Discrete-time Signal Processing

MsCV Vibot - UE4 Digital Signal Processing

Olivier Morel

Le2i Vision Robotics Team - ERL CNRS

December 11, 2017

Preamble

- Main references:

Digital Signal Processing, Principles, Algorithms, and Applications, John G. Proakis, Dimitris G. Manolakis.

Linear Processing for Discrete-Time Signal, Frédéric Truchetet University of Burgundy

Discrete Time Signal Processing, Benoît Champagne, Fabrice Labeau, Dpt of Electrical & Computer Engineering McGill University

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- Course flow:
  - 8x2H Main course mix with tutorials
  - Assessment : closed book

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    $\Rightarrow$ 1 sheet of A4 paper allowed!
    - double-sided
    - manuscript
    - no hard copy

Outline

Introduction

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Quantity that varies as function of time and/or space and has ability to convey information

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  - **Finance**: time variations of a stock value or a market index.
- By extension, any series of measurements of a physical quantity can be considered a signal (temperature measurements for instance)

Types of signals and representations

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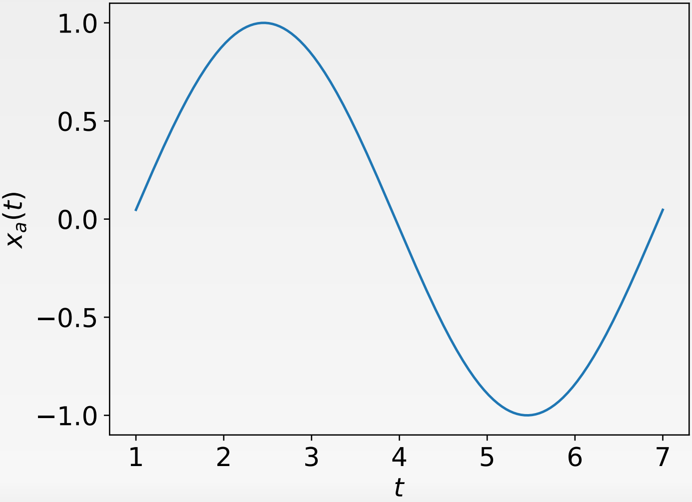
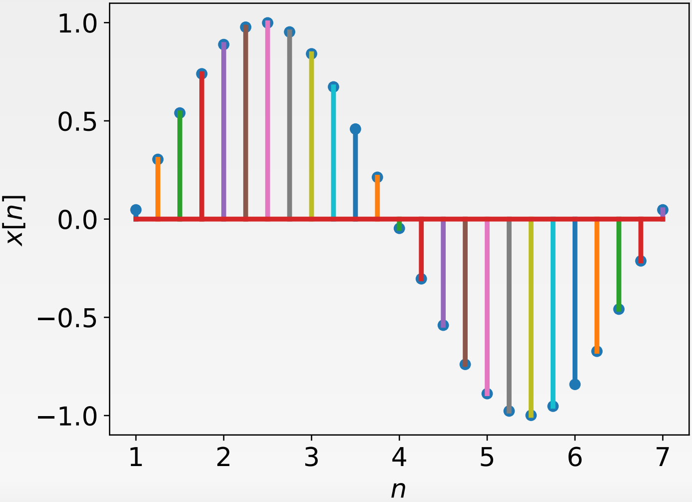
Multi-dimensional signal

$x(t_1, \dots, t_N)$

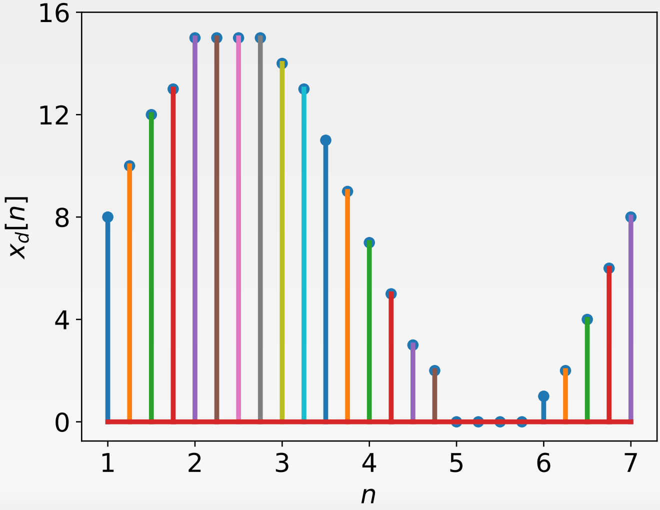
Example: analog signal



Example: discrete signal



Example: digital signal



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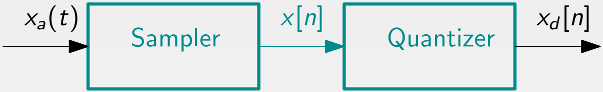
A/D converter



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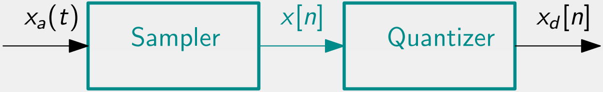

- **Sampler**: analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.

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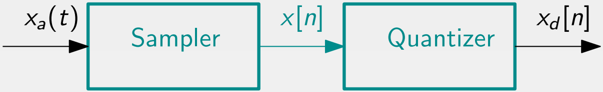

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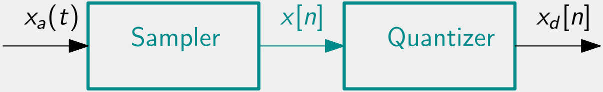

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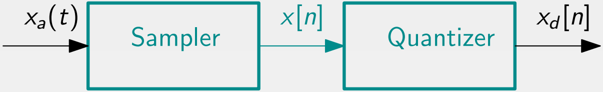

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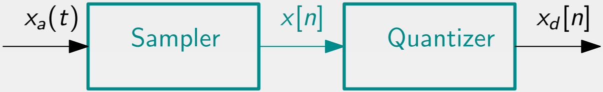

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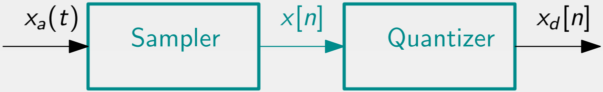

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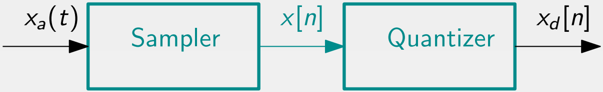

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- WAVE file example:
  - sampling rate: 44.1kHz (sampling rate used for audio CD's)
  - level resolution: 16 bits per sample (some systems use 24 bits)

Digital system

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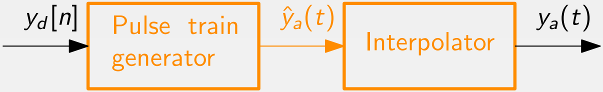
- Operations can be described of mean of an algorithm
- Important distinctions
  - Real-time system: computing associated to each sampling interval can be accomplished in a time $\leq$ the sampling interval
  - Off-line system: requires the use of external data storage units

D/A converter



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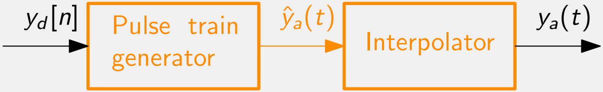

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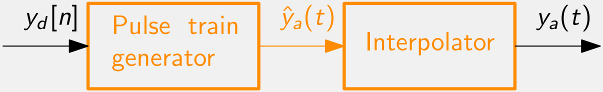

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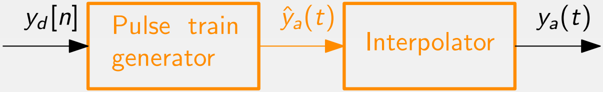

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- One device can generally take care of both steps.

Pros and cons of DSP

Advantages

- Robustness (signal levels can be regenerated)
- Storage capability (can interfaced to low-cost devices for storage)
- Flexibility (software programmable)
- Structure (easy interconnection of DSP blocks)

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Disadvantages

- Cost/complexity added by A/D and D/A conversion
- Input signal bandwidth is technology limited
- Quantization effects

Outline

Discrete-Time signals

Definition

sequence of real or complex numbers, that is, a mapping from the set of integers $\mathbb{Z}$ into $\mathbb{R}$ or $\mathbb{C}$, as in:

$$n \in \mathbb{Z} \to x[n] \in \mathbb{R} \; or \; \mathbb{C}$$

- n is called the discrete-time index
- x[n], the nth number in the sequence, is called a sample

Description

- Sequence notation:

$$x = \{\cdots, 0, \underline{0}, 1, 4, 1, 0, 0, \cdots\},$$

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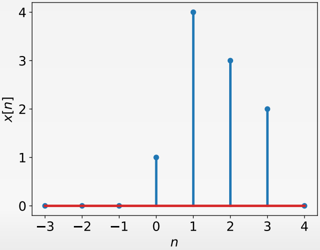
- Graphical:



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- Explicit mathematical expression:

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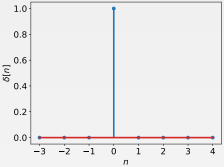
- Recursive approach:

$$x[n] = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ \frac{1}{2}x[n-1] & n > 0. \end{cases}$$

Basic Discrete-time signals

- Unit pulse:

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$



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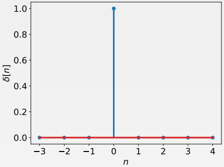
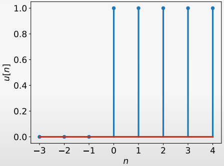


- Unit step:

$$u[n] = \begin{cases} 1 & n \geqslant 0, \\ 0 & n < 0. \end{cases}$$



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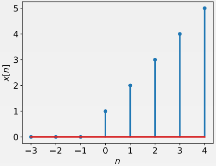
- Ramp function:

$$r[n] = n \cdot u[n].$$



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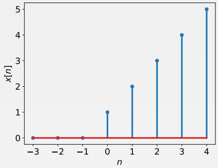
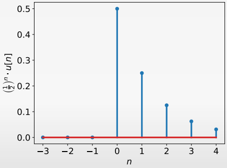


- Exponential sequence:

$$x[n] = a^n \cdot u[n].$$



Uniform sampling

- DT signals are commonly generated via uniform (or periodic) sampling of an analog signal $x_a(t)$:

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where $T_s > 0$: sampling period.

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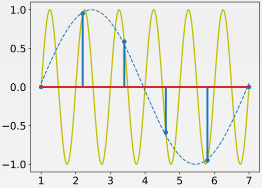
  - $F_s = 1/T_s$: sampling frequency,
  - ω : normalized radian frequency of the DT signal.

Nyquist rate



Nyquist rate: $2 \cdot F$

The sampling frequency F_s must satisfy

$$F_s \gg 2 \cdot F$$

Basic operations on signal

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Property

$\mathcal{S}$ equipped with addition and scaling is a vector space.

Classes of signals

Energy signals

all $x \in \mathcal{S}$ with finite energy:

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Power signals

all $x \in \mathcal{S}$ with finite power:

$$\mathcal{P}_x \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty$$

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all $x \in \mathcal{S}$ that can be bounded:

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Absolutely summable

all $x \in \mathcal{S}$ such that:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Discrete convolution

Discrete convolution of x and y

$$(x * y)[n] = x[n] * y[n] \triangleq \sum_{k=-\infty}^{\infty} x[k] \cdot y[n-k]$$

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Properties

- commutative law: $x * y = y * x$
- associative law: $(x * y) * z = x * (y * z)$
- convolution by unit pulse: $x * \delta = x$

Correlation of DT signals

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  - can be considered as a measure of similarity of two signals
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- **Crosscorrelation**: performed on two signals
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- **Autocorrelation**: performed on one signal
  - indicates how the signal energy (power) is distributed within the signal
  - applications of signal autocorrelation are in radar, sonar, satellite, and wireless communications systems

Crosscorrelation

Definition

$$R_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]y[k-n] = \sum_{k=-\infty}^{\infty} x[k+n]y[k]$$

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Link with convolution

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

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Properties

- Even function:

$$R_{xx}[n] = R_{xx}[-n]$$

- Energy:

$$R_{xx}[0] = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \mathcal{E}_x$$

$$\forall n \in \mathbb{Z},\ R_{xx}[n] < R_{xx}[0]$$

Correlation coefficient

Definition

$$c_{xy} = \frac{R_{xy}[0]}{\sqrt{R_{xx}[0] \cdot R_{yy}[0]}}$$

Correlation coefficient

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Properties

- Similarity measurement of two signals
- $-1 \leq c_{xy} \leq 1$
- Geometrically represents angle between euclidean vectors x and y

$$C_{xy} = \frac{x \cdot y}{\sqrt{|x|^2 \, |y|^2}} = \frac{x \cdot y}{|x| \, |y|} \triangleq \cos(x, y)$$

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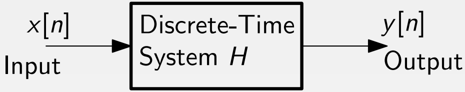
- An also be defined in terms of paramter n:

$$-1 \leq c_{xy}[n] = \frac{R_{xy}[n]}{\sqrt{R_{xx}[0]R_{yy}[0]}} \leq 1$$

Outline

Definition

- A Discrete-Time system is a mapping H from $\mathcal{S}$ into itself :



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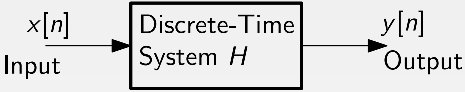

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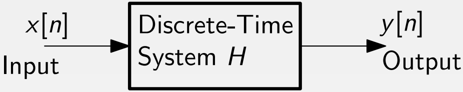


- The system output $y[n]$ generally depends on $x[k]$ for all values of $k \in \mathbb{Z}$
- Notations:

$$y[n] = H\left(x[n]\right) \triangleq \left(Hx\right)[n]$$

Outline

Time reversal

$$y[n] = (Rx)[n] \triangleq x[-n]$$

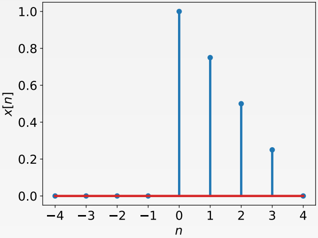
Basic systems

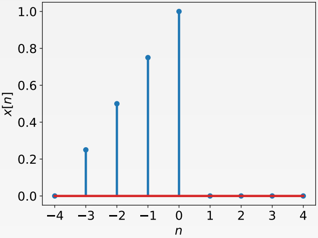
Time reversal

Time reversal

$$y[n] = (Rx)[n] \triangleq x[-n]$$

- Mirror image about origin:





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$$y[n] = (D_k x)[n] \triangleq x[n-k]$$

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Basic systems

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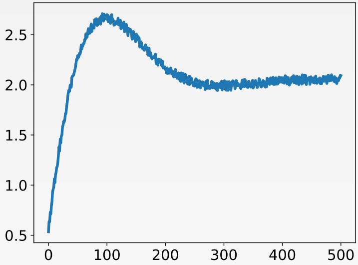

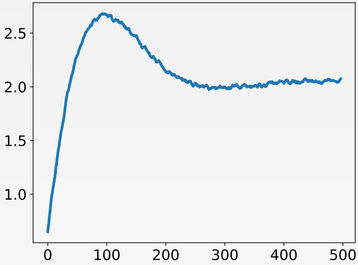

Systems properties

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- Dynamic systems require memory:

$$y[n] = \frac{1}{2}\left(x[n-1] + x[n]\right)$$

Systems properties: causal versus anti-causal

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Anti-causal

y[n] only depends on values x[k] for k>n.

Systems properties: linear or not ?

Linearity

$$\forall (\alpha, \beta) \in \mathbb{C}^2, \forall (x, y) \in \mathcal{S}^2, \; H(\alpha x + \beta y) = \alpha H(x) + \beta H(y)$$

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$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$

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$$y[n]=\frac{1}{3}\,(x[n-1]+x[n]+x[n+1])$$

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Systems properties: time invariant or not ?

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$$\forall (n,k) \in \mathbb{Z}^2,\ (Hx)[n] = y[n] \Rightarrow (Hx)[n-k] = y[n-k]$$

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- Example of system not time invariant:

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x bounded $\Rightarrow y = Hx$ bounded

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if $\forall n \in \mathbb{Z}\, |x[n]| \leq B_x$ then $\exists B_y / \forall n \in \mathbb{Z},\ |y[n]| \leq B_y$

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- A system is stable (Bounded Input Bounded Output) if every bounded input produces a bounded output.

Outline

Linear Time-Invariant (LTI) systems

- DT systems that are both Linear and Time-Invariant play a central role in digital signal processing:
  - Many physical systems are either LTI or approximately so
  - Many efficient tools are available for the analysis and design of LTI systems

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Fundamental property

Let H a LTI system and $y = Hx$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x * h[n],$$

with $h \triangleq H\delta$ known as impulse response of H.

Proof of the fundamental property

First we have:

$$y[n] = (Hx)[n] = H(x[n]).$$

And for any DT signal, we can write:

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Invoking Time-Invariant property:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]HD_k(\delta[n]) = \sum_{k=-\infty}^{\infty} x[k]D_kH(\delta[n]) = \sum_{k=-\infty}^{\infty} x[k]D_kh[n]$$

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- Example with $h = \{\cdots, 0, 1, \underline{0}, -1, 0, \cdots\}$ and $x = u$:

$x[n]$	0	0	0	**0**	1	1	1	1	1
mask			$\rightarrow$	**-1**	0	1	$\rightarrow$		
$y[n]$	0	0	0	1	**1**	0	0	0	0

Characterization of a LTI system

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$\Rightarrow$Clearly, $y[n]$ only depends on values $x[m]$ for $m \leq n$ if and only if $h[k] = 0$ for $k < 0$

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Thus the system is **stable** provided $|\alpha| < 1$

FIR and IIR

FIR system

An LTI system has a Finite Impulse Response (FIR) if we can find integers $N_1 \leq N_2$ such that:

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- The impulse response is often called a **convolution mask**.

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2. Let the LTI system described by:

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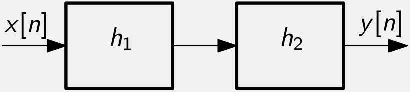
$\Rightarrow$ the system is **IIR** (cannot find any N_2)

Interconnection of LTI systems: cascade



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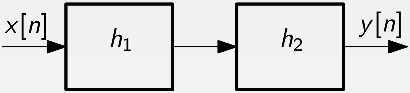

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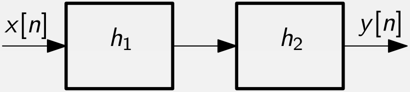

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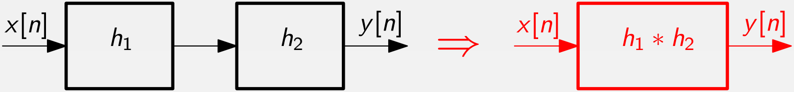

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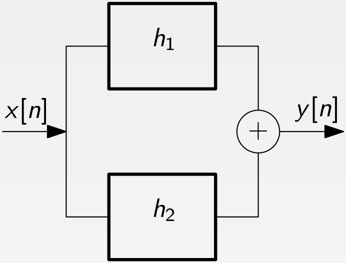

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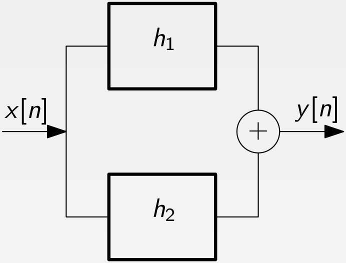

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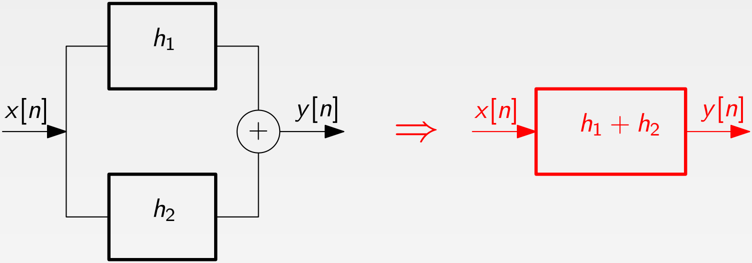

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- Both X(z) and the ROC are needed when specifying a ZT.

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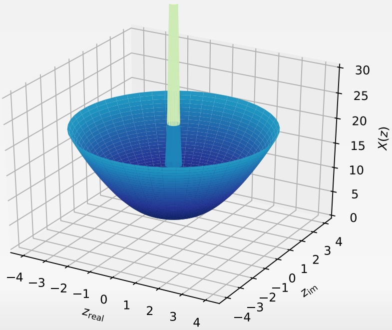

Outline

4 **The Z-Transform (ZT)**

- Definition
- Study of the ROC
- Properties of the ZT
- Rational ZTs
- Inverse ZT

Signal with finite duration

Definition

A signal with finite duration is defined such that:

$$\exists\,(N_1, N_2) \in \mathbb{Z}^2,\ N_1 \leq N_2 / \forall n < N_1 \text{ and } \forall n > N_2,\ x[n] = 0$$

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Theorem

Radius of convergence

To any power series $\sum_{n=0}^{\infty} c_n w^n$, we can associate a radius of convergence

$$R_w = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|,$$

such that:

- if $|w| < R_w \Rightarrow$ the series converges absolutely
- if $|w| > R_w \Rightarrow$ the series diverges

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- Suppose $x[n] = 0$ for $n < 0$:

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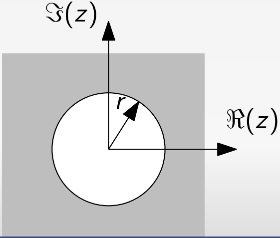

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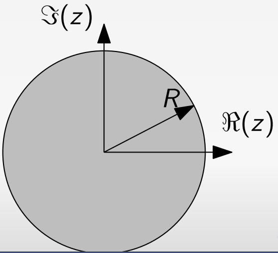

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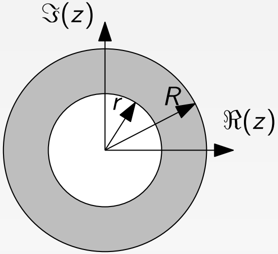

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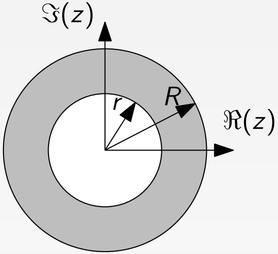

- if $r > R$, the ZT does not exists.

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$$X(z) = \frac{1}{1 - \frac{1}{2z}} - 1 + \frac{1}{1 - \frac{z}{2}} = \frac{2 - \frac{5}{2} z^{-1}}{1 - \frac{5}{2} z^{-1} + z^{-2}}$$

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Since $\{|z| > 2\} \cap \left\{|z| < \frac{1}{2}\right\} = \emptyset$, the ROC is empty and the ZT does not exist.

Outline

Introductory remarks

- Notations for ZT pairs:

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), z \in \mathcal{R}_x$$
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Basic symmetries

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Assuming that the ROC $\mathcal{R}_x$ was defined by: $r < |z| < R$, then the ROC of $X_f(z)$ is: $1/R < |z| < 1/r$

Linearity and Time shift

Linearity

$$\forall\,(a,b)\in\mathbb{C}^2,\; ax[n]+by[n]\overset{z}{\longleftrightarrow} aX(z)+bY(z),\; z\in\mathcal{R}_x\cap\mathcal{R}_y$$

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$$X(z) = \frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}, \ \text{ROC} : |z| > 1$$

Differentiation

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For x[n] causal (i.e. $x[n] = 0 \; for \; n < 0$), we have:

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Initial value (anti-causal case)

For x[n] anti-causal (i.e. $x[n] = 0 \; for \; n > 0$), we have:

$$\lim_{z^{-1} \to \infty} X(z) = x[0]$$

Outline

4 **The Z-Transform (ZT)**

- Definition
- Study of the ROC
- Properties of the ZT
- Rational ZTs
- Inverse ZT

Rational function

Definition

$X(z)$ is a rational function in z (or z^{-1}) if:

$$X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are polynomials in z (resp. z^{-1})

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- Rational ZT plays a central role in DSP
- Essential for the realization of practical IIR filters
- Two important issues related to rational ZT are investigated:
  - Pole-Zero (PZ) characterization
  - Inversion via partial fraction expansion

Poles and zeros

Pole

$X(z)$ has a pole of order L at $z = p_0$ if:

$$X(z) = \frac{\psi(z)}{(z - p_0)^L}, \; 0 < |\psi(p_0)| < \infty$$

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- The order L is sometimes referred to as the **multiplicity** of the pole/zero.

Poles and zeros at ∞

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Consider rational function $X(z) = N(z)/D(z)$:

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Property

number of poles $=$ number of zeros, if poles and zeros at 0 and ∞ are included.

Poles and Zeros of a rational: examples

- Example 1:

$$X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z-1)^2}$$

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$$X(z) = \frac{1 - z^{-4}}{1 + 3z^{-1}} = \frac{z^4 - 1}{z^3 (z+3)}$$

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zeros $z_{k\in[0,3]} = e^{jk\pi/2}0,\ L = 1$

Pole-zero and rational function link

Property

For rational functions $X(z) = N(Z)/D(z)$, knowledge of the poles and zeros (along with their order) completely specify $X(z)$, up to a scaling factor $G \in \mathbb{C}$.

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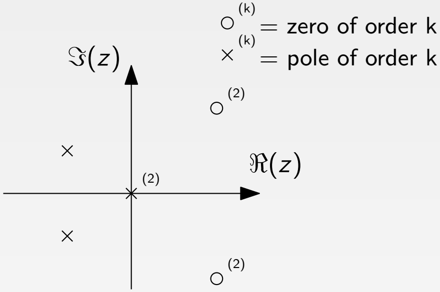
$$X(z) = G\frac{z-1}{z-2} = G\frac{1-z^{-1}}{1-2z^{-1}}$$

Pole-zero (PZ) diagram



- The presence of poles or zeros at ∞ should be mentioned on the diagram
- It is useful to indicate ROC on the PZ-diagram

ROC and PZ diagram

- Consider $x[n] = a^n u[n]$, where $a > 0$:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \text{ ROC: } |z| > a$$

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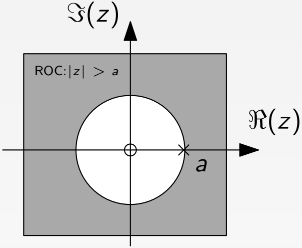

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ROC for rational ZT

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  - because $X(z)$ does not converge at a pole
- ROC can always be extended to nearest pole
- ROC is delimited by poles
  - annular region between poles
- If we are given only $X(z)$, then several possible ROC:

ROC for rational ZT

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Outline

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  - Partial fraction expansion
- Partial fraction is the most useful technique in the context of rational ZTs

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Inverse Z-Transform

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- Cauchy integral theorem:

$$\frac{1}{2\pi j}\oint_{\mathcal{C}} z^{k-1-n}dz = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

Inversion via Partial Fraction Expansion

- Let be a rational ZT defined according to:
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- Under these conditions, $X(z)$ may be expressed as:

$$X(z) = \sum_{k=1}^{K} \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

  - $p_1, \cdots, p_K$ are the distinct poles of $X(z)$
  - $L_1, \cdots, L_K$ are the corresponding orders

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Inversion method

Given $X(z)$ as above with ROC: $r < |z| < R$.

1. Determine the PFE of $X(z)$:

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- Elementary inverse ZTs from 2 simple poles:
  - $\frac{1}{1-az^{-1}} \overset{\mathcal{Z}^{-1}}{\rightarrow} a^n u[n] \text{ since } |z| > |a| \text{ (causal)}$
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- Finally:

$$x[n] = \frac{a^{n+1}}{a-b} u[n] - \frac{b^{n+1}}{b-a} u[-n-1]$$

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Example

ZT of a causal signal

$$X(z) = \frac{-5 + 3z^{-1} + z^{-2}}{3 + 4z^{-1} + z^{-2}}$$

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$$z^{-2} + 4z^{-1} + 3 \;\overline{\big)\; z^{-2} + 3z^{-1} - 5}$$

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$X(z)$ rewrites $X(z) = 1 - \frac{z^{-1}+8}{z^{-2}+4z^{-1}+3}$

Example

$$X(z) = 1 - \frac{z^{-1} + 8}{z^{-2} + 4z^{-1} + 3}$$

- The denominator of the 2nd term has 2 roots, poles at $z = -1/3$ and $z = -1$, hence:

$$X(z) = 1 - \frac{z^{-1} + 8}{3\left(1 + \frac{1}{3}z^{-1}\right)\left(1 + z^{-1}\right)}$$

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Example

$$X(z) = 1 + \frac{5}{6\left(1 + \frac{1}{3}z^{-1}\right)} - \frac{7}{2\left(1 + z^{-1}\right)}$$

- Causality of $x[n]$ determines the ROC of $X(z)$

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  - ROC is supposed to be delimited by circles with radius 1/3 and/or 1
  - causal sequence $\Rightarrow$ ROC must extend outwards from the outermost pole $\Rightarrow$ ROC is $|z| > 1$
- The sequence $x[n]$ is then given by:

$$x[n] = \delta[n] + \frac{5}{6}\left(-\frac{1}{3}\right)^n u[n] - \frac{7}{2}(-1)^n u[n]$$

Use of shift property

- In some cases, a simple multiplication by z^k is sufficient to put $X(z)$ into a suitable format, that is:

$$Y(z) = z^k X(z) = \frac{N(z)}{D(z)}$$

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- The PFE method is then applied to $Y(z)$, yielding a DT signal $y[n]$
- Finally, the shift property is applied to recover $x[n]$:

$$x[n] = y[n-k]$$

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$$y[n] = \frac{1}{2}\left(1 + (-1)^n\right) u[n]$$

- Using linearity and shift property:

$$x[n] = y[n] - y[n - 128]$$

Example

$$X(z) = \frac{1 - z^{-128}}{1 - z^{-2}}, \ |z| > 1$$

- $X(z)$ can be rewritten according to:

$$X(z) = Y(z) - z^{-128}Y(z)$$

where:

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- Therefore:

$$x[n] = \frac{1}{2}\left(1 + (-1)^n\right)\left(u[n] - u[n - 128]\right)$$

Outline

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Definition

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},\ \omega \in \mathbb{R}$$

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- $X(\omega)$ is called the spectrum of $x[n]$: $X(\omega) = |X(w)|\, e^{j\angle X(\omega)}$
  - $|X(w)|$: magnitude spectrum
  - $\angle X(\omega)$: phase spectrum
- The Fourier Transform is a specific case of the ZT taking $z = e^{j\omega}$ with $|z| = 1 \in$ ROC.

Fourier Transform of a sampled continuous signal I

- Let $s_e(t)$ the sampled expression of the continuous signal $s(t)$ with sampling period T_s:

$$s_e(t) = s(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$$

- If we denote $s[k] = s(kTs)$, we have:

$$s_e(t) = \sum_{k=-\infty}^{\infty} s[k]\delta(t - kT_s)$$

- The Fourier transform gives:

$$TF\{s_e(t)\} = \hat{s}_e(\omega) = s[k] \sum_{k=-\infty}^{\infty} s[k] TF\{\delta(t - kT_s)\}$$

Fourier Transform of a sampled continuous signal II

- Applying the delay theorem:

$$\hat{s}_e(\omega) = s[k] \sum_{k=-\infty}^{\infty} s[k] e^{-jkwT_s} TF\{\delta(t)\} = s[k] \sum_{k=-\infty}^{\infty} s[k] e^{-jkwT_s}$$

- Finally:

$$\hat{s}_e(\omega) = S(\omega T_s)$$

with the Nyquist frequency equal to:

$$\omega_N = \frac{\pi}{T_s}$$

Inverse DTFT

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$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega,\ n \in \mathbb{Z}$$

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Proof: Note that $\int_{-\pi}^{\pi} e^{j\omega n}d\omega = 2\pi\delta[n]$:

$$\begin{aligned}
\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega &= \int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\right)e^{j\omega n}d\omega \\
&= \sum_{k=-\infty}^{\infty} x[k]\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \\
&= 2\pi\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \\
&= 2\pi x[n]
\end{aligned}$$

Outline

Convergence of the DTFT

- For the DTFT to exist, the series $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ must converge
- That is, the partial sum

$$X_M(\omega) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$$

  must converge to a limit $X(\omega)$ as $M \to \infty$
- **Absolutely summable signals**
  - $X_M(\omega)$ converges uniformly to $X(\omega)$
  - $X(\omega)$ is continuous
- **Energy signals**
  - $X_M(\omega)$ does not necessarily converge
  - $X(\omega)$ may be discontinuous at certain points
- **Power signals**
  - Most power signals do not have a DTFT
  - Exceptions including: Periodic signals, Unit step

Outline

Properties

- Linearity

$$ax[n] + by[n] \overset{\mathcal{F}}{\leftrightarrow} aX(\omega) + bY(\omega)$$

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$$e^{j\omega_0 n}x[n] \stackrel{\mathcal{F}}{\leftrightarrow} X(\omega - \omega_0)$$

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- Frequency modulation

$$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\leftrightarrow} X(\omega - \omega_0)$$

- Differentiation

$$nx[n] \overset{\mathcal{F}}{\leftrightarrow} j\frac{dX(\omega)}{d\omega}$$

Even and odd component definition

DT signal

$$
\begin{aligned}
x[n] &= x_e[n] + x_o[n] \\
x_e[n] &\triangleq \frac{1}{2}\left(x[n] + x^*[-n]\right) = x_e^*[-n] \\
x_o[n] &\triangleq \frac{1}{2}\left(x[n] - x^*[-n]\right) = -x_o^*[-n]
\end{aligned}
$$

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DTFT

$$X(\omega) = X_e(\omega) + X_o(\omega)$$

$$X_e(\omega) \triangleq \frac{1}{2}\left(X(\omega) + X^*(w)\right) = X_e^*(-\omega)$$

$$X_o(\omega) \triangleq \frac{1}{2}\left(X(\omega) - X^*(w)\right) = -X_o^*(-\omega)$$

Basic symmetries

Real and imaginary parts decomposition

$$x[n] = x_R[n] + jx_I[n]$$
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- $jx_I[n] \overset{\mathcal{F}}{\leftrightarrow} X_o(\omega)$
- $x_e[n] \overset{\mathcal{F}}{\leftrightarrow} X_R(\omega)$
- $x_o[n] \overset{\mathcal{F}}{\leftrightarrow} jX_I(\omega)$

More properties

Convolution

$$x[n]*\mathrm{y}[\mathrm{n}] \overset{\mathcal{F}}{\leftrightarrow} X(\omega)Y(\omega)$$

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Plancherel's relation

$$\sum_{n=-\infty}^{\infty} x[n]y[n]^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)Y^*(\omega)d\omega$$

Outline

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6 **Analysis of LTI systems in the z-Domain**

Linear Constant Coefficient Difference Equations

Definition

A DT system can be described by an LCCDE of order N if:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

where $a_0 \neq 0$ and $a_N \neq 0$.

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where $a_0 \neq 0$ and $a_N \neq 0$.

- If we further assume initial rest conditions, i.e.:

$$\forall n < n_0,\ x[n] = 0 \Rightarrow \forall n < n_0,\ y[n] = 0$$

LCCDE corresponds to unique causal LTI system.

Example: Accumulator system

$$x[n] \to y[n] = \sum_{k=-\infty}^{n} x[k]$$

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This LTI system can be rewritten according to:

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$\Rightarrow$ LCCDE of order $N = 1\,(M = 0,\ a_0 = 1,\ a_1 = -1,\ b_0 = 1)$
LCCDEs lead to efficient recursive implementation:

- Recursive because computation of $y[n]$ make use past output signal values ($y[n-1]$)
- Efficient: in the case of the accumulator it requires only 1 adder and 1 memory unit instead of an infinite number of adders and memory units.

Outline

One-sided Z-Transform

- The two-sided ZT requires that the corresponding signals be specified for entire time range $-\infty < n < \infty$
  - Prevent evaluation of the output of non-relaxed systems
- The one-sided ZT can be used to solve difference equations with initial conditions

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Characteristics

1. Does not contain information about the signal $x[n]$ for negative values of time ($n < 0$)
2. It is unique only for causal signals
3. one-sided ZT of $x[n]$ is identical to the two-sided ZT of $x[n]u[n]$

Properties

Almost all properties for the two-sided ZT carry over to the one-sided ZT with exception of the shifting property.

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Case 1: Time Delay

If

$$x[n] \overset{z^+}{\longleftrightarrow} X^+(z)$$

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$$\forall k > 0,\; x[n-k] \overset{z^+}{\longleftrightarrow} z^{-k}\left(X^+(z) + \sum_{n=1}^{k} x[-n]z^n\right)$$

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Case 2: Time advance

If

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then

$$\forall k > 0,\ x[n+k] \overset{z^+}{\longleftrightarrow} z^{k}\left(X^+(z) - \sum_{n=0}^{k-1} x[n]z^{-n}\right)$$

Example

Determine the step response of the system with IC $y[-1] = 1$:

$$y[n] = \alpha y[n-1] + x[n], \text{ with } -1 < \alpha < 1$$

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3. Solving for $Y^+[n]$:
$$Y^+(z) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{1}{(1 - \alpha z^{-1})(1 - z^{-1})}$$
4. Perform partial-fraction expansion:
$$Y^+(x) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\frac{\alpha}{\alpha - 1}}{1 - \alpha z^{-1}} + \frac{\frac{1}{1-\alpha}}{1 - z^{-1}}$$

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5. Compute inverse ZT:
$$\begin{aligned} y[n] &= \alpha^{n+1}u[n] + \frac{1-\alpha^{n+1}}{1-\alpha}u[n] \\ &= \frac{1}{1-\alpha}\left(1-\alpha^{n+2}\right)u[n] \end{aligned}$$

Final Value Theorem

If

$$x[n] \overset{z^+}{\longleftrightarrow} X^+(z)$$

then

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1) X^+(z)$$

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$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X^+(z)$$

- The limit exists if the ROC of $(z-1)X^+(z)$ includes the unit circle
- Useful when the asymptotic behavior of a signal $x[n]$ is desired knowing its ZT

Outline

The system function

LTI system $\mathcal{H}$ (recall)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \text{ with } h[n] = \mathcal{H}\left\{\delta[n]\right\}$$

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Definition

The system function of $\mathcal{H}$, denoted $H(z)$ is the ZT of $h[n]$:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n},\ z \in \mathcal{R}_H$$

where $\mathcal{R}_H$ denotes the corresponding ROC.

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where $\mathcal{R}_H$ denotes the corresponding ROC.

- If $H(z)$ and $\mathcal{R}_H$ are known, $h[n]$ can be recovered via inverse ZT
- if $z = e^{j\omega} \in \mathcal{R}_H$ (the ROC contains the unit circle) then

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \equiv H(\omega)$$

Properties

Let $\mathcal{H}$ be LTI system with system function $H(z)$ and ROC $\mathcal{R}_H$.

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- LTI system $\mathcal{H}$ is causal iff $\mathcal{R}_H$ is the exterior of a circle (including ∞)
- LTI system $\mathcal{H}$ is stable iff $\mathcal{R}_H$ contains the unit circle:

$$\begin{aligned}\mathcal{H}\,\text{stable} &\Leftrightarrow \sum_n |h[n]| < \infty \\ &\Leftrightarrow e^{j\omega} \in \mathcal{R}_H\end{aligned}$$

$$|H(z)| \leq \sum_n \left|h[n]z^{-n}\right|$$

evaluated on the unit circle: $z = e^{i\omega}$:

$$|H(z)| \leq \sum_n |h[n]| < \infty$$

LCCDE system function

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Taking ZT on both sides:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

LCCDE system function

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Leading to a rational system function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Rational system with real coefficients

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- In many applications, coefficients a_k and b_k are real implying:

$$H^*(z) = H(z^*)$$

- Thus, if z_k is a zero of $H(z)$ then:

$$H(z_k^*) = (H(z_k))^* = 0^* = 0$$

which shows that z_k^* is also a zero of $H(z)$

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- More generally, it can be shown that complex poles (or zeros) occur in complex conjugate pairs:
  - if p_k is a pole of order l of $H(z)$, so is p_k^*
  - if z_k is a zero of order l of $H(z)$, so is z_k^*

Outline

6 Analysis of LTI systems in the z-Domain

- LTI systems described by LCCDE
- One-sided Z-Transform
- The system function
- Response of rational system Functions
- Schur-Cohn Stability test
- Frequency response of rational systems
- Analysis of certain basic systems

Response of rational system functions

- Let be $H(z) = \frac{B(z)}{A(z)}$ the system function of a LCCDE system:
  - roots of $A(z)$ are the poles of $H(z)$
  - roots of $B(z)$ are the zeros of $H(z)$

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- The partial fraction expansion of $Y(z)$ yields

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

if $\forall k \forall m$, $p_k \neq q_m$ and there is no pole-zero cancellation.

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- The inverse transform of $Y(z)$ can be written

$$y[n] = \underbrace{\sum_{k=1}^{N} A_k (p_k)^n u[n]}_{\text{natural response}} + \underbrace{\sum_{k=1}^{L} Q_k (q_k)^n u[n]}_{\text{forced response}}$$

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  - Influence of the input signal on the natural response is through the scale factor $\{A_k\}$
  - Influence of the system on the forced response is through the scale factor $\{Q_k\}$

Response with nonzero initial conditions

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- $x[n]$ is assumed to be causal
- effects of all previous input signals are reflected in the initial conditions $y[-1]$, $y[-2]$, $, \cdots, y[-N]$
- To determine $y[n], \forall n \geq 0$, the one-sided ZT can be used:

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$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-k} \left(Y^+(z) + \sum_{n=1}^{k} y[-n] z^n \right) + \sum_{k=0}^{M} b_k z^{-k} X^+(z)$$

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$$Y^+(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} X^+(z) - \frac{\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y[-n] z^n}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

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Since $x[n]$ is causal $X^+(z) = X(z)$ and the expression can be written:

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- Since $Y_{zi}^+(z) = \frac{N_0(z)}{A(z)}$ its poles are $p_1, \cdots, p_N$ and the zero-input response has the form:

$$y_{zi}[n] = \sum_{k=1}^{N} D_k (p_k)^n u[n]$$

Response with nonzero initial conditions

- The terms involving the poles $\{p_k\}$ can be combined:

$$y[n] = y_{zs}[n] + y_{zi}[n] = \sum_{k=1}^{N} A_k^{'}(p_k)^n u[n] + \sum_{k=1}^{L} Q_k(q_k)^n u[n]$$

with $A_k^{'} = A_k + D_k$.

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- There is no effect on the forced response of the system

Outline

6 Analysis of LTI systems in the z-Domain

Schur-Cohn stability test

Reminder

- LTI system $\mathcal{H}$ is causal iff $\mathcal{R}_H$ is the exterior of a circle (including ∞)
- LTI system $\mathcal{H}$ is stable iff $\mathcal{R}_H$ contains the unit circle

Schur-Cohn stability test

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$\Rightarrow$ A causal system described by its rational system function is stable iff its poles are strictly inside the unit circle.

Let be $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$ the denominator polynomial of $H(z)$. A polynomial of degree m is denoted by:

$$A_m(z) = \sum_{k=0}^{m} a_m[k] z^{-k} \quad a_m(0) = 1$$

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The reverse polynomial $B_m(z)$ of degree m is defined as:

$$B_m(z) = z^{-m} A_m(z^{-1}) = \sum_{k=0}^{m} a_m[m-k] z^{-k}$$

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3. Loop to step 2 until it fails or $m = 1$

Outline

Frequency response of rational systems

- Knowing poles and zeros of $H(z)$ can be expressed as:

$$H(z) = Gz^{-K} \frac{\prod_{k=0}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

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$$\theta_k(\omega) = \angle\left(e^{j\omega} - z_k\right)\phi_k(\omega) = \angle\left(e^{j\omega} - p_k\right)$$

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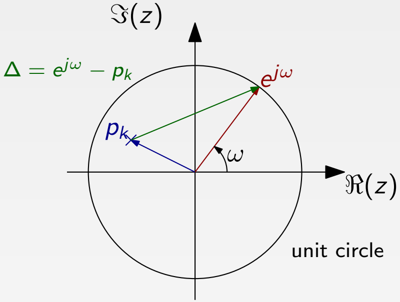
- Phase response:

$$\angle H(\omega) = \angle G - \omega K + \sum_{k=1}^{\infty} \theta_k(\omega) - \sum_{k=1}^{N} \phi_k(\omega)$$

Geometrical interpretation

- Consider pole p_k:



  - $\Delta = e^{j\omega} - p_k$: vector joining p_k to point $e^{j\omega}$ on unit circle
  - $U_k(\omega) = |\Delta|$: length of vector Δ
  - $\phi_k(\omega) = \angle\Delta$: angle between Δ and real axis

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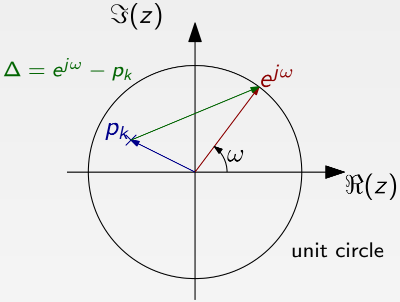

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  - $U_k(\omega) = |\Delta|$: length of vector Δ
  - $\phi_k(\omega) = \angle\Delta$: angle between Δ and real axis
- A similar interpretation holds for the terms $V_k(\omega)$ and $\theta_k(\omega)$ associated to the zeros z_k

Some basic principles

- For stable and causal systems, the poles are located inside the unit circle; zeros can be anywhere.

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Some basic principles

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- Poles near the unit circle at $p = re^{j\omega_0}$ $(r < 1)$ give rise to:
  - peak in $|H(\omega)|$ near ω_0
  - rapid phase variation near ω_0
- Zeros near the unit circle at $z = re^{j\omega_0}$ give rise to:
  - deep notch in $|H(\omega)|$ near ω_0
  - rapid phase variation near ω_0

Outline

First order LTI systems

The system function is given by:

$$H(z) = G\frac{1 - bz^{-1}}{1 - az^{-1}}$$

- Poles and zeros are:
  - pole: $z = a$ (simple)
  - zero: $z = b$ (simple)

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- Practical requirements:
  - causality: ROC: $|z| > |a|$
  - stability: $|a| < 1$
- Impulse response (ROC: $|z| > |a|$):

$$h[n] = G\left(1 - \frac{b}{a}\right) a^n u[n] + G\frac{b}{a}\delta[n]$$

Low-pass case

$$H(z) = G\frac{1 - bz^{-1}}{1 - az^{-1}}$$

- To get a low-pass behavior: $a = 1 - \epsilon$, where $0 < \epsilon \ll 1$
- Additional attenuation of high-frequency is possible by proper placement of the zero $z = b$.

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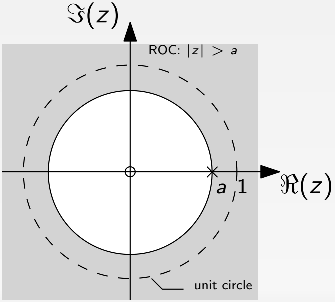
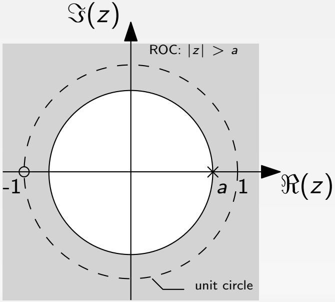
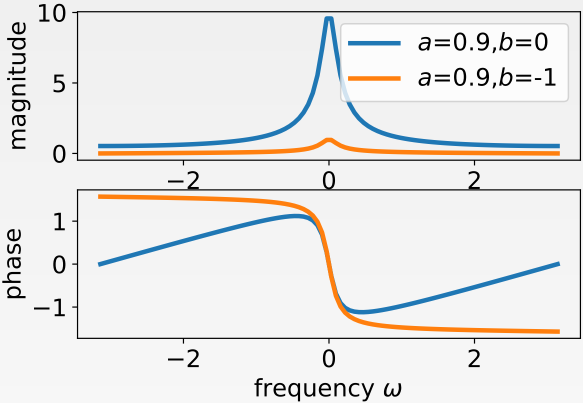


$H_1(z) = G_1\frac{1}{1-az^{-1}}$ (zero: $b=0$)
$G_1 = 1-a \Rightarrow H_1(\omega=0) = 1$



$H_2(z) = G_2\frac{1+z^{-1}}{1-az^{-1}}$ (zero: $b=-1$)
$G_2 = \frac{1-a}{2} \Rightarrow H_2(\omega=0) = 1$

Frequency responses of the corresponding low-pass systems



High-pass case

$$H(z) = G\frac{1 - bz^{-1}}{1 - az^{-1}}$$

- To get a high-pass behavior: $a = -1 + \epsilon$, where $0 < \epsilon \ll 1$
- To get a high attenuation of the DC component, one has to locate the zero at or near $b = 1$

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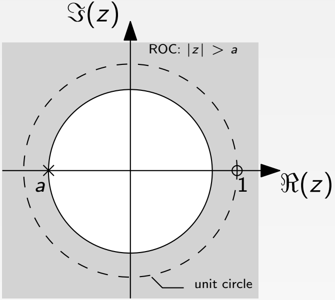
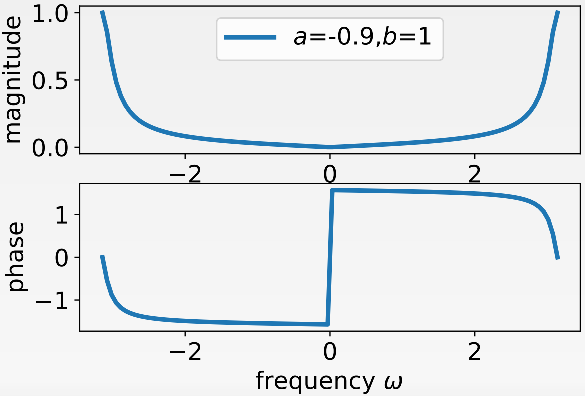


$H_3(z) = G_3 \frac{1-z^{-1}}{1-az^{-1}}$ (zero: $b = 1$)
$G_3 = \frac{1+a}{2} \Rightarrow H_3(\omega = -\pi) = 1$

Frequency response of the corresponding high-pass system

- Frequency response:



Second order systems

System function

$$H(z) = G\frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Second order systems

System function

$$H(z) = G\frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

- Poles
  - if $a_1^2 > 4a_2$: 2 distinct poles (real) at $p_{1,2} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_2}$
  - if $a_1^2 = 4a_2$: double pole (real) at $p_1 = -\frac{a_1}{2}$
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Second order systems

System function

$$H(z) = G\frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

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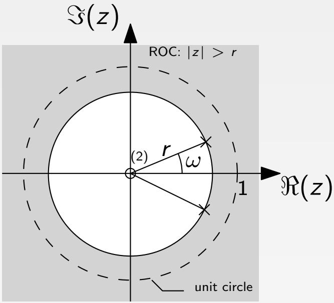
- Practical requirements:
  - causality: ROC: $|z| > \max\{|p_1|, |p_2|\}$
  - stability: $|p_1| < 1$ and $|p_2| < 1 \Leftrightarrow |a_2| < 1$ and $a_2 > |a_1| - 1$

Second order systems: resonator

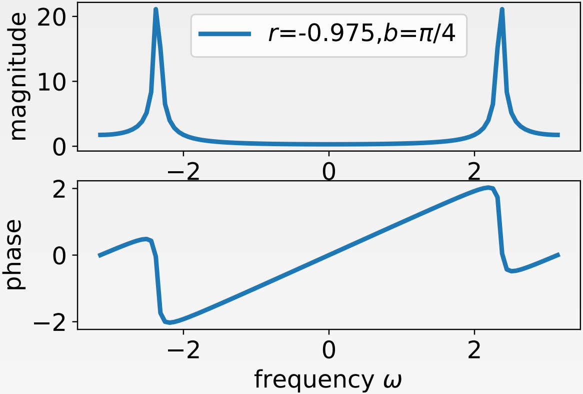


$$p_1 = re^{j\omega_0}$$
$$p_2 = re^{-j\omega_0} = p_1^*$$

$$H(z) = G\frac{1}{\left(1 - re^{j\omega_0 z^{-1}}\right)\left(1 - re^{-j\omega_0 z^{-1}}\right)}$$
$$= G\frac{1}{1 - 2r\cos\left(\omega_0\right)z^{-1} + r^2 z^{-2}}$$

Second order systems: resonator

- Frequency response:



- The frequency response clearly shows peaks around $\pm\omega_0$.
- For r close to 1 (but < 1), $|H(\omega)|$ reaches a maximum at $\pm\omega_0$

FIR filters

System function

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\begin{aligned}
H(z) = B(z) &= b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \\
&= b_0 \left(1 - z_1 z^{-1}\right) \cdots \left(1 - z_M z^{-1}\right)
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- Impulse response:

$$h[n] = \begin{cases} b_n & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

FIR filters: moving average system

- Difference equation:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

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- PZ analysis: roots of the numerator

$$z^M = 1 \Rightarrow z = e^{j2\pi k/M},\ k = 0, 1, \cdots, M-1$$

$\Rightarrow$ there is no pole at $z = 1$ because of PZ cancellation:

$$H(z) = \frac{1}{M}\prod_{k=1}^{M-1}\left(1 - e^{j2\pi k/M} z^{-1}\right)$$

FIR filters: moving average system

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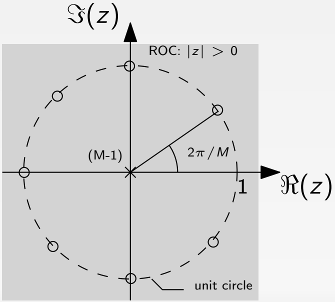
- PZ diagram for $M = 8$



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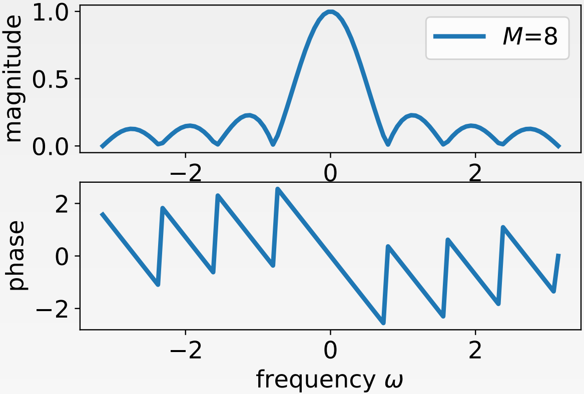

Outline

Introduction

Discrete Time Fourier Transform $\neq$ Discrete Fourier Transform

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Definition of the DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \omega \in [-\pi, \pi]$$

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- the summation over n is infinite
- the variable ω is continuous

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In many situation, it is either not possible, or not necessary to implement the infinite summation:

- only the signal samples $x[n]$ from n to $N-1$ are available
- the signal is known to be zero outside this range; or
- the signal is periodic with period N

Outline

The DFT and its inverse

Definition of the DFT

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\begin{aligned}
X[k] = \mathrm{DFT}_N\{x[n]\} &\triangleq \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N},\ k \in \mathbb{Z} \\
&\triangleq \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n},\ \omega = 2\pi k/N
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- The "D" in DFT stands for **discrete frequency** (i.e. ω_k)

Examples

1. Consider

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1, \ldots, N-1 \end{cases}$$

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case $a = 1$

$$\Rightarrow X[k] = \begin{cases} N & \text{if } k = 0 \\ 0 & \text{if } k = 1, \ldots, N-1 \end{cases}$$

case $a = e^{j2\pi l/N}$

$$\Rightarrow X[k] = \begin{cases} N & \text{if } k = l \text{ modulo } N \\ 0 & \text{otherwise} \end{cases}$$

Inverse DFT (IDFT)

Definition of the IDFT

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  - **$x[n]$ is known to be zero for $n < 0$ and for $n \geq N$**

Outline

7 **Discrete Fourier Transform (DFT)**

- The DFT and its inverse
- Relationship between the DFT and the DTFT
- Properties of the DFT
- Relation between linear and circular convolutions
- The FFT

Introduction

The DFT may be viewed as a finite approximation to the DTFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} \approx X(\omega_k = \frac{2\pi k}{N}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_k n}$$

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- It should not be possible to recover the DTFT exactly from the DFT
  - an arbitrary signal $x[n]$ cannot be recovered entirely from its N-point DFT
- However, in the following two special cases the DTFT can be evaluated exactly at any frequency $\omega \in [-\pi, \pi]$ if the DFT is known:
  - **finite length signals**
  - ***N*-periodic signals**

Finite length signals

Assumption

Suppose $x[n] = 0$ for $n < 0$ and for $n \geq N$

Inverse DFT

In this case $x[n]$ can be recovered entirely from its N-point DFT:

$$\tilde{x}[n] = \text{IDFT}\left\{X[k]\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N},\ n \in \mathbb{Z}$$

- For $n = 0, \ldots, N-1$ the IDFT theorem yields: $x[n] = \tilde{x}[n]$
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$$x[n] = \begin{cases} \tilde{x}[n] & \text{if}\, 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

Finite length signals

Relationship between DFT and DTFT

In this case the DTFT $X(\omega = \omega_k = 2\pi k/N)$ can be completely reconstructed from the N-point DFT $X[k]$:

$$X(\omega_k) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = X[k]$$

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In the general case, i.e. ω arbitrary, the following theorem can be applied.

Theorem

$X(\omega)$ and $X[k]$ respectively denote the DTFT and N-point DFT of signal $x[n]$ ($x[n] = 0$ for for $n < 0$ and for $n \geq N$:

$$X(\omega) = \sum_{k=0}^{N-1} X[k]P(\omega - \omega_k)$$

where

$$P(\omega) \triangleq \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n}$$

Finite length signals

$$X(\omega) = \sum_{k=0}^{N-1} X[k] P(\omega - \omega_k) \text{ with } \mathrm{P}(\omega) \triangleq \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n}$$

Properties of $P(\omega)$:

- The theorem provides a kind of interpolation formula for evaluating $X(\omega)$ in between adjacent values of $X(\omega_k) = X[k]$

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- If $\omega \neq 2\pi l$

$$P(\omega) = \frac{1}{N} \frac{1 - e^{-j\omega N}}{1 - e^{j\omega}} = \frac{1}{N} e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Finite length signals

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- The theorem provides a kind of interpolation formula for evaluating $X(\omega)$ in between adjacent values of $X(\omega_k) = X[k]$
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- If $\omega = 2\pi l \, (l \in \mathbb{Z})$ then $e^{-j\omega n} = e^{-j2\pi l n} = 1$ so that $P(\omega) = 1$
- If $\omega \neq 2\pi l$

$$P(\omega) = \frac{1}{N} \frac{1 - e^{-j\omega N}}{1 - e^{j\omega}} = \frac{1}{N} e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

- Note that at frequency $\omega_k = 2\pi/N$

$$P(\omega_k) = \begin{cases} 1 & k = 0 \\ 0 & k = 1, \ldots, N-1 \end{cases}$$

Periodic signals

Assumption

Suppose $x[n]$ is N-periodic, i.e. $x[n+N] = x[n]$

Inverse DFT

In this case $x[n]$ can be recovered entirely from its N-point DFT:

$$\tilde{x}[n] = \text{IDFT}\{X[k]\} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N},\ n \in \mathbb{Z}$$

- For $n = 0, \ldots, N-1$ the IDFT theorem yields: $x[n] = \tilde{x}[n]$
- Since both $x[n]$ and $\tilde{x}[n]$ are known to be N-periodic, it follows that $x[n] = \tilde{x}[n]$ must also be true for $n < 0$ and for $n \geq N$:

$$x[n] = \tilde{x}[n],\ \forall n \in \mathbb{Z}$$

Periodic signals

Relationship between DFT and DTFT

Since the N-periodic signal $x[n]$ can be recovered completely from its N-point DFT $X[k]$, it should be possible to reconstruct the DTFT $X(\omega)$ from $X[k]$.

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Theorem

$X(\omega)$ and $X[k]$ respectively denote the DTFT and N-point DFT of signal $x[n]$

$$X(\omega) = \frac{2\pi}{N} \sum_{-\infty}^{\infty} X[k]\delta_a(\omega - \omega_k)$$

where $\delta_a(\omega)$ denotes an analog delta function centered at $\omega = 0$

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where $\delta_a(\omega)$ denotes an analog delta function centered at $\omega = 0$

- $X(\omega) \Leftrightarrow$ periodic train of infinite impulses in the ω domain
- When $x[n]$ is N-periodic, the DFT admits a Fourier series interpretation since the IDFT provides an expansion of $x[n]$ as a sum of harmonically related complex exponential signals $e^{j\omega_k n}$:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j\omega_k n}, \; n \in \mathbb{Z}$$

Signal reconstruction via DTFT sampling

1. Let $X(\omega)$ be the DTFT of signal $x[n]$, $n \in \mathbb{Z}$, that is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \; \omega \in \mathbb{R}.$$

2. Consider the sampled values of $X(\omega)$ at uniformly spaced frequencies $\omega_k = 2\pi k/N$ for $k = 0, \ldots, N-1$.
3. Suppose we compute the IDFT of the samples $X(\omega_k)$:

$$\hat{x}[n] = \mathrm{IDFT}\left\{X(\omega_k)\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X(\omega_k)e^{j\omega_k n}$$

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What is the relationship between the original signal $x[n]$ and the reconstructed sequence $\hat{x}[n]$?

- $\hat{x}[n]$ is N-periodic, while $x[n]$ may not be
- Even for $n = 0, \ldots, N-1$ there is no reason for $\hat{x}[n]$ to be equal to $x[n]$

Signal reconstruction via DTFT sampling

Theorem

$$\hat{x} = \mathrm{IDFT}\left\{X(\omega_k)\right\} = \sum_{r=-\infty}^{\infty} x[n - rN]$$

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- Each of these sequences $x[n - rN]$ is a shifted version of $x[n]$ by an integer multiple of N

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  1. **Time limited signal**: suppose $x[n] = 0$ for $n < 0$ and for $n \geq N$. Then there is no temporal overlap of the sequences $x[n-rN]$. We can recover $x[n]$ exactly from one period of $\hat{x}[n]$:

$$x[n] = \begin{cases} \hat{x}[n] & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Signal reconstruction via DTFT sampling

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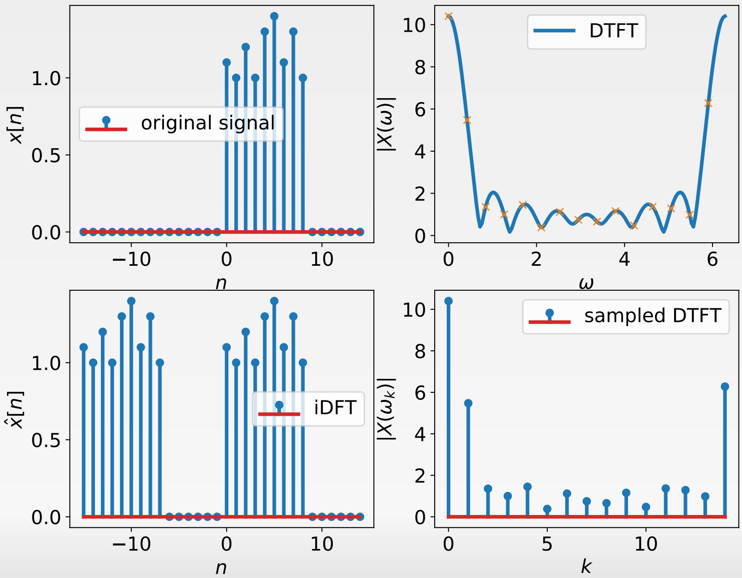
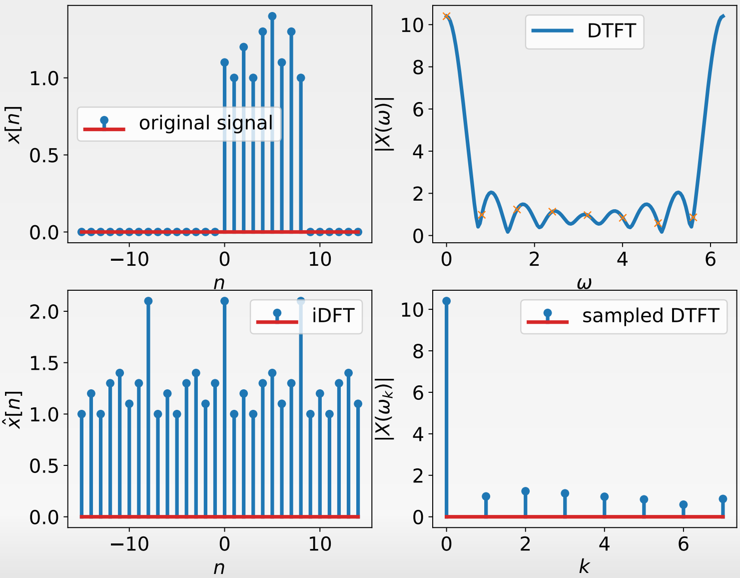
     $$x[n] = \begin{cases} \hat{x}[n] & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

  2. **Non time-limited signal**: suppose that $x[n] \neq 0$ for some $n < 0$ or $n \geq N$. Then, the sequences $x[n - rN]$ for different values of r will overlap in the time-domain. In this case, it is not true that $\hat{x}[n] = x[n]$ for all $0 \leq n \leq N - 1$ $\Rightarrow$ temporal aliasing

Time limited signal



Non time-limited signal



Outline

Properties of the DFT

Notations

$x[n]$ and $y[n]$ are defined over $0 \leq n \leq N-1$:

$$x[n] \overset{\mathrm{DFT}_N}{\leftrightarrow} X[k]$$

$$y[n] \overset{\mathrm{DFT}_N}{\leftrightarrow} Y[k]$$

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Modulo N operation

any integer $n \in \mathbb{Z}$ can be expressed uniquely as $n = k + rN$ where $k \in \{0, \ldots, N-1\}$ and $r \in \mathbb{Z}$:

$$(n)_N = n \text{ modulo N} \triangleq k$$

Time reversal and complex conjugation

Circular time reversal

Given a sequence $x[n]$, $0 \leq n \leq N-1$, its circular reversal (CR) is defined as:

$$\mathrm{CR}\left\{x[n]\right\} = x\left[(-n)_N\right],\ 0 \leq n \leq N-1$$

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n	0	1	2	3	4	5
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$(-n)_6$	0	5	4	3	2	1
$x\left[(-n)_6\right]$	6	1	2	3	4	5

Time reversal and complex conjugation

Interpretation:

- Circular reversal can be seen as an operation on the set of samples $x[0], \dots, x[N-1]$:
  - $x[0]$ is left unchanged
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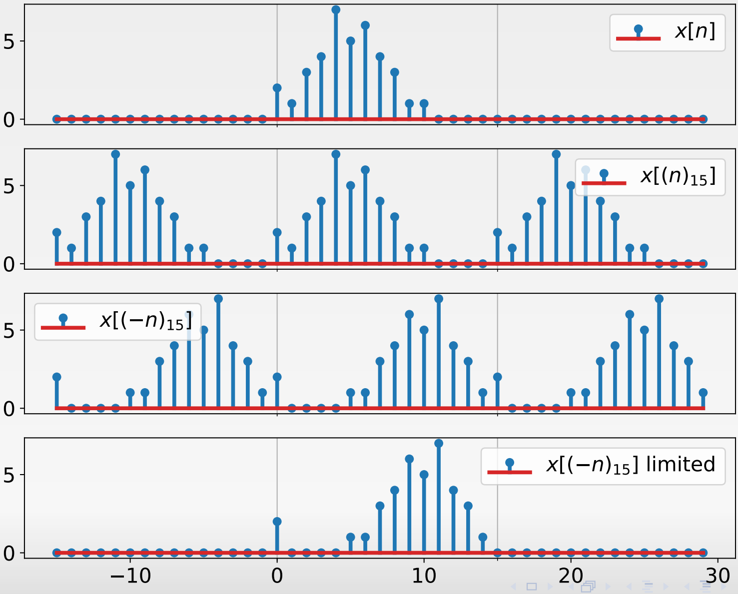
- One can also see this operation consisting in:
  1. periodizing the samples of $x[n]$, $0 \leq n \leq N-1$ with period N
  2. time-reversing the periodized sequence
  3. keeping only the samples between 0 and $N-1$

Time reversal and complex conjugation



Time reversal and complex conjugation

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$$x\left[(-n)_N\right] \overset{\mathrm{DFT}_N}{\leftrightarrow} X[-k]$$

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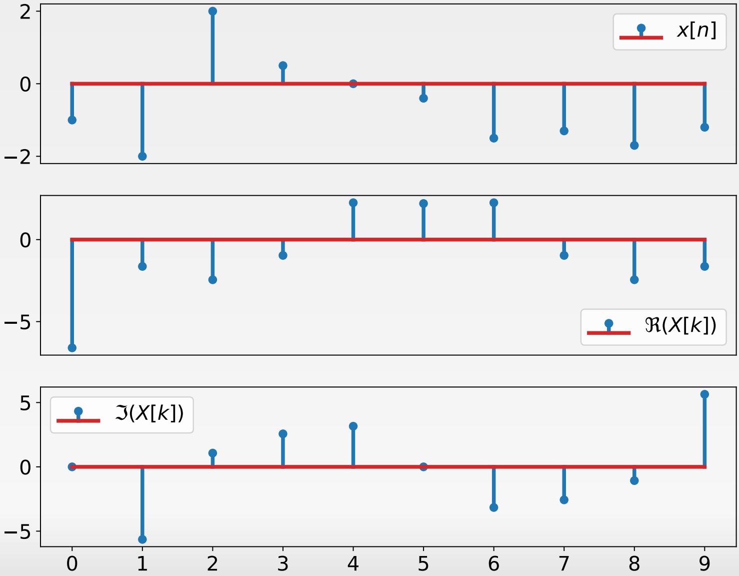
  - $X[N-k] = X^*[k]$ for $1 \leq k \leq N-1$

Time reversal and complex conjugation



Linearity

Linearity

$$ax[n] + by[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} aX[k] + bY[k]$$

Even and odd decomposition

Conjugate symmetric components of finite sequences

$$x_{e,N}[n] \triangleq \frac{1}{2}\left(x[n] + x^*\left[(-n)_N\right]\right)$$

$$x_{e,N}[n] \triangleq \frac{1}{2}\left(x[n] - x^*\left[(-n)_N\right]\right)$$

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$$\Re\{x[n]\} \overset{\mathrm{DFT}_N}{\longleftrightarrow} X_e[k]$$

$$j\Im\{x[n]\} \overset{\mathrm{DFT}_N}{\longleftrightarrow} X_o[k]$$

$$x_{e,N}[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} \Re\{X[k]\}$$

$$x_{o,N}[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} j\Im\{X[k]\}$$

Circular shift

Definition

Given a sequence $x[n]$ defined over the interval $0 \leq n \leq N-1$, we define its circular shift by k as follows:

$$\mathrm{CS}_k\left\{x[n]\right\} = x\left[(n-k)_N\right], \; 0 \leq n \leq N-1$$

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n	0	1	2	3	4	5
$x[n]$	6	5	4	3	2	1
$(n-2)_6$	4	5	0	1	2	3
$x\left[(n-2)_6\right]$	2	1	6	5	4	3

Circular shift

Interpretation:

- Can be seen as an operation on the set of signal samples $x[n]$ in which:
  - signal samples $x[n]$ are shifted as in a conventional shift
  - any signal sample leaving the interval $0 \leq n \leq N-1$ from one end reenters by the other end

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- Alternatively, it may be interpreted as follows:
  1. periodizing the samples of $x[n]$, $0 \leq n \leq N-1$ with period N
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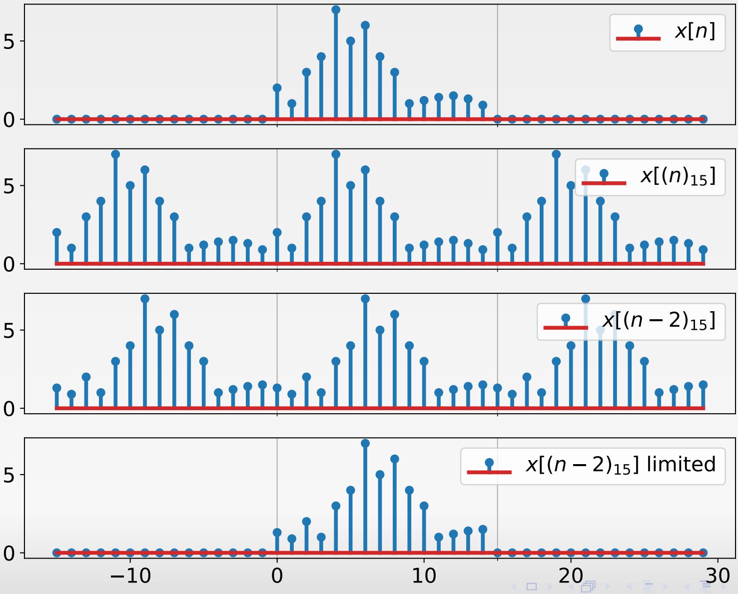
Circular shift



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Circular shift property

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Frequency shift property

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- Since the DFT $X[k]$ is already periodic, the modulo N operation is not needed here, that is: $X\left[(k-m)-N\right] = X[k-m]$.

Circular convolution

Definition

Let $x[n]$ and $y[n]$ be 2 sequences defined over $0 \leq n \leq N-1$:

$$x[n] \circledast y[n] \triangleq \sum_{m=0}^{N-1} x[m] y\left[(n-m)_N\right], \; 0 \leq n \leq N-1$$

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Multiplication Property

$$x[n]y[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} \frac{1}{N} X[k] \circledast Y[k]$$

Other properties

Plancherel's relation

$$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k]$$

Parseval's relation

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

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Plancherel's relation

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Parseval's relation

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

- Parseval's relation is a special case of Plancherel's relation: with $y[n] = x[n]$
- It allows the computation of the energy of the signal samples $x[n]\,(n = 0, \ldots, N-1)$ directly from the DFT samples $X[k]$

Outline

7 **Discrete Fourier Transform (DFT)**

- The DFT and its inverse
- Relationship between the DFT and the DTFT
- Properties of the DFT
- Relation between linear and circular convolutions
- The FFT

Introduction

Linear convolution

Time domain expression:

$$y_l[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k], \; n \in \mathbb{Z}$$

Frequency domain representation via DTFT:

$$Y_l(\omega) = X_1(\omega)X_2(\omega), \; \omega \in [0, 2\pi]$$

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Circular convolution

Time domain expression:

$$y_c[n] = x_1[n] \circledast x_2[n] = \sum_{k=0}^{N-1} x_1[k]x_2\left[(n-k)_N\right],\ 0 \le n \le N-1$$

Frequency domain representation via N-point DFT

$$Y_c[k] = X_1[k]X_2[k],\ k \in \{0, \ldots N-1\}$$

A necessary condition...

Circular convolution and linear convolution are equivalent if:

$$y_l[n] = \begin{cases} y_c[n] & \text{if } 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$true if signals $x_1[n]$ and $x_2[n]$ have both finite length.

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Finite length assumption

Suppose that $x_1[n]$ and $x_2[n]$ are time limited to $0 \leq n < N_1$ and $0 \leq n < N_2$ respectively then the linear convolution is time limited to $0 \leq n < N_1 + N_2 - 1$

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$\Rightarrow$true if signals $x_1[n]$ and $x_2[n]$ have both finite length.

Finite length assumption

Suppose that $x_1[n]$ and $x_2[n]$ are time limited to $0 \leq n < N_1$ and $0 \leq n < N_2$ respectively then the linear convolution is time limited to $0 \leq n < N_1 + N_2 - 1$

Example: Consider $x_1[n] = \{\underline{1}, 1, 1, 1\}$ and $x_2[n] = \{\underline{1}, 1/2, 1/2\}$

- $N_1 = 4$ and $N_2 = 3$
- $y_l[n] = \{1, 1.5, 2, 2, 1, .5\}$
- $\Rightarrow$ $N_3 = 6 = N_1 + N_2 - 1$

...proved to be a sufficient condition

Assuming $N \geq N_1 + N_2 - 1$:

1. Linear convolution gives:

$$y_l[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] = \sum_{k=0}^{n} x_1[k]x_2[n-k],\ 0 \leq n < N$$

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2. Circular convolution gives:

$$\begin{aligned} y_c[n] &= \sum_{k=0}^{N-1} x_1[k]x_2\left[(n-k)_N\right],\ 0 \leq n < N \\ &= \sum_{k=0}^{n} x_1[k]x_2[n-k] + \underbrace{\sum_{k=n+1}^{N-1} x_1[k]x_2[N+n-k]}_{=0} \end{aligned}$$

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Conclusion

The linear and circular convolution are equivalent if and only if:

$$N \geq N_1 + N_2 - 1$$

Relationship between $y_c[n]$ and $y_l[n]$

Assuming that $N \geq \max\{N_1, N_2\}$ the DFT of the 2 sequences $x_1[n]$ and $x_2[n]$ are samples of the corresponding DTFT:

$$N \geq N_1 \Rightarrow X_1[k] = X_1(\omega_k),\ \omega_k = 2\pi k/N$$
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$\Rightarrow$ The circular convolution $y_c[n]$ can be computed as the N-point IDFT of these frequency samples:

$$y_c[n] = \mathrm{IDFT}_N\{Y_c[k]\} = IDFT_N\{Y_l(\omega_k)\}$$

Relationship between $y_c[n]$ and $y_l[n]$

Applying the "signal reconstruction via DTFT sampling" theorem we obtain:

$$y_c[n] = \sum_{r=-\infty}^{\infty} y_l[n - rN],\ 0 \leq n < N$$

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To get $y_c[n] = y_l[n]$ for $0 \leq n < N$, temporal aliasing must be avoided: the length of DFT$\geq$length of $y_l[n]$, i.e. :

$$N \geq N_1 + N_2 - 1$$

Linear convolution via DFT

Linear convolution via DFT can be summarized according to the following steps:

- Suppose that $x_1[n]$ and $x_2[n]$ are time limited to $0 \leq n < N_1$ and $0 \leq n < N_2$ respectively

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$$X_1[k] = \mathrm{DFT}_N\left\{x_1[n]\right\},\, 0 \leq k < N$$
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- Compute the IDTF:

$$x_1[n] * x_2[n] = \begin{cases} \mathrm{IDFT}_N\left\{X_1[k]X_2[k]\right\} & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

Outline

The FFT

Recall of the DFT

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$\Rightarrow$ algorithm complexity $= N^2$

The FFT

Fast Fourier Transform

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- Assuming $N = 2^k$
- Considering even and odd part of the signals the DFT_N is split into 2 $\mathrm{DFT}_{N/2}$
- The FFT leads to:
  - $\frac{N}{2} \log_2 N$ complex multiplications
  - $N \log_2 N$ complex additions
- The algorithm complexity becomes $\frac{N}{2} \log_2 N$