Problem: Given a set of points (x_i, y_i) for $i = 0, 1, 2, ..., n$, where the x_i are distinct values of the independent variable and the y_i are corresponding values of some function f, either

- (**Interpolation**) approximate the value of f at some value of x not listed among the x_i or
- (**Approximation**) determine a function g that in some sense approximates the data.

Types of Interpolation:

- **polynomial**
- **piecewise polynomial (spline)**
- rational
- trigonometric
- exponential.

Objective: Find a polynomial $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ that interpolates the given data (x_i, y_i), $i = 0, 1, 2, ..., n$.
That is, solve the system:

$$\begin{array}{ll} a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n & = y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n & = y_1 \\ \vdots & \vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n & = y_n \end{array}$$

Theorem. Given $n+1$ distinct points $x_0, x_1, ..., x_n$ and $n+1$ arbitrary real values $y_0, y_1, ..., y_n$, there is a unique polynomial p_n of degree $\leq n$ that interpolates the given data. In this case, p_n is called the interpolating polynomial.

The special structure of the Newton form

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)\cdots(x-x_{n-1})$$

leads to a system of equations whose solution can be obtained by forward substitution.

$$\begin{aligned}
a_0 &= f(x_0) \\
a_0 + a_1(x_1 - x_0) &= f(x_1) \\
a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) &= f(x_2) \\
\vdots \quad &\quad \vdots \\
a_0 + a_1(x_n - x_0) + \cdots + a_n(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1}) &= f(x_n)
\end{aligned}$$

Define $f[x_i] := f(x_i)$ and, for $0 < k \leq n$,

$$f[x_i, x_{i+1}, \dots, x_{i+k}] := \frac{f[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - f[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$a_0 = f[x_0]$, $a_1 = f[x_0, x_1]$,..., $a_k = f[x_0, x_1, \dots, x_k]$,...,$a_n = f[x_0, \dots, x_n]$

Therefore,

$$p_n(x) = \sum_{k=0}^{n} f[x_0, x_1, x_2, ..., x_k]\left(\prod_{i=0}^{k-1}(x - x_i)\right).$$

Newton's algorithm table (for $n = 3$):

$$\begin{array}{lllll}
 & \text{Zeroth} & \text{First} & \text{Second} & \text{Third} \\
x_0 & f[x_0] & & & \\
 & & f[x_0, x_1] & & \\
x_1 & f[x_1] & & f[x_0, x_1, x_2] & \\
 & & f[x_1, x_2] & & f[x_0, x_1, x_2, x_3] \\
x_2 & f[x_2] & & f[x_1, x_2, x_3] & \\
 & & f[x_2, x_3] & & \\
x_3 & f[x_3] & & &
\end{array}$$

Numerical example for Newton's algorithm (for $n = 3$):

	Zeroth	First	Second	Third
$x_0 = -2$	$f[x_0] = 6$			
		$f[x_0, x_1] = -5$		
$x_1 = 0$	$f[x_1] = -4$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2] = 6$		$f[x_0, x_1, x_2, x_3]$
$x_2 = 1$	$f[x_2] = 2$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3] = 4$		
$x_3 = 3$	$f[x_3] = 10$			

Objective: Find a polynomial $p_n(x)$ that interpolates the given data (x_i, y_i), $i = 0$, 1, 2, ..., n.
The cardinal function $L_i(x)$ has degree n and is associated with the interpolating point x_i in the sense

$$L_i(x_j) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}.$$

In fact,

$$L_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

Lagrange's Interpolating Polynomial:

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x).$$

Property: $p_n(x_j) = y_j$ for $j = 0, 1, 2, \ldots n$.

Theorem. If $x_0, x_1, x_2, ..., x_n$ are $n+1$ distinct points in $[a, b]$ and f is continuous on $[a, b]$ and has $n+1$ continuous derivatives on (a, b), then for each $x \in [a, b]$ there exists a $\xi(x) \in [a, b]$ such that

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_n),$$

where p_n is the interpolating polynomial.

Proof. Let $\Psi(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$. Define the function

$$e(x) = f(x) - p_n(x) - \Psi(x)\frac{f(t) - p_n(t)}{\Psi(t)}.$$

Observe that $e(x) = 0$ has $n+2$ zeros: x_i, $i = 0, \ldots, n$, and t. Then, $e^{(n+1)}(x)$ has at least one zero, say ξ, for which

$$f(t) = p_n(t) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(t - x_0)(t - x_1)(t - x_2)\cdots(t - x_n).$$