Assignment 6: Systems, Eigenpairs, and Interpolation

Due: Monday, November 18^{th}, 2024, before 2:00 pm

Only two out of the following 10 problems will be graded at instructor's choice. This assignment is also a study guide for the upcoming (third) exam on November 20^{th}. Please write your solutions for each problem in the allocated space and upload your work (single file!) into BrightSpace before the deadline (Monday, November 18^{th}, by 2:00 pm). Do not use MATLAB, unless specified.

1. Exercise 5 (c), Sec. 4.2, page 104. (Gaussian elimination)

$$\begin{cases} 2x+y+z=7 \\ 2x+2y+3z=10 \\ -4x+4y+5z=14 \end{cases}$$

Augmented Matrix

$$\begin{bmatrix} 2 & 1 & 1 & 7 \\ 2 & 2 & 3 & 10 \\ -4 & 4 & 5 & 14 \end{bmatrix} \quad \begin{array}{l} R_2^{new} = R_2^{old} - R_1 \\ R_3^{new} = R_3^{old} + 2R_1 \end{array} \quad \begin{bmatrix} 2 & 1 & 1 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 28 \end{bmatrix}$$

$$R_3^{new} = R_3^{old} - 6R_2 \quad \begin{bmatrix} 2 & 1 & 1 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & 10 \end{bmatrix}$$

In system form: $\begin{cases} 2x+y+z=7 \\ y+2z=3 \\ -5z=10 \end{cases} \xrightarrow{\text{Back substitution}} \begin{cases} x=1 \\ y=7 \\ z=-2 \end{cases}$

2. Let $A = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 8 & 3 \\ 1 & -2 & 5 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, and the initial vector $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) Perform two iterations of the Jacobi method.

$$\begin{cases} x_1^{(k+1)} = \frac{1}{4}\left(1 + x_2^{(k)} - 2x_3^{(k)}\right) \\ x_2^{(k+1)} = \frac{1}{8}\left(-2 + x_1^{(k)} - 3x_3^{(k)}\right) \\ x_3^{(k+1)} = \frac{1}{5}\left(3 - x_1^{(k)} + 2x_2^{(k)}\right) \end{cases}$$

4.5 pts

$$\begin{cases} x_1^{(1)} = \frac{1}{4}(1 + 0 - 2\cdot 0) = \frac{1}{4} \\ x_2^{(1)} = \frac{1}{8}(-2 + 0 - 3\cdot 0) = -\frac{1}{4} \\ x_3^{(1)} = \frac{1}{5}(3 - 0 + 2\cdot 0) = \frac{3}{5} \end{cases} \qquad \begin{cases} x_1^{(2)} = \dots = -\frac{9}{80} \simeq -0.1125 \\ x_2^{(2)} = \dots = -\frac{71}{160} \simeq -0.4438 \\ x_3^{(2)} = \dots = \frac{9}{20} \simeq 0.45 \end{cases}$$

(b) Perform two iterations of the Gauss-Seidel method.

$$\begin{cases} x_1^{(k+1)} = \frac{1}{4}\left(1 + x_2^{(k)} - 2x_3^{(k)}\right) \\ x_2^{(k+1)} = \frac{1}{8}\left(-2 + x_1^{(k+1)} - 3x_3^{(k)}\right) \\ x_3^{(k+1)} = \frac{1}{5}\left(3 - x_1^{(k+1)} + 2x_2^{(k+1)}\right) \end{cases}$$

4.5 pts.

$$\begin{cases} x_1^{(1)} = \frac{1}{4}(1 + 0 - 2\cdot 0) = 0.25 \\ x_2^{(1)} = \frac{1}{8}(-2 + 0.25) = -\frac{7}{32} \simeq -.2188 \\ x_3^{(1)} = \frac{1}{5}\left[3 - \frac{1}{4} + 2\left(-\frac{7}{32}\right)\right] = \frac{37}{80} \simeq .4625 \end{cases} \qquad \begin{cases} x_1^{(2)} = \dots = -\frac{23}{640} \simeq -.0359 \\ x_2^{(2)} = \dots = -\frac{1048}{2449} \simeq -.4279 \\ x_3^{(2)} = \dots = \frac{661}{1516} \simeq .4360 \end{cases}$$

(c) Explain why both methods should converge.

Both methods converge since A is diagonally dominant.

1 pt.

3. Exercise 4, Sec. 14.1, page 421.

$$A = \begin{bmatrix} -2 & 3+m & -2 \\ 1 & 6 & -1 \\ 2+m & 2 & 14 \end{bmatrix}$$

if $m>0$, find all values of m s.t. the Gheschgorin circles are all disjoint.

$C_1 = \{ z : |z+2| \le r_1 = 5+m \}$

$C_2 = \{ z : |z-6| \le r_2 = 2 \}$

$C_3 = \{ z : |z-14| \le r_3 = 4+m \}$

$$\text{Conditions: } \begin{cases} r_1 + r_2 < 8 \\ r_2 + r_3 < 8 \\ m > 0 \end{cases} \iff \begin{cases} 5+m+2 < 8 \\ 2+4+m < 8 \\ m > 0 \end{cases} \iff \begin{cases} m < 1 \\ m < 2 \\ m > 0 \end{cases}$$

$\Rightarrow 0 < m < 1$

<u>Answer:</u> $\boxed{0 < m < 1}$ or $\boxed{m \in (0,1)}$

4. Exercise 5, Sec. 14.1, page 421. Find the characteristic polynomial $p(\lambda)$ and eigenpairs for the following matrices:

(a) $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$; $p(\lambda) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix}$

$= \lambda^3 - 6\lambda^2 + 11\lambda - 6.$

$\sigma(A) = \{1, 2, 3\} \longrightarrow$ 1 pt.

$\underline{\lambda = 1}$: $Null(A - I_3) = ?$

$$\begin{bmatrix} 0 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & -4 & 4 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + \frac{1}{2}x_3 = 0 \\ x_2 - \frac{1}{2}x_3 = 0 \\ x_3 = \text{free variable} \end{cases} \Rightarrow \begin{cases} x_1 = -0.5x_3 \\ x_2 = 0.5x_3 \\ x_3 = \text{free var.} \end{cases}$$

So, $Null(A - I_3) = Span\left\{\begin{bmatrix} -0.5 \\ 0.5 \\ 1 \end{bmatrix}\right\}$

Eigenpairs: $\left(1, c\begin{bmatrix} -0.5 \\ 0.5 \\ 1 \end{bmatrix}\right)$, with $c \neq 0$

$\longrightarrow$ 2 pts.

4(a) $\underline{\lambda=2}$: $Null(A-2I_3)=?$

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 4 & -4 & 3 & 0 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + \frac{1}{2}x = 0 \\ x_2 - \frac{1}{4}x_3 = 0 \\ x_3 = \text{free var} \end{cases} \iff \vec{x} = x_3\begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix} = x_3\begin{bmatrix} -0.5 \\ 0.25 \\ 1 \end{bmatrix}$$

So, $Null(A-2I_3) = Span\left\{\begin{bmatrix} -0.5 \\ 0.25 \\ 1 \end{bmatrix}\right\}$ 2pts.

Eigenpairs: $\left(2, c\begin{bmatrix} -0.5 \\ 0.25 \\ 1 \end{bmatrix}\right)$, with $c \neq 0$.

$\underline{\lambda=3}$: $Null(A-3I_3)=?$

$$\left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 1 & -3 & 1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0.25 & 0 \\ 0 & 1 & -0.25 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 0.25x_3 = 0 \\ x_2 - 0.25x_3 = 0 \\ x_3 = \text{free var} \end{cases} \iff \vec{x} = x_3\begin{bmatrix} -0.25 \\ 0.25 \\ 1 \end{bmatrix}$$

So, $Null(A-3I_3) = Span\left\{\begin{bmatrix} -0.25 \\ 0.25 \\ 1 \end{bmatrix}\right\}$

Eigenpairs: $\left(3, c\begin{bmatrix} -0.25 \\ 0.25 \\ 1 \end{bmatrix}\right)$, with $c \neq 0$.

4(b) $B = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$;

$p(\lambda) = \det(B - \lambda I_3) = \lambda^3 - 3\lambda^2 - 9\lambda + 27$

$\sigma(B) = \{-3, 3, 3\}$ $\longrightarrow$ 1 pt.

$\underline{\lambda = -3}$: $\text{Null}(B + 3I_3) = \text{Span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$

Eigenpairs: $\left(-3, c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$, with $c \neq 0$

$\longrightarrow$ 1.5 pts.

$\underline{\lambda = 3}$: $\text{Null}(B - 3I_3) = ?$

$$\begin{bmatrix} -2 & -2 & 2 & 0 \\ -2 & -2 & 2 & 0 \\ 2 & 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_2, x_3 \text{ free variables} \end{cases} \iff \begin{cases} x_1 = -x_2 + x_3 \\ x_2, x_3 \text{ free} \end{cases} \iff \vec{x} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

So, $\text{Null}(B - 3I_3) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

Eigenpairs: $\left(3, c_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$, with $c_1^2 + c_2^2 \neq 0$.

$\longrightarrow$ 2.5 pts

5. Exercise 6, Sec. 14.1, page 421.

Let $A = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 4 & a+2 \\ b+1 & 1 & 12 \end{bmatrix}$

If $a, b \geq 0$, find the values of a and b s.t. the center circle in the Gerschgorin Theorem will be tangent to the two others.

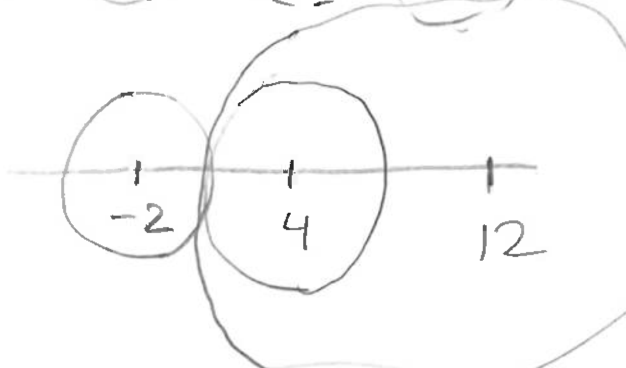
$r_1 = 3,\ C_1 = \{ z \in \mathbb{C} : |z+2| \leq 3 \}$

$r_2 = a+3,\ C_2 = \{ z \in \mathbb{C} : |z-4| \leq a+3 \}$

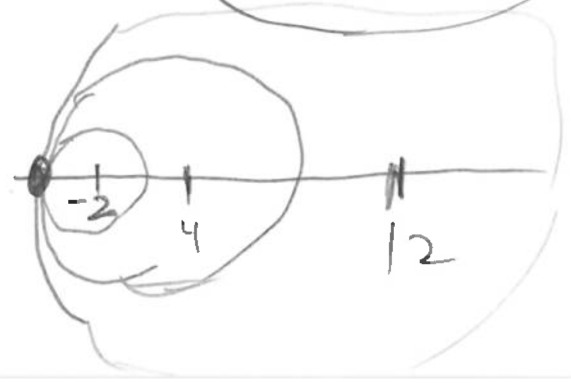
$r_3 = b+2,\ C_3 = \{ z \in \mathbb{C} : |z-12| \leq b+2 \}$

1) $\begin{cases} r_1 + r_2 = 6 \\ r_2 + r_3 = 8 \end{cases} \Rightarrow \boxed{a=0,\ b=3}$

2) $\begin{cases} r_2 = 3 \\ r_3 = 11 \end{cases} \Rightarrow \boxed{a=0,\ b=9}$



3) $\begin{cases} r_2 = 9 \\ r_3 = 17 \end{cases} \Rightarrow \boxed{a=6,\ b=15}$



6. Exercise 2, Sec. 14.2, page 425. Let $(\lambda, \vec{x})$ be an eigenpair of A. If $\lambda \neq 0$, show that $(\frac{1}{\lambda}, \vec{x})$ is an eigenpair of A^{-1}.

Proof: $A\vec{x} = \lambda\vec{x} \Leftrightarrow A^{-1}A\vec{x} = A^{-1}(\lambda\vec{x}) \Leftrightarrow \vec{x} = \lambda A^{-1}\vec{x}$

$\Leftrightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$, that is, $(\frac{1}{\lambda}, \vec{x})$ eigenpair of A^{-1}.

7. Find m_3 and $\mathbf{x}_3$ in the example 14.2 on the Power Method, page 423.

$$\vec{X}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad \vec{z}^{(0)} = A\vec{X}^{(0)} = \begin{bmatrix} -9 & 14 & 4 \\ -7 & 12 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 1 \end{bmatrix}$$

$$\Rightarrow m_1 = 9, \quad \vec{X}^{(1)} = \frac{1}{9}\begin{bmatrix} 9 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1/9 \end{bmatrix}$$

$$\vec{z}^{(1)} = A\vec{x}^{(1)} = \begin{bmatrix} 49/9 \\ 49/9 \\ 1/9 \end{bmatrix}$$

$$\Rightarrow m_2 = \frac{49}{9}, \quad \vec{X}^{(2)} = \frac{1}{m_2}\vec{z}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1/49 \end{bmatrix}$$

$$\vec{z}^{(2)} = A\vec{X}^{(2)} = \begin{bmatrix} 249/49 \\ 249/49 \\ 1/49 \end{bmatrix} \Rightarrow m_3 = \frac{249}{49}, \quad \vec{X}^{(3)} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{249} \end{bmatrix}$$

8. Exercise 6, page 164. Fill in the divided-difference table.

x_i				
1.1	2.45			
		0.609		
			0.079	
2.2	3.12			
		0.782		0.024
3.3	3.98		0.157	
		1.127		
4.4	5.22			

9. Exercise 5, page 176. Let $f(x) = 2x^2 e^x + 1$. Construct a Lagrange polynomial of degree two or less using $x_0 = 0$, $x_1 = 0.5$, and $x_2 = 1$. Approximate $f(0.8)$.

$f(x_0) = f(0) = 1$; $f(x_1) = f(0.5) = 1.82$; $f(x_2) = 6.44$

$$p(x) = 1 \cdot \frac{(x-0.5)(x-1)}{(0-0.5)(0-1)} + 1.82 \frac{x(x-1)}{(0.5-0)(0.5-1)} + 6.44 \frac{x(x-0.5)}{(1-0)(1-0.5)}$$

$$\simeq 7.58x^2 - 2.14x + 1$$

$$f(0.8) \simeq p(0.8) \simeq 4.14$$

10. Exercise 2 from Applied Problems for Chapter 5, page 178. (You may use MATLAB for this exercise.)

$$P_K(T) = -0.2858 \cdot 10^{-11} T^3 + 0.1809 \cdot 10^{-8} T^2 + 0.2638 \cdot 10^{-4} T + 0.7581 \cdot 10^{-2}$$

$$P_\mu(T) = -0.0002 T^3 - 0.0450 T^2 - 4.9780 T + 241$$