Applied Maths Module

Collection of Exams with Solutions

Désiré Sidibé

Associate Professor
Université de Bourgogne-Franche Comté
Laboratoire LE2I, UMR CNRS 6306
12, rue la Fonderie - 71200 Le Creusot, France
ddsidibe@u-bourgogne.fr

Course: Applied Mathematics (D. Sidibé)

MidTerm Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1 (20 Points)

The complete set of solutions to $Ax = b$ is

$$x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

for any arbitrary constants c and d.

1. If A is a $m \times n$ matrix with rank r, give as much true information as possible about the integers m, n and r (i.e. what are the values of m, n and r?).
2. Construct an explicit example of a possible matrix A and a possible right-hand side vector b with the above solution. (There are many acceptable answers, you just need to provide one).

■ PROBLEM 2 (30 Points)

Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

1. What is the rank of C? What is the dimension of the kernel of C? Give a basis for each of the four fundamental subspaces of C.
2. Prove that the linear system $Cx = b$ is compatible, i.e. can be solved, for every $b \in \mathcal{R}^2$.
3. Solve the linear system $Cx = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

■ PROBLEM 3 (20 Points)

Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and prove the following formula

$$A^k = \frac{1}{2} \begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}, \ \forall\, k \geq 1.$$

■ PROBLEM 4 (30 Points)

Say whether the following statements are true or not. If they are true you need to provide a proof, and if they are false you need to give a counter-example.

1. If A is an invertible matrix such that the elements in each row add up to one, then A^{-1} must have the same property.
2. The rank of AB is less than or equal to the ranks of A and B for any A and B.
3. The rank of $A + B$ is less than or equal to the ranks of A and B for any A and B.
4. If the entries in every row of A add to one, then $\det(A - I) = 0$.
5. The square of a Markov matrix is a Markov matrix.

Applied Mathematics : Exam 1 (October 15^{th} 2010)

Solutions by Prof. D. Sidibé (October 18^{th} 2010)

Problem 1 (20 pts)

1) We know that the complete set of solutions to a linear system $Ax = b$ is given by $x = x_p + x_N$

→ So in this example, the dimension of the nullspace is 2.

→ Since we can multiply A on the right by a 3 dimensional vector, A must have 3 columns, hence $\underline{n = 3}$

Therefore the rank of A must be $\underline{r = n - 2 = 1}$

→ Finally there are no restriction on m, except that $\underline{m \geq r = 1}$.

2) NOTE : There many correct answers to this question. Here we construct a minimal system $Ax = b$.

We take $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}_{1\times 3}$

We know that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ should be in the nullspace of A

$$\Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A special solution is $A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$.

In this case $b = Ax = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -1$.

So an example of system is $\underline{\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} x = -1}$

NOTE : The rank of A is clearly $r = 1$, and so the dimension of $N(A)$ is 2.

By construction $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \in N(A)$.

Problem 2 (30 pts)

Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

1) First we reduce C to its rref by Gauss elimination

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} -1 \\ 2 \end{matrix}}_{F} \end{bmatrix}$$

* 2 pivots $\Rightarrow$ $\text{rank}(C) = 2$

* $\dim(N(C)) = \#$ free variables $= n - r = 1$.

* A basis for the column space is given by the pivot columns of C: $\quad C(C) = \alpha \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

* $\dim(C(C)) = \dim(C(C^T))$; so a basis for the rowspace must have 2 vectors; which are also given by the pivot rows of the rref of C.

  $\to$ basis for $C(C^T)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} ; \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

  NOTE: In this case, since the two rows of C are independent, they also form a basis for rowspace

* Since $\dim(N(C)) = 1$, we need to find one non-zero vector in $N(C)$.

  We can put the free variable $x_3 = 1$ and solve $Rx = 0$ or just use the rref of C $\to$ $x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

* Finally, the dimension of the nullspace of C^T is $2 - \text{rank}(C) = 2 - 2 = 0$.

  $\to$ There is no basis for $N(C^T)$!

NOTE: Saying that the zero vector is a basis is not a correct answer ⚠

2) Since rank(C) = 2, the column space of C span the entire vector space $\mathbb{R}^2$. So every $b \in \mathbb{R}^2$ is in $\mathcal{C}(C)$, which means that $Ax = b$ is compatible for every $b \in \mathbb{R}^2$.

NOTE: This the right way to answer that question.

3) The set of solutions to $Cx = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is given by $x = x_p + x_N$, where x_p is a particular solution and $x_N \in N(C)$.

* we already have $N(C)$
* we need to find a particular solution, setting the free variable $x_3 = 0 \Rightarrow x_p = \begin{bmatrix} -3/2 \\ 5/4 \\ 0 \end{bmatrix}$

So the set of solutions is $x = \begin{bmatrix} -3/2 \\ 5/4 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$

NOTE: Do not forget to add the nullspace, otherwise you do not have all solutions.

Problem 3 (20 pts)

* we want to diagonalize $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, so we need to find the eigenvalues and eigenvectors.

→ eigenvalues $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda_1 = 1$ and $\lambda_2 = 3$

→ eigenvectors

For $\lambda_1 = 1 \rightarrow x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda_2 = 3 \rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

NOTE: of course, as we know, for one eigenvalue the are many possible eigenvectors.

→ Put the eigenvectors in $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

we can make S to be orthogonal $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Hence the diagonalization of A is $A = S \Lambda S^T$

$$\Rightarrow \quad A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

* we also know that $\forall\, k \geq 1, \quad A^k = S \Lambda^k S^T$

So $$A^k = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1^k & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \quad \vdots$$

$$A^k = \frac{1}{2} \begin{pmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{pmatrix}$$

Problem 4 (30 pts)

NOTE : Givins a 2x2 example is not a proof !

1) TRUE

Proof : If the rows of A add to one, then $A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

If A^{-1} exist, then multiplying both sides by A^{-1}, we get

$$(A^{-1} A) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow A^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

which means that the rows of A^{-1} also add to one.

2) TRUE

Proof For any vector x, $(AB)x = A(Bx)$

So the column space of AB is included in the column space of A. $\Rightarrow$ rank $(AB) \leq$ rank (A).

Equivalently, $(AB)^T = B^T A^T$ and the column space of $(AB)^T$ is included in the column space of B^T

$$\Rightarrow \quad \text{rank}\left((AB)^T\right) = \text{rank}(AB) \leq \text{rank}(B^T) = \text{rank}(B)$$

So, we have $\begin{cases} \text{rank}(AB) \leq \text{rank}(A) \\ \text{and} \quad \text{rank}(AB) \leq \text{rank}(B) \end{cases}$

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3) FALSE

For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then

$\text{rank}(A) = \text{rank}(B) = 1$

But $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has rank two

$$\Rightarrow \quad \text{rank}(AB) > \text{rank}(A) \text{ and } \text{rank}(B).$$

4) TRUE

If the rows of A add to one, then $A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$$\Rightarrow \quad (A - I)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = A\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0$$

So the $N(A - I)$ is not empty and therefore $\det(A - I) = 0$

5) TRUE

Markov matrix $\Leftrightarrow$ 2 properties : 1) all entries ≥ 0
2) columns add to one.

Let A be a Markov matrix, and $x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

we have $x^T A = x^T$ (property 2)

So $x^T A^2 = x^T A A = (x^T A) A = x^T A = x^T$

$\Rightarrow$ A^2 also has property 2) and of course property 1)

So A^2 is also a Markov matrix.

NOTE : It is always a good idea to think in terms of matrix multiplication !

Course: Applied Mathematics (D. Sidibé)

Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1 (40 Points)

Your classmate Yukti performed the usual elimination steps to convert a matrix A into an upper triangular matrix $U = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

1. What is the rank of A? Give a basis for the nullspace $N(A)$.

2. If $Uy = \begin{bmatrix} 9 \\ -12 \\ 0 \end{bmatrix}$, find the complete solution y (i.e. describe all possible solutions y).

3. Your classmate Pablo gives you a matrix $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$, and tells you that $A = LU$.

   Describe the complete sequence of elimination steps that Pablo performed (assuming he did elimination the usual way, starting with the first column and eliminating downwards).

4. Now, you are asked to solve the equation $Ax = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$.

   Your classmate Corina tells you that a solution is $x_C = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, while your other classmate Raquel tells you that a correct solution is $x_R = \begin{bmatrix} 11 \\ -2 \\ 0 \\ -1 \end{bmatrix}$.

   Is Corina's solution correct, or Raquel's solution, or both correct?

■ PROBLEM 2 (30 Points)

Let A be a 3×3 real symmetric matrix. The trace of A is equal to zero and two of its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, with eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ respectively.

1. What is the third eigenvalue λ_3?
2. Give an eigenvector v_3 for the eigenvalue λ_3 (Hint: what must be true of v_1, v_2 and v_3?).
3. What is A^5?

■ PROBLEM 3 (30 Points)

1. If P is a projection matrix, show that $(I - P)$ is also a projection matrix.

   If P is the projection onto the column space $C(A)$ of a matrix A, then $(I - P)$ is the projection onto which vector space? Explain your result.

2. Suppose A is a 3×5 matrix and the equation $Ax = b$ has a solution for every b.
   - a) What is the rank of A?
   - b) What is the rank of the 6×5 matrix $B = \begin{bmatrix} A \\ A \end{bmatrix}$?

3. Let $A = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$. What are the eigenvalues of A?

   Hint: very little calculation required. You shouldn't need to compute $det(A - \lambda I)$ unless you really want!

Applied Mathematics : Exam - 04th Nov 2011

Solutions by Prof. D. SIDIBE (4th Nov 2011)

Problem 1

Let A be a 3×4 matrix

After elimination, yukti gets $U = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1) • From U, we see that there are 2 pivots, so the rank of A is 2 : $\text{rank}(A) = 2$.

• There are also 2 free variables : x_3 and x_4

So the dimension of the nullspace is $\dim N(A) = 2$

To find a basis for $N(A)$

- Set $\begin{cases} x_3 = 1 \\ x_4 = 0 \end{cases}$ and solve for $Ux = 0 \Rightarrow x_1 = \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

- Set $\begin{cases} x_3 = 0 \\ x_4 = 1 \end{cases}$ and solve for $Ux = 0 \Rightarrow x_2 = \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

Hence $$N(A) = \left\{ \alpha \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} ; \ \alpha, \beta \in \mathbb{R} \right\}$$

2) The complete solution is given by $y = y_p + y_N$, where y_p is a particular solution and $y_N \in N(A)$.

- First, we find a particular solution y_p setting all free variables to zero : $y_3 = y_4 = 0$, and solve $Uy_p = b$

$\Rightarrow$ we find $y = \begin{pmatrix} 33 \\ -6 \\ 0 \\ 0 \end{pmatrix}$

So the complete set of solution is

$$\boxed{y = \begin{pmatrix} 33 \\ -6 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad ; \quad \alpha, \beta \in \mathbb{R}}$$

3) $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 2 & 10 & 0 & 0 \\ -1 & 2 & 7 & -21 \end{bmatrix}$

And we perform elimination with A to get U:

$$\underbrace{\begin{pmatrix} 1 & 4 & -1 & 3 \\ 2 & 10 & 0 & 0 \\ -1 & 2 & 7 & -21 \end{pmatrix}}_{A} \xrightarrow[r_3 + r_1]{r_2 - 2r_1} \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 6 & 6 & -18 \end{pmatrix} \xrightarrow[r_3 - 3r_2]{} \underbrace{\begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 6 \end{pmatrix}}_{U}$$

<u>elimination steps</u>

4) The complete set of solutions to $Ax = b$ is
$x = x_p + x_N$.

we already know, the nullspace of A: $N(A)$ (<u>Question 1</u>)

To find x_p, we set <u>all free variables to zero</u> ($x_3 = x_4 = 0$) and solve $Ux = c$ (where c is the vector obtained when applying the same elimination steps to b!)

Solving $Ux = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ we get $\underline{x_p = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$

we know that $N(A) = \{ \alpha u_1 + \beta u_2 , \alpha, \beta \in \mathbb{R} \}$

with $u_1 = \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

we can see that $\begin{cases} X_c = X_p + u_1 \\ X_R = X_p - u_2 \end{cases}$ with means

that both solutions are correct.

NOTE we can simply answer this question checking

$A X_c = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$ and $A X_R = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$, since we

know $A = \begin{pmatrix} 1 & 4 & -1 & 3 \\ 2 & 10 & 0 & 0 \\ -1 & 2 & 7 & -21 \end{pmatrix}$.

Problem 2

A is a 3×3 symmetric matrix with eigenvalues
$\lambda_1 = 1$ and $\lambda_2 = -1$.

1) we know $\operatorname{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 0$

so $\lambda_3 = -(\lambda_1 + \lambda_2) = 1$ $\quad \lambda_3 = 0$

2) For a symmetric matrix, the eigenvectors must be orthogonal to each other.

So we need $v_3 \perp v_1$ and $v_3 \perp v_2$.

If we put $v_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then we have

$$\begin{cases} a+b+c=0 \\ a-b=0 \end{cases} \Rightarrow \text{ So we can take } \boxed{v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}.$$

3) To easily find A^5, we have to diagonalize A.

$A = Q \Lambda Q^T$ (since A is symmetric and the matrix Q whose columns are the eigenvectors is orthogonal).

$Q = \left[\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3} \right]$: The columns of Q are the eigenvectors of A.

So $Q = \begin{bmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{2}} & 1/\sqrt{6} \\ 1/\sqrt{3} & -\frac{1}{\sqrt{2}} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$ and $\Lambda = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$

we have $A^5 = Q \Lambda^5 Q^T$

with $\Lambda^5 = \begin{pmatrix} (1)^5 & & \\ & (-1)^5 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$

$$\text{So } A^5 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -\frac{1}{\sqrt{2}} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{A^5 = \begin{pmatrix} -1/6 & +5/6 & 1/3 \\ 5/6 & -1/6 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}}$$

Problem 3

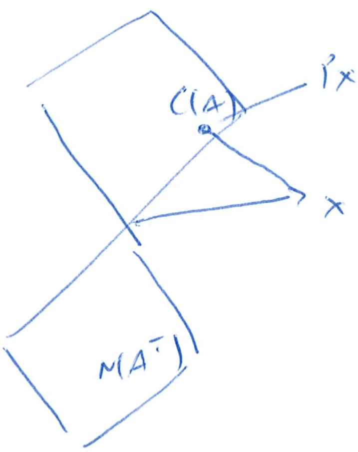
1) P is a projection matrix, then $P^2 = P$ and $P = P^T$

So $(I-P)^T = I - P^T = I - P$

and $(I-P)^2 = I^2 - IP - PI + P^2 = I - 2P + P^2$

$$= I - 2P + P = I - P$$

which proves that $(I-P)$ is also a projection matrix.



$C(A)$ and $N(A^T)$ are orthogonal complements.

If P projects onto $C(A)$, then $(I-P)$ projects onto $N(A^T)$

we can also see that $(I-P)x = x - Px \in N(A^T)$.

2) a) $Ax = b$ has a solution for every $b \in \mathbb{R}^3$ which means every $b \in \mathbb{R}^3$ is in $C(A)$.

So $C(A) = \mathbb{R}^3$ and $\boxed{\text{rank}(A) = 3}$

b) Let $B = \begin{bmatrix} A \\ A \end{bmatrix}_{6\times 5}$

If we perform elimination with B, we will get

$B \longrightarrow \begin{bmatrix} A \\ 0 \end{bmatrix}$, so $\boxed{\text{rank}(B) = 3}$

3) $A = \begin{bmatrix} 0,5 & 0,2 & 0,2 \\ 0,1 & 0,5 & 0,5 \\ 0,4 & 0,3 & 0,3 \end{bmatrix}$

- The last two colums are the same $\rightarrow$ A is singular which also means $\underline{\lambda_1 = 0}$ is one eigenvalue

(You can also do elimination and find there are only two pivots).

- A is a Markov matrix $\rightarrow$ $\underline{\lambda_2 = 1}$ is an eigenvalue

- Finally trace $(A) = \lambda_1 + \lambda_2 + \lambda_3 = 1,3$

$\Rightarrow \lambda_3 = \text{trace} - (\lambda_1 + \lambda_2) \Rightarrow \underline{\lambda_3 = 0,3}$

Course: Applied Mathematics (D. Sidibé)

Mid-Term Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1 (30 Points)

Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}$.

1. What is the rank of A? Give a basis for the column sapce $C(A)$.
2. Find the dimension of the nullspace $N(A)$ and give a basis for $N(A)$.

   What is the set of **all solutions** to $Ax = 0$?
3. For which number b_3 does the system $Ax = \begin{bmatrix} 3 \\ 9 \\ b_3 \end{bmatrix}$ have a solution?

   Find the complete set of solutions for that value of b_3.

■ PROBLEM 2 (30 Points)

We want to find the circle of equation $a(x^2 + y^2) + b(x + y) = 1$ which best fits the following data:

x	0	-1	1	1
y	1	0	-1	1

1. Let $z = \begin{bmatrix} a \\ b \end{bmatrix}$. What is the system $Az = b$ that the vector z must satisfies for the points to be on the circle? In other words, give the matrix A and the vector b.
2. Is the previous system solvable? Justify your answer.
3. Find the linear least squares solution and draw the data points and the obtained circle on a figure.
4. Let $M = A^T A$, where A is the matrix in Question 1. Find the eigenvalues of M.

■ PROBLEM 3 (30 Points)

The matrix $A = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix}$ has three eigenvalues $\lambda_1 = 18$, $\lambda_2 = 9$ and $\lambda_3 = -9$.

1. Find the eigenvectors corresponding to those three eigenvalues.
2. Find an orthogonal matrix Q such that $A = Q\Lambda Q^T$. What is the matrix Λ?
3. Let $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Write x as a linear combination of the three eigenvectors and compute $A^{10}x$.

■ PROBLEM 4 (10 Points)

For which number s is the following matrix positive definite?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix}$$

Exam, october 23^{rd} 2012 / Solutions

Problem I

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}$$

1) we do elimination

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix} \xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[3r_3 + r_2]{} \begin{bmatrix} \underline{1} & 2 & 1 & 4 \\ 0 & 0 & 0 & \underline{-3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- we have 2 pivots, so $\underline{\text{rank}(A) = 2}$

- The pivot columns of A form a basis for $C(A)$

  so $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}$ form a basis for $C(A)$.

2) we have 2 free variables : x_2 and x_3

   so $\underline{\dim N(A) = 2}$

   To find a basis for $N(A)$, we need to find the special solutions to $Ax = 0$, which equivalent to $Ux = 0$

- set $\begin{cases} x_2 = 1 \\ x_3 = 0 \end{cases}$ and solve for $x_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

- set $\begin{cases} x_2 = 0 \\ x_3 = 1 \end{cases}$ and solve for $x_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\Longrightarrow \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis for $N(A)$.

The set of all solutions to $Ax = 0$ is given by

$$\boxed{\alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{with } \alpha, \beta \in \mathbb{R}.}$$

NOTE we want all solutions ⚠

3) The system $Ax = b$ has a solution if $b \in C(A)$.

We can perform the same row operations with b

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 3 & 6 & 3 & 9 & 9 \\ 2 & 4 & 2 & 9 & b_3 \end{array}\right] \xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & b_3 - 6 \end{array}\right]$$

$$\xrightarrow[3r_3 + r_2]{} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 3b_3 - 18 \end{array}\right]$$

So for $b \in C(A)$, we need $3b_3 - 18 = 0$

$$\Rightarrow \boxed{b_3 = 6}$$

The complete set of solutions is given by $x = x_p + x_N$ where x_p is a particular solution, and $x_N \in N(A)$.

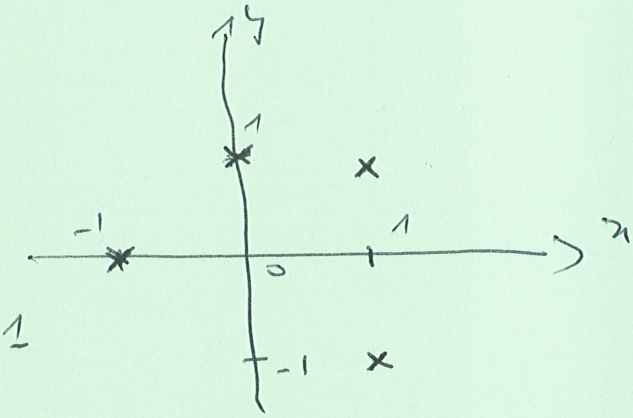
With $b = \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix}$, we see that $x_p = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (Sum of first two columns of A)

So the set of solutions is

$$\boxed{x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad \alpha, \beta \in \mathbb{R}}$$

Problem 2

x	0	-1	1	1
y	1	0	-1	1



We want the circle of equation $a(x^2+y^2)+b(x+y)=1$

1) The 4 points are ~~th~~ on the circle if their coordinates satisfy:

$$a(0^2+1^2)+b(0+1)=1$$
$$a(-1^2+0^2)+b(-1+0)=1$$
$$a(1^2+(-1)^2)+b(1+(-1))=1$$
$$a(1^2+1^2)+b(1+1)=1$$

$$\Leftrightarrow \quad \begin{aligned} a+b&=1\\ a-b&=1\\ 2a+0&=1\\ 2a+2b&=1 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

So, the vector $z = \begin{bmatrix} a \\ b \end{bmatrix}$ must satisfies $Az=b$

with $\boxed{A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$

2) The system $Az = b$ can be solved only if $b \in C(A)$.

To see that, we can perform elimination with A and B

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{array}\right]$$

A b

This tells us that it is not possible to find a linear combination of the columns of A that gives the vector b.

NOTE : If such linear combination exist, let write $\alpha \, Col_1 + \beta \, Col_2 = b$, then we should have $0 \cdot \alpha + 0 \cdot \beta = -1$, which is of course impossible

So $\underline{b \notin C(A)}$ and the system cannot be solved.

NOTE :
- Saying $b \notin C(A)$ because $\operatorname{rank}(A) = 2$ is not correct.
- Just saying $b \notin C(A)$ without proof is not enough ⚠

3) To find the LLS, we form the normal equation $A^T A z = A^T b$.

with $A^T A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 4 & 6 \end{bmatrix}$

$A^T b = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Since the matrix $A^T A$ is invertible ($\det \neq 0$) we can find the LLS by $\boxed{z = (A^T A)^{-1} A^T b}$

First we find $(A^T A)^{-1} = \frac{1}{44}\begin{bmatrix} 6 & -4 \\ -4 & 10 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$

So $z = \frac{1}{22}\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$$\Rightarrow \boxed{z = \begin{bmatrix} 7/11 \\ -1/11 \end{bmatrix}}$$

The equation of the circle is

$$\boxed{\frac{7}{11}(x^2 + y^2) - \frac{1}{11}(x + y) = 1}.$$

4) $M = A^T A = \begin{bmatrix} 10 & 4 \\ 4 & 6 \end{bmatrix}$

To find the eigenvalues of M, we solve the equation $\det(M - \lambda I) = 0$

$$\begin{vmatrix} 10-\lambda & 4 \\ 4 & 6-\lambda \end{vmatrix} = 0 \iff (10-\lambda)(6-\lambda) - 16 = 0$$

$$\iff \lambda^2 - 16\lambda + 60 - 16 = 0$$

$$\iff \lambda^2 - 16\lambda + 44 = 0$$

So $\lambda = \frac{16 \pm \sqrt{16^2 - 4 \times 44}}{2} = \frac{16 \pm 4\sqrt{5}}{2}$

So the eigenvalues of M are $\boxed{\begin{cases} \lambda_1 = 8 + 2\sqrt{5} \\ \lambda_2 = 8 - 2\sqrt{5} \end{cases}}$

NOTE

We can check $\begin{cases} \text{trace}(M) = \lambda_1 + \lambda_2 = 16 \\ \det(M) = \lambda_1 \cdot \lambda_2 = 44 \end{cases}$ OK

Problem III

$$A = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix} \quad \text{has 3 eigenvalues} \quad \begin{cases} \lambda_1 = 18 \\ \lambda_2 = 9 \\ \lambda_3 = -9 \end{cases}$$

1) We know the eigenvalues and want the eigenvectors.
We know eigenvectors are vectors in $N(A - \lambda I)$.

For $\lambda_1 = 18$

$$A - \lambda_1 I = \begin{bmatrix} -16 & 10 & -2 \\ 10 & -13 & 8 \\ -2 & 8 & -7 \end{bmatrix}$$

We perform elimination to find a vector in the nullspace.

$$\begin{bmatrix} -16 & 10 & -2 \\ 10 & -13 & 8 \\ -2 & 8 & -7 \end{bmatrix} \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \underbrace{\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

We solve $Ux = 0$, setting the free variable $x_3 = 1$.

A vector in $N(A - \lambda_1 I)$ is $\boxed{v_1 = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}}$.

NOTE
We can check

$$A v_1 = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix} \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \\ 18 \end{pmatrix} = 18 \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix} = 18\, v_1$$

OK

* You can also continue doing elimination to get the rref of $A - \lambda_1 I$, and see the special solution.
→ Same result.

For $\lambda_2 = 9$

$$A - \lambda_2 I = \begin{bmatrix} -7 & 10 & -2 \\ 10 & -4 & 8 \\ -2 & 8 & 2 \end{bmatrix}$$

we perform elimination to find a vector in the nullspace

$$\begin{pmatrix} -7 & 10 & -2 \\ 10 & -4 & 8 \\ -2 & 8 & 2 \end{pmatrix} \longrightarrow \quad \cdots \quad \longrightarrow \cdots \longrightarrow \underbrace{\begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{U}$$

→ the rank is 2, there is one free variable x_3.

→ we solve $Ux = 0$ setting $x_3 = 1$ and get

$$\boxed{v_2 = \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}}$$

checking

$$Av_2 = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix} \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -9/2 \\ 9 \end{bmatrix} = 9 v_2 = \lambda_2 v_2$$

ok

For $\lambda_3 = -9$

$$A - \lambda_3 I = \begin{bmatrix} 11 & 10 & -2 \\ 10 & 14 & 8 \\ -2 & 8 & 20 \end{bmatrix}$$

Elimination

$$\begin{pmatrix} 11 & 10 & -2 \\ 10 & 14 & 8 \\ -2 & 8 & 20 \end{pmatrix} \longrightarrow \quad - - - - \longrightarrow \cdots \longrightarrow \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

we find $\boxed{v_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}$

2) Since A is symmetric, the eigenvectors are orthogonal.

<u>NOTE</u> we can check

$$v_1^T v_2 = \begin{pmatrix} 1/2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix} = -\frac{1}{2} - \frac{1}{2} + 1 = 0$$

$$v_1^T v_3 = \begin{pmatrix} \frac{1}{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1 - 2 + 1 = 0$$

$$v_2^T v_3 = \begin{pmatrix} -1 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = -2 + 1 + 1 = 0 .$$

<u>ok</u>

- We normalize the eigenvectors and put them as columns of a matrix Q.

$$q_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$q_2 = \frac{V_2}{\|V_2\|} = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

$$q_3 = \frac{V_3}{\|V_3\|} = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

So we have $A = Q \Lambda Q^T$ with

$$\Lambda = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$$

3) The eigenvectors of A are the columns of Q.
And we want a linear combination of the columns of Q that gives $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

So we want $Qy = x \Rightarrow \underline{y = Q^T x}$.

$$y = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}.$$

We can check that $\frac{1}{3} q_1 - \frac{2}{3} q_2 + \frac{2}{3} q_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = x$.

So we have $x = \frac{1}{3} q_1 - \frac{2}{3} q_2 + \frac{2}{3} q_3$

Therefore $A^{10} x = \frac{1}{3} A^{10} q_1 + \frac{2}{3} A^{10} q_2 + \frac{2}{3} A^{10} q_3$

$$\boxed{A^{10} x = \frac{1}{3} \lambda_1^{10} q_1 - \frac{2}{3} \lambda_2^{10} q_2 + \frac{2}{3} \lambda_3^{10} q_3}$$

Problem 4

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix}$$ is positive definitive if

a) All pivots are > 0
b) All eigenvalues are > 0
c) $x^T A x > 0$ for all x.

→ we do elimination and check the pivots are all positive numbers.

$$A = \begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix} \rightsquigarrow \ldots \rightsquigarrow \ldots$$

$$\rightsquigarrow \begin{pmatrix} s & -4 & -4 \\ 0 & s - \frac{16}{s} & -4 - \frac{16}{s} \\ 0 & 0 & \frac{s^2-16}{s} - \left(-4 - \frac{16}{s}\right)^2 / s - \frac{16}{s} \end{pmatrix}$$

So we need:

- $\underline{s > 0}$
- $s - \frac{16}{s} > 0 \longrightarrow \frac{s^2-16}{s} > 0 \longrightarrow s^2 > 16 \longrightarrow \underline{s > 4}$
- $\frac{s^2-16}{s} - \left(-4 - \frac{16}{s}\right)^2 / \left(s - \frac{16}{s}\right) > 0$

$$\longrightarrow (s+4)^2 (s-8) > 0 \longrightarrow \underline{s > 8}$$

$$\boxed{\text{Finally we need } \underline{s > 8}}$$

Course: Applied Mathematics

Second Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1

The population of a city in 2000 was 500,000 while the population of the suburbs of that city in 2000 was 700,000.

Suppose that demographic studies show that each year about 6% of the city's population moves to the suburbs (and 94% stays in the city), while 2% of the suburban population moves to the city (and 98% remains in the suburbs).

For simplicity, we ignore other influences on the population such as births, deaths, and migration into and out of the city/suburban region.

1. Compute the population of the city and of the suburbs in the year 2002, and 2005.
2. Can you find an equilibium state where the population distribution among the city and the suburbs do not vary anymore?

■ PROBLEM 2

Let $\mathbf{u}$ be a unit vector in $\mathcal{R}^2$ ($\|\mathbf{u}\| = 1$). Define a transformation f from $\mathcal{R}^2$ to $\mathcal{R}^2$ by

$$f(\mathbf{x}) = \mathbf{x} - 2(\mathbf{u}^T\mathbf{x})\mathbf{u}$$

1. Show that the linear transformation f is length preserving, i.e. for all $\mathbf{x}$, the length of $\mathbf{x}$ is equal to the length of $f(\mathbf{x})$.
2. Show that $f \ o \ f$ is the identity transformation.
3. Specifically set $\mathbf{u} = \begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix}$ and find the transformation matrix A_f, i.e. a matrix such that $f(\mathbf{x}) = A_f\mathbf{x}$, and show that $A_f^2 = I$.

■ PROBLEM 3

A game in Le Creusot casino involves a device that produces independent samples of a continuous random variable X with probability density function

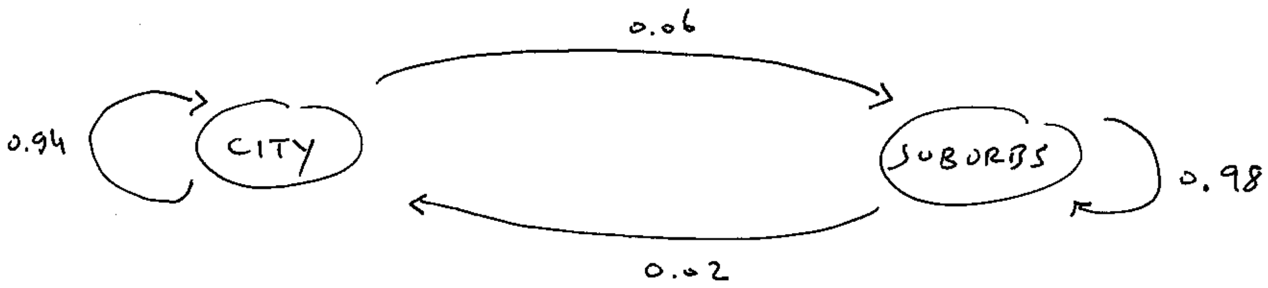
$$f_X(x) = \begin{cases} 3/x^4 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

The game is rather simple: the price to play is 1€, and the payoff in dollars is X.

1. What is your expected profit (profit = X - 1.00€) from one play of the game? Give a numerical answer with proper units.
2. What is the standard deviation of your profit from one play of the game?
3. You decide to play the game repeatedly until you profit at least 9.00€ on a single game (i.e. until the first $X \geq 10$) and then you will stop. Find the expected number of games you will play.

2 Exam (2014)

Problem I



Let defines $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where

$x_1 \leftarrow$ population of the city

$x_2 \leftarrow$ population of the suburbs.

Then we have $X_{n+1} = \underbrace{\begin{bmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{bmatrix}}_{M} X_n$

1) The population in year 2000 is $X_0 = \begin{bmatrix} 500{,}000 \\ 700{,}000 \end{bmatrix}$

The population in 2001 is $X_1 = M X_0$

and the population in year 2002 is $\underline{X_2 = M^2 X_0}$

Equivalently, the population in year 2005 is $\underline{X_5 = M^5 X_0}$

which gives $X_2 = \begin{bmatrix} 469{,}280 \\ 730{,}072 \end{bmatrix}$

and $X_5 = \begin{bmatrix} 431{,}820 \\ 768{,}180 \end{bmatrix}$

2) The population distribution reaches an equilibrium, when $x_{n+1} = x_n$, i.e. $x_n = M x_n$

which means that x_n is an eigenvector of M corresponding to the eigenvalue $\lambda = 1$.

So we have to find the eigenvalues / eigenvectors of M.

* We know $\underline{\lambda = 1}$ is one eigenvalue (M is a Markov matrix)

and we have $\mathrm{trace}(M) = \lambda_1 + \lambda_2 = 0.94 + 0.98 = 1.92$

so $\underline{\lambda_2 = 0.92}$

* Let finds an eigenvector for $\lambda = 1$.

x is an eigenvector for $\lambda = 1$ if $x \in N(M - I)$.

$$M - I = \begin{bmatrix} -0.06 & 0.02 \\ 0.06 & -0.02 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_1 + R_2]{} \begin{bmatrix} -0.06 & 0.02 \\ 0 & 0 \end{bmatrix}$$

$x \in N(M - I)$ if $-0.06\, x_1 + 0.02\, x_2 = 0$

If we take $x_2 = 1$, then $x_1 = \frac{0.02}{0.06} = \frac{1}{3}$

So $x = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ is an eigenvector of M for $\lambda = 1$.

<u>Checking</u>

$$Mx = \begin{bmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \times 0.94 + 0.02 \\ \frac{1}{3} \times 0.06 + 0.98 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$\underline{\underline{\text{OK}}}$

So the equilibrium distribution is $X = \begin{pmatrix} x_1/(x_1+x_2) \\ x_2/(x_1+x_2) \end{pmatrix}$

$$\Rightarrow \quad X_{\text{equilibrium}} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$

That is at a long term, 0.25 of the entire population will live in the city, while 0.75 will be in the suburbs.

Which corresponds to

$$X = \begin{bmatrix} 300,000 \\ 900,000 \end{bmatrix}$$

Problem 2

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$x \longrightarrow x - 2(u^T x)\, u$$

1) $\|f(x)\|^2 = f(x)^T f(x) = (x - 2(u^T x)u)^T (x - 2(u^T x)u)$

$= (x^T - 2(u^T x)u^T)(x - 2(u^T x)u)$

$= x^T x - 2(u^T x)x^T u - 2(u^T x)u^T x + 4(u^T x)^2 u^T u$

But $x^T u = u^T x$ and $u^T u = 1$ since $\|u\| = 1$

So $\|f(x)\|^2 = x^T x - 4(u^T x)^2 + 4(u^T x)^2 = x^T x = \|x\|^2$

$\Rightarrow \boxed{\|f(x)\| = \|x\|}$

2) $f \circ f(x) = f(f(x)) = f(x - 2(u^T x)u)$

$= (x - 2(u^T x)u) - 2(u^T(x - 2(u^T x)u))u$

$= x - 2(u^T x)u - 2(u^T x - 2(u^T x)\underbrace{u^T u}_{=1})u$

$= x - 2(u^T x)u - 2(\underbrace{u^T x - 2(u^T x)}_{-u^T x})u$

$= x - 2(u^T x)u + 2(u^T x)u$

$f \circ f(x) = x$ so $f \circ f$ is identity transformation.

3) $u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$f(x) = x - 2(u^T x)\,u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2\left(x_1\cos\theta + x_2\sin\theta\right)\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 2x_1\cos^2\theta + 2x_2\sin\theta\cos\theta \\ 2x_1\cos\theta\sin\theta + 2x_2\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 2x_1\cos^2\theta + x_2\sin(2\theta) \\ x_1\sin(2\theta) + 2x_2\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} (1-2\cos^2\theta)\,x_1 - x_2\sin(2\theta) \\ -x_1\sin(2\theta) + (1-2\sin^2\theta)\,x_2 \end{bmatrix}$$

$$\boxed{f(x) = \underbrace{\begin{bmatrix} 1-2\cos^2\theta & -\sin(2\theta) \\ -\sin(2\theta) & 1-2\sin^2\theta \end{bmatrix}}_{A_f.}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

It is easy to check that $A_f^2 = I$, which confirms that $f \circ f = \text{identity}$.

Problem 3

$$f_X(x) = \begin{cases} 3/x^4 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

1) profit $= X - 1€$, so our expected profit is

$E_{profit} = E[X] - 1€.$

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^{\infty} x \cdot \frac{3}{x^4}\, dx$$

$$= \int_1^{\infty} \frac{3}{x^3}\, dx = -\frac{3}{2} \cdot x^{-2} \Big|_1^{\infty}$$

$$\underline{E(X) = 1.5}$$

so $\boxed{E_{profit} = 1.5 - 1 = 0.5\ €}.$

2) $\sigma = \sqrt{Var(X-1)} = \sqrt{Var(X)}.$

$Var(X) = E[X^2] - E[X]^2$

We need to find $E[X^2]$.

$$E[X^2] = \int_1^{\infty} x^2 \cdot \frac{3}{x^4}\, dx = \int_1^{\infty} \frac{3}{x^2}\, dx = -3 \cdot x^{-1} \Big|_1^{\infty} = 3$$

So $Var(X) = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}$

and $\boxed{\sigma(\text{profit}) = \sqrt{\frac{3}{4}}\ €}.$

3) Let T be the number of plays until a profit of at least 9 € is achieved.

Then T is a geometric r.v. with probability of success

$$P(X \geq 10) = \int_{10}^{\infty} \frac{3}{x^4}\, dx = \frac{1}{10^3}.$$

and $E[T] = 1/P(X \geq 10) = \underline{1000}$

$\Rightarrow$ We must play at least $\underline{1000 \text{ games}}$.

Course: Applied Mathematics

Mid-Term Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1

Let $A = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}$.

1. Find the dimensions of the four fundamental subspaces of A.
2. Find a basis for $C(A)$ and $N(A)$.
3. Let $b = \begin{pmatrix} 5 \\ -4 \\ 8 \end{pmatrix}$. Is the system $Ax = b$ solvable?

■ PROBLEM 2

Find the matrix X such that $X = AX + B$, where

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}.$$

■ PROBLEM 3

A small company has been in business for three years and has recorded annual profits (in thousands of dollars) as follows.

Years	1	2	3
Profits	7	4	3

Assuming that there is a linear trend in the declining profits, predict the year and the month in which the company begins to lose money.

■ PROBLEM 4

Diagonalize the matrix $A = \begin{pmatrix} -8 & -6 \\ 12 & 10 \end{pmatrix}$.

■ PROBLEM 5

In order to grow a certain crop, it is recommended that each square foot of ground be treated with 10 units of phosphorous, 9 units of potassium, and 19 units of nitrogen. Suppose that there are three brands of fertilizer on the market; say brand $\mathcal{X}$, brand $\mathcal{Y}$, and brand $\mathcal{Z}$.

One pound of brand $\mathcal{X}$ contains 2 units of phosphorous, 3 units of potassium, and 5 units of nitrogen. One pound of brand $\mathcal{Y}$ contains 1 unit of phosphorous, 3 units of potassium, and 4 units of nitrogen. One pound of brand $\mathcal{Z}$ contains only 1 unit of phosphorous and 1 unit of nitrogen.

1. Determine whether or not it is possible to meet exactly the recommendation by applying some combination of the three brands of fertilizer?

2. Take into account the obvious fact that a negative number of pounds of any brand can never be applied, and suppose that because of the way fertilizer is sold only an integral number of pounds of each brand will be applied. Under these constraints, determine all possible combinations of the three brands that can be applied to satisfy the recommendations exactly.

3. Suppose that brand $\mathcal{X}$ costs \$1 per pound, brand $\mathcal{Y}$ costs \$6 per pound, and brand $\mathcal{Z}$ costs \$3 per pound. Determine the least expensive solution that will satisfy the recommendations exactly as well as the constraints.

Problem I

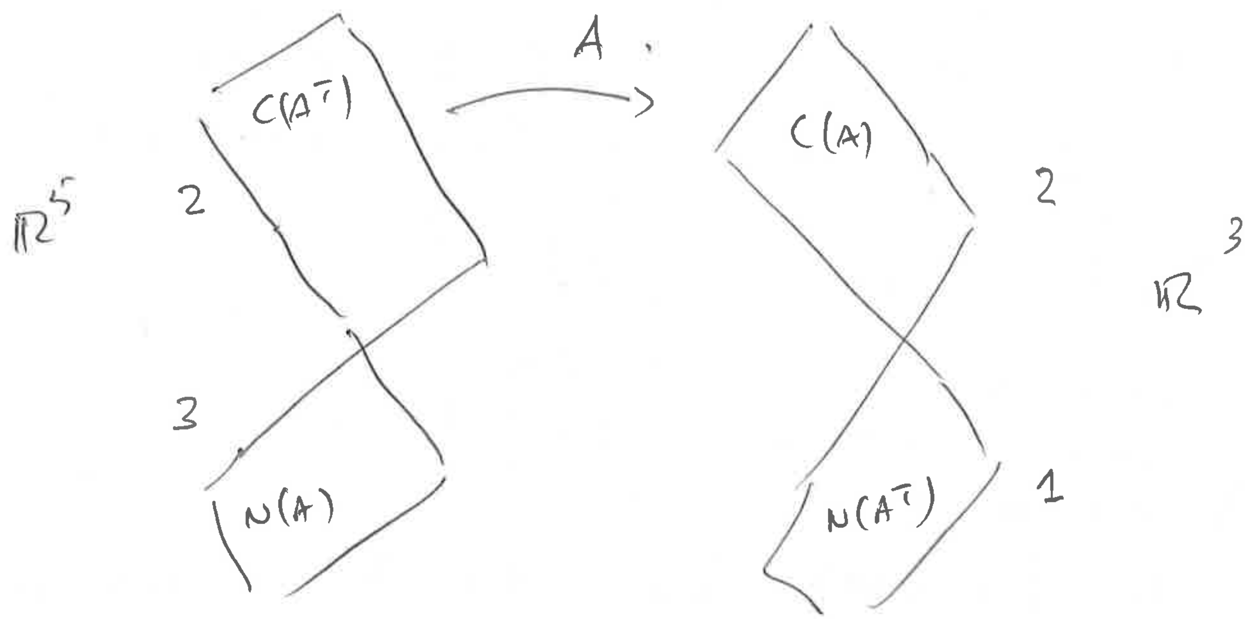
$$A = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix} \longrightarrow \dots \text{ elimination}$$

$$\longrightarrow \underbrace{\begin{pmatrix} \boxed{1} & 2 & 0 & -2 & 1 \\ 0 & 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{rref}(A)}$$

1) We have two pivots, so $\text{rank}(A) = 2$.

Therefore the dimensions of the 4 subspaces are

$$\dim C(A) = \text{rank}(A) = 2$$
$$\dim N(A) = n - \text{rank}(A) = 3$$
$$\dim C(A^T) = \dim C(A) = 2$$
$$\dim N(A^T) = m - \text{rank}(A^T) = 1$$



2) basis for $C(A)$ $\equiv$ pivot columns of A

$\Rightarrow$ $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ form a basis for $C(A)$

we have 3 free variables, from the $\mathrm{rref}(A)$ we find 3 special solutions to $Ax = 0$

$\underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{x_1}$, $\underbrace{\begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}}_{x_2}$ and $\underbrace{\begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix}}_{x_3}$ form a basis for $N(A)$

<u>NOTE</u>: We can check that $Ax_1 = Ax_2 = Ax_3 = 0$.

2) let $b = \begin{pmatrix} 5 \\ -4 \\ 8 \end{pmatrix}$

$Ax = b$ is solvable only if $b \in C(A)$.

That is if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 8 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha + \beta = 5 \\ -2\alpha + 0 = -4 \\ \alpha + 2\beta = 8 \end{cases} \Rightarrow \begin{cases} \alpha = 2 \\ \beta = 3 \end{cases}$$

$\Rightarrow$ So $b \in C(A)$ and $Ax = b$ is solvable.

The solution set is

$$x = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \alpha x_1 + \beta x_2 + \gamma x_3$$

Problem 2

We want X such that $AX + B = X$

That is $X - AX = B \Rightarrow X(I - A) = B$.

If $(I-A)^{-1}$ exists, then $\underline{X = (I-A)^{-1} B}$

Let check if $(I-A)^{-1}$ exists.

$$I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This is an invertible matrix (rank = 3), and

$$(I-A)^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$ (using Gauss-Jordan procedure)

Therefore $X = (I-A)^{-1} B \Rightarrow \boxed{X = \begin{pmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 3 \end{pmatrix}}$

Check

$$AX + B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -3 & -3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 3 \end{pmatrix}$$

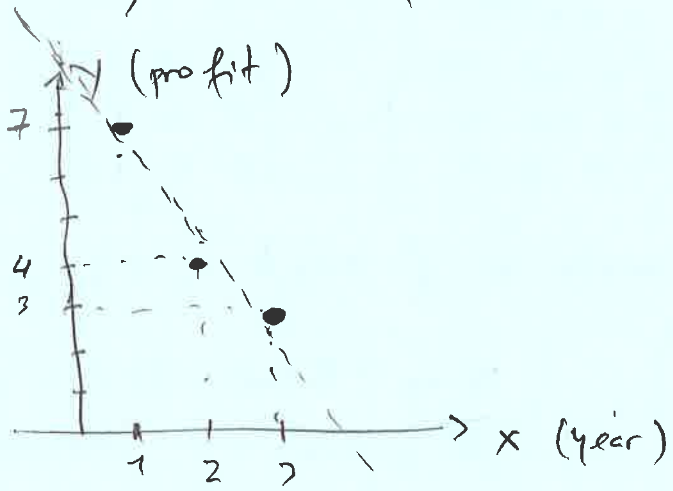
$= X$. $\underline{\text{ok}}$

Problem 3

Years	1	2	3
Profits	7	4	3

We assume a linear model for the profits:

$$y = \alpha x + \beta \quad \text{where} \quad \begin{cases} x \to \text{year} \\ y \to \text{profit for year } x \end{cases}$$



Using given data, we have

$$\begin{cases} 7 = \alpha + \beta \\ 4 = 2\alpha + \beta \\ 3 = 3\alpha + \beta \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{x}$$

We use LLS (linear least - square) $\Rightarrow \hat{x} = (A^T A)^{-1} A^T b$

$$A^T A = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \quad \text{and} \quad A^T b = \begin{pmatrix} 29 \\ 14 \end{pmatrix}$$

So $\hat{x} = \begin{pmatrix} -2 \\ \frac{26}{3} \end{pmatrix}$ and $\boxed{y = -2x + \frac{26}{3}}$

↳ The company begins losing money when $y \leq 0$

$$\Rightarrow -2x + \frac{26}{3} \leq 0 \Rightarrow x \geq \frac{26}{6} \simeq 4.33$$

x is in year, so the company starts losing money $\boxed{\text{after } (4 + \tfrac{1}{3}) \text{ years (i.e. after March of the 4th year}}$

Problem 4

$$A = \begin{pmatrix} -8 & -6 \\ 12 & 10 \end{pmatrix}$$

1) First we compute the eigenvalues of A.

$$\det(A - \lambda I) = 0 \Rightarrow \det \begin{pmatrix} -8-\lambda & -6 \\ 12 & 10-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (-8-\lambda)(10-\lambda) + 72 = 0$$

$$\Rightarrow -80 + 8\lambda - 10\lambda + \lambda^2 + 72 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 8 = 0 \Rightarrow \boxed{\begin{cases} \lambda_1 = 4 \\ \lambda_2 = -2 \end{cases}}$$

2) we find the eigenvectors:

$\underline{\lambda_1 = 4}$ $\quad A x_1 = \lambda_1 x_1 \Rightarrow x_1 \in N(A - \lambda_1 I)$

$$A - \lambda_1 I = \begin{pmatrix} -12 & -6 \\ 12 & 6 \end{pmatrix} \Rightarrow x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

<u>check</u>: $$A x_1 = \begin{pmatrix} -8 & -6 \\ 12 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ 12 \end{pmatrix} + \begin{pmatrix} 12 \\ -20 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix} = 4 x_1.$$

$\underline{\lambda_2 = -2}$ $\quad A x_2 = \lambda_2 x_2 \Rightarrow x_2 \in N(A - \lambda_2 I)$

$$A - \lambda_2 I = \begin{pmatrix} -6 & -6 \\ 12 & 12 \end{pmatrix} \Rightarrow x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

<u>check</u> $$A x_2 = \begin{pmatrix} -8 & -6 \\ 12 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2 x_2.$$

So the eigenvectors are $\boxed{x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

<u>Diagonalization</u> $\quad A = S \Lambda S^{-1}$

with $$S = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

Problem 5

1) Let x, y and z be the units of phosphorus, potassium and nitrogen respectively.

Assume we use a combination of the 3 brands of fertilizer as follows: $\begin{cases} \alpha \text{ pounds of brand } X \\ \beta \text{ pounds of brand } Y \\ \gamma \text{ pounds of brand } Z \end{cases}$

We meet the recommendation exactly, if:

$$\begin{cases} 2\alpha + \beta + \gamma = 10 \\ 3\alpha + 3\beta = 9 \\ 5\alpha + 4\beta + \gamma = 19 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 3 & 3 & 0 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 19 \end{bmatrix}$$

Let's solve this system by elimination:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 3 & 0 & 9 \\ 5 & 4 & 1 & 19 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 3 & -3 & -12 \\ 0 & +3 & -3 & -12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 3 & -3 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is one free variable: γ

If $\underline{\gamma = 1}$, then we have $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is in the nullspace of the matrix.

If $\gamma = 0$, a particular solution ~~is~~ is $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 0 \end{bmatrix}$

So $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 0 \end{pmatrix} + a \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; $a \in \mathbb{R}$ is the set of solutions.

$\Rightarrow$ Therefore, we can say that it is possible to satisfy the requirements, using some combinatio of the 3 brands.

2) given that the solution must be non-negative integers, we must choose the coefficient $a \in \mathbb{R}$ such that

$$\begin{cases} 7 - a \in \mathbb{N} \Rightarrow a \leq 7. \\ -4 + a \in \mathbb{N} \Rightarrow a \geq 4 \\ a \in \mathbb{N} \end{cases}$$

$\Rightarrow$ So possible values for a are $\{4, 5, 6, 7\}$.

which correspond to the combinations:

# pounds of	x	3	2	1	0
	y	0	1	2	3
	z	4	5	6	7
		$a=4$	$a=5$	$a=6$	$a=7$

Cost

3) Cost of each combination : 15\$ 23\$ 31\$ 39\$

$\Rightarrow$ $\boxed{\text{The least expensive solution is 3 pounds of } X + 4 \text{ pounds of } Z \text{; which costs } \underline{15\$}.}$

Course: Applied Mathematics

2nd Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1

Consider the following system:

$$\begin{aligned} 2x + 2y + 3z &= 0, \\ 4x + 8y + 12z &= -4, \\ 6x + 2y + \alpha z &= 4. \end{aligned}$$

- Determine all values of α for which the system is consistent (that is the system is solvable)
- Determine all values of α for which there is a unique solution, and compute the solution for these cases.
- Determine all values of α for which there are infinitely many different solutions, and give the general form of the solution.

■ PROBLEM 2

A particular electronic device consists of a collection of switching circuits that can be either in an ON state or an OFF state. These electronic switches are allowed to change state at regular time intervals called clock cycles. Suppose that at the end of each clock cycle, 30% of the switches currently in the OFF state change to ON, while 90% of those in the ON state revert to the OFF state.

1. Show that the device approaches an equilibrium in the sense that the proportion of switches in each state eventually becomes constant, and determine these equilibrium proportions.
2. Independent of the initial proportions, about how many clock cycles does it take for the device to become essentially stable?

■ PROBLEM 3

Find the SVD of $A = \begin{pmatrix} -4 & -6 \\ 3 & -8 \end{pmatrix}$

■ PROBLEM 4

1. The inhabitants of an island tell the truth one third of the time. They lie with probability 2/3. On an occasion, after one of them made a statement, you ask another "was that statement true?" and he says "yes".

   What is the probability that the statement was indeed true?

2. Three prisoners are informed by their jailer that one of them has been chosen to be executed at random with equal probability, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free.

   The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from 1/3 to 1/2, since he would then be one of two prisoners.

   What do you think of the jailer's reasoning?

3. Let x and y be independent random variables with means μ_x and μ_y, and variances σ_x^2 and σ_y^2 respectively. Show that

$$\mathrm{Var}[xy] = \sigma_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2.$$

NOTE: The three questions of Problem 4 are independent of each other.

Course: Applied Mathematics

1st Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1

Let $A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{bmatrix}$

1. What is the rank of A?
2. Find a basis for the column space and the nullspace of A.
3. Find all solutions to $Ax = b$ for $b = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$.

■ PROBLEM 2

Determine the parabola $y = ax^2 + bx + c$ that best fits the following data.

y	5	2	0	3
x	-2	-1	1	2

Draw a figure with the points and the estimated parabola.
What value will you predict for y, when $x = 5$?

■ PROBLEM 3

Let A be an $n \times n$ matrix. Suppose v and w are orthogonal eigenvectors of $A^T A$.
Show that Av and Aw are orthogonal vectors.

■ PROBLEM 4

Elite NASA engineers determine that if a satellite is placed in orbit starting at point $\mathcal{O}$, it will return exactly to the same point after one orbit of the earth. Unfortunately, if there is a small mistake in the original location of the satellite, which engineers label by a vetcor $X \in \mathbb{R}^3$ with origin at $\mathcal{O}$, after one orbit the satellite will instead return to some other point $Y \in \mathbb{R}^3$.
Computations show that Y is related to X by a matrix A as follows:

$$Y = AX = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 1/2 \\ 1 & 1/2 & 0 \end{bmatrix} X.$$

1. Find the eigenvalues of the above matrix A.
   You are given that $det(A - \lambda I) = \lambda^3 - 1/2\lambda^2 - 3/2\lambda$.
2. Determine all possible eigenvectors for each eigenvalue.
3. Discuss case by case, what will happen to the satellite if initial mistake in its location is in a direction given by an eigenvector.

APPLIED MATHS 2016 – 2017 : 1st EXAM, Oct. 2016

PROBLEM 1

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{bmatrix}$$

1) Perform elimination

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 3 \\ 0 & \boxed{2} & 1 & 0 \\ 0 & 0 & 0 & \boxed{3} \end{bmatrix}$$

3 pivots $\longrightarrow$ $\boxed{\text{rank}(A) = 3}$

2) Basis for column space is given by pivot columns

so $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix}$ form a basis for $C(A)$.

- rank $(A) = 3 \Rightarrow \dim N(A) = 1$.

we set the free variable $x_3 = 1$ and solve $Ax = 0$

which gives $x = \begin{pmatrix} -2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}$

so $N(A) = \alpha \begin{bmatrix} -2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$, $\alpha \in \mathbb{R}$.

3) Solve $Ax = \underbrace{\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}}_{b}$

Perform same row operations with vector b.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 3 & 2 \\ 2 & 4 & 6 & 9 & 7 \\ 2 & 6 & 7 & 6 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 3 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 2 & 1 & 0 & 0 \end{array}\right]$$

So $Ax = b \Rightarrow$
$$\begin{cases} x_1 + 2x_2 + 3x_3 + 3x_4 = 2 \\ 3x_4 = 3 \\ 2x_2 + x_3 = 0 \end{cases}$$

we set the free variable $x_3 = 0$, and get one particular solution

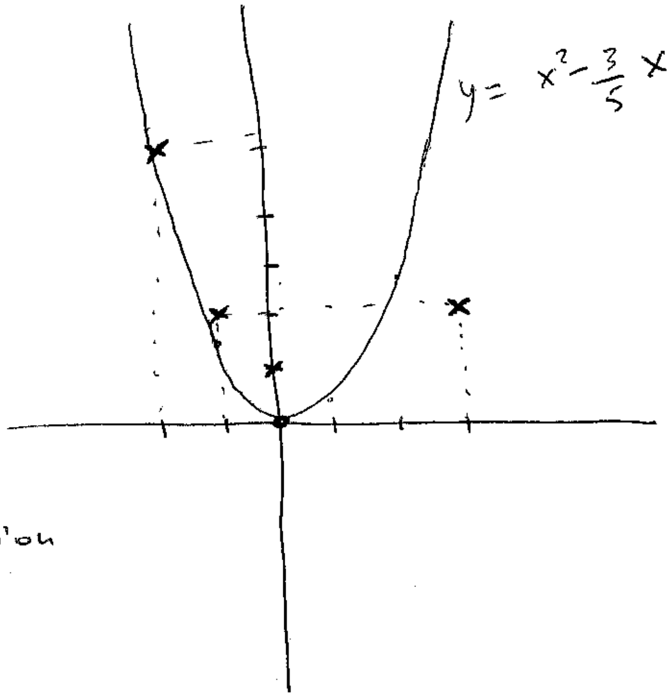
$$x_p = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So the set of solutions to $Ax = b$ is given by

$$\boxed{x = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}, \quad \alpha \in \mathbb{R}}$$

PROBLEM 2

Y	5	2	0	3
X	-2	-1	1	2



we want the best parabola

$$y = ax^2 + bx + c$$

for each point we have one equation

$$5 = 4a - 2b + c$$

$$2 = a - b + c$$

$$0 = a + b + c \quad \Rightarrow \quad Ax = b \quad \text{with}$$

$$3 = 4a + 2b + c$$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

$\rightarrow$ solution is given by $\boxed{\hat{x} = (A^T A)^{-1} A^T b}$. (least squares)

$$A^T A = \begin{bmatrix} 4 & 1 & 1 & 4 \\ -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 4 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 4 & 1 & 1 & 4 \\ -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 34 \\ -6 \\ 10 \end{bmatrix}$$

We need to solve $(A^T A)\hat{x} = A^T b$

$$\Rightarrow \begin{cases} 34a + 10c = 34 \\ 10b = -6 \\ 10a + 4c = 10 \end{cases} \Rightarrow \boxed{\begin{cases} a = 1 \\ b = -3/5 \\ c = 0 \end{cases}}$$

So best parabola is $\boxed{y = x^2 - \frac{3}{5}x}$

For $x = 5$, we predict $\underline{y = 22}$.

PROBLEM 3

v and w are orthogonal eigenvectors of A^TA.

Let $(A^TA)v = \lambda_1 v$ and $(A^TA)w = \lambda_2 w$

we have $(Av)^T(Aw) = (v^TA^T)(Aw) = v^T\underbrace{(A^TA)w}_{\lambda_2 w}$

but $(A^TA)w = \lambda_2 w$, so we have

$(Av)^T(Aw) = v^T(\lambda_2 w) = \lambda_2 v^Tw = 0$

Since v and w are orthogonal vectors $(v^Tw = 0)$.

Therefore $(Av)^T(Aw) = 0$, which show that

Av and Aw are orthogonal.

PROBLEM 4

$$Y = A X = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 1/2 \\ 1 & 1/2 & 0 \end{bmatrix} X$$

1) Eigenvalues of A.

$$\det(A - \lambda I) = \lambda^3 - \frac{1}{2}\lambda^2 - \frac{3}{2}\lambda = \lambda(\lambda+1)\left(\lambda - \frac{3}{2}\right)$$

So eigenvalues are $\boxed{\lambda_1 = 0, \quad \lambda_2 = -1, \quad \lambda_3 = \frac{3}{2}}$

Checking: $\begin{cases} \text{trace}(A) = \frac{1}{2} = \sum \lambda_i \\ \det(A) = 0 = \prod \lambda_i \end{cases}$

2) eigenvectors

*) For $\lambda_1 = 0$

$$A - \lambda_1 I = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 1/2 \\ 1 & 1/2 & 0 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_1]{R_1 \leftarrow R_2} \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 0 & 1/2 & 1 \\ 1 & 1/2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 0 & 1/2 & 1 \\ 0 & -1/2 & -1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

One free variable : x_3

Solve $(A - \lambda_1 I)x = 0 \iff \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$

Taking $x_3 = 1$, we get $X_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Checking $A X_1 = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 1/2 \\ 1 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 X_1$. ok.

$$\boxed{\text{So eigenvectors for } \lambda_1 = 0 \text{ are } X_1 = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \alpha \in \mathbb{R}.}$$

*/ For $\lambda_2 = -1$

$$A - \lambda_2 I = \begin{bmatrix} 1 & 1/2 & 1 \\ 1/2 & 3/2 & 1/2 \\ 1 & 1/2 & 1 \end{bmatrix}$$

eigenvectors are $x_2 \in N(A - \lambda_2 I)$.

$$\begin{bmatrix} 1 & 1/2 & 1 \\ 1/2 & 3/2 & 1/2 \\ 1 & 1/2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - \lambda_2 I) x = 0 \iff \begin{cases} 2x_1 + x_2 + 2x_3 = 0 \\ 5x_2 = 0 \end{cases}$$

Taking the free variable $x_3 = 1$, we get $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

checking $A X_2 = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 1/2 \\ 1 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -X_2$. ok

$$\boxed{\text{So eigenvectors for } \lambda_2 = -1 \text{ are } X_2 = \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \beta \in \mathbb{R}}$$

*/ For $\lambda_3 = 3/2$

$$A - \lambda_3 I = \begin{bmatrix} -3/2 & 1/2 & 1 \\ 1/2 & -1 & 1/2 \\ 1 & 1/2 & -3/2 \end{bmatrix}$$

eigenvectors are $x_3 \in N(A - \lambda_3 I)$

$$\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 1 & 1/2 & -3/2 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 1 & 2 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 1 & 2 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - \lambda_3 I)\, x = 0 \iff \begin{cases} -3x_1 + x_2 + 2x_3 = 0 \\ -5x_2 + 5x_3 = 0 \end{cases}$$

Taking free variable $x_3 = 1$, gives $x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

checking $A x_3 = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 1/2 \\ 1 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 3/2 \end{bmatrix} = \frac{3}{2} x_3$ ok

$$\boxed{\text{So eigenvectors for } \lambda_3 = 3/2 \text{ are } x_3 = \gamma \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \gamma \in \mathbb{R}}$$

3) Cases

- If mistake is in direction of x_1 (eigenvector for $\lambda_1 = 0$)
  then $y = A x_1 = 0$.

  $\rightarrow$ satellite returns to origin (Bien)

- If mistake is in direction of x_2 (eigenvector for $\lambda_2 = -1$)
  then $y = A x_2 = -x_2$

  $\rightarrow$ Satellite moves to the opposite to x, then to x then to $-x$, etc. (Not as good as case 1, but still ok.)

- If mistake is in direction of x_3 (eigenvector for $\lambda_3 = 3/2$)
  then $y = A x_3 = \frac{3}{2} x_3$

  $\rightarrow$ satellite keeps moving in x_3 direction and get lost. (engineers are fired! Pas bien)