Inverse kinematic model for generic 3R positional robots using Conformal Geometric Algebra

Abhilash Nayak[1] and Durgesh Haribhau Salunkhe[2]

[1] abhilash.un@outlook.com
[2] salunkhedurgesh@outlook.com

Abstract. The inverse kinematics of a generic 3R robot has been investigated with multiple approaches in the past. The well-known Pieper's method gave a geometric interpretation of the inverse kinematic model (IKM) allowing researchers to study the robot's global kinematic properties. In this article, we study the IKM of a generic 3R robot, leveraging the advantages of Conformal Geometric Algebra (CGA) to provide further useful insights on its kinematic properties. A univariate polynomial related to θ_2 is obtained by interpreting the IKM as the intersection of two circles, which are elements of CGA.

Keywords: kinematics, conformal geometric algebra, geometry

1 Introduction

Inverse kinematics problem for a positional 3R robot was first addressed by Pieper [1] in 1968. Since then, multiple works have been published that use the geometric interpretation of a conic intersecting a circle to study the number of IKS and cuspidal properties of the robot [2]. Most of the past work depends on obtaining a univariate polynomial in $t_3 = \tan\frac{\theta_3}{2}$ and later obtaining θ_1, θ_2 by backpropagation. Geometric methods are useful in developing an intuition of the robot, allowing the designer to understand the effect of specific link length and arrangement on the global kinematic properties.

Recently, methods using Conformal Geometric Algebra (CGA) to investigate robot kinematics have gained popularity [3, 4]. CGA is a 5-dimensional representation of 3-dimensional Euclidean space. There are several advantages of this embedding: (i) Rotations and translations can be unified as orthogonal transformations, (ii) the algebra is covariant, i.e. the structure of relationships between objects is invariant under the allowed transformations, (iii) circles and spheres are objects of the algebra leading to an intuitive geometric interpretation of inverse kinematics of robots with revolute joints. Some of the recently published works discuss simplified geometries such as the anthropomorphic structure [5, 6] and claims made in [7] cannot be extended to generic 3R robots (In Equation (70), P_2 does not lie on Π_1 and s_1 is ill defined). Furthermore, the framework developed in these works is specific to non-generic serial robots (refer to [8] for

definition).
In this article, we present a novel approach to solve IKM of a generic 3R robot using CGA. The method provides a geometric understanding of the IKM, allowing insight into the number of IKS and their distribution. Furthermore, the method does not study the IKM in different cases, thus unifying the inverse kinematic model of 3R robots.

2 CGA: Notations and basic operations

The 5-dimensional conformal geometric algebra $\mathbb{G}_{4,1}$ is described with the orthogonal unit vectors $\boldsymbol{e}_i^2 = +1$ for $i = 1, ..., 4$ and $\boldsymbol{e}_5^2 = -1$. A fundamental operation in geometric algebra is the geometric product. Given two vectors[3], $\boldsymbol{a}$ and $\boldsymbol{b}$, their geometric product is defined as

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} \tag{1}$$

where $\boldsymbol{a} \cdot \boldsymbol{b}$ is the inner product that yields a scalar and $\boldsymbol{a} \wedge \boldsymbol{b}$ is the outer product that gives a bivector. They follow the properties $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ and $\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a}$.

Orthogonality of the basis vectors implies that $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0,\ i \neq j$.

$$\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{e}_j + \boldsymbol{e}_i \wedge \boldsymbol{e}_j = \begin{cases} \pm 1, & i = j \\ \boldsymbol{e}_i \wedge \boldsymbol{e}_j, \text{ denoted as } \boldsymbol{e}_{ij}, & i \neq j \end{cases} \tag{2}$$

In this algebra, a basis change is done by introducing two null vectors $\boldsymbol{e}_0$ and $\boldsymbol{e}_\infty$ that represent a point at the origin and a point at infinity respectively:

$$\boldsymbol{e}_\infty = \boldsymbol{e}_4 + \boldsymbol{e}_5 \ ; \ \boldsymbol{e}_0 = \frac{1}{2}(\boldsymbol{e}_5 - \boldsymbol{e}_4) \tag{3}$$

From (2), it follows that $\boldsymbol{e}_\infty^2 = \boldsymbol{e}_0^2 = 0$ and $\boldsymbol{e}_\infty \cdot \boldsymbol{e}_0 = -1$. $\mathbb{G}_{4,1}$ has $2^5 = 32$ basis elements called blades listed in Table I in [9]. A multivector $\boldsymbol{A}$ is a linear combination of those basis elements $\{1, \boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{0123\infty}\}$. The conformal pseudoscalar is $\boldsymbol{I} = \boldsymbol{e}_{0123\infty}$. Therefore, the conformal dual of a multivector $\boldsymbol{A}$ is

$$\boldsymbol{A}^* = \boldsymbol{A}\boldsymbol{I}^{-1},\ \boldsymbol{I}^{-1} = \boldsymbol{e}_0\boldsymbol{I}_3^{-1}\boldsymbol{e}_\infty \tag{4}$$

where $\boldsymbol{I}_3 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ and $\boldsymbol{I}_3^{-1} = \boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1$.

Here we use the formulation in [3] to represent geometric objects and their duals. Those objects are in general null space representations with respect to either the inner (IPNS) or the outer (OPNS) product [9]. An Euclidean point $\mathbf{x} \in \mathbb{R}^3$ is embedded as a null vector in CGA using the up() function:

$$\boldsymbol{X} = \mathrm{up}(\mathbf{x}) = \boldsymbol{e}_0 + \mathbf{x} + \frac{1}{2}\mathbf{x}^2\boldsymbol{e}_\infty,\ \boldsymbol{X} \cdot \boldsymbol{X} = 0 \tag{5}$$

[3] Points and circles of Euclidean geometry are represented in capital italic letters while its vectors in small bold letters. CGA elements are represented in capital bold italic letters unless they are vectors in which case we use small bold italic letters.

OPNS is also known as the primal or direct representation of the geometric primitives and IPNS its dual according to (4). For instance, the direct representation of a sphere in CGA is given by $\boldsymbol{S} = \boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \boldsymbol{P}_3 \wedge \boldsymbol{P}_4$ where $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \boldsymbol{P}_4$ are conformal representations of points on the sphere that are not all in the same plane. The dual representation is given by $\boldsymbol{S}^* = \boldsymbol{P}_S - \frac{1}{2}r^2\boldsymbol{e}_\infty$, where $\boldsymbol{P}_S$ is the conformal center point and r is the radius of the sphere. These representations allow us to build CGA elements at our convenience, which is especially beneficial in kinematic analysis. Table 1 lists the direct and dual representations of

CGA primitives	Direct/primal (OPNS)	Dual (OPNS)
Point $\boldsymbol{P}$	$\boldsymbol{X}$ in (5)	$\boldsymbol{X}$
Point pair $\boldsymbol{A}$	$\boldsymbol{P}_1 \wedge \boldsymbol{P}_2$	$\boldsymbol{S}_1 \vee \boldsymbol{S}_2 \vee \boldsymbol{S}_3$
Sphere $\boldsymbol{S}$	$\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \boldsymbol{P}_3 \wedge \boldsymbol{P}_4$	$\boldsymbol{P}_S - \frac{1}{2}r^2\boldsymbol{e}_\infty$
Plane $\boldsymbol{E}$	$\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \boldsymbol{P}_3 \wedge \boldsymbol{e}_\infty$	$\boldsymbol{n} + d\boldsymbol{e}_\infty$
Line $\boldsymbol{L}$	$\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \boldsymbol{e}_\infty$	$\boldsymbol{E}_1 \vee \boldsymbol{E}_2$
Circle $\boldsymbol{C}$	$\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \boldsymbol{P}_3$	$\boldsymbol{S}_1 \vee \boldsymbol{S}_2$ or $\boldsymbol{S}_1 \vee \boldsymbol{E}_1$

Table 1: Geometric primitives in 3D CGA. The outer product '$\wedge$' acts as the join of different elements, its dual is the regressive product represented by '$\vee$'. $\boldsymbol{P}_S$ is the center of the sphere and r is its radius; $\boldsymbol{n}$ is the normal vector of the plane and d is the distance from the origin.

all CGA primitives. Note that to represent a plane, the basis elements of the normal vector $\boldsymbol{n}$ can only be $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ since it just represents a direction and hence we have to forego the $\boldsymbol{e}_0$ and $\boldsymbol{e}_\infty$ elements that constrain it. To obtain this vector from a given line, a grade 3 element, we have to keep only coefficients of $\boldsymbol{e}_{145}$, $\boldsymbol{e}_{245}$ and $\boldsymbol{e}_{345}$ but replace the basis vectors by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, respectively. Translation along a vector $\boldsymbol{n}$, with a distance r is defined by the following versor [3, Section 13.2.2]:

$$\boldsymbol{R}_\infty(r, \boldsymbol{n}) = 1 - \frac{r}{2}(\boldsymbol{n} \wedge \boldsymbol{e}_\infty) \tag{6}$$

Rotations in CGA are given by the following versor [3, Sections 7.2 and 13.2.2]:

$$\boldsymbol{R}(\phi, \boldsymbol{B}) = \exp^{-\boldsymbol{B}\frac{\phi}{2}} = \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\boldsymbol{B} \tag{7}$$

Transformation between frames is accomplished using the following motors:

$$\boldsymbol{M}(\theta) = \boldsymbol{R}(\theta, \boldsymbol{e}_{12}),\ \boldsymbol{G}(a, d, \alpha) = \boldsymbol{R}_\infty(d, \boldsymbol{e}_3)\boldsymbol{R}(\alpha, \boldsymbol{e}_{23})\boldsymbol{R}_\infty(a, \boldsymbol{e}_1) \tag{8}$$

The angle between any two given lines, $\boldsymbol{L}_1, \boldsymbol{L}_2$, in CGA can be calculated as:

$$\cos(\theta) = \frac{\boldsymbol{L}_1 \cdot \boldsymbol{L}_2}{\sqrt{\boldsymbol{L}_1 \cdot \boldsymbol{L}_1}\sqrt{\boldsymbol{L}_2 \cdot \boldsymbol{L}_2}} \tag{9}$$

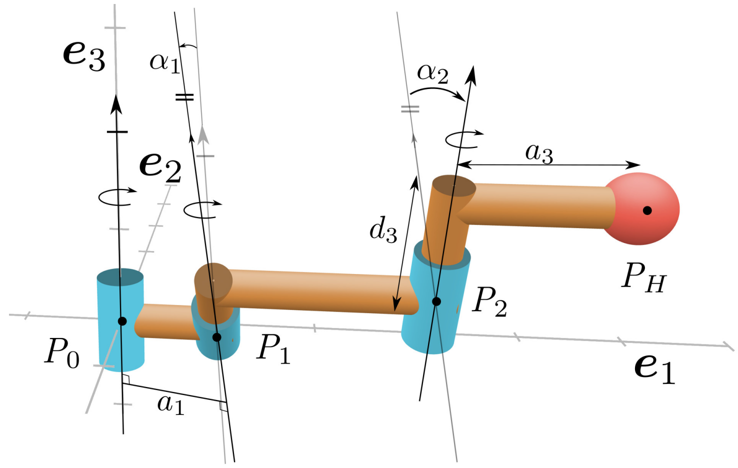

Fig. 1: The schematic of a generic 3R robot and annotations relevant to CGA

3 Inverse kinematic model

A generic positional 3R robot is shown in Fig. 1. $P_i,\ i = \{0, 1, 2, H\}$ are points where the links are connected using revolute joints. P_0 is the origin and P_H is where the end-effector lies when the robot is said to be in its 'Home' position with joint angles $\theta_i = 0,\ i = \{1, 2, 3\}$. Inverse kinematics involves finding θ_i of the end-effector, given an arbitrary position, $P = (x, y, z)$.

We will follow Selig's inverse kinematics approach for 3R robots [10, Section 5.2]. In this approach, the necessary joint angles that move the end-effector from the home position P_H to a target position P are obtained by solving the equations:

$$e^{\theta_1 S_1} e^{\theta_2 S_2} e^{\theta_3 S_3} \begin{pmatrix} \mathbf{p}_H \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} \tag{10}$$

where $\mathbf{p}_H$ and $\mathbf{p}$ are Euclidean vectors representing P_H and P, $S_i,\ i = \{1, 2, 3\}$ is the Lie algebra element representing the ith joint. The equations are further written as:

$$e^{\theta_2 S_2} \begin{pmatrix} \mathbf{a} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} \text{ with } \begin{pmatrix} \mathbf{a} \\ 1 \end{pmatrix} = e^{\theta_3 S_3} \begin{pmatrix} \mathbf{p}_H \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} = e^{-\theta_1 S_1} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} \tag{11}$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors representing points on circles parametrized by the first and last joint angles. Selig continues with an algebraic approach to solve these equations. The geometric interpretation is stated as finding the intersection of a pair of conics expressed in terms of θ_1 or θ_3, similar to Pieper's interpretation of IKM [1]. However, CGA allows a better geometric intuition on how these solutions might be represented in the workspace by leveraging the fact that interactions between circles and their rotors are easy in CGA. We show that the inverse kinematic analysis reduces to finding the intersection between a circle and a torus - two geometric manifolds that naturally appear as a link between the robot's workspace and joint space.

In (11), let the circles on which $\mathbf{a}$ and $\mathbf{b}$ lie be named C_A and C_B. In terms of 3D-CGA, they are calculated as follows. The center of C_B lies on z-axis and contains P. As listed in Table 1, C_B in CGA is represented as the intersection of a sphere and a plane:

$$\boldsymbol{C}_B = \boldsymbol{S}_B \vee \boldsymbol{E}_B, \text{ with } \boldsymbol{S}_B^* = \boldsymbol{p}_{\boldsymbol{S}_B} - \frac{1}{2} r_B^2 \boldsymbol{e}_\infty \text{ and } \boldsymbol{E}_B^* = \boldsymbol{n}_B + d_B \boldsymbol{e}_\infty \tag{12}$$

where $\boldsymbol{S}_B^*$ is the dual representation (cf. Table 1) of the sphere with center $\boldsymbol{p}_{\boldsymbol{S}_B} = \boldsymbol{p}_0, \boldsymbol{p}_0 = \text{up}(\mathbf{p}_0)$ and radius $r_b = ||\mathbf{p}||$. Plane $\boldsymbol{E}_B^*$ is also dually represented (cf. Table 1) with its normal $\boldsymbol{n}_B = \boldsymbol{e}_3$ being the vector representing the z-axis and $d_B = z$.
Similarly, C_A is obtained as the intersection of the following sphere and plane although expressing it as a CGA element is more nuanced:

$$\boldsymbol{C}_A = \boldsymbol{S}_A \vee \boldsymbol{E}_A, \text{ with } \boldsymbol{S}_A^* = \boldsymbol{p}_{\boldsymbol{S}_A} - \frac{1}{2} r_A^2 \boldsymbol{e}_\infty \text{ and } \boldsymbol{E}_A^* = \boldsymbol{n}_A + d_A \boldsymbol{e}_\infty \tag{13}$$

where the center of the dual sphere $\boldsymbol{S}_A^*$ is $\boldsymbol{p}_{\boldsymbol{S}_A} = \boldsymbol{p}_2$, the null vector of P_2 and its radius $r_A = \sqrt{a_3^2 + d_3^2}$. The plane $\boldsymbol{E}_A^*$ is again dually represented whose normal vector $\boldsymbol{n}_A$ denotes the third joint axis obtained by transforming the unit vector, $\boldsymbol{e}_3$, using the following motors (cf. (8)):

$$\boldsymbol{n}_A = \boldsymbol{T}_3 \boldsymbol{e}_3 \boldsymbol{T}_3^{-1} \text{ with } \boldsymbol{T}_3 = \boldsymbol{M}(\theta_1)\boldsymbol{G}(\alpha_1, a_1, d_1)\boldsymbol{M}(\theta_2)\boldsymbol{G}(\alpha_2, a_2, d_2). \tag{14}$$

Additionally, the distance of the plane from the origin $d_A = \boldsymbol{n}_A \cdot \boldsymbol{p}_0$.

Solving (11) implies rotating $\boldsymbol{C}_A$ about the second joint axis by θ_2 to meet $\boldsymbol{C}_B$. Applying a rotor $\boldsymbol{R}_2$ to it, we obtain the rotated circle as $\boldsymbol{C}_{A_{\theta_2}} = \boldsymbol{R}_2 \boldsymbol{C}_A \boldsymbol{R}_2^{-1}$. $\boldsymbol{R}_2$ as shown in (7) needs a bivector to represent the plane of rotation perpendicular to the rotation axis. This is determined by transforming the bivector $\boldsymbol{e}_{12}$ using the following motors (cf. (8)):

$$\boldsymbol{B}_{j_2} = \boldsymbol{T}_2 \boldsymbol{e}_{12} \boldsymbol{T}_2^{-1} \text{ with } \boldsymbol{T}_2 = \boldsymbol{M}(\theta_1)\boldsymbol{G}(\alpha_1, a_1, d_1). \tag{15}$$

Thus, from (7), $\boldsymbol{R}_2 = \exp^{-\boldsymbol{B}_{j_2} \frac{\theta_2}{2}}$ and the rotated circle $\boldsymbol{C}_{A_{\theta_2}}$ is a function of trigonometric functions of θ_2. Therefore the intersection $\boldsymbol{x} = \boldsymbol{C}_{A_{\theta_2}} \vee \boldsymbol{C}_B$ is also a function of θ_2. For this intersection to be a single point, $\boldsymbol{x} \cdot \boldsymbol{x} = 0$ must be satisfied [4], leading to an univariate quartic polynomial in $t_2 = \tan \frac{\theta_2}{2}$.

Calculating θ_1: For a chosen θ_2 value, the corresponding point $\boldsymbol{x}$ on circles $\boldsymbol{C}_{A_{\theta_2}}$ and $\boldsymbol{C}_B$ can be determined. If we consider $\boldsymbol{C}_B$, it follows from (11) that θ_1 is the angle made by the directed arc on C_B that connects $\boldsymbol{p}$ to the point $\boldsymbol{x}$. To calculate this angle, let us consider two lines, $\boldsymbol{L}_{11} = \boldsymbol{p}_0 \wedge \boldsymbol{p} \wedge \boldsymbol{e}_\infty$ and $\boldsymbol{L}_{12} = \boldsymbol{p}_0 \wedge \boldsymbol{x} \wedge \boldsymbol{e}_\infty$. By projecting these lines onto plane $\boldsymbol{E}_B$, we can find the angle between them using (9). Finding arccos() of the above value yields

θ_1, however the sign of the obtained value depends on the sign of the bivector obtained by vectorizing lines $\boldsymbol{L}_{11}$ and $\boldsymbol{L}_{12}$ as explained in Section 2 and taking the outer product of the resulting vectors $\boldsymbol{B}_1 = -\mathrm{vec}_{\boldsymbol{L}_{11}} \wedge \mathrm{vec}_{\boldsymbol{L}_{12}}$. This bivector represents the positive rotation about the first joint axis. The negative sign comes from (11). The sign of θ_1 obtained from (9) must be adjusted to match the sign of $\boldsymbol{B}_1$.

Calculating θ_3: Similar to the calculation of θ_1, we look for two points on $\boldsymbol{C}_A$ whose connecting directed arc makes an angle θ_3 about its center. One such point is $\boldsymbol{p}_H$, to find the other one, we have to rotate the obtained $\boldsymbol{x}$, currently on $\boldsymbol{C}_{A_{\theta_2}}$ back to $\boldsymbol{C}_A$ by its corresponding θ_2: $\boldsymbol{x}_{-\theta_2} = \boldsymbol{R}_2^{-1}\boldsymbol{x}\boldsymbol{R}_2$ with $\boldsymbol{R}_2 = \exp^{-\boldsymbol{B}_{j_2}\frac{\theta_2}{2}}$. It follows from (11) that θ_3 is the angle made by the directed arc on C_A that connects $\boldsymbol{x}_{-\theta_2}$ to the point $\boldsymbol{p}_H$. To calculate this angle, let us again consider two lines, $\boldsymbol{L}_{31} = \boldsymbol{p}_2 \wedge \boldsymbol{p}_H \wedge \boldsymbol{e}_\infty$ and $\boldsymbol{L}_{32} = \boldsymbol{p}_2 \wedge \boldsymbol{x}_{-\theta_2} \wedge \boldsymbol{e}_\infty$. By projecting these lines onto the plane $\boldsymbol{E}_A$, we can find the angle between them using (9). In this case, the sign of θ_3 depends on the sign of the bivector $\boldsymbol{B}_3 = -\mathrm{vec}_{\boldsymbol{L}_{31}} \wedge \mathrm{vec}_{\boldsymbol{L}_{32}}$, vectors corresponding to $\boldsymbol{L}_{31}$ and $\boldsymbol{L}_{32}$. $\boldsymbol{B}_3$ represents the positive rotation about the third joint axis. The sign of θ_3 obtained from (9) must be adjusted to match the sign of $\boldsymbol{B}_3$.

Example

Let us consider a generic 3R robot shown in Fig. 1 with D-H parameters as $\mathbf{d} = [0,1,1], \mathbf{a} = [1,2,1.5], \alpha = [\pi/4, -\pi/6, 0]$. We find its inverse kinematics solutions for a given arbitrary end-effector position: $\mathbf{p} = [-1.62, 0.465, 2.21]$. For demonstration purposes, $\mathbf{p}$ is chosen such that we have 4 IKS. In terms of CGA, $\boldsymbol{p} = \mathrm{up}(\mathbf{p}) = -1.62\boldsymbol{e}_1 + 0.465\boldsymbol{e}_2 + 2.21\boldsymbol{e}_3 + 3.36\boldsymbol{e}_4 + 4.36\boldsymbol{e}_5$. In home position, $\boldsymbol{p}_0 = -0.5\boldsymbol{e}_4 + 0.5\boldsymbol{e}_5,\ \boldsymbol{p}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_5,\ \boldsymbol{p}_2 = 3\boldsymbol{e}_1 - 0.707\boldsymbol{e}_2 + 0.707\boldsymbol{e}_3 + 4.5\boldsymbol{e}_4 + 5.5\boldsymbol{e}_5,\ \boldsymbol{p}_H = 4.5\boldsymbol{e}_1 - 0.966\boldsymbol{e}_2 + 1.67\boldsymbol{e}_3 + 11.5\boldsymbol{e}_4 + 12.5\boldsymbol{e}_5$ From (12),

$$\begin{aligned}
\boldsymbol{S}_B^* &= \boldsymbol{p}_0 - \frac{1}{2}2.78^2\boldsymbol{e}_\infty \Rightarrow \boldsymbol{S}_B = -3.36\boldsymbol{e}_{1234} - 4.36\boldsymbol{e}_{1235} \\
\boldsymbol{E}_B^* &= \boldsymbol{e}_3 + 2.21\boldsymbol{e}_\infty \Rightarrow \boldsymbol{E}_B = 2.21\boldsymbol{e}_{1234} + 2.21\boldsymbol{e}_{1235} - \boldsymbol{e}_{1245} \\
\boldsymbol{C}_B &= \boldsymbol{S}_B \vee \boldsymbol{E}_B = 2.21\boldsymbol{e}_{123} + 3.36\boldsymbol{e}_{124} + 4.36\boldsymbol{e}_{125}
\end{aligned}$$

From (13),

$$\begin{aligned}
\boldsymbol{S}_A^* &= \boldsymbol{p}_2 - \frac{1}{2}1.802^2\boldsymbol{e}_\infty \\
\boldsymbol{S}_A &= 3.87\boldsymbol{e}_{1234} + 2.87\boldsymbol{e}_{1235} - 0.707\boldsymbol{e}_{1245} - 0.707\boldsymbol{e}_{1345} - 3\boldsymbol{e}_{2345} \\
\boldsymbol{n}_A &= -0.26\boldsymbol{e}_2 + 0.96\boldsymbol{e}_3 \\
\boldsymbol{E}_A &= \boldsymbol{n}_A + (\boldsymbol{n}_A \cdot \boldsymbol{p}_H)\boldsymbol{e}_\infty = 1.87\boldsymbol{e}_{1234} + 1.87\boldsymbol{e}_{1235} - 0.966\boldsymbol{e}_{1245} - 0.259\boldsymbol{e}_{1345} \\
\boldsymbol{C}_A &= \boldsymbol{S}_A \vee \boldsymbol{E}_A = 1.87\boldsymbol{e}_{123} - 2.42\boldsymbol{e}_{124} - 1.46\boldsymbol{e}_{125} + 0.317\boldsymbol{e}_{134} + 0.575\boldsymbol{e}_{135} \\
&\qquad - 0.5\boldsymbol{e}_{145} - 5.6\boldsymbol{e}_{234} - 5.6\boldsymbol{e}_{235} - 2.9\boldsymbol{e}_{245} - 0.776\boldsymbol{e}_{345}
\end{aligned}$$

from (15), $\boldsymbol{B}_{j_2} = 0.707(\boldsymbol{e}_{12} + \boldsymbol{e}_{13} - \boldsymbol{e}_{24} - \boldsymbol{e}_{25} - \boldsymbol{e}_{34} - \boldsymbol{e}_{35})$.
The condition for the intersection of this circle with $\boldsymbol{C}_B$ to be a point,

$$\boldsymbol{x} \cdot \boldsymbol{x} = -4.60 \sin(\theta_2) - 0.95 \sin(2\theta_2) + 1.09 \cos(\theta_2) - 1.99 \cos(2\theta_2) + 2.61 = 0$$

Solving the above equation gives four solutions, $\theta_2 = \{2.0, 0.326, 1.56, -2.998\}$. Fig. 2 shows the four rotated circles corresponding to the obtained θ_2 solutions.

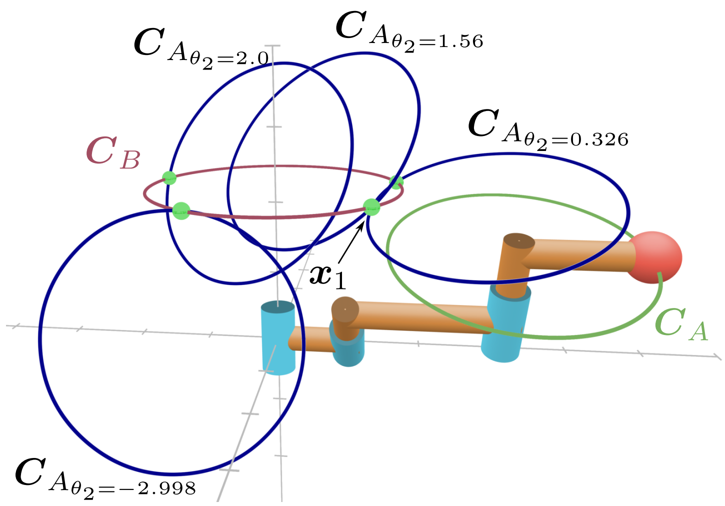

Fig. 2: An example illustration of four IKS of a generic 3R robot interpreted as intersection between a fixed and rotating circle.

If we consider one of those circles, $\boldsymbol{C}_{A_{\theta_2=1.56}}$, its intersection with $\boldsymbol{C}_B$ gives:

$$\boldsymbol{x}_1 = \boldsymbol{C}_{A_{\theta_2=1.56}} \vee \boldsymbol{C}_B = 1.3\boldsymbol{e}_1 - 1.07\boldsymbol{e}_2 + 2.21\boldsymbol{e}_3 + 3.36\boldsymbol{e}_4 + 4.36\boldsymbol{e}_5$$

Note that in $\boldsymbol{x}_1$ the coefficient of $\boldsymbol{e}_3$ matches that of $\boldsymbol{p}$ as both of them lie in the plane of circle $\boldsymbol{C}_B$. In fact, this should hold for all intersection points of $\boldsymbol{C}_{A_{\theta_2}}$ and $\boldsymbol{C}_B$.
To find θ_1, we calculate the projections of $\boldsymbol{L}_{11}$ and $\boldsymbol{L}_{12}$ onto $\boldsymbol{E}_B$:

$$\boldsymbol{L}_{11_{\boldsymbol{E}_B}} = -3.58\boldsymbol{e}_{134} - 3.58\boldsymbol{e}_{135} + 1.62\boldsymbol{e}_{145} + 1.03\boldsymbol{e}_{234} + 1.03\boldsymbol{e}_{235} - 0.465\boldsymbol{e}_{245}$$
$$\boldsymbol{L}_{12_{\boldsymbol{E}_B}} = 2.87\boldsymbol{e}_{134} + 2.87\boldsymbol{e}_{135} - 1.3\boldsymbol{e}_{145} - 2.37\boldsymbol{e}_{234} - 2.37\boldsymbol{e}_{235} + 1.07\boldsymbol{e}_{245}$$

From (9), θ_1 could be ± 2.731. Its sign is determined by calculating the bivector $\boldsymbol{B}_1 = -1.13\boldsymbol{e}_{12}$, which is negative. Therefore $\theta_1 = -2.731$. To find θ_3, we calculate the projections of $\boldsymbol{L}_{31}$ and $\boldsymbol{L}_{32}$ onto $\boldsymbol{E}_A$:

$$\boldsymbol{L}_{31_{\boldsymbol{E}_A}} = -1.45\boldsymbol{e}_{124} - 1.45\boldsymbol{e}_{125} + 2.51\boldsymbol{e}_{134} + 2.51\boldsymbol{e}_{135} - 1.5\boldsymbol{e}_{145}$$
$$\begin{aligned}\boldsymbol{L}_{32_{\boldsymbol{E}_A}} = {} & 3.79\boldsymbol{e}_{124} + 3.79\boldsymbol{e}_{125} - 1.29\boldsymbol{e}_{134} - 1.29\boldsymbol{e}_{135} + 1.19\boldsymbol{e}_{145} - 1.7\boldsymbol{e}_{234} \\ & - 1.7\boldsymbol{e}_{235} + 0.881\boldsymbol{e}_{245} + 0.236\boldsymbol{e}_{345}\end{aligned}$$

From (9), θ_3 could be ± 2.488. Its sign is determined by calculating the bivector $\boldsymbol{B}_3 = -1.32\boldsymbol{e}_{12} - 0.354\boldsymbol{e}_{13}$, which is negative. Therefore $\theta_3 = -2.488$. Following this procedure for the remaining θ_2 solutions, the complete set of IKS for the given problem is $(0.0, 2.0, 1.0)$, $(2.58, 0.326, 2.138)$, $(-2.731, 1.56, -2.488)$, $(-1.341, -2.998, -1.753)$.

4 Conclusions

A generic IKM for 3R robots was presented in this article as the intersection of two circles as elements of Conformal Geometric Algebra. CGA also allows us to extend the motion description to include orientations. This fact will be used to study the IKM for a generic 6R robot and its kinematic properties.

A (very) strange approach to inverse kinematics of 3R robots: It is possible to further accelerate the computation of generic 3R robots by combining previously published methods. We have a univariate quartic polynomial in t_3 from [1], in t_1 from [10] and we obtain a quartic univariate polynomial in t_2 from Section 3, where $t_i = \tan\frac{\theta_i}{2}$. As all of them are quartic polynomials, they independently give us a closed form solution for each angle. A matching exercise can replace the backpropagation in individual approaches to obtain the inverse kinematic solution of a generic 3R robot.

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