

EE3900-Assignment 3

W Vaishnavi
AI20BTECH11025

Download all latex-tikz codes from

<https://github.com/vaishnavi-w/EE3900/blob/main/Assignment3/latex3.tex>

and python codes from

<https://github.com/vaishnavi-w/EE3900/blob/main/Assignment3/codes>

1 RAMSEY 4.4 SYSTEMS OF CIRCLES Q.3

Find the equation of a circle which cuts orthogonally the two circles

$$S_1 = \mathbf{x}^T \mathbf{x} - \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (1.0.1)$$

$$S_2 = \mathbf{x}^T \mathbf{x} + \begin{pmatrix} -3 & 6 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (1.0.2)$$

and passes through the point $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

2 SOLUTION

Lemma 2.1. *Tangent to a circle : Consider a circle*

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

Given a point \mathbf{q} on the circle, the tangent at that point is given as,

$$(\mathbf{q} + \mathbf{c})^T \mathbf{x} + \mathbf{c}^T \mathbf{q} + f = 0 \quad (2.0.2)$$

Lemma 2.2. *Orthogonality of circles : Two circles are said to be orthogonal if they meet at right angles i.e the tangents at their points of intersection are perpendicular to each other. That implies, tangents to one circle at the points of contact are normals to the other circle. Given two circles,*

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{c}_1^T \mathbf{x} + f_1 = 0 \quad (2.0.3)$$

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{c}_2^T \mathbf{x} + f_2 = 0 \quad (2.0.4)$$

They are orthogonal if

$$2\mathbf{c}_1^T \mathbf{c}_2 = f_1 + f_2 \quad (2.0.5)$$

Proof. Let the two circles (2.0.3) and (2.0.4) meet at a point \mathbf{q} i.e \mathbf{q} satisfies the equation of the circles

$$\mathbf{q}^T \mathbf{q} + 2\mathbf{c}_1^T \mathbf{q} + f_1 = 0 \quad (2.0.6)$$

$$\mathbf{q}^T \mathbf{q} + 2\mathbf{c}_2^T \mathbf{q} + f_2 = 0 \quad (2.0.7)$$

Eliminating quadratic term,

$$2(\mathbf{c}_1^T - \mathbf{c}_2^T) \mathbf{q} + f_1 - f_2 = 0 \quad (2.0.8)$$

Given the point of contact \mathbf{q} , the equation of tangent to circle (2.0.3) is

$$(\mathbf{q} + \mathbf{c}_1)^T \mathbf{x} + \mathbf{c}_1^T \mathbf{q} + f_1 = 0 \quad (2.0.9)$$

As it is a normal to the second circle (2.0.4), it passes through the center of it

$$(\mathbf{q} + \mathbf{c}_1)^T (-\mathbf{c}_2) + \mathbf{c}_1^T \mathbf{q} + f_1 = 0 \quad (2.0.10)$$

$$\Rightarrow (\mathbf{c}_1^T - \mathbf{c}_2^T) \mathbf{q} + f_1 - \mathbf{c}_1^T \mathbf{c}_2 = 0 \quad (2.0.11)$$

$$\Rightarrow (\mathbf{c}_1^T - \mathbf{c}_2^T) \mathbf{q} = \mathbf{c}_1^T \mathbf{c}_2 - f_1 \quad (2.0.12)$$

Substituting (2.0.12) in (2.0.8),

$$2(\mathbf{c}_1^T \mathbf{c}_2 - f_1) + f_1 - f_2 = 0 \quad (2.0.13)$$

$$\Rightarrow 2\mathbf{c}_1^T \mathbf{c}_2 = f_1 + f_2 \quad (2.0.14)$$

□

Solution : Let the equation of the circle be

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (2.0.15)$$

It passes through the point $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, substituting it

$$\begin{pmatrix} 6 & -4 \end{pmatrix} \mathbf{c} - f = 13 \quad (2.0.16)$$

It is also orthogonal to the circles (1.0.1) and (1.0.2)

$$\begin{pmatrix} -2 & -2 \end{pmatrix} \mathbf{c} - f = 1 \quad (2.0.17)$$

$$\begin{pmatrix} -3 & 6 \end{pmatrix} \mathbf{c} - f = -2 \quad (2.0.18)$$

Expressing in the form of a matrix

$$\begin{pmatrix} -2 & -2 & -1 \\ -3 & 6 & -1 \\ 6 & -4 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 13 \end{pmatrix} \quad (2.0.19)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -2 & -2 & -1 & 1 \\ -3 & 6 & -1 & -2 \\ 6 & -4 & -1 & 13 \end{pmatrix} \quad (2.0.20)$$

$$\begin{matrix} \xleftarrow{R_2 \rightarrow 2R_2 - 3R_1} \\ \xrightarrow{R_3 \rightarrow R_3 + 3R_1} \end{matrix} \begin{pmatrix} -2 & -2 & -1 & 1 \\ 0 & 18 & 1 & -7 \\ 0 & -10 & -4 & 16 \end{pmatrix} \quad (2.0.21)$$

$$\begin{matrix} \xleftarrow{R_2 \rightarrow 2R_2 + \frac{R_3}{2}} \\ \xrightarrow{R_1 \rightarrow 2R_1 - \frac{R_3}{2}, R_3 \rightarrow \frac{R_3}{2}} \end{matrix} \begin{pmatrix} -4 & 1 & 0 & -6 \\ 0 & 31 & 0 & -6 \\ 0 & -5 & -2 & 8 \end{pmatrix} \quad (2.0.22)$$

$$\begin{matrix} \xleftarrow{R_3 \rightarrow 31R_3 + 5R_2} \\ \xrightarrow{R_1 \rightarrow 31R_1 - R_2} \end{matrix} \begin{pmatrix} -124 & 0 & 0 & -180 \\ 0 & 31 & 0 & -6 \\ 0 & 0 & -62 & 218 \end{pmatrix} \quad (2.0.23)$$

$$\mathbf{c} = \begin{pmatrix} \frac{45}{31} \\ \frac{31}{-6} \\ \frac{-6}{31} \end{pmatrix} \quad (2.0.24)$$

$$f = \frac{-109}{31} \quad (2.0.25)$$

The required equation of circle,

$$S = \mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} \frac{45}{31} & \frac{-6}{31} \end{pmatrix} \mathbf{x} - \frac{109}{31} = 0 \quad (2.0.26)$$

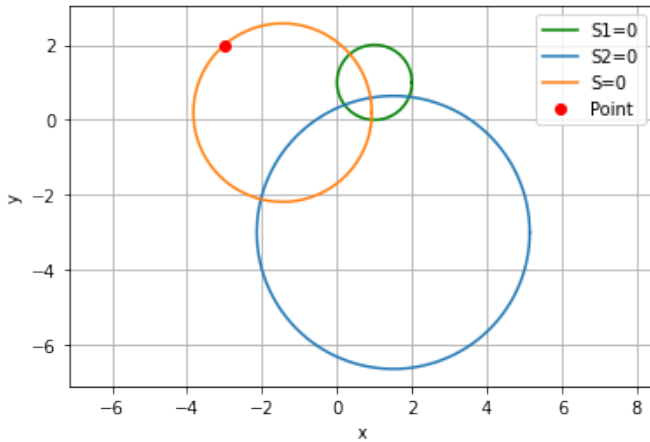


Fig. 0: Plot of circles