

# AI1103-Assignment 6

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Download all latex-tikz codes from

<https://github.com/vaishnavi-w/AI1103/blob/main/Assignment6/latex6.tex>

## QUESTION

Which of the following conditions imply independence of random variables  $X$  and  $Y$ ?

- 1)  $\Pr(X > a | Y > a) = \Pr(X > a)$  for all  $a \in R$
- 2)  $\Pr(X > a | Y < b) = \Pr(X > a)$  for all  $a, b \in R$
- 3)  $X$  and  $Y$  are uncorrelated
- 4)  $E[(X - a)(Y - b)] = E[X - a]E[Y - b]$  for all  $a, b \in R$

## SOLUTION

- 1) Two random variables  $X$  and  $Y$  are independent when the joint probability distribution of random variables is product of their individual probability distributions i.e for all sets  $A, B$

$$\Pr(X \in A, Y \in B) = \Pr(X \in A) \Pr(Y \in B) \quad (0.0.1)$$

Consider the CDFs,

$$F_X(a) = \Pr(X < a) \quad (0.0.2)$$

$$F_Y(b) = \Pr(Y < b) \quad (0.0.3)$$

$$F_{X,Y}(a, b) = \Pr(X < a, Y < b) \quad (0.0.4)$$

Let  $F_{X,Y}(a, b) = F_X(a)F_Y(b)$  be true  
Partial derivative w.r.t  $a$ ,

$$\frac{\partial F_{X,Y}(a, b)}{\partial a} = \frac{\partial F_X(a)}{\partial a} F_Y(b) \quad (0.0.5)$$

Partial derivative w.r.t  $b$ ,

$$\frac{\partial^2 F_{X,Y}(a, b)}{\partial b \partial a} = \frac{\partial F_X(a)}{\partial a} \frac{\partial F_Y(b)}{\partial b} \quad (0.0.6)$$

$$\implies p_{X,Y}(a, b) = p_X(a) p_Y(b) \quad (0.0.7)$$

when  $X, Y$  are discrete. And,

$$\implies f_{X,Y}(a, b) = f_X(a) f_Y(b) \quad (0.0.8)$$

when  $X, Y$  are continuous, for all  $a, b \in R$ .

$\therefore$  Two random variables are independent if the joint CDF can be expressed as the product of individual CDFs i.e for all  $a, b \in R$

$$F_{X,Y}(a, b) = F_X(a) F_Y(b) \quad (0.0.9)$$

Consider,

$$\Pr(X > a | Y > a) = \frac{\Pr(X > a, Y > a)}{\Pr(Y > a)} \quad (0.0.10)$$

Given that  $\Pr(X > a | Y > a) = \Pr(X > a)$ ,

$$\implies \Pr(X > a, Y > a) = \Pr(X > a) \Pr(Y > a) \quad (0.0.11)$$

for all  $a \in R$ .

$$\begin{aligned} 1 - F_X(a) - F_Y(a) &= \Pr(X > a) - \Pr(Y < a) \\ &= \Pr(X > a, Y > a) + \Pr(X > a, Y < a) \\ &\quad - \Pr(X > a, Y < a) - \Pr(X < a, Y < a) \end{aligned} \quad (0.0.12)$$

$$\begin{aligned} 1 - F_X(a) - F_Y(a) &= \Pr(X > a, Y > a) \\ &\quad - F_{X,Y}(a, a) \end{aligned} \quad (0.0.13)$$

Substituting in,

$$\begin{aligned} 1 - F_X(a) - F_Y(a) + F_{X,Y}(a, a) &= \\ (1 - F_X(a))(1 - F_Y(a)) \end{aligned} \quad (0.0.14)$$

$$\implies F_{X,Y}(a, a) = F_X(a) F_Y(a) \quad (0.0.15)$$

On comparing with (0.0.9) in this case, it is true only under the condition that  $b = a$ . It may not be true for other values of  $b$ . The spectrum of conditions for independence is underrepresented.

Hence, the condition does not imply independence of  $X$  and  $Y$ .

**Counterexample:** Consider two random variables  $X, Y \in \{0, 1, 2\}$  with the probabilities of

the ordered pairs  $(X, Y)$  given in the Table1

$(X, Y)$	Pr	$(X, Y)$	Pr	$(X, Y)$	Pr
(0,0)	0.2	(1,0)	0.2	(2,0)	0.1
(0,1)	0.1	(1,1)	0.1	(2,1)	0.1
(0,2)	0.1	(1,2)	0.05	(2,2)	0.05

TABLE 1:  $\Pr(X, Y)$

Case 1:  $a < 0$

$$\Pr(X > a|Y > a) = 1 = \Pr(X > a) \quad (0.0.16)$$

Case 2:  $0 \leq a < 1$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} = \frac{0.3}{0.5} = 0.6 \quad (0.0.17)$$

$$\Pr(X > a) = \Pr(X = 1) + \Pr(X = 2) = 0.6 \quad (0.0.18)$$

Case 3:  $1 \leq a < 2$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} = \frac{0.05}{0.2} = 0.25 \quad (0.0.19)$$

$$\Pr(X > a) = \Pr(X = 2) = 0.25 \quad (0.0.20)$$

Case 4:  $a \geq 2$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} \quad (0.0.21)$$

is not defined as  $\Pr(Y > a) = 0$

In all the cases,  $\Pr(X > a|Y > a) = \Pr(X > a)$  is true.

Consider,

$$\Pr(X = 1, Y = 2) = 0.05 \quad (0.0.22)$$

$$\begin{aligned} \Pr(X = 1) \Pr(Y = 2) &= 0.35 \times 0.2 = 0.7 \\ &\neq \Pr(X = 1, Y = 2) \end{aligned} \quad (0.0.23)$$

Clearly,  $X$  and  $Y$  are not independent.

**Option 1 is incorrect**

2) From Bayes theorem,

$$\Pr(X > a|Y < b) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \quad (0.0.24)$$

Given that  $\Pr(X > a|Y < b) = \Pr(X > a)$ ,

$$\implies \Pr(X > a, Y < b) = \Pr(X > a) \Pr(Y < b) \quad (0.0.25)$$

for all  $a, b \in R$ . Consider

$$F_Y(b) = \Pr(X > a, Y < b) + \Pr(X < a, Y < b) \quad (0.0.26)$$

$$\implies F_Y(b) - F_{X,Y}(a, b) = \Pr(X > a, Y < b) \quad (0.0.27)$$

Substituting in,

$$F_Y(b) - F_{X,Y}(a, b) = (1 - F_X(a)) F_Y(b) \quad (0.0.28)$$

$$\implies F_{X,Y}(a, b) = F_X(a) F_Y(b) \quad (0.0.29)$$

for all  $a, b \in R$ . Thus,  $X$  and  $Y$  are independent.

**Option 2 is correct**

3) Two random variables  $X$  and  $Y$  are uncorrelated if their covariance is zero.

$$\text{cov}[X, Y] = E[XY] - E[X]E[Y] = 0 \quad (0.0.30)$$

Uncorrelatedness does not imply independence.

**Counterexample:** Let  $X \sim U[-1, 1]$  be a uniformly distributed random variable.

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.31)$$

$$E[X] = \int_{-1}^1 xf(x) dx = 0 \quad (0.0.32)$$

Let  $Y = X^2$  be another random variable.

$X$  and  $Y$  are dependent.

$$\text{cov}[X, Y] = E[XY] - E[X]E[Y] \quad (0.0.33)$$

$$= E[X^3] - 0 \times E[Y] \quad (0.0.34)$$

$$= \int_{-1}^1 x^3 f(x) dx = 0 \quad (0.0.35)$$

$X$  and  $Y$  are uncorrelated but not independent.

**Option 3 is incorrect**

4) Given that,

$$E[(X - a)(Y - b)] = E[X - a]E[Y - b] \quad (0.0.36)$$

$$\begin{aligned} \operatorname{cov}[(X - a), (Y - b)] &= E[(X - a)(Y - b)] \\ &\quad - E[X - a]E[Y - b] \quad (0.0.37) \end{aligned}$$

$$\Rightarrow \operatorname{cov}[(X - a)(Y - b)] = 0 = \operatorname{cov}[X, Y] \quad (0.0.38)$$

From option 3, it follows that  $X$  and  $Y$  are not necessarily independent.

**Option 4 is incorrect.**