AI1103-Assignment 5

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Download all latex-tikz codes from

https://github.com/vaishnavi−w/AI1103/blob/main/Assignment5/latex5.tex

Question

Let X_1 and X_2 be i.i.d. with probability mass function $f_\theta(x) = \theta^x (1-\theta)^{1-x}$; $x = 0, 1$ where $\theta \in (0, 1)$. Which of the following statements are true?

1) $X_1 + 2X_2$ is a sufficient statistic
2) $X_1 - X_2$ is a sufficient statistic
3) $X_1^2 + X_2^2$ is a sufficient statistic
4) $X_1^2 + X_2$ is a sufficient statistic

Solution

A statistic $t = T(X)$ is sufficient for a parameter θ if the conditional probability distribution of the data, given the statistic $t = T(X)$ does not depend on the parameter θ. i.e,

$$P_\theta(X_1 = x_1, X_2 = x_2 | T = t) \tag{0.0.1}$$

is independent of θ for all x_1, x_2 and t

1) Let $T = X_1 + 2X_2$

$$\Pr(X_1 = 0, X_2 = 0 | T = t) = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

$$\Pr(X_1 = 1, X_2 = 0 | T = t) = \begin{cases} 1 & t = 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.3}$$

$$\Pr(X_1 = 0, X_2 = 1 | T = t) = \begin{cases} 1 & t = 2 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.4}$$

$$\Pr(X_1 = 1, X_2 = 1 | T = t) = \begin{cases} 1 & t = 3 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.5}$$

$\therefore$ All the conditional probabilities are independent of θ, $X_1 + 2X_2$ is a sufficient statistic.

2) Let $T = X_1 - X_2$

$$\Pr(T = 0) = \Pr(X_1 = 0, X_2 = 0) + \Pr(X_1 = 1, X_2 = 1) \tag{0.0.6}$$

As X_1 and X_2 are independent

$$= \Pr(X_1 = 0)\Pr(X_2 = 0) + \Pr(X_1 = 1)\Pr(X_2 = 1) \tag{0.0.7}$$

$$= (1-\theta)^2 + \theta^2 \tag{0.0.8}$$

Consider,

$$\Pr(X_1 = 0, X_2 = 0 | T = 0) = \frac{\Pr(X_1 = 0, X_2 = 0)}{\Pr(T = 0)} = \frac{(1-\theta)^2}{(1-\theta)^2 + \theta^2} \tag{0.0.9}$$

depends on θ.
$\therefore X_1 - X_2$ is not a sufficient statistic.

3) Let $T = X_1^2 + X_2^2$

$$\Pr(X_1 = 0, X_2 = 0 | T = t) = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.10}$$

$$\Pr(X_1 = 1, X_2 = 0 | T = t) = \begin{cases} \frac{1}{2} & t = 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.11}$$

$$\Pr(X_1 = 0, X_2 = 1 | T = t) = \begin{cases} \frac{1}{2} & t = 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.12}$$

$$\Pr(X_1 = 1, X_2 = 1 | T = t) = \begin{cases} 1 & t = 2 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.13}$$

$\therefore$ All the conditional probabilities are independent of θ, $X_1^2 + X_2^2$ is a sufficient statistic.

4) Let $T = X_1^2 + X_2$

$$\Pr\left(X_1 = 0, X_2 = 0 | T = t\right) = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.14}$$

$$\Pr\left(X_1 = 1, X_2 = 0 | T = t\right) = \begin{cases} \frac{1}{2} & t = 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.15}$$

$$\Pr\left(X_1 = 0, X_2 = 1 | T = t\right) = \begin{cases} \frac{1}{2} & t = 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.16}$$

$$\Pr\left(X_1 = 1, X_2 = 1 | T = t\right) = \begin{cases} 1 & t = 2 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.17}$$

$\therefore$ All the conditional probabilities are independent of θ, $X_1^2 + X_2$ is a sufficient statistic.

Answer : Options 1,3,4