

AI1103-Assignment 6

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Download all latex-tikz codes from

<https://github.com/vaishnavi-w/AI1103/blob/main/Assignment6/latex6.tex>

QUESTION

Which of the following conditions imply independence of random variables X and Y ?

- 1) $\Pr(X > a|Y > a) = \Pr(X > a)$ for all $a \in R$
- 2) $\Pr(X > a|Y < b) = \Pr(X > a)$ for all $a, b \in R$
- 3) X and Y are uncorrelated
- 4) $E[(X - a)(Y - b)] = E[X - a]E[Y - b]$ for all $a, b \in R$

SOLUTION

- 1) Two random variables X and Y are independent when the joint probability of random variables is product of their individual probabilities

$$\Pr(X \in A, Y \in B) = \Pr(X \in A) \Pr(Y \in B) \quad (0.0.1)$$

In other words conditional probability follows

$$\Pr(X \in A|Y \in B) = \Pr(X \in A) \quad (0.0.2)$$

for all sets A and B .

In the case of $\Pr(X > a|Y > a) = \Pr(X > a)$ the sets $A = B$. The spectrum of conditions for independence is underrepresented. Hence, the condition does not imply independence of X and Y .

Counterexample: Consider two random variables $X, Y \in \{0, 1, 2\}$ with the probabilities of the ordered pairs (X, Y) given in the Table 1

(X, Y)	Pr	(X, Y)	Pr	(X, Y)	Pr
(0,0)	0.2	(1,0)	0.2	(2,0)	0.1
(0,1)	0.1	(1,1)	0.1	(2,1)	0.1
(0,2)	0.1	(1,2)	0.05	(2,2)	0.05

TABLE 1: $\Pr(X, Y)$

Consider

$$\Pr(X > 1, Y > 1) = \Pr(X = 2, Y = 2) = 0.05 \quad (0.0.3)$$

$$\begin{aligned} \Pr(X > 1) \Pr(Y > 1) &= \Pr(X = 2) \Pr(Y = 2) \\ &= 0.25 \times 0.2 = 0.05 \quad (0.0.4) \end{aligned}$$

Similarly, we can find that $\Pr(X > a, Y > a) = \Pr(X > a) \Pr(Y > a)$ for all $a \in \{0, 1, 2\}$

Consider,

$$\Pr(X = 1, Y = 2) = 0.05 \quad (0.0.5)$$

$$\begin{aligned} \Pr(X = 1) \Pr(Y = 2) &= 0.35 \times 0.2 = 0.07 \\ &\neq \Pr(X = 1, Y = 2) \quad (0.0.6) \end{aligned}$$

Clearly, X and Y are not independent.

Option 1 is incorrect

- 2) From Bayes theorem,

$$\Pr(X > a|Y < b) = \frac{\Pr((X > a), (Y < b))}{\Pr(Y < b)} \quad (0.0.7)$$

Given that $\Pr(X > a|Y < b) = \Pr(X > a)$,

$$\frac{\Pr((X > a), (Y < b))}{\Pr(Y < b)} = \Pr(X > a) \quad (0.0.8)$$

$$\Pr((X > a), (Y < b)) = \Pr(X > a) \Pr(Y < b) \quad (0.0.9)$$

for all $a, b \in R$.

From (0.0.1), X and Y are independent.

Option 2 is correct

- 3) Two random variables X and Y are uncorrelated if their covariance is zero.

$$\text{cov}[X, Y] = E[XY] - E[X]E[Y] = 0 \quad (0.0.10)$$

Uncorrelatedness does not imply independence.

Counterexample: Let $X \sim U[-1, 1]$ be a uniformly distributed random variable.

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.11)$$

$$E[X] = \int_{-1}^1 xf(x) dx = 0 \quad (0.0.12)$$

Let $Y = X^2$ be another random variable.

X and Y are dependent.

$$\text{cov}[X, Y] = E[XY] - E[X]E[Y] \quad (0.0.13)$$

$$= E[X^3] - 0 \times E[Y] \quad (0.0.14)$$

$$= \int_{-1}^1 x^3 f(x) dx = 0 \quad (0.0.15)$$

X and Y are uncorrelated but not independent.

Option 3 is incorrect

4) Given that,

$$E[(X - a)(Y - b)] = E[X - a]E[Y - b] \quad (0.0.16)$$

$$\begin{aligned} \text{cov}[(X - a), (Y - b)] &= E[(X - a)(Y - b)] \\ &\quad - E[X - a]E[Y - b] \end{aligned} \quad (0.0.17)$$

$$\implies \text{cov}[(X - a)(Y - b)] = 0 \quad (0.0.18)$$

From option 3, it follows that X and Y are not necessarily independent.

Option 4 is incorrect.