AI1103-Assignment 3

W Vaishnavi
AI20BTECH11025

Download all python codes from

```
https://github.com/vaishnavi−w/AI1103/blob/main/
    Assignment3/code3.py
```

and latex-tikz codes from

```
https://github.com/vaishnavi−w/AI1103/blob/main/
    Assignment3/latex3.tex
```

Question

Probability density function $p(x)$ of random variable x is as shown below. The value of a is

A) $\frac{2}{c}$
B) $\frac{1}{c}$
C) $\frac{2}{(b+c)}$
D) $\frac{1}{(b+c)}$

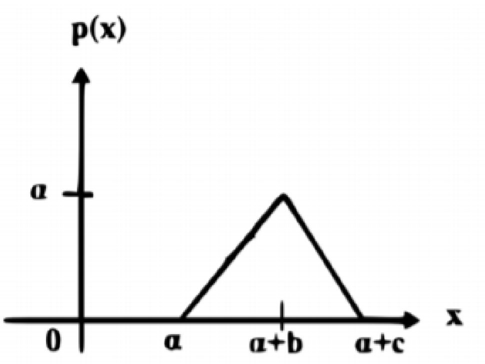

Fig. 4: PDF

Solution

Let Y_1 and Y_2 be two independent and identically distributed (IID) random variables.
Let X be a random variable such that

$$X = Y_1 + Y_2 \tag{0.0.1}$$

Let

$$p_{Y_1}(y) = \Pr(Y_1 = y) \tag{0.0.2}$$
$$p_{Y_2}(y) = \Pr(Y_2 = y) \tag{0.0.3}$$
$$p_X(x) = \Pr(X = x) \tag{0.0.4}$$

be the probability densities of random variables Y_1, Y_2 and X.
Y_1 and Y_2 lie in the range $\left(\frac{-c}{4}, \frac{c}{4}\right)$, therefore,

$$\int_{\frac{-c}{4}}^{\frac{c}{4}} p_{Y_1}(y)\, dy = 1 \tag{0.0.5}$$
$$\frac{c}{2} \times p_{Y_1}(y) = 1 \tag{0.0.6}$$
$$p_{Y_1}(y) = \frac{2}{c} \tag{0.0.7}$$

The PDF for Y_1 and Y_2,

$$p_{Y_1}(y) = p_{Y_2}(y) = \begin{cases} \frac{2}{c} & \frac{-c}{4} \le y \le \frac{c}{4} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.8}$$

The density of X is obtained by convolution of Y_1 and Y_2

$$p_X(x) = p_{Y_1}(x) * p_{Y_2}(x) \tag{0.0.9}$$
$$= \int_{-\infty}^{\infty} p_{Y_1}(x-y)\, p_{Y_2}(y)\, dy \tag{0.0.10}$$

where $*$ denotes the convolution operation. Since convolution operation is time invariant,

$$\begin{aligned} p_X(x-t) &= p_{Y_1}(x-t) * p_{Y_2}(x) \\ &= p_{Y_1}(x) * p_{Y_2}(x-t) \end{aligned} \tag{0.0.11}$$

$$p_X(x-t) = \int_{-\infty}^{\infty} p_{Y_1}(x-y-t)\, p_{Y_2}(y)\, dy \tag{0.0.12}$$

On time shifting Y_1 by shifting factor $t = a + \frac{c}{4}$,

$$p_X\left(x - \left(a + \frac{c}{4}\right)\right) = \int_{-\infty}^{\infty} p_{Y_1}\left(x - y - \left(a + \frac{c}{4}\right)\right) p_{Y_2}(y)\, dy \tag{0.0.13}$$

Let $p_X\left(x - \left(a + \frac{c}{4}\right)\right) = p'_X(x)$

We know that,

$$\frac{-c}{4} \leq y \leq \frac{c}{4} \quad (0.0.14)$$

$$\frac{-c}{4} \leq x - y - \left(a + \frac{c}{4}\right) \leq \frac{c}{4} \quad (0.0.15)$$

$$x - \left(a + \frac{3c}{4}\right) \leq y \leq x - \left(a + \frac{c}{4}\right) \quad (0.0.16)$$

When $a \leq x \leq a + \frac{c}{2}$

$$p'_X(x) = \int_{\frac{-c}{4}}^{x-a-\frac{c}{4}} p_{Y_1}(x - y - \left(a + \frac{c}{4}\right)) p_{Y_2}(y)\, dy \quad (0.0.17)$$

$$= \int_{\frac{-c}{4}}^{x-a-\frac{c}{4}} \frac{2}{c} \times \frac{2}{c}\, dy \quad (0.0.18)$$

$$= \frac{4}{c^2}(x - a) \quad (0.0.19)$$

Similarly, when $a + \frac{c}{2} \leq x \leq a + c$

$$p'_X(x) = \int_{x-a-\frac{3c}{4}}^{\frac{c}{4}} p_{Y_1}(x - y - \left(a + \frac{c}{4}\right)) p_{Y_2}(y)\, dy \quad (0.0.20)$$

$$= \frac{4}{c^2}(a + c - x) \quad (0.0.21)$$

The PDF of time shifted X is,

$$p'_x = \begin{cases} \frac{4}{c^2}(x-a) & a \leq x \leq a + \frac{c}{2} \\ \frac{4}{c^2}(a+c-x) & a + \frac{c}{2} \leq x \leq a + c \\ 0 & \text{otherwise} \end{cases} \quad (0.0.22)$$

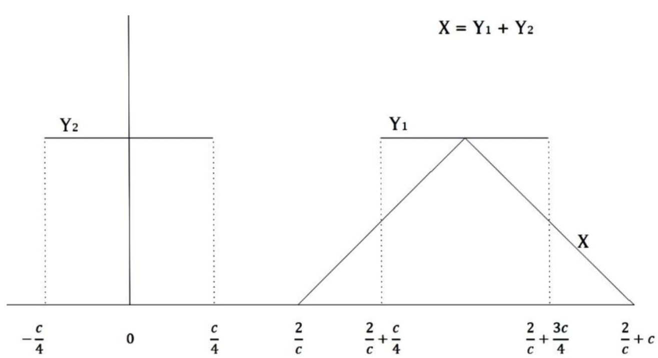

Fig. 4: PDF of time shifted X

On comparing the parameters of PDF of time shifted X with that in the question, we have

$$b = \frac{c}{2} \quad (0.0.23)$$

$$a = \frac{2}{c} \quad (0.0.24)$$

Answer : Option A

Verification

Verifying the result obtained with numerical values: Let $a = 1$.

$\because a = \frac{2}{c} \implies c = 2$

The PDF of X is,

$$p_x = \begin{cases} (x-1) & 1 \leq x \leq 2 \\ (3-x) & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.25)$$

CDF of X is defined as,

$$F_X(x) = \Pr(X \leq x) \quad (0.0.26)$$

The CDF of X,

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2-2x+1}{2} & x \leq 2 \\ \frac{6x-x^2-7}{2} & x \leq 3 \\ 1 & x > 3 \end{cases} \quad (0.0.27)$$

The plots for CDF and PDF of X are given in Figure 4 and Figure 4

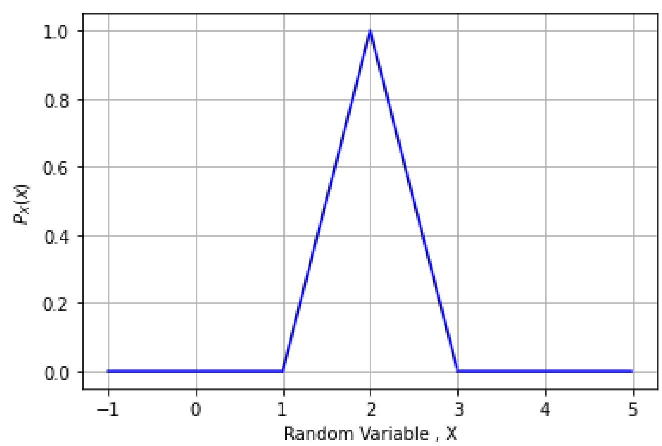

Fig. 4: PDF of X

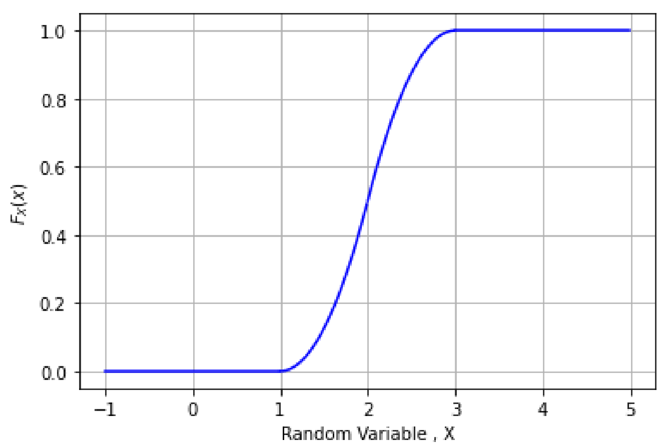

Fig. 4: CDF of X