# AI1103-Assignment 6

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Download all latex-tikz codes from

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https://github.com/vaishnavi−w/AI1103/blob/main/
    Assignment6/latex6.tex
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## Question

Which of the following conditions imply independence of random variables $X$ and $Y$?

1) $\Pr(X > a|Y > a) = \Pr(X > a)$ for all $a \in R$
2) $\Pr(X > a|Y < b) = \Pr(X > a)$ for all $a, b \in R$
3) $X$ and $Y$ are uncorrelated
4) $E[(X - a)(Y - b)] = E[X - a] E[Y - b]$ for all $a, b \in R$

## Solution

1) Two random variables $X$ and $Y$ are independent when

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B) \tag{0.0.1}$$

In other words conditional probability follows

$$\Pr(X \in A|Y \in B) = \Pr(X \in A) \tag{0.0.2}$$

for all sets $A$ and $B$.
In the case of $\Pr(X > a|Y > a) = \Pr(X > a)$ the sets $A = B$. The spectrum of conditions for independence is underrepresented. Hence, the condition does not imply independence of $X$ and $Y$.
**Option 1 is incorrect**

2) From Bayes theorem,

$$\Pr(X > a|Y < b) = \frac{\Pr((X > a), (Y < b))}{\Pr(Y < b)} \tag{0.0.3}$$

Given that $\Pr(X > a|Y < b) = \Pr(X > a)$,

$$\frac{\Pr((X > a), (Y < b))}{\Pr(Y < b)} = \Pr(X > a) \tag{0.0.4}$$

$$\Pr((X > a), (Y < b)) = \Pr(X > a)\Pr(Y < b) \tag{0.0.5}$$

for all $a, b \in R$.
As the joint probability of random variables is product of their individual probabilities, $X$ and $Y$ are independent.
**Option 2 is correct**

3) Two random variables $X$ and $Y$ are uncorrelated if their covariance is zero.

$$cov[X, Y] = E[XY] - E[X] E[Y] = 0 \tag{0.0.6}$$

Uncorrelatedness does not imply independence.
Counterexample: Let $X \sim U[-1, 1]$ be a uniformly distributed random variable.

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{0.0.7}$$

$$E[X] = \int_{-1}^{1} x f(x)\, dx = 0 \tag{0.0.8}$$

Let $Y = X^2$ be another random variable.
$X$ and $Y$ are dependent.

$$cov[X, Y] = E[XY] - E[X] E[Y] \tag{0.0.9}$$
$$= E\left[X^3\right] - 0 \times E[Y] \tag{0.0.10}$$
$$= \int_{-1}^{1} x^3 f(x)\, dx = 0 \tag{0.0.11}$$

$X$ and $Y$ are uncorrelated but not independent.
**Option 3 is incorrect**

4) Given that,

$$E[(X - a)(Y - b)] = E[X - a] E[Y - b] \tag{0.0.12}$$

$$cov[(X - a), (Y - b)] = E[(X - a)(Y - b)] - E[X - a] E[Y - b] \tag{0.0.13}$$

$$\implies cov[(X - a)(Y - b)] = 0 \tag{0.0.14}$$

From option 3, it follows that $X$ and $Y$ are not necessarily independent.
**Option 4 is incorrect.**