Linear systems of equations arise in many problems in engineering and science:

- **Electrical Circuits:** Used to analyze and solve for currents and voltages in complex circuits using Kirchhoff's laws.
- **Structural Analysis:** Helps in determining the forces and moments in structures like bridges and buildings.
- **Control Systems:** Essential for designing and analyzing control systems in mechanical and electrical engineering.
- **Mechanical Systems Design:** Used in the design and analysis of mechanical components and systems, such as gears and linkages.
- **Fluid Dynamics:** Applied in modeling and solving problems related to fluid flow in pipes and channels.

- **Mathematics:** Used to find numerical solutions of boundary-value problems and partial differential equations.
- **Physics:** Used in quantum mechanics to solve Schrodinger's equation and in classical mechanics for solving motion equations.
- **Chemistry:** Helps in balancing chemical equations and in the analysis of reaction kinetics.
- **Biology:** Applied in modeling population dynamics and in the analysis of biological networks.
- **Economics:** Used in input-output models to analyze economic systems and in optimization problems.
- **Environmental Science:** Helps in modeling the spread of pollutants and in the analysis of ecological systems.

In 1949, Prof. Wassily Leontief used Harvard's Mark II computer to solve a very large system of linear equations modeling the U.S. economy, by using the information produced by the U.S. Bureau of Labor Statistics (more than 250,000 pieces of information). Leontief had divided the U.S. economy into 500 "sectors," such as the coal industry, the automotive industry, communications, and so on. For each sector, he had written a linear equation that described how the sector distributed its output to the other sectors of the economy. Because the Mark II could not handle the resulting system of 500 equations in 500 unknowns, Leontief has distilled the problem into a system of 42 equations in 42 unknowns.

Programming the Mark II computer for Leontief's 42 equations had required several months of effort. The Mark II produced a solution after 56 hours of humming and blinking. Leontief was awarded the 1973 Nobel Prize in Economic Science for opening the door to a new era in mathematical modeling in economics.

Before studying systems of equations, we need to be familiar with some algebra associated with matrices (see Section 4.1 for a brief review).

Solve the system:

$$\begin{cases} x_1 - 5x_2 + 7x_3 = 12 \\ 3x_1 + 2x_2 - x_3 = 4 \\ 2x_1 - x_2 + 4x_3 = 12 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 10 \\ 2x_1 + 3x_2 + x_3 + 5x_4 = 31 \\ -x_1 + x_2 - 5x_3 + 3x_4 = -2 \\ 3x_1 + x_2 + 7x_3 - 2x_4 = 18 \end{cases}$$

Consider the system:

$$\begin{cases} \frac{2}{3}x_1 + \frac{2}{7}x_2 + \frac{1}{5}x_3 = \frac{43}{15} \\ \frac{1}{3}x_1 + \frac{1}{7}x_2 - \frac{1}{2}x_3 = \frac{5}{6} \\ \frac{1}{5}x_1 - \frac{3}{7}x_2 + \frac{2}{5}x_3 = -\frac{12}{5} \end{cases}$$

Exact solution: $x_1 = 1$, $x_2 = 7$, $x_3 = 1$.

Same system solved in 4 decimal digit rounding arithmetic:

$$\begin{cases} 0.6667x_1 + 0.2857x_2 + 0.2000x_3 = 2.867 \\ 0.3333x_1 + 0.1429x_2 - 0.5000x_3 = 0.8333 \\ 0.2000x_1 - 0.4286x_2 + 0.4000x_3 = -2.400 \end{cases}$$

Solution: $x_1 = 2.715$, $x_2 = 3.000$, $x_3 = 1.000$.

Why is the solution so inaccurate?

Answer: The troublemakers: cancellation errors which produce small pivots.

First pass of Gaussian Elimination:

$$\begin{cases} 0.6667x_1 + & 0.2857x_2 + 0.2000x_3 = 2.867 \\ & 0.0001x_2 - 0.6000x_3 = -0.5997 \\ & -0.5143x_2 + 0.3400x_3 = -3.260 \end{cases}$$

Second pass of Gaussian Elimination:

$$\begin{cases} 0.6667x_1 + & 0.2857x_2 + 0.2000x_3 = 2.867 \\ & 0.0001x_2 - 0.6000x_3 = -0.5997 \\ & -3086x_3 = -3087 \end{cases}$$

To avoid small pivots, one needs to employ *pivoting strategies*.

Partial Pivoting Strategy:

- during the i^{th} pass of Gaussian elimination, find

$$M_i = \max_{i \le j \le n} |a_{ji}|$$

- let j_0 be the smallest value of j for which this maximum occurs;
- if $j_0 > i$, then interchange rows i and j_0.

Previous system: $\begin{cases} 0.6667x_1 + 0.2857x_2 + 0.2000x_3 = 2.867 \\ 0.3333x_1 + 0.1429x_2 - 0.5000x_3 = 0.8333 \\ 0.2000x_1 - 0.4286x_2 + 0.4000x_3 = -2.400 \end{cases}$

1st pass of GE: $\begin{cases} 0.6667x_1 + 0.2857x_2 + 0.2000x_3 = 2.867 \\ \quad 0.0001x_2 - 0.6000x_3 = -0.5997 \\ \quad -0.5143x_2 + 0.3400x_3 = -3.260 \end{cases}$

Partial pivoting: $\begin{cases} 0.6667x_1 + 0.2857x_2 + 0.2000x_3 = 2.867 \\ \quad -0.5143x_2 + 0.3400x_3 = -3.260 \\ \quad 0.0001x_2 - 0.6000x_3 = -0.5997 \end{cases}$

2nd pass of GE: $\begin{cases} 0.6667x_1 + 0.2857x_2 + 0.2000x_3 = 2.867 \\ \quad -0.5143x_2 + 0.3400x_3 = -3.260 \\ \quad\quad -0.5999x_3 = -0.6003 \end{cases}$

Back substitution produces: $x_1 = 1.001$, $x_2 = 7.000$, $x_3 = 1.000$.

Idea:

- convert $\mathbf{Ax} = \mathbf{b}$ into $\mathbf{x} = \mathbf{Tx} + \mathbf{c}$;
- compute $\mathbf{x}^{(k+1)} = \mathbf{Tx}^{(k)} + \mathbf{c}$ starting from $\mathbf{x}^{(0)}$;
- stop when the difference between successive approximations $\mathbf{x}^{(k)}$ and $\mathbf{x}^{(k+1)}$ falls below a specified threshold.

Questions:
Does the fixed point problem have a unique solution?
Does the iterative process converge to this unique solution?
If yes, how quickly does it converge?
How to choose $\mathbf{T}$ and $\mathbf{c}$?

Definition. The pair of $n \times n$ matrices $(\mathbf{M}, \mathbf{N})$, with $\mathbf{M}$ nonsingular, is called a splitting of $\mathbf{A}$ if $\mathbf{A} = \mathbf{M} + \mathbf{N}$.

Note the equivalence

$$\mathbf{Ax} = \mathbf{b} \quad \Longleftrightarrow \quad \mathbf{x} = -\mathbf{M}^{-1}\mathbf{Nx} + \mathbf{M}^{-1}\mathbf{b}.$$

Natural splitting: $\mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{U}$, where $\mathbf{D}$ is the diagonal part of $\mathbf{A}$, $\mathbf{L}$ is the strictly lower triangular part of $\mathbf{A}$, and $\mathbf{U}$ is the strictly upper triangular part of $\mathbf{A}$.

The Jacobi Method: is based on the splitting $\mathbf{A} = \mathbf{D} + (\mathbf{L} + \mathbf{U})$.

Iteration scheme:

$$\mathbf{x}^{(k+1)} = \mathbf{T}_{jac}\mathbf{x}^{(k)} + \mathbf{c}_{jac},$$

where $\mathbf{T}_{jac} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$ and $\mathbf{c}_{jac} = \mathbf{D}^{-1}\mathbf{b}$.

By individual components:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right], \quad i = 1 : n.$$

Example. Consider the system of equations:

$$\begin{aligned} 9x_1 + x_2 + 2x_3 &= 18 \\ -3x_1 + 7x_2 + x_3 &= -14 \\ x_1 + 2x_2 + 5x_3 &= 5. \end{aligned} \qquad \mathbf{A} = \begin{bmatrix} 9 & 1 & 2 \\ -3 & 7 & 1 \\ 1 & 2 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 18 \\ -14 \\ 5 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}; \mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}; \mathbf{U} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{T}_{jac} = -D^{-1}(L+U) = \begin{bmatrix} 0 & -1/9 & -2/9 \\ 3/7 & 0 & -1/7 \\ -1/5 & -2/5 & 0 \end{bmatrix}; \mathbf{c}_{jac} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Jacobi's method for the system:

$$
\begin{aligned}
9x_1 + x_2 + 2x_3 &= 18 \\
-3x_1 + 7x_2 + x_3 &= -14 \\
x_1 + 2x_2 + 5x_3 &= 5.
\end{aligned}
$$

$$
\begin{aligned}
x_1^{(k+1)} &= \frac{1}{9}(18 - x_2^{(k)} - 2x_3^{(k)}) \\
x_2^{(k+1)} &= \frac{1}{7}(-14 + 3x_1^{(k)} - x_3^{(k)}) \\
x_3^{(k+1)} &= \frac{1}{5}(5 - x_1^{(k)} - 2x_2^{(k)})
\end{aligned}
$$

Natural splitting: $\mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{U}$, where $\mathbf{D}$ is the diagonal part of $\mathbf{A}$, $\mathbf{L}$ is the strictly lower triangular part of $\mathbf{A}$, and $\mathbf{U}$ is the strictly upper triangular part of $\mathbf{A}$.

The Gauss-Seidel Method: is based on the splitting $\mathbf{A} = (\mathbf{D} + \mathbf{L}) + \mathbf{U}$.

Iteration scheme:

$$\mathbf{x}^{(k+1)} = \mathbf{T}_{gs}\mathbf{x}^{(k)} + \mathbf{c}_{gs},$$

where $\mathbf{T}_{gs} = -(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$ and $\mathbf{c}_{gs} = (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b}$.

By individual components:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right], \quad i = 1:n.$$

Example. Consider the system of equations:

$$\begin{aligned} 9x_1 + x_2 + 2x_3 &= 18 \\ -3x_1 + 7x_2 + x_3 &= -14 \\ x_1 + 2x_2 + 5x_3 &= 5. \end{aligned} \qquad \mathbf{A} = \begin{bmatrix} 9 & 1 & 2 \\ -3 & 7 & 1 \\ 1 & 2 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 18 \\ -14 \\ 5 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}; \mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}; \mathbf{U} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{T}_{gs} = -(D+L)^{-1}U = \begin{bmatrix} 0 & -35/315 & -70/315 \\ 0 & 15/315 & -15/315 \\ 0 & 13/315 & 8/315 \end{bmatrix};$$

$$\mathbf{c}_{gs} = (D+L)^{-1}\mathbf{b} = \begin{bmatrix} 630/315 \\ -900/315 \\ -171/315 \end{bmatrix}.$$

The Gauss-Seidel method for the system:

$$\begin{aligned} 9x_1 + x_2 + 2x_3 &= 18 \\ -3x_1 + 7x_2 + x_3 &= -14 \\ x_1 + 2x_2 + 5x_3 &= 5. \end{aligned}$$

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{9}(18 - x_2^{(k)} - 2x_3^{(k)}) \\ x_2^{(k+1)} &= \frac{1}{7}(-14 + 3x_1^{(k+1)} - x_3^{(k)}) \\ x_3^{(k+1)} &= \frac{1}{5}(5 - x_1^{(k+1)} - 2x_2^{(k+1)}). \end{aligned}$$

Definition. An $n \times n$ matrix $\mathbf{A}$ is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad \text{for } i = 1 : n.$$

Theorem. If $\mathbf{A}$ is a strictly diagonal dominant matrix, then the Jacobi and Gauss-Seidel methods will converge for any choice of the initial vector $\mathbf{x}^{(0)}$.