EE3900-Assignment 5

W Vaishnavi
AI20BTECH11025

Download all latex-tikz codes from

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https://github.com/vaishnavi−w/EE3900/blob/main/
    Assignment5/latex5.tex
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and python codes from

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https://github.com/vaishnavi−w/EE3900/blob/main/
    Assignment5/hyperbola.py
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1 Quadratic Forms Q.30

Find the equation of a hyperbola with the vertices $\begin{pmatrix} 0 \\ \pm\frac{\sqrt{11}}{2} \end{pmatrix}$ and foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$

2 Solution

Lemma 2.1. *The equation of a conic with directrix* $\mathbf{n}^\top \mathbf{x} = c$*, eccentricity* e *and focus* $\mathbf{F}$ *is given by*

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{2.0.2}$$
$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{2.0.3}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{2.0.4}$$

Lemma 2.2. *For* $|\mathbf{V}| \neq 0$*, the length of the semi-major axis and eccentricity of the conic in* (2.0.1) *are given by*

$$a = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \tag{2.0.5}$$
$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{2.0.6}$$

Lemma 2.3. *Given (closest, for* $|\mathbf{V}| \neq 0$*) vertex* $\mathbf{B}$ *and focus* $\mathbf{F}$ *of conic* (2.0.1)*, equation of directrix is given as* $\mathbf{n}^\top(\mathbf{x} - \mathbf{P}) = 0$ *where*

$$\mathbf{P} = \mathbf{B} + \frac{(\mathbf{B} - \mathbf{F})}{e} \tag{2.0.7}$$
$$\mathbf{n} = \mathbf{B} - \mathbf{F} \tag{2.0.8}$$

and eccentricity

$$e = \begin{cases} 1 & |\mathbf{V}| = 0 \\ \frac{\|\mathbf{F_1} - \mathbf{F_2}\|}{\|\mathbf{B_1} - \mathbf{B_2}\|} & |\mathbf{V}| \neq 0 \end{cases} \tag{2.0.9}$$

Proof. For $|\mathbf{V}| \neq 0$,

Distance between the vertices is equal to the length of the major axis. That gives,

$$\|\mathbf{B_1} - \mathbf{B_2}\| = 2\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = 2a \tag{2.0.10}$$

Distance between the foci given as,

$$\|\mathbf{F_1} - \mathbf{F_2}\| = 2\sqrt{\frac{(\mathbf{u}^T \mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}} = 2ae \tag{2.0.11}$$

Distance between the closest vertex and focus

$$\begin{aligned}\|\mathbf{B} - \mathbf{F}\| &= \left|\frac{\|\mathbf{B_1} - \mathbf{B_2}\|}{2} - \frac{\|\mathbf{F_1} - \mathbf{F_2}\|}{2}\right| \\ &= \left|\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} - \sqrt{\frac{(\mathbf{u}^T \mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}}\right|\end{aligned} \tag{2.0.12}$$

Dividing (2.0.10) and (2.0.11) gives e

The directrix is perpendicular to the line joining vertex and focus. Thus normal vector for directrix is

$$\implies \mathbf{n} = \mathbf{B} - \mathbf{F} \tag{2.0.13}$$

For $|\mathbf{V}| = 0$, the vertex is the mid-point of line joining $\mathbf{P}$ and $\mathbf{F}$

$$\mathbf{B} = \frac{\mathbf{P} + \mathbf{F}}{2} \tag{2.0.14}$$
$$\mathbf{P} = 2\mathbf{B} + \mathbf{F} = \mathbf{B} + \frac{(\mathbf{B} - \mathbf{F})}{1} \tag{2.0.15}$$

$\mathbf{m}$ is the unit vector in the direction of $\mathbf{FB}$

$$\mathbf{m} = \frac{\mathbf{B} - \mathbf{F}}{\|\mathbf{B} - \mathbf{F}\|} \tag{2.0.16}$$

For $|\mathbf{V}| \neq 0$, the directrix passes through a point $\mathbf{P}$,

$$\mathbf{P} = \mathbf{B} + \mathbf{m}\frac{a(1-e)}{e} \tag{2.0.17}$$

Substituting (2.0.12), (2.0.16) gives the lemma □

Solution: Let the equation of the hyperbola be

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.18}$$

Let $\mathbf{B_1}, \mathbf{B_2}$ be the vertices and $\mathbf{F_1}, \mathbf{F_2}$ be the foci

$$\|\mathbf{B_1} - \mathbf{B_2}\| = \frac{\sqrt{11}}{2} \tag{2.0.19}$$

$$\|\mathbf{F_1} - \mathbf{F_2}\| = 3 \tag{2.0.20}$$

From (2.0.9) eccentricity,

$$e = \frac{\|\mathbf{F_1} - \mathbf{F_2}\|}{\|\mathbf{B_1} - \mathbf{B_2}\|} = \frac{6}{\sqrt{11}} \tag{2.0.21}$$

Considering $\mathbf{B} = \begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, the normal vector $\mathbf{n}$ of directrix

$$\mathbf{n} = \left(\begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{\sqrt{11}-6}{2} \end{pmatrix} \tag{2.0.22}$$

The directrix passes through the point $\mathbf{P}$,

$$\mathbf{P} = \begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix} + \frac{\sqrt{11}}{6}\left(\begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix} \tag{2.0.23}$$

Equation of the directrix can be given as

$$\begin{pmatrix} 0 & \frac{\sqrt{11}-6}{2} \end{pmatrix}\left(x - \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix}\right) = 0 \tag{2.0.24}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \frac{11}{12} \tag{2.0.25}$$

Calculating $\mathbf{V}, \mathbf{u}$ and f,

$$\mathbf{V} = 1^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\frac{6}{\sqrt{11}}\right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \tag{2.0.26}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{-25}{11} \end{pmatrix} \tag{2.0.27}$$

$$\mathbf{u} = \frac{11}{12}\left(\frac{6}{\sqrt{11}}\right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.28}$$

$$f = 3^2 - \left(\frac{11}{12} \times \frac{6}{\sqrt{11}}\right)^2 = \frac{25}{4} \tag{2.0.29}$$

Equation of the hyperbola,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{-25}{11} \end{pmatrix}\mathbf{x} + \frac{25}{4} = 0 \tag{2.0.30}$$

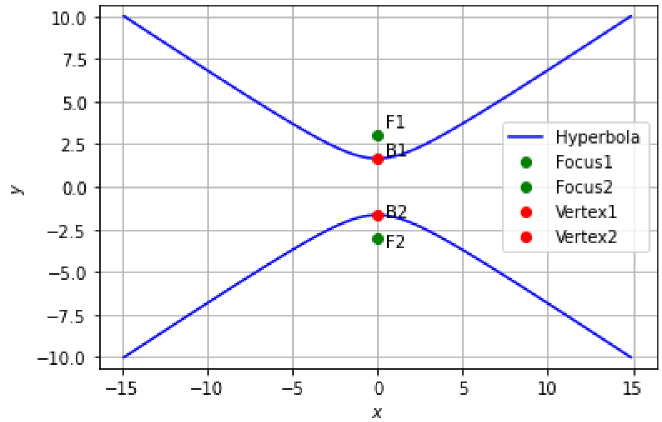

Fig. 0: Plot of Hyperbola