Final Exam Name:________________________________

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There are five equally weighted partial credit questions. Please show all your work for full credit, displaying only the final answer will earn you very little, if anything.

Very important: throughout this exam, wherever they appear, $\alpha =$ 1 $\beta =$ 2 $\gamma =$ 3

1. **(a)** Determine the internal representation of the decimal number $\alpha\beta\gamma.9375$ in the normalized floating-point number system of a 32-bit word length microcomputer.

sign of mantissa (1 bit)	exponent (7 bits)	normalized mantissa (24 bits)

First, state your number: $\alpha\beta\gamma.9375 = 123.9375 = (1111011.1111)_2 = .11110111111_2 \times 2^{(111)_2}$

$123 \div 2 = 61 + 1$
$61 \div 2 = 30 + 1$
$30 \div 2 = 15 + 0$
$15 \div 2 = 7 + 1$
$7 \div 2 = 3 + 1$
$3 \div 2 = 1 + 1$
$1 \div 2 = 0 + 1$

$0.9375 \times 2 = 1 + 0.875$
$0.875 \times 2 = 1 + 0.75$
$0.75 \times 2 = 1 + 0.5$
$0.5 \times 2 = 1 + 0$

Answer:

0	0000111	1111011111100 - - - 0

→ 3 pts.

(b) Determine the internal representation of the **same** decimal number $\alpha\beta\gamma.9375$ from part **(a)** in the IEEE single-precision format

sign (1 bit)	biased exp. c (8 bits)	fractional part f of the normalized mantissa (23 bits)

$$123.9375 = (1.\underbrace{1110111111})_2 \times 2^6$$

$$c - 127 = 6 \Rightarrow c = 133 = (10000101)_2$$

Answer:

0	10000101	11101111110 0 - - - 0

→ 3 pts.

(c) Convert the following (single-precision) IEEE 32-bit machine number to decimal

s	$10001e_3e_21$	$1m_20m_40000000000000000000$

where $s = 0$, $e_3 = 0$, $e_2 = 0$, $m_2 = 0$, $m_4 = 0$

$$c = (10001001)_2 = 1\cdot 2^7 + 1\cdot 2^3 + 1\cdot 2^0 = 137$$

$$\text{Floating-point number} = (-1)^s \cdot (1.f)_2 \cdot 2^{c-127}$$

$$= (-1)^0 \cdot (1.1000)_2 \cdot 2^{137-127}$$

$$= 1 \cdot \left(1 + \frac{1}{2}\right) \cdot 2^{10} = 1536$$

Answer: 1536

→ 4 pts.

2. **(a)** Prove that the equation $\alpha\cos(\pi x) = \beta x^2 + \gamma \ln(x+1)$ has a unique root in the interval $[0, 1]$.

$f(x) = \cos(\pi x) - 2x^2 - 3\ln(x+1) = 0$

Existence: $f(0)\cdot f(1) = 1\cdot(-1-2-3\ln 2) < 0$

$\Rightarrow$ there exists (at least) one root. 2pts

Uniqueness: $f'(x) = -\pi\sin(\pi x) - 4x - \frac{3}{x+1} < 0$ in $[0,1]$

$\Rightarrow f\downarrow \Rightarrow$ there exists (at most) one root

2 pts

From Existence & Uniqueness $\Rightarrow f(x) = 0$ has exactly one root in $[0,1]$

(b) Perform the **bisection method** to determine c_1, the second approximation to the location of the root.

a/sign of f(a)	c/sign of f(c)	b/sign of f(b)
0 / +	$c_0 = 0.5$ / −	1 / −
0	$c_1 = 0.25$	0.5

Answer: $c_1 = 0.25$ → 2pts.

(c) Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-3}.

$$|\alpha - c_n| \le \frac{b-a}{2^{n+1}} = \frac{1-0}{2^{n+1}} \le 10^{-3}$$

$$\Rightarrow 2^{n+1} \ge 10^3 \Rightarrow n+1 \ge \ln(10^3)/\ln 2$$

Answer: $n+1 = \left[\frac{3\ln 10}{\ln 2}\right] + 1 = 10$ → 4pts.

3. The graph of a polynomial $p(x) = -2x^3 + a_2x^2 + a_1x + a_0$ passes through the following three points:

x	$\overset{\overset{1}{\shortparallel}}{\alpha}$	$\overset{\overset{2}{\shortparallel}}{\alpha+1}$	$\overset{\overset{3}{\shortparallel}}{\alpha+2}$
p(x)	-2α	β	γ

By imposing the interpolating conditions, you should obtain a system of three linear equations in a_2, a_1, and a_0. Using the (naive) Gaussian elimination with back substitution, find the coefficients a_2, a_1, and a_0.

$$p(1) = -2 + a_2 + a_1 + a_0 = -2$$
$$p(2) = -16 + 4a_2 + 2a_1 + a_0 = 2$$
$$p(3) = -54 + 9a_2 + 3a_1 + a_0 = 3$$

$$\Rightarrow \begin{cases} a_2 + a_1 + a_0 = 0 \\ 4a_2 + 2a_1 + a_0 = 18 \\ 9a_2 + 3a_1 + a_0 = 57 \end{cases}$$

$\searrow$ 3pts.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 18 \\ 9 & 3 & 1 & 57 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 18 \\ 0 & -6 & -8 & 57 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 18 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{cases} a_2 + a_1 + a_0 = 0 \\ -2a_1 - 3a_0 = 18 \\ a_0 = 3 \end{cases} \longrightarrow 4\text{pts.}$$

Answer: $a_2 = 10.5$, $a_1 = -13.5$, $a_0 = 3$ $\longrightarrow$ 3pts.

4. (a) Find the eigenvalues and corresponding eigenvectors of the matrix.

$$A = \begin{bmatrix} \beta+\gamma & \alpha \\ 0 & \beta-\gamma \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & -1 \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 5-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) = 0 \Rightarrow \lambda = -1, 5$$

$\boxed{\lambda = -1}$ $\left[\begin{array}{cc|c} 6 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} 6v_1 + v_2 = 0 \\ v_2 = \text{free} \end{cases} \Rightarrow \begin{cases} v_1 = -\frac{1}{6} v_2 \\ v_2 = \text{free} \end{cases}$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix}$$

$$E_{\lambda=-1} = \left\{ c \begin{bmatrix} -1/6 \\ 1 \end{bmatrix}, c \neq 0 \right\} = \left\{ c \begin{bmatrix} -1 \\ 6 \end{bmatrix}, c \neq 0 \right\}$$

$\boxed{\lambda = 5}$ $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} v_2 = 0 \\ v_1 = \text{free} \end{cases}$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$E_{\lambda=5} = \left\{ c \begin{bmatrix} 1 \\ 0 \end{bmatrix}, c \neq 0 \right\}$$

$\rightarrow$ 4 pts.

Answer: $\lambda_1 = -1$, $\mathbf{v}_1 = c \begin{bmatrix} -1 \\ 6 \end{bmatrix}$; $\lambda_2 = 5$, $\mathbf{v}_2 = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $c \neq 0$

(b) Without computing the eigenvalues of the following matrix, can the complex number $\lambda = (\alpha + \beta + \gamma) - 5i$ be one of them?

4 pts.

$\lambda = 6 - 5i$

$$B = \begin{bmatrix} -5 & -1 & 1 & -1 & 1 \\ -2 & -3 & 3 & 0 & 4 \\ 1 & -2 & 0 & -2 & 2 \\ 0 & 2 & 3 & 2 & 1 \\ -1 & 1 & -2 & 4 & 5 \end{bmatrix}$$

Justify your answer.

$C_1 = \{z \in \mathbb{C} : |z+5| \leq 4\}$ $\quad |\lambda + 5| = \sqrt{11^2 + (-5)^2} > 4$ No

$C_2 = \{z \in \mathbb{C} : |z+3| \leq 9\}$ $\quad |\lambda + 3| = \sqrt{9^2 + (-5)^2} > 9$ No

$C_3 = \{z \in \mathbb{C} : |z-0| \leq 7\}$ $\quad |\lambda - 0| = \sqrt{6^2 + (-5)^2} > 7$ No

$C_4 = \{z \in \mathbb{C} : |z-2| \leq 6\}$ $\quad |\lambda - 2| = \sqrt{4^2 + (-5)^2} > 6$ No

$C_5 = \{z \in \mathbb{C} : |z-5| \leq 8\}$ $\quad |\lambda - 5| = \sqrt{1^2 + (-5)^2} < 8$ Yes

Notice that λ belongs to C_5, so the answer is: Yes.

Answer: Yes? No?

(c) Without computing the eigenvalues of the matrix B above, find an upper bound for its spectral radius $\rho(B) = \max\{|\lambda| : \lambda \text{ eigenvalue of } B\}$. **Justify your answer.**

Since $\sigma(B) \subset \bigcup_{i=1}^{5} C_i \Rightarrow |\lambda| \leq 13$ because the point $z = 13 + 0 \cdot i$ is the most distant point of $\bigcup_{i=1}^{5} C_i$ from the origin.

Answer: $\rho = 13$

2 pts

$$f(x) = \sin(\pi x) + 8x^2 + 6x + 1$$

5. An interpolating polynomial is to be used to approximate $f(x) = \alpha\sin(\pi x) + 4\beta x^2 + 2\gamma x + 1$ with three equally spaced nodes in $[0, 1]$, i.e., $x_0 = 0$, $x_1 = 1/2$, and $x_2 = 1$.

(a) Form a divided-difference table and obtain Newton's interpolating polynomial. Then, approximate $f(2/3)$.

a_0 $\quad$ a_1 $\quad$ a_2

$x_0 = 0 \quad f(x_0) = \boxed{1}$

$f[x_0, x_1] = \boxed{12}$

$x_1 = \frac{1}{2} \quad f(x_1) = 7$ $\qquad f[x_1, x_2, x_3] = \boxed{4}$

$f[x_1, x_2] = 16$

$x_2 = 1 \quad f(x_2) = 15$

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$
$$= 1 + 12(x - 0) + 4(x - 0)(x - 0.5) = 1 + 12x + 4x(x - 0.5)$$
$$f\left(\tfrac{2}{3}\right) \simeq P_2\left(\tfrac{2}{3}\right) = 1 + 12 \cdot \tfrac{2}{3} + 4 \cdot \tfrac{2}{3}\left(\tfrac{2}{3} - 0.5\right) = \cdots$$

5 pts. $\quad$ 1 pt.

Answer: $p_2(x) =$ $1 + 12(x - 0) + 4(x - 0)(x - \frac{1}{2})$, $f(2/3) \approx$

(b) What bound can be placed on the error? *Hint: see Section 5.3 (eq. (5.19), page 168) or the course slides on BrightSpace for the theoretical bound. That is, use* $|f(x) - p_n(x)| \leq \dfrac{\|f^{(n+1)}\|_\infty}{4(n+1)}\left(\dfrac{b-a}{n}\right)^{n+1}$

$$|f(x) - P_2(x)| \leq \frac{1}{4 \cdot 3}\|f^{(3)}\|_\infty \cdot \left(\frac{1}{2}\right)^3 = (*)$$

$f'(x) = \pi\cos(\pi x) + 16x + 6$; $f''(x) = -\pi^2\sin(\pi x) + 16$

$f'''(x) = -\pi^3\cos(\pi x)$. Thus, $\|f^{(3)}\|_\infty = \pi^3$

Answer: $(*) = \dfrac{\pi^3}{96}$ $\longrightarrow$ 4 pts.

1	2	3	4	5	Total