# Assignment 5 Presentation

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## Equation of a conic

The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity $e$ and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{2}$$
$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{3}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{4}$$

For $|V| \neq 0$, the length of the semi-major axis of the conic in (1) is given by

$$a = \sqrt{\frac{u^\top V^{-1} u - f}{\lambda_1}} \tag{5}$$

The eccentricity of conic (1) is given by,

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{6}$$

# Eccentricity

For $|\mathbf{V}| \neq 0$, given vertices $B_1, B_2$ and foci $F_1, F_2$ eccentricity of conic (1) is given by,

$$e = \frac{\|\mathbf{F}_1 - \mathbf{F}_2\|}{\|\mathbf{B}_1 - \mathbf{B}_2\|} \tag{7}$$

## Proof.

Distance between the vertices is equal to the length of the major axis.

$$\|\mathbf{B}_1 - \mathbf{B}_2\| = 2\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = 2a \tag{8}$$

Distance between the foci given as,

$$\|\mathbf{F}_1 - \mathbf{F}_2\| = 2\sqrt{\frac{(\mathbf{u}^T \mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}} = 2ae \tag{9}$$

Dividing (8) and (9) gives $e$ $\square$

# Directrix

For $|\mathbf{V}| \neq 0$, given vertices $B_1, B_2$ and foci $F_1, F_2$ of conic (1), equation of directrix is given as $\mathbf{n}^\top(\mathbf{x} - \mathbf{P}) = 0$ where

$$\mathbf{P} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} + \left(\frac{\|\mathbf{B}_1 - \mathbf{B}_2\|}{\|\mathbf{F}_1 - \mathbf{F}_2\|}\right)^2 \frac{(\mathbf{F}_1 - \mathbf{F}_2)}{2} \tag{10}$$

$$\mathbf{n} = \mathbf{F}_1 - \mathbf{F}_2 \tag{11}$$

## Proof.

The directrix is perpendicular to the line joining foci. Thus normal vector for directrix is

$$\implies \mathbf{n} = \mathbf{F}_1 - \mathbf{F}_2 \tag{12}$$

Let c be the centre of the conic

$$\mathbf{c} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} \tag{13}$$

m is the unit direction vector of line joining the foci

$$\mathbf{m} = \frac{\mathbf{F}_1 - \mathbf{F}_2}{\|\mathbf{F}_1 - \mathbf{F}_2\|} \tag{14}$$

The directrix passes through a point P,

$$\mathbf{P} = \mathbf{c} + \mathbf{m}\frac{a}{e} \tag{15}$$

Substituting $a, e$ from (8), (9) gives the lemma $\square$

# Question

## Quadratic Forms Q.30

Find the equation of a hyperbola with the vertices $\begin{pmatrix} 0 \\ \pm\frac{\sqrt{11}}{2} \end{pmatrix}$ and foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$

# Solution

Let the equation of the hyperbola be

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{16}$$

Let $\mathbf{B}_1, \mathbf{B}_2$ be the vertices and $\mathbf{F}_1, \mathbf{F}_2$ be the foci

$$\|\mathbf{B}_1 - \mathbf{B}_2\| = \frac{\sqrt{11}}{2} \tag{17}$$

$$\|\mathbf{F}_1 - \mathbf{F}_2\| = 3 \tag{18}$$

From (7) eccentricity,

$$e = \frac{\|\mathbf{F}_1 - \mathbf{F}_2\|}{\|\mathbf{B}_1 - \mathbf{B}_2\|} = \frac{6}{\sqrt{11}} \tag{19}$$

# Solution Contd.

Let n be the normal vector of directrix,

$$n = \left(\begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{20}$$

The directrix passes through the point P,

$$P = \frac{1}{2}\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix}\right) + \left(\frac{\sqrt{11}/2}{3}\right)^2 \frac{1}{2}\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix} \tag{21}$$

Equation of the directrix can be given as

$$\begin{pmatrix} 0 & 6 \end{pmatrix}\left(x - \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix}\right) = 0 \tag{22}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} x = \frac{11}{12} \tag{23}$$

# Solution Contd.

Calculating $\mathbf{V}, \mathbf{u}$ and $f$,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\frac{6}{\sqrt{11}}\right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{-25}{11} \end{pmatrix} \tag{25}$$

$$\mathbf{u} = \frac{11}{12} \left(\frac{6}{\sqrt{11}}\right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{26}$$

$$f = 3^2 - \left(\frac{11}{12} \times \frac{6}{\sqrt{11}}\right)^2 = \frac{25}{4} \tag{27}$$

Equation of the hyperbola,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{-25}{11} \end{pmatrix} \mathbf{x} + \frac{25}{4} = 0 \tag{28}$$

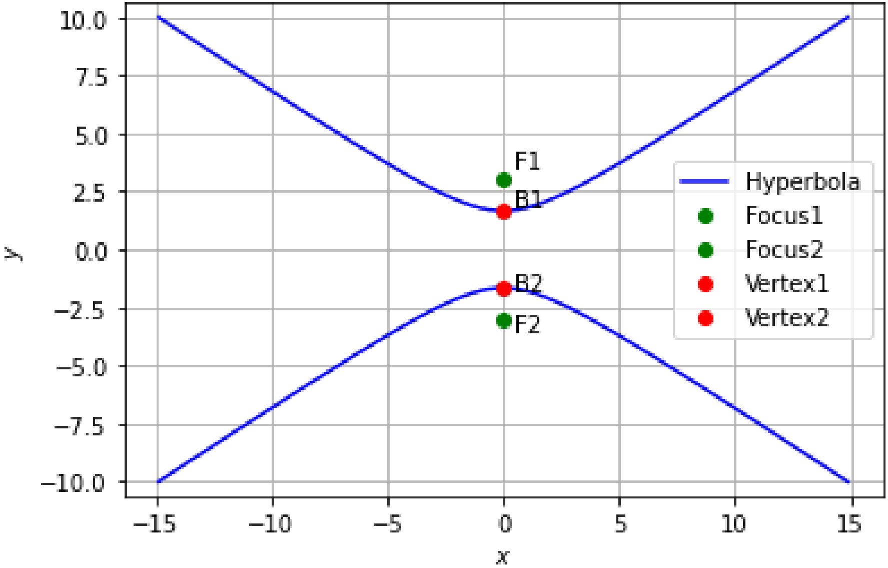


Figure: Plot of Hyperbola