EE3900-Assignment 5

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Download all latex-tikz codes from

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https://github.com/vaishnavi−w/EE3900/blob/main/
    Assignment5/latex5.tex
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and python codes from

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https://github.com/vaishnavi−w/EE3900/blob/main/
    Assignment5/hyperbola.py
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1 Quadratic Forms Q.30

Find the equation of a hyperbola with the vertices $\begin{pmatrix} 0 \\ \pm\frac{\sqrt{11}}{2} \end{pmatrix}$ and foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$

2 Solution

Theorem 2.1. *The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by*

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{2.0.2}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{2.0.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{2.0.4}$$

Theorem 2.2. *For $|\mathbf{V}| \neq 0$, the length of the semi-major axis of the conic in* (2.0.1) *is given by*

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \tag{2.0.5}$$

Theorem 2.3. *The eccentricity of conic* (2.0.1) *is given by,*

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{2.0.6}$$

Solution: Let the equation of the hyperbola be

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.7}$$

Let $\mathbf{B_1}, \mathbf{B_2}$ be the vertices and $\mathbf{F_1}, \mathbf{F_2}$ be the foci

$$\|\mathbf{B_1} - \mathbf{B_2}\| = \frac{\sqrt{11}}{2} \tag{2.0.8}$$

$$\|\mathbf{F_1} - \mathbf{F_2}\| = 3 \tag{2.0.9}$$

Distance between the vertices is equal to the length of the major axis. That gives,

$$\|\mathbf{B_1} - \mathbf{B_2}\| = 2\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \tag{2.0.10}$$

Distance between the foci given as,

$$\|\mathbf{F_1} - \mathbf{F_2}\| = 2\sqrt{\frac{(\mathbf{u}^T \mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}} \tag{2.0.11}$$

Dividing (2.0.10) and (2.0.11) to get the eccentricity,

$$\sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_2}} = \frac{6}{\sqrt{11}} = e \tag{2.0.12}$$

The directrix of the hyperbola is perpendicular to the y-axis and also passes through the point

$$\begin{pmatrix} 0 \\ \frac{1}{e}\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix} \tag{2.0.13}$$

Equation of the directrix can be given as

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\left(x - \begin{pmatrix} 0 \\ \frac{11}{12} \end{pmatrix}\right) = 0 \tag{2.0.14}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \frac{11}{12} \tag{2.0.15}$$

Comparing it with $\mathbf{n}^\top \mathbf{x} = c$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = \frac{11}{12} \tag{2.0.16}$$

$$\implies \|\mathbf{n}\| = 1 \tag{2.0.17}$$

Calculating $\mathbf{V}, \mathbf{u}$ and f,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\frac{6}{\sqrt{11}} \right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{2.0.18}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{-25}{11} \end{pmatrix} \tag{2.0.19}$$

$$\mathbf{u} = \frac{11}{12} \left(\frac{6}{\sqrt{11}} \right)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.20}$$

$$f = 3^2 - \left(\frac{11}{12} \times \frac{6}{\sqrt{11}} \right)^2 = \frac{25}{4} \tag{2.0.21}$$

Equation of the hyperbola,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{-25}{11} \end{pmatrix} \mathbf{x} + \frac{25}{4} = 0 \tag{2.0.22}$$

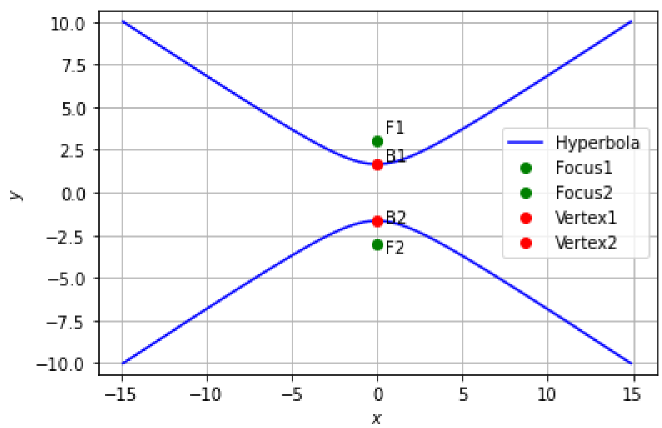

Fig. 0: Plot of Hyperbola