EE3900-Assignment 5

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Download all latex-tikz codes from

```
https://github.com/vaishnavi−w/EE3900/blob/main/
    Assignment5/latex5.tex
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and python codes from

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https://github.com/vaishnavi−w/EE3900/blob/main/
    Assignment5/hyperbola.py
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1 Quadratic Forms Q.30

Find the equation of a hyperbola with the vertices $\begin{pmatrix} 0 \\ \pm\frac{\sqrt{11}}{2} \end{pmatrix}$ and foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$

2 Solution

Lemma 2.1. *The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by*

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{2.0.2}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{2.0.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{2.0.4}$$

Solution : Let the focus and vertex be,

$$\mathbf{F} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix} \tag{2.0.5}$$

$$\implies \|\mathbf{F}\| = 3 \tag{2.0.6}$$

Let e be the eccentricity of the hyperbola and b the length of the semi-major axis. From observation,

$$b = \frac{\sqrt{11}}{2}, be = 3 \tag{2.0.7}$$

$$\implies e = \frac{3}{\sqrt{11}/2} = \frac{6}{\sqrt{11}} \tag{2.0.8}$$

The directrix of the hyperbola passes through the point $\begin{pmatrix} 0 \\ \frac{b}{e} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix}$ and perpendicular to the y-axis. It is given as,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix} \right) = 0 \tag{2.0.9}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \frac{11}{12} \tag{2.0.10}$$

Comparing it with $\mathbf{n}^\top \mathbf{x} = c$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = \frac{11}{12} \tag{2.0.11}$$

$$\implies \|\mathbf{n}\| = 1 \tag{2.0.12}$$

Calculating $\mathbf{V}, \mathbf{u}$ and f,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\frac{6}{\sqrt{11}} \right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{-25}{11} \end{pmatrix} \tag{2.0.13}$$

$$\mathbf{u} = \frac{11}{12} \left(\frac{6}{\sqrt{11}} \right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.14}$$

$$f = 3^2 - \left(\frac{11}{12} \times \frac{6}{\sqrt{11}} \right)^2 = \frac{25}{4} \tag{2.0.15}$$

Equation of the hyperbola,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{-25}{11} \end{pmatrix} \mathbf{x} + \frac{25}{4} = 0 \tag{2.0.16}$$

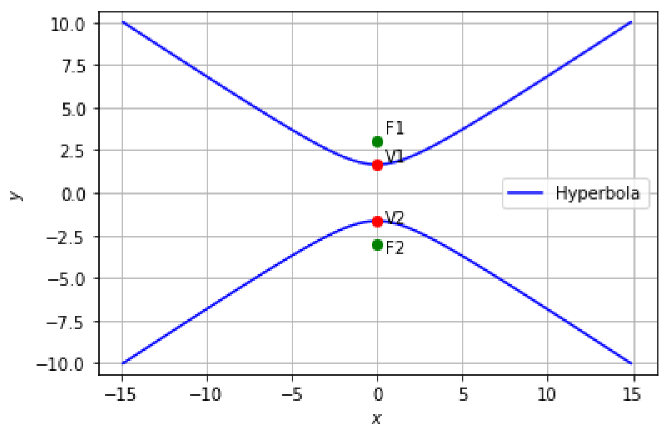

Fig. 0: Plot of Hyperbola