Example 1: Alison and Kevin plan to open a savings account in which they will save the payment for the purchase of a car. They already have \$10,000 in cash. After examining their budget, they feel they can comfortably deposit an extra \$300 into the account at the end of each month. What is the minimum annual interest rate, compounded on a monthly basis, that the couple must earn on their investment to reach their goal of accumulating \$50,000 within five years?

Example 2: Consider a sphere of solid material floating in water. Archimedes' principle states that the buoyancy force is equal to the weight of the replaced liquid. Let $V_s = (4/3)\pi r^3$ be the volume of the sphere, and let $V_h = (1/3)\pi h^2(3r - h)$ be the volume of water displaced when it is submerged to a depth h. Given values of r and the densities of water and sphere material, find h corresponding to the static equilibrium.

Example 3: A segment of circle is the region enclosed by an arc and its chord. If r is the radius of the circle and θ the angle subtended at the center of the circle, then it can be shown that the area A of the segment is

$$A = \frac{1}{2}r^2(\theta - \sin\theta),$$

where θ is in radian. Given $A = 10$ and $r = 2$, find the value of θ.

Definition. A root α of the equation $f(x) = 0$ is said to be a root of multiplicity m if

$$f(x) = (x - \alpha)^m q(x),$$

where $\lim_{x\to\alpha} q(x) \neq 0$.

Theorem. Let f be a continuous function with m continuous derivatives. The equation $f(x) = 0$ has a root of multiplicity m at $x = \alpha$ iff $f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$ but $f^{(m)}(\alpha) \neq 0$.

Rootfinding Techniques:

- enclosure methods (e.g., the bisection method and the method of false position);
- fixed point iteration schemes (e.g., Newton's method and the secant method).

All simple enclosure methods are based on the

Intermediate Value Theorem. Let f be a continuous function over the closed interval $[a, b]$, and let k be any number that lies between the values $f(a)$ and $f(b)$. Then there exists c with $a < c < b$ such that $f(c) = k$.

The basic idea of these methods is to find an interval which contains a root and then systematically shrinking the size of that interval.

The Bisection Method for finding solutions to $f(x) = 0$:

- locate an interval that contains a zero;
- determine which half of the interval contains a root;
- repeat the process on that half.

Example. Verify that $x^5 + 2x - 1 = 0$ has a root on the interval $[0, 1]$. Then, perform the bisection method to determine c_2 (i.e., the third approximation to the location of the root).

Convergence Analysis:

Let (a_n, b_n) be the enclosing interval during the $(n+1)^{th}$ iteration and

$$c_n = \frac{a_n + b_n}{2}$$

denote the midpoint of the interval $[a_n, b_n]$.

Theorem. Let f be continuous on the closed interval $[a, b]$ and suppose that $f(a) \cdot f(b) < 0$. The bisection method generates a sequence of approximations $\{c_n\}$ which converges to a root $\alpha \in (a, b)$ with the property

$$|\alpha - c_n| \leq \frac{b-a}{2^{n+1}}.$$

The method of false position iteratively determines a sequence of root enclosing intervals, (a_n, b_n), and a sequence of approximations $\{c_n\}$. Steps:

- locate an interval that contains a zero of f, say (a, b);
- select c, the x-intercept of the line which passes through the points $(a, f(a))$ and $(b, f(b))$;
- determine whether the root lies on the subinterval (a, c) or on the subinterval (c, b);
- repeat the process on that subinterval.

Sequence of approximations:

$$c_n = b_n - f(b_n)\frac{b_n - a_n}{f(b_n) - f(a_n)} = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

Example. Verify that $x^5 + 2x - 1 = 0$ has a root on the interval $[0, 1]$.Then, perform the method of false position to determine c_2, the third approximation to the location of the root.

In this method, the first point x_2, of the iteration is the intersection point of the x axis and the secant line passing through $(x_0, f(x_0))$ and $(x_1, f(x_1))$. The next point, x_3, is generated by the intersection of the secant line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ with the x-axis, and so on.
For $n \geq 2$,

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Note that **two** initial guesses, x_0 and x_1, must be provided to generate x_2, x_3, ...

Example. If the secant method is used to find a zero of $f(x) = x^3 - 3x^2 + 2x - 6$ with $x_0 = 1.5$ and $x_1 = 2$, what is x_2?

Definition. A fixed point of the function g is any real number, α, for which $g(\alpha) = \alpha$.

Definition. A fixed point iteration scheme to approximate the fixed point, α, of a function g, generates the sequence $\{x_n\}$ by the rule $x_{n+1} = g(x_n)$, for all $n \geq 0$, given a starting approximation x_0.

Theorem. Let $g : [a, b] \to [a, b]$ be a continuous function.

- Then g has at least a fixed point in $[a, b]$.

Furthermore, suppose that g is differentiable on the open interval (a, b) and there exists a positive constant $k < 1$ such that $|g'(x)| \leq k < 1$ for all $x \in (a, b)$. Then:

- $\{x_n\}$ **converges** to the **unique** fixed point α for any $x_0 \in [a, b]$
- $|x_{n+1} - x_n| \leq k^n |x_1 - x_0|$
- $|x_n - \alpha| \leq \frac{k^n}{1-k} |x_1 - x_0|$

Example: Solve $x = \frac{4}{x^2+3}$, in $[0, 2]$, using the fixed point iteration method.

Goal: solve $f(x) = 0$.

Definition. Newton's Method is the fixed point iteration scheme based on the iteration function

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

That is, starting from an initial approximation, x_0, the sequence x_n is generated by $x_{n+1} = g(x_n)$.

Theorem. Let f be a twice differentiable function on the interval $[a, b]$ with $\alpha \in (a, b)$ and $f(\alpha) = 0$. Further, suppose that $f'(\alpha) \neq 0$. Then, there exists $\delta > 0$ such that for any $x_0 \in [\alpha - \delta, \alpha + \delta]$, the sequence $\{x_n\}$ generated by Newton's method converges to α.

Example. Verify that $x^5 + 2x - 1 = 0$ has a root on the interval $(0, 1)$. Then, perform two iterations of Newton's method to determine x_2, the third approximation to the location of the root.

Definition. Let x_0, x_1,... be a sequence that converges to a number α, and set $e_n = \alpha - x_n$. If there exists a number k and a positive constant C such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^k} = C$$

then k is called the order of convergence of the sequence and C the asymptotic error constant.

Example: Find the order of convergence and the asymptotic error constant for the convergent sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \quad x_0 = 1.$$

Theorem. If α is a simple root, that is $f'(\alpha) \neq 0$, then Newton's method converges quadratically.

Theorem. Let g be a continuous function on $[a, b]$ with $k \geq 2$ continuous derivatives on the open interval (a, b). Further, let $\alpha \in (a, b)$ be a fixed point of g. If

$$g'(\alpha) = g''(\alpha) = \cdots = g^{(k-1)}(\alpha) = 0,$$

but $g^{(k)}(\alpha) \neq 0$, then there exists a $\delta > 0$ such that for any $x_0 \in [\alpha - \delta, \alpha + \delta]$ the sequence $x_{n+1} = g(x_n)$ converges to the fixed point α of order k with asymptotic error constant

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^k} = \frac{|g^{(k)}(\alpha)|}{k!}.$$

Remark. If $f'(\alpha) = 0$ (i.e., α is a root of $f(x) = 0$ with multiplicity $m \geq 2$), then Newton's method provides only linear convergence.

Example. Verify that $f(x) = x(1 - \cos x)$ has a root of multiplicity three at $x = 0$. Use both Newton's method (with $x_0 = 1$) and the bisection method (with starting interval [-2, 1]) to compute the first ten iterations. Compare the results. Same for $f(x) = e^x \sin^2 x - x^2$.

Modified Newton's Method:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}.$$

`fzero` is a MATLAB built-in function. For example, `x = fzero(fun,x0)` tries to find a zero of the function `fun` near `x0`. For more information, type `help fzero` at the prompt.

3.7 Newton's Method for Nonlinear Systems

Suppose we need to solve the system of n nonlinear equations:

$$\begin{cases} f_1(x_1, x_2, \dots, x_m) = 0 \\ f_2(x_1, x_2, \dots, x_m) = 0 \\ \vdots \\ f_m(x_1, x_2, \dots, x_m) = 0. \end{cases}$$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ and $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_m) \\ f_2(x_1, x_2, \dots, x_m) \\ \vdots \\ f_m(x_1, x_2, \dots, x_m) \end{bmatrix}$.

The *Jacobian matrix* is

$$\mathbf{J}(\mathbf{x}) := \mathbf{F}'(\mathbf{x}) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_m \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_m \\ \vdots & \vdots & \vdots & \vdots \\ \partial f_m/\partial x_1 & \partial f_m/\partial x_2 & \cdots & \partial f_m/\partial x_m \end{bmatrix}.$$

3.7 Newton's Method for Nonlinear Systems

Newton's method for a *system of equations* takes form

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - [\mathbf{J}(\mathbf{x}^{(n)})]^{-1}\mathbf{F}(\mathbf{x}^{(n)}).$$

It is costly to compute $[\mathbf{J}(\mathbf{x}^{(n)})]^{-1}$. Instead, one defines

$$\Delta\mathbf{x}^{(n)} = -[\mathbf{J}(\mathbf{x}^{(n)})]^{-1}\mathbf{F}(\mathbf{x}^{(n)}),$$

and then solve

$$[\mathbf{J}(\mathbf{x}^{(n)})]\Delta\mathbf{x}^{(n)} = -\mathbf{F}(\mathbf{x}^{(n)})$$

for $\Delta\mathbf{x}^{(n)}$. Then, the next iterate is

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \Delta\mathbf{x}^{(n)}.$$

Example: Use one iteration of Newton's method for systems with initial guess $(0.5, 0.5)^T$ on $x_1^3 + x_2 - 1 = 0$, $x_1 - x_2^3 - 1 = 0$.
Use `fsolve(f,x0)` too.(`f=@ (x)[f1;f2],x=[x(1) x(2)]`)