Chapter 6: Interpolation with Spline Functions

Problem with polynomial interpolation: the oscillatory nature of high-degree polynomials can induce large fluctuations over the entire range when approximating a set of data points.

Solution: divide the interval and construct a lower degree approximating polynomial on each subinterval (i.e., use **piecewise polynomial interpolation**).

Let $f : [a, b] \to \mathbf{R}$, and let $a = x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$ be the n points at which f is to be interpolated.

Definition: A function S is called a **spline of degree k** for f on $[a, b]$ if it satisfies the following conditions:

1. S interpolates f at x_1, x_2,...,x_n.
2. S is a polynomial of degree $\leq k$ on each subinterval $[x_i, x_{i+1}]$, $i = 1, 2, \ldots, n-1$.
3. $S^{(r)}$ is continuous on $[a, b]$ for $0 \leq r \leq k-1$.

6.1 Piecewise Linear Interpolation

Let $f : [a, b] \to \mathbf{R}$, and let $a = x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$ be the n points at which f is to be interpolated.

The piecewise linear interpolant of f relative to the above partition is a function S that satisfies

1. S interpolates f at x_1, x_2,...,x_n.
2. on each subinterval $[x_i, x_{i+1}]$, $i = 1, 2, ..., n-1$, S coincides with the linear polynomial $S(x) = S_i(x) = a_i + b_i(x - x_i)$.
3. S is continuous on $[a, b]$.

Here, $a_i = f(x_i)$ and $b_i = [f(x_{i+1}) - f(x_i)]/(x_{i+1} - x_i)$, $i = 1 : n-1$.

Error in Piecewise Linear Interpolation:

$$\max_{x\in[a,b]} |f(x) - S(x)| \leq \frac{1}{8}h^2 \max_{x\in[a,b]} |f''(x)|,$$

where $h = \max_{1\leq i\leq n-1}(x_{i+1} - x_i)$.

Quadratic Spline

Let $f : [a, b] \to \mathbf{R}$, and let $a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be the n points at which f is to be interpolated.

A quadratic spline interpolant of f relative to the above partition is a function Q that satisfies

1. Q interpolates f at x_1, x_2, ..., x_n.
2. on each subinterval $[x_i, x_{i+1}]$, $i = 1, 2, ..., n-1$, Q coincides with the quadratic polynomial

$$Q(x) = Q_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2.$$

3. Q and Q' are continuous on $[a, b]$.

Here, for $i = 1, 2, \ldots, n-1$,

$$a_i = f(x_i),\ b_{i+1} = -b_i + 2f[x_{i+1}, x_i],\ c_i = \frac{b_{i+1} - b_i}{2(x_{i+1} - x_i)},$$

with b_1 arbitrary.

Solution Procedure:

1. Calculate $a_i = f(x_i)$, $i = 1 : n-1$.
2. Choose b_1 (e.g., $b_1 = 0$).
3. Calculate

$$b_{i+1} = -b_i + 2\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

and

$$c_i = \frac{b_{i+1} - b_i}{2(x_{i+1} - x_i)}$$

for $i = 1 : n-1$.

6.2 Natural Cubic Splines

Let $f : [a, b] \to \mathbf{R}$, and let $a = x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$ be the n points at which f is to be interpolated.

A cubic spline interpolant of f relative to the above partition is a function S that satisfies

1. S interpolates f at x_1, x_2,...,x_n.
2. on each subinterval $[x_i, x_{i+1}]$, $i = 1, 2, ..., n - 1$, S coincides with the cubic polynomial

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3.$$

3. S, S', and S'' are continuous on $[a, b]$.

6.2 Natural Cubic Splines

The definition of a cubic spline provides $n + 3(n-2) = 4n - 6$ equations. Hereafter, $h_i = x_{i+1} - x_i$.

1. Interpolation: $S_i(x_i) = a_i = f(x_i)$, $i = 1 : n$.
2. Continuity of spline: $a_{i+1} = a_i + b_i h_i + c_i h_i^2 + d_i h_i^3$, $i = 1 : n-2$.
3. Cont. of spl. derivative: $b_{i+1} = b_i + 2c_i h_i + 3d_i h_i^2$, $i = 1 : n-2$.
4. Cont. of spline sec. derivative: $c_{i+1} = c_i + 3d_i h_i$, $i = 1 : n-2$.

4.$\rightarrow d_i = \frac{c_{i+1}-c_i}{3h_i} \rightarrow a_{i+1} = a_i + b_i h_i + \frac{c_{i+1}+2c_i}{3} h_i^2$

$\rightarrow b_i = \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(c_{i+1} + 2c_i)$

3.$\rightarrow$

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1}),$$

for $i = 2, 3, ..., n-1$.

We need two more equations. The simplest choice is $S''(x_1) = S''(x_n) = 0 \rightarrow c_1 = c_n = 0$.

6.2 Natural Cubic Splines

Solution Procedure:

1. Calculate $a_i = f(x_i)$, $h_i = x_{i+1} - x_i$, $i = 1 : n-1$.
2. Solve the system

$$\begin{bmatrix} 1 & 0 & & & 0 \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & & & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ \frac{3(a_3-a_2)}{h_2} - \frac{3(a_2-a_1)}{h_1} \\ \vdots \\ \frac{3(a_n-a_{n-1})}{h_{n-1}} - \frac{3(a_{n-1}-a_{n-2})}{h_{n-2}} \\ 0 \end{bmatrix}$$ for $c_1, c_2, \ldots, c_n$.

3. Calculate

$b_i = \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(c_{i+1} + 2c_i)$, $d_i = \frac{c_{i+1}-c_i}{3h_i}$, $i = 1 : n-1$.