Quiz 2
Information and Communication (Spring 2023)
Time : 40 mins, Total Marks: 20

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Instructions:

- This is a closed book, traditional, exam.
- Malpractice will directly result in 0 and further academic action will be initiated.
- You can request (not demand) for an additional hint from the course instructor (not from anyone else), who should be around. The discretion of providing (or not providing) the hint for a particular question will be left to the instructor. If the instructor is absent, no hint will be provided. *No debate or discussion will be there during the time of evaluation or post-evaluation regarding these hints.*

Questions:

1. Assume the following fact: For A, B, C being any three non-empty **collections** of random variables, the conditional mutual information $I(A;B|C)$ is non-negative, and equal to 0 if and only if the collections A and B are independent of each other, given C (that is $p_{A,B|C}(a,b|c) = p_{A|C}(a|c) \cdot p_{B|C}(b|c), \forall a, b, c$).

   (a) (4 marks) For any three collections of random variables X, Y and Z, show that $H(X|Y,Z) \leq H(X|Z)$. Give the conditions for when equality will hold in this inequality, and the interpretation of those conditions in some Communication setup.

   (b) (3+5 marks) Consider random variables $X_i : i \in \{1, \cdots, n\}$ and Y. Give the definition for the conditional joint entropy $H(X_1, \cdots, X_n|Y)$ (that is, state the mathematical expression to calculate this conditional joint entropy). State and prove the chain rule for $H(X_1, \cdots, X_n|Y)$. Discuss the conditions under which $H(X_1, \cdots, X_n|Y)$ will be equal to $\sum_{i=1}^{n} H(X_i|Y)$.

2. (8 marks) Let $X \sim \text{Bern}(0.5)$. Let the alphabet for Y be $\mathcal{Y} = \{0, \varepsilon, 1\}$. Let $p_{Y|X}(0|0) = p_{Y|X}(1|1) = 1-p$, and $p_{Y|X}(\varepsilon|0) = p_{Y|X}(\varepsilon|1) = p$. Calculate the mutual information between X and Y.

3. (4 marks) Construct intuitively a short-length prefix-free code for a random variable $X \sim \text{Binom}(n = 4, p = 0.1)$. (Note: The word 'Construct' means that you have to assign the codewords for each value that X can take. Think about how to assign so that the average length is short).