EE3900-Assignment 3

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Download all latex-tikz codes from

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https://github.com/vaishnavi-w/EE3900/blob/main/
    Assignment3/latex3.tex
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and python codes from

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https://github.com/vaishnavi-w/EE3900/blob/main/
    Assignment3/codes
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1 Ramsey 4.4 Systems of circles Q.3

Find the equation of a circle which cuts orthogonally the two circles

$$S_1 = \mathbf{x}^\top \mathbf{x} - \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{1.0.1}$$

$$S_2 = \mathbf{x}^\top \mathbf{x} + \begin{pmatrix} -3 & 6 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{1.0.2}$$

and passes through the point $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

2 Solution

Lemma 2.1. *Orthogonality of circles : Two circles are said to be orthogonal if they meet at right angles i.e the tangents at their points of intersection are perpendicular to each other. Given two circles,*

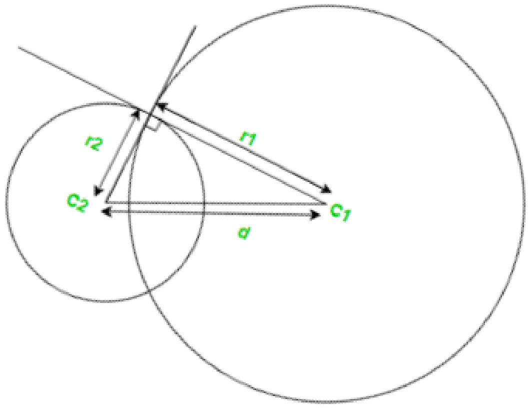

Fig. 0: Orthogonal circles

$$\mathbf{x}^\top \mathbf{x} - 2\mathbf{c_1}^\top \mathbf{x} + f_1 = 0 \tag{2.0.1}$$

$$\mathbf{x}^\top \mathbf{x} - 2\mathbf{c_2}^\top \mathbf{x} + f_2 = 0 \tag{2.0.2}$$

They are orthogonal if

$$2\mathbf{c_1}^\top \mathbf{c_2} = f_1 + f_2 \tag{2.0.3}$$

Proof. The radii drawn at the point of intersection and the line joining the centres form a right angled triangle. From Pythagorean theorem,

$$r_1^2 + r_2^2 = d^2 \tag{2.0.4}$$

where r_1, r_2 are the radii and d is the distance between the centre. They are given as

$$r_1 = \sqrt{\|\mathbf{c_1}\|^2 - f_1} \tag{2.0.5}$$

$$r_2 = \sqrt{\|\mathbf{c_2}\|^2 - f_2} \tag{2.0.6}$$

$$d = \|\mathbf{c_1} - \mathbf{c_2}\| \tag{2.0.7}$$

Substituting the values

$$\begin{aligned} \|\mathbf{c_1}\|^2 - f_1 + \|\mathbf{c_2}\|^2 - f_2 &= \|\mathbf{c_1} - \mathbf{c_2}\|^2 \\ &= \|\mathbf{c_1}\|^2 + \|\mathbf{c_2}\|^2 - 2\mathbf{c_1}^\top \mathbf{c_2} \end{aligned} \tag{2.0.8}$$

$$\implies 2\mathbf{c_1}^\top \mathbf{c_2} = f_1 + f_2 \tag{2.0.9}$$

□

Solution : Let the equation of the circle be

$$\mathbf{x}^\top \mathbf{x} - 2\mathbf{c}^\top \mathbf{x} + f = 0 \tag{2.0.10}$$

It passes through the point $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, substituting it

$$\begin{pmatrix} -6 & 4 \end{pmatrix} \mathbf{c} - f = 13 \tag{2.0.11}$$

It is also orthogonal to the circles (1.0.1) and (1.0.2)

$$\begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{c} - f = 1 \tag{2.0.12}$$

$$\begin{pmatrix} 3 & -6 \end{pmatrix} \mathbf{c} - f = -2 \tag{2.0.13}$$

Expressing in the form of a matrix

$$\begin{pmatrix} 2 & 2 & -1 \\ 3 & -6 & -1 \\ -6 & 4 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 13 \end{pmatrix} \tag{2.0.14}$$

Row reducing the augumented matrix,

$$\begin{pmatrix} 2 & 2 & -1 & 1 \\ 3 & -6 & -1 & -2 \\ -6 & 4 & -1 & 13 \end{pmatrix} \tag{2.0.15}$$

$$\xleftrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to 2R_2 - 3R_1} \begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & -18 & 1 & -7 \\ 0 & 10 & -4 & 16 \end{pmatrix} \tag{2.0.16}$$

$$\xleftrightarrow[R_1 \to 2R_1 - \frac{R_3}{2}, R_3 \to \frac{R_3}{2}]{R_2 \to 2R_2 + \frac{R_3}{2}} \begin{pmatrix} 4 & -1 & 0 & -6 \\ 0 & -31 & 0 & -6 \\ 0 & 5 & -2 & 8 \end{pmatrix} \tag{2.0.17}$$

$$\xleftrightarrow[R_1 \to 31R_1 - R_2]{R_3 \to 31R_3 + 5R_2} \begin{pmatrix} 124 & 0 & 0 & -180 \\ 0 & -31 & 0 & -6 \\ 0 & 0 & -62 & 218 \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{c} = \begin{pmatrix} \frac{-45}{31} \\ \frac{6}{31} \end{pmatrix} \tag{2.0.19}$$

$$f = \frac{-109}{31} \tag{2.0.20}$$

The required equation of circle,

$$S = \mathbf{x}^\top \mathbf{x} - 2\begin{pmatrix} \frac{-45}{31} & \frac{6}{31} \end{pmatrix} \mathbf{x} - \frac{109}{31} = 0 \tag{2.0.21}$$

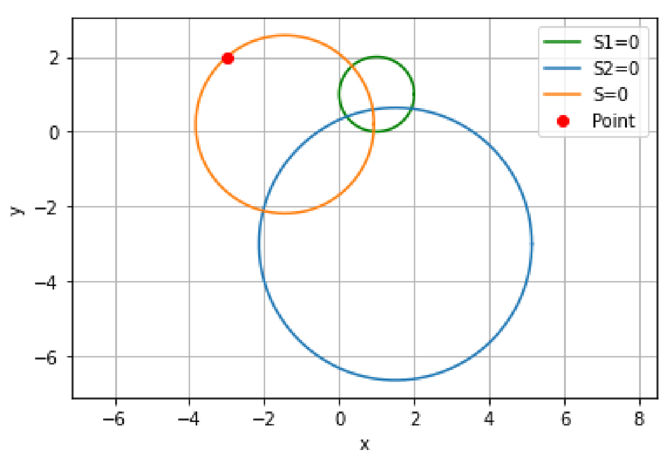

Fig. 0: Plot of circles