Section 5.1 Polynomial Interpolation Theory (supplemental).

<u>Thm.</u> Given $n+1$ distinct points $x_0, x_1, \dots, x_n$ and $n+1$ arbitrary real values $y_0, y_1, \dots, y_n$, there is a unique polynomial $p_n(x) = a_0 + a_1 x + \dots + a_n x^n$ of degree $\leq n$ that interpolates the given data.

<u>Proof:</u> We have the system

$$\begin{cases} a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^n = y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 + \dots + a_n x_1^n = y_1 \\ \vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \dots + a_n x_n^n = y_n \end{cases}$$

unknowns: $a_0, a_1, \dots, a_n$ (the coefficients of p_n)

$$X = \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ \vdots & & \vdots & \\ 1 & x_n & \cdots & x_n^n \end{bmatrix}$$ matrix of the system ($n+1 \times n+1$)

if $\det(X) \neq 0 \Rightarrow$ unique solution.

X is a Vandermonde matrix.

$$\det(X) = \prod_{0 \leq j < i \leq n} (x_i - x_j) \neq 0.$$ Let's prove it.

Consider the function:

$$V(x) = \det \begin{bmatrix} 1 & x_0 & \cdots & x_0^n & x_0^{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^n & x_n^{n+1} \\ 1 & x & \cdots & x^n & x^{n+1} \end{bmatrix} \leftarrow \text{polynomial of degree } n+1$$

Observe that: $V(x_0) = V(x_1) = \ldots = V(x_n) = 0 \Rightarrow$

$\Rightarrow V(x) = a(x-x_0)(x-x_1)\ldots(x-x_n)$, where

a is the leading coefficient, that is:

$V(x) = a x^{n+1} + \ldots$

Observe that $a = \det \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix}$

We prove the statement by mathematical induction:

<u>Base step</u>: $\det \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} = x_1 - x_0 = \prod_{0 \le j < i \le 1} (x_i - x_j)$

<u>induction step</u>: Assume $\det \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} = \prod_{0 \le j < i \le n} (x_i - x_j)$.

Then, $\det \begin{bmatrix} 1 & x_0 & \cdots & x_0^{n+1} \\ 1 & x_1 & \cdots & x_1^{n+1} \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^{n+1} \\ 1 & x_{n+1} & \cdots & x_{n+1}^{n+1} \end{bmatrix} = V(x_{n+1}) = \prod_{0 \le j < i \le n} (x_i - x_j)(x_{n+1} - x_0) \cdots (x_{n+1} - x_n)$

$= \prod_{0 \le j < i \le n+1} (x_i - x_j)$ ✓

Explicit Vandermonde determinants:

$$\det\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_0^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} = \prod_{0\le j<i\le 2} (x_i - x_j) = (x_1-x_0)(x_2-x_1)(x_2-x_0)$$

$$\det\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} = \prod_{0\le j<i\le 3} (x_i - x_j) =$$

$$= (x_1-x_0)(x_2-x_1)(x_2-x_0)(x_3-x_2)(x_3-x_1)(x_3-x_0)$$

Example: Find the polynomial that interpolates the table:

x	0	1.5	2
y	3	3.75	3

$$p_2(x) = a_0 + a_1 x + a_2 x^2$$

$$\begin{cases} a_0 + a_1\cdot 0 + a_2\cdot 0^2 = 3 \\ a_0 + a_1\cdot 1.5 + a_2\cdot 1.5^2 = 3.75 \\ a_0 + a_1\cdot 2 + a_2\cdot 2^2 = 3 \end{cases} \Rightarrow \begin{cases} a_0 = 3 \\ a_1 = 2 \\ a_2 = -1 \end{cases}$$

Answer: $p_2(x) = 3 + 2x - x^2$

5.2 Newton's Divided Differences Interpolating Polynomial (supplemental)

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots$$
$$+ a_n(x-x_0)\cdots(x-x_{n-1})$$

$$a_0 = f(x_0) \overset{not.}{=} f[x_0]$$

$$a_1 = f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$a_2 = f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}, \text{ where}$$

$$f[x_1,x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1}$$

$$f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$$

$$a_3 = f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$$

$$\vdots$$

$$a_n = f[x_0,\dots,x_n] = \frac{f[x_1,x_2,\dots,x_n]-f[x_0,x_1,\dots,x_{n-1}]}{x_n-x_0}$$

<u>Example</u>: Find Newton's interpolation polynomial for the table

	x_0	x_1	x_2
x	0	1.5	2
y	3	3.75	3

$$P_2(x) = a_0 + a_1(x-0) + a_2(x-0)(x-1.5)$$

$$a_0 = f[x_0] = f[0] = 3$$

$$a_1 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(1.5) - f(0)}{1.5 - 0} = \frac{3.75 - 3}{1.5}$$

$$= 0.5$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\frac{f[x_2] - f[x_1]}{x_2 - x_1} - \frac{f[x_1] - f[x_0]}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\frac{3 - 3.75}{2 - 1.5} - \frac{3.75 - 3}{1.5 - 0}}{2 - 0} = -1$$

So, $\boxed{P_2(x) = 3 + 0.5(x-0) - 1\cdot(x-0)(x-1.5)}$

$$\parallel$$

$$3 + 2x - x^2$$

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	x	y
$x_0 =$	0	$\boxed{3 = f[x_0]}$
$x_1 =$	1.5	$3.75 = f[x_1]$
$x_2 =$	2	$3 = f[x_2]$

a_0

a_1

$$\boxed{f[x_0, x_1] = \frac{3.75-3}{1.5-0} = 0.5}$$

a_3

$$\boxed{f[x_0, x_1, x_2] = -1}$$

$$f[x_1, x_2] = \frac{3-3.75}{2-1.5} = -1.5$$

$$f[x_0, x_1, x_2] = \frac{-1.5-0.5}{2-0} = -1$$