AI1103-Assignment 6

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Download all latex-tikz codes from

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https://github.com/vaishnavi−w/AI1103/blob/main/
    Assignment6/latex6.tex
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Question

Which of the following conditions imply independence of random variables X and Y?

1) $\Pr(X > a|Y > a) = \Pr(X > a)$ for all $a \in R$
2) $\Pr(X > a|Y < b) = \Pr(X > a)$ for all $a, b \in R$
3) X and Y are uncorrelated
4) $E[(X - a)(Y - b)] = E[X - a]E[Y - b]$ for all $a, b \in R$

Solution

1) Two random variables X and Y are independent when the joint probability distribution of random variables is product of their individual probability distributions i.e for all sets A,B

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B) \tag{0.0.1}$$

Consider the CDFs,

$$F_X(a) = \Pr(X < a) \tag{0.0.2}$$
$$F_Y(b) = \Pr(Y < b) \tag{0.0.3}$$
$$F_{X,Y}(a, b) = \Pr(X < a, Y < b) \tag{0.0.4}$$

Let $F_{X,Y}(a, b) = F_X(a)F_Y(b)$ be true. Partial derivative w.r.t a,

$$\frac{\partial F_{X,Y}(a, b)}{\partial a} = \frac{\partial F_X(a)}{\partial a}F_Y(b) \tag{0.0.5}$$

Partial derivative w.r.t b,

$$\frac{\partial^2 F_{X,Y}(a, b)}{\partial b \partial a} = \frac{\partial F_X(a)}{\partial a}\frac{\partial F_Y(b)}{\partial b} \tag{0.0.6}$$
$$\implies p_{X,Y}(a, b) = p_X(a)p_Y(b) \tag{0.0.7}$$

when X,Y are discrete. And,

$$\implies f_{X,Y}(a, b) = f_X(a)f_Y(b) \tag{0.0.8}$$

when X,Y are continuous, for all $a, b \in R$.
$\therefore$ Two random variables are independent if the joint CDF can be expressed as the product of individual CDFs i.e for all $a, b \in R$

$$F_{X,Y}(a, b) = F_X(a)F_Y(b) \tag{0.0.9}$$

Consider,

$$\Pr(X > a|Y > a) = \frac{\Pr(X > a, Y > a)}{\Pr(Y > a)} \tag{0.0.10}$$

Given that $\Pr(X > a|Y > a) = \Pr(X > a)$,

$$\implies \Pr(X > a, Y > a) = \Pr(X > a)\Pr(Y > a) \tag{0.0.11}$$

for all $a \in R$.

$$\begin{aligned} 1 - F_X(a) - F_Y(a) &= \Pr(X > a) - \Pr(Y < a) \\ &= \Pr(X > a, Y > a) + \Pr(X > a, Y < a) \\ &\quad - \Pr(X > a, Y < a) - \Pr(X < a, Y < a) \end{aligned} \tag{0.0.12}$$

$$\begin{aligned} 1 - F_X(a) - F_Y(a) = \Pr(X > a, Y > a) \\ - F_{X,Y}(a, a) \end{aligned} \tag{0.0.13}$$

Substituting in (0.0.11),

$$\begin{aligned} 1 - F_X(a) - F_Y(a) + F_{X,Y}(a, a) = \\ (1 - F_X(a))(1 - F_Y(a)) \end{aligned} \tag{0.0.14}$$

$$\implies F_{X,Y}(a, a) = F_X(a)F_Y(a) \tag{0.0.15}$$

On comparing with (0.0.9) in this case, it is true only under the condition that $b = a$. It may not be true for other values of b. The spectrum of conditions for independence is underrepresented. Hence, the condition does not imply independence of X and Y.

Counterexample: Consider two random variables $X, Y \in \{0, 1, 2\}$ with the probabilities of the ordered pairs (X, Y) given in the Table1

(X,Y)	Pr	(X,Y)	Pr	(X,Y)	Pr
(0,0)	0.2	(1,0)	0.2	(2,0)	0.1
(0,1)	0.1	(1,1)	0.1	(2,1)	0.1
(0,2)	0.1	(1,2)	0.05	(2,2)	0.05

TABLE 1: $\Pr(X,Y)$

Case 1: $a < 0$

$$\Pr(X > a|Y > a) = 1 = \Pr(X > a) \tag{0.0.16}$$

Case 2: $0 \le a < 1$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} = \frac{0.3}{0.5} = 0.6 \tag{0.0.17}$$

$$\Pr(X > a) = \Pr(X = 1) + \Pr(X = 2) = 0.6 \tag{0.0.18}$$

Case 3: $1 \le a < 2$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} = \frac{0.05}{0.2} = 0.25 \tag{0.0.19}$$

$$\Pr(X > a) = \Pr(X = 2) = 0.25 \tag{0.0.20}$$

Case 4: $a \ge 2$

$$\Pr(X > a|Y > a) = \frac{\Pr(X, Y > a)}{\Pr(Y > a)} \tag{0.0.21}$$

is not defined as $\Pr(Y > a) = 0$. In all the cases, $\Pr(X > a|Y > a) = \Pr(X > a)$ is true.
Consider,

$$\Pr(X = 1, Y = 2) = 0.05 \tag{0.0.22}$$

$$\Pr(X = 1)\Pr(Y = 2) = 0.35 \times 0.2 = 0.7 \neq \Pr(X = 1, Y = 2) \tag{0.0.23}$$

Clearly, X and Y are not independent.
Option 1 is incorrect

2) From Bayes theorem,

$$\Pr(X > a|Y < b) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \tag{0.0.24}$$

Given that $\Pr(X > a|Y < b) = \Pr(X > a)$,

$$\implies \Pr(X > a, Y < b) = \Pr(X > a)\Pr(Y < b) \tag{0.0.25}$$

for all $a, b \in R$. Consider

$$F_Y(b) = \Pr(X > a, Y < b) + \Pr(X < a, Y < b) \tag{0.0.26}$$

$$\implies F_Y(b) - F_{X,Y}(a, b) = \Pr(X > a, Y < b) \tag{0.0.27}$$

Substituting in (0.0.25),

$$F_Y(b) - F_{X,Y}(a, b) = (1 - F_X(a))F_Y(b) \tag{0.0.28}$$

$$\implies F_{X,Y}(a, b) = F_X(a)F_Y(b) \tag{0.0.29}$$

for all $a, b \in R$. Thus, X and Y are independent.
Option 2 is correct

3) Two random variables X and Y are uncorrelated if their covariance is zero.

$$cov[X, Y] = E[XY] - E[X]E[Y] = 0 \tag{0.0.30}$$

Uncorrelatedness does not imply independence.
Counterexample: Let $X \sim U[-1, 1]$ be a uniformly distributed random variable.

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{0.0.31}$$

$$E[X] = \int_{-1}^{1} xf(x)\,dx = 0 \tag{0.0.32}$$

Let $Y = X^2$ be another random variable. X and Y are dependent.

$$cov[X, Y] = E[XY] - E[X]E[Y] \tag{0.0.33}$$
$$= E\left[X^3\right] - 0 \times E[Y] \tag{0.0.34}$$
$$= \int_{-1}^{1} x^3 f(x)\,dx = 0 \tag{0.0.35}$$

X and Y are uncorrelated but not independent.
Option 3 is incorrect

4) Given that,

$$E[(X - a)(Y - b)] = E[X - a]E[Y - b] \tag{0.0.36}$$

$$cov[(X - a), (Y - b)] = E[(X - a)(Y - b)] - E[X - a]E[Y - b] \tag{0.0.37}$$

$$\implies cov\left[(X-a)(Y-b)\right] = 0 = cov\left[X, Y\right] \tag{0.0.38}$$

From option 3, it follows that X and Y are not necessarily independent.
Option 4 is incorrect.