

Discrete-time Signal Processing

MsCV Vibot - UE4 Digital Signal Processing

Olivier Morel

Le2i Vision Robotics Team - ERL CNRS

November 29, 2017

Preamble

■ Main references:



Digital Signal Processing, Principles, Algorithms, and Applications,
John G. Proakis, Dimitris G. Manolakis.



Linear Processing for Discrete-Time Signal, Frédéric Truchetet
University of Burgundy



Discrete Time Signal Processing, Benoît Champagne, Fabrice
Labeau, Dpt of Electrical & Computer Engineering McGill University

Preamble

- Main references:



Digital Signal Processing, Principles, Algorithms, and Applications,
John G. Proakis, Dimitris G. Manolakis.



Linear Processing for Discrete-Time Signal, Frédéric Truchetet
University of Burgundy



Discrete Time Signal Processing, Benoît Champagne, Fabrice
Labeau, Dpt of Electrical & Computer Engineering McGill University

- Course flow:

- 8x2H Main course mix with tutorials
- Assessment : closed book

Preamble

■ Main references:



Digital Signal Processing, Principles, Algorithms, and Applications,
John G. Proakis, Dimitris G. Manolakis.



Linear Processing for Discrete-Time Signal, Frédéric Truchetet
University of Burgundy



Discrete Time Signal Processing, Benoît Champagne, Fabrice
Labeau, Dpt of Electrical & Computer Engineering McGill University

■ Course flow:

- 8x2H Main course mix with tutorials

- Assessment : closed book

⇒ 1 sheet of A4 paper allowed!

- double-sided

- manuscript

- no hard copy

Outline

- 1 Introduction**
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT
- 8 Implementation of DT systems

Introduction

Signal

Quantity that varies as function of **time** and/or **space** and has ability to convey information

Introduction

Signal

Quantity that varies as function of **time** and/or **space** and has ability to convey information

- Signals are ubiquitous in science and engineering:

Signal

Quantity that varies as function of **time** and/or **space** and has ability to convey information

- Signals are ubiquitous in science and engineering:
 - **Electrical signals**: currents and voltages in AC circuits, radio communications audio and video signals.

Signal

Quantity that varies as function of **time** and/or **space** and has ability to convey information

- Signals are ubiquitous in science and engineering:
 - **Electrical signals**: currents and voltages in AC circuits, radio communications audio and video signals.
 - **Mechanical signals**: sound or pressure waves, vibrations in a structure, earthquakes.

Signal

Quantity that varies as function of **time** and/or **space** and has ability to convey information

- Signals are ubiquitous in science and engineering:
 - **Electrical signals**: currents and voltages in AC circuits, radio communications audio and video signals.
 - **Mechanical signals**: sound or pressure waves, vibrations in a structure, earthquakes.
 - **Biomedical signals**: electro-encephalogram, lung and heart monitoring, X-ray and other types of images.

Signal

Quantity that varies as function of **time** and/or **space** and has ability to convey information

- Signals are ubiquitous in science and engineering:
 - **Electrical signals**: currents and voltages in AC circuits, radio communications audio and video signals.
 - **Mechanical signals**: sound or pressure waves, vibrations in a structure, earthquakes.
 - **Biomedical signals**: electro-encephalogram, lung and heart monitoring, X-ray and other types of images.
 - **Finance**: time variations of a stock value or a market index.

Signal

Quantity that varies as function of **time** and/or **space** and has ability to convey information

- Signals are ubiquitous in science and engineering:
 - **Electrical signals**: currents and voltages in AC circuits, radio communications audio and video signals.
 - **Mechanical signals**: sound or pressure waves, vibrations in a structure, earthquakes.
 - **Biomedical signals**: electro-encephalogram, lung and heart monitoring, X-ray and other types of images.
 - **Finance**: time variations of a stock value or a market index.
- By extension, any series of measurements of a physical quantity can be considered a signal (temperature measurements for instance)

Types of signals and representations

Analog signal

$$t \in \mathbb{R} \rightarrow x_a(t) \in \mathbb{R} \text{ or } \mathbb{C}$$

Types of signals and representations

Analog signal

$$t \in \mathbb{R} \rightarrow x_a(t) \in \mathbb{R} \text{ or } \mathbb{C}$$

Discrete signal

$$n \in \mathbb{Z} \rightarrow x[n] \in \mathbb{R} \text{ or } \mathbb{C}$$

Types of signals and representations

Analog signal

$$t \in \mathbb{R} \rightarrow x_a(t) \in \mathbb{R} \text{ or } \mathbb{C}$$

Discrete signal

$$n \in \mathbb{Z} \rightarrow x[n] \in \mathbb{R} \text{ or } \mathbb{C}$$

Digital signal

$$n \in \mathbb{Z} \rightarrow x_d[n] \in A, \text{ where } A \text{ represents a finite set of signal levels.}$$

Types of signals and representations

Analog signal

$$t \in \mathbb{R} \rightarrow x_a(t) \in \mathbb{R} \text{ or } \mathbb{C}$$

Discrete signal

$$n \in \mathbb{Z} \rightarrow x[n] \in \mathbb{R} \text{ or } \mathbb{C}$$

Digital signal

$$n \in \mathbb{Z} \rightarrow x_d[n] \in A, \text{ where } A \text{ represents a finite set of signal levels.}$$

Multi-channel signal

$$x(t) = (x_1(t), \dots, x_N(t))$$

Types of signals and representations

Analog signal

$$t \in \mathbb{R} \rightarrow x_a(t) \in \mathbb{R} \text{ or } \mathbb{C}$$

Discrete signal

$$n \in \mathbb{Z} \rightarrow x[n] \in \mathbb{R} \text{ or } \mathbb{C}$$

Digital signal

$$n \in \mathbb{Z} \rightarrow x_d[n] \in A, \text{ where } A \text{ represents a finite set of signal levels.}$$

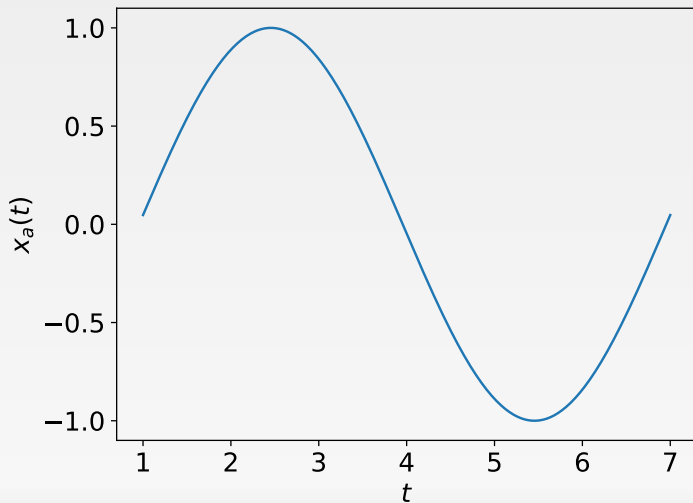
Multi-channel signal

$$x(t) = (x_1(t), \dots, x_N(t))$$

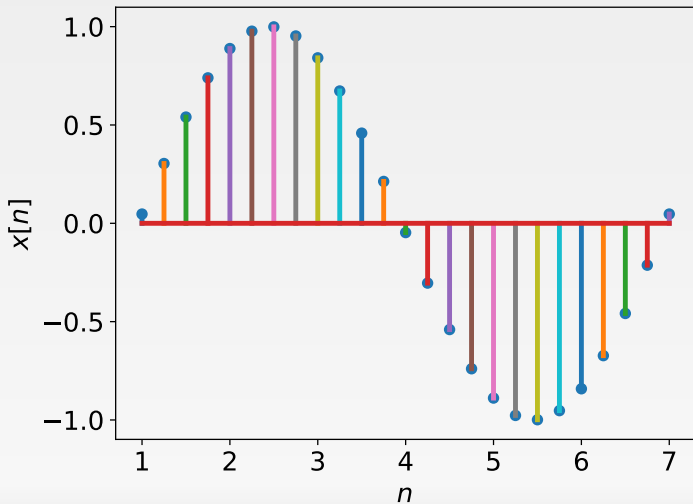
Multi-dimensional signal

$$x(t_1, \dots, t_N)$$

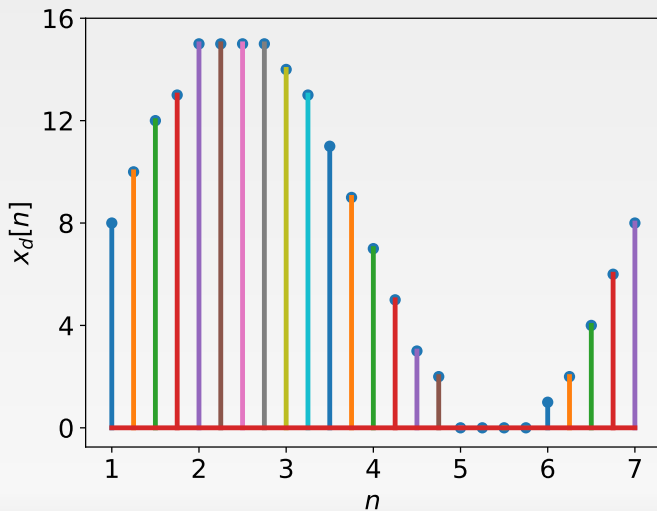
Example: analog signal



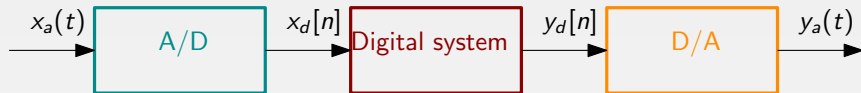
Example: discrete signal



Example: digital signal



Generic structure of a DSP

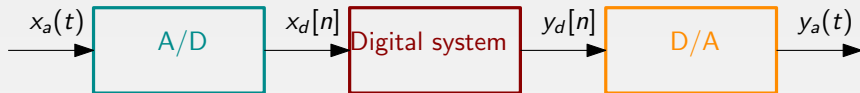


Generic structure of a DSP



1. The A/D (analog-to-digital) converter transforms the analog signal $x_a(t)$ into a digital signal (sampler + quantizer) $x_d[n]$

Generic structure of a DSP



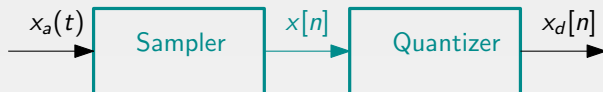
1. The A/D (analog-to-digital) converter transforms the analog signal $x_a(t)$ into a digital signal (sampler + quantizer) $x_d[n]$
2. The DSP (Digital Signal Processing) performs the desired operations on the digital signal $x_d[n] \rightarrow y_d[n]$

Generic structure of a DSP



1. The A/D (analog-to-digital) converter transforms the analog signal $x_a(t)$ into a digital signal (sampler + quantizer) $x_d[n]$
2. The DSP (Digital Signal Processing) performs the desired operations on the digital signal $x_d[n] \rightarrow y_d[n]$
3. The D/A (digital-to-analog) converter transforms the digital output into an analog signal $y_a(t)$

A/D converter



A/D converter



- **Sampler:** analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.

A/D converter



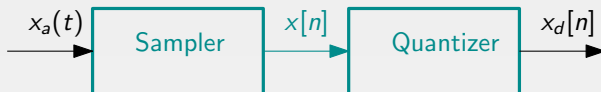
- **Sampler:** analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.
- **Quantizer:** the DT signal is approximated by a digital signal $x_d[n]$ with only a finite set of possible levels.

A/D converter



- **Sampler:** analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.
- **Quantizer:** the DT signal is approximated by a digital signal $x_d[n]$ with only a finite set of possible levels.
- The number of representation levels is generally equal to 2^b where b is the number of bits in a word

A/D converter



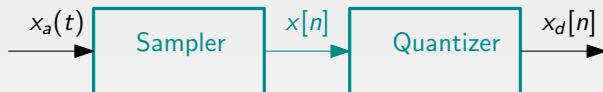
- **Sampler:** analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.
- **Quantizer:** the DT signal is approximated by a digital signal $x_d[n]$ with only a finite set of possible levels.
- The number of representation levels is generally equal to 2^b where b is the number of bits in a word
- In most systems, the set of discrete levels is uniformly spaced

A/D converter



- **Sampler:** analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.
- **Quantizer:** the DT signal is approximated by a digital signal $x_d[n]$ with only a finite set of possible levels.
- The number of representation levels is generally equal to 2^b where b is the number of bits in a word
- In most systems, the set of discrete levels is uniformly spaced
- WAVE file example:

A/D converter



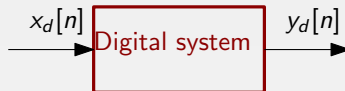
- **Sampler:** analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.
- **Quantizer:** the DT signal is approximated by a digital signal $x_d[n]$ with only a finite set of possible levels.
- The number of representation levels is generally equal to 2^b where b is the number of bits in a word
- In most systems, the set of discrete levels is uniformly spaced
- WAVE file example:
 - sampling rate: 44.1kHz (sampling rate used for audio CD's)

A/D converter

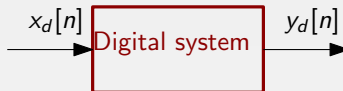


- **Sampler:** analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.
- **Quantizer:** the DT signal is approximated by a digital signal $x_d[n]$ with only a finite set of possible levels.
- The number of representation levels is generally equal to 2^b where b is the number of bits in a word
- In most systems, the set of discrete levels is uniformly spaced
- WAVE file example:
 - sampling rate: 44.1kHz (sampling rate used for audio CD's)
 - level resolution: 16 bits per sample (some systems use 24 bits)

Digital system

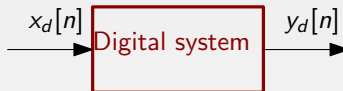


Digital system



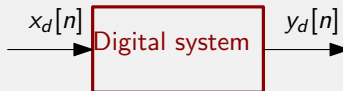
- Similar to a microprocessor: ability to perform mathematical operations and store intermediate results in internal memory

Digital system



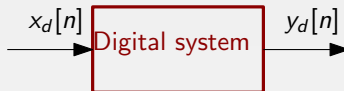
- Similar to a microprocessor: ability to perform mathematical operations and store intermediate results in internal memory
- Operations can be described of mean of an algorithm

Digital system



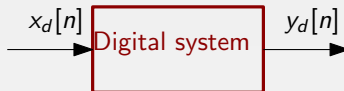
- Similar to a microprocessor: ability to perform mathematical operations and store intermediate results in internal memory
- Operations can be described of mean of an algorithm
- Important distinctions

Digital system



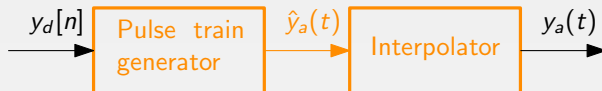
- Similar to a microprocessor: ability to perform mathematical operations and store intermediate results in internal memory
- Operations can be described of mean of an algorithm
- Important distinctions
 - **Real-time system**: computing associated to each sampling interval can be accomplished in a time \leq the sampling interval

Digital system

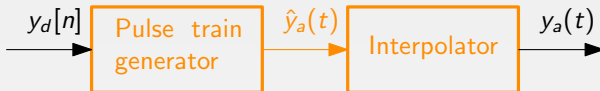


- Similar to a microprocessor: ability to perform mathematical operations and store intermediate results in internal memory
- Operations can be described of mean of an algorithm
- Important distinctions
 - **Real-time system**: computing associated to each sampling interval can be accomplished in a time \leq the sampling interval
 - **Off-line system**: requires the use of external data storage units

D/A converter

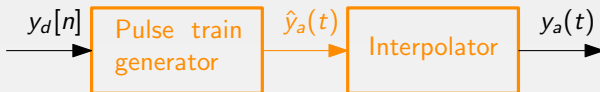


D/A converter



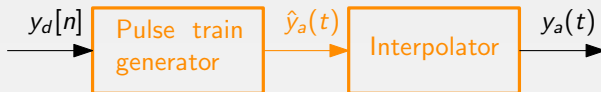
- **Pulse train generator:** the digital signal $y_d[n]$ is transformed into a sequence of scaled, analog pulses

D/A converter



- **Pulse train generator:** the digital signal $y_d[n]$ is transformed into a sequence of scaled, analog pulses
- **Interpolator:** the high frequency components of $\hat{y}_a(t)$ are removed via low-pass filtering to produce a smooth analog output $y_a(t)$

D/A converter



- **Pulse train generator:** the digital signal $y_d[n]$ is transformed into a sequence of scaled, analog pulses
- **Interpolator:** the high frequency components of $\hat{y}_a(t)$ are removed via low-pass filtering to produce a smooth analog output $y_a(t)$
- One device can generally take care of both steps.

Pros and cons of DSP

Advantages

- Robustness (signal levels can be regenerated)
- Storage capability (can interfaced to low-cost devices for storage)
- Flexibility (software programmable)
- Structure (easy interconnection of DSP blocks)

Pros and cons of DSP

Advantages

- Robustness (signal levels can be regenerated)
- Storage capability (can interfaced to low-cost devices for storage)
- Flexibility (software programmable)
- Structure (easy interconnection of DSP blocks)

Disadvantages

- Cost/complexity added by A/D and D/A conversion
- Input signal bandwidth is technology limited
- Quantization effects

Outline

- 1 Introduction
- 2 Discrete-Time signals**
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT
- 8 Implementation of DT systems

Definition

sequence of real or complex numbers, that is, a mapping from the set of integers \mathbb{Z} into \mathbb{R} or \mathbb{C} , as in:

$$n \in \mathbb{Z} \rightarrow x[n] \in \mathbb{R} \text{ or } \mathbb{C}$$

- n is called the discrete-time index
- $x[n]$, the n th number in the sequence, is called a sample

Description

- Sequence notation:

$$x = \{\dots, 0, \underline{0}, 1, 4, 1, 0, 0, \dots\},$$

where underline indicates origin of time: $n = 0$

Description

- Sequence notation:

$$x = \{\dots, 0, \underline{0}, 1, 4, 1, 0, 0, \dots\},$$

where underline indicates origin of time: $n = 0$

- Table:

n	\dots	-2	-1	0	1	2	3	4	5	\dots
$x[n]$	\dots	0	0	0	1	4	1	0	0	\dots

Description

- Sequence notation:

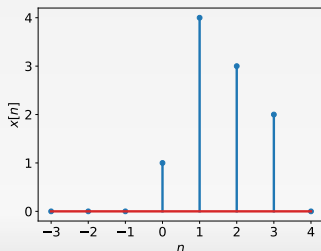
$$x = \{\dots, 0, \underline{0}, 1, 4, 1, 0, 0, \dots\},$$

where underline indicates origin of time: $n = 0$

- Table:

n	\dots	-2	-1	0	1	2	3	4	5	\dots
$x[n]$	\dots	0	0	0	1	4	1	0	0	\dots

- Graphical:



- Explicit mathematical expression:

$$x[n] = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ 1/n & n > 0. \end{cases}$$

- Explicit mathematical expression:

$$x[n] = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ 1/n & n > 0. \end{cases}$$

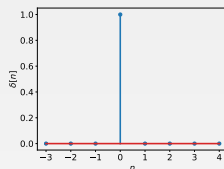
- Recursive approach:

$$x[n] = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ \frac{1}{2}x[n-1] & n > 0. \end{cases}$$

Basic Discrete-time signals

- Unit pulse:

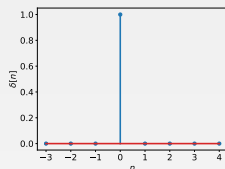
$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$



Basic Discrete-time signals

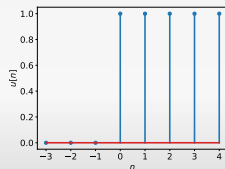
■ Unit pulse:

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$



■ Unit step:

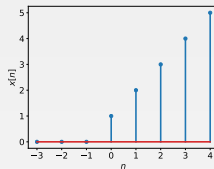
$$u[n] = \begin{cases} 1 & n \geq 0, \\ 0 & n < 0. \end{cases}$$



Basic Discrete-time signals

- Ramp function:

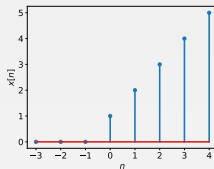
$$r[n] = n \cdot u[n].$$



Basic Discrete-time signals

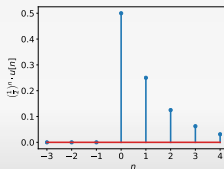
- Ramp function:

$$r[n] = n \cdot u[n].$$



- Exponential sequence:

$$x[n] = a^n \cdot u[n].$$



Uniform sampling

- DT signals are commonly generated via uniform (or periodic) sampling of an analog signal $x_a(t)$:

$$x[n] = x_a(nT_s), \quad n \in \mathbb{Z},$$

where $T_s > 0$: sampling period.

Uniform sampling

- DT signals are commonly generated via uniform (or periodic) sampling of an analog signal $x_a(t)$:

$$x[n] = x_a(nT_s), \quad n \in \mathbb{Z},$$

where $T_s > 0$: sampling period.

- Example of a sine wave with frequency F defined by $x_a(t) = \sin 2\pi Ft$

$$x[n] = \sin(2\pi FnT_s) = \sin\left(2\pi \frac{F}{F_s} n\right) = \sin(\omega n),$$

Uniform sampling

- DT signals are commonly generated via uniform (or periodic) sampling of an analog signal $x_a(t)$:

$$x[n] = x_a(nT_s), \quad n \in \mathbb{Z},$$

where $T_s > 0$: sampling period.

- Example of a sine wave with frequency F defined by $x_a(t) = \sin 2\pi Ft$

$$x[n] = \sin(2\pi FnT_s) = \sin\left(2\pi \frac{F}{F_s} n\right) = \sin(\omega n),$$

- $F_s = 1/T_s$: sampling frequency,

Uniform sampling

- DT signals are commonly generated via uniform (or periodic) sampling of an analog signal $x_a(t)$:

$$x[n] = x_a(nT_s), \quad n \in \mathbb{Z},$$

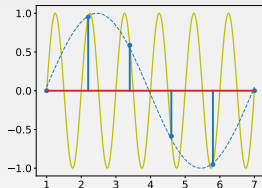
where $T_s > 0$: sampling period.

- Example of a sine wave with frequency F defined by $x_a(t) = \sin 2\pi Ft$

$$x[n] = \sin(2\pi FnT_s) = \sin\left(2\pi \frac{F}{F_s} n\right) = \sin(\omega n),$$

- $F_s = 1/T_s$: sampling frequency,
- ω : normalized radian frequency of the DT signal.

Nyquist rate



Nyquist rate: $2 \cdot F$

The sampling frequency F_s must satisfy

$$F_s \gg 2 \cdot F$$

Basic operations on signal

In the set \mathcal{S} of all DT signals the following operations can be defined:

scaling

$$(\alpha x)[n] = \alpha \cdot x[n]$$

Basic operations on signal

In the set \mathcal{S} of all DT signals the following operations can be defined:

scaling

$$(\alpha x)[n] = \alpha \cdot x[n]$$

addition

$$(x + y)[n] = x[n] + y[n]$$

Basic operations on signal

In the set \mathcal{S} of all DT signals the following operations can be defined:

scaling

$$(\alpha x)[n] = \alpha \cdot x[n]$$

addition

$$(x + y)[n] = x[n] + y[n]$$

multiplication

$$(xy)[n] = x[n] \cdot y[n]$$

Basic operations on signal

In the set \mathcal{S} of all DT signals the following operations can be defined:

scaling

$$(\alpha x)[n] = \alpha \cdot x[n]$$

addition

$$(x + y)[n] = x[n] + y[n]$$

multiplication

$$(xy)[n] = x[n] \cdot y[n]$$

Property

\mathcal{S} equipped with addition and scaling is a vector space.

Classes of signals

Energy signals

all $x \in \mathcal{S}$ with finite energy:

$$\mathcal{E}_x \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Classes of signals

Energy signals

all $x \in \mathcal{S}$ with finite energy:

$$\mathcal{E}_x \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Power signals

all $x \in \mathcal{S}$ with finite power:

$$\mathcal{P}_x \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2 < \infty$$

Classes of signals

Bounded signals

all $x \in \mathcal{S}$ that can be bounded:

$$\exists B_x \in \mathbb{R}^+ / \forall n \in \mathbb{Z}, |x[n]| \leq B_x$$

Classes of signals

Bounded signals

all $x \in \mathcal{S}$ that can be bounded:

$$\exists B_x \in \mathbb{R}^+ / \forall n \in \mathbb{Z}, |x[n]| \leq B_x$$

Absolutely summable

all $x \in \mathcal{S}$ such that:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Discrete convolution

Discrete convolution of x and y

$$(x * y)[n] = x[n] * y[n] \triangleq \sum_{k=-\infty}^{\infty} x[k] \cdot y[n - k]$$

Discrete convolution

Discrete convolution of x and y

$$(x * y)[n] = x[n] * y[n] \triangleq \sum_{k=-\infty}^{\infty} x[k] \cdot y[n - k]$$

Properties

- commutative law: $x * y = y * x$
- associative law: $(x * y) * z = x * (y * z)$
- convolution by unit pulse: $x * \delta = x$

Correlation of DT signals

- Signal correlation is an operation similar to signal convolution with different physical meaning

Correlation of DT signals

- Signal correlation is an operation similar to signal convolution with different physical meaning
- Can be applied to **energy signals**

Correlation of DT signals

- Signal correlation is an operation similar to signal convolution with different physical meaning
- Can be applied to **energy signals**
- **Crosscorrelation**: performed on two signals
 - can be considered as a measure of similarity of two signals
 - application when the signal is corrupted by noise

Correlation of DT signals

- Signal correlation is an operation similar to signal convolution with different physical meaning
- Can be applied to **energy signals**
- **Crosscorrelation**: performed on two signals
 - can be considered as a measure of similarity of two signals
 - application when the signal is corrupted by noise
- **Autocorrelation**: performed on one signal
 - indicates how the signal energy (power) is distributed within the signal
 - applications of signal autocorrelation are in radar, sonar, satellite, and wireless communications systems

Crosscorrelation

Definition

$$R_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]y[k-n] = \sum_{k=-\infty}^{\infty} x[k+n]y[k]$$

Crosscorrelation

Definition

$$R_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]y[k-n] = \sum_{k=-\infty}^{\infty} x[k+n]y[k]$$

Property

$$R_{xy}[n] = R_{yx}[-n]$$

Crosscorrelation

Definition

$$R_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]y[k-n] = \sum_{k=-\infty}^{\infty} x[k+n]y[k]$$

Property

$$R_{xy}[n] = R_{yx}[-n]$$

Link with convolution

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

$$R_{xy}[n] = x[n] * y[-n]$$

Definition

$$R_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[k-n] = \sum_{k=-\infty}^{\infty} x[k+n]x[k]$$

Definition

$$R_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[k-n] = \sum_{k=-\infty}^{\infty} x[k+n]x[k]$$

Properties

- Even function:

$$R_{xx}[n] = R_{xx}[-n]$$

- Energy:

$$R_{xx}[0] = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \mathcal{E}_x$$

$$\forall n \in \mathbb{Z}, R_{xx}[n] < R_{xx}[0]$$

Correlation coefficient

Definition

$$c_{xy} = \frac{R_{xy}[0]}{\sqrt{R_{xx}[0] \cdot R_{yy}[0]}}$$

Correlation coefficient

Definition

$$c_{xy} = \frac{R_{xy}[0]}{\sqrt{R_{xx}[0] \cdot R_{yy}[0]}}$$

Properties

- Similarity measurement of two signals
- $-1 \leq c_{xy} \leq 1$
- Geometrically represents angle between euclidean vectors x and y

$$C_{xy} = \frac{x \cdot y}{\sqrt{|x|^2 |y|^2}} = \frac{x \cdot y}{|x| |y|} \triangleq \cos(x, y)$$

Correlation coefficient

- $c_{xy} \simeq 1 \Rightarrow x$ and y are very similar (almost overlap)

Correlation coefficient

- $c_{xy} \simeq 1 \Rightarrow x$ and y are very similar (almost overlap)
- $c_{xy} \simeq 0 \Rightarrow x$ and y are very different (orthogonal)

Correlation coefficient

- $c_{xy} \simeq 1 \Rightarrow x$ and y are very similar (almost overlap)
- $c_{xy} \simeq 0 \Rightarrow x$ and y are very different (orthogonal)
- $c_{xy} \simeq -1 \Rightarrow x$ and y are asimilar (opposite direction, but almost the same sample values)

Correlation coefficient

- $c_{xy} \simeq 1 \Rightarrow x$ and y are very similar (almost overlap)
- $c_{xy} \simeq 0 \Rightarrow x$ and y are very different (orthogonal)
- $c_{xy} \simeq -1 \Rightarrow x$ and y are asimilar (opposite direction, but almost the same sample values)
- An also be defined in terms of paramter n :

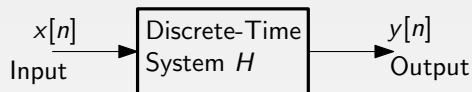
$$-1 \leq c_{xy}[n] = \frac{R_{xy}[n]}{\sqrt{R_{xx}[0]R_{yy}[0]}} \leq 1$$

Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems**
 - Basic systems
 - Linear Time-Invariant (LTI) systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT
- 8 Implementation of DT systems

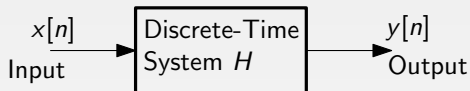
Definition

- A Discrete-Time system is a mapping H from \mathcal{S} into itself :



Definition

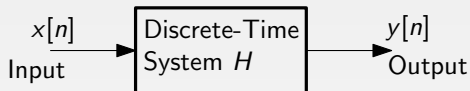
- A Discrete-Time system is a mapping H from \mathcal{S} into itself :



- The system output $y[n]$ generally depends on $x[k]$ for all values of $k \in \mathbb{Z}$

Definition

- A Discrete-Time system is a mapping H from \mathcal{S} into itself :



- The system output $y[n]$ generally depends on $x[k]$ for all values of $k \in \mathbb{Z}$
- Notations:

$$y[n] = H(x[n]) \triangleq (Hx)[n]$$

Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems**
 - Basic systems
 - Linear Time-Invariant (LTI) systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT
- 8 Implementation of DT systems

Basic systems

Time reversal

Time reversal

$$y[n] = (Rx)[n] \triangleq x[-n]$$

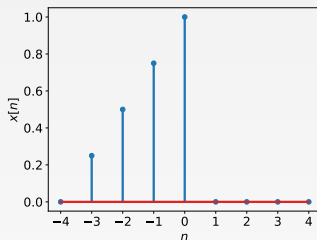
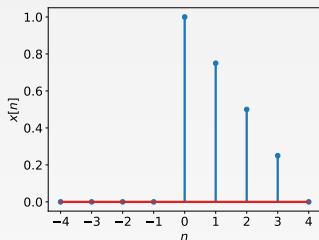
Basic systems

Time reversal

Time reversal

$$y[n] = (Rx)[n] \triangleq x[-n]$$

- Mirror image about origin:



Basic systems

Delay or shift by integer k

Delay or shift by integer k

$$y[n] = (D_k x)[n] \triangleq x[n - k]$$

Basic systems

Delay or shift by integer k

Delay or shift by integer k

$$y[n] = (D_k x)[n] \triangleq x[n - k]$$

■ Interpretation:

- $k \geq 0 \Rightarrow$ graph of $x[n]$ shifted by k units to the right
- $k \leq 0 \Rightarrow$ graph of $x[n]$ shifted by $|k|$ units to the left

Basic systems

Delay or shift by integer k

Delay or shift by integer k

$$y[n] = (D_k x)[n] \triangleq x[n - k]$$

- Interpretation:
 - $k \geq 0 \Rightarrow$ graph of $x[n]$ shifted by k units to the right
 - $k \leq 0 \Rightarrow$ graph of $x[n]$ shifted by $|k|$ units to the left
- Application: any signal $x \in \mathcal{S}$ can be expressed as a linear combination of shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$$

Basic systems

Delay or shift by integer k

Delay or shift by integer k

$$y[n] = (D_k x)[n] \triangleq x[n - k]$$

- Interpretation:
 - $k \geq 0 \Rightarrow$ graph of $x[n]$ shifted by k units to the right
 - $k \leq 0 \Rightarrow$ graph of $x[n]$ shifted by $|k|$ units to the left
- Application: any signal $x \in \mathcal{S}$ can be expressed as a linear combination of shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$$

$$x[n] = (x * \delta)[n]$$

Basic systems

Moving average system

Moving average system

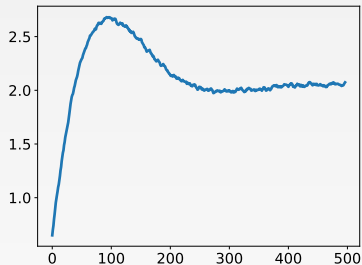
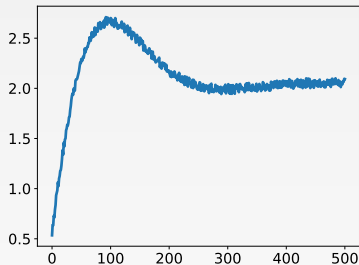
$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^N x[n-k]$$

Basic systems

Moving average system

Moving average system

$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^N x[n-k]$$



Systems properties

1. Static or dynamic (memoryless or not)

Systems properties

1. Static or dynamic (memoryless or not)
2. Causal versus anti-causal

Systems properties

1. Static or dynamic (memoryless or not)
2. Causal versus anti-causal
3. Linear or non-linear

Systems properties

1. Static or dynamic (memoryless or not)
2. Causal versus anti-causal
3. Linear or non-linear
4. Time invariant or not

Systems properties

1. Static or dynamic (memoryless or not)
2. Causal versus anti-causal
3. Linear or non-linear
4. Time invariant or not
5. Stable or not

Systems properties: static or dynamic ?

Static

$y[n] = (Hx)[n]$ is a function of $x[n]$ only.

Systems properties: static or dynamic ?

Static

$y[n] = (Hx)[n]$ is a function of $x[n]$ only.

- Static systems are memoryless:

$$y[n] = (x[n])^2$$

Systems properties: static or dynamic ?

Static

$y[n] = (Hx)[n]$ is a function of $x[n]$ only.

- Static systems are memoryless:

$$y[n] = (x[n])^2$$

- Dynamic systems require memory:

$$y[n] = \frac{1}{2} (x[n-1] + x[n])$$

Systems properties: causal versus anti-causal

Causal

$y[n]$ only depends on values $x[k]$ for $k \leq n$.

Systems properties: causal versus anti-causal

Causal

$y[n]$ only depends on values $x[k]$ for $k \leq n$.

- Present output depend only on past and present inputs
- Example:

$$y[n] = \sum_{k=-\infty}^n x[k]$$

Systems properties: causal versus anti-causal

Causal

$y[n]$ only depends on values $x[k]$ for $k \leq n$.

- Present output depend only on past and present inputs
- Example:

$$y[n] = \sum_{k=-\infty}^n x[k]$$

Anti-causal

$y[n]$ only depends on values $x[k]$ for $k > n$.

Systems properties: linear or not ?

Linearity

$$\forall (\alpha, \beta) \in \mathbb{C}^2, \forall (x, y) \in \mathcal{S}^2, H(\alpha x + \beta y) = \alpha H(x) + \beta H(y)$$

Systems properties: linear or not ?

Linearity

$$\forall (\alpha, \beta) \in \mathbb{C}^2, \forall (x, y) \in \mathcal{S}^2, H(\alpha x + \beta y) = \alpha H(x) + \beta H(y)$$

- Example of a linear system:

$$y[n] = \frac{1}{3} (x[n-1] + x[n] + x[n+1])$$

Systems properties: linear or not ?

Linearity

$$\forall (\alpha, \beta) \in \mathbb{C}^2, \forall (x, y) \in \mathcal{S}^2, H(\alpha x + \beta y) = \alpha H(x) + \beta H(y)$$

- Example of a linear system:

$$y[n] = \frac{1}{3} (x[n-1] + x[n] + x[n+1])$$

- Example of a non-linear system:

$$y[n] = (x[n])^2$$

Systems properties: time invariant or not ?

Time-invariant

$$\forall (n, k) \in \mathbb{Z}^2, (Hx)[n] = y[n] \Rightarrow (Hx)[n - k] = y[n - k]$$

Systems properties: time invariant or not ?

Time-invariant

$$\forall (n, k) \in \mathbb{Z}^2, (Hx)[n] = y[n] \Rightarrow (Hx)[n - k] = y[n - k]$$

- Example of a time invariant system: the moving average system.

$$y[n] = \frac{1}{2N + 1} \sum_{k=-N}^N x[n - k]$$

Systems properties: time invariant or not ?

Time-invariant

$$\forall (n, k) \in \mathbb{Z}^2, (Hx)[n] = y[n] \Rightarrow (Hx)[n - k] = y[n - k]$$

- Example of a time invariant system: the moving average system.

$$y[n] = \frac{1}{2N + 1} \sum_{k=-N}^N x[n - k]$$

- Example of system not time invariant:

$$y[n] = x[2n]$$

Systems properties: stable or not ?

Stable

x bounded $\Rightarrow y = Hx$ bounded

Systems properties: stable or not ?

Stable

x bounded $\Rightarrow y = Hx$ bounded

Stable

if $\forall n \in \mathbb{Z} |x[n]| \leq B_x$ then $\exists B_y / \forall n \in \mathbb{Z}, |y[n]| \leq B_y$

Systems properties: stable or not ?

Stable

x bounded $\Rightarrow y = Hx$ bounded

Stable

if $\forall n \in \mathbb{Z} |x[n]| \leq B_x$ then $\exists B_y / \forall n \in \mathbb{Z}, |y[n]| \leq B_y$

- A system is stable (Bounded Input Bounded Output) if every bounded input produces a bounded output.

Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems**
 - Basic systems
 - Linear Time-Invariant (LTI) systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT
- 8 Implementation of DT systems

Linear Time-Invariant (LTI) systems

- DT systems that are both **Linear** and **Time-Invariant** play a central role in digital signal processing:
 - Many physical systems are either LTI or approximately so
 - Many efficient tools are available for the analysis and design of LTI systems

Linear Time-Invariant (LTI) systems

- DT systems that are both **Linear** and **Time-Invariant** play a central role in digital signal processing:
 - Many physical systems are either LTI or approximately so
 - Many efficient tools are available for the analysis and design of LTI systems

Fundamental property

Let H a LTI system and $y = Hx$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x * h[n],$$

with $h \triangleq H\delta$ known as impulse response of H .

Proof of the fundamental property

First we have:

$$y[n] = (Hx)[n] = H(x[n]).$$

And for any DT signal, we can write:

$$x[n] = x * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Proof of the fundamental property

First we have:

$$y[n] = (Hx)[n] = H(x[n]).$$

And for any DT signal, we can write:

$$x[n] = x * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Invoking linearity:

$$y[n] = H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) = \sum_{k=-\infty}^{\infty} x[k]H(\delta[n-k])$$

Proof of the fundamental property

First we have:

$$y[n] = (Hx)[n] = H(x[n]).$$

And for any DT signal, we can write:

$$x[n] = x * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Invoking linearity:

$$y[n] = H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) = \sum_{k=-\infty}^{\infty} x[k]H(\delta[n-k])$$

Invoking Time-Invariant property:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]HD_k(\delta[n]) = \sum_{k=-\infty}^{\infty} x[k]D_kH(\delta[n]) = \sum_{k=-\infty}^{\infty} x[k]D_kh[n]$$

Graphical interpretation

- to compute the sample values of $y[n]$ according to $y[n] = h * x[n]$:

Graphical interpretation

- to compute the sample values of $y[n]$ according to $y[n] = h * x[n]$:
 1. Time reverse sequence of $h[k]$:
 $\Rightarrow h[-k]$

Graphical interpretation

- to compute the sample values of $y[n]$ according to $y[n] = h * x[n]$:
 1. Time reverse sequence of $h[k]$:
 $\Rightarrow h[-k]$
 2. Shift $h[-k]$ by n samples:
 $\Rightarrow h[-(k - n)] = h[n - k]$

Graphical interpretation

- to compute the sample values of $y[n]$ according to $y[n] = h * x[n]$:
 1. Time reverse sequence of $h[k]$:
 $\Rightarrow h[-k]$
 2. Shift $h[-k]$ by n samples:
 $\Rightarrow h[-(k - n)] = h[n - k]$
 3. Multiply sequences $x[k]$ and $h[n - k]$ and sum over k :
 $\Rightarrow y[n]$

Graphical interpretation

- to compute the sample values of $y[n]$ according to $y[n] = h * x[n]$:
 1. Time reverse sequence of $h[k]$:
 $\Rightarrow h[-k]$
 2. Shift $h[-k]$ by n samples:
 $\Rightarrow h[-(k - n)] = h[n - k]$
 3. Multiply sequences $x[k]$ and $h[n - k]$ and sum over k :
 $\Rightarrow y[n]$

Graphical interpretation

- to compute the sample values of $y[n]$ according to $y[n] = h * x[n]$:

1. Time reverse sequence of $h[k]$:
 $\Rightarrow h[-k]$
2. Shift $h[-k]$ by n samples:
 $\Rightarrow h[-(k - n)] = h[n - k]$
3. Multiply sequences $x[k]$ and $h[n - k]$ and sum over k :
 $\Rightarrow y[n]$

- Example with $h = \{\dots, 0, 1, \underline{0}, -1, 0, \dots\}$ and $x = u$:

$x[n]$	0	0	0	0	1	1	1	1	1
mask			\rightarrow	-1	0	1	\rightarrow		
$y[n]$	0	0	0	1	1	0	0	0	0

Characterization of a LTI system

A LTI system is fully characterized by the knowledge of its impulse response $h = H\delta$. For any other input x we have:

$$Hx = x * h.$$

Characterization of a LTI system

A LTI system is fully characterized by the knowledge of its impulse response $h = H\delta$. For any other input x we have:

$$Hx = x * h.$$

- Example of the accumulator system defined by $y[n] = \sum_{k=-\infty}^n x[k]$:
 - Let's define as input x the unit pulse δ :

$$h[n] = \sum_{k=-\infty}^n \delta[k]$$

Characterization of a LTI system

A LTI system is fully characterized by the knowledge of its impulse response $h = H\delta$. For any other input x we have:

$$Hx = x * h.$$

- Example of the accumulator system defined by $y[n] = \sum_{k=-\infty}^n x[k]$:
 - Let's define as input x the unit pulse δ :

$$h[n] = \sum_{k=-\infty}^n \delta[k]$$

$$h[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

Characterization of a LTI system

A LTI system is fully characterized by the knowledge of its impulse response $h = H\delta$. For any other input x we have:

$$Hx = x * h.$$

- Example of the accumulator system defined by $y[n] = \sum_{k=-\infty}^n x[k]$:
 - Let's define as input x the unit pulse δ :

$$h[n] = \sum_{k=-\infty}^n \delta[k]$$

$$h[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

$$h[n] = u[n]$$

Causality of a LTI system

Causality

A LTI system is causal if and only if:

$$\forall n < 0, h[n] = 0$$

Causality of a LTI system

Causality

A LTI system is causal if and only if:

$$\forall n < 0, h[n] = 0$$

■ Proof:

$$y[n] = h * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Causality of a LTI system

Causality

A LTI system is causal if and only if:

$$\forall n < 0, h[n] = 0$$

■ Proof:

$$y[n] = h * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[n] = \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] \cdots$$

Causality of a LTI system

Causality

A LTI system is causal if and only if:

$$\forall n < 0, h[n] = 0$$

■ Proof:

$$y[n] = h * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[n] = \dots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] \dots$$

\Rightarrow Clearly, $y[n]$ only depends on values $x[m]$ for $m \leq n$ if and only if $h[k] = 0$ for $k < 0$

Stability of a LTI system

Stability

A LTI system is stable if and only if the sequence $h[n]$ is absolutely summable, that is:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Stability of a LTI system

Stability

A LTI system is stable if and only if the sequence $h[n]$ is absolutely summable, that is:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Example: consider LTI system with impulse response $h[n] = \alpha^n u[n]$:

Stability of a LTI system

Stability

A LTI system is stable if and only if the sequence $h[n]$ is absolutely summable, that is:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Example: consider LTI system with impulse response $h[n] = \alpha^n u[n]$:
- Causality: $h[n] = 0$ for $n < 0 \Rightarrow$ **causal**

Stability of a LTI system

Stability

A LTI system is stable if and only if the sequence $h[n]$ is absolutely summable, that is:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Example: consider LTI system with impulse response $h[n] = \alpha^n u[n]$:
- Causality: $h[n] = 0$ for $n < 0 \Rightarrow$ **causal**
- Stability:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |\alpha|^n$$

Stability of a LTI system

Stability

A LTI system is stable if and only if the sequence $h[n]$ is absolutely summable, that is:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Example: consider LTI system with impulse response $h[n] = \alpha^n u[n]$:
- Causality: $h[n] = 0$ for $n < 0 \Rightarrow$ **causal**
- Stability:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |\alpha|^n$$

clearly the sum diverges if $|\alpha| \geq 1$ while it converges if $|\alpha| < 1$:

$$\sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1 - \alpha} < \infty$$

Stability of a LTI system

Stability

A LTI system is stable if and only if the sequence $h[n]$ is absolutely summable, that is:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Example: consider LTI system with impulse response $h[n] = \alpha^n u[n]$:
- Causality: $h[n] = 0$ for $n < 0 \Rightarrow$ **causal**
- Stability:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |\alpha|^n$$

clearly the sum diverges if $|\alpha| \geq 1$ while it converges if $|\alpha| < 1$:

$$\sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1 - \alpha} < \infty$$

Thus the system is **stable** provided $|\alpha| < 1$

FIR system

An LTI system has a **Finite Impulse Response (FIR)** if we can find integers $N_1 \leq N_2$ such that:

$$h[n] = 0 \text{ when } n < N_1 \text{ or } n > N_2$$

FIR system

An LTI system has a **Finite Impulse Response (FIR)** if we can find integers $N_1 \leq N_2$ such that:

$$h[n] = 0 \text{ when } n < N_1 \text{ or } n > N_2$$

- Otherwise the LTI system has an **Infinite Impulse Response (IIR)**.

FIR system

An LTI system has a **Finite Impulse Response (FIR)** if we can find integers $N_1 \leq N_2$ such that:

$$h[n] = 0 \text{ when } n < N_1 \text{ or } n > N_2$$

- Otherwise the LTI system has an **Infinite Impulse Response (IIR)**.
- **FIR systems are necessarily stable:**

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=N_1}^{N_2} |h[n]| < \infty$$

FIR system

An LTI system has a **Finite Impulse Response (FIR)** if we can find integers $N_1 \leq N_2$ such that:

$$h[n] = 0 \text{ when } n < N_1 \text{ or } n > N_2$$

- Otherwise the LTI system has an **Infinite Impulse Response (IIR)**.
- **FIR systems are necessarily stable:**

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=N_1}^{N_2} |h[n]| < \infty$$

- The impulse response is often called a **convolution mask**.

FIR and IIR system examples

1. Let the LTI system described by:

$$h[n] = u[n] - u[n - N]$$

FIR and IIR system examples

1. Let the LTI system described by:

$$h[n] = u[n] - u[n - N]$$

$$h[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases}$$

\Rightarrow the system is **FIR** with $N_1 = 0$ and $N_2 = N - 1$

FIR and IIR system examples

1. Let the LTI system described by:

$$h[n] = u[n] - u[n - N]$$

$$h[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases}$$

\Rightarrow the system is **FIR** with $N_1 = 0$ and $N_2 = N - 1$

2. Let the LTI system described by:

$$h[n] = \alpha^n u[n]$$

FIR and IIR system examples

1. Let the LTI system described by:

$$h[n] = u[n] - u[n - N]$$

$$h[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases}$$

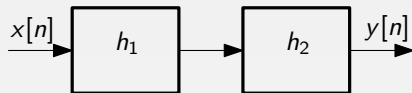
\Rightarrow the system is **FIR** with $N_1 = 0$ and $N_2 = N - 1$

2. Let the LTI system described by:

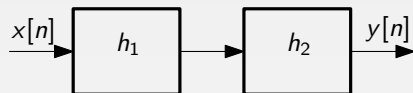
$$h[n] = \alpha^n u[n]$$

\Rightarrow the system is **IIR** (cannot find any N_2)

Interconnection of LTI systems: cascade

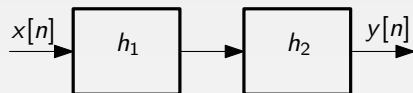


Interconnection of LTI systems: cascade



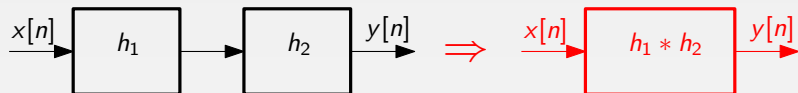
$$y = h_2 * (h_1 * x)$$

Interconnection of LTI systems: cascade



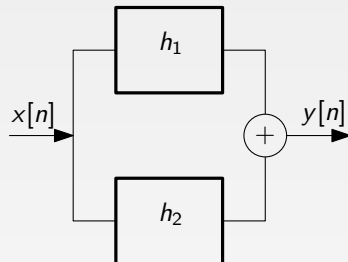
$$y = h_2 * (h_1 * x) = (h_2 * h_1) * x$$

Interconnection of LTI systems: cascade

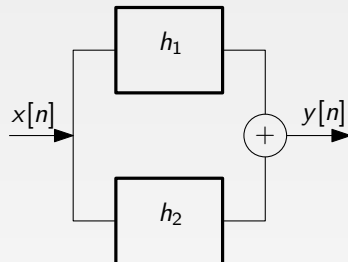


$$y = h_2 * (h_1 * x) = (h_2 * h_1) * x = (h_1 * h_2) * x$$

Interconnection of LTI systems: parallel

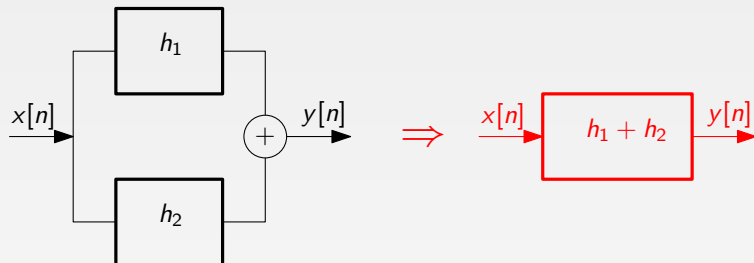


Interconnection of LTI systems: parallel



$$y = (h_1 * x) + (h_2 * x)$$

Interconnection of LTI systems: parallel



$$y = (h_1 * x) + (h_2 * x) = (h_1 + h_2) * x$$

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)**
 - Definition
 - Study of the ROC
 - Properties of the ZT
 - Rational ZTs
 - Inverse ZT
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)**
 - Definition
 - Study of the ROC
 - Properties of the ZT
 - Rational ZTs
 - Inverse ZT
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain

Introduction

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z , defined as:

$$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}, z \in \mathbb{C}$$

Introduction

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z , defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

- The domain of $x(z)$ is the set of all $z \in \mathbb{C}$ such that the series converges absolutely, that is:

$$\text{Dom}(X) = \left\{ z \in \mathbb{C} / \sum_{n=-\infty}^{\infty} |x[n]z|^{-n} < \infty \right\}$$

Introduction

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z , defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

- The domain of $x(z)$ is the set of all $z \in \mathbb{C}$ such that the series converges absolutely, that is:

$$\text{Dom}(X) = \left\{ z \in \mathbb{C} / \sum_{n=-\infty}^{\infty} |x[n]z|^{-n} < \infty \right\}$$

- The domain of $X(z)$ is called the **Region Of Convergence (ROC)**.

Introduction

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z , defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

- The domain of $x(z)$ is the set of all $z \in \mathbb{C}$ such that the series converges absolutely, that is:

$$\text{Dom}(X) = \left\{ z \in \mathbb{C} / \sum_{n=-\infty}^{\infty} |x[n]z|^{-n} < \infty \right\}$$

- The domain of $X(z)$ is called the **Region Of Convergence (ROC)**.
- The ROC **only depends on $|z|$** : if $z \in \text{ROC}$, so is $ze^{i\phi}$ for any angle ϕ

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z , defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

- The domain of $x(z)$ is the set of all $z \in \mathbb{C}$ such that the series converges absolutely, that is:

$$\text{Dom}(X) = \left\{ z \in \mathbb{C} / \sum_{n=-\infty}^{\infty} |x[n]z|^{-n} < \infty \right\}$$

- The domain of $X(z)$ is called the **Region Of Convergence (ROC)**.
- The ROC **only depends on $|z|$** : if $z \in \text{ROC}$, so is $ze^{i\phi}$ for any angle ϕ
- Within the ROC, $X(z)$ is an analytic function of complex variable z :

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z , defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

- The domain of $x(z)$ is the set of all $z \in \mathbb{C}$ such that the series converges absolutely, that is:

$$\text{Dom}(X) = \left\{ z \in \mathbb{C} / \sum_{n=-\infty}^{\infty} |x[n]z|^{-n} < \infty \right\}$$

- The domain of $X(z)$ is called the **Region Of Convergence (ROC)**.
- The ROC **only depends on $|z|$** : if $z \in \text{ROC}$, so is $ze^{i\phi}$ for any angle ϕ
- Within the ROC, $X(z)$ is an analytic function of complex variable z :
 - $X(z)$ is smooth,

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z , defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

- The domain of $x(z)$ is the set of all $z \in \mathbb{C}$ such that the series converges absolutely, that is:

$$\text{Dom}(X) = \left\{ z \in \mathbb{C} / \sum_{n=-\infty}^{\infty} |x[n]z|^{-n} < \infty \right\}$$

- The domain of $X(z)$ is called the **Region Of Convergence (ROC)**.
- The ROC **only depends on $|z|$** : if $z \in \text{ROC}$, so is $ze^{i\phi}$ for any angle ϕ
- Within the ROC, $X(z)$ is an analytic function of complex variable z :
 - $X(z)$ is smooth,
 - derivative exists, etc.

Introduction

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z , defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

- The domain of $x(z)$ is the set of all $z \in \mathbb{C}$ such that the series converges absolutely, that is:

$$\text{Dom}(X) = \left\{ z \in \mathbb{C} / \sum_{n=-\infty}^{\infty} |x[n]z|^{-n} < \infty \right\}$$

- The domain of $X(z)$ is called the **Region Of Convergence (ROC)**.
- The ROC **only depends on $|z|$** : if $z \in \text{ROC}$, so is $ze^{i\phi}$ for any angle ϕ
- Within the ROC, $X(z)$ is an analytic function of complex variable z :
 - $X(z)$ is smooth,
 - derivative exists, etc.
- **Both $X(z)$ and the ROC are needed** when specifying a ZT.

ZT of unit step

- Unit step: $x[n] = u[n]$

1. ZT:

$$X(z) = \sum_{n=0}^{\infty} z^{-n}, z \in \mathbb{C}$$

ZT of unit step

- Unit step: $x[n] = u[n]$

1. ZT:

$$X(z) = \sum_{n=0}^{\infty} z^{-n}, z \in \mathbb{C}$$

$$x(z) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \frac{1}{1 - \frac{1}{z}}$$

ZT of unit step

- Unit step: $x[n] = u[n]$

1. ZT:

$$X(z) = \sum_{n=0}^{\infty} z^{-n}, z \in \mathbb{C}$$

$$x(z) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \frac{1}{1 - \frac{1}{z}}$$

2. ROC:

$$\left| \frac{1}{z} \right| < 1$$

ZT of unit step

- Unit step: $x[n] = u[n]$

1. ZT:

$$X(z) = \sum_{n=0}^{\infty} z^{-n}, z \in \mathbb{C}$$

$$x(z) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \frac{1}{1 - \frac{1}{z}}$$

2. ROC:

$$\left| \frac{1}{z} \right| < 1$$

$$|z| > 1$$

ZT of a signal defined by a sequence

- Signal defined by: $x = \{\dots, 1, 1, \underline{1}, 1, 1, \dots\}$

1. ZT:

$$X(z) = z^{-2} + z^{-1} + 1 + z + z^2$$

ZT of a signal defined by a sequence

- Signal defined by: $x = \{\dots, 1, 1, \underline{1}, 1, 1, \dots\}$

1. ZT:

$$X(z) = z^{-2} + z^{-1} + 1 + z + z^2$$

2. ROC:

$$0 < |z| < \infty$$

ZT of a signal defined by a sequence

- Signal defined by: $x = \{\dots, 1, 1, \underline{1}, 1, 1, \dots\}$

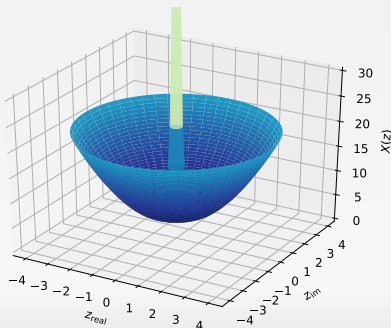
1. ZT:

$$X(z) = z^{-2} + z^{-1} + 1 + z + z^2$$

2. ROC:

$$0 < |z| < \infty$$

3. Modulus of ZT:



Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)**
 - Definition
 - Study of the ROC
 - Properties of the ZT
 - Rational ZTs
 - Inverse ZT
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain

Signal with finite duration

Definition

A signal with finite duration is defined such that:

$$\exists (N_1, N_2) \in \mathbb{Z}^2, N_1 \leq N_2 / \forall n < N_1 \text{ and } \forall n > N_2, x[n] = 0$$

Signal with finite duration

Definition

A signal with finite duration is defined such that:

$$\exists (N_1, N_2) \in \mathbb{Z}^2, N_1 \leq N_2 / \forall n < N_1 \text{ and } \forall n > N_2, x[n] = 0$$

- ZT is defined by:

$$\begin{aligned} X(z) &= \sum_{n=N_1}^{N_2} x[n]z^{-n} \\ &= x[N_1]z^{-N_1} + x[N_1 + 1]z^{-N_1-1} + \dots + x[N_2]z^{-N_2} \end{aligned}$$

Signal with finite duration

Definition

A signal with finite duration is defined such that:

$$\exists (N_1, N_2) \in \mathbb{Z}^2, N_1 \leq N_2 / \forall n < N_1 \text{ and } \forall n > N_2, x[n] = 0$$

- ZT is defined by:

$$\begin{aligned} X(z) &= \sum_{n=N_1}^{N_2} x[n]z^{-n} \\ &= x[N_1]z^{-N_1} + x[N_1 + 1]z^{-N_1-1} + \dots + x[N_2]z^{-N_2} \end{aligned}$$

- ZT exists $\forall z \in \mathbb{C}$, except possibly at $z = 0$ and $z = \infty$:
 - $N_2 > 0 \Rightarrow$
 - $N_1 < 0 \Rightarrow$

Signal with finite duration

Definition

A signal with finite duration is defined such that:

$$\exists (N_1, N_2) \in \mathbb{Z}^2, N_1 \leq N_2 / \forall n < N_1 \text{ and } \forall n > N_2, x[n] = 0$$

- ZT is defined by:

$$\begin{aligned} X(z) &= \sum_{n=N_1}^{N_2} x[n]z^{-n} \\ &= x[N_1]z^{-N_1} + x[N_1 + 1]z^{-N_1-1} + \dots + x[N_2]z^{-N_2} \end{aligned}$$

- ZT exists $\forall z \in \mathbb{C}$, except possibly at $z = 0$ and $z = \infty$:
 - $N_2 > 0 \Rightarrow z = 0 \notin \text{ROC}$
 - $N_1 < 0 \Rightarrow$

Signal with finite duration

Definition

A signal with finite duration is defined such that:

$$\exists (N_1, N_2) \in \mathbb{Z}^2, N_1 \leq N_2 / \forall n < N_1 \text{ and } \forall n > N_2, x[n] = 0$$

- ZT is defined by:

$$\begin{aligned} X(z) &= \sum_{n=N_1}^{N_2} x[n]z^{-n} \\ &= x[N_1]z^{-N_1} + x[N_1 + 1]z^{-N_1-1} + \dots + x[N_2]z^{-N_2} \end{aligned}$$

- ZT exists $\forall z \in \mathbb{C}$, except possibly at $z = 0$ and $z = \infty$:
 - $N_2 > 0 \Rightarrow z = 0 \notin \text{ROC}$
 - $N_1 < 0 \Rightarrow z = \infty \notin \text{ROC}$

Theorem

Radius of convergence

To any power series $\sum_{n=0}^{\infty} c_n w^n$, we can associate a **radius of convergence**

$$R_w = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|,$$

such that:

- if $|w| < R_w \Rightarrow$ the series converges absolutely
- if $|w| > R_w \Rightarrow$ the series diverges

Causal signals

- Suppose $x[n] = 0$ for $n < 0$:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} c_n w^n$$

Causal signals

- Suppose $x[n] = 0$ for $n < 0$:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} c_n w^n$$

- The ROC is given by:

$$|w| < R_w = \lim_{n \rightarrow \infty} \left| \frac{x[n]}{x[n+1]} \right|$$

$$|z| > \frac{1}{R_w} \equiv r$$

Causal signals

- Suppose $x[n] = 0$ for $n < 0$:

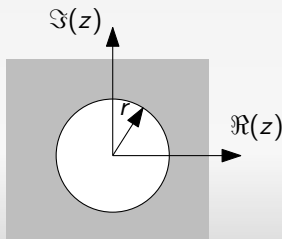
$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} c_n w^n$$

- The ROC is given by:

$$|w| < R_w = \lim_{n \rightarrow \infty} \left| \frac{x[n]}{x[n+1]} \right|$$

$$|z| > \frac{1}{R_w} \equiv r$$

- The ROC is the **exterior of a circle of radius r** :



Causal signals: example

- Consider the causal sequence:

$$x[n] = a^n u[n]$$

Causal signals: example

- Consider the causal sequence:

$$x[n] = a^n u[n]$$

- ZT:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Causal signals: example

- Consider the causal sequence:

$$x[n] = a^n u[n]$$

- ZT:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = \frac{1}{1 - az^{-1}}, \text{ provided } |az^{-1}| < 1$$

Causal signals: example

- Consider the causal sequence:

$$x[n] = a^n u[n]$$

- ZT:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = \frac{1}{1 - az^{-1}}, \text{ provided } |az^{-1}| < 1$$

- The ROC is the exterior of a circle of radius $r = |a|$

Anti-causal signals

- Suppose $x[n] = 0$ for $n > 0$:

$$X(z) = \sum_{n=-\infty}^0 x[n]z^{-n} = \sum_{n=0}^{\infty} x[-n]z^n = \sum_{n=0}^{\infty} c_n w^n$$

Anti-causal signals

- Suppose $x[n] = 0$ for $n > 0$:

$$X(z) = \sum_{n=-\infty}^0 x[n]z^{-n} = \sum_{n=0}^{\infty} x[-n]z^n = \sum_{n=0}^{\infty} c_n w^n$$

- The ROC is given by:

$$|w| < R_w = \lim_{n \rightarrow \infty} \left| \frac{x[-n]}{x[-n-1]} \right|$$

$$|z| < R_w \equiv R$$

Anti-causal signals

- Suppose $x[n] = 0$ for $n > 0$:

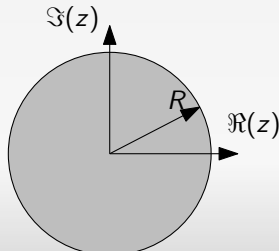
$$X(z) = \sum_{n=-\infty}^0 x[n]z^{-n} = \sum_{n=0}^{\infty} x[-n]z^n = \sum_{n=0}^{\infty} c_n w^n$$

- The ROC is given by:

$$|w| < R_w = \lim_{n \rightarrow \infty} \left| \frac{x[-n]}{x[-n-1]} \right|$$

$$|z| < R_w \equiv R$$

- The ROC is the **interior of a circle of radius R** :



Anti-causal signals: example

- Consider the anti-causal sequence:

$$x[n] = -a^n u[-n - 1]$$

Anti-causal signals: example

- Consider the anti-causal sequence:

$$x[n] = -a^n u[-n - 1]$$

- ZT:

$$X(z) = - \sum_{n=-\infty}^1 a^n z^{-n} = - \sum_{n=1}^{\infty} (a^{-1}z)^n$$

Anti-causal signals: example

- Consider the anti-causal sequence:

$$x[n] = -a^n u[-n - 1]$$

- ZT:

$$X(z) = - \sum_{n=-\infty}^1 a^n z^{-n} = - \sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$\begin{aligned} X(z) &= - \frac{a^{-1}z}{1 - a^{-1}z}, \text{ provided } |a^{-1}z| < 1 \\ &= \frac{1}{1 - az^{-1}} \end{aligned}$$

Anti-causal signals: example

- Consider the anti-causal sequence:

$$x[n] = -a^n u[-n - 1]$$

- ZT:

$$X(z) = - \sum_{n=-\infty}^1 a^n z^{-n} = - \sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$\begin{aligned} X(z) &= - \frac{a^{-1}z}{1 - a^{-1}z}, \text{ provided } |a^{-1}z| < 1 \\ &= \frac{1}{1 - az^{-1}} \end{aligned}$$

- The ROC is the interior of a circle of radius $r = |a|$

Arbitrary signals

- The series $X(z)$ can be decomposed as:

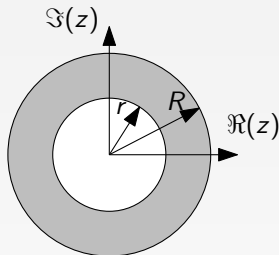
$$X(z) = \underbrace{\sum_{n=-\infty}^{-1} x[n]z^{-n}}_{\text{needs } |z| < R} + \underbrace{\sum_{n=0}^{\infty} x[n]z^{-n}}_{\text{needs } |z| > r}$$

Arbitrary signals

- The series $X(z)$ can be decomposed as:

$$X(z) = \underbrace{\sum_{n=-\infty}^{-1} x[n]z^{-n}}_{\text{needs } |z| < R} + \underbrace{\sum_{n=0}^{\infty} x[n]z^{-n}}_{\text{needs } |z| > r}$$

- If $r < R$, the ZT exists and ROC: $r < |z| < R$:

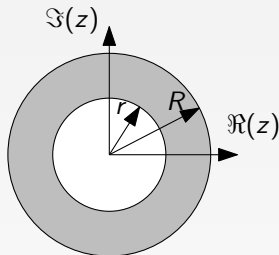


Arbitrary signals

- The series $X(z)$ can be decomposed as:

$$X(z) = \underbrace{\sum_{n=-\infty}^{-1} x[n]z^{-n}}_{\text{needs } |z| < R} + \underbrace{\sum_{n=0}^{\infty} x[n]z^{-n}}_{\text{needs } |z| > r}$$

- If $r < R$, the ZT exists and ROC: $r < |z| < R$:



- if $r > R$, the ZT does not exist.

Arbitrary signals: example 1

Consider the DT signal:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n - 1]$$

Arbitrary signals: example 1

Consider the DT signal:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

Arbitrary signals: example 1

Consider the DT signal:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n}_{\text{needs } |z| > \frac{1}{2}} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n}_{\text{needs } |z| < 2}$$

Arbitrary signals: example 1

Consider the DT signal:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n}_{\text{needs } |z| > \frac{1}{2}} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n}_{\text{needs } |z| < 2}$$

The two series will converge iff ROC: $\frac{1}{2} < |z| < 2$

Arbitrary signals: example 1

Consider the DT signal:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n}_{\text{needs } |z| > \frac{1}{2}} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n}_{\text{needs } |z| < 2}$$

The two series will converge iff ROC: $\frac{1}{2} < |z| < 2$

$$X(z) = \frac{1}{1 - \frac{1}{2z}} - 1 + \frac{1}{1 - \frac{z}{2}} = \frac{2 - \frac{5}{2}z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}}$$

Arbitrary signals: example 2

Consider the DT signal:

$$x[n] = 2^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

Arbitrary signals: example 2

Consider the DT signal:

$$x[n] = 2^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} (2)^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n}$$

Arbitrary signals: example 2

Consider the DT signal:

$$x[n] = 2^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} (2)^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n}$$

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n}_{\text{needs } |z| > 2} + \underbrace{\sum_{n=1}^{\infty} (2z)^n}_{\text{needs } |z| < \frac{1}{2}}$$

Arbitrary signals: example 2

Consider the DT signal:

$$x[n] = 2^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} (2)^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n}$$

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n}_{\text{needs } |z| > 2} + \underbrace{\sum_{n=1}^{\infty} (2z)^n}_{\text{needs } |z| < \frac{1}{2}}$$

Since $\{|z| > 2\} \cap \{|z| < \frac{1}{2}\} = \emptyset$, the ROC is empty and the ZT does not exist.

Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)**
 - Definition
 - Study of the ROC
 - Properties of the ZT**
 - Rational ZTs
 - Inverse ZT
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain

Introductory remarks

- Notations for ZT pairs:

$$x[n] \xleftrightarrow{z} X(z), z \in \mathcal{R}_x$$

$$y[n] \xleftrightarrow{z} Y(z), z \in \mathcal{R}_y$$

\mathcal{R}_x and \mathcal{R}_y denote the ROC of $X(z)$ and $Y(z)$ respectively.

Introductory remarks

- Notations for ZT pairs:

$$x[n] \xleftrightarrow{z} X(z), z \in \mathcal{R}_x$$

$$y[n] \xleftrightarrow{z} Y(z), z \in \mathcal{R}_y$$

\mathcal{R}_x and \mathcal{R}_y denote the ROC of $X(z)$ and $Y(z)$ respectively.

- When stating a property, the corresponding ROC must also be specified

Introductory remarks

- Notations for ZT pairs:

$$x[n] \xleftrightarrow{z} X(z), z \in \mathcal{R}_x$$

$$y[n] \xleftrightarrow{z} Y(z), z \in \mathcal{R}_y$$

\mathcal{R}_x and \mathcal{R}_y denote the ROC of $X(z)$ and $Y(z)$ respectively.

- When stating a property, the corresponding ROC must also be specified
- In some cases, the true ROC may be larger than the one indicated

Basic symmetries

Basic symmetries

$$x[-n] \xleftrightarrow{z} X(z^{-1}), z^{-1} \in \mathcal{R}_x$$

$$x^*[n] \xleftrightarrow{z} X^*(z^*), z \in \mathcal{R}_x$$

Basic symmetries

Basic symmetries

$$x[-n] \xleftrightarrow{z} X(z^{-1}), z^{-1} \in \mathcal{R}_x$$

$$x^*[n] \xleftrightarrow{z} X^*(z^*), z \in \mathcal{R}_x$$

■ Proof:

Let be $x_f[n] = x[-n]$:

$$X_f(z) = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$$

Basic symmetries

$$x[-n] \xleftrightarrow{z} X(z^{-1}), z^{-1} \in \mathcal{R}_x$$

$$x^*[n] \xleftrightarrow{z} X^*(z^*), z \in \mathcal{R}_x$$

■ Proof:

Let be $x_f[n] = x[-n]$:

$$X_f(z) = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$$

$$X_f(z) = \sum_{n=-\infty}^{\infty} x[n]z^n = X(z^{-1})$$

Basic symmetries

Basic symmetries

$$x[-n] \xleftrightarrow{z} X(z^{-1}), z^{-1} \in \mathcal{R}_x$$

$$x^*[n] \xleftrightarrow{z} X^*(z^*), z \in \mathcal{R}_x$$

■ Proof:

Let be $x_f[n] = x[-n]$:

$$X_f(z) = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$$

$$X_f(z) = \sum_{n=-\infty}^{\infty} x[n]z^n = X(z^{-1})$$

Assuming that the ROC \mathcal{R}_x was defined by: $r < |z| < R$, then the ROC of $X_f(z)$ is: $1/R < |z| < 1/r$

Linearity and Time shift

Linearity

$$\forall (a, b) \in \mathbb{C}^2, ax[n] + by[n] \xleftrightarrow{z} aX(z) + bY(z), z \in \mathcal{R}_x \cap \mathcal{R}_y$$

Linearity and Time shift

Linearity

$$\forall (a, b) \in \mathbb{C}^2, ax[n] + by[n] \xleftrightarrow{z} aX(z) + bY(z), z \in \mathcal{R}_x \cap \mathcal{R}_y$$

Time shift

$$\forall d \in \mathbb{Z}, x[n - d] \xleftrightarrow{z} z^{-d}X(z), z \in \mathcal{R}_x$$

Linearity and Time shift

Linearity

$$\forall (a, b) \in \mathbb{C}^2, ax[n] + by[n] \xleftrightarrow{z} aX(z) + bY(z), z \in \mathcal{R}_x \cap \mathcal{R}_y$$

Time shift

$$\forall d \in \mathbb{Z}, x[n-d] \xleftrightarrow{z} z^{-d}X(z), z \in \mathcal{R}_x$$

- Let's denote $X_d(z)$ the ZT of the DT x shifted (or delayed) by d :

$$X_d(z) = \sum_{n=-\infty}^{\infty} x[n-d]z^{-n}$$

Linearity and Time shift

Linearity

$$\forall (a, b) \in \mathbb{C}^2, ax[n] + by[n] \xleftrightarrow{z} aX(z) + bY(z), z \in \mathcal{R}_x \cap \mathcal{R}_y$$

Time shift

$$\forall d \in \mathbb{Z}, x[n-d] \xleftrightarrow{z} z^{-d}X(z), z \in \mathcal{R}_x$$

- Let's denote $X_d(z)$ the ZT of the DT x shifted (or delayed) by d :

$$X_d(z) = \sum_{n=-\infty}^{\infty} x[n-d]z^{-n}$$

$$X_d(z) = \sum_{l=-\infty}^{\infty} x[l]z^{-d}z^{-l}, \text{ with } l = n - d$$

Exponential modulation

Exponential modulation (scaling)

$$a^n x[n] \xleftrightarrow{z} X(z/a), \quad z/a \in \mathcal{R}_x$$

Exponential modulation

Exponential modulation (scaling)

$$a^n x[n] \xleftrightarrow{z} X(z/a), \quad z/a \in \mathcal{R}_x$$

■ Proof:

$$\sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] (a^{-1}z)^{-n}$$

Exponential modulation

Exponential modulation (scaling)

$$a^n x[n] \xleftrightarrow{z} X(z/a), \quad z/a \in \mathcal{R}_x$$

■ Proof:

$$\sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] (a^{-1}z)^{-n}$$

■ Assuming that the ROC \mathcal{R}_x was defined by: $r < |z| < R$, then:

$$r < |a^{-1}z| < R$$

Exponential modulation: example

$$x[n] = \cos(\omega_0 n) u[n]$$

Exponential modulation: example

$$x[n] = \cos(\omega_0 n) u[n]$$

$$x[n] = \frac{1}{2} e^{j\omega_0 n} u[n] + \frac{1}{2} e^{-j\omega_0 n} u[n]$$

Exponential modulation: example

$$x[n] = \cos(\omega_0 n) u[n]$$

$$x[n] = \frac{1}{2} e^{j\omega_0 n} u[n] + \frac{1}{2} e^{-j\omega_0 n} u[n]$$

$$X(z) = \frac{1}{2} ZT \{ e^{j\omega_0 n} u[n] \} + \frac{1}{2} ZT \{ e^{-j\omega_0 n} u[n] \}$$

Exponential modulation: example

$$x[n] = \cos(\omega_0 n) u[n]$$

$$x[n] = \frac{1}{2} e^{j\omega_0 n} u[n] + \frac{1}{2} e^{-j\omega_0 n} u[n]$$

$$X(z) = \frac{1}{2} ZT \{ e^{j\omega_0 n} u[n] \} + \frac{1}{2} ZT \{ e^{-j\omega_0 n} u[n] \}$$

$$X(z) = \underbrace{\frac{1}{2} \frac{1}{1 - e^{j\omega_0} z^{-1}}}_{|z| > |e^{j\omega_0}| = 1} + \underbrace{\frac{1}{2} \frac{1}{1 - e^{-j\omega_0} z^{-1}}}_{|z| > |e^{-j\omega_0}| = 1}$$

Exponential modulation: example

$$x[n] = \cos(\omega_0 n) u[n]$$

$$x[n] = \frac{1}{2} e^{j\omega_0 n} u[n] + \frac{1}{2} e^{-j\omega_0 n} u[n]$$

$$X(z) = \frac{1}{2} ZT \{ e^{j\omega_0 n} u[n] \} + \frac{1}{2} ZT \{ e^{-j\omega_0 n} u[n] \}$$

$$X(z) = \underbrace{\frac{1}{2} \frac{1}{1 - e^{j\omega_0} z^{-1}}}_{|z| > |e^{j\omega_0}| = 1} + \underbrace{\frac{1}{2} \frac{1}{1 - e^{-j\omega_0} z^{-1}}}_{|z| > |e^{-j\omega_0}| = 1}$$

$$X(z) = \frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}, \text{ ROC : } |z| > 1$$

Differentiation

$$nx[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz}, z \in \mathcal{R}_x$$

Differentiation

$$nx[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz}, z \in \mathcal{R}_x$$

■ Proof:

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n] \frac{dz^{-n}}{dz}$$

Differentiation

$$nx[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz}, z \in \mathcal{R}_x$$

■ Proof:

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n] \frac{dz^{-n}}{dz}$$

$$\frac{dX(z)}{dz} = - \sum_{n=-\infty}^{\infty} nx[n] z^{-n-1} = -z^{-1} \sum_{n=-\infty}^{\infty} nx[n] z^{-n}$$

Differentiation: example

$$x[n] = na^n u[n]$$

Differentiation: example

$$x[n] = na^n u[n]$$

$$X(z) = -z \frac{d}{dz} \left(\frac{1}{1 - az^{-1}} \right), |z| > |a|$$

Differentiation: example

$$x[n] = na^n u[n]$$

$$X(z) = -z \frac{d}{dz} \left(\frac{1}{1 - az^{-1}} \right), |z| > |a|$$

$$X(z) = -z (-az^{-2}) \left(\frac{1}{(1 - az^{-1})^2} \right), \text{ROC} : |z| > |a|$$

Differentiation: example

$$x[n] = na^n u[n]$$

$$X(z) = -z \frac{d}{dz} \left(\frac{1}{1 - az^{-1}} \right), |z| > |a|$$

$$X(z) = -z (-az^{-2}) \left(\frac{1}{(1 - az^{-1})^2} \right), \text{ROC} : |z| > |a|$$

$$Z(x) = \frac{az^{-1}}{(1 - az^{-1})^2}, \text{ROC} : |z| > |a|$$

Convolution

$$x[n] * y[n] \xleftrightarrow{z} X(z)Y(z), \quad z \in \mathcal{R}_x \cap \mathcal{R}_y$$

Convolution

$$x[n] * y[n] \xleftrightarrow{z} X(z)Y(z), z \in \mathcal{R}_x \cap \mathcal{R}_y$$

- Let be $c[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$:

$$C(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y[n-k]z^{-n}$$

Convolution

$$x[n] * y[n] \xleftrightarrow{z} X(z)Y(z), z \in \mathcal{R}_x \cap \mathcal{R}_y$$

- Let be $c[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$:

$$C(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y[n-k]z^{-n}$$

$$C(z) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y[l]z^{-l-k}, \text{ with } l = n - k$$

Convolution

$$x[n] * y[n] \xleftrightarrow{z} X(z)Y(z), \quad z \in \mathcal{R}_x \cap \mathcal{R}_y$$

- Let be $c[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$:

$$C(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y[n-k]z^{-n}$$

$$C(z) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y[l]z^{-l-k}, \text{ with } l = n - k$$

$$C(z) = \sum_{l=-\infty}^{\infty} y[l]z^{-l} \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

Initial value

Initial value (causal case)

For $x[n]$ causal (i.e. $x[n] = 0$ for $n < 0$), we have:

$$\lim_{z \rightarrow \infty} X(z) = x[0]$$

Initial value

Initial value (causal case)

For $x[n]$ causal (i.e. $x[n] = 0$ for $n < 0$), we have:

$$\lim_{z \rightarrow \infty} X(z) = x[0]$$

Initial value (anti-causal case)

For $x[n]$ anti-causal (i.e. $x[n] = 0$ for $n > 0$), we have:

$$\lim_{z^{-1} \rightarrow \infty} X(z) = x[0]$$

Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)**
 - Definition
 - Study of the ROC
 - Properties of the ZT
 - Rational ZTs**
 - Inverse ZT
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain

Rational function

Definition

$X(z)$ is a rational function in z (or z^{-1}) if:

$$X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are polynomials in z (resp. z^{-1})

Definition

$X(z)$ is a rational function in z (or z^{-1}) if:

$$X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are polynomials in z (resp. z^{-1})

- Rational ZT plays a central role in DSP
- Essential for the realization of practical IIR filters
- Two important issues related to rational ZT are investigated:
 - Pole-Zero (PZ) characterization
 - Inversion via partial fraction expansion

Poles and zeros

Pole

$X(z)$ has a **pole** of order L at $z = p_0$ if:

$$X(z) = \frac{\psi(z)}{(z - p_0)^L}, \quad 0 < |\psi(p_0)| < \infty$$

Poles and zeros

Pole

$X(z)$ has a **pole** of order L at $z = p_0$ if:

$$X(z) = \frac{\psi(z)}{(z - p_0)^L}, \quad 0 < |\psi(p_0)| < \infty$$

Zero

$X(z)$ has a **zero** of order L at $z = z_0$ if:

$$X(z) = (z - z_0)^L \psi(z), \quad 0 < |\psi(z_0)| < \infty$$

Poles and zeros

Pole

$X(z)$ has a **pole** of order L at $z = p_0$ if:

$$X(z) = \frac{\psi(z)}{(z - p_0)^L}, \quad 0 < |\psi(p_0)| < \infty$$

Zero

$X(z)$ has a **zero** of order L at $z = z_0$ if:

$$X(z) = (z - z_0)^L \psi(z), \quad 0 < |\psi(z_0)| < \infty$$

- The order L is sometimes referred as the **multiplicity** of the pole/zero.

Poles and zeros at ∞

Poles at ∞

$X(z)$ has a pole of order L at $z = \infty$ if:

$$X(z) = z^L \psi(z), \quad 0 < |\psi(\infty)| < \infty$$

Poles and zeros at ∞

Poles at ∞

$X(z)$ has a pole of order L at $z = \infty$ if:

$$X(z) = z^L \psi(z), \quad 0 < |\psi(\infty)| < \infty$$

Zeros at ∞

$X(z)$ has a zero of order L at $z = \infty$ if :

$$X(z) = \frac{\psi(z)}{z^L}, \quad 0 < |\psi(\infty)| < \infty$$

Poles and zeros at ∞

Poles at ∞

$X(z)$ has a pole of order L at $z = \infty$ if:

$$X(z) = z^L \psi(z), \quad 0 < |\psi(\infty)| < \infty$$

Zeros at ∞

$X(z)$ has a zero of order L at $z = \infty$ if :

$$X(z) = \frac{\psi(z)}{z^L}, \quad 0 < |\psi(\infty)| < \infty$$

- Let be $X(z) = N(z)/D(z)$ (expressed in “simplified” form):

Poles and zeros at ∞

Poles at ∞

$X(z)$ has a pole of order L at $z = \infty$ if:

$$X(z) = z^L \psi(z), \quad 0 < |\psi(\infty)| < \infty$$

Zeros at ∞

$X(z)$ has a zero of order L at $z = \infty$ if :

$$X(z) = \frac{\psi(z)}{z^L}, \quad 0 < |\psi(\infty)| < \infty$$

- Let be $X(z) = N(z)/D(z)$ (expressed in “simplified” form):
 - If $\text{order}(N(z)) - \text{order}(D(z)) = L > 0 \Rightarrow X(z)$ has a pole of order L at $z = \infty$

Poles and zeros at ∞

Poles at ∞

$X(z)$ has a pole of order L at $z = \infty$ if:

$$X(z) = z^L \psi(z), \quad 0 < |\psi(\infty)| < \infty$$

Zeros at ∞

$X(z)$ has a zero of order L at $z = \infty$ if :

$$X(z) = \frac{\psi(z)}{z^L}, \quad 0 < |\psi(\infty)| < \infty$$

- Let be $X(z) = N(z)/D(z)$ (expressed in “simplified” form):
 - If $\text{order}(N(z)) - \text{order}(D(z)) = L > 0 \Rightarrow X(z)$ has a pole of order L at $z = \infty$
 - If $\text{order}(N(z)) - \text{order}(D(z)) = L < 0 \Rightarrow X(z)$ has a zero of order L at $z = \infty$

Poles and Zeros of a rational $X(z)$

Consider rational function $X(z) = N(z)/D(z)$:

- Roots of $N(z) \Rightarrow$ zeros of $X(z)$

Poles and Zeros of a rational $X(z)$

Consider rational function $X(z) = N(z)/D(z)$:

- Roots of $N(z) \Rightarrow$ zeros of $X(z)$
- Roots of $D(z) \Rightarrow$ poles of $X(z)$

Poles and Zeros of a rational $X(z)$

Consider rational function $X(z) = N(z)/D(z)$:

- Roots of $N(z) \Rightarrow$ zeros of $X(z)$
- Roots of $D(z) \Rightarrow$ poles of $X(z)$
- Must take into account pole-zero cancellation:

Poles and Zeros of a rational $X(z)$

Consider rational function $X(z) = N(z)/D(z)$:

- Roots of $N(z) \Rightarrow$ zeros of $X(z)$
- Roots of $D(z) \Rightarrow$ poles of $X(z)$
- Must take into account pole-zero cancellation:
 - common roots of $N(z)$ and $D(z)$ do not count as zeros and poles.

Poles and Zeros of a rational $X(z)$

Consider rational function $X(z) = N(z)/D(z)$:

- Roots of $N(z) \Rightarrow$ zeros of $X(z)$
- Roots of $D(z) \Rightarrow$ poles of $X(z)$
- Must take into account pole-zero cancellation:
 - common roots of $N(z)$ and $D(z)$ do not count as zeros and poles.
- Repeated roots in $N(z)$ (or $D(z)$) lead to multiple zeros (respectively poles).

Poles and Zeros of a rational $X(z)$

Consider rational function $X(z) = N(z)/D(z)$:

- Roots of $N(z) \Rightarrow$ zeros of $X(z)$
- Roots of $D(z) \Rightarrow$ poles of $X(z)$
- Must take into account pole-zero cancellation:
 - common roots of $N(z)$ and $D(z)$ do not count as zeros and poles.
- Repeated roots in $N(z)$ (or $D(z)$) lead to multiple zeros (respectively poles).

Property

number of poles = number of zeros, if poles and zeros at 0 and ∞ are included.

Poles and Zeros of a rational: examples

■ Example 1:

$$X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z - 1)^2}$$

Poles and Zeros of a rational: examples

■ Example 1:

$$X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z - 1)^2}$$

Corresponding poles and zeros:

poles $p_1 = 1, L = 2$

Poles and Zeros of a rational: examples

■ Example 1:

$$X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z - 1)^2}$$

Corresponding poles and zeros:

poles $p_1 = 1, L = 2$

zeros $z_1 = 0, L = 1$
 $z_2 = \infty, L = 1$

Poles and Zeros of a rational: examples

■ Example 1:

$$X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z - 1)^2}$$

Corresponding poles and zeros:

poles $p_1 = 1, L = 2$

zeros $z_1 = 0, L = 1$
 $z_2 = \infty, L = 1$

■ Example 2:

$$X(z) = \frac{1 - z^{-4}}{1 + 3z^{-1}} = \frac{z^4 - 1}{z^3(z + 3)}$$

Poles and Zeros of a rational: examples

■ Example 1:

$$X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z - 1)^2}$$

Corresponding poles and zeros:

poles $p_1 = 1, L = 2$

zeros $z_1 = 0, L = 1$
 $z_2 = \infty, L = 1$

■ Example 2:

$$X(z) = \frac{1 - z^{-4}}{1 + 3z^{-1}} = \frac{z^4 - 1}{z^3(z + 3)}$$

Corresponding poles and zeros:

poles $p_1 = 0, L = 3$
 $p_2 = -3, L = 1$

Poles and Zeros of a rational: examples

■ Example 1:

$$X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z - 1)^2}$$

Corresponding poles and zeros:

poles $p_1 = 1, L = 2$

zeros $z_1 = 0, L = 1$
 $z_2 = \infty, L = 1$

■ Example 2:

$$X(z) = \frac{1 - z^{-4}}{1 + 3z^{-1}} = \frac{z^4 - 1}{z^3(z + 3)}$$

Corresponding poles and zeros:

poles $p_1 = 0, L = 3$
 $p_2 = -3, L = 1$

zeros $z_{k \in [0,3]} = e^{jk\pi/2}, L = 1$

Pole-zero and rational function link

Property

For rational functions $X(z) = N(z)/D(z)$, knowledge of the poles and zeros (along with their order) completely specify $X(z)$, up to a scaling factor $G \in \mathbb{C}$.

Pole-zero and rational function link

Property

For rational functions $X(z) = N(z)/D(z)$, knowledge of the poles and zeros (along with their order) completely specify $X(z)$, up to a scaling factor $G \in \mathbb{C}$.

■ Example:

poles $p_1 = 2, L = 1$

zeros $z_1 = 1, L = 1$

Pole-zero and rational function link

Property

For rational functions $X(z) = N(z)/D(z)$, knowledge of the poles and zeros (along with their order) completely specify $X(z)$, up to a scaling factor $G \in \mathbb{C}$.

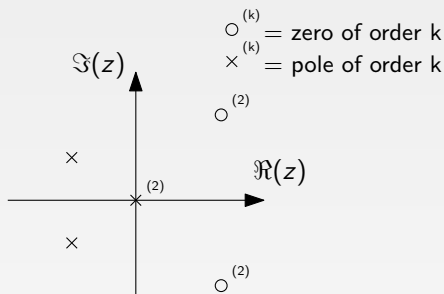
■ Example:

poles $p_1 = 2, L = 1$

zeros $z_1 = 1, L = 1$

$$X(z) = G \frac{z - 1}{z - 2} = G \frac{1 - z^{-1}}{2 - z^{-2}}$$

Pole-zero (PZ) diagram



- The presence of poles or zeros at ∞ should be mentioned on the diagram
- It is useful to indicate ROC on the PZ-diagram

ROC and PZ diagram

- Consider $x[n] = a^n u[n]$, where $a > 0$:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \text{ ROC: } |z| > a$$

ROC and PZ diagram

- Consider $x[n] = a^n u[n]$, where $a > 0$:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \text{ ROC: } |z| > a$$

poles $p_1 = a, L = 1$

zeros $z_1 = 0, L = 1$

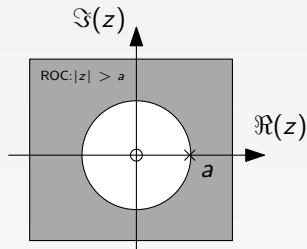
ROC and PZ diagram

- Consider $x[n] = a^n u[n]$, where $a > 0$:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \text{ ROC: } |z| > a$$

poles $p_1 = a, L = 1$

zeros $z_1 = 0, L = 1$



ROC for rational ZT

- ROC does not contain poles

ROC for rational ZT

- ROC does not contain poles
 - because $X(z)$ does not converge at a pole

ROC for rational ZT

- ROC does not contain poles
 - because $X(z)$ does not converge at a pole
- ROC can always be extended to nearest pole

ROC for rational ZT

- ROC does not contain poles
 - because $X(z)$ does not converge at a pole
- ROC can always be extended to nearest pole
- ROC is delimited by poles

ROC for rational ZT

- ROC does not contain poles
 - because $X(z)$ does not converge at a pole
- ROC can always be extended to nearest pole
- ROC is delimited by poles
 - annular region between poles

ROC for rational ZT

- ROC does not contain poles
 - because $X(z)$ does not converge at a pole
- ROC can always be extended to nearest pole
- ROC is delimited by poles
 - annular region between poles
- If we are given only $X(z)$, then several possible ROC:

ROC for rational ZT

- ROC does not contain poles
 - because $X(z)$ does not converge at a pole
- ROC can always be extended to nearest pole
- ROC is delimited by poles
 - annular region between poles
- If we are given only $X(z)$, then several possible ROC:
 - any annular region between two poles of increasing magnitude

ROC for rational ZT

- ROC does not contain poles
 - because $X(z)$ does not converge at a pole
- ROC can always be extended to nearest pole
- ROC is delimited by poles
 - annular region between poles
- If we are given only $X(z)$, then several possible ROC:
 - any annular region between two poles of increasing magnitude
 - accordingly, several possible DT signals $x[n]$

Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)**
 - Definition
 - Study of the ROC
 - Properties of the ZT
 - Rational ZTs
 - Inverse ZT
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain

Introduction

- The inverse Z-Transform consists in finding $x[n]$ given its ZT $X(z)$ and its corresponding ROC.

Introduction

- The inverse Z-Transform consists in finding $x[n]$ given its ZT $X(z)$ and its corresponding ROC.
- Several methods exist:

Introduction

- The inverse Z-Transform consists in finding $x[n]$ given its ZT $X(z)$ and its corresponding ROC.
- Several methods exist:
 - Contour integration via residue theorem

Introduction

- The inverse Z-Transform consists in finding $x[n]$ given its ZT $X(z)$ and its corresponding ROC.
- Several methods exist:
 - Contour integration via residue theorem
 - Power series expansion

Introduction

- The inverse Z-Transform consists in finding $x[n]$ given its ZT $X(z)$ and its corresponding ROC.
- Several methods exist:
 - Contour integration via residue theorem
 - Power series expansion
 - Partial fraction expansion

Introduction

- The inverse Z-Transform consists in finding $x[n]$ given its ZT $X(z)$ and its corresponding ROC.
- Several methods exist:
 - Contour integration via residue theorem
 - Power series expansion
 - Partial fraction expansion
- **Partial fraction** is the most useful technique in the context of **rational ZTs**

Contour integration

Inverse Z-Transform

$$x[k] = \frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz$$

Contour integration

Inverse Z-Transform

$$x[k] = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{k-1} dz$$

$$\begin{aligned} \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{k-1} dz &= \frac{1}{2\pi j} \oint_{\mathcal{C}} \sum_{n=-\infty}^{\infty} x[n] z^{-n} z^{k-1} dz \\ &= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi j} \oint_{\mathcal{C}} z^{k-1-n} dz \end{aligned}$$

Contour integration

Inverse Z-Transform

$$x[k] = \frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz$$

$$\begin{aligned} \frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz &= \frac{1}{2\pi j} \oint_C \sum_{n=-\infty}^{\infty} x[n] z^{-n} z^{k-1} dz \\ &= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi j} \oint_C z^{k-1-n} dz \end{aligned}$$

- Cauchy integral theorem:

$$\frac{1}{2\pi j} \oint_C z^{k-1-n} dz = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

Inversion via Partial Fraction Expansion

- Let be a rational ZT defined according to:
 - $X(z) = \frac{N(z)}{D(z)}$
 - $N(z)$ and $D(z)$ are polynomials in z^{-1}
 - degree of $D(z) >$ degree of $N(z)$

Inversion via Partial Fraction Expansion

- Let be a rational ZT defined according to:
 - $X(z) = \frac{N(z)}{D(z)}$
 - $N(z)$ and $D(z)$ are polynomials in z^{-1}
 - degree of $D(z) >$ degree of $N(z)$
- Under these conditions, $X(z)$ may be expressed as:

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

- p_1, \dots, p_K are the distinct poles of $X(z)$
- L_1, \dots, L_K are the corresponding orders

Expression of the constants A_{kl}

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

Constants A_{kl} can be computed as follows:

Expression of the constants A_{kl}

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

Constants A_{kl} can be computed as follows:

- simple poles ($L_k = 1$):

$$A_{kl} \equiv (1 - p_k z^{-1}) X(z) \Big|_{z=p_k}$$

Expression of the constants A_{kl}

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

Constants A_{kl} can be computed as follows:

- simple poles ($L_k = 1$):

$$A_{kl} \equiv (1 - p_k z^{-1}) X(z) \Big|_{z=p_k}$$

- multiple poles ($L_k > 1$):

$$A_{kl} \equiv \frac{1}{(L_k - l)! (-p_k)^{L_k - l}} \left\{ \frac{d^{L_k - l}}{(dz^{-1})^{L_k - l}} (1 - p_k z^{-1})^{L_k} X(z) \right\} \Big|_{z=p_k}$$

Inversion method

Given $X(z)$ as above with ROC: $r < |z| < R$.

1. Determine the PFE of $X(z)$:

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

Inversion method

Given $X(z)$ as above with ROC: $r < |z| < R$.

1. Determine the PFE of $X(z)$:

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

2. Invoking linearity of the ZT, express $x[n]$ as:

$$x[n] = \sum_{k=1}^K \sum_{l=1}^{L_K} A_{kl} \mathcal{Z}^{-1} \left\{ \frac{1}{(1 - p_k z^{-1})^l} \right\}$$

Inversion method

Given $X(z)$ as above with ROC: $r < |z| < R$.

1. Determine the PFE of $X(z)$:

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

2. Invoking linearity of the ZT, express $x[n]$ as:

$$x[n] = \sum_{k=1}^K \sum_{l=1}^{L_K} A_{kl} \mathcal{Z}^{-1} \left\{ \frac{1}{(1 - p_k z^{-1})^l} \right\}$$

3. Evaluate the elementary inverse ZTs:

Inversion method

Given $X(z)$ as above with ROC: $r < |z| < R$.

1. Determine the PFE of $X(z)$:

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

2. Invoking linearity of the ZT, express $x[n]$ as:

$$x[n] = \sum_{k=1}^K \sum_{l=1}^{L_K} A_{kl} \mathcal{Z}^{-1} \left\{ \frac{1}{(1 - p_k z^{-1})^l} \right\}$$

3. Evaluate the elementary inverse ZTs:

- simple poles ($L_k = 1$):

$$\frac{1}{1 - p_k z^{-1}} \xrightarrow{\mathcal{Z}^{-1}} \begin{cases} p_k^n u[n] & \text{if } |p_k| \leq r \\ -p_k^n u[-n-1] & \text{if } |p_k| \geq R \end{cases}$$

Inversion method

Given $X(z)$ as above with ROC: $r < |z| < R$.

1. Determine the PFE of $X(z)$:

$$X(z) = \sum_{k=1}^K \sum_{l=1}^{L_k} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

2. Invoking linearity of the ZT, express $x[n]$ as:

$$x[n] = \sum_{k=1}^K \sum_{l=1}^{L_k} A_{kl} \mathcal{Z}^{-1} \left\{ \frac{1}{(1 - p_k z^{-1})^l} \right\}$$

3. Evaluate the elementary inverse ZTs:

- simple poles ($L_k = 1$):

$$\frac{1}{1 - p_k z^{-1}} \xrightarrow{\mathcal{Z}^{-1}} \begin{cases} p_k^n u[n] & \text{if } |p_k| \leq r \\ -p_k^n u[-n-1] & \text{if } |p_k| \geq R \end{cases}$$

- multiple poles ($L_k > 1$):

$$\frac{1}{(1 - p_k z^{-1})^l} \xrightarrow{\mathcal{Z}^{-1}} \begin{cases} \binom{n+l-1}{l-1} p_k^n u[n] & \text{if } |p_k| \leq r \\ -\binom{n+l-1}{l-1} p_k^n u[-n-1] & \text{if } |p_k| \geq R \end{cases}$$

Example

$$X(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})}, \quad |a| < |z| < |b|$$

Example

$$X(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})}, \quad |a| < |z| < |b|$$

- PFE can be written as:

$$X(z) = \frac{A_1}{(1 - az^{-1})} + \frac{A_2}{(1 - bz^{-1})},$$

with :

- $A_1 \equiv (1 - az^{-1}) X(z) \Big|_{z=a} = \frac{a}{a-b}$
- $A_2 \equiv (1 - bz^{-1}) X(z) \Big|_{z=b} = \frac{b}{b-a}$

Example

$$X(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})}, \quad |a| < |z| < |b|$$

- PFE can be written as:

$$X(z) = \frac{A_1}{(1 - az^{-1})} + \frac{A_2}{(1 - bz^{-1})},$$

with :

- $A_1 \equiv (1 - az^{-1}) X(z) \Big|_{z=a} = \frac{a}{a-b}$
- $A_2 \equiv (1 - bz^{-1}) X(z) \Big|_{z=b} = \frac{b}{b-a}$
- Elementary inverse ZTs from 2 simple poles:

- $\frac{1}{1-az^{-1}} \xrightarrow{\mathcal{Z}^{-1}} a^n u[n]$ since $|z| > |a|$ (causal)
- $\frac{1}{1-bz^{-1}} \xrightarrow{\mathcal{Z}^{-1}} -b^n u[-n-1]$ since $|z| < |b|$ (anti-causal)

Example

$$X(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})}, \quad |a| < |z| < |b|$$

- PFE can be written as:

$$X(z) = \frac{A_1}{(1 - az^{-1})} + \frac{A_2}{(1 - bz^{-1})},$$

with :

- $A_1 \equiv (1 - az^{-1}) X(z) \Big|_{z=a} = \frac{a}{a-b}$
- $A_2 \equiv (1 - bz^{-1}) X(z) \Big|_{z=b} = \frac{b}{b-a}$
- Elementary inverse ZTs from 2 simple poles:
 - $\frac{1}{1-az^{-1}} \xrightarrow{\mathcal{Z}^{-1}} a^n u[n]$ since $|z| > |a|$ (causal)
 - $\frac{1}{1-bz^{-1}} \xrightarrow{\mathcal{Z}^{-1}} -b^n u[-n-1]$ since $|z| < |b|$ (anti-causal)
- Finally:

$$x[n] = \frac{a^{n+1}}{a-b} u[n] - \frac{b^{n+1}}{b-a} u[-n-1]$$

Putting $X(z)$ in a suitable form

- When applying the above PFE method to $X(z) = N(z)/D(z)$, it is essential that:

Putting $X(z)$ in a suitable form

- When applying the above PFE method to $X(z) = N(z)/D(z)$, it is essential that:
 - $N(z)$ and $D(z)$ be polynomials in z^{-1}

Putting $X(z)$ in a suitable form

- When applying the above PFE method to $X(z) = N(z)/D(z)$, it is essential that:
 - $N(z)$ and $D(z)$ be polynomials in z^{-1}
 - degree of $D(z) >$ degree of $N(z)$

Putting $X(z)$ in a suitable form

- When applying the above PFE method to $X(z) = N(z)/D(z)$, it is essential that:
 - $N(z)$ and $D(z)$ be polynomials in z^{-1}
 - degree of $D(z) >$ degree of $N(z)$
- If either one of the above conditions are not satisfied, further algebraic manipulations must be applied to $X(z)$

Putting $X(z)$ in a suitable form

- When applying the above PFE method to $X(z) = N(z)/D(z)$, it is essential that:
 - $N(z)$ and $D(z)$ be polynomials in z^{-1}
 - degree of $D(z) >$ degree of $N(z)$
- If either one of the above conditions are not satisfied, further algebraic manipulations must be applied to $X(z)$
- There are two common types of manipulations:

Putting $X(z)$ in a suitable form

- When applying the above PFE method to $X(z) = N(z)/D(z)$, it is essential that:
 - $N(z)$ and $D(z)$ be polynomials in z^{-1}
 - degree of $D(z) >$ degree of $N(z)$
- If either one of the above conditions are not satisfied, further algebraic manipulations must be applied to $X(z)$
- There are two common types of manipulations:
 - polynomial division

Putting $X(z)$ in a suitable form

- When applying the above PFE method to $X(z) = N(z)/D(z)$, it is essential that:
 - $N(z)$ and $D(z)$ be polynomials in z^{-1}
 - degree of $D(z) >$ degree of $N(z)$
- If either one of the above conditions are not satisfied, further algebraic manipulations must be applied to $X(z)$
- There are two common types of manipulations:
 - polynomial division
 - use of shift property

Polynomial division

- Find $Q(z)$ and $R(z)$, such that:

$$\frac{N(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)}$$

Polynomial division

- Find $Q(z)$ and $R(z)$, such that:

$$\frac{N(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)}$$

- $Q(z)$ is a polynomial in z^{-1} ,

Polynomial division

- Find $Q(z)$ and $R(z)$, such that:

$$\frac{N(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)}$$

- $Q(z)$ is a polynomial in z^{-1} ,
- $R(z)$ is a polynomial in z^{-1} with degree less than that of $D(z)$.

Polynomial division

- Find $Q(z)$ and $R(z)$, such that:

$$\frac{N(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)}$$

- $Q(z)$ is a polynomial in z^{-1} ,
- $R(z)$ is a polynomial in z^{-1} with degree less than that of $D(z)$.
- $Q(z)$ and $R(z)$ are determined using a division table:

$$\begin{array}{r} Q(z) \\ D(z) \overline{) \begin{array}{l} N(z) \\ -Q(z)D(z) \\ \hline R(z) \end{array}} \end{array}$$

Polynomial division

- Find $Q(z)$ and $R(z)$, such that:

$$\frac{N(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)}$$

- $Q(z)$ is a polynomial in z^{-1} ,
- $R(z)$ is a polynomial in z^{-1} with degree less than that of $D(z)$.
- $Q(z)$ and $R(z)$ are determined using a division table:

$$\begin{array}{r} Q(z) \\ D(z) \overline{) N(z)} \\ \underline{-Q(z)D(z)} \\ R(z) \end{array}$$

- To increase the z^{-1} power in $N(z)$, $D(z)$ and $N(z)$ are expressed in decreasing powers of z (e.g. $D(z) = 1 + 2z^{-1} + z^{-2}$)

Polynomial division

- Find $Q(z)$ and $R(z)$, such that:

$$\frac{N(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)}$$

- $Q(z)$ is a polynomial in z^{-1} ,
- $R(z)$ is a polynomial in z^{-1} with degree less than that of $D(z)$.
- $Q(z)$ and $R(z)$ are determined using a division table:

$$\begin{array}{r} Q(z) \\ D(z) \overline{) N(z)} \\ \underline{-Q(z)D(z)} \\ R(z) \end{array}$$

- To increase the z^{-1} power in $N(z)$, $D(z)$ and $N(z)$ are expressed in decreasing powers of z (e.g. $D(z) = 1 + 2z^{-1} + z^{-2}$)
- To decrease the z^{-1} power in $N(z)$, $D(z)$ and $N(z)$ are expressed in increasing powers of z (e.g. $D(z) = z^{-2} + 2z^{-1} + 1$)

Example

ZT of a causal signal

$$X(z) = \frac{-5 + 3z^{-1} + z^{-2}}{3 + 4z^{-1} + z^{-2}}$$

Example

ZT of a causal signal

$$X(z) = \frac{-5 + 3z^{-1} + z^{-2}}{3 + 4z^{-1} + z^{-2}}$$

- Use long division to make the degree of numerator smaller than the degree of the denominator
⇒ decrease the z^{-1} power in $N(z)$

Example

ZT of a causal signal

$$X(z) = \frac{-5 + 3z^{-1} + z^{-2}}{3 + 4z^{-1} + z^{-2}}$$

- Use long division to make the degree of numerator smaller than the degree of the denominator
⇒ decrease the z^{-1} power in $N(z)$

$$z^{-2} + 4z^{-1} + 3 \overline{) z^{-2} + 3z^{-1} - 5}$$

Example

ZT of a causal signal

$$X(z) = \frac{-5 + 3z^{-1} + z^{-2}}{3 + 4z^{-1} + z^{-2}}$$

- Use long division to make the degree of numerator smaller than the degree of the denominator
⇒ decrease the z^{-1} power in $N(z)$

$$z^{-2} + 4z^{-1} + 3 \overline{) \begin{array}{r} 1 \\ z^{-2} + 3z^{-1} - 5 \\ - (z^{-2} + 4z^{-1} + 3) \end{array}}$$

Example

ZT of a causal signal

$$X(z) = \frac{-5 + 3z^{-1} + z^{-2}}{3 + 4z^{-1} + z^{-2}}$$

- Use long division to make the degree of numerator smaller than the degree of the denominator
 \Rightarrow decrease the z^{-1} power in $N(z)$

$$z^{-2} + 4z^{-1} + 3 \overline{) \begin{array}{r} 1 \\ z^{-2} + 3z^{-1} - 5 \\ - (z^{-2} + 4z^{-1} + 3) \\ \hline -z^{-1} - 8 \end{array}}$$

Example

ZT of a causal signal

$$X(z) = \frac{-5 + 3z^{-1} + z^{-2}}{3 + 4z^{-1} + z^{-2}}$$

- Use long division to make the degree of numerator smaller than the degree of the denominator
⇒ decrease the z^{-1} power in $N(z)$

$$\begin{array}{r} z^{-2} + 4z^{-1} + 3 \overline{) \begin{array}{r} 1 \\ z^{-2} + 3z^{-1} - 5 \\ - (z^{-2} + 4z^{-1} + 3) \\ \hline -z^{-1} - 8 \end{array}} \end{array}$$

$X(z)$ rewrites $X(z) = 1 - \frac{z^{-1}+8}{z^{-2}+4z^{-1}+3}$

Example

$$X(z) = 1 - \frac{z^{-1} + 8}{z^{-2} + 4z^{-1} + 3}$$

- The denominator of the 2nd term has 2 roots, poles at $z = -1/3$ and $z = -1$, hence:

$$X(z) = 1 - \frac{z^{-1} + 8}{3 \left(1 + \frac{1}{3}z^{-1}\right) (1 + z^{-1})}$$

Example

$$X(z) = 1 - \frac{z^{-1} + 8}{z^{-2} + 4z^{-1} + 3}$$

- The denominator of the 2nd term has 2 roots, poles at $z = -1/3$ and $z = -1$, hence:

$$X(z) = 1 - \frac{z^{-1} + 8}{3 \left(1 + \frac{1}{3}z^{-1}\right) (1 + z^{-1})}$$

- The PFE gives:

$$X(z) = 1 - \frac{1}{3} \left(\frac{A_1}{1 + \frac{1}{3}z^{-1}} + \frac{A_2}{1 + z^{-1}} \right)$$

with:

Example

$$X(z) = 1 - \frac{z^{-1} + 8}{z^{-2} + 4z^{-1} + 3}$$

- The denominator of the 2nd term has 2 roots, poles at $z = -1/3$ and $z = -1$, hence:

$$X(z) = 1 - \frac{z^{-1} + 8}{3 \left(1 + \frac{1}{3}z^{-1}\right) (1 + z^{-1})}$$

- The PFE gives:

$$X(z) = 1 - \frac{1}{3} \left(\frac{A_1}{1 + \frac{1}{3}z^{-1}} + \frac{A_2}{1 + z^{-1}} \right)$$

with:

- $A_1 = \left. \frac{z^{-1} + 8}{1 + z^{-1}} \right|_{z=-1/3} = -\frac{5}{2}$

Example

$$X(z) = 1 - \frac{z^{-1} + 8}{z^{-2} + 4z^{-1} + 3}$$

- The denominator of the 2nd term has 2 roots, poles at $z = -1/3$ and $z = -1$, hence:

$$X(z) = 1 - \frac{z^{-1} + 8}{3 \left(1 + \frac{1}{3}z^{-1}\right) (1 + z^{-1})}$$

- The PFE gives:

$$X(z) = 1 - \frac{1}{3} \left(\frac{A_1}{1 + \frac{1}{3}z^{-1}} + \frac{A_2}{1 + z^{-1}} \right)$$

with:

- $A_1 = \left. \frac{z^{-1} + 8}{1 + z^{-1}} \right|_{z=-1/3} = -\frac{5}{2}$

- $A_2 = \left. \frac{z^{-1} + 8}{1 + \frac{1}{3}z^{-1}} \right|_{z=-1} = \frac{21}{2}$

Example

$$X(z) = 1 + \frac{5}{6(1 + \frac{1}{3}z^{-1})} - \frac{7}{2(1 + z^{-1})}$$

- Causality of $x[n]$ determines the ROC of $X(z)$

Example

$$X(z) = 1 + \frac{5}{6(1 + \frac{1}{3}z^{-1})} - \frac{7}{2(1 + z^{-1})}$$

- Causality of $x[n]$ determines the ROC of $X(z)$
 - ROC is supposed to be delimited by circles with radius $1/3$ and/or 1

Example

$$X(z) = 1 + \frac{5}{6(1 + \frac{1}{3}z^{-1})} - \frac{7}{2(1 + z^{-1})}$$

- Causality of $x[n]$ determines the ROC of $X(z)$
 - ROC is supposed to be delimited by circles with radius $1/3$ and/or 1
 - causal sequence \Rightarrow ROC must extend outwards from the outermost pole \Rightarrow ROC is $|z| > 1$

Example

$$X(z) = 1 + \frac{5}{6(1 + \frac{1}{3}z^{-1})} - \frac{7}{2(1 + z^{-1})}$$

- Causality of $x[n]$ determines the ROC of $X(z)$
 - ROC is supposed to be delimited by circles with radius $1/3$ and/or 1
 - causal sequence \Rightarrow ROC must extend outwards from the outermost pole \Rightarrow ROC is $|z| > 1$
- The sequence $x[n]$ is then given by:

$$x[n] = \delta[n] + \frac{5}{6} \left(-\frac{1}{3}\right)^n u[n] - \frac{7}{2} (-1)^n u[n]$$

Use of shift property

- In some cases, a simple multiplication by z^k is sufficient to put $X(z)$ into a suitable format, that is:

$$Y(z) = z^k X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ satisfy previous conditions

Use of shift property

- In some cases, a simple multiplication by z^k is sufficient to put $X(z)$ into a suitable format, that is:

$$Y(z) = z^k X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ satisfy previous conditions

- The PFE method is then applied to $Y(z)$, yielding a DT signal $y[n]$

Use of shift property

- In some cases, a simple multiplication by z^k is sufficient to put $X(z)$ into a suitable format, that is:

$$Y(z) = z^k X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ satisfy previous conditions

- The PFE method is then applied to $Y(z)$, yielding a DT signal $y[n]$
- Finally, the shift property is applied to recover $x[n]$:

$$x[n] = y[n - k]$$

Example

$$X(z) = \frac{1 - z^{-128}}{1 - z^{-2}}, \quad |z| > 1$$

Example

$$X(z) = \frac{1 - z^{-128}}{1 - z^{-2}}, \quad |z| > 1$$

Example

$$X(z) = \frac{1 - z^{-128}}{1 - z^{-2}}, |z| > 1$$

- $X(z)$ can be rewritten according to:

$$X(z) = Y(z) - z^{-128} Y(z)$$

where:

$$Y(z) = \frac{1}{1 - z^{-2}} = \frac{1}{(1 - z^{-1})(1 + z^{-1})}$$

Example

$$X(z) = \frac{1 - z^{-128}}{1 - z^{-2}}, |z| > 1$$

- $X(z)$ can be rewritten according to:

$$X(z) = Y(z) - z^{-128} Y(z)$$

where:

$$Y(z) = \frac{1}{1 - z^{-2}} = \frac{1}{(1 - z^{-1})(1 + z^{-1})}$$

- ZT of $Y(z)$ is:

$$y[n] = \frac{1}{2} (1 + (-1)^n) u[n]$$

Example

$$X(z) = \frac{1 - z^{-128}}{1 - z^{-2}}, |z| > 1$$

- $X(z)$ can be rewritten according to:

$$X(z) = Y(z) - z^{-128} Y(z)$$

where:

$$Y(z) = \frac{1}{1 - z^{-2}} = \frac{1}{(1 - z^{-1})(1 + z^{-1})}$$

- ZT of $Y(z)$ is:

$$y[n] = \frac{1}{2} (1 + (-1)^n) u[n]$$

- Using **linearity** and **shift property**:

$$x[n] = y[n] - y[n - 128]$$

Example

$$X(z) = \frac{1 - z^{-128}}{1 - z^{-2}}, |z| > 1$$

- $X(z)$ can be rewritten according to:

$$X(z) = Y(z) - z^{-128} Y(z)$$

where:

$$Y(z) = \frac{1}{1 - z^{-2}} = \frac{1}{(1 - z^{-1})(1 + z^{-1})}$$

- ZT of $Y(z)$ is:

$$y[n] = \frac{1}{2} (1 + (-1)^n) u[n]$$

- Using **linearity** and **shift property**:

$$x[n] = y[n] - y[n - 128]$$

- Therefore:

$$x[n] = \frac{1}{2} (1 + (-1)^n) (u[n] - u[n - 128])$$

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals**
 - Definition
 - Convergence of the DTFT
 - Properties
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals**
 - Definition
 - Convergence of the DTFT
 - Properties
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in \mathbb{R}$$

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in \mathbb{R}$$

- $X(\omega)$ is continuous and 2π periodic:

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in \mathbb{R}$$

- $X(\omega)$ is **continuous** and **2π periodic**:
 - $X(\omega) = X(\omega + 2\pi)$

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in \mathbb{R}$$

- $X(\omega)$ is **continuous** and **2π periodic**:
 - $X(\omega) = X(\omega + 2\pi)$
- Nyquist frequency is defined by: $\omega_N = \pi$

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in \mathbb{R}$$

- $X(\omega)$ is **continuous** and **2π periodic**:
 - $X(\omega) = X(\omega + 2\pi)$
- Nyquist frequency is defined by: $\omega_N = \pi$
- $X(\omega)$ is called the spectrum of $x[n]$: $X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in \mathbb{R}$$

- $X(\omega)$ is **continuous** and **2π periodic**:
 - $X(\omega) = X(\omega + 2\pi)$
- Nyquist frequency is defined by: $\omega_N = \pi$
- $X(\omega)$ is called the spectrum of $x[n]$: $X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$
 - $|X(\omega)|$: magnitude spectrum

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in \mathbb{R}$$

- $X(\omega)$ is **continuous** and **2π periodic**:
 - $X(\omega) = X(\omega + 2\pi)$
- Nyquist frequency is defined by: $\omega_N = \pi$
- $X(\omega)$ is called the spectrum of $x[n]$: $X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$
 - $|X(\omega)|$: magnitude spectrum
 - $\angle X(\omega)$: phase spectrum

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in \mathbb{R}$$

- $X(\omega)$ is **continuous** and **2π periodic**:
 - $X(\omega) = X(\omega + 2\pi)$
- Nyquist frequency is defined by: $\omega_N = \pi$
- $X(\omega)$ is called the spectrum of $x[n]$: $X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$
 - $|X(\omega)|$: magnitude spectrum
 - $\angle X(\omega)$: phase spectrum
- The Fourier Transform is a specific case of the ZT taking $z = e^{j\omega}$ with $|z| = 1 \in \text{ROC}$.

Fourier Transform of a sampled continuous signal I

- Let $s_e(t)$ the sampled expression of the continuous signal $s(t)$ with sampling period T_s :

$$s_e(t) = s(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$$

- If we denote $s[k] = s(kT_s)$, we have:

$$s_e(t) = \sum_{k=-\infty}^{\infty} s[k] \delta(t - kT_s)$$

- The Fourier transform gives:

$$TF\{s_e(t)\} = \hat{s}_e(\omega) = s[k] \sum_{k=-\infty}^{\infty} s[k] TF\{\delta(t - kT_s)\}$$

Fourier Transform of a sampled continuous signal II

- Applying the delay theorem:

$$\hat{s}_e(\omega) = s[k] \sum_{k=-\infty}^{\infty} s[k] e^{-jk\omega T_s} TF\{\delta(t)\} = s[k] \sum_{k=-\infty}^{\infty} s[k] e^{-jk\omega T_s}$$

- Finally:

$$\hat{s}_e(\omega) = S(\omega T_s)$$

with the Nyquist frequency equal to:

$$\omega_N = \frac{\pi}{T_s}$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega, \quad n \in \mathbb{Z}$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega, \quad n \in \mathbb{Z}$$

Proof: Note that $\int_{-\pi}^{\pi} e^{j\omega n} d\omega = 2\pi \delta[n]$:

$$\begin{aligned} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega &= \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \right) e^{j\omega n} d\omega \\ &= \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega \\ &= 2\pi \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \\ &= 2\pi x[n] \end{aligned}$$

Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals**
 - Definition
 - **Convergence of the DTFT**
 - Properties
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT

Convergence of the DTFT

- For the DTFT to exist, the series $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ must converge
- That is, the partial sum

$$X_M(\omega) = \sum_{n=-M}^M x[n]e^{-j\omega n}$$

must converge to a limit $X(\omega)$ as $M \rightarrow \infty$

- **Absolutely summable signals**

- $X_M(\omega)$ converges uniformly to $X(\omega)$
- $X(\omega)$ is continuous

- **Energy signals**

- $X_M(\omega)$ does not necessarily converge
- $X(\omega)$ may be discontinuous at certain points

- **Power signals**

- Most power signals do not have a DTFT
- Exceptions including: Periodic signals, Unit step

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals**
 - Definition
 - Convergence of the DTFT
 - Properties
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT

- Linearity

$$ax[n] + by[n] \xleftrightarrow{\mathcal{F}} aX(\omega) + bY(\omega)$$

Properties

- Linearity

$$ax[n] + by[n] \xleftrightarrow{\mathcal{F}} aX(\omega) + bY(\omega)$$

- Time shift

$$x[n - d] \xleftrightarrow{\mathcal{F}} e^{-j\omega d} X(\omega)$$

Properties

- Linearity

$$ax[n] + by[n] \xleftrightarrow{\mathcal{F}} aX(\omega) + bY(\omega)$$

- Time shift

$$x[n - d] \xleftrightarrow{\mathcal{F}} e^{-j\omega d} X(\omega)$$

- Frequency modulation

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$$

Properties

- Linearity

$$ax[n] + by[n] \xleftrightarrow{\mathcal{F}} aX(\omega) + bY(\omega)$$

- Time shift

$$x[n - d] \xleftrightarrow{\mathcal{F}} e^{-j\omega d} X(\omega)$$

- Frequency modulation

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$$

- Differentiation

$$nx[n] \xleftrightarrow{\mathcal{F}} j \frac{dX(\omega)}{d\omega}$$

Even and odd component definition

DT signal

$$x[n] = x_e[n] + x_o[n]$$

$$x_e[n] \triangleq \frac{1}{2} (x[n] + x^*[-n]) = x_e^*[-n]$$

$$x_o[n] \triangleq \frac{1}{2} (x[n] - x^*[-n]) = -x_o^*[-n]$$

Even and odd component definition

DT signal

$$x[n] = x_e[n] + x_o[n]$$

$$x_e[n] \triangleq \frac{1}{2} (x[n] + x^*[-n]) = x_e^*[-n]$$

$$x_o[n] \triangleq \frac{1}{2} (x[n] - x^*[-n]) = -x_o^*[-n]$$

DTFT

$$X(\omega) = X_e(\omega) + X_o(\omega)$$

$$X_e(\omega) \triangleq \frac{1}{2} (X(\omega) + X^*(\omega)) = X_e^*(-\omega)$$

$$X_o(\omega) \triangleq \frac{1}{2} (X(\omega) - X^*(\omega)) = -X_o^*(-\omega)$$

Basic symmetries

Real and imaginary parts decomposition

$$\begin{aligned}x[n] &= x_R[n] + jx_I[n] \\X(\omega) &= X_R(\omega) + jX_I(\omega)\end{aligned}$$

Basic symmetries

Real and imaginary parts decomposition

$$\begin{aligned}x[n] &= x_R[n] + jx_I[n] \\ X(\omega) &= X_R(\omega) + jX_I(\omega)\end{aligned}$$

■ $x[-n] \xleftrightarrow{\mathcal{F}} X(-\omega)$

Basic symmetries

Real and imaginary parts decomposition

$$\begin{aligned}x[n] &= x_R[n] + jx_I[n] \\X(\omega) &= X_R(\omega) + jX_I(\omega)\end{aligned}$$

- $x[-n] \xleftrightarrow{\mathcal{F}} X(-\omega)$
- $x^*[n] \xleftrightarrow{\mathcal{F}} X^*(-\omega)$

Basic symmetries

Real and imaginary parts decomposition

$$\begin{aligned}x[n] &= x_R[n] + jx_I[n] \\X(\omega) &= X_R(\omega) + jX_I(\omega)\end{aligned}$$

- $x[-n] \xleftrightarrow{\mathcal{F}} X(-\omega)$
- $x^*[n] \xleftrightarrow{\mathcal{F}} X^*(-\omega)$
- $x_R[n] \xleftrightarrow{\mathcal{F}} X_e(\omega)$

Basic symmetries

Real and imaginary parts decomposition

$$\begin{aligned}x[n] &= x_R[n] + jx_I[n] \\X(\omega) &= X_R(\omega) + jX_I(\omega)\end{aligned}$$

- $x[-n] \xleftrightarrow{\mathcal{F}} X(-\omega)$
- $x^*[n] \xleftrightarrow{\mathcal{F}} X^*(-\omega)$
- $x_R[n] \xleftrightarrow{\mathcal{F}} X_e(\omega)$
- $jx_I[n] \xleftrightarrow{\mathcal{F}} X_o(\omega)$

Basic symmetries

Real and imaginary parts decomposition

$$\begin{aligned}x[n] &= x_R[n] + jx_I[n] \\X(\omega) &= X_R(\omega) + jX_I(\omega)\end{aligned}$$

- $x[-n] \xleftrightarrow{\mathcal{F}} X(-\omega)$
- $x^*[n] \xleftrightarrow{\mathcal{F}} X^*(-\omega)$
- $x_R[n] \xleftrightarrow{\mathcal{F}} X_e(\omega)$
- $jx_I[n] \xleftrightarrow{\mathcal{F}} X_o(\omega)$
- $x_e[n] \xleftrightarrow{\mathcal{F}} X_R(\omega)$

Basic symmetries

Real and imaginary parts decomposition

$$\begin{aligned}x[n] &= x_R[n] + jx_I[n] \\X(\omega) &= X_R(\omega) + jX_I(\omega)\end{aligned}$$

- $x[-n] \xleftrightarrow{\mathcal{F}} X(-\omega)$
- $x^*[n] \xleftrightarrow{\mathcal{F}} X^*(-\omega)$
- $x_R[n] \xleftrightarrow{\mathcal{F}} X_e(\omega)$
- $jx_I[n] \xleftrightarrow{\mathcal{F}} X_o(\omega)$
- $x_e[n] \xleftrightarrow{\mathcal{F}} X_R(\omega)$
- $x_o[n] \xleftrightarrow{\mathcal{F}} jX_I(\omega)$

More properties

Convolution

$$x[n]*y[n] \xleftrightarrow{\mathcal{F}} X(\omega)Y(\omega)$$

More properties

Convolution

$$x[n]*y[n] \xleftrightarrow{\mathcal{F}} X(\omega)Y(\omega)$$

Multiplication

$$x[n]y[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\phi)Y(\omega - \phi)d\phi$$

More properties

Convolution

$$x[n]*y[n] \xleftrightarrow{\mathcal{F}} X(\omega)Y(\omega)$$

Multiplication

$$x[n]y[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\phi)Y(\omega - \phi)d\phi$$

Parseval's relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

More properties

Convolution

$$x[n]*y[n] \xleftrightarrow{\mathcal{F}} X(\omega)Y(\omega)$$

Multiplication

$$x[n]y[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\phi)Y(\omega - \phi)d\phi$$

Parseval's relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Plancherel's relation

$$\sum_{n=-\infty}^{\infty} x[n]y[n]^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)Y^*(\omega)d\omega$$

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain**
 - LTI systems described by LCCDE
 - One-sided Z-Transform
 - The system function
 - Response of rational system Functions
 - Schur-Cohn Stability test
 - Frequency response of rational systems

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain**
 - LTI systems described by LCCDE
 - One-sided Z-Transform
 - The system function
 - Response of rational system Functions
 - Schur-Cohn Stability test
 - Frequency response of rational systems

Linear Constant Coefficient Difference Equations

Definition

A DT system can be described by an LCCDE of order N if:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

where $a_0 \neq 0$ and $a_N \neq 0$.

Linear Constant Coefficient Difference Equations

Definition

A DT system can be described by an LCCDE of order N if:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

where $a_0 \neq 0$ and $a_N \neq 0$.

- If we further assume initial **rest conditions**, i.e.:

$$\forall n < n_0, x[n] = 0 \Rightarrow \forall n < n_0, y[n] = 0$$

LCCDE corresponds to unique causal LTI system.

Example: Accumulator system

$$x[n] \rightarrow y[n] = \sum_{k=-\infty}^n x[k]$$

Example: Accumulator system

$$x[n] \rightarrow y[n] = \sum_{k=-\infty}^n x[k]$$

This LTI system can be rewritten according to:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n-1} x[k] + x[n] \\ &= y[n-1] + x[n] \end{aligned}$$

Example: Accumulator system

$$x[n] \rightarrow y[n] = \sum_{k=-\infty}^n x[k]$$

This LTI system can be rewritten according to:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n-1} x[k] + x[n] \\ &= y[n-1] + x[n] \end{aligned}$$

\Rightarrow LCCDE of order $N = 1$ ($M = 0$, $a_0 = 1$, $a_1 = -1$, $b_0 = 1$)

Example: Accumulator system

$$x[n] \rightarrow y[n] = \sum_{k=-\infty}^n x[k]$$

This LTI system can be rewritten according to:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n-1} x[k] + x[n] \\ &= y[n-1] + x[n] \end{aligned}$$

\Rightarrow LCCDE of order $N = 1$ ($M = 0$, $a_0 = 1$, $a_1 = -1$, $b_0 = 1$)

LCCDEs lead to efficient recursive implementation:

- Recursive because computation of $y[n]$ make use past output signal values ($y[n-1]$)
- Efficient: in the case of the accumulator it requires only 1 adder and 1 memory unit instead of an infinite number of adders and memory units.

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain**
 - LTI systems described by LCCDE
 - **One-sided Z-Transform**
 - The system function
 - Response of rational system Functions
 - Schur-Cohn Stability test
 - Frequency response of rational systems

One-sided Z-Transform

- The two-sided ZT requires that the corresponding signals be specified for entire time range $-\infty < n < \infty$
 - Prevent evaluation of the output of non-relaxed systems
- The one-sided ZT can be used to solve difference equations with initial conditions

One-sided Z-Transform

- The two-sided ZT requires that the corresponding signals be specified for entire time range $-\infty < n < \infty$
 - Prevent evaluation of the output of non-relaxed systems
- The one-sided ZT can be used to solve difference equations with initial conditions

Definition

$$X^+(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

One-sided Z-Transform

- The two-sided ZT requires that the corresponding signals be specified for entire time range $-\infty < n < \infty$
 - Prevent evaluation of the output of non-relaxed systems
- The one-sided ZT can be used to solve difference equations with initial conditions

Definition

$$X^+(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Characteristics

1. Does not contain information about the signal $x[n]$ for negative values of time ($n < 0$)
2. It is **unique** only for **causal** signals
3. one-sided ZT of $x[n]$ is identical to the two-sided ZT of $x[n]u[n]$

Properties

Almost all properties for the two-sided ZT carry over to the one-sided ZT with exception of the **shifting property**.

Properties

Almost all properties for the two-sided ZT carry over to the one-sided ZT with exception of the **shifting property**.

Case 1: Time Delay

If

$$x[n] \xleftrightarrow{z^+} X^+(z)$$

then

$$\forall k > 0, x[n - k] \xleftrightarrow{z^+} z^{-k} \left(X^+(z) + \sum_{n=1}^k x[-n]z^n \right)$$

Properties

Almost all properties for the two-sided ZT carry over to the one-sided ZT with exception of the **shifting property**.

Case 1: Time Delay

If

$$x[n] \xleftrightarrow{z^+} X^+(z)$$

then

$$\forall k > 0, x[n - k] \xleftrightarrow{z^+} z^{-k} \left(X^+(z) + \sum_{n=1}^k x[-n]z^n \right)$$

Case 2: Time advance

If

$$x[n] \xleftrightarrow{z^+} X^+(z)$$

then

$$\forall k > 0, x[n + k] \xleftrightarrow{z^+} z^k \left(X^+(z) - \sum_{n=0}^{k-1} x[n]z^{-n} \right)$$

Example

Determine the step response of the system with IC $y[-1] = 1$:

$$y[n] = \alpha y[n-1] + x[n], \text{ with } -1 < \alpha < 1$$

Example

Determine the step response of the system with IC $y[-1] = 1$:

$$y[n] = \alpha y[n-1] + x[n], \text{ with } -1 < \alpha < 1$$

1. By taking the one-sided ZT of both sides:

$$Y^+(z) = \alpha z^{-1} (Y^+(z) + y[-1]z) + X^+(z).$$

Example

Determine the step response of the system with IC $y[-1] = 1$:

$$y[n] = \alpha y[n-1] + x[n], \text{ with } -1 < \alpha < 1$$

1. By taking the one-sided ZT of both sides:

$$Y^+(z) = \alpha z^{-1} (Y^+(z) + y[-1]z) + X^+(z).$$

2. Compute one-sided ZT of $x[n]$:

$$x[n] = u[n] \xleftrightarrow{z^+} X^+(z) = \frac{1}{1 - z^{-1}}$$

Example

Determine the step response of the system with IC $y[-1] = 1$:

$$y[n] = \alpha y[n-1] + x[n], \text{ with } -1 < \alpha < 1$$

1. By taking the one-sided ZT of both sides:

$$Y^+(z) = \alpha z^{-1} (Y^+(z) + y[-1]z) + X^+(z).$$

2. Compute one-sided ZT of $x[n]$:

$$x[n] = u[n] \xleftrightarrow{z^+} X^+(z) = \frac{1}{1 - z^{-1}}$$

3. Solving for $Y^+[n]$:

$$Y^+(z) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{1}{(1 - \alpha z^{-1})(1 - z^{-1})}$$

Example

Determine the step response of the system with IC $y[-1] = 1$:

$$y[n] = \alpha y[n-1] + x[n], \text{ with } -1 < \alpha < 1$$

1. By taking the one-sided ZT of both sides:

$$Y^+(z) = \alpha z^{-1} (Y^+(z) + y[-1]z) + X^+(z).$$

2. Compute one-sided ZT of $x[n]$:

$$x[n] = u[n] \xleftrightarrow{z^+} X^+(z) = \frac{1}{1 - z^{-1}}$$

3. Solving for $Y^+[n]$:

$$Y^+(z) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{1}{(1 - \alpha z^{-1})(1 - z^{-1})}$$

4. Perform partial-fraction expansion:

$$Y^+(x) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\frac{\alpha}{\alpha-1}}{1 - \alpha z^{-1}} + \frac{\frac{1}{1-\alpha}}{1 - z^{-1}}$$

Example

Determine the step response of the system with IC $y[-1] = 1$:

$$y[n] = \alpha y[n-1] + x[n], \text{ with } -1 < \alpha < 1$$

1. By taking the one-sided ZT of both sides:

$$Y^+(z) = \alpha z^{-1} (Y^+(z) + y[-1]z) + X^+(z).$$

2. Compute one-sided ZT of $x[n]$:

$$x[n] = u[n] \xleftrightarrow{z^+} X^+(z) = \frac{1}{1 - z^{-1}}$$

3. Solving for $Y^+[n]$:

$$Y^+(z) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{1}{(1 - \alpha z^{-1})(1 - z^{-1})}$$

4. Perform partial-fraction expansion:

$$Y^+(x) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\frac{\alpha}{\alpha-1}}{1 - \alpha z^{-1}} + \frac{\frac{1}{1-\alpha}}{1 - z^{-1}}$$

5. Compute inverse ZT:

$$\begin{aligned} y[n] &= \alpha^{n+1} u[n] + \frac{1 - \alpha^{n+1}}{1 - \alpha} u[n] \\ &= \frac{1}{1 - \alpha} (1 - \alpha^{n+2}) u[n] \end{aligned}$$

Final Value Theorem

If

$$x[n] \xleftrightarrow{z^+} X^+(z)$$

then

$$\lim_{n \rightarrow \infty} x[n] = \lim_{z \rightarrow 1} (z - 1)X^+(z)$$

Final Value Theorem

If

$$x[n] \xleftrightarrow{z^+} X^+(z)$$

then

$$\lim_{n \rightarrow \infty} x[n] = \lim_{z \rightarrow 1} (z - 1)X^+(z)$$

- The limit exists if the ROC of $(z - 1)X^+(z)$ includes the unit circle
- Useful when the asymptotic behavior of a signal $x[n]$ is desired knowing its ZT

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain**
 - LTI systems described by LCCDE
 - One-sided Z-Transform
 - **The system function**
 - Response of rational system Functions
 - Schur-Cohn Stability test
 - Frequency response of rational systems

The system function

LTI system \mathcal{H} (recall)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \text{ with } h[n] = \mathcal{H}\{\delta[n]\}$$

The system function

LTI system \mathcal{H} (recall)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \text{ with } h[n] = \mathcal{H}\{\delta[n]\}$$

Definition

The system function of \mathcal{H} , denoted $H(z)$ is the ZT of $h[n]$:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}, z \in \mathcal{R}_H$$

where \mathcal{R}_H denotes the corresponding ROC.

The system function

LTI system \mathcal{H} (recall)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \text{ with } h[n] = \mathcal{H}\{\delta[n]\}$$

Definition

The system function of \mathcal{H} , denoted $H(z)$ is the ZT of $h[n]$:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}, \quad z \in \mathcal{R}_H$$

where \mathcal{R}_H denotes the corresponding ROC.

- If $H(z)$ and \mathcal{R}_H are known, $h[n]$ can be recovered via inverse ZT
- if $z = e^{j\omega} \in \mathcal{R}_H$ (the ROC contains the unit circle) then

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \equiv H(\omega)$$

Properties

Let \mathcal{H} be LTI system with system function $H(z)$ and ROC \mathcal{R}_H .

- If $y[n]$ denotes the response of \mathcal{H} to arbitrary input $x[n]$, then:

Properties

Let \mathcal{H} be LTI system with system function $H(z)$ and ROC \mathcal{R}_H .

- If $y[n]$ denotes the response of \mathcal{H} to arbitrary input $x[n]$, then:

$$Y(z) = H(z)X(z)$$

Properties

Let \mathcal{H} be LTI system with system function $H(z)$ and ROC \mathcal{R}_H .

- If $y[n]$ denotes the response of \mathcal{H} to arbitrary input $x[n]$, then:

$$Y(z) = H(z)X(z)$$

- LTI system \mathcal{H} is causal iff \mathcal{R}_H is the exterior of a circle (including ∞)

Properties

Let \mathcal{H} be LTI system with system function $H(z)$ and ROC \mathcal{R}_H .

- If $y[n]$ denotes the response of \mathcal{H} to arbitrary input $x[n]$, then:

$$Y(z) = H(z)X(z)$$

- LTI system \mathcal{H} is causal iff \mathcal{R}_H is the exterior of a circle (including ∞)
- LTI system \mathcal{H} is stable iff \mathcal{R}_H contains the unit circle:

$$\begin{aligned}\mathcal{H} \text{ stable} &\Leftrightarrow \sum_n |h[n]| < \infty \\ &\Leftrightarrow e^{j\omega} \in \mathcal{R}_H\end{aligned}$$

$$|H(z)| \leq \sum_n |h[n]z^{-n}|$$

evaluated on the unit circle: $z = e^{j\omega}$:

$$|H(z)| \leq \sum_n |h[n]| < \infty$$

LCCDE system function

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

LCCDE system function

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Taking ZT on both sides:

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

LCCDE system function

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Taking ZT on both sides:

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

Leading to a rational system function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

Rational system with real coefficients

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

- In many applications, coefficients a_k and b_k are real implying:

$$H^*(z) = H(z^*)$$

- Thus, if z_k is a zero of $H(z)$ then:

$$H(z_k^*) = (H(z_k))^* = 0^* = 0$$

which shows that z_k^* is also a zero of $H(z)$

Rational system with real coefficients

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

- In many applications, coefficients a_k and b_k are real implying:

$$H^*(z) = H(z^*)$$

- Thus, if z_k is a zero of $H(z)$ then:

$$H(z_k^*) = (H(z_k))^* = 0^* = 0$$

which shows that z_k^* is also a zero of $H(z)$

- More generally, it can be shown that complex poles (or zeros) occur in complex conjugate pairs:
 - if p_k is a pole of order l of $H(z)$, so is p_k^*
 - if z_k is a zero of order l of $H(z)$, so is z_k^*

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain**
 - LTI systems described by LCCDE
 - One-sided Z-Transform
 - The system function
 - Response of rational system Functions**
 - Schur-Cohn Stability test
 - Frequency response of rational systems

Response of rational system functions

- Let be $H(z) = \frac{B(z)}{A(z)}$ the system function of a LCCDE system:
 - roots of $A(z)$ are the poles of $H(z)$
 - roots of $B(z)$ are the zeros of $H(z)$

Response of rational system functions

- Let be $H(z) = \frac{B(z)}{A(z)}$ the system function of a LCCDE system:
 - roots of $A(z)$ are the poles of $H(z)$
 - roots of $B(z)$ are the zeros of $H(z)$
- Let assume that the input signal $x[n]$ has a rational ZT of the form
$$X(z) = \frac{N(z)}{Q(z)}$$

Response of rational system functions

- Let be $H(z) = \frac{B(z)}{A(z)}$ the system function of a LCCDE system:
 - roots of $A(z)$ are the poles of $H(z)$
 - roots of $B(z)$ are the zeros of $H(z)$
- Let assume that the input signal $x[n]$ has a rational ZT of the form $X(z) = \frac{N(z)}{Q(z)}$
- If the system is initially relaxed the initial conditions ($y[-1] = y[-2] = \dots = y[-N] = 0$) the output of the system has the form:

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

Response of rational system functions

- Let be $H(z) = \frac{B(z)}{A(z)}$ the system function of a LCCDE system:
 - roots of $A(z)$ are the poles of $H(z)$
 - roots of $B(z)$ are the zeros of $H(z)$
- Let assume that the input signal $x[n]$ has a rational ZT of the form $X(z) = \frac{N(z)}{Q(z)}$
- If the system is initially relaxed the initial conditions ($y[-1] = y[-2] = \dots = y[-N] = 0$) the output of the system has the form:

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

- The partial fraction expansion of $Y(z)$ yields

$$Y(z) = \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^L \frac{Q_k}{1 - q_k z^{-1}}$$

if $\forall k \forall m, p_k \neq q_m$ and there is no pole-zero cancellation.

Response of rational system functions

$$Y(z) = \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^L \frac{Q_k}{1 - q_k z^{-1}}$$

Response of rational system functions

$$Y(z) = \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^L \frac{Q_k}{1 - q_k z^{-1}}$$

- The inverse transform of $Y(z)$ can be written

$$y[n] = \underbrace{\sum_{k=1}^N A_k (p_k)^n u[n]}_{\text{natural response}} + \underbrace{\sum_{k=1}^L Q_k (q_k)^n u[n]}_{\text{forced response}}$$

Response of rational system functions

$$Y(z) = \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^L \frac{Q_k}{1 - q_k z^{-1}}$$

- The inverse transform of $Y(z)$ can be written

$$y[n] = \underbrace{\sum_{k=1}^N A_k (p_k)^n u[n]}_{\text{natural response}} + \underbrace{\sum_{k=1}^L Q_k (q_k)^n u[n]}_{\text{forced response}}$$

- Influence of the input signal on the **natural response** is through the scale factor $\{A_k\}$
- Influence of the system on the **forced response** is through the scale factor $\{Q_k\}$

Response with nonzero initial conditions

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

- $x[n]$ is assumed to be causal
- effects of all previous input signals are reflected in the initial conditions $y[-1], y[-2], \dots, y[-N]$
- To determine $y[n], \forall n \geq 0$, the one-sided ZT can be used:

Response with nonzero initial conditions

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

- $x[n]$ is assumed to be causal
- effects of all previous input signals are reflected in the initial conditions $y[-1], y[-2], \dots, y[-N]$
- To determine $y[n], \forall n \geq 0$, the one-sided ZT can be used:

$$Y^+(z) = - \sum_{k=1}^N a_k z^{-k} \left(Y^+(z) + \sum_{n=1}^k y[-n] z^n \right) + \sum_{k=0}^M b_k z^{-k} X^+(z)$$

Response with nonzero initial conditions

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

- $x[n]$ is assumed to be causal
- effects of all previous input signals are reflected in the initial conditions $y[-1], y[-2], \dots, y[-N]$
- To determine $y[n], \forall n \geq 0$, the one-sided ZT can be used:

$$Y^+(z) = - \sum_{k=1}^N a_k z^{-k} \left(Y^+(z) + \sum_{n=1}^k y[-n] z^n \right) + \sum_{k=0}^M b_k z^{-k} X^+(z)$$

$$Y^+(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} X^+(z) - \frac{\sum_{k=1}^N a_k z^{-k} \sum_{n=1}^k y[-n] z^n}{1 + \sum_{k=1}^N a_k z^{-k}}$$

Response with nonzero initial conditions

Since $x[n]$ is causal $X^+(z) = X(z)$ and the expression can be written:

$$Y^+(z) = \underbrace{H(z)X(z)}_{Y_{zs}(z)} + \underbrace{\frac{N_0(z)}{A(z)}}_{Y_{zi}^+}$$

Response with nonzero initial conditions

Since $x[n]$ is causal $X^+(z) = X(z)$ and the expression can be written:

$$Y^+(z) = \underbrace{H(z)X(z)}_{Y_{zs}(z)} + \underbrace{\frac{N_0(z)}{A(z)}}_{Y_{zi}^+}$$

- $Y_{zs}(z) = H(z)X(z)$ is the **zero-state** response of the system
- $Y_{zi}^+(z) = \frac{N_0(z)}{A(z)}$ is the **zero-input** response of the system

Response with nonzero initial conditions

Since $x[n]$ is causal $X^+(z) = X(z)$ and the expression can be written:

$$Y^+(z) = \underbrace{H(z)X(z)}_{Y_{zs}(z)} + \underbrace{\frac{N_0(z)}{A(z)}}_{Y_{zi}^+}$$

- $Y_{zs}(z) = H(z)X(z)$ is the **zero-state** response of the system
- $Y_{zi}^+(z) = \frac{N_0(z)}{A(z)}$ is the **zero-input** response of the system
- The zero-state response of the system remains the same and gives:

$$y_{zs}[n] = \sum_{k=1}^N A_k(p_k)^n u[n] + \sum_{k=1}^L Q_k(q_k)^n u[n]$$

Response with nonzero initial conditions

Since $x[n]$ is causal $X^+(z) = X(z)$ and the expression can be written:

$$Y^+(z) = \underbrace{H(z)X(z)}_{Y_{zs}(z)} + \underbrace{\frac{N_0(z)}{A(z)}}_{Y_{zi}^+}$$

- $Y_{zs}(z) = H(z)X(z)$ is the **zero-state** response of the system
- $Y_{zi}^+(z) = \frac{N_0(z)}{A(z)}$ is the **zero-input** response of the system
- The zero-state response of the system remains the same and gives:

$$y_{zs}[n] = \sum_{k=1}^N A_k(p_k)^n u[n] + \sum_{k=1}^L Q_k(q_k)^n u[n]$$

- Since $Y_{zi}^+(z) = \frac{N_0(z)}{A(z)}$ its poles are p_1, \dots, p_N and the zero-input response has the form:

$$y_{zi}[n] = \sum_{k=1}^N D_k(p_k)^n u[n]$$

Response with nonzero initial conditions

- The terms involving the poles $\{p_k\}$ can be combined:

$$y[n] = y_{zs}[n] + y_{zi}[n] = \sum_{k=1}^N A'_k (p_k)^n u[n] + \sum_{k=1}^L Q_k (q_k)^n u[n]$$

with $A'_k = A_k + D_k$.

Response with nonzero initial conditions

- The terms involving the poles $\{p_k\}$ can be combined:

$$y[n] = y_{zs}[n] + y_{zi}[n] = \sum_{k=1}^N A'_k (p_k)^n u[n] + \sum_{k=1}^L Q_k (q_k)^n u[n]$$

with $A'_k = A_k + D_k$.

- Effect of initial conditions alter the natural response of the system through modification of the scale factors

Response with nonzero initial conditions

- The terms involving the poles $\{p_k\}$ can be combined:

$$y[n] = y_{zs}[n] + y_{zi}[n] = \sum_{k=1}^N A'_k (p_k)^n u[n] + \sum_{k=1}^L Q_k (q_k)^n u[n]$$

with $A'_k = A_k + D_k$.

- Effect of initial conditions alter the natural response of the system through modification of the scale factors
- No new poles are introduced by the nonzero initial conditions

Response with nonzero initial conditions

- The terms involving the poles $\{p_k\}$ can be combined:

$$y[n] = y_{zs}[n] + y_{zi}[n] = \sum_{k=1}^N A'_k (p_k)^n u[n] + \sum_{k=1}^L Q_k (q_k)^n u[n]$$

with $A'_k = A_k + D_k$.

- Effect of initial conditions alter the natural response of the system through modification of the scale factors
- No new poles are introduced by the nonzero initial conditions
- There is no effect on the forced response of the system

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain**
 - LTI systems described by LCCDE
 - One-sided Z-Transform
 - The system function
 - Response of rational system Functions
 - Schur-Cohn Stability test**
 - Frequency response of rational systems

Schur-Cohn stability test

Reminder

- LTI system \mathcal{H} is causal iff \mathcal{R}_H is the exterior of a circle (including ∞)
- LTI system \mathcal{H} is stable iff \mathcal{R}_H contains the unit circle

Schur-Cohn stability test

Reminder

- LTI system \mathcal{H} is causal iff \mathcal{R}_H is the exterior of a circle (including ∞)
- LTI system \mathcal{H} is stable iff \mathcal{R}_H contains the unit circle

\Rightarrow A causal system described by its rational system function is stable iff its poles are strictly inside the unit circle.

Schur-Cohn stability test

Reminder

- LTI system \mathcal{H} is causal iff \mathcal{R}_H is the exterior of a circle (including ∞)
- LTI system \mathcal{H} is stable iff \mathcal{R}_H contains the unit circle

\Rightarrow A causal system described by its rational system function is stable iff its poles are strictly inside the unit circle.

Let be $A(z) = 1 + a_1z^{-1} + a_2z^{-2} + \dots + a_Nz^{-N}$ the denominator polynomial of $H(z)$. A polynomial of degree m is denoted by:

$$A_m(z) = \sum_{k=0}^m a_m[k]z^{-k} \quad a_m(0) = 1$$

Schur-Cohn stability test

Reminder

- LTI system \mathcal{H} is causal iff \mathcal{R}_H is the exterior of a circle (including ∞)
- LTI system \mathcal{H} is stable iff \mathcal{R}_H contains the unit circle

\Rightarrow A causal system described by its rational system function is stable iff its poles are strictly inside the unit circle.

Let be $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}$ the denominator polynomial of $H(z)$. A polynomial of degree m is denoted by:

$$A_m(z) = \sum_{k=0}^m a_m[k] z^{-k} \quad a_m(0) = 1$$

The reverse polynomial $B_m(z)$ of degree m is defined as:

$$B_m(z) = z^{-m} A_m(z^{-1}) = \sum_{k=0}^m a_m[m-k] z^{-k}$$

Schur-Cohn stability test

The set of coefficients K_1, K_2, \dots, K_N must satisfy the condition $|K_m| < 1$

Schur-Cohn stability test

The set of coefficients K_1, K_2, \dots, K_N must satisfy the condition $|K_m| < 1$

1. First iteration:

$$A_N(z) = A(z)$$

$$\text{and } K_N = a_N(N)$$

Schur-Cohn stability test

The set of coefficients K_1, K_2, \dots, K_N must satisfy the condition $|K_m| < 1$

1. First iteration:

$$A_N(z) = A(z)$$

$$\text{and } K_N = a_N(N)$$

2. Compute the lower-degree polynomial:

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

$$\text{and } K_m = a_m(m)$$

Schur-Cohn stability test

The set of coefficients K_1, K_2, \dots, K_N must satisfy the condition $|K_m| < 1$

1. First iteration:

$$A_N(z) = A(z)$$

and $K_N = a_N(N)$

2. Compute the lower-degree polynomial:

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

and $K_m = a_m(m)$

3. Loop to step 2 until it fails or $m = 1$

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain**
 - LTI systems described by LCCDE
 - One-sided Z-Transform
 - The system function
 - Response of rational system Functions
 - Schur-Cohn Stability test
 - **Frequency response of rational systems**

Frequency response of rational systems

- Knowing poles and zeros of $H(z)$ can be expressed as:

$$H(z) = Gz^{-K} \frac{\prod_{k=0}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)}$$

Frequency response of rational systems

- Knowing poles and zeros of $H(z)$ can be expressed as:

$$H(z) = Gz^{-K} \frac{\prod_{k=0}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)}$$

- Frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = Ge^{-j\omega K} \frac{\prod_{k=0}^M (j\omega - z_k)}{\prod_{k=1}^N (j\omega - p_k)}$$

Frequency response of rational systems

- Knowing poles and zeros of $H(z)$ can be expressed as:

$$H(z) = Gz^{-K} \frac{\prod_{k=0}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)}$$

- Frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = Ge^{-j\omega K} \frac{\prod_{k=0}^M (j\omega - z_k)}{\prod_{k=1}^N (j\omega - p_k)}$$

- Define:

$$V_k(\omega) = |e^{j\omega} - z_k| \quad U_k(\omega) = |e^{j\omega} - p_k|$$
$$\theta_k(\omega) = \angle(e^{j\omega} - z_k) \quad \phi_k(\omega) = \angle(e^{j\omega} - p_k)$$

Frequency response of rational systems

- Knowing poles and zeros of $H(z)$ can be expressed as:

$$H(z) = Gz^{-K} \frac{\prod_{k=0}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)}$$

- Frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = Ge^{-j\omega K} \frac{\prod_{k=0}^M (j\omega - z_k)}{\prod_{k=1}^N (j\omega - p_k)}$$

- Define:

$$V_k(\omega) = |e^{j\omega} - z_k| \quad U_k(\omega) = |e^{j\omega} - p_k|$$
$$\theta_k(\omega) = \angle(e^{j\omega} - z_k) \quad \phi_k(\omega) = \angle(e^{j\omega} - p_k)$$

- Magnitude response:

$$|H(\omega)| = |G| \frac{V_1(\omega) \cdots V_M(\omega)}{U_1(\omega) \cdots U_N(\omega)}$$

Frequency response of rational systems

- Knowing poles and zeros of $H(z)$ can be expressed as:

$$H(z) = Gz^{-K} \frac{\prod_{k=0}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)}$$

- Frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = Ge^{-j\omega K} \frac{\prod_{k=0}^M (j\omega - z_k)}{\prod_{k=1}^N (j\omega - p_k)}$$

- Define:

$$V_k(\omega) = |e^{j\omega} - z_k| \quad U_k(\omega) = |e^{j\omega} - p_k|$$
$$\theta_k(\omega) = \angle(e^{j\omega} - z_k) \quad \phi_k(\omega) = \angle(e^{j\omega} - p_k)$$

- Magnitude response:

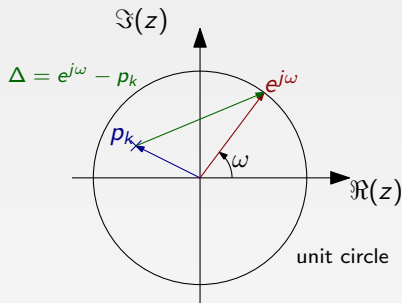
$$|H(\omega)| = |G| \frac{V_1(\omega) \cdots V_M(\omega)}{U_1(\omega) \cdots U_N(\omega)}$$

- Phase response:

$$\angle H(\omega) = \angle G - \omega K + \sum_{k=1}^{\infty} \theta_k(\omega) - \sum_{k=1}^N \phi_k(\omega)$$

Geometrical interpretation

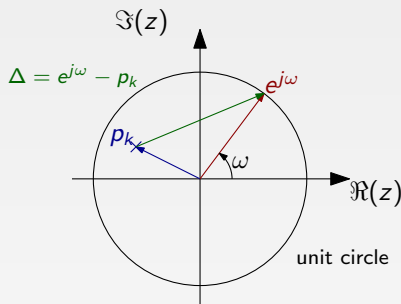
- Consider pole p_k :



- $\Delta = e^{j\omega} - p_k$: vector joining p_k to point $e^{j\omega}$ on unit circle
- $U_k(\omega) = |\Delta|$: length of vector Δ
- $\phi_k(\omega) = \angle \Delta$: angle between Δ and real axis

Geometrical interpretation

- Consider pole p_k :



- $\Delta = e^{j\omega} - p_k$: vector joining p_k to point $e^{j\omega}$ on unit circle
- $U_k(\omega) = |\Delta|$: length of vector Δ
- $\phi_k(\omega) = \angle \Delta$: angle between Δ and real axis
- A similar interpretation holds for the terms $V_k(\omega)$ and $\theta_k(\omega)$ associated to the zeros z_k

Some basic principles

- For stable and causal systems, the poles are located inside the unit circle; zeros can be anywhere.

Some basic principles

- For stable and causal systems, the poles are located inside the unit circle; zeros can be anywhere.
- Poles near the unit circle at $p = re^{j\omega_0}$ ($r < 1$) give rise to:
 - peak in $|H(\omega)|$ near ω_0
 - rapid phase variation near ω_0

Some basic principles

- For stable and causal systems, the poles are located inside the unit circle; zeros can be anywhere.
- Poles near the unit circle at $p = re^{j\omega_0}$ ($r < 1$) give rise to:
 - peak in $|H(\omega)|$ near ω_0
 - rapid phase variation near ω_0
- Zeros near the unit circle at $z = re^{j\omega_0}$ give rise to:
 - deep notch in $|H(\omega)|$ near ω_0
 - rapid phase variation near ω_0

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain**
 - LTI systems described by LCCDE
 - One-sided Z-Transform
 - The system function
 - Response of rational system Functions
 - Schur-Cohn Stability test
 - Frequency response of rational systems

First order LTI systems

The system function is given by:

$$H(z) = G \frac{1 - bz^{-1}}{1 - az^{-1}}$$

- Poles and zeros are:
 - pole: $z = a$ (simple)
 - zero: $z = b$ (simple)

First order LTI systems

The system function is given by:

$$H(z) = G \frac{1 - bz^{-1}}{1 - az^{-1}}$$

- Poles and zeros are:
 - pole: $z = a$ (simple)
 - zero: $z = b$ (simple)
- Practical requirements:
 - causality: ROC: $|z| > |a|$
 - stability: $|a| < 1$

First order LTI systems

The system function is given by:

$$H(z) = G \frac{1 - bz^{-1}}{1 - az^{-1}}$$

- Poles and zeros are:
 - pole: $z = a$ (simple)
 - zero: $z = b$ (simple)
- Practical requirements:
 - causality: ROC: $|z| > |a|$
 - stability: $|a| < 1$
- Impulse response (ROC: $|z| > |a|$):

$$h[n] = G \left(1 - \frac{b}{a} \right) a^n u[n] + G \frac{b}{a} \delta[n]$$

Low-pass case

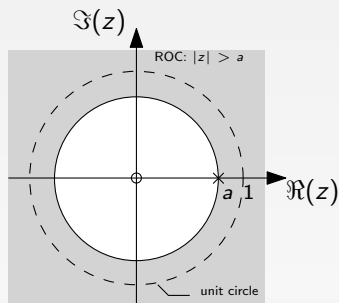
$$H(z) = G \frac{1 - bz^{-1}}{1 - az^{-1}}$$

- To get a low-pass behavior: $a = 1 - \epsilon$, where $0 < \epsilon \ll 1$
- Additional attenuation of high-frequency is possible by proper placement of the zero $z = b$.

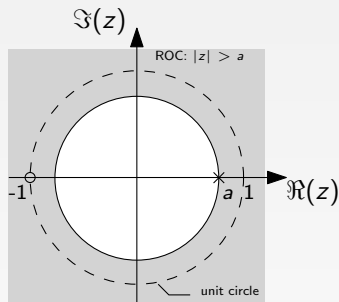
Low-pass case

$$H(z) = G \frac{1 - bz^{-1}}{1 - az^{-1}}$$

- To get a low-pass behavior: $a = 1 - \epsilon$, where $0 < \epsilon \ll 1$
- Additional attenuation of high-frequency is possible by proper placement of the zero $z = b$.

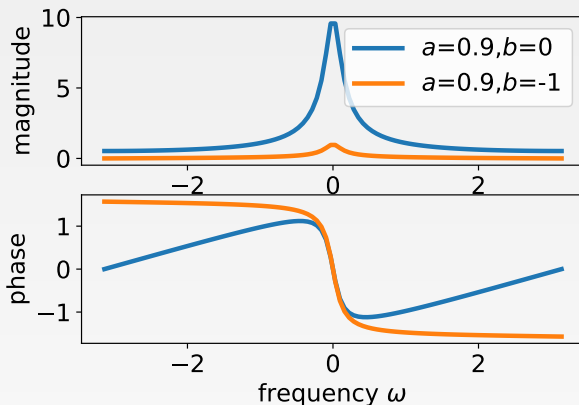


$$H_1(z) = G_1 \frac{1}{1 - az^{-1}} \quad (\text{zero: } b = 0)$$
$$G_1 = 1 - a \Rightarrow H_1(\omega = 0) = 1$$



$$H_2(z) = G_2 \frac{1 + z^{-1}}{1 - az^{-1}} \quad (\text{zero: } b = -1)$$
$$G_2 = \frac{1-a}{2} \Rightarrow H_2(\omega = 0) = 1$$

Frequency responses of the corresponding low-pass systems



High-pass case

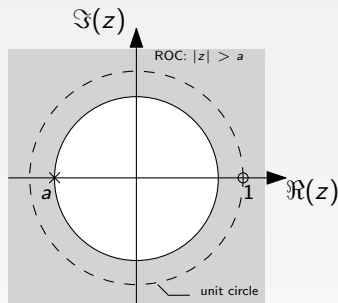
$$H(z) = G \frac{1 - bz^{-1}}{1 - az^{-1}}$$

- To get a high-pass behavior: $a = -1 + \epsilon$, where $0 < \epsilon \ll 1$
- To get a high attenuation of the DC component, one has to locate the zero at or near $b = 1$

High-pass case

$$H(z) = G \frac{1 - bz^{-1}}{1 - az^{-1}}$$

- To get a high-pass behavior: $a = -1 + \epsilon$, where $0 < \epsilon \ll 1$
- To get a high attenuation of the DC component, one has to locate the zero at or near $b = 1$

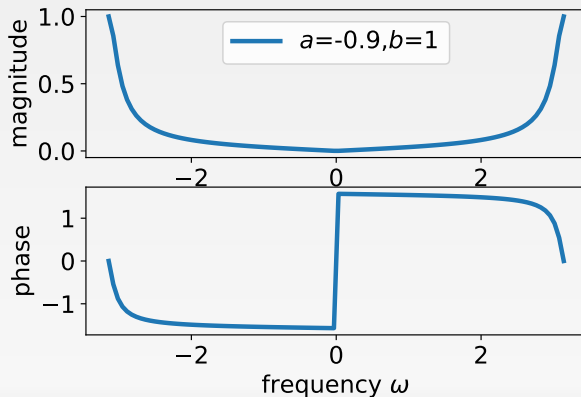


$$H_3(z) = G_3 \frac{1 - z^{-1}}{1 - az^{-1}} \quad (\text{zero: } b = 1)$$

$$G_3 = \frac{1+a}{2} \Rightarrow H_3(\omega = -\pi) = 1$$

Frequency response of the corresponding high-pass system

- Frequency response:



Second order systems

System function

$$H(z) = G \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Second order systems

System function

$$H(z) = G \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

■ Poles

- if $a_1^2 > 4a_2$: 2 distinct poles (real) at $p_{1,2} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_2}$
- if $a_1^2 = 4a_2$: double pole (real) at $p_1 = -\frac{a_1}{2}$
- if $a_1^2 < 4a_2$: 2 distinct poles (complex) at $p_{1,2} = -\frac{a_1}{2} \pm j\frac{1}{2}\sqrt{4a_2 - a_1^2}$

Second order systems

System function

$$H(z) = G \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

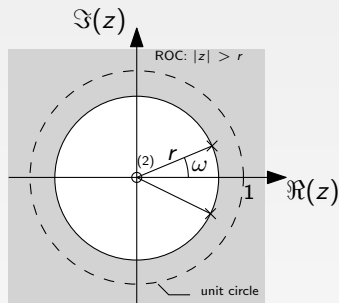
■ Poles

- if $a_1^2 > 4a_2$: 2 distinct poles (real) at $p_{1,2} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_2}$
- if $a_1^2 = 4a_2$: double pole (real) at $p_1 = -\frac{a_1}{2}$
- if $a_1^2 < 4a_2$: 2 distinct poles (complex) at $p_{1,2} = -\frac{a_1}{2} \pm j\frac{1}{2}\sqrt{4a_2 - a_1^2}$

■ Practical requirements:

- causality: ROC: $|z| > \max\{|p_1|, |p_2|\}$
- stability: $|p_1| < 1$ and $|p_2| < 1 \Leftrightarrow |a_2| < 1$ and $a_2 > |a_1| - 1$

Second order systems: resonator



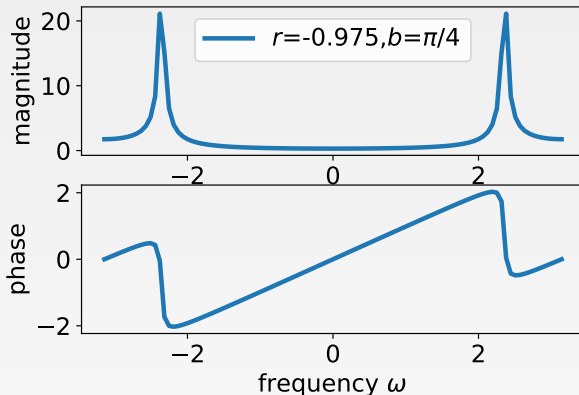
$$p_1 = re^{j\omega_0}$$

$$p_2 = re^{-j\omega_0} = p_1^*$$

$$\begin{aligned} H(z) &= G \frac{1}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})} \\ &= G \frac{1}{1 - 2r \cos(\omega_0)z^{-1} + r^2z^{-2}} \end{aligned}$$

Second order systems: resonator

■ Frequency response:



- The frequency response clearly shows peaks around $\pm\omega_0$.
- For r close to 1 (but < 1), $|H(\omega)|$ reaches a maximum at $\pm\omega_0$

System function

$$\begin{aligned} H(z) &= B(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \\ &= b_0 (1 - z_1 z^{-1}) \dots (1 - z_M z^{-1}) \end{aligned}$$

System function

$$\begin{aligned}H(z) &= B(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \\&= b_0 (1 - z_1 z^{-1}) \dots (1 - z_M z^{-1})\end{aligned}$$

- This is zeroth order rational system: $A(z) = 1$

System function

$$\begin{aligned} H(z) &= B(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \\ &= b_0 (1 - z_1 z^{-1}) \dots (1 - z_M z^{-1}) \end{aligned}$$

- This is zeroth order rational system: $A(z) = 1$
 - The M zeros z_k can be anywhere in the complex plane

System function

$$\begin{aligned} H(z) &= B(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \\ &= b_0 (1 - z_1 z^{-1}) \dots (1 - z_M z^{-1}) \end{aligned}$$

- This is zeroth order rational system: $A(z) = 1$
 - The M zeros z_k can be anywhere in the complex plane
 - There is multiple pole of order M at $z = 0$

System function

$$\begin{aligned} H(z) &= B(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \\ &= b_0 (1 - z_1 z^{-1}) \dots (1 - z_M z^{-1}) \end{aligned}$$

- This is zeroth order rational system: $A(z) = 1$
 - The M zeros z_k can be anywhere in the complex plane
 - There is multiple pole of order M at $z = 0$
- Practical requirement: none

System function

$$\begin{aligned} H(z) &= B(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \\ &= b_0 (1 - z_1 z^{-1}) \dots (1 - z_M z^{-1}) \end{aligned}$$

- This is zeroth order rational system: $A(z) = 1$
 - The M zeros z_k can be anywhere in the complex plane
 - There is multiple pole of order M at $z = 0$
- Practical requirement: none
 - Above system is always causal and stable

System function

$$\begin{aligned} H(z) &= B(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \\ &= b_0 (1 - z_1 z^{-1}) \dots (1 - z_M z^{-1}) \end{aligned}$$

- This is zeroth order rational system: $A(z) = 1$
 - The M zeros z_k can be anywhere in the complex plane
 - There is multiple pole of order M at $z = 0$
- Practical requirement: none
 - Above system is always causal and stable
- Impulse response:

$$h[n] = \begin{cases} b_n & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

FIR filters: moving average system

- Difference equation:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

FIR filters: moving average system

- Difference equation:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

- System function:

$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} = \frac{1}{M} \frac{1 - z^{-M}}{1 - z^{-1}}$$

FIR filters: moving average system

- Difference equation:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

- System function:

$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} = \frac{1}{M} \frac{1 - z^{-M}}{1 - z^{-1}}$$

- PZ analysis: roots of the numerator

$$z^M = 1 \Rightarrow z = e^{j2\pi k/M}, k = 0, 1, \dots, M-1$$

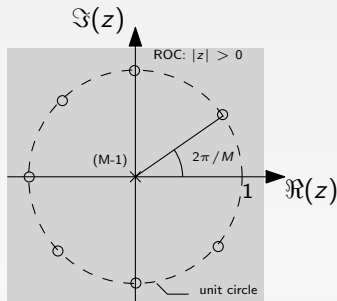
\Rightarrow there is no pole at $z = 1$ because of PZ cancellation:

$$H(z) = \frac{1}{M} \prod_{k=1}^{M-1} \left(1 - e^{j2\pi k/M} z^{-1}\right)$$

FIR filters: moving average system

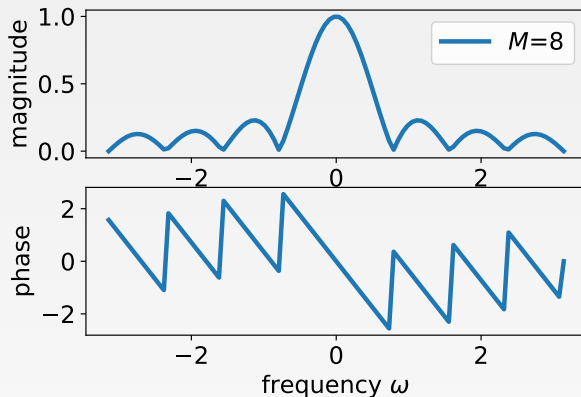
$$H(z) = \frac{1}{M} \prod_{k=1}^{M-1} \left(1 - e^{j2\pi k/M} z^{-1} \right)$$

- PZ diagram for $M = 8$



FIR filters: moving average system

- Frequency response:



Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT**
- 8 Implementation of DT systems

Outline

- 1 Introduction
- 2 Discrete-Time signals
- 3 Discrete-Time systems
- 4 The Z-Transform (ZT)
- 5 Fourier Transform of DT signals
- 6 Analysis of LTI systems in the z-Domain
- 7 The DFT
- 8 Implementation of DT systems**