AI1103-Assignment 6

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Download all latex-tikz codes from

https://github.com/vaishnavi−w/AI1103/blob/main/Assignment6/latex6.tex

Question

Which of the following conditions imply independence of random variables X and Y?

1) $\Pr(X > a|Y > a) = \Pr(X > a)$ for all $a \in R$
2) $\Pr(X > a|Y < b) = \Pr(X > a)$ for all $a, b \in R$
3) X and Y are uncorrelated
4) $E[(X - a)(Y - b)] = E[X - a]E[Y - b]$ for all $a, b \in R$

Solution

1) Two random variables X and Y are independent when the joint probability of random variables is product of their individual probabilities

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B) \tag{0.0.1}$$

In other words conditional probability follows

$$\Pr(X \in A|Y \in B) = \Pr(X \in A) \tag{0.0.2}$$

for all sets A and B.
In the case of $\Pr(X > a|Y > a) = \Pr(X > a)$ the sets $A = B$. The spectrum of conditions for independence is underrepresented. Hence, the condition does not imply independence of X and Y.
Counterexample: Consider two random variables $X, Y \in \{0, 1, 2\}$ with the probabilities of the ordered pairs (X, Y) given in the Table1

(X,Y)	Pr	(X,Y)	Pr	(X,Y)	Pr
(0,0)	0.2	(1,0)	0.2	(2,0)	0.1
(0,1)	0.1	(1,1)	0.1	(2,1)	0.1
(0,2)	0.1	(1,2)	0.05	(2,2)	0.05

TABLE 1: $\Pr(X, Y)$

Consider

$$\Pr(X > 1|Y > 1) = \frac{\Pr(X > 1, Y > 1)}{\Pr(Y > 1)}$$
$$= \frac{\Pr(X = 2, Y = 2)}{\Pr(Y = 2)} = \frac{0.05}{0.2} = 0.25 \tag{0.0.3}$$

$$\Pr(X > 1) = \Pr(X = 2) = 0.25 \tag{0.0.4}$$

Similarly, we can find that $\Pr(X > a|Y > a) = \Pr(X > a)$ for all $a \le 2$. ($\because \Pr(Y > a) = 0$ for all $a > 2$)
Consider,

$$\Pr(X = 1, Y = 2) = 0.05 \tag{0.0.5}$$

$$\Pr(X = 1)\Pr(Y = 2) = 0.35 \times 0.2 = 0.7$$
$$\neq \Pr(X = 1, Y = 2) \tag{0.0.6}$$

Clearly, X and Y are not independent.
Option 1 is incorrect

2) From Bayes theorem,

$$\Pr(X > a|Y < b) = \frac{\Pr((X > a), (Y < b))}{\Pr(Y < b)} \tag{0.0.7}$$

Given that $\Pr(X > a|Y < b) = \Pr(X > a)$,

$$\frac{\Pr((X > a), (Y < b))}{\Pr(Y < b)} = \Pr(X > a) \tag{0.0.8}$$

$$\Pr((X > a), (Y < b)) = \Pr(X > a)\Pr(Y < b) \tag{0.0.9}$$

for all $a, b \in R$.
From (0.0.1), X and Y are independent.
Option 2 is correct

3) Two random variables X and Y are uncorrelated if their covariance is zero.

$$cov[X, Y] = E[XY] - E[X]E[Y] = 0 \tag{0.0.10}$$

Uncorrelatedness does not imply independence.

Counterexample: Let $X \sim U[-1,1]$ be a uniformly distributed random variable.

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{0.0.11}$$

$$E[X] = \int_{-1}^{1} x f(x)\, dx = 0 \tag{0.0.12}$$

Let $Y = X^2$ be another random variable.
X and Y are dependent.

$$cov[X,Y] = E[XY] - E[X]E[Y] \tag{0.0.13}$$
$$= E\left[X^3\right] - 0 \times E[Y] \tag{0.0.14}$$
$$= \int_{-1}^{1} x^3 f(x)\, dx = 0 \tag{0.0.15}$$

X and Y are uncorrelated but not independent.
Option 3 is incorrect

4) Given that,

$$E[(X-a)(Y-b)] = E[X-a]E[Y-b] \tag{0.0.16}$$

$$cov[(X-a),(Y-b)] = E[(X-a)(Y-b)] - E[X-a]E[Y-b] \tag{0.0.17}$$

$$\implies cov[(X-a)(Y-b)] = 0 \tag{0.0.18}$$

From option 3, it follows that X and Y are not necessarily independent.
Option 4 is incorrect.