Workshop on image processing, computer vision and graphics
Graph cut optimization
class:35 36, 02.07.2010, time: 10am to 1.15 pm

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Outline

- Motivation of selecting MAP-MRF framework
- MAP-MRF approach
- Mapping of MAP-MRF to graph-cut problem
- What Energy functions can be minimized with Graph-cuts?
- Some examples of MAP-MRF framework using graph-cut
- MAP-MRF framework for Super Resolution
- Summary

Motivation of selecting MAP-MRF framework

- Bayesian framework suitable for problems in Computer Vision
- MAP-MRF with Gibbs gives easy implementation and formulation.
- Problems: High computational cost or Standard methods used are very slow.
- Boykov et.al proposed methods to solve MAP-MRF using graph-cut algorithms -MAP-MRF estimation is equivalent to min-cut problem on a graph
- Applied to many vision problems

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MAP-MRF framework

- MRF framework: Given set of pixels $S = \{s_1 \ldots s_m\}$ and set of labels $\Lambda = \{l_1 \ldots l_L\}$ and neighborhood system N, Find mapping of S to Λ.
- Let F be the configuration for labels $F = \{f_i \ldots f_N\}, f_i \in \Lambda$ is the label for s_i
- F is MRF with respect to N iff
  - Positivity: $P(F = f) \geq 0 \quad \forall \quad f \in F$
  - Markovianity:

$$P(F_s = f_s / F_r = f_r, \forall r \neq s) = P(F_s = f_s / F_r = f_r, \forall r \in N_s)$$

- Easy Implementability

MAP-MRF framework contd..

Let G be the observed image

- $G = \phi(H(F)) + N$
  where $H =$Camera Transfer Function and $\phi =$Recorder distortion
  H is assumed to be LSI and ϕ is invertible nonlinear function and N is additive noise assumed to be iid
- In the framework of Restoration :Given G What is F?
  - $P(F = f/G = g)$, Maximum likelihood of $F = f$ given $G = g$
  - From Bays rule
    $P(F = f/G = g) \propto P(G = g/F = f)P(F = f)$
    where $P(G = g/F = f)$=Data model, $P(F = f)$ =Prior and $P(F = f/G = g)$=Aposteriori distribution
  - Need to maximize aposteriori(MAP) distribution

Gibbs Distribution

- Geman and Geman proved equivalence between MRF and Gibbs distribution

$$P(f) = \frac{1}{Z} exp(-U(f)/T)$$

where $U(f) = \sum_{c \in N} V_c(f)$= Energy function, V_c=Clique potential, $Z = \sum_f exp(-U(f)/T)$ =Partition function and T=Temperature

- Hammersely Clifford Theorem:

F is MRF on S with respect to N
if and only if
F is Gibbs random field on S with respect to N

- Relates the conditional distribution(local characteristics) and joint distribution(Gibbs measure)

MAP-MRF

- $\hat{f} = \underset{f}{\mathbf{argmax}} \; P(f/g)$
- From Bays Rule
  $\hat{f} = \underset{f}{\mathbf{argmax}} \; P(g/f)\dot{P}(f)$
  I-term: Likelihood function and II-term:Prior Model
- $\hat{f} = \underset{f}{\mathbf{argmax}} \; exp\{\sum_{p} \in (p(g/f_p)) - \sum_{p,q \in N} V_{p,q}(f_p, f_q)\}$
- MAP estimate of f given g is equivalent to minimizing energy function with prior and data model
  $\hat{f} = \underset{f}{\mathbf{argmin}} \; \{-\sum_{p} ln(p(g/f_p)) + \sum_{p,q \in N} V_{p,q}(f_p, f_q)\}$

MAP-MRF

- The Energy Function has data term and regularization term
- $U(f) = \left\{\sum_i (g_i - \phi(H(f_i)) + \sum_{i,j \in N} V_{i,j}(f_i, f_j)\right\}$
- Different ways of defining Clique potential which defines the regularization term or smoothness term in the energy function and describes the prior probability of a particular realization of the elements of the clique.
- Data model should capture the cost of assigning the label
- MAP-MRF is usually solved using SA which is very slow but guarantees the global minima for any arbitrary energy function
- Boykov et.al suggested max-flow/min-cut graph algorithms to solve some class of energy functions with MAP-MRF framework within a known factor of global minimum

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Mapping of MAP-MRF to graph-cut

- Vision problems as image labeling:Depth(stereo), Object Index(Segmentation), Original Intensity(Restoration)
- Labeling problem can be cast in terms of energy minimization
  - Labeling of pixels
  - Penalty for pixel labeling
  - Interaction between neighboring pixels:Smoothness term
- All pixels and labels are considered as vertices, edge and edge weights are calculated dynamically
- Min-cut on G has unique binary segmentation
- Segmentation associated with min-cut that satisfies user defined constraints minimizes the energy function

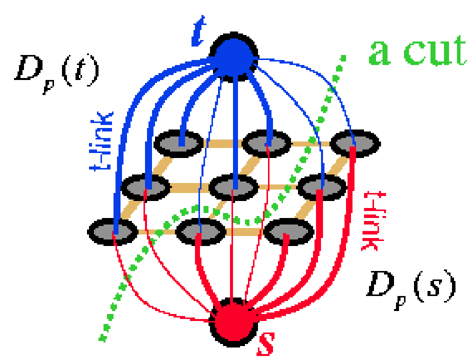
Energy Minimization

- Global minimum can be found in polynomial time if the energy function
  - is convex,
  - or has only two labels, eg. Icing model.
- Discontinuity- preserving energy function is not convex, eg. Potts model. Thus global minimization is NP- hard, takes exponential time,
- Thus global minimization Approximation algorithm to find local minimum
  - EM, Belief Propagation, Graph-Cuts
- **What is Graph-Cuts?**
  - Minimize an energy function with non binary variables by repeatedly minimizing an energy function with binary variables using Max- flow/ min- cut method

s-t graph-cuts for Binary Energy Minimization

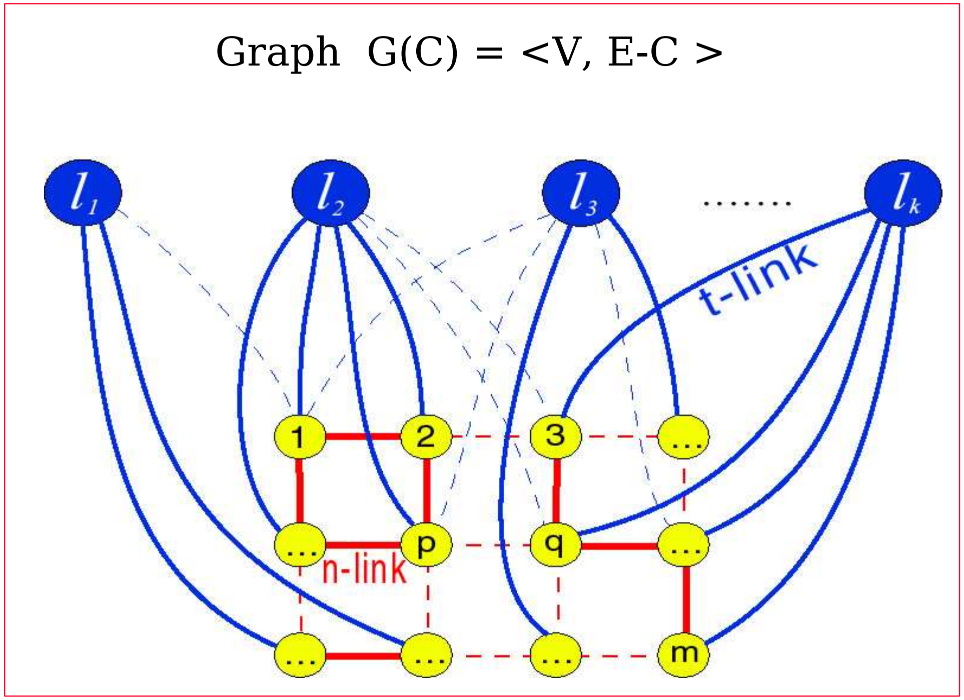
- Posterior energy (MRF)
- Complete characterization of binary energies that can be minimized with s-t graph cuts.
- $U(f) = \sum_{p}(D_p(f_p)) + \sum_{pq \in N} V(f_p, f_q)$
- $U(f)$ can be minimized by graph-cuts $\Longleftrightarrow V(s,s) + V(t,t) \leqslant V(s,t) + V(t,s)$


Example: binary image restoration



s-t graph-cuts for multi label problems

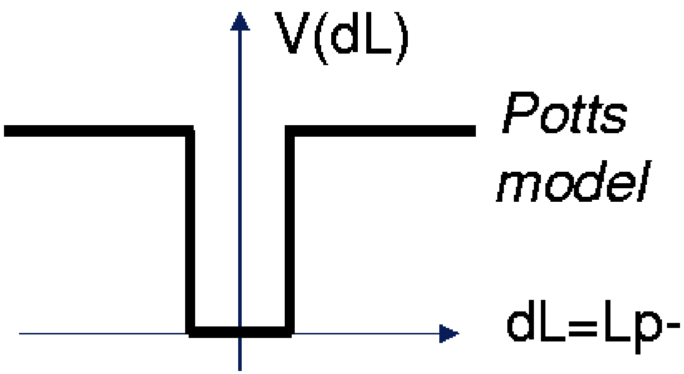
- Class of Energy that can be minimized exactly : Energies with convex interactions
  - excludes robust discontinuity-preserving interactions
- Guaranteed quality approximation algorithms for multi-label energies with discontinuity-preserving interactions like Potts model of interactions and Metric interactions

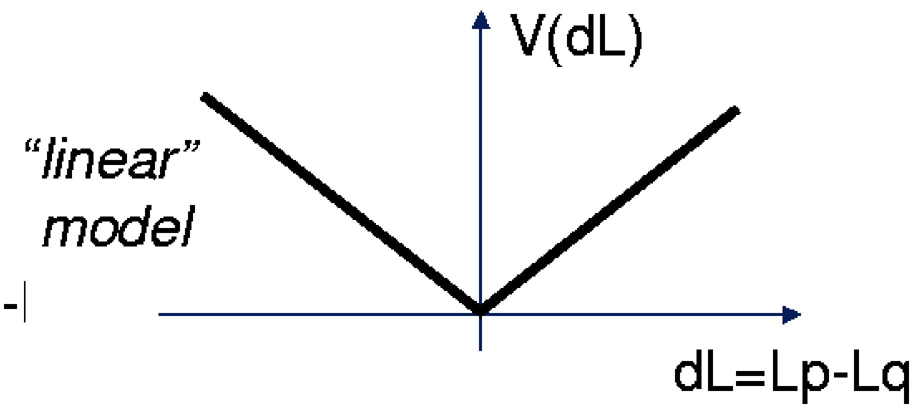


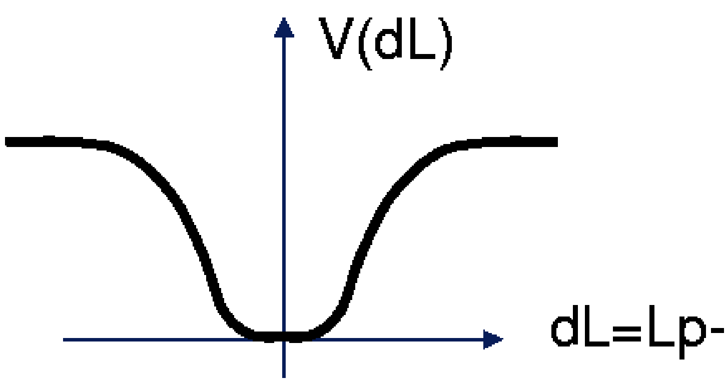
Different types of Pixel Interactions

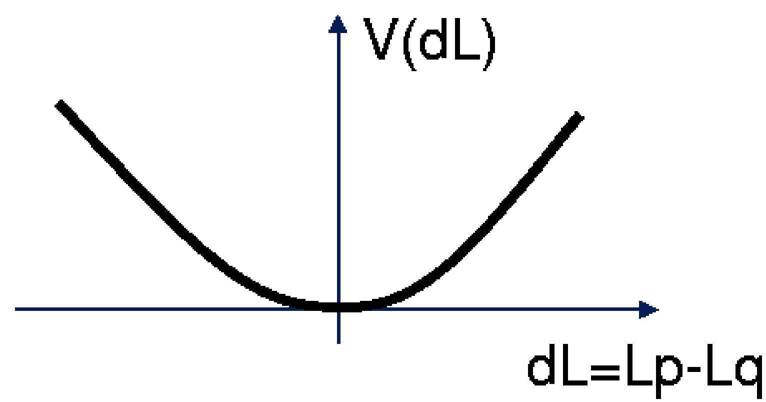
Discontinuity preserving interactions:

Convex interactions: Linear Models

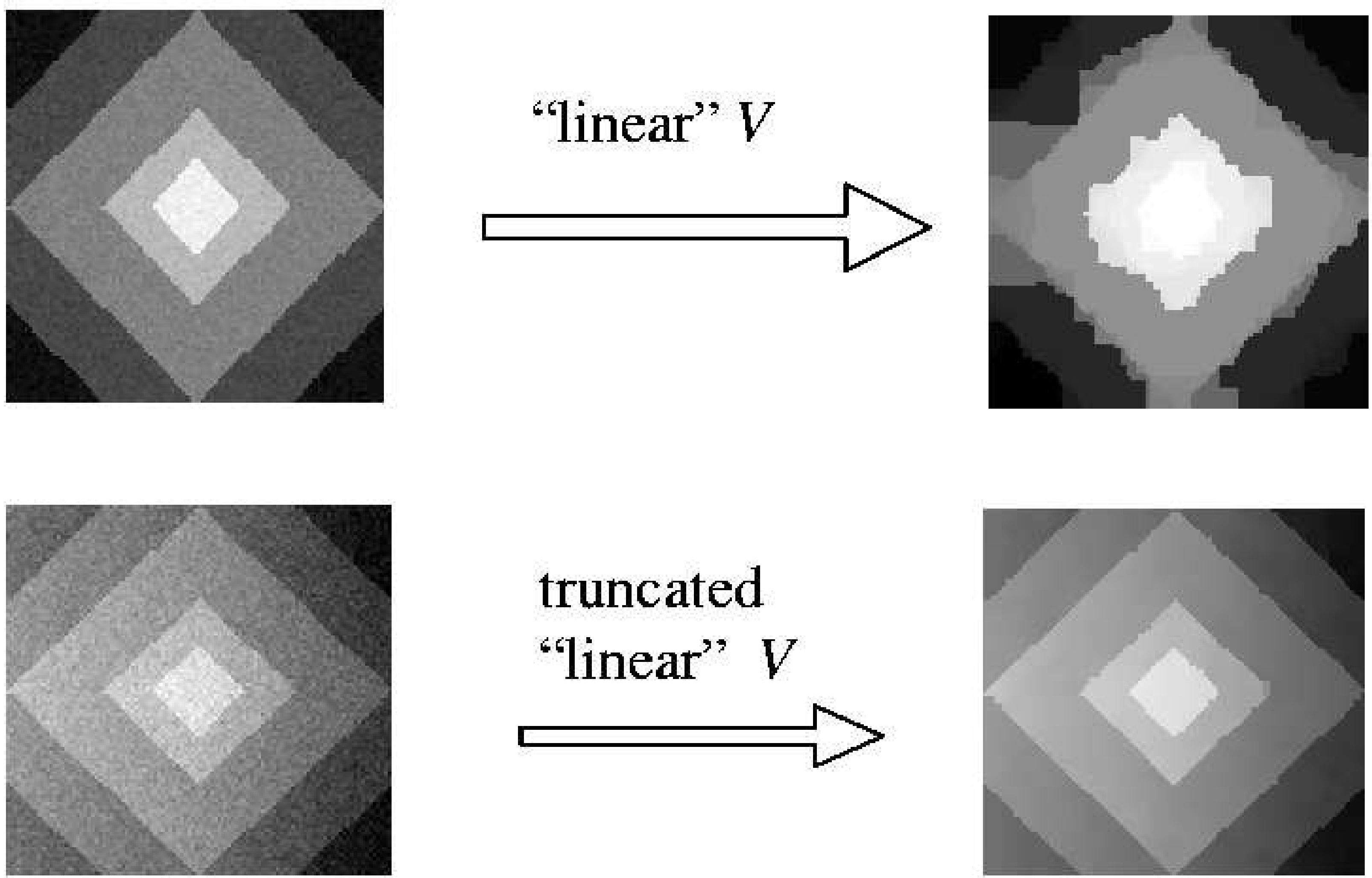




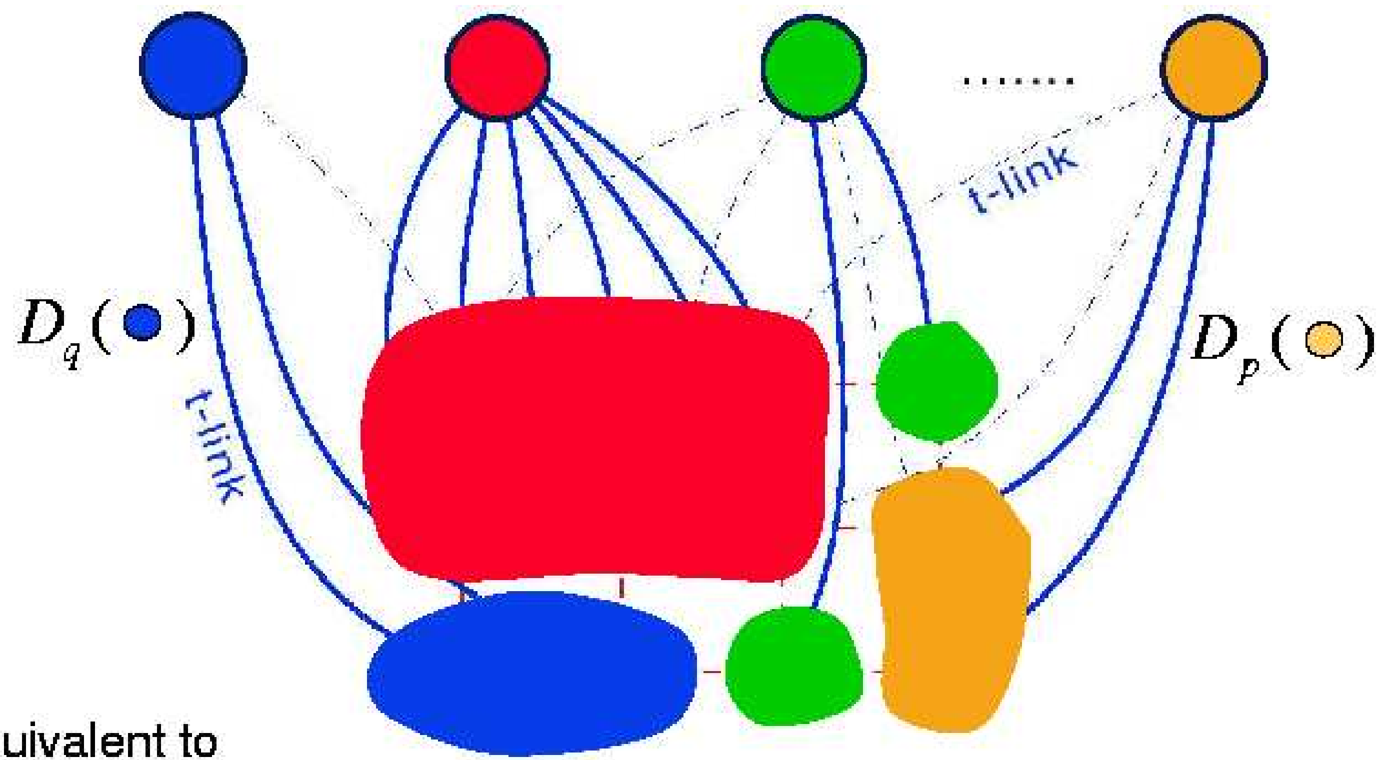




Convex vs. Discontinuity-preserving



Multi way Graph-cut



Equivalent to minimization of **the Potts energy** of labeling *L*

$$E(L) \;=\; \sum_{p} \overset{\text{t-links}}{-D_p(L_p)} \;+\; \sum_{pq\in N} \overset{\text{n-links}}{w_{pq}\cdot\delta_{Lp\neq Lq}}$$

Multi way Graph-cut algorithms by Boykov et.al

- Equivalent to Potts energy minimization
- NP-hard problem (3 or more labels)
  - two labels can be solved via s-t cuts (Greig et. al., 1989)
- Two approximation algorithms (Boykov et.al 1998,2001)
  Basic Idea:break multi-way cut computation into a sequence of binary s-t cuts.
  - α- Expansion
    Each label competes with the other labels for space in the image
  - $\alpha - \beta$ Swap : Define a move which allows to change pixels from α to β and β to α

α-*Expansion approximation algorithm*

Guaranteed quality approximation

- within a factor of 2 from Global minimum (Potts Model)
- applies to a wide class of energies with robust interactions
- Potts model (BVZ 1989), Metric interactions (BVZ 2001), Sub modular interactions (KZ 2002,2004)

Algorithm

1. Start with any arbitrary labeling f
2. Set $success = 0$
3. For each label $\alpha \in L$ (random order)
   (a) find $\hat{f}$ = $argmin \;\; U(f^1)$ among f^1 within one α-expansion f
   (b) $If \;\; U(\hat{f}) < U(f)$, set $f = \hat{f}$ and $success = 1$
4. If $success = 1$ go to step 2
5. return f

$\alpha - \beta$ *Swap approximation algorithm*

Handles more general energy functions

- Experimentally proved results

Algorithm

1. Start with any arbitrary labeling f
2. Set $success = 0$
3. For each pair of labels $\{\alpha, \beta\} \in L$ (random order)
   (a) find $\hat{f}$ = $argmin\ U(f^1)$ among f^1 within one $\alpha - \beta$ swap of f
   (b) $If U(\hat{f}) < U(f)$, set $f = \hat{f}$ and $success = 1$
4. If $success = 1$ go to step 2
5. return f

Finding optimal expansion move

3.a step in algo. The structure of the graph is dynamically determined by the current position P and label α.

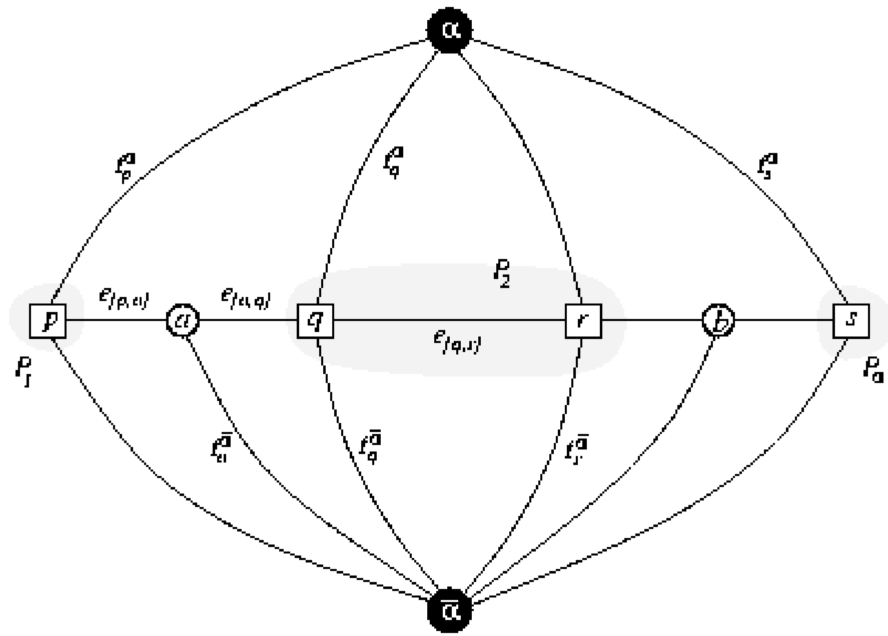


Figure 6: An example of $\mathcal{G}_\alpha$ for a 1D image. The set of pixels in the image is $\mathcal{P} = \{p, q, r, s\}$ and the current partition is $\mathbf{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_\alpha\}$ where $\mathcal{P}_1 = \{p\}$, $\mathcal{P}_2 = \{q, r\}$, and $\mathcal{P}_\alpha = \{s\}$. Two auxiliary nodes $a = a_{\{p,q\}}$, $b = a_{\{r,s\}}$ are introduced between neighboring pixels separated in the current partition. Auxiliary nodes are added at the boundary of sets $\mathcal{P}_l$.

Expansion move-assignment of weights

<table>
<tr><th>edge</th><th>weight</th><th>for</th></tr>
<tr><td>$t_p^{\bar{\alpha}}$</td><td>∞</td><td>$p \in \mathcal{P}_\alpha$</td></tr>
<tr><td>$t_p^{\bar{\alpha}}$</td><td>$D_p(f_p)$</td><td>$p \notin \mathcal{P}_\alpha$</td></tr>
<tr><td>t_p^{α}</td><td>$D_p(\alpha)$</td><td>$p \in \mathcal{P}$</td></tr>
<tr><td>$e_{\{p,a\}}$</td><td>$V(f_p, \alpha)$</td><td rowspan="3">$\{p, q\} \in \mathcal{N},\ f_p \neq f_q$</td></tr>
<tr><td>$e_{\{a,q\}}$</td><td>$V(\alpha, f_q)$</td></tr>
<tr><td>$t_a^{\bar{\alpha}}$</td><td>$V(f_p, f_q)$</td></tr>
<tr><td>$e_{\{p,q\}}$</td><td>$V(f_p, \alpha)$</td><td>$\{p, q\} \in \mathcal{N},\ f_p = f_q$</td></tr>
</table>

Finding optimal swap move

3.1 step in algo. The structure of the graph is dynamically determined by the current position P and labels α, β.

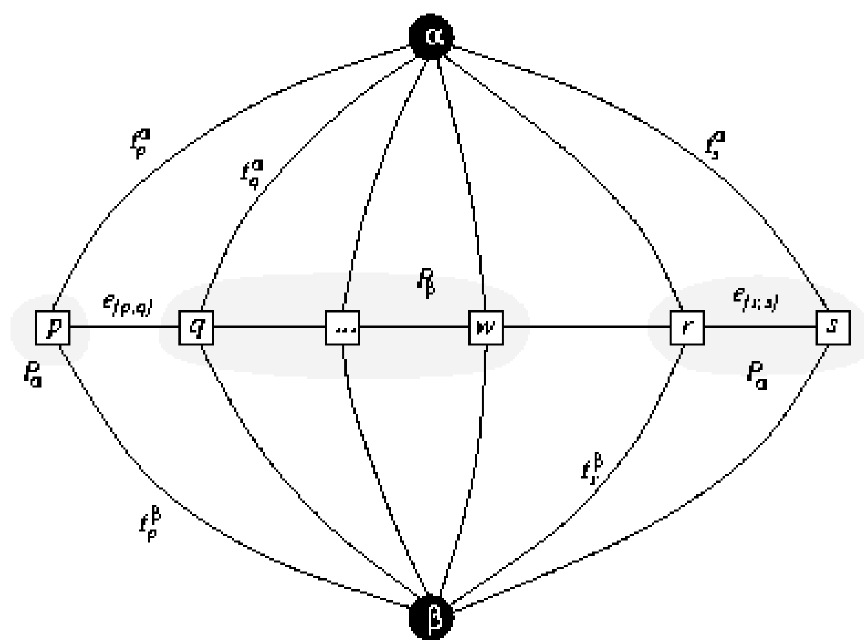


Figure 4: An example of the graph $\mathcal{G}_{\alpha\beta}$ for a 1D image. The set of pixels in the image is $\mathcal{P}_{\alpha\beta} = \mathcal{P}_\alpha \cup \mathcal{P}_\beta$ where $\mathcal{P}_\alpha = \{p, r, s\}$ and $\mathcal{P}_\beta = \{q, \ldots, w\}$.

Optimal swap move-assignment of weights

edge	weight	for
t_p^{α}	$D_p(\alpha) + \sum_{\substack{q \in \mathcal{N}_p \\ q \notin \mathcal{P}_{\alpha\beta}}} V(\alpha, f_q)$	$p \in \mathcal{P}_{\alpha\beta}$
t_p^{β}	$D_p(\beta) + \sum_{\substack{q \in \mathcal{N}_p \\ q \notin \mathcal{P}_{\alpha\beta}}} V(\beta, f_q)$	$p \in \mathcal{P}_{\alpha\beta}$
$e_{\{p,q\}}$	$V(\alpha, \beta)$	$\substack{\{p,q\} \in \mathcal{N} \\ p,q \in \mathcal{P}_{\alpha\beta}}$

α-*Expansion move*

initial solution

-expansion

-expansion

-expansion

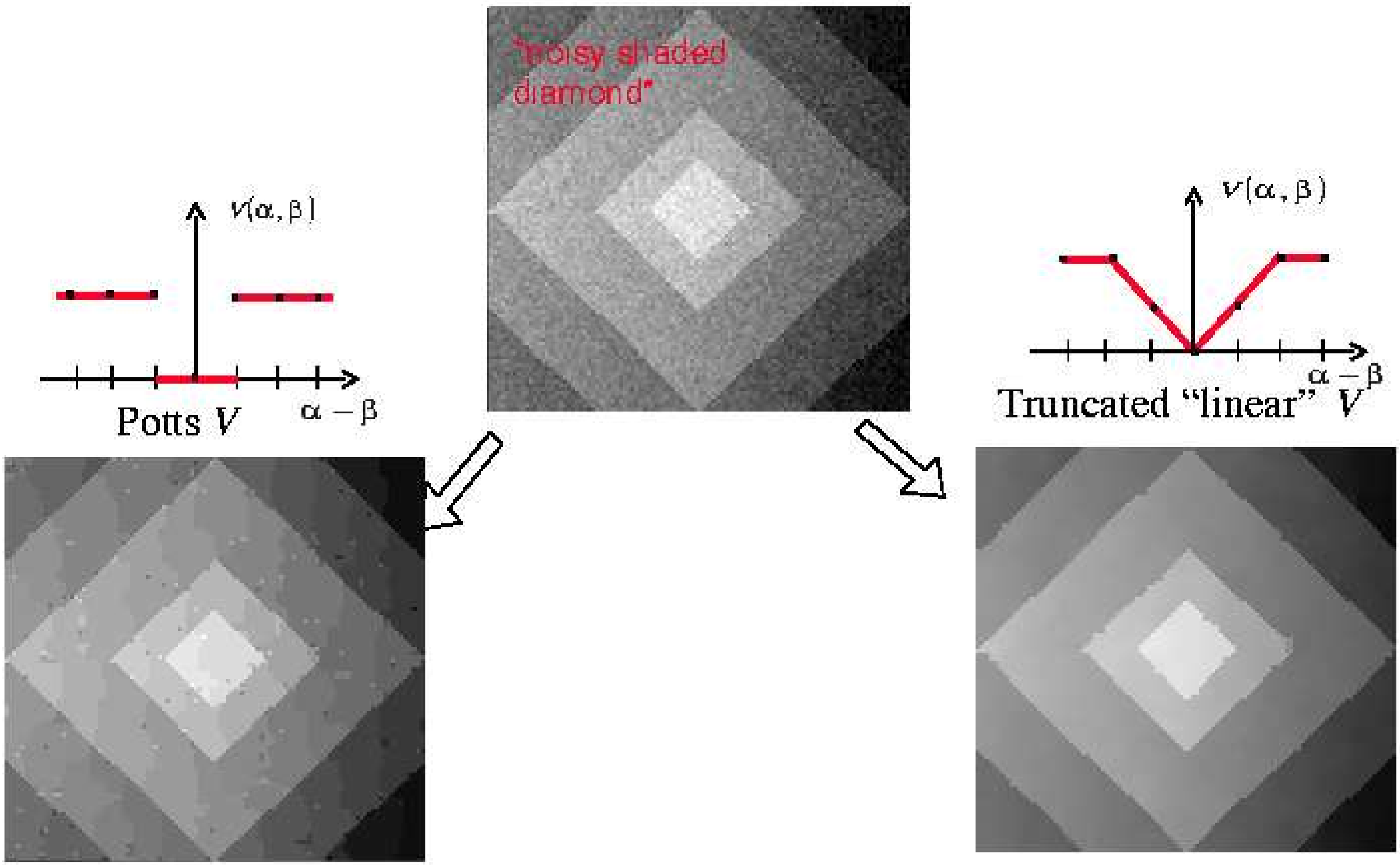
-expansion

-expansion

-expansion

-expansion

Example for Metric Interactions

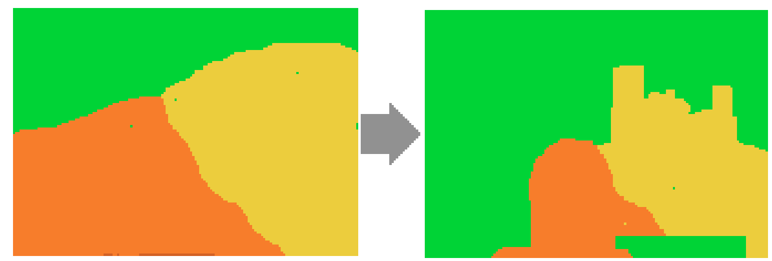


Comparison

single “one-pixel” move
(simulated annealing, ICM,...)

- Only one pixel can change its label at a time
- Finding an optimal move is computationally trivial

single *a*-expansion move



- Large number of pixels can change their labels simultaneously
- Finding an optimal move is computationally intensive O(2^n) (*s*-*t* cuts)

Comparisons contd..

simulated annealing

- Finds local minimum of energy with respect to small one-pixel moves
- Initialization is important
- solution could be arbitrarily far from the global minima
- May not know when to stop. Practical complexity may be worse than exhaustive search
- Can be applied to any-thing

α-Expansion

- Finds local minimum of energy with respect to very strong moves
- In practice, results do not depend on initialization
- solution is within the factor of 2 from the global minima
- In practice, one cycle through all labels gives sufficiently good results
- Applies to a restricted class of energies

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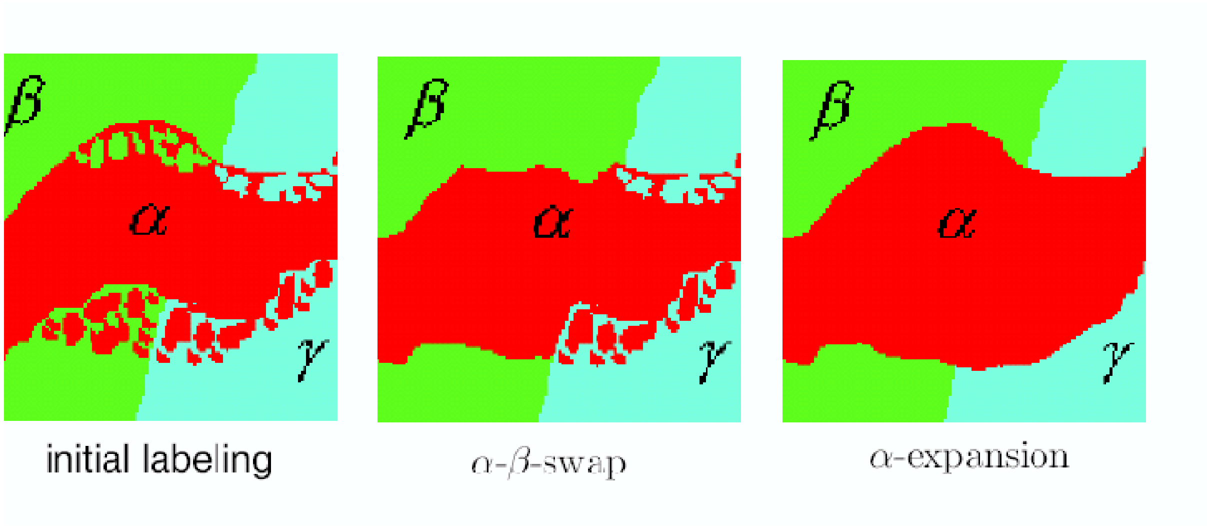
What Energy functions can be minimized with Graph-cuts?

- α-expansion algorithm can be applied to pairwise interactions that are metric on the space of labels
  - $V(a,a) = 0$
  - $V(a,b) \geq 0$
  - $V(a,b) \leq V(a,c) + V(c,b)$
- Any truncated metric is also a metric(includes robust interactions)
- α-expansion algorithm further generalizes to submodular pair-wise interactions
- $V(c,c) + V(a.b) \leq V(a,c) + V(c,b)$
- $\alpha - \beta$ swap can be applied to pairwise interactions which are semi-metric on the space of labels
- Let E be a function of binary variables. If E is not regular, then E is not graph-representable.

Regular functions

- All functions of one variable are regular
- A function V of two variables is called regular if $V(0,0)+V(1,1) \;\leq\; V(0,1)+V(1,0)$
- A function V of more than two variables is called regular if all projections of V of two variables are regular.
- Let V be a function of n binary variables from F^3, ie. $V(x_1,\ldots x_n) = \sum_i V^i(x_i) + \sum_{i<j} V^{i,j}(x_i,x_j) + \sum_{i<j<k} V^{i,j,k}(x_i,x_j,x_k)$. Then, V is graph-representable if and only if V is regular
- Any projection of a graph-representable function is graph-representable.

Moves



If V is Metric, then each expansion move is regular
$E(0,0)+E(1,1)=V(\beta,\gamma)+V(\alpha,\alpha) \;\; \leq \;\; V(\beta,\alpha)+V(\alpha,\gamma)=E(0,1)+E(1,0)$
If V is Semi-metric, then each swap move is regular
$E(0,0)+E(1,1)=V(\beta,\beta)+V(\alpha,\alpha) \;\; \leq \;\; V(\beta,\alpha)+V(\alpha,\beta)=E(0,1)+E(1,0)$

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Some examples of MAP-MRF using graph-cut

- Image segmentation 1: Jiangjian Xiao and Mubarak Shah CVPR 2005
  - Motion cue to segment using graph-cut
  - refine the segmentation by alpha matte
  - Example 1
- Image segmentation/Object Extraction: Yuri Boykov and Vladimir Kolmogorov 2004
  - Combine both active contours and graph-cuts
  - Reduces the metrification error
  - Example 2

Examples contd..

- Texture Synthesis :Image quilting by Efros and Freeman, 2001
  Example
- Video Texture :3D generalization of Image quilting by Kwatra, Schodl, Essa, Bobick 2003
  Process
  Source
  Synthesized
- Stereo: Boykov et.al 98, 2002
  Example
- Multiview reconstruction by Boykov et.al 2004
  Example
- Interactive Digital photomontage by microsoft research lab
  Example

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MAP-MRF framework for Super Resolution

- The Energy function is given by
- $E(f) = \sum_p D_p(f_p) + \sum_{p,q \in N_p} V_{p,q}(f_p, f_q) + \sum_{(p,q)_{\downarrow d} \in N_p} V_{(p,q)_{\downarrow d}}(g_{p\downarrow d}, g_{q\downarrow d})$
  where $D = g - DHf$ is a data model term and next two terms are regularization terms. g=observed image, D=decimation function and H= Camera transfer function.
- f is the Super Resolution Image - needs to estimate
- Regularization terms are truncated linear models
- $V_{p,q}(f_p, f_q) = min(K, |f_p - f_q|)$
- $V_{(p,q)_{\downarrow d}}(g_{p\downarrow d}, g_{q\downarrow d}) = min(K, |g_{p\downarrow d} - g_{q\downarrow d}|)$
- Once we have the MAP-MRF framework for SR, we can apply Graph-cuts to estimate f.

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Summary

- MAP-MRF framework
- What energy functions can be minimized with Graph-cuts
- $\alpha-$Expansion and $\alpha-\beta$ Swap algorithms in Graph-cuts
- Examples using Graph-cuts
- Framework of SR

THANK YOU