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Markov Data-Based LQG Control[1]

In this paper the Markov data-based LQG control problem is considered. The Markov data-based LQG control problem is to find the optimal control sequence which minimizes a quadratic cost function over some finite interval [0, N]. To solve this problem, we show that a complete input-output description of the system is not necessary. Obviously, a complete state space model is not necessary for this problem either. The main contributions of this paper include: (i) develop a new data-based LQG controller in a recursive form and a batch-form, (ii) derive a closed-form expression for the system's optimal performance in terms of the Markov parameters, (iii) develop an algorithm for choosing the output weighting matrix, and (iv) demonstrate that the amount of information about the system required by the data-based controller design is less than the amount required to construct the full state space model. A numerical example is given to show the effectiveness of the data-based design method. [S0022-0434(00)02503-X]

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1 Introduction

In spite of the advanced state of control theory, the most fundamental issue in control remains open: What model is appropriate for control design? The complexity of this question is illustrated by the fact that the set of models that yield exactly the same optimal controller is large. Furthermore, members of this ''controller equivalent'' set of models can be arbitrarily far apart by *open-loop* criteria. For example, see [1–3]. While robust control takes a *given* nominal model and asks how large can the *open-loop* errors be (for acceptable *closed-loop* performance), performance is usually compromised to obtain the robustness. This compromise leaves two concerns: (i) a ''controller equivalent'' model (which would *not* compromise performance) might not lie as close to the nominal model as the perturbed model used for robust control, and (ii) the philosophy of robust control is suspect, since it is not necessary to force the system to tolerate an *open-loop* measure of error, but a closed-loop measure of model error. These are not new observations. Skelton [1] shows that unbounded open-loop errors do not suggest a bad model for control design, and arbitrarily small open loop errors do not suggest a good model for control design. We believe the determination of the ''nominal'' model for control design deserves much more attention.

Mathematical models of complex physical systems are not always reliable, and this has severely limited the acceptability of model-based methods in industry. Even adaptive and robust control methods have not completely overcome the danger (and fear in potential users) of modeling errors. While the model-based theory promises good performance when the model is accurate, it can deliver much worse performance and even instability when the model upon which the controller is based is not accurate. Moreover, since the modeling and control design problems are not independent problems it is not clear how to improve a model for control design, since the goodness of the model must be judged by closed loop performance of the real system (or at least a higher order truth model) [1]. It was shown in [1] that an arbitrarily small model error by an open-loop criterion can still yield destabilizing model-based controllers.

This paper considers two questions: In finite horizon optimal control, how much information about the system is really necessary to implement the exact optimal control? Second, how can this information be obtained? We categorize controllers as either model-based or data-based. A model-based controller requires a transfer function or a state space model of the plant. A data-based controller requires neither. A major impact of Kalman's work [4] was the introduction of state space concepts building the foundation of a rigorous model-based control theory. Model-based control methods (including *LQG*, H_∞, and *MPC*) have been well developed since the 1960s. Models may be parametrized in two ways: (i) The construction of a model from first principles, or (ii) system identification using input-output data. Identification is further divided into two categories, (iia) the determination of specific parameter values within a specified model structure (this is the ''gray-box'' approach), or (iib) the determination of a state space model without structural constraints (this is the ''black-box'' approach). Any adaptive control that requires the update of a black or gray-box model is considered a model-based theory of control. Fuzzy control is usually data-based, but could also be model-based. One can interpret PID controllers as ''model-based,'' assuming a second-order model of the plant, or as ''data-based'' by empirical tuning without knowledge of a model.

For SISO systems, the number of Markov parameters required to completely define a proper nth order system is $2n$ [3]. We shall require a smaller set of the Markov parameters (which can be constructed from almost any input-output sequence). Of course, if enough Markov parameters are given, a complete state space model can be constructed. Ho and Kalman [5] gave the important principles of a minimal realization theory and an algorithm to construct a state space model of a linear system, which is referred to as Kalman-Ho algorithm. In [2,3], an algorithm is given to generate state space models using Markov parameters and covariance parameters. The algorithm is called q-Markov COVER algorithm. The main feature of this technique is that the realizations match the first q Markov parameters and the first q covariance parameters of the system.

A complete state space model might be more than enough information for optimal control. Consider a fourth-order dynamic system with a unit pulse response sequence: $\{0.9990, -0.4681, 0.2953, -0.2206, 0.1763, \ldots\}$. This example will be further illustrated in Section 5 where initial conditions and system disturbance information are provided. The first eight Markov parameters are needed to construct a state space model. However, only the first four Markov parameters are required to compute the exact an optimal control sequence which minimizes $y_3^2+\Sigma_{k=0}^2(y_k^2$

[1]This work was done when the first author was a Ph.D. researcher with the Structural Systems and Control Laboratory, Purdue University, West Lafayette, Indiana, before September 1997.

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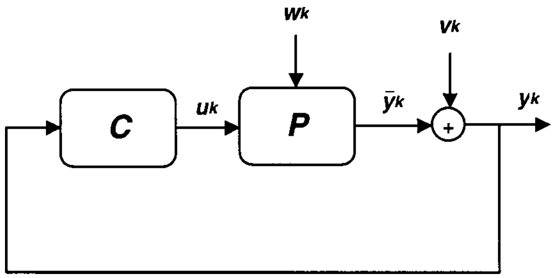

Fig. 1 Data-based LQG control

$+0.001\,u_k^2)$, where y_k is the system output and u_k is the system input. The optimal control sequence computed from the first four Markov parameters is $\{0, -0.4829, 0.2561\}$. This example shows that with only the first four Markov parameters, the complete system model cannot be constructed, but the exact optimal control can be constructed. The data-based control method developed in this paper can be used to design the optimal control without knowledge of the complete system model.

These deficiencies of model-based control theory motivate the development of a data-based control theory. Our data-based control system is based upon the fact that a finite number of Markov parameters can be computed from the knowledge of input-output data, and hence any controller based only on a finite number of the Markov parameters is data-based. Markov parameters can be directly calculated from step response [6]. It can also be calculated from well-conditioned time response [7]. In this paper the attention is focused on digital control systems. We state the problem as follows.

Consider the system block diagram depicted by Fig. 1. In the figure, the block **P** is the unknown plant, w_k is the system external disturbance, y_k is the output of interest, which is corrupted by the measurement disturbance v_k,$\overline{y_k}$ is the uncorrupted output, u_k is the control signal, and the block **C** is an optimal digital controller to be designed. Let the finite Markov parameter sequences be: $[M_1, M_2, \ldots, M_{N-1}]$ which represents the relation between u_k and $\overline{y_k}$, and $[H_1, H_2, \ldots, H_{N+1}]$ which represents the relation between w_k and $\overline{y_k}$. Then the data-based control problem can be stated as follows: Given only the finite Markov parameter data sequences M_i, $i=1, 2, \ldots, N-1$, H_i, $i=1, 2, \ldots, N+1$ and the characteristics of the disturbances w_k and v_k, can one compute the optimal control signal sequence $\{u_1, u_2, \ldots, u_{N-1}\}$ to minimize the cost function (2.3)?

The significance of this control problem should be clear. It avoids the modeling (model reduction) steps of the conventional design procedure and hence modeling errors. The system Markov parameters can be obtained much more directly from system sequence data. Furthermore, it appears that a scheme consisting of on-line update of the Markov parameters plus a controller which adapts optimally to updated Markov parameters should be possible for highly robust control.

The idea of using system Markov parameters to design an optimal controller is not new. In [8], the Pulse Response Based Control (PRBC) problem was addressed for disturbance-free linear systems and an algorithm for solving the PRBC problem was presented. Recently, Furuta et al. [9,10] presented the first result associated with the data-based LQG control. But the data-based controller in [9] uses an infinite number of Markov parameters for the finite horizon LQG problem. This is equivalent to having enough data to compute the complete model. Hence, it is still model-based. The work [10] was the continuation of the earlier work [9], and assumes zero initial state covariance. The revised LQG controller is expressed in terms of finite number of Markov parameters and all past input and output data. Their controller has a batch-from implementation.

The contributions of this paper are the following: (i) Develop a new data-based finite horizon LQG controller in a recursive form, which uses the first $N+1$ Markov parameters. The controller can also be implemented in a batch-form; (ii) Derive a closed-form expression for the system's optimal performance in terms of the Markov parameters; (iii) Develop an algorithm for choosing the output weighting matrix; (iv) Demonstrate that the amount of information about the system required by the data-based controller design is less than the amount required to construct the full state space model. An example is given to show the effectiveness of the data-based design method. The preliminary results of this research was shown in [11].

The paper is organized as follows. In the following section the model-based LQG control problem is briefly reviewed and the batch-form solutions to the difference Riccati equations are given. In Section 3, the data-based LQG control problem is formulated and solved. Section 4 addresses a data-based ''Output Variance Constraint (OVC)'' problem and an algorithm is given for choosing the weighting matrices for the LQG cost function. Section 5 presents a numerical example to illustrate the effectiveness of the data-based design method. Proofs are given in Appendices. Finally, conclusions are given.

Notation. For matrices A and B, $A>0$ means A is a positive definite symmetric matrix, and $A>B$ means $A-B>0$. A^T denotes the transpose of A. A^{-1} denotes the inverse of A if it exists. $\mathbf{E}\{\circ\}$ is the mathematical expectation operator. $\mathbf{P}[M]$ means 0 if the matrix $M\leqslant 0$, otherwise, $\mathbf{P}[M]=U_+E_+U_+^T$ from the matrix spectral decomposition of $M=U_+E_+U_+^T+U_-E_-U_-^T$, where the diagonal elements of E_+ contain the positive eigenvalues of M and the diagonal elements of E_- contain the negative eigenvalues of M. For a vector signal y, $y_{k,i}$ denotes the sample at time k of the ith channel.

2 Model-Based LQG Control Theory

In what follows we shall briefly review the LQG model-based control theory which is the basis for the development of data-based control theory. It is also possible to derive the data-based control theory without using the model-based theory.

2.1 Model-Based LQG Control. Consider a linear discrete-time system

$$x_{k+1}=Ax_k+Bu_k+Dw_k$$
$$y_k=Cx_k+v_k \tag{2.1}$$

where x_k is the n dimensional state vector, u_k is the n_u dimensional control input vector, y_k is the n_y dimensional output vector of interest, x_0 is the initial random state vector, w_k is the n_w dimensional disturbance vector to the system, and v_k is the n_y dimensional disturbance vector corrupting the output. It is assumed that w_k, v_k, and x_0 are zero mean, uncorrelated white noises. The covariance matrices for the disturbances w_k and v_k are given by $W>0$ and $V>0$, respectively. The system matrices A, B, C, and D are real constant with appropriate dimensions.

The model-based LQG control problem is the problem of finding the functional

$$u_k=f(u_0,\ u_1,\ \ldots,\ u_{k-1},\ y_0,\ y_1,\ \ldots,\ y_{k-1}) \tag{2.2}$$

such that the quadratic cost

$$J=\mathbf{E}\left\{y_N^TQy_N+\sum_{k=0}^{N-1}(y_k^TQy_k+u_k^TRu_k)\right\} \tag{2.3}$$

is minimized subject to system model (2.1) and known characteristics of the initial conditions and the disturbances (x_0, w_k, v_k), where Q and R are positive definite symmetric weighting matrices.

Notice that the cost function (2.3) can be rewritten as follows

$$J=\mathbf{E}\left\{x_N^T C^T Q C x_N+\sum_{k=0}^{N-1}(x_k^T C^T Q C x_k+u_k^T R u_k)\right\} \tag{2.4}$$

The solution of the model-based LQG control problem is well known. The optimal input is given by

$$u_k=-(R+B^T X_{k+1}B)^{-1}B^T X_{k+1}A\hat{x}_k,\quad k=0,\ 1,\ 2,\ \ldots,\ N-1 \tag{2.5}$$

where X_{k+1} is the solution of the difference Riccati equation

$$X_k=C^T QC+A^T X_{k+1}A-A^T X_{k+1}B(R+B^T X_{k+1}B)^{-1}B^T X_{k+1}A,$$
$$X_N=C^T QC \tag{2.6}$$

The optimal state estimation, $\hat{x}_k$, can be obtained from

$$\hat{x}_{k+1}=A\hat{x}_k+Bu_k+L_k(y_k-C\hat{x}_k),\ \ \hat{x}_0=0 \tag{2.7}$$

where the estimator gain L_k is given by

$$L_k=AY_kC^T(V+CY_kC^T)^{-1} \tag{2.8}$$

where Y_k is the solution of the following difference Riccati equation

$$Y_{k+1}=DWD^T+AY_kA^T-AY_kC^T[V+CY_kC^T]^{-1}CY_kA^T,$$
$$Y_0=DW_0D^T \tag{2.9}$$

Remark 2.1. It is assumed that the initial conditions of the plant lie in the range space of the disturbance matrix D. That is, $x_0=D\hat{w}_0$ for some $\hat{w}_0$ with known covariance $W_0>0$. Then the boundary condition for the estimator difference Riccati equation (2.9) is given by

$$Y_0=\mathbf{E}\{(x_0-\hat{x}_0)(x_0-\hat{x}_0)^T\}=E\{x_0x_0^T\}=DW_0D^T \tag{2.10}$$

In the next subsection the batch-form solutions for the difference Riccati equations (2.6) and (2.9) will be given.

2.2 Batch-Form Solutions to the Riccati Equations. The difference Riccati equations play crucial roles in linear quadratic optimal control for discrete-time systems. Many effective numerical methods for solving the Riccati equations have been developed. Most often the recursive forms of the Riccati equations (2.6) and (2.9) are used. The batch-form, i.e., closed-form, solutions have received less attention, although such solutions are available in [9,10,12]. In what follows, we shall present the batch-form solutions of the difference Riccati equations which will be used in derivation of the data-based controller.

Lemma 2.1. The difference Riccati equation (2.6) is equivalent to the following batch-form expression

$$\mathbf{X}_k=\mathbf{C}_k^T(\mathbf{Q}_k^{-1}+\mathbf{S}_k\mathbf{R}_k^{-1}\mathbf{S}_k^T)^{-1}\mathbf{C}_k,\quad k=2,3,\ldots,N \tag{2.11}$$

where

$$\mathbf{C}_k=\begin{bmatrix} C\\ CA\\ CA^2\\ \vdots\\ CA^{N-k}\end{bmatrix},$$

$$\mathbf{S}_k=\begin{bmatrix} 0 & & & & \\ CB & 0 & & & \\ CAB & CB & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ CA^{N-k-1}B & CA^{N-k-2}B & \cdots & CB & 0\end{bmatrix},$$

$$\mathbf{S}_N=0 \tag{2.12}$$

$$\mathbf{Q}_k=diag(Q,\ Q,\ \ldots,\ Q),\quad \mathbf{R}_k=diag(R,\ R,\ \ldots,\ R) \tag{2.13}$$

and $\mathbf{Q}_k$ and $\mathbf{R}_k$ contain $N-k+1$ diagonal blocks, respectively.

Proof. This lemma can be proved using backward induction and the Matrix Inversion Lemma.

Lemma 2.2. The difference Riccati equation (2.9) is equivalent to the following batch-form expression

$$Y_k=\mathbf{D}_k(\mathbf{W}_k^{-1}+\mathbf{T}_k^T\mathbf{V}_k^{-1}\mathbf{T}_k)^{-1}\mathbf{D}_k^T,\quad k=0,\ 1,\ \ldots,\ N-2 \tag{2.14}$$

where

$$\mathbf{D}_k=[D\ \ AD\ \ \ldots\ \ A^kD],$$

$$\mathbf{T}_k=\begin{bmatrix} 0 & CD & CAD & \cdots & CA^{k-1D}\\ & 0 & CD & \cdots & CA^{k-2}D\\ & & \ddots & \ddots & \vdots\\ & & & \ddots & CD\\ & & & & 0\end{bmatrix} \tag{2.15}$$

$$\mathbf{W}_k=diag\{W,\ \ldots\ W,\ W_0\},\quad \mathbf{V}_k=diag\{V,\ \ldots\ V,\ V\} \tag{2.17}$$

and $\mathbf{W}_k$ and $\mathbf{V}_k$ contain $k+1$ diagonal blocks, respectively.

Proof. This lemma is the dual version of Lemma 2.1.

Remark 2.2. For a system with realization (2.1), $M_i=CA^{i-1}B$, $i=1,\ 2,\ldots$ are referred to as the input-output Markov parameters, $H_i=CA^{i-1}D$, $i=1,\ 2,\ldots$ are referred to as the disturbance-output Markov parameters. Notice that $\mathbf{S}_k$ contains only the input-output Markov parameters and $\mathbf{T}_k$ contains only the disturbance-output Markov parameters.

3 Data-Based LQG Control Theory

The model-based LQG control requires an explicit state space model (2.1). In this section, we will consider the data-based LQG control problem which requires only the first $N+1$ Markov parameters.

The data-based LQG control problem: Consider the discrete-time system depicted in Fig. 1. Suppose we are given the finite Markov parameter data sequences: M_i, $i=1,2,\ \ldots,N-1$, H_i, $i=1,2,\ \ldots,N+1$, the disturbance covariance matrices W, V, and W_0. The data-based LQG control problem is to find the optimal control sequence

$$u_k=f(M_i,\ H_i,\ W,\ V,\ Q,\ R,\ u_{k-1},\ y_{k-1}) \tag{3.1}$$
$$k=0,\ 1,\ 2,\ \ldots,\ N-1$$

such that the cost function (2.3) is minimized. Notice that the data-based LQG control problem is equivalent to the finite horizon model-based LQG control problem in the sense of optimality.

3.1 Data-Based Optimal Estimation. We shall first define the controller state vector which gives the optimal estimation from the past observations. Then we shall present an algorithm to compute the controller state recursively in terms of the Markov data sequences and the past observations. Note that in the model-based control, $\hat{x}_k$, is the optimal state estimation and also serves as the controller state vector. In our data-based control, we define the controller state vector as follows:

$$\overline{x}_k^{N-k+1}\triangleq\mathbf{C}_k\hat{x}_k=\begin{bmatrix} C\\ CA\\ \vdots\\ CA^{N-k}\end{bmatrix}\hat{x}_k \tag{3.2}$$

where $\hat{x}_k$ is the optimal estimation of the plant states x_k, the superscript $N-k+1$ indicates that, $\overline{x}_k^{N-k+1}$, has the dimension of $(N-k+1)n_y$, and the subscript k is the time index. It should be pointed out that $\hat{x}_k$ is computed using the state space model. In the following we will show that the data-based controller state vector,

$\overline{x}_k^{N-k+1}$, can be computed using only the system Markov parameters. It is not necessary to compute $\hat{x}_k$. The following theorem gives our first result.

Theorem 3.1. Consider the plant **P** with the known Markov parameter sequences:

$$M_l,\ \ i=1,\ 2,\ \ldots,\ N-1 \quad \text{and}\ H_i,\ \ i=1,\ 2,\ \ldots,\ N+1 \tag{3.3}$$

Then the data-based controller state equation is given in terms of the Markov parameter sequences as follows:

$$\overline{x}_k^{N-k+1}=\mathbf{A}_k\overline{x}_{k-1}^{N-k+2}+\mathbf{B}_k u_{k-1}+\mathbf{F}_k y_{k-1} \tag{3.4}$$

$$\overline{x}_0^{N+1}=0$$

where $\mathbf{A}_k,\mathbf{B}_k,\mathbf{F}_k$ are time-varying gain matrices and given by:

$$\mathbf{A}_k=[-\mathbf{F}_k\ \ \mathbf{I}_{(N-k+1)n_y}],\quad \mathbf{F}_k=\mathbf{H}_k\mathbf{P}_k\mathbf{N}_k^T(V+\mathbf{N}_k\mathbf{P}_k\mathbf{N}_\mathbf{k}^T)^{-1} \tag{3.5}$$

$$\mathbf{B_k}=\begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{N-k+1}\end{bmatrix},$$

$$\mathbf{H}_k=\begin{bmatrix} H_2 & H_3 & \cdots & H_{k+1} \\ H_3 & H_4 & \cdots & H_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-k+2} & H_{N-k+3} & \vdots & H_{N+1}\end{bmatrix} \tag{3.6}$$

$$\mathbf{P}_k=(\mathbf{W}_k^{-1}+\mathbf{T}_k^T\mathbf{V}_k^{-1}\mathbf{T}_k)^{-1},\quad \mathbf{N}_k=[H_1\ \ H_2\ \ \ldots\ \ H_k] \tag{3.7}$$

$$\mathbf{T}_k=\begin{bmatrix} 0 & H_1 & H_2 & \cdots & H_{k-1} \\ & 0 & H_1 & \cdots & H_{k-2} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & H_1 \\ & & & & 0\end{bmatrix} \tag{3.8}$$

$$\mathbf{W}_k=diag\{W,\ \ \ldots\ \ W,\ W_0\},\quad \mathbf{V}_k=diag\{V,\ \ \ldots\ \ V,\ V\} \tag{3.9}$$

and $\mathbf{W}_k$ and $\mathbf{V}_k$ contain k diagonal blocks, respectively.

Proof. See Appendix A.

It should be noted that the optimal estimation in Theorem 3.1 does not utilize an explicit model of the plant. All of the coefficient matrices are functions of the first $N+1$ Markov parameters of the plant. Note also that the dimension of the controller states is time-varying. For instance, when $k=1$, $\overline{x}_1^N$ is a Nn_y dimensional vector, and when $k=N-1$, $\overline{x}_{N-1}^2$ is a $2n_y$ dimensional vector. The intermediate $\mathbf{H}_k$ is a Hankel matrix and $\mathbf{T}_k$ is a Toeplitz matrix. During the control horizon $1\rightarrow N-1$, the following Markov parameter data is used: $H_1,\ \ldots,\ H_{N+1}$ and $M_1,\ \ldots,\ M_{N-1}$. In the calculation of $\overline{x}_1^N$, $\mathbf{B}_0$ is not used since $u_0=0$.

3.2 Data-Based LQG Controller Synthesis. In the previous subsection, a generalized controller state vector $\overline{x}_k^{N-k+1}$ has been introduced, which is a function of the system Markov parameters and the past input-output data. The following result provides an optimal control law in terms of the vector, $\overline{x}_k^{N-k+1}$.

Theorem 3.2. Consider the system **P** with its Markov parameter sequences M_i and H_j, $i=1,\ 2,\ \ldots,\ N-1$ and $j=1,\ 2,\ \ldots,\ N+1$. The optimal data-based LQG control law associated with the cost function (2.3) is given by

$$u_k=\mathbf{G}_k\overline{x}_k^{N-k+1} \tag{3.10}$$

where $\mathbf{G}_k$ is referred to as the data-based control gain, $\overline{x}_k^{N-k+1}$ is the data-based controller state vector derived in Theorem 3.1, and

$$\begin{aligned}\mathbf{G}_k=&-(R+\mathbf{B}_{k+1}^T(\mathbf{Q}_{k+1}^{-1}+\mathbf{S}_{k+1}R_{k+1}^{-1}S_{k+1}^T)^{-1}\mathbf{B}_{k+1})^{-1}\mathbf{B}_{k+1}^T\\ &\cdot(\mathbf{Q}_{k+1}^{-1}+\mathbf{S}_{k+1}\mathbf{R}_{k+1}^{-1}\mathbf{S}_{k+1}^T)^{-1}[O_{n_y}\ \ \mathbf{I}_{(N-k)n_y}]\end{aligned} \tag{3.11}$$

$$\mathbf{B}_{k+1}=\begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{N-k}\end{bmatrix},$$

$$\mathbf{S}_{k+1}=\begin{bmatrix} 0 & & & & \\ M_1 & 0 & & & \\ M_2 & M_1 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ M_{N-k-1} & M_{N-k-2} & \cdots & M_1 & 0\end{bmatrix} \tag{3.12}$$

$$\mathbf{Q}_{k+1}=diag(Q,\ Q,\ \ldots,\ Q),\quad \mathbf{R}_{k+1}=diag(R,\ R,\ \ldots,\ R) \tag{3.13}$$

and $\mathbf{Q}_{k+1}$ and $\mathbf{R}_{k+1}$ contain $N-k$ diagonal blocks, respectively.

Proof. See Appendix B.

Remark 3.1. The data-based controller gain $\mathbf{G}_k\in\mathcal{R}^{n_u\times(N-k)n_y}$ given in Theorem 3.2 is a function of only the first $N+1$ system Markov parameters. Notice also that the controller has a dynamic structure. When $k=1$, $\mathbf{S}_2$ is a $(N-1)$ by $(N-1)$ Toeplitz matrix. When $k=N-1$, $\mathbf{S}_N=0$. During the control horizon $1\rightarrow N-1$, the system Markov parameter sequence M_i, $i=1,\ 2,\ \ldots,\ N-1$, is used for the gain calculation. Theorem 3.1 and Theorem 3.2 provide the data-based LQG control design approach. In the next subsection, we shall discuss the implementation of the controller.

3.3 Data-Based Controller Implementation. The data-based LQG optimal controller can be put into two forms: (i) The recursive form in (3.4) and (3.10). For convenience, we put them together as follows:

$$\overline{x}_k^{N-k+1}=\mathbf{A}_k\overline{x}_{k-1}^{N-k+2}+\mathbf{B}_k u_{k-1}+\mathbf{F}_k y_{k-1},\ \ \overline{x}_0^{N+1}=0$$

$$u_k=\mathbf{G}_k\overline{x}_k^{N-k+1} \tag{3.14}$$

(ii) A batch form: the current control input is calculated using all the past input and output data directly. Now we eliminate the intermediate controller state vector and obtain a control law form in a batch form.

From (3.4) we have

$$\begin{aligned}\overline{x}_0^{N+1}&=0\\ \overline{x}_1^{N}&=\mathbf{A}_1x_0^{N+1}+\mathbf{B}_1u_0+\mathbf{F}_1y_0=\mathbf{F}_1y_0\\ \overline{x}_2^{N-1}&=\mathbf{A}_2x_1^N+\mathbf{B}_2u_1+\mathbf{F}_2y_1=\mathbf{B}_2u_1+\mathbf{A}_2^NF_1y_0+\mathbf{F}_2y_1\\ \overline{x}_3^{N-2}&=\mathbf{A}_3x_2^{N-1}+\mathbf{B}_3u_2+\mathbf{F}_3y_2\\ &=\mathbf{A}_3\mathbf{B}_2u_1+\mathbf{B}_3u_2+\mathbf{A}_3\mathbf{A}_2\mathbf{F}_1y_0+\mathbf{A}_3\mathbf{F}_2y_1+\mathbf{F}_3y_2\\ &\vdots\end{aligned}$$

$$\begin{aligned}\overline{x}_{N-1}^2=&\mathbf{A}_{N-1}\cdots\mathbf{A}_3\mathbf{B}_2u_1+\cdots+\mathbf{A}_{N-1}\mathbf{F}_{N-2}u_{N-3}+\mathbf{F}_{N-1}u_{N-2}\\ &+\mathbf{A}_{N-1}\cdots\mathbf{A}_3\mathbf{A}_2\mathbf{F}_1y_0+\cdots+\mathbf{A}_{N-1}\mathbf{F}_{N-2}y_{N-3}+\mathbf{F}_{N-1}y_{N-2}\end{aligned}$$

Then using (3.10) we obtain

$$\begin{aligned}u_0&=0\\ u_1&=\mathbf{G}_1\mathbf{F}_1y_0\\ u_2&=\mathbf{G}_2\mathbf{B}_2u_1+\mathbf{G}_2\mathbf{A}_2\mathbf{F}_1y_0+\mathbf{G}_2\mathbf{F}_2y_1\\ u_3&=\mathbf{G}_3\mathbf{A}_3\mathbf{B}_2u_1+\mathbf{G}_3\mathbf{B}_3u_2+\mathbf{G}_3\mathbf{A}_3\mathbf{A}_2\mathbf{F}_1y_0+\mathbf{G}_3\mathbf{A}_3\mathbf{F}_2y_1+\mathbf{F}_3y_2\\ &\vdots\end{aligned}$$

$$u_{N-1}=\mathbf{G}_{N-1}\mathbf{A}_{N-1}\cdots\mathbf{A}_3\mathbf{B}_2u_1+\cdots+\mathbf{G}_{N-1}\mathbf{A}_{N-1}\mathbf{F}_{N-2}u_{N-3}+\mathbf{G}_{N-1}\mathbf{F}_{N-1}u_{N-2}+\mathbf{G}_{N-1}\mathbf{A}_{N-1}\cdots\mathbf{A}_3\mathbf{A}_2\mathbf{F}_1y_0+\cdots+\mathbf{G}_{N-1}\mathbf{A}_{N-1}F_{N-2}y_{N-3}+\mathbf{G}_{N-1}\mathbf{F}_{N-1}y_{N-2} \tag{3.15}$$

which can be rewritten in a compact form

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & F_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & F_{N-1,2} & \cdots & F_{N-1,N-1} \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-2} \end{bmatrix}+\begin{bmatrix} G_{11} & 0 & \cdots & 0 \\ G_{21} & G_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{N-1,1} & G_{N-1,2} & \cdots & G_{N-1,N-1} \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-2} \end{bmatrix} \tag{3.16}$$

where the gain matrices $F_{i,j}$ and $G_{i,j}$ are defined accordingly. This controller implementation (3.16) will also be used to derive the system's optimal performance in the following subsection.

3.4 Optimal Performance. The optimal data-based LQG control sequence has been derived in Theorems 3.1 and 3.2. An alternative implementation is given in (3.16). We seek to know the minimal cost resulted from the optimal control sequence. In model-based control, the system's optimal performance is computed using the Riccati equations and a Lyapunov equation. For data-based LQG control, we shall present a closed-form expression for the system's optimal performance in terms of only the first $N+1$ Markov parameters.

Theorem 3.3. Consider a dynamic system with the given Markov parameters M_i, $i=1, 2, \ldots, N-1$ and H_i, $i=1, 2, \ldots, N-1$, the noise covariance matrices $W_0>0$, $W>0$, and $V>0$, and the weighting matrices $Q>0$ and $R>0$. Then the optimal data-based LQG control sequence obtained in (3.4) and (3.10) yields the minimum cost, which is given by

$$J_{\text{opt}}=\text{tr}(\mathbf{Q}_p\mathbf{Y}_p)+\text{tr}(\mathbf{U}_p^T\mathbf{R}_p\mathbf{U}_p\mathbf{Y}_p) \tag{3.17a}$$

where

$$\mathbf{Q}_p=diag\{Q,\ Q,\ \ldots,\ Q\} \quad \text{with } N+1 \text{ diagonal blocks and} \tag{3.17b}$$

$$\mathbf{R}_p=diag\{R,\ R,\ \ldots,\ R\} \quad \text{with N}-1 \text{ diagonal blocks,} \tag{3.17c}$$

$$\mathbf{Y}_p=\mathbf{T}_{p1}W_0\mathbf{T}_{p1}^T+\mathbf{T}_{p2}\mathbf{W}_p\mathbf{T}_{p2}^T+\mathbf{T}_{p3}\mathbf{V}_p\mathbf{T}_{p3}^T \tag{3.17d}$$

$$\mathbf{W}_p=diag\{W,\ W,\ \ldots,\ W\} \quad \text{with } N \text{ diagonal blocks,} \tag{3.17e}$$

$$\mathbf{V}_p=diag\{V,\ V,\ \ldots,\ V\} \quad \text{with } N+1 \text{ diagonal blocks, and} \tag{3.17f}$$

$$\mathbf{T}_{p1}=\mathbf{T}_{p3}\mathbf{H}_{p1},\ \ \mathbf{T}_{p2}=\mathbf{T}_{p3}\mathbf{H}_{p2},\quad \text{and}\ \ \mathbf{T}_{p3}=(\mathbf{I}-\mathbf{M}_p\mathbf{U}_p)^{-1} \tag{3.17g}$$

$\mathbf{H}_{p1}$, $\mathbf{H}_{p2}$, $\mathbf{M}_p$, and $\mathbf{U}_p$ are defined in $(C.2)$–$(C.6)$ in Appendix C.

Proof. See Appendix C.

The significance of Theorem 3.3 should be clear. It gives the minimum cost prior to the control sequence implementation and is computed directly from the Markov parameters. More importantly, it establishes the foundation for solving the output variance constraint control, which will be considered in the next section.

4 Data-Based Output Variance Constraint Control

The model-based Output Variance Constraint problem (OVC) was considered in [13] and references therein. It is an optimal control problem to minimize control effort subject to multiple performance constraints on output variances. The basic result provided an algorithm to iterate the two model-based Riccati equations with a certain update of the weighting matrix. In this section we consider the data-based OVC problem. The solution to this problem also provides a method for selecting the output weighting matrix in the quadratic cost function. Without loss of generality, we state the problem as follows.

Data-Based OVC Control Problem: Consider a dynamic system with known Markov parameters: M_i, $i=1, 2, \ldots, N-1$, H_i, $i=1, 2, \ldots, N+1$, the disturbance covariance matrices W, V and W_0. Let σ_i, $i=1, 2, \ldots, n_y$, be prescribed positive scalars. Find the optimal control sequence, u_k, $k=1, 2, \ldots, N-1$, such that

$$J_u=\mathbf{E}\left\{\sum_{k=0}^{N-1} u_k^TRu_k\right\}, \tag{4.1}$$

where $R>0$ is a given weighting matrix on the system input, is minimized subject to the following output constraints:

$$J_i=\mathbf{E}\left\{\sum_{k=0}^{N} y_{k,i}^2\right\}\leq\sigma_i,\quad i=1,\ 2,\ldots,\ n_y \tag{4.2}$$

where $y_{k,i}$ is the ith channel output of the system.

Define $E_i=[0\cdots 0\ 1\ 0\cdots 0]$ where the number in the ith position is 1 and the rest are zero. Then we have $y_{k,i}=E_iy_k$. We also define $\mathbf{E}_i=diag\{E_i,\ E_i,\ \ldots,\ E_i\}$ with $N+1$ diagonal blocks. The function of the matrix $\mathbf{E}_i$ is to pick the i-th channel signal from the overall output vector $\mathbf{y}_p$ in (C.6). Multiplying the left side of (C.6) by $\mathbf{E}_i$ yields the ith channel output signal

$$\begin{bmatrix} y_{0,i} \\ y_{1,i} \\ \vdots \\ y_{N,i} \end{bmatrix}=\mathbf{E}_i\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}=\mathbf{E}_i\mathbf{T}_{p1}\hat{w}_0+\mathbf{E}_i\mathbf{T}_{p2}\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}+\mathbf{E}_i\mathbf{T}_{p3}\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \tag{4.3}$$

Recall that $\hat{w}_0$, w_k, and v_k are uncorrelated. Considering (4.2) and (4.3), we have

$$J_i=tr(\mathbf{E}_i\mathbf{Y}_p\mathbf{E}_i^T) \tag{4.4}$$

which gives the output variance for the ith output channel. The following is an algorithm for solving the data-based OVC problem.

Data-Based OVC Control Design Algorithm: Consider the OVC data-based control problem. The design procedure is given as follows:

Step 1. Choose the initial diagonal weighting matrix $Q(i)>0$, positive scalars α and $0<\beta<1$.

Step 2. Design a data-based LQG controller using Theorems 3.1–3.2.

Step 3. Calculate J_i, $i=1, 2, \ldots, n_y$ using (4.4) and form a performance matrix

$$\mathbf{J}=diag\{J_1,\ J_2,\ldots,\ J_{n_y}\} \tag{4.5}$$

Step 4. Update the weighting matrix $Q(i)$:

$$Q(i)=\beta Q(i-1)+(1-\beta)\mathbf{P}[Q(i-1)+\alpha(\mathbf{J}-\Phi)] \tag{4.6}$$

where $\Phi=diag\{\sigma_1,\ \sigma_2,\ldots,\ \sigma_{n_y}\}$, the operator $\mathbf{P}[M]$ is defined in Section 1.

Step 5. Go to Step 1 until $\|Q(i)-Q(i-1)\|<\varepsilon$ where $\varepsilon>0$ is a prescribed tolerance.

Remark 4.1. The update Eq. (4.6) for the weighting Q originates from the model-based OVC problem [13] and the convergence of algorithm for the model-based design is guaranteed. The convergence rate depends on the selection of α and β. The investigation of the convergence property of the above algorithm remains for future research.

5 Numerical Example

Consider a discrete time dynamic system

$$x_{k+1}=\begin{bmatrix} -0.2868 & 0.0753 & -0.0426 & 0.1038 \\ 0.0753 & -0.4127 & 0.0863 & -0.2259 \\ -0.0426 & 0.0863 & -0.2948 & 0.1156 \\ 0.1038 & -0.2259 & 0.1156 & -0.6007 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.0593 \\ 0.4451 \\ 0.9180 \end{bmatrix}(u_k+w_k)$$

$$y_k=[0 \;\; 0.8585 \;\; 0.9972 \;\; 0.5493]x_k+v_k \tag{5.1}$$

where the disturbance covariances are $W=0.1679$, $W_0=0.1679$, and $V=0.1038$. The system initial condition is $x_0=[0.0, 0.0, 1.0, 3.0]^T$. We choose the control horizon $N=5$. The first $N+1$ Markov parameters are: {0.9990, −0.4681, 0.2953, −0.2206, 0.1763, −0.1442} (calculated from the known model in this case, but normally would be calculated from data). Let the weightings $Q=1$ and $R=0.001$. To see the initial state response, we set $w_k=0$ and $v_k=0$. Then we use only the first six given Markov parameters to design an optimal data-based controller using the results of Theorems 3.1–3.2. The optimal control sequence is {0, −0.4829, 0.2562, −0.1698, 0.1182}. It should be pointed out that the state space model (5.1) is not used (it contains more information than the first six Markov parameters). The closed-loop system response versus the open-loop system response with the same initial conditions is given by Fig. 2. Note that the first sample represents the initial condition and the closed-loop system and the open-loop system have the same amplitude. At the second sample, the two cases still have the same amplitude because the system has one sample transport delay from the input to the output.

To verify the equivalence of optimality between the data-based control and the model-based control, the LQG problem is also solved using the model-based design in Section 2. Two design methods generate the same optimal control sequence. But the state space model contains more system information than the first six Markov parameters.

Next, we compare the data-based design with the model-based design using the same amount of data. Suppose only the first five Markov parameters are available, we will find an (erroneous) state space model from this data and design a model-based control sequence. There are many methods for the state space realization. Here we will use the Kalman-Ho algorithm [5]. Using the Kalman-Ho algorithm, we first obtain a third-order linear system model

$$\tilde{x}_{k+1}=\begin{bmatrix} -0.5715 & -0.2836 & 0.0002 \\ -0.2836 & -0.5112 & -0.0115 \\ 0.0032 & 0.0048 & 0.0001 \end{bmatrix}\tilde{x}_k + \begin{bmatrix} 0.9809 \\ -0.4908 \\ 0.3230 \end{bmatrix}(u_k+w_k)$$

$$\tilde{y}_k=[0.9830 \;\; -0.1809 \;\; 0.0004]\tilde{x}_k+v_k \tag{5.2}$$

It should be emphasized that the third order model is the highest order model we can obtain since only a 3×3 Hankel matrix can be formed with the first five Markov parameters. We also note that the first five Markov parameters of this model do not match those of the real system. With model (5.2), a model-based LQG controller is designed using the algorithm in Section 2. The system (5.1) is considered as the real system on which the simulation is conducted with the same initial conditions and weighting matrices. The system responses with both the model-based control and the data-based control are shown in Fig. 3. As can be seen, the data-based control delivers better performance than the model-based control associated with a Kalman-Ho realization algorithm generated with the limited data.

Next we will use q-Markov COVariance Equivalent Realizations theory [14] to obtain a state space model from the given five Markov parameters. The main feature of the method is that it can find realizations to match the first q-Markov parameters and covariance parameters. To use this method, we must have the co-

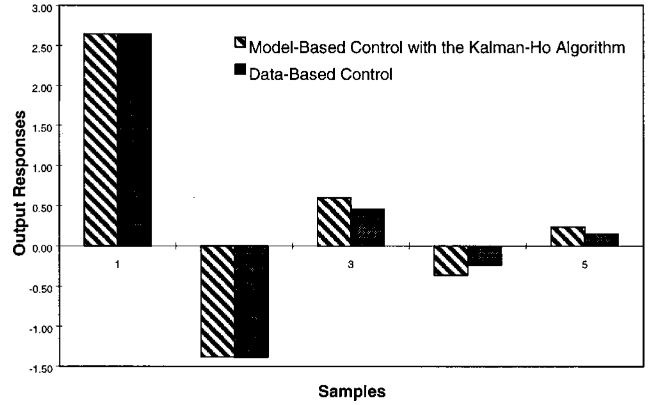

Fig. 3 Data-based control versus model-based control

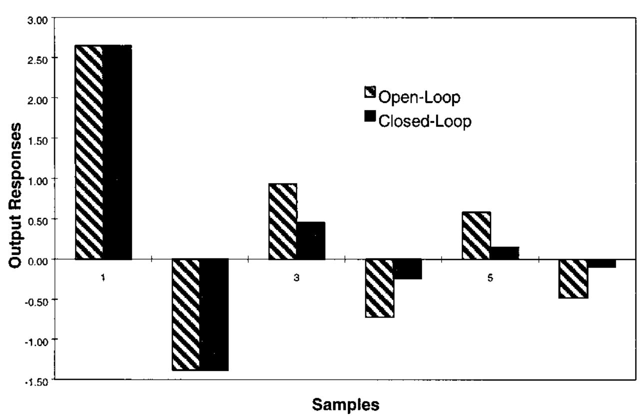

Fig. 2 Data-based control of system (5.1)

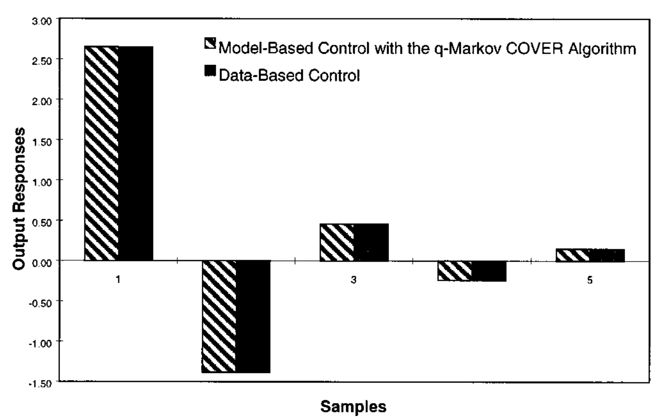

Fig. 4 Data-based control versus model-based control

variance parameters. Our goal is to find a model such that its first five Markov parameters match the given ones. In this case, it does not matter what covariance parameters we use as long as the ''data matrix'' is positive in the q-Markov COVER algorithm [14]. Here we arbitrarily choose the covariance parameters as $\{10, 0, \ldots\}$. That is, the first covariance parameter is 10 and the rest are zeros. Then we apply the algorithm to obtain a sixth-order model (5.3).

$$A=\begin{bmatrix} 0 & 0.2926 & -0.4983 & -0.5494 & -0.4648 & -0.2203 \\ 0 & 0.8622 & 0.4306 & -0.0336 & 0.1507 & -0.0705 \\ 0 & -0.3971 & 0.4956 & -0.6284 & 0.1011 & -0.2006 \\ 0 & -0.0270 & 0.5465 & -0.0114 & -0.6511 & 0.1708 \\ 0 & -0.0973 & 0.0707 & 0.5239 & -0.4676 & -0.4167 \\ 0 & -0.0291 & 0.0897 & 0.0878 & 0.2663 & -0.7892 \end{bmatrix}, \quad B=\begin{bmatrix} 0.3159 \\ -0.0053 \\ 0.0656 \\ 0.0285 \\ 0.1341 \\ 0.1627 \end{bmatrix}$$

$$C=[3.1623 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0] \tag{5.3}$$

The pulse response of the 6-th order model and the original system is $\{0.9990, -0.4681, 0.2953, -0.2206, 0.1763, 0.0549, -0.0236\}$. As can be seen, the first five Markov parameters of this model match those of the real system, and the remainder of the Markov parameters are far away from those of the real system. The model-based LQG design is performed with the sixth-order model. The simulation conditions are the same as before. The resulting performance is exactly the same as the data-based control as shown in Fig. 4. It should be pointed out that there exist an infinite number of models which can match the first five Markov parameters and (5.3) is just one of them. We conclude that if a model matches the first five Markov parameters, it will result in the same control action and hence lead to the optimal performance.

These examples have demonstrated that: (i) The data-based control design delivers the optimal performance, (ii) When the horizon is $N=4$, the minimum information needed to design the optimal control sequence is the first five Markov parameters, and (iii) Given only the first $N+1$ Markov parameters of a system, the q-Markov COVER algorithm can be used to find an infinite number of state space models which match the given Markov parameters, and hence the model-based LQG design associated with any q-Markov COVER model yields optimal control.

Conclusions

In this paper, we have presented an LQG data-based control design approach which only uses the first $N+1$ Markov parameters, as opposed to model-based LQG control design where an explicit state space model is required. The Markov data-based controller has the following features: (i) The new design is independent of (and hence does not require knowledge of) the order of the controlled system; (ii) The new design uses less system information than the full model-based methods; (iii) The data-based controller delivers the optimal control action, same as the full-model-based LQG design, (iv) The closed-form expression for the optimal system performance is derived, and (v) The data-based OVC control problem is addressed and an algorithm is suggested to solve the problem. The convergence of the algorithm remains for future research. The robustness of the developed numerical algorithm is not treated. The determination algorithms of the Markov parameters are not covered. The results of this paper can be extended to H_∞ control.

Appendix A: Proof of Theorem 3.1

We will begin with the model-based LQG control and end up with the data-based LQG control. Recall that the state space estimator is given by

$$\hat{x}_{k+1}=A\hat{x}_k+Bu_k+L_k(y_k-C\hat{x}_k) \tag{A.1}$$

$$L_k=AY_kC^T(V+CY_kC^T)^{-1} \tag{A.2}$$

Then it follows from the definition of the controller state (3.2) that

$$\begin{aligned}
\overline{x}_k^{N-k+1}&=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-k} \end{bmatrix}\hat{x}_k \\
&=\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-k+1} \end{bmatrix}\hat{x}_{k-1}+\begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{N-k}B \end{bmatrix}u_{k-1} \\
&\quad+\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-k} \end{bmatrix}L_{k-1}(y_{k-1}-C\hat{x}_{k-1}) \\
&=\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-k+1} \end{bmatrix}\hat{x}_{k-1}+\begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{N-k+1} \end{bmatrix}u_{k-1}+\mathbf{F}_k(y_{k-1} \\
&\quad-C\hat{x}_{k-1}) \\
&=[-\mathbf{F}_k \quad \mathbf{I}_{(N-k+1)n_y}]\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-k+1} \end{bmatrix}\hat{x}_{k-1}+\mathbf{B}_k u_{k-1} \\
&\quad+\mathbf{F}_k y_{k-1} \\
&=\mathbf{A}_k\overline{x}_{k-1}+\mathbf{B}_k u_{k-1}+\mathbf{F}_k y_{k-1}
\end{aligned} \tag{A.3}$$

where

$$\mathbf{A}_k\triangleq[-\mathbf{F}_k \quad \mathbf{I}_{(N-k+1)n_y}], \quad \mathbf{B}_k\triangleq\begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{N-k+1} \end{bmatrix},$$

$$\mathbf{F}_k\triangleq\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-k} \end{bmatrix}L_{k-1}. \tag{A.4}$$

Next we shall express the gain matrix, $\mathbf{F}_k$, in terms of the Markov parameters. Recall the batch form expression of the difference Riccati Eq. (2.14) and the estimator gain, L_k, in (2.8). It follows from (A.4) that

$$\mathbf{F}_k=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-k} \end{bmatrix} L_{k-1}=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-k} \end{bmatrix} AY_{k-1}C^T(V+CY_{k-1}C^T)^{-1}$$

$$=\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-k+1} \end{bmatrix}\mathbf{D}_{k-1}(\mathbf{W}_{k-1}^{-1}+T_{k-1}^TV_{k-1}^{-1}T_{k-1})^{-1}\mathbf{D}_{\mathbf{k-1}}^TC^T$$

$$\cdot(V+CD_{k-1}(W_{k-1}^{-1}+T_{k-1}^TV_{k-1}^{-1}T_{k-1})^{-1}\mathbf{D}_{k-\mathbf{1}}^TC^T)^{-1}$$

$$=\mathbf{H}_k\mathbf{P}_k\mathbf{N}_k^T(V+\mathbf{N}_k\mathbf{P}_k\mathbf{N}_{\mathbf{k}}^T)^{-1} \tag{A.5}$$

where

$$\mathbf{P}_k \triangleq (\mathbf{W}_{k-1}^{-1}+\mathbf{T}_{k-1}^T\mathbf{V}_{k-1}^{-1}\mathbf{T}_{k-1})^{-1} \tag{A.6}$$

$$\mathbf{N}_k \triangleq C\mathbf{D}_{k-1}=[CD \quad CAD\cdots CA^{k-1}D]=[H_1 \quad H_2\cdots H_k] \tag{A.7}$$

and

$$\mathbf{H}_k \triangleq \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-k+1} \end{bmatrix}\mathbf{D}_{k-1}$$

$$=\begin{bmatrix} CAD & CA^2D & \cdots & CA^kD \\ CA^2D & CA^3D & \cdots & CA^{k+1}D \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-k+1}D & CA^{N-k+2}D & \cdots & CA^ND \end{bmatrix}$$

$$=\begin{bmatrix} H_2 & H_3 & \cdots & H_{k+1} \\ H_3 & H_4 & \cdots & H_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-k+2} & H_{N-k+3} & \cdots & H_{N+1} \end{bmatrix} \tag{A.8}$$

We have expressed the gain matrices in terms of the Markov parameters and this completes the proof.

Appendix B: Proof of Theorem 3.2

Recall that the model-based LQG optimal control law is given by

$$u_k=-(R+B^TX_{k+1}B)^{-1}B^TX_{k+1}A\hat{x}_k \tag{B.1}$$

In what follows, we shall express u_k in terms of only the Markov parameters and the controller state $\bar{x}_k^{N-k+1}$. Note from Lemma 2.1 that X_{k+1} has the closed-form expression (2.11). Substitution of X_{k+1} into (B.1) yields

$$u_k=-(R+B^T\mathbf{C}_{k+1}^T(\mathbf{Q}_{k+1}^{-1}+\mathbf{S}_{k+1}\mathbf{R}_{k+1}^{-1}\mathbf{S}_{k+1}^T)^{-1}\mathbf{C}_{k+1}B)^{-1}$$

$$\times B^T\mathbf{C}_{k+1}^T\cdot(\mathbf{Q}_{k+1}^{-1}+\mathbf{S}_{k+1}\mathbf{R}_{k+1}^{-1}\mathbf{S}_{k+1}^T)^{-1}\mathbf{C}_{k+1}A\hat{x}_k$$

$$=-(R+\mathbf{B}_{k+1}^T(\mathbf{Q}_{k+1}^{-1}+\mathbf{S}_{k+1}\mathbf{R}_{k+1}^{-1}\mathbf{S}_{k+1}^T)^{-1}\mathbf{B}_{k+1})^{-1}\mathbf{B}_{k+1}^T\cdot(\mathbf{Q}_{k+1}^{-1}$$

$$+\mathbf{S}_{k+1}\mathbf{R}_{k+1}^{-1}\mathbf{S}_{k+1}^T)^{-1}\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-k} \end{bmatrix}\hat{\mathbf{x}}_k$$

$$=-(R+\mathbf{B}_{k+1}^T(\mathbf{Q}_{k+1}^{-1}+\mathbf{S}_{k+1}\mathbf{R}_{k+1}^{-1}\mathbf{S}_{k+1}^T)^{-1}\mathbf{B}_{k+1})^{-1}\mathbf{B}_{k+1}^T$$

$$\cdot(\mathbf{Q}_{k+1}^{-1}+\mathbf{S}_{k+1}\mathbf{R}_{k+1}^{-1}\mathbf{S}_{k+1}^T)^{-1}[0_{n_y} \quad \mathbf{I}_{(N-k)n_y}]\hat{x}_k^{N-k+1}$$

$$=\mathbf{G}_k\bar{\mathbf{x}}_k^{N-k+1} \tag{B.2}$$

where

$$\mathbf{B}_{\mathbf{k}+1}=\mathbf{C}_{\mathbf{k}+1}B=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-k-1} \end{bmatrix}B=\begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{N-k} \end{bmatrix} \tag{B.3}$$

This completes the proof.

Appendix C: Proof of Theorem 3.3

It is straightforward to show from the state space model (2.1) that

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}=\mathbf{H}_{p1}\omega_0+\mathbf{M}_p\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}+\mathbf{H}_{p2}\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}+\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \tag{C.1}$$

where the assumption $x_0=D\omega_0$ in Remark 2.1 is used and

$$\mathbf{H}_{p1}=\begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_{N+1} \end{bmatrix}, \quad \mathbf{M}_p=\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ M_1 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ M_{N-1} & \cdots & M_1 \end{bmatrix},$$

$$\mathbf{H}_{p2}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ H_1 & 0 & \cdots & 0 \\ H_2 & H_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ H_N & \cdots & H_2 & H_1 \end{bmatrix} \tag{C.2}$$

Also, it follows from (3.16) that

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}=\mathbf{F}_p\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}+\mathbf{G}_p\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \tag{C3}$$

where

$$\mathbf{F}_p=\begin{bmatrix} 0 & 0 & & \\ F_{22} & 0 & \ddots & \\ \vdots & \ddots & \ddots & 0 \\ F_{N-1,2} & \cdots & F_{N-1,N-1} & 0 \end{bmatrix},$$

$$\mathbf{G}_p=\begin{bmatrix} G_{11} & 0 & & & \\ G_{21} & G_{22} & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ G_{N-1,1} & \cdots & G_{N-1,N-1} & 0 & 0 \end{bmatrix} \tag{C.4}$$

Then we have from (C.3)

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}=\mathbf{U}_p\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \tag{C.5}$$

where $\mathbf{U}_p=(\mathbf{I}-\mathbf{F}_p)^{-1}\mathbf{G}_p$. We now substitute ($C$.5) into ($C$.1)

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \mathbf{T}_{p1}\omega_0 + \mathbf{T}_{p2}\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix} + \mathbf{T}_{p3}\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \qquad (C.6)$$

where $\mathbf{T}_{p1} = \mathbf{T}_{p3}\mathbf{H}_{p1}$, $\mathbf{T}_{p2} = \mathbf{T}_{p3}\mathbf{H}_{p2}$, and $\mathbf{T}_{p3} = (\mathbf{I} - \mathbf{M}_p\mathbf{U}_p)^{-1}$. Note that ω_0, w_i and v_i are uncorrelated. Similar to the model-based control, Eq. ($C.6$) can be considered as the closed-loop system equation. With Eqs. ($C.5$) and ($C.6$), we shall give the closed-form expression for the minimum cost of the closed-loop system.

The cost function (2.3) can be rewritten as

$$\begin{aligned} J &= \mathbf{E}\left\{\sum_{k=0}^{N} y_k^T Q y_k\right\} + \mathbf{E}\left\{\sum_{k=1}^{N-1} u_k^T Q u_k\right\} \\ &= \mathbf{E}\{tr(\mathbf{Q}_p\mathbf{y}_p\mathbf{y}_p^T)\} + \mathbf{E}\{tr(\mathbf{R}_p\mathbf{u}_p\mathbf{u}_p^T)\} \end{aligned} \qquad (C.7)$$

where $\mathbf{Q}_p$ and $\mathbf{R}_p$ are defined in (3.17), and

$$\mathbf{y}_p = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad \mathbf{u}_p = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} \qquad (C.8)$$

Substitution of ($C.5$) and ($C.6$) into ($C.7$) yields the minimum cost

$$J_{opt} = tr(\mathbf{Q}_p\mathbf{Y}_p) + tr(\mathbf{U}_p^T\mathbf{R}_p\mathbf{U}_p\mathbf{Y}_p) \qquad (C.9)$$

where $\mathbf{Y}_p = \mathbf{T}_{p1}W_0\mathbf{T}_{p1}^T + \mathbf{T}_{p2}\mathbf{W}_p\mathbf{T}_{p2}^T + \mathbf{T}_{p3}\mathbf{V}_p\mathbf{T}_{p3}^T$, and

$$\mathbf{W}_p = diag\{W, \ W, \ \ldots, \ W\} \quad \text{with } N \text{ diagonal blocks,}$$

$$\mathbf{V}_p = diag\{V, \ V, \ \ldots, \ V\} \quad \text{with } N+1 \text{ diagonal blocks.}$$

This completes the proof.

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