Assignment 5 - Part 1

#1. (*) Existence: $f(0) \cdot f(2) = -3 \cdot (4\sin 2 + 5) < 0$

$\Rightarrow f$ has at least one root in $[0, 2]$

(**) Uniqueness: $f'(x) = 2x\sin x + x^2\cos x + 4$

$> x^2 \cos x + 4$

> 0 in $(0, 2) \Rightarrow f \uparrow$

$\Rightarrow f$ has at most one root in $[0, 2]$

From Existence & Uniqueness, it follows that f has exactly one root in $[0, 2]$.

$[a_0, b_0] = [0, 2] \rightarrow c_0 = 1$

$f(c_0) \simeq 1.841 > 0$

$[a_1, b_1] = [0, 1] \rightarrow c_1 = 0.5$

$f(c_1) \simeq -0.88 < 0$

$[a_2, b_2] = [0.5, 1] \Rightarrow c_2 = 0.75$

Error $= |\alpha - c_n| \leq \frac{b-a}{2^{n+1}} = \frac{2-0}{2^{n+1}} = \frac{1}{2^n} \leq 10^{-4}$

$\Rightarrow 2^n \geq 10^4 \Rightarrow n \geq 4/\log 2$ Answer: $\boxed{n = 14}$

#2. Existence: $f(-2)\,f(-1) < 0 \Rightarrow f$ has at least one root in $[-2,-1]$

Uniqueness: $f'(x) = -\sin x - 6x + 2$

$\geq -1 + 6 + 2 = 7 > 0$

$\Rightarrow f\uparrow$, and so f has at most one root in $[-2,-1]$

From Existence & Uniqueness, it follows that f has exactly one root in $[-2,-1]$

$[a_0, b_0] = [-2,-1] \Rightarrow c_0 = \dfrac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)} \simeq -1.0452$

$f(c_0) \simeq 0.134.. > 0$

$[a_1, b_1] = [-2, -1.0452] \Rightarrow c_1 = \dfrac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)} \simeq -1.0563$

$f(c_1) \simeq 0.032.. > 0$

$[a_2, b_2] = [-2, -1.0563] \Rightarrow c_2 = \dfrac{a_2 f(b_2) - b_2 f(a_2)}{f(b_2) - f(a_2)} \simeq -1.0589$

Answer: $c_0 = -1.0452$

$c_1 \simeq -1.0563$

$c_2 \simeq -1.0589$

#3. $x_0 = 0.5$; $x_1 = 0.6$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \simeq 0.61468$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \simeq 0.61791$$

<u>Answer</u>: $x_2 \simeq 0.61468$; $x_3 \simeq 0.61791$

#4. $x = g(x)$, with $g(x) = 1 + e^{-x}$

Notice that $g : [1,2] \to [1,2]$ continuous

Obviously, $1 \le g(x) \le 2$ for $1 \le x \le 2$,

since $1 \le 1 + e^{-x} \le 2 \Leftrightarrow 0 \le e^{-x} \le 1$ in $[1,2]$.

Also, $|g'(x)| = |-e^{-x}| = \frac{1}{e^x} \le \frac{1}{e}$ for $1 \le x \le 2$.

Thus, $x_{n+1} = g(x_n)$ converges for any x_0 in $[1,2]$.

$$|x_n - \alpha| \le \frac{\left(\frac{1}{e}\right)^n}{1 - \frac{1}{e}} |x_1 - x_0| \le \frac{e^{-n}}{1 - e^{-1}} \le 10^{-6}$$

$$\Rightarrow e^{-n} \le 10^{-6}(1 - e^{-1}) \Leftrightarrow n \ge -\ln(10^{-6}(1 - e^{-1}))$$

$$\simeq 14.27$$

<u>Answer</u>: $\boxed{\text{16 iterations}}$ (that is, $x_0, \dots, x_{15}$)

#5. $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{\frac{1}{x_n} - 5}{-\frac{1}{x_n^2}}$

$= x_n + \left(\dfrac{1}{x_n} - 5\right) x_n^2 = 2x_n - 5x_n^2$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 \cdot 0.25 - 5 \cdot 0.25^2 = 0.1875$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2 \cdot 0.1875 - 5 \cdot 0.1875^2 = 0.1992$

#6 $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n + e^{-x_n} - 1$

$\Rightarrow -\dfrac{f(x_n)}{f'(x_n)} = e^{-x_n} - 1 = \dfrac{1 - e^{x_n}}{e^{x_n}}$

$\Rightarrow \dfrac{f'(x_n)}{f(x_n)} = \dfrac{e^{x_n}}{e^{x_n} - 1}$

Thus, we are looking for $f(x)$ such that:

$\dfrac{f'(x)}{f(x)} = \dfrac{e^x}{e^x - 1} \Rightarrow \displaystyle\int \frac{f'(x)}{f(x)}\,dx = \int \frac{e^x}{e^x - 1}\,dx \Leftrightarrow$

$\ln|f(x)| = \ln|e^x - 1| + C \Rightarrow \boxed{f(x) = C(e^x - 1), \text{ with } C \neq 0}$

7. (a) $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^1} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{(n+1)^k}\right|}{\left|0 - \frac{1}{n^k}\right|}$

$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^k} = \frac{1}{1} = 1$

Hence, order of convergence $= 1$
asymptotic error constant $= 1$

(b) $e_{n+1} = 1 - x_{n+1} = 1 - \frac{x_n^2}{2x_n - 1}$

$= \frac{2x_n - 1 - x_n^2}{2x_n - 1}$

$= \frac{-(1-x_n)^2}{2x_n - 1} = -\frac{e_n^2}{2x_n - 1}$

Therefore, $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n\to\infty} \frac{1}{|2x_n - 1|} = \frac{1}{2\cdot 1 - 1}$

$= 1$

Thus, order of convergence $= 2$
asymptotic error constant $= 1$

#8. $\begin{cases} x^2+y^3=1 \\ x^3-y^2=-\frac{1}{4} \end{cases}$ $\qquad \vec{X}^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$$\vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2+y^3-1 \\ x^3-y^2+\frac{1}{4} \end{bmatrix}; \quad \vec{F}(\vec{X}^{(0)}) = \begin{bmatrix} -\frac{5}{8} \\ \frac{1}{8} \end{bmatrix}$$

$$JF\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x & 3y^2 \\ 3x^2 & -2y \end{bmatrix}; \quad JF(\vec{X}^{(0)}) = \begin{bmatrix} 1 & \frac{3}{4} \\ \frac{3}{4} & -1 \end{bmatrix}$$

Let $\vec{\Delta}x^{(0)} = \begin{bmatrix} \Delta x^{(0)} \\ \Delta y^{(0)} \end{bmatrix}$ be the solution to the system:

$$\begin{bmatrix} 1 & \frac{3}{4} \\ \frac{3}{4} & -1 \end{bmatrix} \begin{bmatrix} \Delta x^{(0)} \\ \Delta y^{(0)} \end{bmatrix} = -\begin{bmatrix} -\frac{5}{8} \\ \frac{1}{8} \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta x^{(0)} \\ \Delta y^{(0)} \end{bmatrix} = \begin{bmatrix} \frac{17}{50} \\ \frac{19}{50} \end{bmatrix}$$

The next approximation is:

$$\vec{X}^{(1)} = \vec{X}^{(0)} + \vec{\Delta}x^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 17/50 \\ 19/50 \end{bmatrix}$$

$$= \begin{bmatrix} 21/25 \\ 22/25 \end{bmatrix} = \begin{bmatrix} 0.84 \\ 0.88 \end{bmatrix}$$