AI1103-Assignment 3

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Download all python codes from

https://github.com/vaishnavi−w/AI1103/blob/main/Assignment3/code3.py

and latex-tikz codes from

https://github.com/vaishnavi−w/AI1103/blob/main/Assignment3/latex3.tex

Question

Probability density function $p(x)$ of random variable x is as shown below. The value of a is

A) $\frac{2}{c}$
B) $\frac{1}{c}$
C) $\frac{2}{(b+c)}$
D) $\frac{1}{(b+c)}$

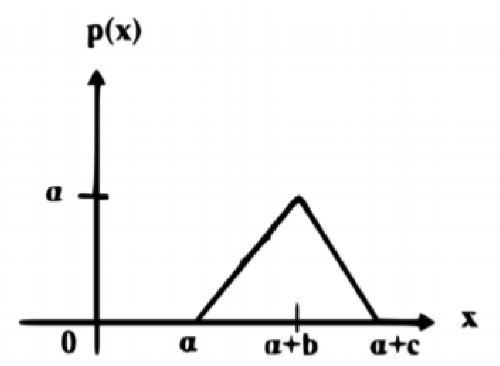

Fig. 4: PDF

Solution

Let Y_1 and Y_2 be two independent and identically distributed (IID) random variables.
Let X be a random variable such that

$$X = Y_1 + Y_2 \tag{0.0.1}$$

Let

$$p_{Y_1}(y) = \Pr(Y_1 = y) \tag{0.0.2}$$
$$p_{Y_2}(y) = \Pr(Y_2 = y) \tag{0.0.3}$$
$$p_X(x) = \Pr(X = x) \tag{0.0.4}$$

be the probability densities of random variables Y_1, Y_2 and X.
Y_1 and Y_2 lie in the range $(\frac{a}{2}, \frac{a+c}{2})$, therefore,

$$\int_{\frac{a}{2}}^{\frac{a+c}{2}} p_{Y_1}(y)\, dy = 1 \tag{0.0.5}$$
$$\frac{c}{2} \times p_{Y_1}(y) = 1 \tag{0.0.6}$$
$$p_{Y_1}(y) = \frac{2}{c} \tag{0.0.7}$$

The PDF for Y_1 and Y_2,

$$p_{Y_1}(y) = p_{Y_2}(y) = \begin{cases} \frac{2}{c} & \frac{a}{2} \le y \le \frac{a+c}{2} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.8}$$

The density of X is obtained by convolution of Y_1 and Y_2

$$p_X(x) = \int_{-\infty}^{\infty} p_{Y_1}(x-y) p_{Y_2}(y)\, dy \tag{0.0.9}$$

We have,

$$X = Y_1 + Y_2 \implies x = 2y \tag{0.0.10}$$
$$\frac{a}{2} \le y \le \frac{a+c}{2} \tag{0.0.11}$$
$$\frac{a}{2} \le x - y \le \frac{a+c}{2} \tag{0.0.12}$$
$$x - \frac{a+c}{2} \le y \le x - \frac{a}{2} \tag{0.0.13}$$

From (0.0.11) and (0.0.13)

$$\max\left(\frac{a}{2}, x - \frac{a+c}{2}\right) \le y \le \min\left(\frac{a+c}{2}, x - \frac{a}{2}\right) \tag{0.0.14}$$

When $a \le x \le \frac{a+c}{2}$

$$p_X(x) = \int_{\frac{a}{2}}^{x-\frac{a}{2}} p_{Y_1}(x-y)p_{Y_2}(y)\,dy \tag{0.0.15}$$

$$= \int_{\frac{a}{2}}^{x-\frac{a}{2}} \frac{2}{c} \times \frac{2}{c}\,dy \tag{0.0.16}$$

$$= \frac{4}{c^2}(x-a) \tag{0.0.17}$$

Similarly, when $\frac{a+c}{2} \le x \le a+c$

$$p_X(x) = \int_{x-\frac{a+c}{2}}^{\frac{a+c}{2}} p_{Y_1}(x-y)p_{Y_2}(y)\,dy \tag{0.0.18}$$

$$= \frac{4}{c^2}(a+c-x) \tag{0.0.19}$$

The PDF of X is,

$$p_x = \begin{cases} \frac{4}{c^2}(x-a) & a \le x \le a+\frac{c}{2} \\ \frac{4}{c^2}(a+c-x) & a+\frac{c}{2} \le x \le a+c \\ 0 & \text{otherwise} \end{cases} \tag{0.0.20}$$

Thus, the PDF of X is,

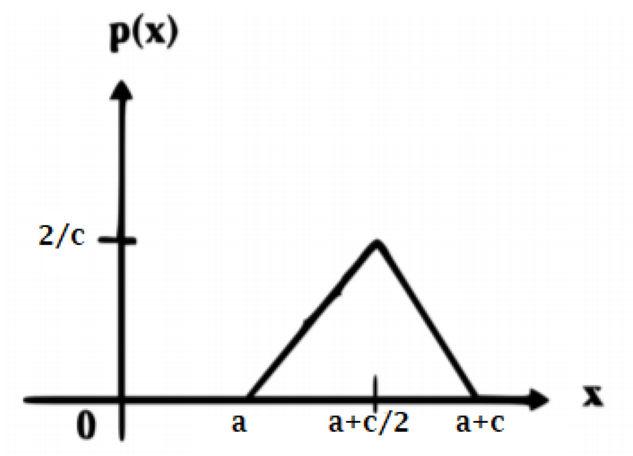

Fig. 4: PDF of X

The PDF of X from Figure.4 in the question,

$$p_x = \begin{cases} \frac{a}{b}(x-a) & a \le x \le a+b \\ \frac{a}{b-c}(a+c-x) & a+b \le x \le a+c \\ 0 & \text{otherwise} \end{cases} \tag{0.0.21}$$

On comparing the parameters, we have

$$b = \frac{c}{2} \tag{0.0.22}$$

$$a = \frac{2}{c} \tag{0.0.23}$$

Answer : Option A

Verification

Verifying the result obtained with numerical values:
Let $a = 1$.
$\because a = \frac{2}{c} \implies c = 2$

The PDF of X is,

$$p_x = \begin{cases} (x-1) & 1 \le x \le 2 \\ (3-x) & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.24}$$

CDF of X is defined as,

$$F_X(x) = \Pr(X \le x) \tag{0.0.25}$$

For $x \le 2$,

$$\Pr(X \le x) = \int_{-\infty}^{x} p_X(x)\,dx \tag{0.0.26}$$

$$= \int_{1}^{x} x - 1\,dx \tag{0.0.27}$$

$$= \frac{x^2 - 2x + 1}{2} \tag{0.0.28}$$

For $x \le 3$,

$$\Pr(X \le x) = \int_{-\infty}^{x} p_X(x)\,dx \tag{0.0.29}$$

$$= \frac{1}{2} + \int_{2}^{x} 3 - x\,dx \tag{0.0.30}$$

$$= \frac{6x - x^2 - 7}{2} \tag{0.0.31}$$

The CDF of X,

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2-2x+1}{2} & x \le 2 \\ \frac{6x-x^2-7}{2} & x \le 3 \\ 1 & x > 3 \end{cases} \tag{0.0.32}$$

The plots for CDF and PDF of X are given in Figure 4 and Figure 4

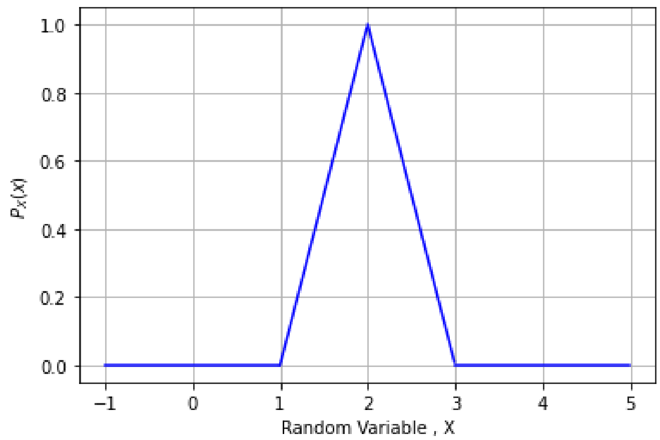

Fig. 4: PDF of X

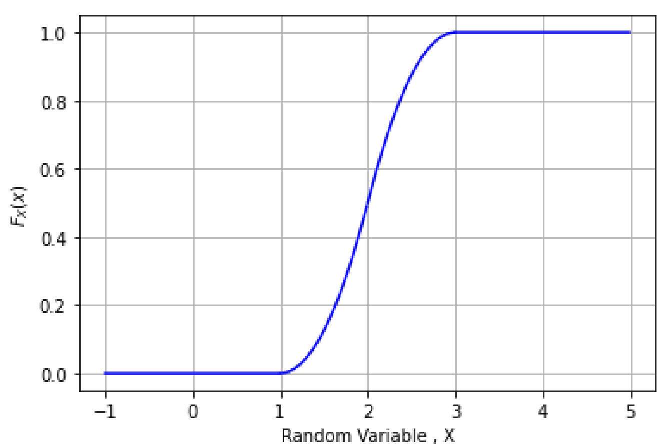

Fig. 4: CDF of X