Exam II Name: ______________________________

MA 47700/MA 57700, Instructor: Nicolae Tarfulea
October 23, 2024

There are five partial credit questions. Please show all your work for full credit; displaying only the final answer will earn you very little, if anything.

Very important: throughout this exam, wherever they appear,

$$\alpha = \qquad \beta = \qquad \gamma =$$

1) (a) Prove thet the equation $\overbrace{\alpha x^2 + 2\beta x - \gamma e^{-x}}^{f(x)} = 0$ has a unique root in the interval [0, 1].

(*) Existence: $f(0)\cdot f(1) = -\gamma\cdot(\alpha + 2\beta - \gamma/e) < 0$

(**) Uniqueness: $f'(x) = 2\alpha x + 2\beta + \gamma e^{-x} > 0 \Rightarrow f\uparrow$

From (*) and (**), the equation has a unique root in the interval $[0,1]$. $\longrightarrow$ 4 pts.

(b) Perform the **bisection method** to determine c_1, the second approximation to the location of the root for the equation in part **(a)**.

$$\left.\begin{array}{l} [a_0, b_0] = [0,1]; \quad c_0 = 0.5, \quad f(c_0) = \cdots \\ [a_1, b_1] = [\quad]; \quad c_1 = \dfrac{a_1 + b_1}{2} \end{array}\right\} \text{2 pts.}$$

(c) Use the theoretical error bound $|\alpha - c_n| \leq \dfrac{b-a}{2^{n+1}}$ to obtain a theoretical bound on the number of iterations needed to approximate the root of the equation in part **(a)** to within 10^{-6}.

$$\frac{1-0}{2^{n+1}} \leq 10^{-6} \Rightarrow 2^{n+1} \geq 10^6 \Rightarrow n+1 \geq \frac{6\ln 10}{\ln 2} \simeq 19.93$$

Answer: $n+1 = 20$ $\longrightarrow$ 4 pts.

2) Consider the function $g(x) = \dfrac{e^{-\alpha x}}{\alpha+\beta+\gamma}$.

(a) Prove that g has a unique fixed point on the interval $\left[0, \dfrac{1}{\alpha+\beta+\gamma}\right]$. That is, prove that the range of g is included in $\left[0, \dfrac{1}{\alpha+\beta+\gamma}\right]$ and $|g'(x)| \leq k$, with $k < 1$ (find it!) on $\left[0, \dfrac{1}{\alpha+\beta+\gamma}\right]$.

$g : \left[0, \dfrac{1}{\alpha+\beta+\gamma}\right] \longrightarrow \left[0, \dfrac{1}{\alpha+\beta+\gamma}\right]$ continuous.

$$0 \leq \frac{e^{-\alpha x}}{\alpha+\beta+\gamma} \leq \frac{e^{-\alpha \cdot 0}}{\alpha+\beta+\gamma} = \frac{1}{\alpha+\beta+\gamma} \text{ for all } x.$$

$$|g'(x)| = \left|\frac{-\alpha e^{-\alpha x}}{\alpha+\beta+\gamma}\right| \leq \frac{\alpha}{\alpha+\beta+\gamma} = K < 1$$

Hence, g has a unique fixed point.

$\longrightarrow$ 4 pts.

(b) Use the iteration scheme $x_{n+1} = g(x_n)$ with $x_0 = 0$ to determine x_2.

$$x_1 = g(0) = \frac{1}{\alpha+\beta+\gamma}$$

$$x_2 = g\left(\frac{1}{\alpha+\beta+\gamma}\right) = \frac{e^{-\alpha/(\alpha+\beta+\gamma)}}{\alpha+\beta+\gamma}$$

2 pts.

Answer: $x_1 = \dfrac{1}{\alpha+\beta+\gamma}$ $\qquad x_2 = e^{-\alpha/(\alpha+\beta+\gamma)}/(\alpha+\beta+\gamma)$

(c) Let p be the unique fixed point of g. Use the theoretical error bound $|x_n - p| \leq \dfrac{k^n}{1-k}|x_1 - x_0|$ to obtain a theoretical bound on the number of iterations needed to approximate the fixed point p to within 10^{-6} if the starting approximation is $x_0 = 0$.

From part(a), $K = \dfrac{\alpha}{\alpha+\beta+\gamma} < 1$

$$\frac{K^n}{1-K}|x_1 - x_0| = \frac{K^n}{1-K}\,\frac{1}{\alpha+\beta+\gamma} = \frac{\alpha^n}{(\alpha+\beta+\gamma)^{n+1}}\,\frac{1}{1-\frac{\alpha}{\alpha+\beta+\gamma}}$$

$$= \left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^n \cdot \frac{1}{\beta+\gamma} \leq 10^{-6}$$

$$\Rightarrow n \ln\left(\frac{\alpha}{\alpha+\beta+\gamma}\right) \leq \ln\left[(\beta+\gamma)10^{-6}\right]$$

$$\Rightarrow n \geq \left\{\ln\left[(\beta+\gamma)10^{-6}\right]\right\} / \ln\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)$$

Answer: Number of iterations =

$$n+1 = \text{floor}(\;) + 2$$

4 pts.

3) It is known (no proof needed) that the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n(2 - \gamma x_n), \quad \text{for } n \geq 0,$$

converges to a **nonzero** limit whenever the starting point x_0 is chosen so that $0 < x_0 < 2/\gamma$.

(a) Find L= $\lim_{n\to\infty} x_n$. (Hint: By letting $n \to \infty$ on both sides of the iterative definition of the sequence one obtains a quadratic equation for L.)

$$L = L(2 - \gamma L) \Rightarrow 1 = 2 - \gamma L \Rightarrow L = \frac{1}{\gamma}$$

Answer: $L =$ $1/\gamma$ $\longrightarrow$ 4 pts.

(b) What are the order of convergence and the asymptotic error constant?

$$g(x) = x(2 - \gamma x) = 2x - \gamma x^2$$

$$g\left(\tfrac{1}{\gamma}\right) = 2 \cdot \frac{1}{\gamma} - \gamma \frac{1}{\gamma^2} = \frac{1}{\gamma}$$

$$g'(x) = 2 - 2\gamma x \; ; \; g'\left(\tfrac{1}{\gamma}\right) = 2 - 2 \cdot \gamma \cdot \frac{1}{\gamma} = 0$$

$$g''(x) = -2\gamma \; ; \; g''\left(\tfrac{1}{\gamma}\right) = -2\gamma.$$

$$C = \frac{\left|g''\left(\frac{1}{\gamma}\right)\right|}{2!} = \frac{2\gamma}{2} = \gamma \longrightarrow 6 \text{ pts.}$$

Answer: Order of convergence $k =$ 2 Asymptotic error constant $C =$ γ

4) Let $f(x) = Ax^4 - x^2 + x + B$, where A and B are constants. If $x_0 = 0$ as the initial approximation, what values of A and B should be chosen so that Newton's method produces $x_1 = 0.\beta$ and $x_2 = 1.\gamma$?

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{B}{1} = -B \Rightarrow B = -0.\beta$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.\beta - \frac{A\,0.\beta^4 - 0.\beta^2 + 0.\beta - 0.\beta}{4A\,0.\beta^3 - 2\cdot 0.\beta + 1}$$

$$= 1.\gamma$$

$$\Rightarrow -A\,0.\beta^4 + 0.\beta^2 = (1.\gamma - 0.\beta)(4A\,0.\beta^3 - 2\cdot 0.\beta + 1)$$

$$\Rightarrow \left[(1.\gamma - 0.\beta)\cdot 4\cdot 0.\beta^3 + 0.\beta^4\right]A = 0.\beta^2 + (1.\gamma - 0.\beta)\cdot (2\cdot 0.\beta - 1)$$

$$\Rightarrow A = \frac{-0.\beta^2 + 0.\beta + 1.\gamma(2\cdot 0.\beta - 1)}{4\cdot 1.\gamma\cdot 0.\beta^3 - 3\cdot 0.\beta^4}$$

Answer: $A =$ $\qquad$ $B =$ $-0.\beta$

6 pts. $\qquad$ 4 pts.

5) Use one iteration of Newton's method for systems with initial guess $\mathbf{x}^{(0)} = [0,\, 0]^T$ on

$$\alpha x_1 - \beta \cos x_2 = 0,\ \ \sin x_1 + \gamma x_2 = 1.$$

$$\vec{F}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 - \beta \cos x_2 \\ \sin x_1 + \gamma x_2 - 1 \end{bmatrix} ; \quad \vec{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{F}(\vec{x}^{(0)}) = \begin{bmatrix} -\beta \\ -1 \end{bmatrix}$$

$$JF\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha & \beta \sin x_2 \\ \cos x_1 & \gamma \end{bmatrix}$$

$$JF(\vec{x}^{(0)}) = \begin{bmatrix} \alpha & 0 \\ 1 & \gamma \end{bmatrix} \longrightarrow 5 \text{ pts.}$$

$$\begin{bmatrix} \alpha & 0 \\ 1 & \gamma \end{bmatrix} \begin{bmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \end{bmatrix} = -\begin{bmatrix} -\beta \\ -1 \end{bmatrix} = \begin{bmatrix} \beta \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \Delta x_1^{(0)} = \frac{\beta}{\alpha}, \qquad \Delta x_2^{(0)} = \frac{\alpha - \beta}{\alpha \gamma}$$

<u>Answer</u>: $\vec{x}^{(1)} = \vec{x}^{(0)} + \begin{bmatrix} \frac{\beta}{\alpha} \\ \frac{\alpha-\beta}{\alpha\gamma} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha} \\ \frac{\alpha-\beta}{\alpha\gamma} \end{bmatrix}$

5 pts.

1	2	3	4	5	Total