

End-Semester Examination

(MA6.102) Probability and Random Processes, Monsoon 2023

29th November, 2023

Max. Duration: 3 Hours

Special Instruction: The solution to each question should begin on a new page.

Question 1. (a) (2 Marks) For three events A, B, C , prove that

$$P(A|B) = P(A|B \cap C)P(C|B) + P(A|B \cap C^c)P(C^c|B).$$

Hint: Conditional total probability theorem.

(b) (3 Marks) For two events A and B , is it true that $P(A|A \cup B) \geq P(A|B)$? If not, give a counterexample.

Question 2 (5 Marks). For an event A , let \mathbb{I}_A denote the indicator random variable of A , i.e., $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{I}_A(\omega) = 0$ if $\omega \in A^c$. For any two events A and B , show that the following are equivalent.

- The events A and B are independent.
- The random variables \mathbb{I}_A and \mathbb{I}_B are independent.

$$Y = \begin{cases} 0 \end{cases}$$

Question 3 (5 Marks). Let X be a discrete random variable that is uniformly distributed over $\{a, a+1, \dots, b-1, b\}$, where a and b are integers with $a < 0 < b$. Let $Y = \max\{0, X\}$ and $Z = \min\{0, X\}$. Find the PMFs P_Y and P_Z .

Question 4 (5 Marks). A permutation on the numbers in $[1 : n]$ can be represented as a function $\pi : [1 : n] \rightarrow [1 : n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation. A fixed point of a permutation $\pi : [1 : n] \rightarrow [1 : n]$ is a value x for which $\pi(x) = x$. Let X be number of fixed points of a permutation chosen uniformly at random from all permutations. Find $\mathbb{E}[X]$.

Hint: Express X as a sum of indicator random variables.

Question 5 (5 Marks). For any two random variables X and Y , Cauchy-Schwarz inequality states that

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

with equality if and only if $X = \alpha Y$, for some constant $\alpha \in \mathbb{R}$. Prove this and use it to show that $|\rho(X, Y)| \leq 1$, where $\rho(X, Y)$ is the correlation coefficient of X and Y given by

$$\rho(X, Y) = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{var}(X)\text{var}(Y)}}.$$

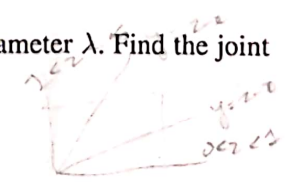
Question 6. (a) (2 Marks) Show that X and Y are independent continuous random variables if and only if their joint PDF f_{XY} factorizes as the product of the functions of the single variables x and y alone, i.e., $f_{XY}(x, y) = g(x)h(y)$, for all x, y .

(b) (3 Marks) Let X and Y be independent exponential random variables with parameter λ . Find the joint PDF of

$$Z = X + Y \text{ and } W = \frac{X}{Y}$$

$$P(2 \leq Z) \\ X+Y \geq 2 \\ Y \geq 2-X$$

$$P(W \leq w) \\ X \leq wY$$



and show that they are independent.

Question 7. Let X and Y be two random variables with the associated MGFs $M_X(s)$ and $M_Y(s)$, respectively. Let Z be a random variable MGF

$$M_Z(s) = M_X(s)^2 M_Y(s)^3.$$

Find $\mathbb{E}[Z]$ and $\text{var}(Z)$ in terms of $\mathbb{E}[X]$, $\text{var}(X)$, $\mathbb{E}[Y]$, and $\text{var}(Y)$.

Question 8 (5 Marks). Consider two sequences of random variables X_1, X_2, \dots and Y_1, Y_2, \dots which converge in probability to x and y , respectively. That is, for every $\epsilon > 0$, we have

$$\lim_{n \rightarrow \infty} P(|X_n - x| \geq \epsilon) = 0,$$

$$\lim_{n \rightarrow \infty} P(|Y_n - y| \geq \epsilon) = 0.$$

Prove that the sequence $X_1 Y_1, X_2 Y_2, \dots$ converges in probability to xy .

Hint: Show that $\lim_{n \rightarrow \infty} P(|(X_n - x)(Y_n - y)| \geq \epsilon) = 0$.

Question 9 (5 Marks). Let $X_1, Y_1, X_2, Y_2, \dots$ are independent random variables and uniformly distributed over the interval $[0, 1]$, and let

$$W = \frac{\sum_{i=1}^{16} X_i - \sum_{i=1}^{16} Y_i}{16}.$$

Find an approximate value to the quantity $P(|W - \mathbb{E}[W]| < 0.001)$ in terms of the CDF of standard Gaussian random variable $\mathcal{N}(0, 1)$.

Question 10 (5 Marks). Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a periodic function with period T , i.e., $g(t + T) = g(t)$, for all $t \in \mathbb{R}$. Consider the random process

$$X(t) = g(t + U), \text{ for all } t \in \mathbb{R},$$

where U is a random variable uniformly distributed over the interval $[0, T]$. Is $X(t)$ a wide-sense stationary (WSS) process?

$$\begin{aligned} \mu(t) &= \mu(t+T) = \mu(t) \\ R_X(t_1, t_2) &= R_X(t_1+T, t_2+T) \end{aligned}$$

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