

Linear Algebra (UG1, Spring 2023)

Midsem [20 marks]; Time: 90 mins (+45 mins)

April 29, 2023

Notations are from class lectures unless stated otherwise. Each step of the proof should be clear. Appropriate reasoning for your claims are must.

Question A [9 marks]

1. Suppose V_1, V_2, \dots, V_m are subspaces of a vector space V defined over the field F . Prove that $V_1 + V_2 + \dots + V_m$ is the smallest subspace of V containing V_1, V_2, \dots, V_m . [3 marks]
2. Suppose the set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is linearly independent in the vector space V over field F . Prove that if the set of vectors $\vec{v}_1 + \vec{w}, \vec{v}_2 + \vec{w}, \dots, \vec{v}_m + \vec{w}$ is linearly dependent in V , then \vec{w} is spanned by the set of linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$. [3 marks]
3. Prove that a vector space V defined over a field F is infinite-dimensional if and only if there is a sequence $\vec{v}_1, \vec{v}_2, \dots$ of vectors in V such that $\vec{v}_1, \dots, \vec{v}_m$ is linearly independent for every positive integer m . [3 marks]

Question B [6 marks]

Let $M_{2 \times 2}(\mathbb{R})$ be the vector space of 2×2 matrices defined over the field \mathbb{R} of real numbers. If $T : M_{2 \times 2}(\mathbb{R}) \rightarrow \mathbb{R}$ is the trace map $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$, i.e., T is the sum of the diagonal entries of a square matrix. Then,

- Show that T is a linear transformation. [1.5 marks]
- Find the nullity and the rank of T . [3 marks]
- State the nullity-rank theorem. Verify whether the theorem holds for T or not. [1.5 marks]

Question C [5 marks]

Using notations from the previous question, consider the map $P : \mathbb{R}^4 \rightarrow M_{2 \times 2}(\mathbb{R})$ given by $P(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$.

- Show that the map P is a linear transformation. [2 marks]
- Find the null space and the range of P . [1 marks]
- Does the inverse of P exist? If yes, what is the inverse of P ? If no, then why not? [2 marks]