**Quant Concepts: Session 3: Numbers + General Algebra**

**NUMBERS**
**For the purpose of the GMAT, all numbers are real.**

*REAL numbers are basically of two types:*

1. **Rational numbers:** A rational number can always be represented by a fraction of the form p/q where p and q are integers and q ≠ 0. Examples: finite decimal numbers, infinite repeating decimals, whole numbers, integers, fractions i.e. 3/5, 16/9, 2, 0.666.....∞ = 2/3 etc.

2. **Irrational numbers:** Any number which cannot be represented in the form p/q where p and q are integers and q ≠ 0 is an irrational number. AN INFINITE NON–RECURRING DECIMAL IS AN IRRATIONAL NUMBER. Examples – √2, π, √5, √7.

**INTEGERS:** The set of Integers I = {0, ±1, ±2, ±3, … ∞}

**EVEN NUMBERS:** The numbers divisible by 2 are even numbers. E.g., 0, ±2, ±4, ±6, ±8, ±10....... Even numbers are expressible in the form 2n where n is an integer other than zero. **Thus –2, –6 etc. are also even numbers.** *Remember that '0' is an even number.*

**ODD NUMBERS:** The numbers not divisible by 2 are odd numbers e.g. ±1, ±3, ±5, ±7, ±9..... Odd numbers are expressible in the form (2n + 1) where n is an integer other than zero (not necessarily prime). Thus, –1, –3, –9 etc. are all odd numbers.

You must remember:

| | | | |
|---|---|---|---|
| Even ± Even = Even | Even ± Odd = Odd | Odd ± Odd = Even | Odd ± Even = Odd |
| Even × Even = Even | Even × Odd = Even | Odd × Odd = Odd | |

**POSITIVE INTEGERS:** The numbers 1, 2, 3, 4, 5..... are known as positive integers.
- 0 is neither positive nor negative.
- 0 is an even number.
- 0 is not a factor of any integer
- 0 is a multiple of all integers.

**Prime numbers:** A natural number which has no other factors besides itself and unity is a prime number. Examples: 2, 3, 5, 7, 11, 13, 17, 19 ......

- If a number has no factor equal to or less than its square root, then the number is prime. This is a test to judge whether a number is prime or not.
- The only even prime number is 2
- 1 is neither prime nor composite (by definition)
- The smallest composite number is 4.
- The product of r consecutive integers is divisible by r !
- If p is a prime number, then 1 + (p – 1) ! is divisible by p.
    - Example. 16 ! + 1 i.e.,(17–1) ! + 1 is divisible by 17.

**Composite numbers:** A composite number has other factors besides itself and unity, e.g., 8, 72, 39 etc. Alternatively, we might say that a natural number greater than 1 that is not prime is a composite number.

**Problems:**

1. If x, y, and z are integers and xy + z is an odd integer, is x an even integer?
   (1) xy + xz is an even integer (2) y + xz is an odd integer

2. If n is an integer between 10 and 99 is n < 80?
   (1) The sum of the two digits of n is a prime number.
   (2) Each of the two digits of n is a prime number.

3. For all positive integers m, [m] = 3m when m is odd and [m] = ½ m when m is even. Which of the following is equivalent to [9] × [6]?
   [81] [54] [36] [27] [18]

## FACTORS / HCF (GCD / GCF) & LCM OF NUMBERS

Prime factors:

A composite number can be uniquely expressed as a product of prime factors.

Ex. $12 = 2 \times 6 = 2 \times 2 \times 3 = 2^2 \times 3^1$ $20 = 4 \times 5 = 2 \times 2 \times 5 = 2^2 \times 5^1$

$124 = 2 \times 62 = 2 \times 2 \times 31 = 2^2 \times 31$ etc.

**If k and n are both integers greater than 1 and if k is a factor of n, k cannot be a factor of (n + 1).**

***NOTE:***

The number of divisors (factors) of a given number N (including one and the number itself) where $N = a^m \times b^n \times c^p$ ..... where a, b, c are prime numbers is given by (1 + m) (1 + n) (1 + p) ......

e.g. (1) $90 = 2 \times 3 \times 3 \times 5 = 2^1 \times 3^2 \times 5^1$

Hence here a = 2 b = 3 c = 5, m = 1 n = 2 p = 1

Number of divisors = (1 + m)(1 + n)(1 + p) .... = 2 x 3 x 2 = 12

Number of factors of 90 = 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90 = 12

HCF: It is the greatest factor common to two or more given numbers. It is also called GCF OR GCD (greatest common factor or greatest common divisor). e.g. HCF of 10 & 15 = 5, HCF of 55 & 200 = 5, HCF of 64 & 36 = 4

To find the HCF of given numbers, resolve the numbers into their prime factors and then pick the common term(s) from them and multiply them. This is the required HCF.

LCM: Lowest common multiple of two or more numbers is the smallest number which is exactly divisible by all of them.

e.g. LCM of 5, 7, 10 = 70, LCM of 2, 4, 5 = 20, LCM of 11, 10, 3 = 330

To find the LCM resolve all the numbers into their prime factors and then pick all the quantities (prime factors) but not more than once and multiply them. This is the LCM.

***NOTE:***

1. LCM x HCF = Product of two numbers (valid only for "two")

2. HCF of fractions = HCF of numerators ÷ LCM of denominators

3. LCM of fractions = LCM of numerators ÷ HCF of denominators

Q. Find the LCM of 25 and 35 if their HCF is 5. LCM = 25 x 35/5 = 175

**Calculating LCM:** After expressing the numbers in terms of prime factors, the LCM is the product of highest powers of all factors.

Q. Find the LCM of 40, 120, and 380.
$40 = 4 \times 10 = 2 \times 2 \times 2 \times 5 = 2^3 \times 5^1$,
$120 = 4 \times 30 = 2 \times 2 \times 2 \times 5 \times 3 = 2^3 \times 5^1 \times 3^1$
$380 = 2 \times 190 = 2 \times 2 \times 95 = 2 \times 2 \times 5 \times 19 = 2^2 \times 5^1 \times 19^1$
Required LCM = $2^3 \times 5^1 \times 3^1 \times 19^1 = 2280$.

**Calculating HCF:** After expressing the numbers in term of the prime factors, the HCF is product of COMMON factors.

Ex. Find HCF of 88, 24, and 124
$88 = 2 \times 44 = 2 \times 2 \times 22 = 2 \times 2 \times 2 \times 11 = 2^3 \times 11^1$
$24 = 2 \times 12 = 2 \times 2 \times 6 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3^1$
$124 = 2 \times 62 = 2 \times 2^1 \times 31^1 = 2^2 \times 31^1$ HCF = $2^2$

**Problems:**

1. The integers m and p are such that 2 < m < p, and m is not a factor of p. If r is the remainder when p is divided by m, is r > 1?
(1) the greatest common factor of m and p is 2
(2) the least common multiple of m and p is 30

2. If the integer n is greater than 1, is n equal to 2?
(1) n has exactly two positive factors
(2) The difference between any two distinct positive factors is odd.

3. The function f is defined for all positive integers n by the following rule: f(n) is the number of positive integers each of which is less than n and also has no positive factor in common with n other than 1. If p is a prime number then f(p) = ?
p – 1    p – 2    (p + 1) / 2    (p – 1) / 2    2

4. For every positive even integer n, the function h(n) is defined to be the product of all the even integers from 2 to n, inclusive. If p is the smallest prime factor of h(100) + 1, then p is
A. between 2 and 10    B. between 10 and 20    C. between 20 and 30
D. between 30 and 40    E. greater than 40

5. Is the integer n odd?
(1) n is divisible by 3    (2) 2n is divisible by twice as many positive integers as n

6. How many different prime numbers are factors of the positive integer n?
   (1) four different prime numbers are factors of 2n
   (2) four different prime numbers are factors of $n^2$.

7. Does the integer k have a factor p such that $1 < p < k$?
   (1) $k > 4!$ (2) $13! + 2 \le k \le 13! + 13$.

8. The positive integer k has exactly two positive prime factors, 3 and 7. If K has a total of 6 positive factors, including 1 and k, what is the value of K?
   (1) $3^2$ is a factor of k (2) $7^2$ is NOT a factor of k

**Divisibility / Remainders**

**TESTS FOR DIVISIBILITY:**

1. A number is divisible by 2 if its unit's digit is even or zero e.g. 128, 146, 34 etc.
2. A number is divisible by 3 if the sum of its digits is divisible by 3 e.g. 102, 192, 99 etc.
3. A number is divisible by 4 when the number formed by last two right hand digits is divisible by '4' e.g. 576, 328, 144 etc.
4. A number is divisible by 5 when its unit's digit is either five or zero: e.g. 1111535, 3970, 145 etc.
5. A number is divisible by 6 when it's divisible by 2 and 3 both. e.g. 714, 509796, 1728 etc.
6. A number is divisible by 8 when the number formed by the last three right hand digits is divisible by '8'. e.g. 512, 4096, 1304 etc.
7. A number is divisible by 9 when the sum of its digits is divisible by 9 e.g. 1287, 11583, 2304 etc.
8. A number is divisible by 10 when its unit's digit is zero. e.g. 100, 170, 10590 etc.
9. A number is divisible by 11 when the difference between the sums of digits in the odd and even places is either zero or a multiple of 11. e.g. 17259, 62468252, 12221 etc. For the number 17259: Sum of digits in even places = 7 + 5 = 12, Sum of digits in the odd places = 1 + 2 + 9 = 12 Hence 12 – 12 = 0.
10. A number is divisible by 12 when it is divisible by 3 & 4 both. e.g. 672, 8064 etc.
11. A number is divisible by 25 when the number formed by the last two Right hand digits is divisible by 25, e.g., 1025, 3475, 55550 etc.

***NOTE:***

1. When any number with even number of digits is added to its reverse, the sum is always divisible by 11. e.g. 2341 + 1432 = 3773 which is divisible by 11.
2. If X is a prime number then for any whole number "a" ($a^X - a$) is divisible by X e.g.
   Let X = 3 and a = 5. Then according to our rule $5^3 - 5$ should be divisible by 3.
   Now ($5^3 - 5$) = 120 which is divisible by 3.

**Problems:**

1. If t is a positive integer and r is the remainder when $t^2 + 5t + 6$ is divided by 7, what is the value of r?
   (1) when t is divided by 7, the remainder is 6 (2) when $t^2$ is divided by 7, the remainder is 1

2. If p is a positive odd integer, what is the remainder when p is divided by 4?
   (1) When p is divided by 8, the remainder is 5.
   (2) p is the sum of the squares of two positive integers.

3. If n is a positive integer and r is the remainder when $(n - 1)(n + 1)$ is divided by 24, what is the value of r?
   (1) n is not divisible by 2 (2) n is not divisible by 3

4. If N is a positive integer, is $(N^3 – N)$ divisible by 4?
(1) n = 2k + 1, where K is an integer. (2) $n^2 + n$ is divisible by 6

**Power of a Prime Number in a Factorial:** If we have to find the power of a prime number p in n!, it is found using a general rule, which is $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+........$, where $\left[\frac{n}{p}\right]$ denotes the greatest integer ≤ to $\left[\frac{n}{p}\right]$ etc.

For example power of 3 in 100 ! = $\left[\frac{100}{3}\right]+\left[\frac{100}{3^2}\right]+\left[\frac{100}{3^3}\right]+\left[\frac{100}{3^4}\right]+\left[\frac{100}{3^5}\right]+....$ = 33 + 11 + 3 + 1 + 0 = 48.

For example power of 5 in 200 ! = $\left[\frac{200}{5}\right]+\left[\frac{200}{5^2}\right]+\left[\frac{200}{5^3}\right]+...$ = 40 + 8 + 1 + 0 = 49.

**Number of Zeroes at the end of a Factorial:** It is given by the power of 5 in the number.
Actually, the number of zeroes will be decided by the power of 10, but 10 is not a prime number, we have 10 = 5 × 2, and hence we check power of 5.
For example, the number of zeroes at the end of 100 ! = = 20 + 4 = 24.
The number of zeroes at the end of 500 ! = = 100 + 20 + 4 = 124.
The number of zeroes at the end of 1000 ! = = 200 + 40 +8 + 1 = 249.

**Unit's digits in powers:** Every digit has a cyclicity of 4. The fifth power of any single digit number has the same right hand digit as the number itself.

**Example:** What will be the unit's digit in $128^{96}$?
In all such questions, divide the power by 4 and check the remainder.
If the remainder is 1, 2 or 3, then convert the question to LAST DIGIT RAISED TO REMAINDER.
If the remainder is 0, convert the question to LAST DIGIT RAISED TO FOUR.
In this question, 96/4 = 0, so the question converts to $8^4 = 8^2 \times 8^2 = 64 \times 64 = 4 \times 4 = 16 = 6$

**Problems:**

1. If d is a positive integer, f is the product of the first 30 positive integers, what is the value of d?
(1) $10^d$ is a factor of f (2) d > 6

2. If n and m are positive integers, what is the remainder when $3^{4n+2} + m$ is divided by 10?
(1) n = 2 (2) m = 1

**ARITHMETIC PROGRESSION (A.P.)**

It is a series of numbers in which every term after the first can be derived from the term immediately preceding it by adding to it a fixed quantity called COMMON DIFFERENCE.

***Ex.*** 1, 5, 9, 13 ....; 1, –2, –5, –8, .....; a, a + d, a + 2d, a + 3d, are in A.P.
If in an A.P. a = the first term, d = common difference, $T_n$ = the nth term, $l$ = the last term, $S_n$ = Sum of n terms, We have $T_n = a + (n-1)d \quad S_n = \frac{n}{2}(a+l) \quad S_n = \frac{n}{2}[2a + (n-1)d]$

Sum of first n natural numbers = $\frac{n(n+1)}{2}$

Sum of first n odd natural numbers = $n^2$

Sum of first n even natural numbers = $n(n+1)$

Sum of squares of first n natural numbers = $\frac{n(n+1)(2n+1)}{6}$

Sum of cubes of first n natural numbers = $\left[\frac{n(n+1)}{2}\right]^2$

**GEOMETRIC PROGRESSION (G.P.)**

A series in which each term is formed from the preceding by multiplying it by a constant factor is called a Geometric Progression or G.P. The constant factor is called the common ratio and is formed by dividing any term by the term which precedes it. Ex.: 1,2,4,8,16.....; 3,9,27,81,243.... etc.

The General form of a G.P. with n terms is a, ar, $ar^2$, .... $ar^{n-1}$

Thus if a = the first term, r = the common ratio, then the nth term $T_n = ar^{n-1}$

The sum to n terms $S_n = \frac{a(r^n - 1)}{r - 1}$ ... If a G.P. has infinite terms and –1 < r < 1, the sum to infinity

$S_\infty = \frac{a}{1-r}$

**QUADRATIC EQUATIONS**

The quadratic form is generally represented by $aX^2 + bX + C = 0$ where a ≠ 0, and a, b, c are constants
e. g., $X^2 + 4X - 12 = 0$ $3X^2 - 3X + 2 = 0$

**Factorization method:** Ex. Solve $X^2 - 4X + 3 = 0$
We have $X^2 - 4X + 3 = (X - 1)(X - 3) = 0 \Rightarrow$ either $(X - 1) = 0$ or $(X - 3) = 0 \Rightarrow X = 1$ and $X = 3$

**General Method:** The general solution to $AX^2 + BX + C = 0$ is given as $X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

***Ex.*** Solve $2X^2 - 7X + 6 = 0$ Here we have A = 2, B = –7, C = 6 ⇒ X = [– (–7)² ± (4 x 2 x 6)] / (2 x 2)
⇒ X = [7 ± √(49 – 48)] / (2 x 2) = (7 ± 1) /4 Hence X = 2, 3/2

If the equation has roots $x_1$ and $x_2$ then we may write the equation as $(x - x_1)(x - x_2) = 0$ or
$x^2 - (x_1 + x_2)x + x_1x_2 = 0$ or $x^2 - Sx + P = 0$, where S = sum of roots and P = product of roots.

Comparing with the original equation, we get, Sum of roots $S = -\frac{b}{a}$ and Product of roots $P = \frac{c}{a}$.

**Problems:**

1. If each term in the sum *a1 + a2+... an* is either 7 or 77 and the sum equals 350, which of the following could be the value of n?
   38 39 40 41 42

2. When a certain tree was first planted, it was 4 feet tall, and the height of the tree increased by a constant amount each year for the next 6 years. At the end of the 6th year, the tree was 1/5 taller than it was at the end of the 4th year. By how many feet did the height of the tree increased each year?
   3/10 2/5 ½ 2/3 6/5

3. For a finite sequence of nonzero numbers, the number of variations in sign is designed as the number of pairs of consecutive terms of the sequence for which the product of the two consecutive terms is negative. What is the number of variations in sign for the sequence 1, –3, 2, 5, –4, –6?
   One Two Three Four Five

4. $2 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8$?
   $2^9$ $2^{10}$ $2^{16}$ $2^{35}$ $2^{37}$

5. A certain list contains several different integers. Is the product of the integers in the list positive?
   (1) The product of the greatest and the smallest of the integers in the list is positive
   (2) There is an even number of integers in the list

6. If there are more than two numbers in a certain list, is each of the numbers in the list equal to 0?
   (1) The product of any two numbers in the list is equal to 0.
   (2) The sum of any two numbers in the list is equal to 0.

**DECIMALS and FRACTIONS**

**Recurring Decimals (Conversion to a Rational Number):** If in a decimal fractions a figure or a set of figures is repeated continually, then such a number is called a recurring decimal.

(i) 2/3 = 0.6666.... (ii) 22/7 = 3.142857142857 …

Rule: Write the recurring figures only one in the numerator and take as many nines in the denominator as the number of repeating figures.

Ex. (1) 0.666666666 … = 6/9 = 2/3 (2) 0.234234234234 … = 234/999

**Rounding Off**

| Number | Nearest tenth | Nearest hundredth | Nearest thousandth |
|---|---|---|---|
| 1.2346 | 1.2 | 1.23 | 1.235 |
| 31.6479 | 31.6 | 31.65 | 31.648 |
| 9.7462 | 9.7 | 9.75 | 9.746 |

**Whether a fraction will result in a terminating decimal or not?** To determine this, express the fraction in the lowest form and then express the denominator in terms of Prime Factors. If the denominator contains powers of only 2 and 5, it is terminating. If the denominator contains any power of any other prime number, it is non-terminating.

SOME FORMULAS

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$
2. $(a + b)^2 - (a - b)^2 = 4ab$
3. $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$
4. $a^2 - b^2 = (a + b)(a - b)$
5. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

6. (1) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ (2) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
7. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$
8. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$ so if $(a + b + c) = 0$ then $a^3 + b^3 + c^3 = 3abc$.
9. If p is a prime number then for any whole number a, $a^p - a$ is divisible by p.

**Factor Theorem:** If f(x) is completely divisible by (x – a), then f(a) = 0. So, (x – a) **is a factor of** f(x), f(a) = 0

**Check whether (x + 1) is a factor of f(x) = $4x^2 + 3x - 1$.** Putting x + 1 = 0, i.e., x = –1 in the given expression we get f(–1) = 0. So, (x + 1) is a factor of f(x).

**Remainder Theorem:** If an expression f(x) is divided by (x – a), then the remainder is f(a).

**Let f(x) = $x^3 + 3x^2 - 5x + 4$ be divided by (x – 1). Find the remainder.**
Remainder = $f(1) = 1^3 + 3 \times 1^2 - 5 \times 1 + 4 = 3$.

**Some properties of square numbers:**
- A square number always has odd number of factors.
- A square number cannot end with 2, 3, 7, 8 or an odd number of zeroes.
- Every square number is a multiple of 3, or exceeds a multiple of 3 by unity.
- Every square number is a multiple of 4 or exceeds a multiple of 4 by unity.
- If a square number ends in 9, the preceding digit is even.

**Questions based on Number Line**

1. If n denotes a number to the left of 0 on the number line such that the square of n is less than 1/100, then the reciprocal of n must be
A. less than –10 B. between –1 and –1/10 C. between –1/10 and 0
D. between 0 and –1/10 E. greater than 10

2. If s and t are two different numbers on the number line, is s + t = 0?
(1) The distance between s and 0 is the same as the distance between t and 0
(2) 0 is between s and t

**Miscellaneous questions**

1. The symbol * represents one of the four arithmetic operations: addition, subtraction, multiplication, and division. Is (5 * 6) * 2 = 5 * (6 * 2)? (1) 5 * 6 = 6 * 5 (2) 2 * 0 = 2

2. If k is a positive integer and the ten's digit of k + 5 is 4, what is the ten's digit of k?
(1) k > 35 (2) The units digit of k is greater than 5.

3. If the operation ^ is one of the four arithmetic operations addition, subtraction, multiplication, and division, is (6 ^ 2) ^ 4 = 6 ^ (2 ^ 4)? (1) 3 ^ 2 > 3 (2) 3 ^ 1 = 3

4. Is the hundredth digit of decimal d greater than 5?
(1) The tenth digit of 10d is 7 (2) The thousandth digit of d/10 is 7

5. What is the tens digit of the positive integer r?
(1) The tens digit of r/10 is 3. (2) The hundreds digit of 10r is 6.

6. In the table above, $z = 20q$?

| | | | |
|---|---|---|---|
| $q$ | $q$ | $q$ | $q$ |
| $q$ | $r$ | $s$ | $t$ |
| $q$ | $u$ | $v$ | $w$ |
| $q$ | $x$ | $y$ | $z$ |

(1) $q = 3$
(2) Each value in the table other than q is equal to the sum of the value immediately above it in the table and the value immediately to its left in the table.

7. For which of the following functions is f(a+b)=f(a)+f(b) for al positive numbers a and b?
$f(x)=x^2$ $f(x)=x+1$ $f(x)=\sqrt{x}$ $f(x)=2/3$ $f(x)=-3x$

8. What is the result when x is rounded to the nearest hundredth?
(1) When x is rounded to the nearest thousandth the result is 0.455
(2) The thousandth digit is 5