

EE3900-Assignment 5

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Download all latex-tikz codes from

<https://github.com/vaishnavi-w/EE3900/blob/main/Assignment5/latex5.tex>

and python codes from

<https://github.com/vaishnavi-w/EE3900/blob/main/Assignment5/hyperbola.py>

1 QUADRATIC FORMS Q.30

Find the equation of a hyperbola with the vertices $\begin{pmatrix} 0 \\ \pm \frac{\sqrt{11}}{2} \end{pmatrix}$ and foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$

2 SOLUTION

Lemma 2.1. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus \mathbf{F} is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (2.0.1)$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top, \quad (2.0.2)$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \quad (2.0.3)$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (2.0.4)$$

Lemma 2.2. For $|\mathbf{V}| \neq 0$, the length of the semi-major axis of the conic in (2.0.1) is given by

$$a = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \quad (2.0.5)$$

Lemma 2.3. The eccentricity of conic (2.0.1) is given by,

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \quad (2.0.6)$$

Lemma 2.4. For $|\mathbf{V}| \neq 0$, given vertices B_1, B_2 and foci F_1, F_2 eccentricity of conic (2.0.1) is given by,

$$e = \frac{\|\mathbf{F}_1 - \mathbf{F}_2\|}{\|\mathbf{B}_1 - \mathbf{B}_2\|} \quad (2.0.7)$$

Proof. Distance between the vertices is equal to the length of the major axis. That gives,

$$\|\mathbf{B}_1 - \mathbf{B}_2\| = 2 \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = 2a \quad (2.0.8)$$

Distance between the foci given as,

$$\|\mathbf{F}_1 - \mathbf{F}_2\| = 2 \sqrt{\frac{(\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}} = 2ae \quad (2.0.9)$$

Dividing (2.0.8) and (2.0.9) gives e \square

Lemma 2.5. Distance between the closest vertex and focus

$$\begin{aligned} \|\mathbf{B} - \mathbf{F}\| &= \left| \frac{\|\mathbf{B}_1 - \mathbf{B}_2\|}{2} - \frac{\|\mathbf{F}_1 - \mathbf{F}_2\|}{2} \right| \\ &= \left| \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} - \sqrt{\frac{(\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}} \right| \end{aligned} \quad (2.0.10)$$

Lemma 2.6. Given vertex \mathbf{B} and focus \mathbf{F} of conic (2.0.1), equation of directrix is given as $\mathbf{n}^\top (\mathbf{x} - \mathbf{P}) = 0$ where

$$\mathbf{P} = \mathbf{B} + \frac{(\mathbf{B} - \mathbf{F})}{e} \quad (2.0.11)$$

$$\mathbf{n} = \mathbf{B} - \mathbf{F} \quad (2.0.12)$$

Proof. The directrix is perpendicular to the line joining vertex and focus. Thus normal vector for directrix is

$$\implies \mathbf{n} = \mathbf{B} - \mathbf{F} \quad (2.0.13)$$

\mathbf{m} is the unit vector in the direction of \mathbf{FB}

$$\mathbf{m} = \frac{\mathbf{B} - \mathbf{F}}{\|\mathbf{B} - \mathbf{F}\|} \quad (2.0.14)$$

For $|\mathbf{V}| = 0$, the vertex is the mid-point of line

joining \mathbf{P} and \mathbf{F}

$$\mathbf{B} = \frac{\mathbf{P} + \mathbf{F}}{2} \quad (2.0.15)$$

$$\mathbf{P} = 2\mathbf{B} + \mathbf{F} = \mathbf{B} + \frac{(\mathbf{B} - \mathbf{F})}{1} \quad (2.0.16)$$

For $|\mathbf{V}| \neq 0$, the directrix passes through a point \mathbf{P} ,

$$\mathbf{P} = \mathbf{B} + \mathbf{m} \frac{a(1-e)}{e} \quad (2.0.17)$$

Substituting (2.0.10), (2.0.14) gives the lemma \square

Solution: Let the equation of the hyperbola be

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.18)$$

Let $\mathbf{B}_1, \mathbf{B}_2$ be the vertices and $\mathbf{F}_1, \mathbf{F}_2$ be the foci

$$\|\mathbf{B}_1 - \mathbf{B}_2\| = \frac{\sqrt{11}}{2} \quad (2.0.19)$$

$$\|\mathbf{F}_1 - \mathbf{F}_2\| = 3 \quad (2.0.20)$$

From (2.0.7) eccentricity,

$$e = \frac{\|\mathbf{F}_1 - \mathbf{F}_2\|}{\|\mathbf{B}_1 - \mathbf{B}_2\|} = \frac{6}{\sqrt{11}} \quad (2.0.21)$$

Considering $\mathbf{B} = \begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, the normal vector \mathbf{n} of directrix

$$\mathbf{n} = \left(\begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \frac{\sqrt{11}-6}{2} \end{pmatrix} \quad (2.0.22)$$

The directrix passes through the point \mathbf{P} ,

$$\mathbf{P} = \begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix} + \frac{\sqrt{11}}{6} \left(\begin{pmatrix} 0 \\ \frac{\sqrt{11}}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix} \quad (2.0.23)$$

Equation of the directrix can be given as

$$\left(0 \quad \frac{\sqrt{11}-6}{2} \right) \left(x - \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix} \right) = 0 \quad (2.0.24)$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \frac{11}{12} \quad (2.0.25)$$

Calculating \mathbf{V}, \mathbf{u} and f ,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\frac{6}{\sqrt{11}} \right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \quad (2.0.26)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -\frac{25}{11} \end{pmatrix} \quad (2.0.27)$$

$$\mathbf{u} = \frac{11}{12} \left(\frac{6}{\sqrt{11}} \right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.28)$$

$$f = 3^2 - \left(\frac{11}{12} \times \frac{6}{\sqrt{11}} \right)^2 = \frac{25}{4} \quad (2.0.29)$$

Equation of the hyperbola,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -\frac{25}{11} \end{pmatrix} \mathbf{x} + \frac{25}{4} = 0 \quad (2.0.30)$$

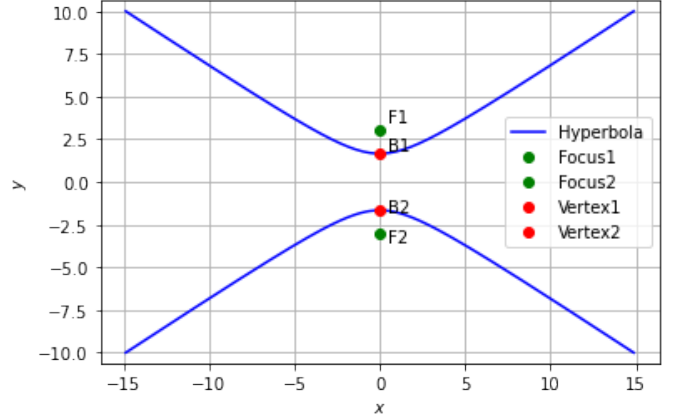


Fig. 0: Plot of Hyperbola