Model Free Prediction : Additional Topics

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Overview

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Review

Multi-step TD

- One-step TD

$$\begin{aligned} V^\pi(s) &= \mathbb{E}_\pi \left[r_{t+1} + \gamma V^\pi(s_{t+1}) | s_t = s \right] \\ V(s_t) &\leftarrow V(s_t) + \alpha_t [r_{t+1} + \gamma V(s_{t+1}) - V(s_t)] \end{aligned}$$

- Two-step TD

$$\begin{aligned} V^\pi(s) &= \mathbb{E}_\pi \left[r_{t+1} + \gamma r_{t+2} + \gamma^2 V^\pi(s_{t+2}) | s_t = s \right] \\ V(s_t) &\leftarrow V(s_t) + \alpha_t [r_{t+1} + \gamma r_{t+2} + \gamma^2 V(s_{t+2}) - V(s_t)] \end{aligned}$$

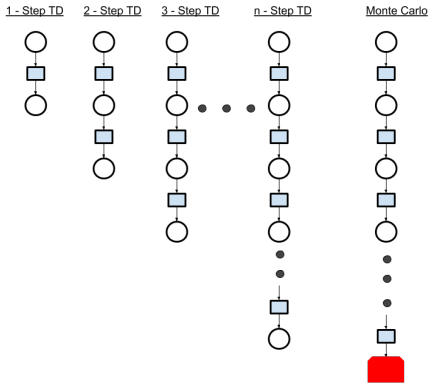
- More generally, define the n-step return

$$G_t^{(n)} \stackrel{\text{def}}{=} r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^{n-1} r_{t+n} + \gamma^n V(s_{t+n})$$

- n-step TD

$$V(s_t) \leftarrow V(s_t) + \alpha_t [G_t^{(n)} - V(s_t)]$$

Why Multi-Step TD ?



- Multi-step TD methods tend to have better bias-variance tradeoff

 Figure Source: Sutton and Barto

λ-Return

- What if we average some or all the n-step returns?

Example : Target could be

$$\frac{1}{2}G_t^{(1)} + \frac{1}{2}G_t^{(4)}$$

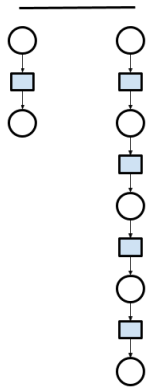


 Figure Source: Sutton and Barto

λ-Return

- Any set of returns can be averaged, even an infinite set, as long as the weights on the component returns are positive and sum to 1
- Choose $\lambda \in [0, 1]$, and define the λ-return at time t as

$$G_t^\lambda \stackrel{\text{def}}{=} (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

- λ-return algorithm: use λ-return as the target

$$V(s_t) \leftarrow V(s_t) + \alpha_t [G_t^\lambda - V(s_t)]$$

- If the episode ends at $T > t$, define $G_t^{(n)}$ to be G_t for all $n > T - t$. Then

$$G_t^\lambda = (1-\lambda) \sum_{n=1}^{T-t} \lambda^{n-1} G_t^{(n)} + \lambda^{T-t} G_t$$

- $\lambda = 0$ gives 1-step TD, $\lambda = 1$ gives MC update

λ-Return

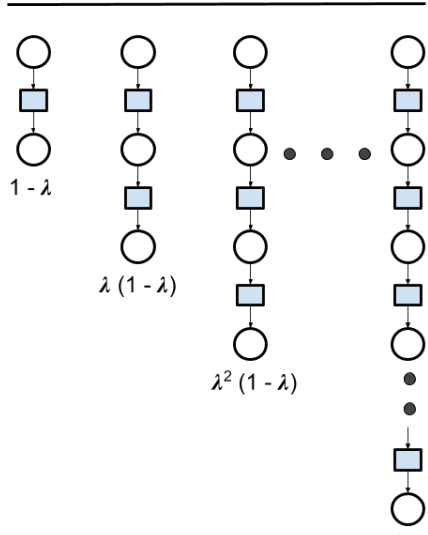
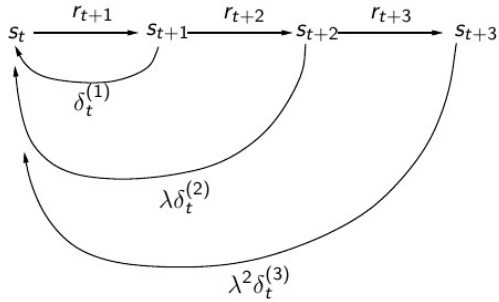


Figure Source: Sutton and Barto

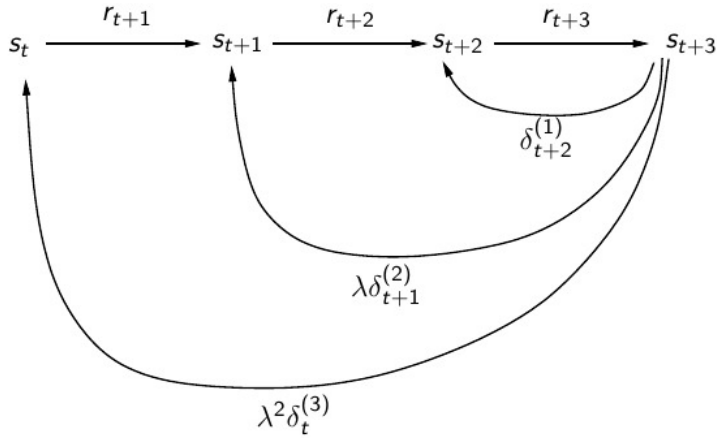
Forward View

$$G_t^\lambda - V(s_t) = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}[G_t^{(n)} - V(s_t)] = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}\delta_t^{(n)}$$



- Not suitable for online implementation;

A Possible Online Implementation



- Requires storing all rewards from the episode

A Rearrangement of n-step TD Errors

$$
\begin{aligned}
\delta_t^{(n)} &\stackrel{\text{def}}{=} r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^{n-1} r_{t+n} + \gamma^n V(s_{t+n}) - V(s_t) \\
&= \gamma^0 [r_{t+1} + \gamma V(s_{t+1}) - V(s_t)] \\
&\quad + \gamma^1 [r_{t+2} + \gamma V(s_{t+2}) - V(s_{t+1})] \\
&\vdots \\
&\quad + \gamma^{n-1} [r_{t+n} + \gamma V(s_{t+n}) - V(s_{t+n-1})] \\
&= \sum_{i=t}^{t+n-1} \gamma^{i-t} \delta_i^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
G_t^\lambda - V(s_t) &= (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} \delta_t^{(n)} = (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} \sum_{i=t}^{t+n-1} \gamma^{i-t} \delta_i^{(1)} \\
&= \sum_{i=0}^{\infty} (\lambda\gamma)^i \delta_{t+i}^{(1)}
\end{aligned}
$$

Online Implementation

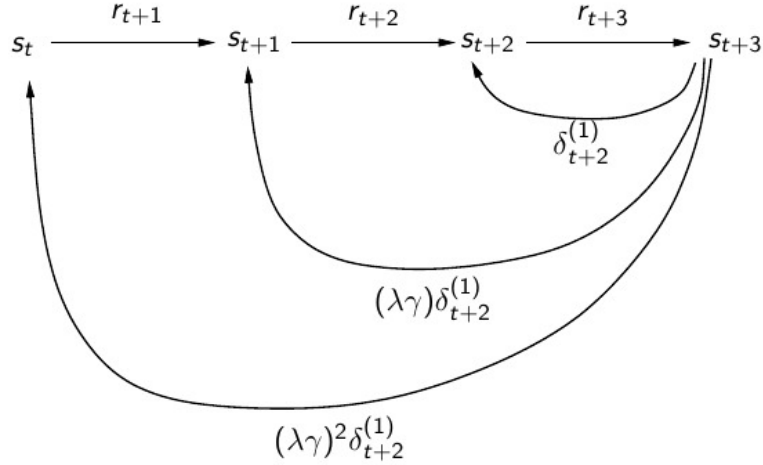


Figure shows shortcuts to previously encountered states

TD(λ) Algorithm

Eligibility Traces

- The eligibility trace of a state $s \in \mathcal{S}$ at time t is defined recursively by

$$
\begin{aligned}
e_0(s) &= 0 \\
e_t(s) &= \begin{cases} (\lambda\gamma)e_{t-1}(s), & s_t \neq s \\ (\lambda\gamma)e_{t-1}(s) + 1, & s_t = s \end{cases}
\end{aligned}
$$

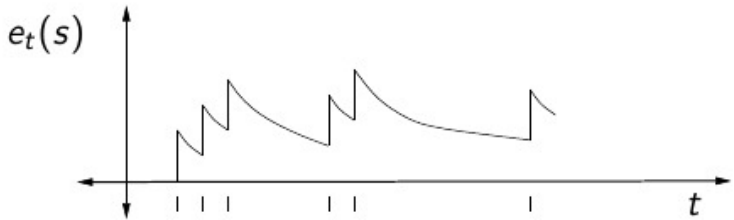


 Figure Source: Sutton and Barto

Algorithm : TD(λ)

Algorithm TD(λ) : Algorithm

Initialize $e(s) = 0$ for all s, $V(s)$ arbitrarily
for For each episode **do**
    Let s be a start state for episode k
    for For each step of the episode **do**
        Take action a recommended by policy π from state s
        Collect reward r and reach next state s'
        Form the one-step TD error $\delta \leftarrow r + \gamma V(s') - V(s)$
        Increment eligibility trace of state s, $e(s) \leftarrow e(s) + 1$
        for For all states $S \in \mathcal{S}$ **do**
            Update $V(S)$: $V(S) \leftarrow V(S) + \alpha e(S)\delta$
            Update eligibility trace: $e(S) \leftarrow \lambda\gamma e(S)$
        end for
        Move to next state: $s \leftarrow s'$
    end for
end for

Certainity Equivalence Estimate

Certainity Equivalence Estimate

- Model based policy evaluation with model estimted from samples
- Given an experience quadruple (s, a, r, s'), we can estimate
  - ★ Compute MLE for model using (s, a, s')

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

  - ★ Compute MLE estimate of the reward function

$$\hat{R}(s,a,s') = \frac{1}{N(s,a,s')} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')\ r_{t,k}$$

- Once MLE estimates of $\hat{P}(s'|s,a)$ and $\hat{R}(s,a,s')$ use a known policy evaluation method to compute a MLE based estimate for V^{π}.

Off Policy Learning

Off-Policy Learning

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- Can we estimate the value V^π or Q^π of a *target policy* π ...
- While following a *behavior policy* μ?
  - ★ Trajectories or transitions are sampled from μ
  - ★ Expected values have to be estimated w.r.t. π
- Possible benefits:
  - ★ Learn by observing other agents
  - ★ Re-use previous experience generated from earlier policies
  - ★ Learn about optimal policy while following exploratory policy
  - ★ Learn about multiple policies while following one policy

Importance Sampling : Review

Let $P(x)$ be the target distribution and $Q(x)$ be the behaviour distribution for some random variable x

$$\begin{aligned}\mathbb{E}_{X\sim P}[f(X)] &= \sum_i P(x_i)f(x_i) \\ &= \sum_i Q(x_i)\left[\frac{P(x_i)}{Q(x_i)}f(x_i)\right] \\ &= \mathbb{E}_{X\sim Q}\left[\frac{P(X)}{Q(X)}f(X)\right]\end{aligned}$$

- We have samples of X drawn from Q, but we wish to estimate expectation under P

$$\mathbb{E}_{X\sim P}[f(X)] \simeq \frac{1}{n}\sum_{i=1}^{n}\left[\frac{P(X_i)}{Q(X_i)}f(X_i)\right],\ X_i \sim Q$$

- Caveat: Q should not assign zero probability to any outcome that is assigned non-zero probability by P

Importance Sampling : Review

The ratio $\frac{P(x)}{Q(x)}$ is the importance sampling (IS) weight for x.

What about the variance of IS estimator $\widehat{\mu}_Q \approx \frac{1}{N}\sum_{x_i \in D}\left[\frac{P(x_i)}{Q(x_i)} f(x_i)\right]$?

$$\begin{aligned}
\text{var}_Q\left[\widehat{\mu}_Q\right] &= \text{var}_Q\left[\frac{P(x)}{Q(x)} f(x)\right] \\
&= \left[\mathbb{E}_Q\left[\left(\frac{P(x)}{Q(x)} f(x)\right)^2\right] - \mathbb{E}_Q\left(\left[\frac{P(x)}{Q(x)} f(x)\right]\right)^2\right] \\
&= \mathbb{E}_P\left[\left(\frac{P(x)}{Q(x)} f(x)^2\right)\right] - \mathbb{E}_P\left(f(x)\right)^2
\end{aligned}$$

If the likelihood ratio $\frac{P(x)}{Q(x)}$ is large, the variance of the estimator explodes

Off-Policy TD using Importance Sampling

- Evaluate target policy π using TD targets $r_{t+1} + \gamma V(s_{t+1})$ generated from μ
- Weigh each TD target by the importance sampling factor $\pi(s_t, a_t)/\mu(s_t, a_t)$

$$V(s_t) \leftarrow V(s_t) + \alpha_t \left[\frac{\pi(s_t|a_t)}{\mu(s_t|a_t)} (r_{t+1} + \gamma V(s_{t+1})) - V(s_t) \right]$$

- π may be deterministic and greedy, μ should be stochastic and exploratory
- The case $\mu = \pi$ is called *on-policy* learning
- On Policy Learning : Learn about policy π from experience sampled from π
- Off Policy Learning : Learn about policy π from experience sampled from μ

Towards Model Free Control

Problem and Motivation

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- **Goal** : How can we learn a good policy?
- **Motivation** : Many real world applications can be modelled as MDP
  - ★ Games like Backgammon and Go
  - ★ Robot Locomotion
  - ★ Inventory or supply chain management
- For almost all these problems, model is unknown or computationally infeasible; but sampling experiences is possible
- Learning better policies through experiences is model free control

Towards Model Free Control

DP algorithms for control

- Value Iteration
- Policy Iteration

Question : How can we do model free control ?

- Value iteration may not come in handy because it requires knowledge of model; so not suitable

$$V_{k+1}(s) \leftarrow \max_a \left[\sum_{s' \in \mathcal{S}} \mathcal{P}^a_{ss'} \left(\mathcal{R}^a_{ss'} + \gamma V_k(s') \right) \right]$$

- How about policy iteration (PI) ?

PI is a two step provess

- ★ Policy evaluation
- ★ Policy improvement

Policy Iteration : Recap

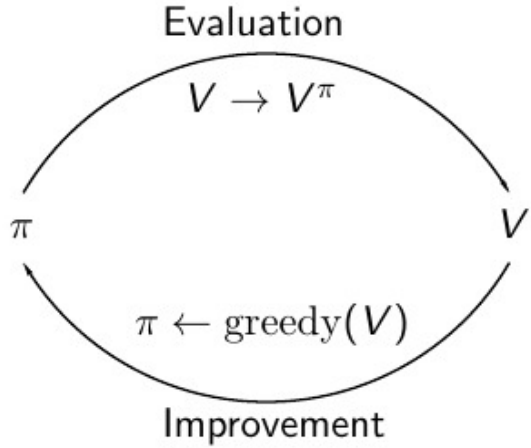


 Figure Source: Sutton and Barto

On Policy Improvement From Samples

- (Greedy) Policy improvement

$$\pi(s) \quad = \quad \arg\max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V^\pi(s') \right]$$

- Generally, model free control is not done with V as greedy policy improvement over V requires the knowledge of the model
- (Greedy) policy improvement over Q is model free

$$\pi(s) = \arg\max_a Q^\pi(s, a)$$

- For model-free policy improvement, we use Q^π, not V^π

Core Idea behind Model Free Control

- Initialize a policy π
- Repeat
  - ★ Policy Evaluation : Find Q^{π}
  - ★ Policy Improvement : Get an improved policy from evaluation of Q^{π}

Policy Evaluation : Action Value Function

- We now need to evaluate Q^π instead of V^π
- Recall that the state-action value function of a policy π is given by,

$$\begin{aligned} Q^\pi(s,a) &\stackrel{\text{def}}{=} \mathbb{E}_\pi(G_t|s_t=s,a_t=a) \\ &= \mathbb{E}_\pi\left(\sum_{k=0}^{\infty}\gamma^k r_{t+k+1}|s_t=s,a_t=a\right) \\ &= \mathbb{E}_\pi(r_{t+1}+\gamma Q^\pi(s_{t+1},a_{t+1})|s_t=s,a_t=a) \end{aligned}$$

- We can use MC or TD methods to evaluate Q^π using samples

First Visit Monte Carlo : Action Value Function

- To evaluate $Q^{\pi}(s,a)$ for some given state s and action a, repeat over several episodes
  - ★ The first time t that $s_t = s$ and $\pi(s) = a$ in the episode
    1. Increment counter for number of visits to s: $N(s,a) \leftarrow N(s,a) + 1$
    2. Increment running sum of total returns with return from current episode: $S(s,a) \leftarrow S(s,a) + G_t$
- Monte Carlo estimate of value function $Q(s,a) \leftarrow S(s,a)/N(s,a)$

The main drawback of this algorithm is

- Many state action pairs may never be visited
- If policy π is deterministic, things get even worse

Exploring Starts Assumption

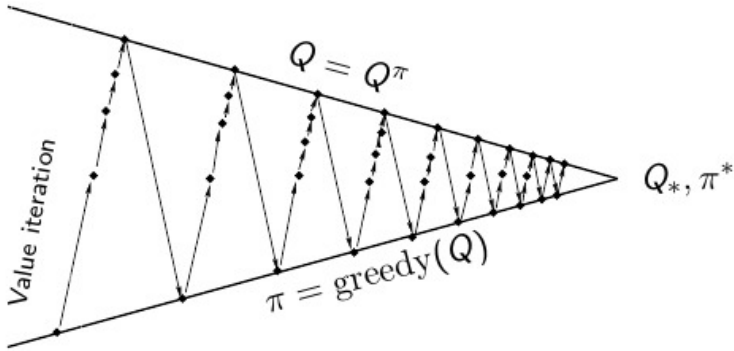
Exploring Starts (ES) Assumption

- First step of each episode start at a state-action pair, and that every such pair has non-zero probability of being selected at start
- Guarantees that all state-action pairs will be visited an infinite number of times in the limit of an infinite number of episodes

Not a realistic assumption at all !! But let's assume it for a while

- With ES assumption, first or every visit MC algorithm will evaluate Q^{π}

Policy Iteration with Action Value Function



- Monte Carlo Policy Evaluation, $Q = Q^\pi$
- Greedy policy improvement , $\pi' = \arg\max_a Q^\pi(s, a)$

Monte Carlo Control with ES

Algorithm Monte Carlo Control with ES

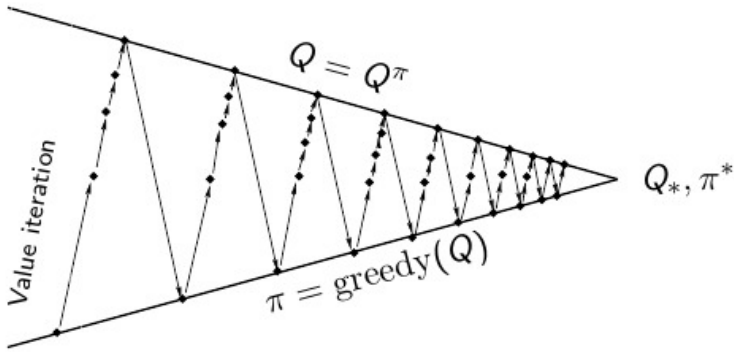
1: Start with an initial policy π_1;
2: **for** $k = 1, 2, \cdots, K$ **do**
3: Policy Evaluation Step : Evaluate Q^{π_k} using first or every visit MC
4: Policy Improvement Step :

$$\pi_{k+1} = \arg\max_a Q^{\pi_k}(s, a)$$

5: **end for**

- Convergence of policy evaluation to Q^π is assured only under the ES assumption
- Once ES assumption is made, to understand convergence to Q_* and π_* one can use the same kind of arguments as we had in the policy iteration algorithm in the DP setting

Policy Iteration with Action Value Function



- Is it good to be always greedy ?
- Should we patiently wait until policy evaluation step converges ?

 Figure Source: Sutton and Barto