AI1103-Assignment 6

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Download all latex-tikz codes from

https://github.com/vaishnavi−w/AI1103/blob/main/Assignment6/latex6.tex

Question

Which of the following conditions imply independence of random variables X and Y?

1) $\Pr(X > a|Y > a) = \Pr(X > a)$ for all $a \in R$
2) $\Pr(X > a|Y < b) = \Pr(X > a)$ for all $a, b \in R$
3) X and Y are uncorrelated
4) $E[(X-a)(Y-b)] = E[X-a]E[Y-b]$ for all $a, b \in R$

Solution

1) Two random variables X and Y are independent when the joint probability distribution of random variables is product of their individual probability distributions i.e for all sets A,B

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B) \tag{0.0.1}$$

Consider the CDFs,

$$F_X(a) = \Pr(X < a) \tag{0.0.2}$$
$$F_Y(b) = \Pr(Y < b) \tag{0.0.3}$$
$$F_{X,Y}(a,b) = \Pr(X < a, Y < b) \tag{0.0.4}$$

Let $F_{X,Y}(a,b) = F_X(a)F_Y(b)$ be true. Partial derivative w.r.t a,

$$\frac{\partial F_{X,Y}(a,b)}{\partial a} = \frac{\partial F_X(a)}{\partial a}F_Y(b) \tag{0.0.5}$$

Partial derivative w.r.t b,

$$\frac{\partial^2 F_{X,Y}(a,b)}{\partial b \partial a} = \frac{\partial F_X(a)}{\partial a}\frac{\partial F_Y(b)}{\partial b} \tag{0.0.6}$$
$$\implies p_{X,Y}(a,b) = p_X(a)p_Y(b) \tag{0.0.7}$$

when X,Y are discrete. And,

$$\implies f_{X,Y}(a,b) = f_X(a)f_Y(b) \tag{0.0.8}$$

when X,Y are continuous, for all $a, b \in R$.
$\therefore$ Two random variables are independent if the joint CDF can be expressed as the product of individual CDFs i.e for all $a, b \in R$

$$F_{X,Y}(a,b) = F_X(a)F_Y(b) \tag{0.0.9}$$

Consider,

$$\Pr(X > a|Y > a) = \frac{\Pr(X > a, Y > a)}{\Pr(Y > a)} \tag{0.0.10}$$

Given that $\Pr(X > a|Y > a) = \Pr(X > a)$,

$$\implies \Pr(X > a, Y > a) = \Pr(X > a)\Pr(Y > a) \tag{0.0.11}$$

for all $a \in R$.

$$\begin{aligned} 1 - F_X(a) - F_Y(a) &= \Pr(X > a) - \Pr(Y < a) \\ &= \Pr(X > a, Y > a) + \Pr(X > a, Y < a) \\ &\quad - \Pr(X > a, Y < a) - \Pr(X < a, Y < a) \end{aligned} \tag{0.0.12}$$

$$\begin{aligned} 1 - F_X(a) - F_Y(a) = \Pr(X > a, Y > a) \\ - F_{X,Y}(a,a) \end{aligned} \tag{0.0.13}$$

Substituting in (0.0.11),

$$\begin{aligned} 1 - F_X(a) - F_Y(a) + F_{X,Y}(a,a) = \\ (1 - F_X(a))(1 - F_Y(a)) \end{aligned} \tag{0.0.14}$$

$$\implies F_{X,Y}(a,a) = F_X(a)F_Y(a) \tag{0.0.15}$$

On comparing with (0.0.9) in this case, it is true only under the condition that $b = a$. It may not be true for other values of b. The spectrum of conditions for independence is underrepresented. Hence, the condition does not imply independence of X and Y.
Counterexample: Consider two random variables $X, Y \in \{0, 1, 2\}$ with the probabilities of the ordered pairs (X, Y) given in the Table1

X \ Y	0	1	2
0	0.2	0.1	0.1
1	0.2	0.1	0.05
2	0.1	0.1	0.05

TABLE 1: $\Pr(X, Y)$

Case 1: $a < 0$

$$\Pr(X > a|Y > a) = 1 = \Pr(X > a) \tag{0.0.16}$$

Case 2: $0 \le a < 1$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} = \frac{0.3}{0.5} = 0.6 \tag{0.0.17}$$

$$\Pr(X > a) = \Pr(X = 1) + \Pr(X = 2) = 0.6 \tag{0.0.18}$$

Case 3: $1 \le a < 2$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} = \frac{0.05}{0.2} = 0.25 \tag{0.0.19}$$

$$\Pr(X > a) = \Pr(X = 2) = 0.25 \tag{0.0.20}$$

Case 4: $a \ge 2$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} \tag{0.0.21}$$

is not defined as $\Pr(Y > a) = 0$. In all the cases, $\Pr(X > a|Y > a) = \Pr(X > a)$ is true.
Consider,

$$\Pr(X = 1, Y = 2) = 0.05 \tag{0.0.22}$$

$$\Pr(X = 1)\Pr(Y = 2) = 0.35 \times 0.2 = 0.7 \neq \Pr(X = 1, Y = 2) \tag{0.0.23}$$

Clearly, X and Y are not independent.
Option 1 is incorrect

2) From Bayes theorem,

$$\Pr(X > a|Y < b) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \tag{0.0.24}$$

Given that $\Pr(X > a|Y < b) = \Pr(X > a)$,

$$\implies \Pr(X > a, Y < b) = \Pr(X > a)\Pr(Y < b) \tag{0.0.25}$$

for all $a, b \in R$. Consider

$$F_Y(b) = \Pr(X > a, Y < b) + \Pr(X < a, Y < b) \tag{0.0.26}$$

$$\implies F_Y(b) - F_{X,Y}(a, b) = \Pr(X > a, Y < b) \tag{0.0.27}$$

Substituting in (0.0.25),

$$F_Y(b) - F_{X,Y}(a, b) = (1 - F_X(a))F_Y(b) \tag{0.0.28}$$

$$\implies F_{X,Y}(a, b) = F_X(a)F_Y(b) \tag{0.0.29}$$

for all $a, b \in R$. Thus, X and Y are independent.
Option 2 is correct

3) Two random variables X and Y are uncorrelated if their covariance is zero.

$$cov[X, Y] = E[XY] - E[X]E[Y] = 0 \tag{0.0.30}$$

Uncorrelatedness does not imply independence.
Counterexample: Let $X \sim U[-1, 1]$ be a uniformly distributed random variable.

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{0.0.31}$$

$$E[X] = \int_{-1}^{1} x f(x)\,dx = 0 \tag{0.0.32}$$

Let $Y = X^2$ be another random variable. X and Y are dependent.

$$cov[X, Y] = E[XY] - E[X]E[Y] \tag{0.0.33}$$

$$= E\left[X^3\right] - 0 \times E[Y] \tag{0.0.34}$$

$$= \int_{-1}^{1} x^3 f(x)\,dx = 0 \tag{0.0.35}$$

X and Y are uncorrelated but not independent.
Option 3 is incorrect

4) Given that,

$$E[(X - a)(Y - b)] = E[X - a]E[Y - b] \tag{0.0.36}$$

$$cov\left[(X-a),(Y-b)\right] = E\left[(X-a)(Y-b)\right] - E\left[X-a\right]E\left[Y-b\right] \quad (0.0.37)$$

$$\implies cov\left[(X-a)(Y-b)\right] = 0 = cov\left[X,Y\right] \quad (0.0.38)$$

From option 3, it follows that X and Y are not necessarily independent.
Option 4 is incorrect.