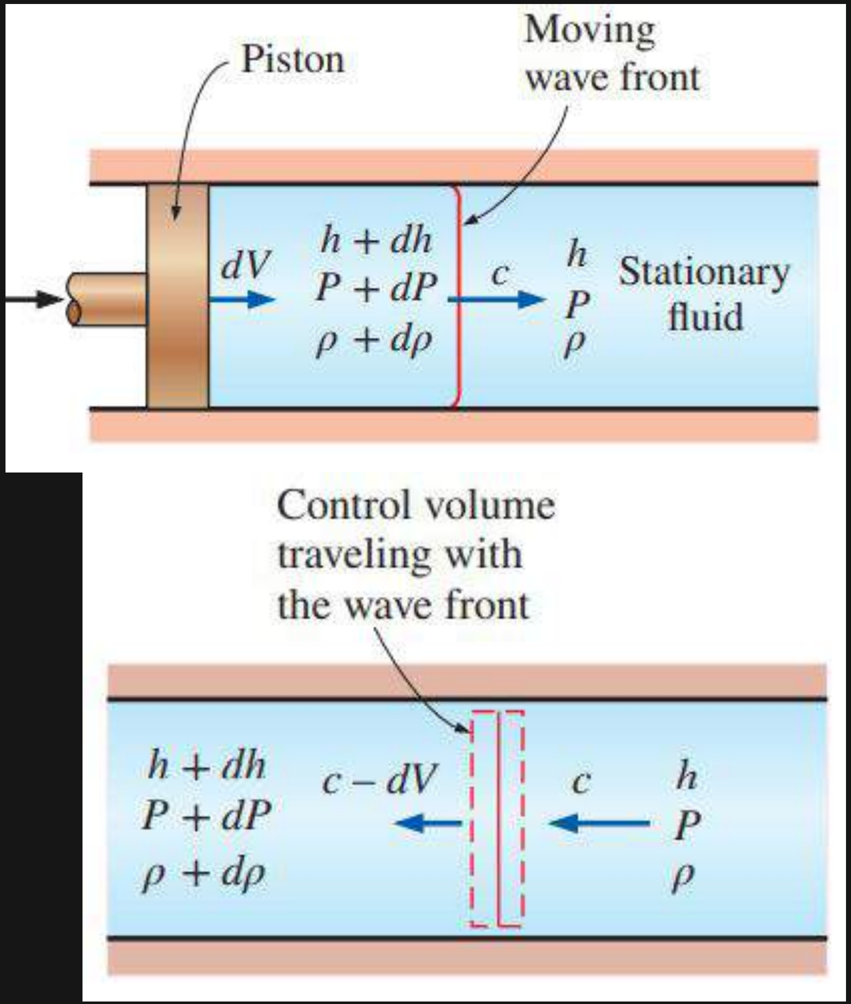
- Putting $dV = c\, d\rho/(\rho + d\rho)$ into the previous equation

$$dp = \rho c\, dV$$
$$\frac{dp}{\rho c} = c\frac{d\rho}{\rho + d\rho}$$
$$c^2 = \frac{dp}{d\rho}\left(1 + \frac{d\rho}{\rho}\right)$$

- For infinitesimal pulse $d\rho/\rho \ll 1$ and we get speed of sound as

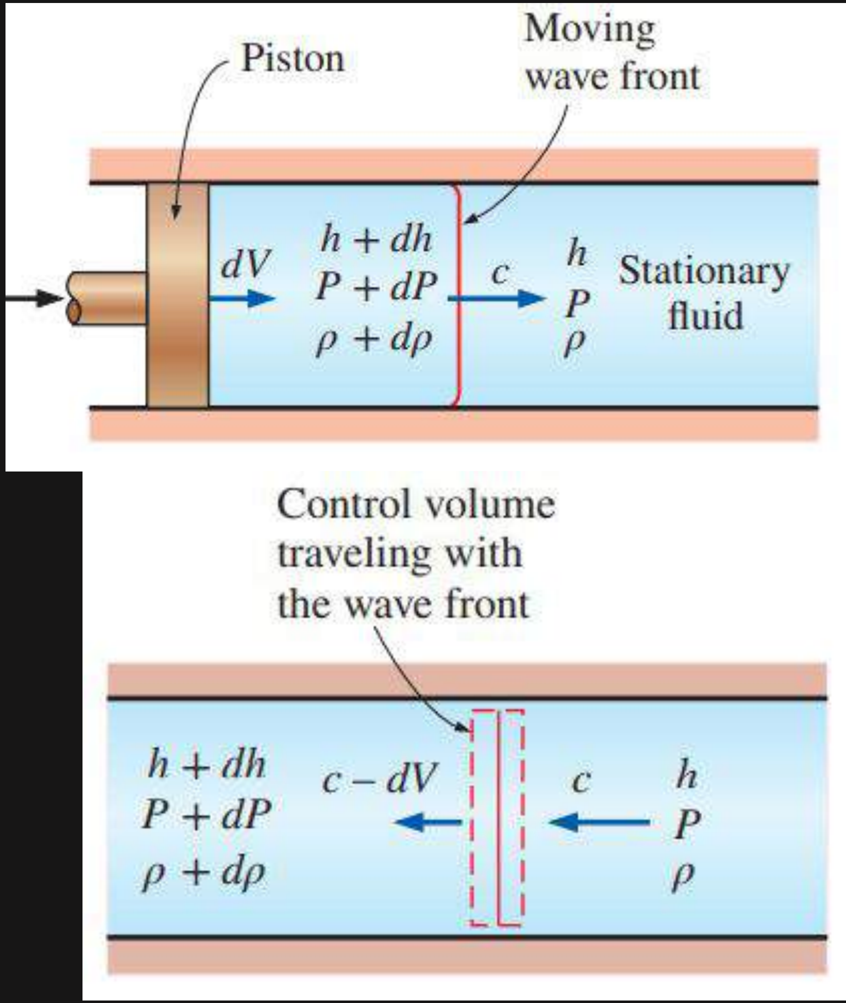
$$c^2 = \frac{dp}{d\rho}$$

- Since the pressure (density) changes are infinitesimal the process is reversible.
- The process of infinitesimal change is also quick to allow any heat transfer to occur→Adiabatic process.
- Therefore, propagation of sound is an isentropic process.



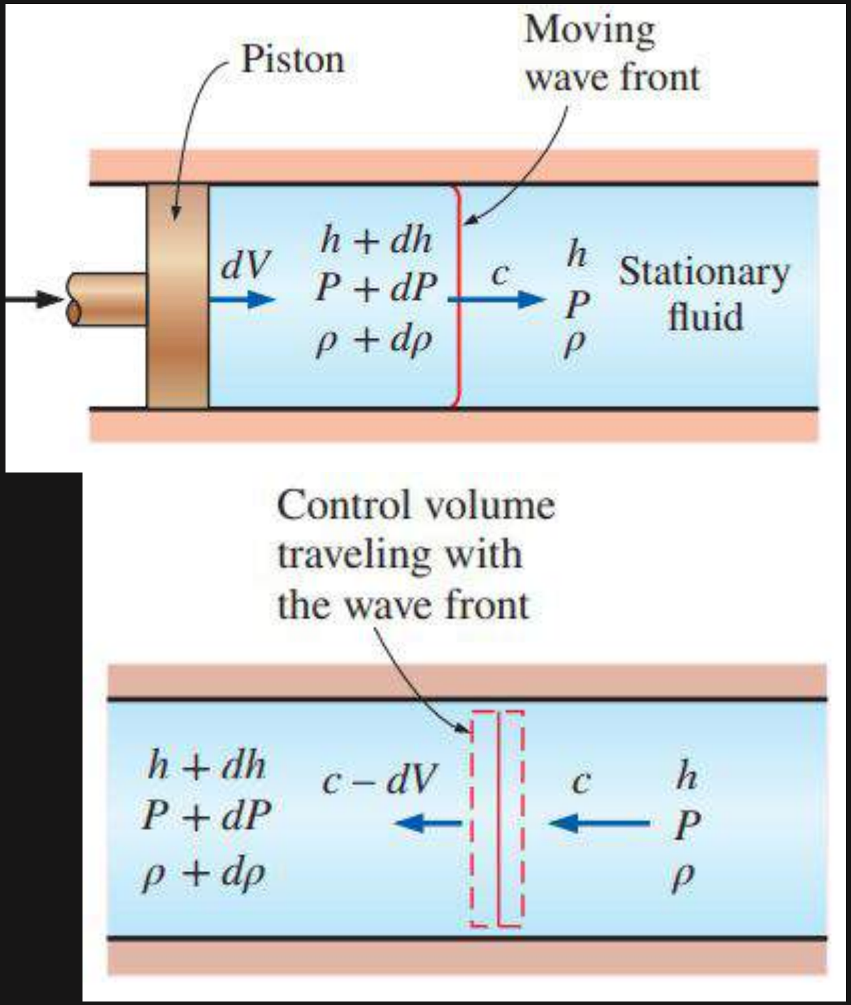
Speed of Sound

- $$\Rightarrow c^2 = \frac{\partial p}{\partial \rho}\bigg|_{s=constant}$$
- For isentropic process
$$pv^k = \text{constant}$$
$$\Rightarrow \frac{p}{\rho^k} = \text{constant}$$
- Differentiating with respect to ρ
$$\Rightarrow \frac{1}{\rho^k}\frac{\partial p}{\partial \rho}\bigg|_s + p\left(-\frac{k}{\rho^{k+1}}\right) = 0$$
$$\Rightarrow \frac{\partial p}{\partial \rho}\bigg|_s = k\frac{p}{\rho} = kRT$$
- For isothermal process (from ideal gas equation)
$$\frac{p}{\rho} = \text{constant}$$



Speed of Soun

- $$\Rightarrow \frac{1}{\rho}\frac{\partial p}{\partial \rho}\Big|_T - p\frac{1}{\rho^2} = 0$$
$$\Rightarrow \frac{\partial p}{\partial \rho}\Big|_T = \frac{p}{\rho} = RT$$
- Therefore,
$$\frac{\partial p}{\partial \rho}\Big|_s = k\frac{\partial p}{\partial \rho}\Big|_T = kRT$$
$$C = \sqrt{\frac{\partial p}{\partial \rho}\Big|_s} = \sqrt{k\frac{\partial p}{\partial \rho}\Big|_T} = \sqrt{kRT} \text{ (For ideal gas)}$$
- For air at $T = 300$ K, $c = 347.2$ m/s.



ME 221: Fluid Mechanics II

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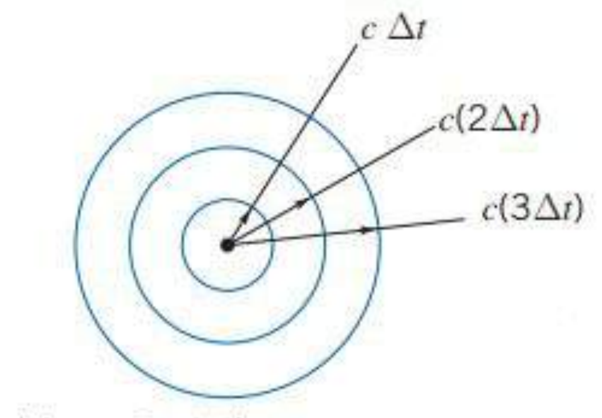
Section3: Compressible flows

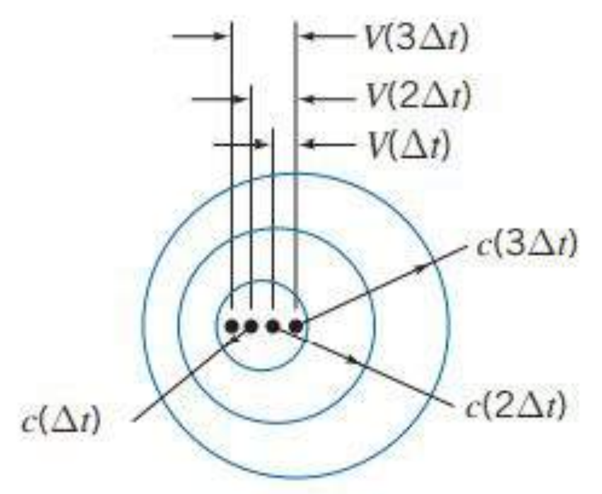
Lecture 15: Stagnation properties

Mach cone

- Consider a point source of sound that emits a pulse every Δt second.
- Each pulse will spread radially outwards at speed c.
- At any time t after being emitted, a pulse will be a sphere of radius ct with centre located at its origination point.
- What happen when source is moving with some speed V?
- Case1: $V = 0 \rightarrow$ stationary source.
- The pulses form concentric spheres.
- Case 2: $V < C \rightarrow$ moving subsonic source (towards left).
- Non-concentric spheres with clustering of wave fronts in front of source.
- An observer in front of source receives pulses at a higher frequency as compared to the source frequency $f = 1/\Delta t$ (Doppler effect).
- Observer hears the sound before the source reaches the observer.



(a) $V = 0$: stationary source



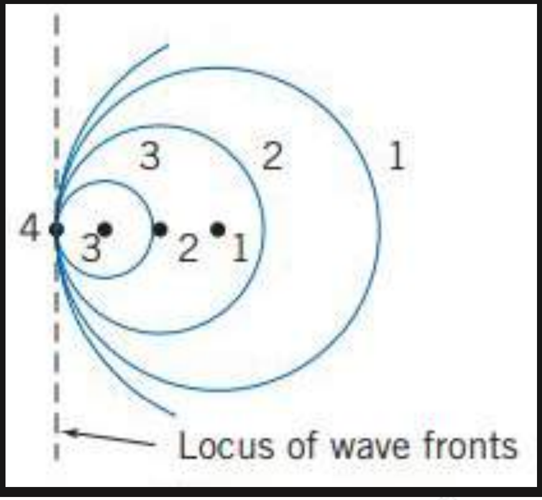
(b) $V < c$: Doppler shift

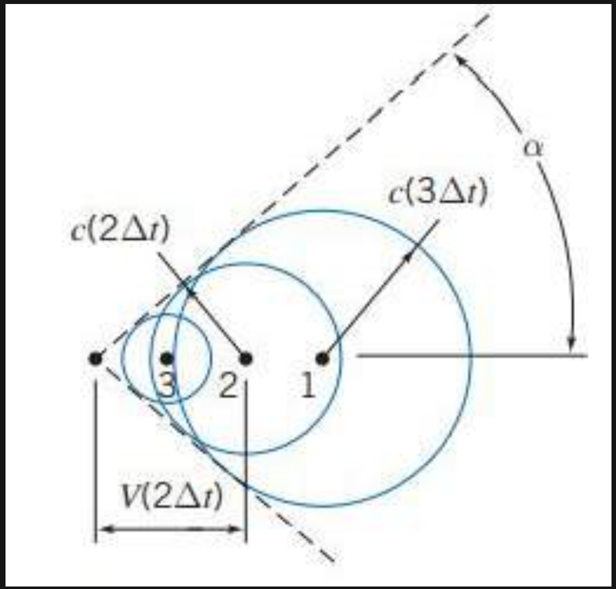
Mach cone

- Case 3: $V = C \rightarrow$ The source moves at sonic speed.
- The wave fronts of the sphere becomes tangent in the front.
- Huge pressure will be build up in front of the source.
- An observer who is ahead of source will hear the sound when the source reaches it.
- Case 4: $V > C \rightarrow$ The source moves at supersonic speed.
- An observer inside the cone can only hear the sound.
- The spheres generate what is called a Mach cone tangent to each sphere.
- Cone angle

$$\sin\alpha = \frac{C}{V} = \frac{1}{M}$$

$$\Rightarrow \alpha = \sin^{-1}\frac{1}{M}$$





Stagnation Properties

- Recall the steady 1D energy equation

$$\left(h + \frac{1}{2}V^2 + gz\right)_{out} = \left(h + \frac{1}{2}V^2 gz\right)_{in} + q - W_s - W_v$$

- Assuming adiabatic condition and no shaft work. Also no change in the elevation. Then

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2$$

- We define stagnation enthalpy

$$h_0 = h + \frac{1}{2}V^2$$

- Such that the energy equation becomes

$$h_{01} = h_{02}$$

- In the absence of any heat and work interaction and any changes in the potential energy, the stagnation enthalpy of a fluid remains constant during a steady flow process.

Stagnation Properties

- h will be called static enthalpy to distinguish it from the stagnation enthalpy.
- Stagnation enthalpy represents the enthalpy of a fluid when it is brought to rest adiabatically.
- During the stagnation process, the kinetic energy of a fluid is converted to its internal energy plus flow energy.
- Properties of a fluid at the stagnation state are called stagnation properties (stagnation temperature, stagnation pressure, etc.)
- If the stagnation process is reversible than it is called isentropic stagnation.
- For an ideal gas with constant specific heat, its enthalpy can be expressed as $h = C_p T$ then

$$C_p T_0 = C_p T + \frac{1}{2} V^2$$

$$\Rightarrow T_0 = T + \frac{V^2}{2C_p}$$

- Where, T_o is the stagnation temperature, and $V^2/2C_p$ is the dynamic temperature.

Stagnation Properties

- Stagnation temperature

$$\frac{T_0}{T} = 1 + \frac{V^2}{2C_pT}$$

- Using $c_p = kR/k - 1$ and $c = \sqrt{kRT}$

$$c_pT = \frac{kR}{k-1} \times \frac{c^2}{kR}$$

$$= \frac{c^2}{k-1}$$

- The expression for stagnation temperature (for adiabatic stagnation) changes to

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\frac{V^2}{c^2}$$

$$\Rightarrow \frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

Stagnation Properties

- For isentropic stagnation process

$$T\,p^{\frac{1-k}{k}} = \text{constant}$$
$$\Rightarrow\ T^{\frac{k}{1-k}}p = \text{constant}$$
$$\Rightarrow\ T^{\frac{k}{1-k}}p = T_0^{\frac{k}{1-k}}p_0$$
$$\Rightarrow \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}}$$

- Similarly

$$T\rho^{1-k} = \text{constant}$$
$$T\rho^{1-k} = T_0\rho_0^{1-k}$$
$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}}$$

Stagnation Properties

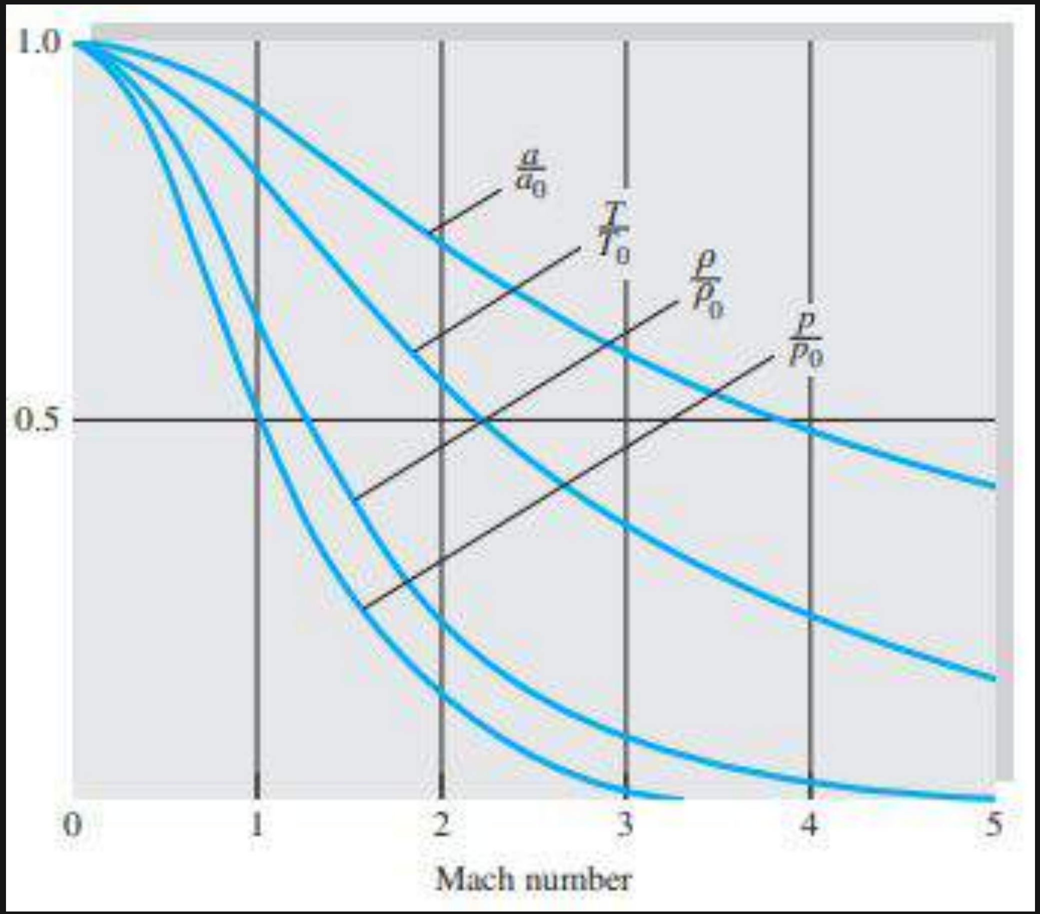
- Where p_0 and ρ_0 are the isentropic stagnation properties.
- Using adiabatic stagnation temperature expression, the isentropic stagnation pressure, density and adiabatic stagnation speed of sound are given as

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{k}{k-1}}$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{k-1}}$$

$$\frac{c_0}{c} = \left(\frac{T_0}{T}\right)^{1/2} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{2}}$$

- In an adiabatic non-isentropic flow, p_0 and ρ_0 retain their local meaning but vary throughout the flow as the entropy changes due to friction or shock waves.



For air

Stagnation Properties

- Note that h_0, T_0, and c_0 are constant in adiabatic flow.
- Recall

$$Tds = dh - \frac{dp}{\rho}$$

- For isentropic flow $ds = 0 \rightarrow dh = dp/\rho$
- For adiabatic process

$$h + \frac{1}{2}V^2 = \text{constant}$$
$$\Rightarrow dh + VdV = 0$$

- Therefore,

$$\frac{dp}{\rho} + VdV = 0 \quad (\text{Bernoulli's equation})$$

- Thus the isentropic flow assumption is equivalent to the frictionless momentum equation.