CSIR UGC NET EXAM (June 2012), Q.104

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Independence of Random Variables

Two random variables X and Y are independent when the joint probability distribution of random variables is product of their individual probability distributions i.e for all sets A,B

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B) \tag{1}$$

Independence of Random Variables

Consider the CDFs,

$$F_X(a) = \Pr(X < a) \tag{2}$$

$$F_Y(b) = \Pr(Y < b) \tag{3}$$

$$F_{X,Y}(a, b) = \Pr(X < a, Y < b) \tag{4}$$

Let $F_{X,Y}(a, b) = F_X(a) F_Y(b)$ be true.
Partial derivative w.r.t a and then w.r.t b,

$$\frac{\partial^2 F_{X,Y}(a, b)}{\partial b \partial a} = \frac{\partial F_X(a)}{\partial a} \frac{\partial F_Y(b)}{\partial b} \tag{5}$$

$$\implies p_{X,Y}(a, b) = p_X(a) p_Y(b) \tag{6}$$

when X,Y are discrete. And,

$$\implies f_{X,Y}(a, b) = f_X(a) f_Y(b) \tag{7}$$

when X,Y are continuous, for all $a, b \in R$.

Independence of Random Variables

Two random variables are independent if the joint CDF can be expressed as the product of individual CDFs i.e for all $a, b \in R$

$$F_{X,Y}(a, b) = F_X(a) F_Y(b) \tag{8}$$

Question

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Which of the following conditions imply independence of random variables X and Y?

1. $\Pr(X > a | Y > a) = \Pr(X > a)$ for all $a \in R$
2. $\Pr(X > a | Y < b) = \Pr(X > a)$ for all $a, b \in R$
3. X and Y are uncorrelated
4. $E[(X - a)(Y - b)] = E[X - a] E[Y - b]$ for all $a, b \in R$

Option 1

Consider,

$$\Pr(X > a | Y > a) = \frac{\Pr(X > a, Y > a)}{\Pr(Y > a)} \tag{9}$$

Given that $\Pr(X > a | Y > a) = \Pr(X > a)$,

$$\implies \Pr(X > a, Y > a) = \Pr(X > a)\Pr(Y > a) \tag{10}$$

for all $a \in R$

Solution contd.

$$\begin{aligned}1 - F_X(a) - F_Y(a) &= \Pr(X > a) - \Pr(Y < a)\\ &= \Pr(X > a, Y > a) + \Pr(X > a, Y < a) - \Pr(X > a, Y < a)\\ &\quad - \Pr(X < a, Y < a) \quad (11)\end{aligned}$$

$$1 - F_X(a) - F_Y(a) = \Pr(X > a, Y > a) - F_{X,Y}(a, a) \quad (12)$$

Substituting in (10),

$$1 - F_X(a) - F_Y(a) + F_{X,Y}(a, a) = (1 - F_X(a))(1 - F_Y(a)) \quad (13)$$

$$\implies F_{X,Y}(a, a) = F_X(a) F_Y(a) \quad (14)$$

for all $a \in R$

Solution contd.

On comparing with (8) in this case, it is true only under the condition that $b = a$. It may not be true for other values of b. The spectrum of conditions for independence is underrepresented. Hence, the condition does not imply independence of X and Y.

Counterexample: Consider two random variables $X, Y \in \{0, 1, 2\}$ with the probabilities of the ordered pairs (X, Y) given in the Table1

$X \backslash Y$	0	1	2
0	0.2	0.1	0.1
1	0.2	0.1	0.05
2	0.1	0.1	0.05

Table: $\Pr(X, Y)$

Solution contd.

In all the cases, $\Pr(X > a | Y > a) = \Pr(X > a)$ is true.
Consider,

$$\Pr(X = 1, Y = 2) = 0.05 \tag{15}$$

$$\Pr(X = 1)\Pr(Y = 2) = 0.35 \times 0.2 = 0.7 \neq \Pr(X = 1, Y = 2) \tag{16}$$

Clearly, X and Y are not independent.

Option 2

$$\Pr\left(X > a | Y < b\right) = \frac{\Pr\left(X > a, Y < b\right)}{\Pr\left(Y < b\right)} \tag{17}$$

Given that $\Pr\left(X > a | Y < b\right) = \Pr\left(X > a\right)$,

$$\implies \Pr\left(X > a, Y < b\right) = \Pr\left(X > a\right) \Pr\left(Y < b\right) \tag{18}$$

for all $a, b \in R$. Consider

$$F_Y\left(b\right) = \Pr\left(X > a, Y < b\right) + \Pr\left(X < a, Y < b\right) \tag{19}$$

$$\implies F_Y\left(b\right) - F_{X,Y}\left(a, b\right) = \Pr\left(X > a, Y < b\right) \tag{20}$$

Substituting in (18),

$$F_Y\left(b\right) - F_{X,Y}\left(a, b\right) = \left(1 - F_X\left(a\right)\right) F_Y\left(b\right) \tag{21}$$

$$\implies F_{X,Y}\left(a, b\right) = F_X\left(a\right) F_Y\left(b\right) \tag{22}$$

for all $a, b \in R$. Thus, X and Y are independent.

Option 3

Uncorrelatedness

Two random variables X and Y are uncorrelated if their covariance is zero.

$$cov[X, Y] = E[XY] - E[X]E[Y] = 0 \tag{23}$$

Uncorrelatedness does not imply independence.

Solution contd.

Counterexample: Let $X \sim U[-1, 1]$ be a uniformly distributed random variable.

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{24}$$

$$E[X] = \int_{-1}^{1} xf(x)\, dx = 0 \tag{25}$$

Let $Y = X^2$ be another random variable. X and Y are dependent.

$$cov[X, Y] = E[XY] - E[X]E[Y] \tag{26}$$

$$= E\left[X^3\right] - 0 \times E[Y] \tag{27}$$

$$= \int_{-1}^{1} x^3 f(x)\, dx = 0 \tag{28}$$

X and Y are uncorrelated but not independent.

Option 4

Given that,

$$E[(X-a)(Y-b)] = E[X-a]\,E[Y-b] \tag{29}$$
$$E[XY - aY - bX + ab] = (E[X]-a)(E[Y]-b) \tag{30}$$
$$E[XY] - aE[Y] - bE[X] + ab = (E[X]-a)(E[Y]-b) \tag{31}$$
$$\implies E[XY] = E[X]\,E[Y] \tag{32}$$

$$cov[X,Y] = E[XY] - E[X]\,E[Y] = 0 \tag{33}$$

From option 3, it follows that X and Y are not necessarily independent.