Pattern Recognition And Machine Learning: Assignment #2

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Problem 1

Show that the set X of all integers with metric defined by $d(m, n) =| m - n |$ is a complete metric space.

Solution

Set of all integers X form a discrete space.Now the metric defined in it $d(m, n) =| m-n |$. Lets prove (X,d) form a valid metric space
1. d(x,y) $\geq$ 0 and finite
2. d(x,y) = d(y,x)and d(x,y) = 0 iff x=y ,ie if x=y, d(x,y) = d(x,x) = $|x - x| = 0$
3.To prove triangle inequality, lets take x,y,z $\in$ X.
$d(x, z) = |x - z|$
$= |x - y + y - z| \leq |x - y| + |y - z|$
$= d(x, y) + d(y, z)$
$\Rightarrow d(x, z) \leq d(x, y) + d(y, z)$
So X is a metric space.

Proving Set of all Integers is Complete in the given metric space

Unravelling the definition for a Cauchy sequence, we get that :
$\forall \epsilon > 0 : \exists N : \forall m, n > N : d(a_n, a_m) < \epsilon$

and for $\epsilon = 1/2$ and with the given metric definition $d(m, n) =| m - n |$, we note that this must mean $d(a_n, a_m) = 0$ (since otherwise it exceeds 1) i.e $a_n = a_m$ for some N and all $m, n \geq N$.That is, a_n is eventually constant.

Now there is an obvious guess for the limit of eventually constant sequences, and we conclude that set of integers with the Euclidean metric is complete.

Problem 2

Show that $d(x, y) = \sqrt{|x - y|}$ defines a metric on the set of all real numbers.

Solution

Lets prove (X,d) form a valid metric space

1. d(x,y) $\geq$ 0 and finite
2. d(x,y) = d(y,x)and d(x,y) = 0 iff x=y ,ie if x=y, d(x,y) = d(x,x) = $\sqrt{|x - x|} = 0$

3.To prove triangle inequality, lets take x,y,z $\in$ X.

$d(x, z) = \sqrt{|x - z|}$
$= \sqrt{|x - y + y - z|}$
$\leq \sqrt{|x - y| + |y - z|}$
$\leq \sqrt{|x - y|} + \sqrt{|y - z|}$
$= d(x, y) + d(y, z)$
$\Rightarrow d(x, z) \leq d(x, y) + d(y, z)$

So X is a metric space.

Problem 3

Show that the closure $\bar{Y}$ of a subspace $\boldsymbol{Y}$ of a normed space $\boldsymbol{X}$ is again a vector space.

Solution

It is sufficient to prove that $\alpha x + \beta y \in \bar{Y}$ where α and β are in the underlying field $\mathbb{F}$ and x,y $\in \bar{Y}$.
Since x,y $\in \bar{Y} \exists x_j, y_j \in X$ such that $x_j \to x$ and $y_j \to$ y. Since multiplication and addition are continuous $\alpha x_j + \beta y_j \to \alpha x + \beta y$.

So, $\alpha x + \beta y \in \bar{Y}$.
Therefore $\bar{Y}$ is a subspace.

Problem 4

Show that in an inner product space, $x \perp y$ iff $\| x+\alpha y \| \geq \| x \| \; \forall \alpha \in \mathbb{R}$

Solution

$\|x + \alpha y\|^2 = \langle x + \alpha y, x + \alpha y\rangle$
$= \langle x, x\rangle + \alpha\langle x, y\rangle + \alpha\langle y, x\rangle + \alpha^2\langle y, y\rangle$
i.e ,
$\|x + \alpha y\|^2 = \langle x, x\rangle + 2\alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle$

$= \|x\|^2 + 2\alpha\langle x, y\rangle + \alpha^2$—(1)

if x $\perp, \langle x, y\rangle = 0$

$(1) \Rightarrow \|x + \alpha y\|^2 = \|x\|^2 + \alpha^2|y\|^2$
$\Rightarrow \|x + \alpha y\|^2 \geq \|x\|^2$
$\Rightarrow \|x + \alpha y\| \geq \|x\|$

hence proved the if part

if $\|x + \alpha y\|^2 \geq\| x^2 \| \; \forall \alpha \varepsilon \Re$

$\Rightarrow 2\alpha\langle x, y\rangle + \alpha^2 \geq 0$,which is true **if** $\langle x, y\rangle = 0$

hence proved the only if part

Problem 5

Find $\langle u, v \rangle$, where $v = (1+2i, 3-i)^T$, $u = (-2+i, 4)^T$.

Solution $langleu, v\rangle = \langle(-2+i, 4), (1+2i, 3-i)\rangle$
for complex numbers $\langle(x1, x2)(y1, y2)\rangle = x1 * \overline{y1} + x2 * \overline{y2}$
$= (-2+i)(1-2i) + 4(3+i)$
$= -2 + 4i + i + 2 + 12 + 4i$
$= 9i + 12$

Problem 6

Which of the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$? [**x**=$(\eta_1, \eta_2, \eta_3)^T$]

(a) All x with $\eta_1 = \eta_2$ and $\eta_3 = 0$.

(b) All x with $\eta_1 = \eta_2 + 1$

Solution

a)
Let Z={All x with $\eta_1 = \eta_2$ and $\eta_3 = 0$}.
Consider X=$(x, x, 0)$,Y=$(y, y, 0) \in Z$
$X + Y = (x + y, x + y, 0) \in Z$
$\alpha X = (\alpha x, \alpha x, 0) \in Z$
Thus Z is closed under addition and scalar multiplication,hence it is a subspace of $\mathbb{R}^3$.
b)
Let Z={All x with $\eta_1 = \eta_2 + 1$}.
Consider X=$(x + 1, x, p)$,Y=$(y + 1, y, q) \in Z$ where p,q$\in \mathbb{R}$.
$X + Y = (x + y + 2, x + y, p + q)) \notin Z$
because $\eta_1 \neq \eta_2 + 1$ is violated here. Hence Z is not closed under addition. So it is not a subspace of $\mathbb{R}^3$.

Problem 7

Show that $\|x\|$ is the distance from x to 0.

Solution
Considering $\|x\|_2$, d(x,y)= $\|x - y\|$ is clearly a metric space.

Let y=0.
Therefore d(x,0)= $\|x - y\|$ = distance between x and origin = $\|x\|$.

Problem 8

If an inner product space $\langle x, u \rangle = \langle x, v \rangle$ for all x, show that u=v.

Solution
Given $\langle x, u \rangle = \langle x, v \rangle$
i.e, $\langle x, u \rangle$ - $\langle x, v \rangle = 0$
$\Rightarrow \langle x, u - v \rangle = 0$—–(1)

we have to prove its true $\forall x$
take x= u-v
(1)$\Rightarrow \langle u - v, u - v \rangle = 0$
$\Rightarrow \| u - v \|^2 = 0$
$\Rightarrow \| u - v \| = 0$

$\Rightarrow u = v$
Hence Proved

Problem 9

Prove that $\|T_1T_2\| \leq \|T_1\|\|T_2\|$;$\|T^n\| \leq \|T\|^n$

Solutions

To Prove $\|T_1T_2\| \leq \|T_1\|\|T_2\|$
A vector norm and its induced matrix norm satisfy the inequality
$\|Tx\| \leq \|T\|\|x\|$
Replace x by T_2x,
$\|T_1T_2x\| \leq \|T_1\|\|T_2\|\|x\|$
For nonzero x we can divide both sides by the positive number $\|x\|$ and can conclude that
$\|T_1T_2\| = max_x \frac{\|T_1T_2x\|}{\|x\|} \leq \|T_1\|\|T_2\|$
Hence Proved

To prove $\|T^n\| \leq \|T\|^n$

if $T_1 = T_2$
$\|TT\| = \|T^2\| \leq \|T\|^2$
similiarly, $\|T.......T\| \leq \|T\|\|T........T\|$
$\|T\|^n \leq \|T\|\|T\|\|T........T\|$
Therefore $\|T^n\| \leq \|T\|^n$ Hence, PROVED.

Problem 10

For a real inner product space prove that $\langle x, y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$

Solution
$\|x+y\|^2$
$= \langle x+y, x+y\rangle$
$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$
$= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$
$= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$ ——**(1)**
similarly,
$\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$ —**-(2)**

(1)-(2) $\Rightarrow \|x+y\|^2 - \|x-y\|^2 = 4\langle x,y\rangle$

$\Rightarrow \langle x,y\rangle = 1/4(\|x+y\|^2 - \|x-y\|^2)$

Problem 11

11. Define T: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by T(x,y)=(x,0). Is T a linear operator ?

An linear operator satisties the following conditions, $i.f(x+y) = f(x) + f(y), \forall x, y \epsilon Z, Z is the vector space$
$ii.f(\alpha x) = \alpha f(x), \forall \alpha \epsilon F(field), x \epsilon Z$
Checking the conditions on the operator defined,
$f(x, y) = (x, 0)$
$i.f(x_1, y_1) + f(x_2, y_2) = (x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) = f(x_1 + x_2, y_1 + y_2)$
$ii.f(\alpha x, y) = (\alpha x, 0) = \alpha(x, 0) = \alpha f(x, 0) = \alpha f(x, y)$

Thus T a linear operator .

Problem 12

Show that a discrete metric space is complete.

Solution
Let (X,d) be the discrete metric space. The standard discrete metric d is defined as d(x,y) = 0 if x=y d(x,y) = 1 otherwise
Unravelling the definition for a Cauchy sequence, we get that :
$\forall \epsilon > 0 : \exists N : \forall m, n > N : d(a_n, a_m) < \epsilon$

and for $\epsilon = 1/2$ and with the given metric definition **d(m,n) = 0 if m=n and d(m,n) = 1 otherwise**, we note that this must mean $d(a_n, a_m) = 0$ (since otherwise it exceeds 1) i.e $a_n = a_m$ for some N and all $m, n \geq N$.That is, a_n is eventually constant.

Now there is an obvious guess for the limit of eventually constant sequences, and we conclude that a discrete metric space is always complete.

Problem 13

Describe Weistrass approximation theorem.

Solution

The set W of all polynomials with real coefficients is dense in the real space C[a,b].
Hence for every x $\in$ [a,b] and given $\in > 0$ $\exists$ a polynomial p such that $|x(t) - p(t)| < \in$ for all t $\in$ [a,b]