AI1103-Assignment 5

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Download all latex-tikz codes from

https://github.com/vaishnavi−w/AI1103/blob/main/Assignment5/latex5.tex

Question

Let X_1 and X_2 be i.i.d. with probability mass function $f_\theta(x) = \theta^x (1-\theta)^{1-x}$; $x = 0, 1$ where $\theta \in (0,1)$. Which of the following statements are true?

1) $X_1 + 2X_2$ is a sufficient statistic
2) $X_1 - X_2$ is a sufficient statistic
3) $X_1^2 + X_2^2$ is a sufficient statistic
4) $X_1^2 + X_2$ is a sufficient statistic

Solution

Given that, X_1 and X_2 are i.i.d. with probability mass function

$$f(x) = \begin{cases} (1-\theta) & x = 0 \\ \theta & x = 1 \end{cases} \tag{0.0.1}$$

A statistic $t = T(X)$ is sufficient for a parameter θ if the conditional probability distribution of the data, given the statistic $t = T(X)$ does not depend on the parameter θ. i.e,

$$P_\theta(X_1 = x_1, X_2 = x_2 | T = t) \tag{0.0.2}$$

is independent of θ for all x_1, x_2 and t

1) Let $T = X_1 + 2X_2$
Consider a case where $x_1 = 0, x_2 = 0$ and $t = 0$

$$\Pr(T = 0) = \Pr(X_1 + 2X_2 = 0) \tag{0.0.3}$$
$$= \Pr(X_1 = 0, X_2 = 0) \tag{0.0.4}$$

As X_1 and X_2 are independent

$$\Pr(T = 0) = \Pr(X_1 = 0)\Pr(X_2 = 0)$$
$$= (1-\theta)^2 \tag{0.0.5}$$

The conditional probability,

$$\Pr(X_1 = 0, X_2 = 0 | T = 0)$$
$$= \frac{\Pr((X_1 = 0, X_2 = 0) \cap (T = 0))}{\Pr(T = 0)} \tag{0.0.6}$$

From (0.0.4), $(X_1 = 0, X_2 = 0) \subseteq (T = 0)$

$$= \frac{\Pr(X_1 = 0, X_2 = 0)}{\Pr(T = 0)} = \frac{(1-\theta)^2}{(1-\theta)^2} = 1 \tag{0.0.7}$$

Similarly, conditional probabilities for other values of x_1, x_2 and t are given in table 1

x_1	x_2	t $t = X_1 + 2X_2$	Conditional probability $P_\theta(X_1 = x_1, X_2 = x_2 \mid T = t)$
0	0	0 otherwise	1 0
1	0	1 otherwise	1 0
0	1	2 otherwise	1 0
1	1	3 otherwise	1 0

TABLE 1: Conditional Probabilities

From table 1, all the conditional probabilities are independent of θ
$\therefore$ $X_1 + 2X_2$ is a sufficient statistic.

2) Let $T = X_1 - X_2$
Consider a case where $x_1 = 0, x_2 = 0$ and $t = 0$

$$\Pr(T = 0) = \Pr(X_1 - X_2 = 0)$$
$$= \Pr(X_1 = 0, X_2 = 0) + \Pr(X_1 = 1, X_2 = 1) \tag{0.0.8}$$

As X_1 and X_2 are independent

$$= \Pr(X_1 = 0)\Pr(X_2 = 0)$$
$$+ \Pr(X_1 = 1)\Pr(X_2 = 1) = (1-\theta)^2 + \theta^2 \tag{0.0.9}$$

The conditional probability,

$$\Pr(X_1 = 0, X_2 = 0|T = 0) = \frac{\Pr((X_1 = 0, X_2 = 0) \cap (T = 0))}{\Pr(T = 0)} \quad (0.0.10)$$

From (0.0.8), $(X_1 = 0, X_2 = 0) \subseteq (T = 0)$

$$= \frac{\Pr(X_1 = 0, X_2 = 0)}{\Pr(T = 0)} = \frac{(1-\theta)^2}{(1-\theta)^2 + \theta^2} \quad (0.0.11)$$

depends on θ.
$\therefore X_1 - X_2$ is not a sufficient statistic.

3) Let $T = X_1^2 + X_2^2$
Consider a case where $x_1 = 1, x_2 = 0$ and $t = 1$

$$\Pr(T = 1) = \Pr\left(X_1^2 + X_2^2 = 1\right)$$
$$= \Pr(X_1 = 1, X_2 = 0) + \Pr(X_1 = 0, X_2 = 1)$$
$$= \theta(1-\theta) + (1-\theta)\theta = 2\theta(1-\theta) \quad (0.0.12)$$

The conditional probability,

$$\Pr(X_1 = 1, X_2 = 0|T = 1) = \frac{\Pr((X_1 = 1, X_2 = 0) \cap (T = 0))}{\Pr(T = 1)} \quad (0.0.13)$$

From (0.0.12), $(X_1 = 1, X_2 = 0) \subseteq (T = 1)$

$$= \frac{\Pr(X_1 = 1, X_2 = 0)}{\Pr(T = 1)} = \frac{\theta(1-\theta)}{2\theta(1-\theta)} = \frac{1}{2} \quad (0.0.14)$$

Similarly, conditional probabilities for other values of x_1, x_2 and t are given in table 3

x_1	x_2	t $t = X_1^2 + X_2^2$	Conditional probability $P_\theta(X_1 = x_1, X_2 = x_2 \mid T = t)$
0	0	0 otherwise	1 0
1	0	1 otherwise	$\frac{1}{2}$ 0
0	1	1 otherwise	$\frac{1}{2}$ 0
1	1	2 otherwise	1 0

TABLE 3: Conditional Probabilities

From table 3, all the conditional probabilities are independent of θ
$\therefore X_1^2 + X_2^2$ is a sufficient statistic.

4) Let $T = X_1^2 + X_2$
Consider a case where $x_1 = 1, x_2 = 0$ and $t = 1$

$$\Pr(T = 1) = \Pr\left(X_1^2 + X_2 = 1\right)$$
$$= \Pr(X_1 = 1, X_2 = 0) + \Pr(X_1 = 0, X_2 = 1)$$
$$= \theta(1-\theta) + (1-\theta)\theta = 2\theta(1-\theta) \quad (0.0.15)$$

The conditional probability,

$$\Pr(X_1 = 1, X_2 = 0|T = 1) = \frac{\Pr((X_1 = 1, X_2 = 0) \cap (T = 0))}{\Pr(T = 1)} \quad (0.0.16)$$

From (0.0.15), $(X_1 = 1, X_2 = 0) \subseteq (T = 1)$

$$= \frac{\Pr(X_1 = 1, X_2 = 0)}{\Pr(T = 1)} = \frac{\theta(1-\theta)}{2\theta(1-\theta)} = \frac{1}{2} \quad (0.0.17)$$

Similarly, conditional probabilities for other values of x_1, x_2 and t are given in table 4

x_1	x_2	t $t = X_1^2 + X_2$	Conditional probability $P_\theta(X_1 = x_1, X_2 = x_2 \mid T = t)$
0	0	0 otherwise	1 0
1	0	1 otherwise	$\frac{1}{2}$ 0
0	1	1 otherwise	$\frac{1}{2}$ 0
1	1	2 otherwise	1 0

TABLE 4: Conditional Probabilities

From table 4, all the conditional probabilities are independent of θ
$\therefore X_1^2 + X_2$ is a sufficient statistic.

Answer : Options 1,3,4