Exam III Name:__________________________________

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There are four equally weighted partial credit questions. Please show all your work for full credit; displaying only the final answer will earn you very little, if anything.

Very important: throughout this exam, wherever they appear, $\alpha =$ $\beta =$ $\gamma =$

1. Consider the linear system

$$\begin{cases} -x_1 + 4x_2 + 2x_3 = 3\alpha - 5\beta + 2\gamma \\ 2x_1 + x_2 + 5x_3 = 3\alpha + \beta + 5\gamma \\ 3x_1 - x_2 + x_3 = 2\alpha + 4\beta + \gamma \end{cases}$$

(a) Solve the system using Gaussian elimination.

Augmented Matrix

$$\begin{bmatrix} -1 & 4 & 2 & 3\alpha - 5\beta + 2\gamma \\ 2 & 1 & 5 & 3\alpha + \beta + 5\gamma \\ 3 & -1 & 1 & 2\alpha + 4\beta + \gamma \end{bmatrix} \xrightarrow[R_3^{new} = R_3^{old} + 3R_1]{R_2^{new} = R_2^{old} + 2R_1}$$

$$\begin{bmatrix} -1 & 4 & 2 & 3\alpha - 5\beta + 2\gamma \\ 0 & 9 & 9 & 9\alpha - 9\beta + 9\gamma \\ 0 & 11 & 7 & 11\alpha - 11\beta + 7\gamma \end{bmatrix} \xrightarrow{R_2^{new} = R_2^{old} \div 9}$$

$$\begin{bmatrix} -1 & 4 & 2 & 3\alpha - 5\beta + 2\gamma \\ 0 & 1 & 1 & \alpha - \beta + \gamma \\ 0 & 11 & 7 & 11\alpha - 11\beta + 7\gamma \end{bmatrix} \xrightarrow{R_3^{new} = R_3^{old} - 11R_2}$$

$$\begin{bmatrix} -1 & 4 & 2 & 3\alpha - 5\beta + 2\gamma \\ 0 & 1 & 1 & \alpha - \beta + \gamma \\ 0 & 0 & -4 & -4\gamma \end{bmatrix} \Rightarrow \begin{cases} -x_1 + 4x_2 + 2x_3 = 3\alpha - 5\beta + 2\gamma \\ x_2 + x_3 = \alpha - \beta + \gamma \\ -4x_3 = -4\gamma \end{cases}$$

Back substitution ⇓ → 3pts

Answer: $x_1 = \alpha + \beta$, $x_2 = \alpha - \beta$, $x_3 = \gamma$

(b) Write out the **individual components** of Jacobi iteration equations for solving the system. Then, starting with the initial vector $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, perform two iterations (i.e., find $x^{(2)}$).

$$\begin{cases} x_1^{(k+1)} = -3\alpha + 5\beta - 2\gamma + 4x_2^{(k)} + 2x_3^{(k)} \\ x_2^{(k+1)} = 3\alpha + \beta + 5\gamma - 2x_1^{(k)} - 5x_3^{(k)} \\ x_3^{(k+1)} = 2\alpha + 4\beta + \gamma - 3x_1^{(k)} + x_2^{(k)} \end{cases}$$

$$\begin{cases} x_1^{(1)} = -3\alpha + 5\beta - 2\gamma \\ x_2^{(1)} = 3\alpha + \beta + 5\gamma \\ x_3^{(1)} = 2\alpha + 4\beta + \gamma \end{cases} \qquad \begin{cases} x_1^{(2)} = \text{---} \\ x_2^{(2)} = \text{---} \\ x_3^{(2)} = \text{---} \end{cases}$$

→ 3.5 pts

(c) Write out the **individual components** of Gauss-Seidel iteration equations for solving the system. Then, starting with the initial vector $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, perform two iterations (i.e., find $x^{(2)}$).

$$\begin{cases} x_1^{(k+1)} = -3\alpha + 5\beta - 2\gamma + 4x_2^{(k)} + 2x_3^{(k)} \\ x_2^{(k+1)} = 3\alpha + \beta + 5\gamma - 2x_1^{(k+1)} - 5x_3^{(k)} \\ x_3^{(k+1)} = 2\alpha + 4\beta + \gamma - 3x_1^{(k+1)} + x_2^{(k+1)} \end{cases}$$

3.5 pts.

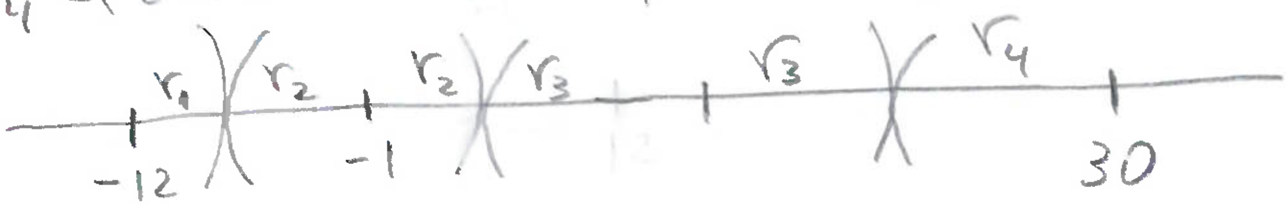
$$\begin{cases} x_1^{(1)} = -3\alpha + 5\beta - 2\gamma \\ x_2^{(1)} = 3\alpha + \beta + 5\gamma + 6\alpha - 10\beta + 4\gamma = 9\alpha - 9\beta + 9\gamma \\ x_3^{(1)} = 2\alpha + 4\beta + \gamma + 9\alpha - 15\beta + 6\gamma + 9\alpha - 9\beta + 9\gamma \\ \quad = 20\alpha - 20\beta + 16\gamma \end{cases} \qquad \begin{cases} x_1^{(2)} = \text{--} \\ x_2^{(2)} = \text{--} \\ x_3^{(2)} = \text{--} \end{cases}$$

2. Let

$$\mathbf{A} = \begin{bmatrix} -12 & a+b & c & \alpha \\ -a & -1 & c & \beta \\ b & b+c & 12 & -a \\ \gamma & b & c & 30 \end{bmatrix}.$$

Find a, b, and c **nonnegative numbers**, <u>or conclude that there is no such triple of numbers</u>, such that the circles in the Gerschgorin Theorem will form a chain of four tangent circles with **mutually disjoint interiors**.

$$\left.\begin{aligned} C_1 &= \{z \in \mathbb{C} : |z+12| \le r_1 = a+b+c+\alpha\} \\ C_2 &= \{z \in \mathbb{C} : |z+1| \le r_2 = a+c+\beta\} \\ C_3 &= \{z \in \mathbb{C} : |z-12| \le r_3 = a+2b+c\} \\ C_4 &= \{z \in \mathbb{C} : |z-30| \le r_4 = b+c+\gamma\} \end{aligned}\right\} \text{2 pts.}$$



$$\text{4 pts.} \begin{cases} r_1 + r_2 = 11 \\ r_2 + r_3 = 13 \\ r_3 + r_4 = 18 \end{cases} \iff \begin{cases} 2a+b+2c = 11-\alpha-\beta \\ 2a+2b+2c = 13-\beta \\ a+3b+2c = 18-\gamma \end{cases}$$

$$\Rightarrow \left.\begin{cases} a = \dots \\ b = \dots \\ c = \dots \end{cases}\right\} \text{3 pts} \Rightarrow \underline{\text{Conclusion}} \longrightarrow \text{1 pt.}$$

3. Let $\mathbf{A} = \begin{bmatrix} \alpha & \beta^2 \\ \gamma^2 & \alpha \end{bmatrix}$.

(a) Find all eigenpairs (that is, the eigenvalues and corresponding eigenvectors) of the matrix

$$p(B) = \beta^2\gamma^2 B^2 - \beta\gamma B + \alpha I_2, \text{ where } B = (A - \alpha I_2)^{-1},$$

without calculating B **or** $p(B)$.

1 pt

$$P_A(\lambda) = \begin{vmatrix} \alpha-\lambda & \beta^2 \\ \gamma^2 & \alpha-\lambda \end{vmatrix} = (\alpha-\lambda)^2 - \beta^2\gamma^2 = 0 \Rightarrow \lambda = \alpha \pm \beta\gamma$$

$\boxed{\lambda_1 = \alpha - \beta\gamma}$: $Null(A-(\alpha-\beta\gamma)I_2) = E_{\lambda_1}$ → 2 pts

$$\left[\begin{array}{cc|c} \beta\gamma & \beta^2 & 0 \\ \gamma^2 & \beta\gamma & 0 \end{array}\right] \xrightarrow[R_2^{new} = \frac{1}{\gamma}R_2^{old}]{R_1^{new} = \frac{1}{\beta}R_1^{old}} \left[\begin{array}{cc|c} \gamma & \beta & 0 \\ \gamma & \beta & 0 \end{array}\right] \xrightarrow{R_2^{new} = R_2^{old} - R_1} \left[\begin{array}{cc|c} \gamma & \beta & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} \gamma x_1 + \beta x_2 = 0 \\ x_2 = \text{free variable} \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{\beta}{\gamma} x_2 \\ x_2 = \text{free} \end{cases} \Rightarrow E_{\lambda_1} = Span\left\{\begin{bmatrix} -\beta/\gamma \\ 1 \end{bmatrix}\right\}$$

2 pts

$\boxed{\lambda_2 = \alpha + \beta\gamma}$ $E_{\lambda_2} = Null(A-(\alpha+\beta\gamma)I_2) = Span\left\{\begin{bmatrix} \beta/\gamma \\ 1 \end{bmatrix}\right\}$

Eigenpairs of A: $\left(\alpha-\beta\gamma, c\begin{bmatrix} -\frac{\beta}{\gamma} \\ 1 \end{bmatrix}\right); \left(\alpha+\beta\gamma, c\begin{bmatrix} \frac{\beta}{\gamma} \\ 1 \end{bmatrix}\right)$; $c \neq 0$

Eigenpairs of B: $\left(\frac{1}{-\beta\gamma}, c\begin{bmatrix} -\frac{\beta}{\gamma} \\ 1 \end{bmatrix}\right); \left(\frac{1}{\beta\gamma}, c\begin{bmatrix} \beta/\gamma \\ 1 \end{bmatrix}\right)$; $c \neq 0$

Eigenpairs of p(B): $\left(\underbrace{p\left(\frac{1}{-\beta\gamma}\right)}_{=\alpha+2}, c\begin{bmatrix} -\beta/\gamma \\ 1 \end{bmatrix}\right); \left(\underbrace{p\left(\frac{1}{\beta\gamma}\right)}_{=\alpha}, c\begin{bmatrix} \beta/\gamma \\ 1 \end{bmatrix}\right)$, $c \neq 0$

where $p(x) = \beta^2\gamma^2 x^2 - \beta\gamma x + \alpha$

$p\left(\frac{1}{-\beta\gamma}\right) = 1 + 1 + \alpha = \alpha + 2$; $p\left(\frac{1}{\beta\gamma}\right) = 1 - 1 + \alpha = \alpha$. → 2 pts

(b) If $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, perform two iterations (i.e., find m_2 and $\mathbf{x}^{(2)}$) of the power method for A.

$$\vec{z}^{(0)} = A\vec{x}^{(0)} = \begin{bmatrix} \alpha & \beta^2 \\ \gamma^2 & \alpha \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha + \beta^2 \\ \gamma^2 + \alpha \end{bmatrix}$$

$$m_1 = \max\{\alpha + \beta^2, \gamma^2 + \alpha\}$$

$$\vec{x}_1^{(1)} = \frac{1}{m_1} \begin{bmatrix} \alpha + \beta^2 \\ \gamma^2 + \alpha \end{bmatrix}$$

1.5 pts.

$$\vec{z}^{(1)} = A\vec{x}^{(1)} = \frac{1}{m_1} \begin{bmatrix} \alpha & \beta^2 \\ \gamma^2 & \alpha \end{bmatrix} \begin{bmatrix} \alpha + \beta^2 \\ \gamma^2 + \alpha \end{bmatrix} = \ldots$$

$$m_2 = \max\{\ldots, \ldots\}$$

$$\vec{x}^{(2)} = \frac{1}{m_2} \vec{z}^{(1)} = \ldots$$

1.5 pts.

4. A radar was used to record the speed of an object during the first 3 seconds of motion (see the table below).

Time t (sec)	0	1	2	3
v (m/sec)	0	α	$\alpha+\beta$	$\alpha+\beta+\gamma$

(a) Construct the divided difference table for the data set, and then write out the Newton form of the interpolating polynomial. Use the resulting polynomial to approximate the value of $v(1.5)$.

0	$\boxed{0} = a_0$			
		$\boxed{\alpha} = a_1$		
1	α		$\boxed{\frac{\beta-\alpha}{2}} = a_2$	
		β		$\boxed{\frac{\gamma-2\beta+\alpha}{6}} = a_3$
2	$\alpha+\beta$		$\frac{\gamma-\beta}{2}$	
		γ		
3	$\alpha+\beta+\gamma$			

4 pts.

$$P_N(t) = 0 + \alpha t + \frac{\beta-\alpha}{2} t(t-1) + \frac{\gamma-2\beta+\alpha}{6} t(t-1)(t-2)$$

$$v(1.5) \simeq p(1.5) = \text{----}$$

$\longrightarrow$ 0.5 pts.

(b) Construct the Lagrange form of the interpolating polynomial. Use the resulting polynomial to approximate the value of $v(1.5)$; it should be the same as in part **(a)** (**why?**).

$$P_L(t) = 0\cdot\frac{(t-1)(t-2)(t-3)}{(0-1)(0-2)(0-3)} + \alpha\frac{(t-0)(t-2)(t-3)}{(1-0)(1-2)(1-3)}$$

$$+(\alpha+\beta)\frac{(t-0)(t-1)(t-3)}{(2-0)(2-1)(2-3)} + (\alpha+\beta+\gamma)\frac{(t-0)(t-1)(t-2)}{(3-0)(3-1)(3-2)}$$

$$= \frac{\alpha}{2}t(t-2)(t-3) + \frac{\alpha+\beta}{-2}t(t-1)(t-3)$$

$$+ \frac{\alpha+\beta+\gamma}{6}t(t-1)(t-2)$$

4 pts.

$$v(1.5) \simeq P_L(1.5) = \cdots$$

$\longrightarrow$ 0.5 pts.

$v(1.5)$ is approximated by the same value in (a) and (b) because $P_N(t)$ and $P_L(t)$ are just different forms of the <u>same</u> interpolating polynomial

1 pt.

1	2	3	4	Total