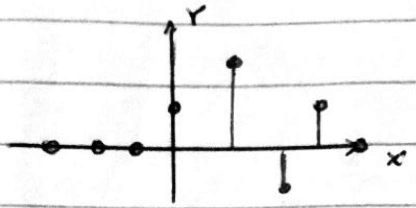
Discrete time signal processing

TDsl - Discrete time signal

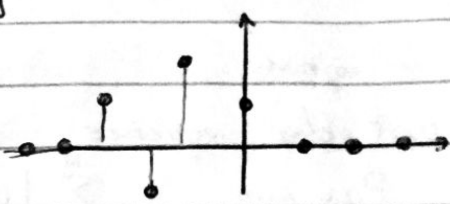
1. Let $x[n]$ be the DT signal represented by the sequence $\{\dots, 0, \underline{1}, 2, -1, 1, 0, \dots\}$

Draw:

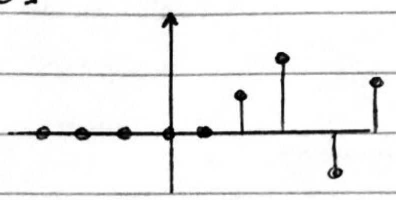
a) $x[n]$



b) $x[-n]$

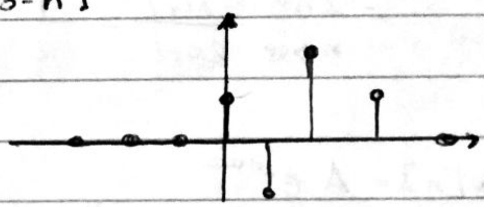


c) $x[n-2]$



$y[2] = x[2-2] = x[0]$

d) $x[3-n]$

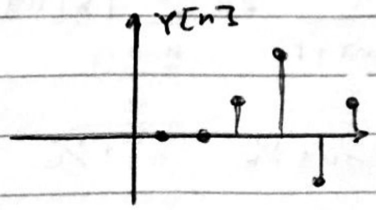


e) $$x[n] * \delta[n-3] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k-3] = \sum_{k=-\infty}^{\infty} x[n-k]\,\delta[k-3]$$

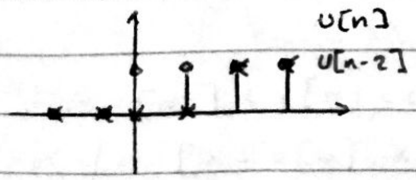
$$= \sum_{k'=-\infty}^{\infty} x[n-k'-3]\,\delta[k'] = x[n-3]$$

$k' = k-3$

It only exist when $k' = 0$ so it is not necessary to calculate for other values of k' where the value will be $\emptyset$ for δ



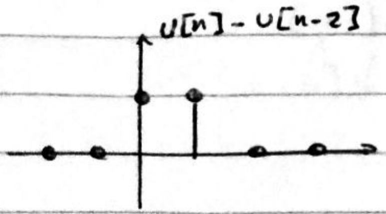
f) $x[n] * (u[n] - u[n-2]) = y[n]$

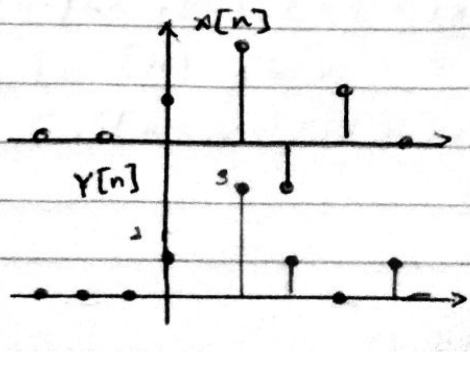


$$y[n] = \sum_{k=-\infty}^{\infty} (u[k] - u[k-2]) \cdot x[n-k]$$

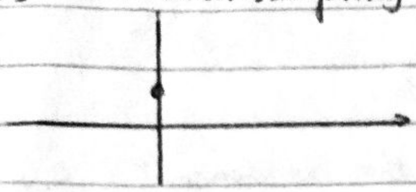
This term will be 1 just for $k=0$ and $k=1$

$$y[n] = \sum_{k=0}^{1} x[n-k] = x[n] + x[n-1]$$





g) $x[2n] \Rightarrow$ downsampling



② Power signals

a) Unit step sequence

$$P_U = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |u[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1$$

$$= \lim_{n\to\infty} \frac{N+1}{2N+1} = \frac{1}{2}$$

b) $x[n] = A\, e^{jn w_0}$

because its the modulus

$$P_x = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} A^2 = \lim_{n\to\infty} \frac{A^2(2N+1)}{2N+1} = A^2$$

c) $x[n]$ is a periodic signal with T as the period

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \qquad N = KT$$

$$P_x = \lim_{K\to\infty} \frac{1}{2KT+1} \sum_{n=-KT}^{KT} |x[n]|^2 = \lim_{k\to\infty} \frac{1}{2kT+1}\, 2k \sum_{n=0}^{T} |x[n]|^2$$

$$\frac{2k/k}{2kT/k + 1/k} = \frac{2}{2T + 1/k} = \frac{1}{T}$$

$$P_x = \frac{1}{T} \sum_{n=0}^{T} |x[n]|^2$$

③ Decompose $x[n]$ as the sum of an even and odd signal

$$x[n] = x_e[n] + x_o[n]$$

$$x_e[n] = x_e[-n] = \tfrac{1}{2}\left(x[n] + x[-n]\right)$$

$$x_o[n] = -x_o[-n] = \tfrac{1}{2}\left(x[n] - x[-n]\right)$$

$$x[n] + x[-n] = x_e[n] + x_e[-n] + x_o[n] + x_o[-n]$$

$$= 2x_e[n]$$

$= -x_o[-n]$

$$x[n] - x[-n] = 2x_o[n]$$

TD2.

① Property of the cross correlation

a) Compute the energy of $x[n] + b y[n-k]$

$$\mathcal{E} = \sum_{l=-\infty}^{\infty} (x[l] + b y[l-k])^2 = \sum_{l=-\infty}^{\infty} x^2[l] + b^2 y^2[l-k] + 2b\, x[l] y[l-k]$$

$$= \sum_{l=-\infty}^{\infty} x^2[l] + b^2 \sum_{l=-\infty}^{\infty} y^2[l-k] + 2b \sum_{l=-\infty}^{\infty} x[l] y[l-k]$$

$m = l-k$

$$= R_{xx}[0] + b^2 R_{yy}[0] + 2b\, R_{xy}[k] \geq 0$$

b) $P(b) = R_{yy}[0] \cdot b^2 + 2 R_{xy}[k] * b + R_{xx}[0]$

Δ = discriminant = $4 R_{xy}^2[k] - 4 R_{yy}[0] R_{xx}[0] \leq 0$ → It has to be less than zero to have no roots and make the function ≥ 0

$b^2 - 4ac$

$$\boxed{R_{xy}^2[k] \leq R_{xx}[0] R_{yy}[0]}$$

↳ Conclusion

② Determine the autocorrelation of:

a) $\delta[n]$

$$R_{\delta\delta}[n] = \sum_{k=-\infty}^{\infty} \delta[k]\, \delta[k-n] = \delta[n] = \delta[-n]$$

b) $u[n]$

$$R_{uu}[n] = \sum_{k=-\infty}^{\infty} u[k]\, u[k-n] = \sum_{k=0}^{\infty} u[k-n]$$

$m = k-n$

$$= \sum_{m=-n}^{\infty} u[m] = \sum_{l=0}^{\infty} 1 \longrightarrow +\infty$$

③ Determine the autocorrelation $z[n] = x[n] + y[n]$

$$R_{zz}[n] = \sum_{k=-\infty}^{\infty} (x[k] + y[k])(x[k-n] + y[k-n])$$

$$= \sum_{-\infty}^{\infty} x[k] x[k-n] + \sum_{-\infty}^{\infty} x[k] y[k-n] + \sum_{-\infty}^{\infty} y[k] x[k-n] + \sum_{-\infty}^{\infty} y[k] y[k-n]$$

$$= R_{xx}[n] + R_{xy}[n] + R_{yx}[n] + R_{yy}[n]$$

$$= R_{xx}[n] + R_{xy}[n] + R_{xy}[-n] + R_{yy}[n]$$

⑤ $y[n] = x[n-k]$

$$R_{yy}[n] = \sum_{l=-\infty}^{\infty} y[l]\, y[l-n] = \sum_{l=-\infty}^{\infty} x[l-k]\, x[l-k-n]$$

$$\underset{m=l-k}{=} \sum_{m=-\infty}^{\infty} x[m]\, x[m-n] = R_{xx}[n]$$

b) $x[n]$ and $y[n] = x[n-k]$

$$R_{xy}[n] = \sum_{l=-\infty}^{\infty} x[l]\, y[l-k] = \sum_{m=-\infty}^{\infty} x[m+(n)]\, y[m]$$ similar

$$\underset{m=l-n}{=} \sum_{m=-\infty}^{\infty} x[m+n]\ [m-k] = \sum_{\substack{l=-\infty \\ l=m-k}}^{\infty} x[l+(k+n)]\, x[l]$$

$$= R_{xx}[k+n]$$

TD3

1. Stability study

$$y[n] = x[n] - y[n+1]$$

a) Express $y[n]$ as a summation of $x[n-k]$ terms

$$y[n+1] = x[n] - y[n]$$

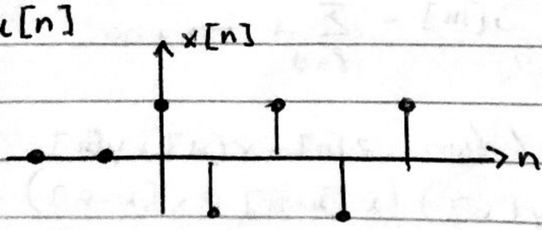
$$y[n] = x[n-1] - y[n-1] \qquad y[n-1] = x[n-2] - y[n-2]$$

$$y[n] = x[n-1] - x[n-2] + y[n-2] \qquad y[n-2] = x[n-3] - y[n-3]$$

$$y[n] = x[n-1] - x[n-2] + x[n-3] - x[n-4] \ldots$$

$$y[n] = \sum_{k=1}^{\infty} (-1)^{k+1} x[n-k]$$

b) $x[n] = (-1)^n u[n]$



x is bounded, since $\forall n$, $|x[n]| \leq 1$

$$y[n] = \sum_{k=1}^{\infty} (-1)^{k+1} (-1)^{n-k} u[n-k] = (-1)^{1+n} \sum_{k=1}^{\infty} u[n-k]$$

$$y[n] = \left| \sum_{k=1}^{\infty} u[n-k] \right| \Rightarrow$$ not bounded system not stable

② LTI systems and convolution

$x[n] \rightarrow \boxed{H} \rightarrow y[n] = h[n] * x[n]$

$h[n]$

$x[n] = \{\underline{1}, 1, 1, 1, 1\}$ an input signal

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

a) $h[n] = \{1, \underline{2}, 1\}$

$h[-n] = \{1, 2, 1\}$

$x[n] = \{0, 0, 0, 1, 1, 1, 1, 1, 0, 0\}$

$[1 \; 2 \; 1]$

$y[n] = \{0 \; 0 \; 1 \; \underline{3} \; 4 \; 4 \; 4 \; 3 \; 1 \; 0\}$

b) $h[n] = \{1, \underline{2}, -1\}$

$h[-n] = \{-1 \; \underline{2} \; 1\}$

$y[n] = \{0 \; 1 \; 3 \; 2 \; 2 \; 2 \; 1 \; -1 \; 0\}$

③ Autocorrelation of LTI systems

a) Prove that por LTI system described by the impulse response $h[n]$

$$R_{yy}[n] = R_{hh}[n] * R_{xx}[n]$$

$$R_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]\, x[k-n] = x[n] * x[-n]$$

$$R_{yy}[n] = \sum_{k=-\infty}^{\infty} y[k]\, y[k-n] = y[n] * y[-n] = \big(h[n] * x[n]\big) * \big(h[-n] * x[-n]\big)$$

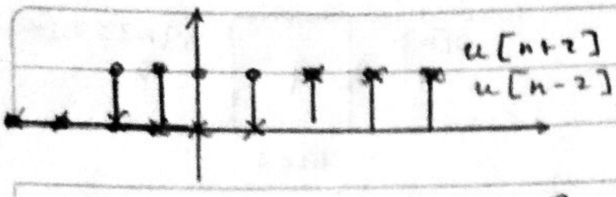
$$= h[n] * h[-n] * x[n] * x[-n]$$

$$= R_{hh}[n] * R_{xx}[n]$$

b) $h[n] = \{\underline{1}, -1\}$, compute the autocorrelation of the system

$R_{hh}[n] = \{-1 \;\; \underline{2} \;\; -1\}$

c) Determine the autocorrelation of $x[n] = u[n+2] - u[n-2]$



$x[n] = \{0\ 0\ 1\ 1\ \underline{1}\ 1\ 0\ 0\}$

$x[n] \to [1, 1, \underline{1}, 1]$

$R_{xx}[n] = \{1\ 2\ 3\ \underline{4}\ 3\ 2\ 1\} \triangleq R_{xx}(0)$

d) Compute the autocorrelation of $y[n]$

$R_{yy}[n] = R_{hh}[n] * R_{xx}[n]$

$R_{xx}[n] = \{1\ 2\ 3\ \underline{4}\ 3\ 2\ 1\}$

$R_{hh}[-n] = \{-1\ \underline{2}\ -1\}$

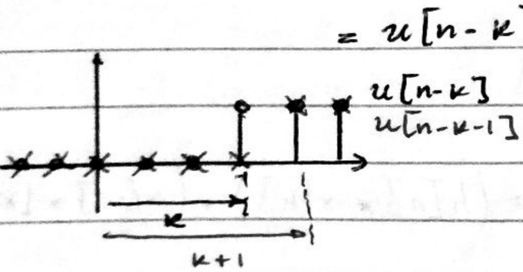
$R_{yy}[n] = \{-1\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ -1\}$

e) $h[n] * u[n-k] = \sum_{l=-\infty}^{\infty} (\delta[l] - \delta[l-1])\, u[n-k-l]$

$h[n] = \delta[n] - \delta[n-1]$

$= \sum_{l=-\infty}^{\infty} \delta[l]\, u[n-k-l] - \sum_{l=-\infty}^{\infty} \delta[l-1]\, u[n-k-l]$

$= u[n-k] - u[n-k-1] = \delta[n-k]$



$y[n] = h[n] * x[n]$

$= h[n] * (u[n+2] - u[n-2]) = h[n] * u[n+2] - h[n] * u[n-2]$

$y[n] = \delta[n+2] - \delta[n-2]$

$y[n] = \{1\ 0\ \underline{0}\ 0\ -1\}$

$y[n] = [1\ 0\ \underline{0}\ 0\ -1]$

$R_{yy}[n] = \{-1\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ -1\}$

TD4

① Determine the z-transform

a) $x_1[n] = \{\underline{1}, 2, 5, 7, 0, 1\}$ $\qquad TZ\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$x_1[n] = \delta[n] + 2\delta[n-1] + 5\delta[n-2] + 7\delta[n-3] + \delta[n-5]$

$X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

ROC: Complex plane $- \{0\}$

↳ because $\frac{1}{z} = \infty$ when $z = 0$

b) $x_2[n] = \{1, 2, \underline{5}, 7, 0, 1\}$

$x_2[n] = x_1[n+2]$

$X_2[z] = z^2 X_1(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$

ROC: Complex plane $- \{0, \infty\}$

c) $x_3[n] = \{1, 2, 5, 7, 0, \underline{1}\}$

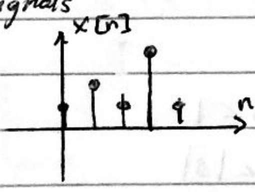
$x_3[n] = x_1[n+5]$

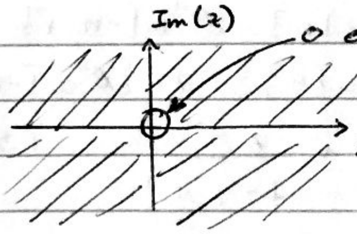
$X_3(z) = z^5 X_1(z) = z^5 + 2z^4 + 5z^3 + 7z^2 + 1$

ROC: Complex plane $- \{\infty\}$

Finite duration signals:

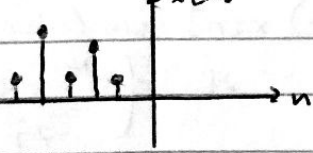
- Causal signals

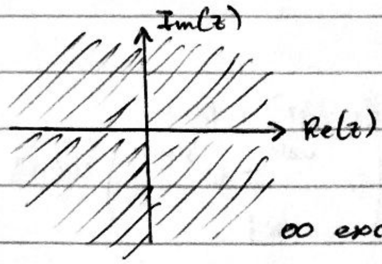




0 excluded

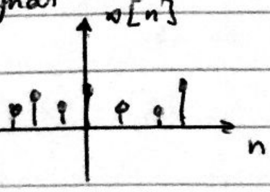
- Anti-causal signal

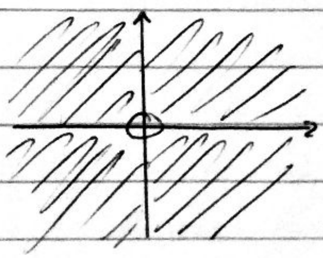




∞ excluded

- Signal





0 and ∞ excluded

d) $X_4[z] = 1$ $\quad$ ROC = complex plane

e) $X_5[z] = z^{-k}$ $\quad$ ROC $= \mathbb{C} - \{0\}$

f) $X_6(z) = z^{k}$ $\quad$ ROC $= \mathbb{C} - \{\infty\}$

② Determine the z-transform

a) $x[n] = \alpha^n u[n]$ with ROC:

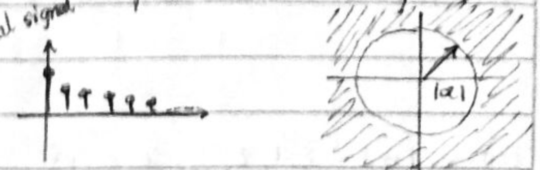
$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=-\infty}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}} \qquad |\alpha z^{-1}| < 1$$

$$\rightarrow |z| > |\alpha|$$

b) $x[n] = -\alpha^n u[n-1]$

Infinite duration signals

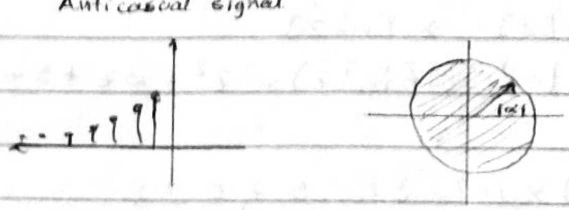
casual signal



$$X(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = \sum_{n=1}^{\infty} -(z\alpha^{-1})^n$$

$$= -\sum_{n=0}^{\infty} (z\alpha^{-1})^n + 1$$

Anticasual signal



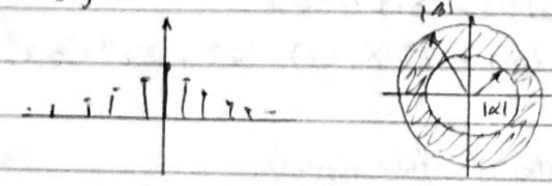
$$= -\frac{1}{1-\alpha^{-1}z} + 1 = \frac{-1+1-z\alpha^{-1}}{1-z\alpha^{-1}}$$

$$X(z) = \frac{-z\alpha^{-1}}{1-z\alpha^{-1}} \times \frac{\alpha z^{-1}}{\alpha z^{-1}} \text{ with ROC}$$

$$X(z) = \frac{-1}{\alpha z^{-1}-1} = \frac{1}{1-\alpha z^{-1}} \qquad |z\alpha^{-1}| < 1$$

$$|z| < |\alpha|$$

signal



c) $x[n] = \alpha^n u[n] + \beta^n u[-n-1]$

$$X(z) = ZT\{\alpha^n u[n]\} + ZT\{\beta^n u[-n-1]\}$$

$$X(z) = \frac{1}{1-\alpha z^{-1}} + \frac{z\beta^{-1}}{1-z\beta^{-1}} \qquad \text{ROC: } |\alpha| < |z| < |\beta|$$

with $|\alpha| < |\beta|$

③

$$X(z) = \frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}} \qquad \text{ROC } |z| > 1$$

a) $x[n] = \cos(\omega_0 n) u[n]$

$$x[n] = \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right) u[n]$$

b) $x[n] = \sin(\omega_0 n) u[n]$

$$x[n] = \left(\frac{e^{j\omega_0 n} - e^{j\omega_0 n}}{2j}\right) u[n]$$

Inverse

$$x(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} \qquad |a|<|z|<|b|$$

$$x(z) = \frac{A_1}{(1-az^{-1})} + \frac{A_2}{(1-bz^{-1})}$$

If we

$$x(z)(1-az^{-1})\Big|_{z=a} \equiv A_1 = \frac{1}{1-bz^{-1}}\Big|_{z=a} = \frac{1}{1-b/a} = \frac{a}{a-b}$$

$$x(z)(1-bz^{-1})\Big|_{z=b} \equiv A_2 = \frac{1}{1-az^{-1}}\Big|_{z=b} = \frac{1}{1-a/b} = \frac{b}{b-a}$$

4) $x(z) = \dfrac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$

5) *We assume we have the causal signal $x[n]$*

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} = \frac{A_1}{1+z^{-1}} + \frac{A_{21}}{1-z^{-1}} + \frac{A_{22}}{(1-z^{-1})^2}$$

$$x(z)(1+z^{-1})\Big|_{z=-1} = A_1 = \frac{1}{(1-z^{-1})^2}\Big|_{z=-1} = 1/4$$

$P_k = 1$, $L_k = 2$, $\ell = 2$

$$A_{22} \stackrel{\downarrow}{=} \frac{1}{(L_k-1)!(-P_k)^{L_k-1}} = \frac{1}{0!(-1)^0}(1-z^{-1})^2 x(z)\Big|_{z=1} = \frac{1}{(1+z^{-1})}\Big|_{z=1} = 1/2$$

$P_k = 1$, $L_k = 2$, $\ell = 1$

$$A_{21} = \frac{1}{1!(-1)^{2-1}}\Big\{\underbrace{\frac{d}{dz^{-1}}(1-\overset{1}{\cancel{p_k}}z^{-1})^2 x(z)}_{Qz^{-1}}\Big\}\Big|_{z=1} = 1/4$$

$$Q(z^{-1}) = Q(x) = \frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2}$$

$$Q(z^{-1}=1) = Q(x=1) = -1/4$$

Or for A_{21}

$x(0) = 1 = A_1 + A_{21} + A_{22}$

$z^{-1} = 0$
$z = \infty$

$$A_{21} = 1 - A_1 - A_{22} = 1 - 1/4 - 1/2 = 1/4$$

• Causal signal $\Rightarrow |z| > 1$

$\frac{n!}{k!(n-k)!}$

$$x[n] = (-1)^n u[n] * 1/4 + 1/4 \binom{n}{0} 1^n u[n] + 1/2 \binom{n+1}{1} 1^n u[n]$$

$$x[n] = \left(\frac{(-1)^n}{4} + \frac{1}{4} + \frac{1}{2} \frac{(n+1)!}{1!(n)!} \right) u[n]$$

$$x[n] = \left(\frac{(-1)^n}{4} + \frac{1}{4} + \frac{n+1}{2} \right) u[n]$$

$\mathcal{DF}$

Discrete aperiodic $x[n]$ → continuous 2π periodic $X(\omega)$

Discrete → Periodic

Periodic → Discrete

Time domain — Frecuency

$\mathcal{F}$

Aperiodic Continuous $\xrightarrow{\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}$ Aperiodic Continuous

Periodic Continuos $\xrightarrow{f(t) = \sum_{-\infty}^{\infty} C_n e^{j\omega t}}$ Discrete Aperiodic

$\mathcal{DF}$ DTFT

Discrete Aperiodic $\xrightarrow{\sum_{-\infty}^{\infty} x[n] e^{-j\omega n}}$ Continuous Periodic

Discrete Periodic $\xrightarrow{DFT}$ Periodic Discrete

Determine the step response of the system with IC $y[-1]=1$
$y[n]=\alpha y[n-1]+x[n]$ with $-1<\alpha<1$

$$\begin{cases} y[n]=\alpha y[n-1]+x[n] \\ y[-1]=1 \end{cases}$$

$$x[n-k] \xleftrightarrow{ZT^+} z^{-k}\left(X^+(z)+\sum_{n=1}^{k} x[-n]z^n\right)$$

$$Y^+(z)=\alpha z^{-1}\left(Y^+(z)+y[-1]z\right)+X^+(z)$$

$$Y^+(z)=\alpha z^{-1}Y^+(z)+\alpha z^{-1}z+X^+(z)$$

$$Y^+(z)(1-\alpha z^{-1})=\alpha+X^+(z)$$

$x[n]=u[n]$

$$X^+(z) \underset{\text{causal}}{=} X(z)=\frac{1}{1-z^{-1}}$$

$$Y^+(z)=\frac{\alpha}{1-\alpha z^{-1}}+\underbrace{\frac{1}{(1-\alpha z^{-1})(1-z^{-1})}}_{R(z^{-1})}=\frac{\alpha}{1-\alpha z^{-1}}+\underbrace{\left(\frac{A_1}{1-\alpha z^{-1}}+\frac{A_2}{1-z^{-1}}\right)}_{R(z^{-1})}$$

$$R(z^{-1})(1-\alpha z^{-1})\Big|_{z=\alpha}=A_1=\frac{1}{1-\alpha^{-1}}=\frac{\alpha}{\alpha-1}$$

$$R(z^{-1})(1-z^{-1})\Big|_{z=1}=A_2=\frac{1}{1-\alpha}$$

$$y[n]=\alpha\,\alpha^n u[n]+\frac{\alpha}{\alpha-1}\alpha^n u[n]+\frac{1}{1-\alpha}u[n]$$

$$=\alpha^{n+1}u[n]+\frac{\alpha^{n+1}-1}{\alpha-1}u[n]$$

- Response with non zero initial condition

$$\sum_{k=0}^{N} a_k y[n-k]=\sum_{k=0}^{M} b_k x[n-k]$$

$$a_0 y[n]+\sum_{k=1}^{N} a_k y[n-k]=\sum_{k=0}^{M} b_k x[n-k]$$

$$y[n]=-\sum_{k=1}^{N}\frac{a_k}{a_0}y[n-k]+\sum_{k=0}^{M}\frac{b_k}{a_0}x[n-k]$$

$$Y^+(z)=-\sum_{k=1}^{N}a_k z^{-k}\left(Y^+(z)+\sum_{n=1}^{k}y[-n]z^n\right)+\sum_{k=0}^{M}b_k z^{-k}X^+(z)$$

$$\left(\sum_{k=1}^{N}a_k z^{-k}+1\right)Y^+(z)=-\sum_{k=1}^{N}a_k z^{-k}\sum_{n=1}^{N}y[-n]z^n+\sum_{k=0}^{N}b_k z^{-k}X^+(z)$$

TD5.

a) $X(z) = \dfrac{z^{-6} + z^{-7}}{1 - z^{-1}}$ Determine the causal signal

$$X(z) = (z^{-6} + z^{-7}) \cdot \frac{1}{1 - z^{-1}} \leadsto z^{-6} Y(z) + z^{-7} Y(z)$$
$$= y[n-6] + y[n-7]$$

$$Y(z) = \frac{1}{1 - z^{-1}} \leadsto y[n] = u[n]$$
by formula

$x[n] = u[n-6] + u[n-7] \rightarrow$ Time shift property

b) $X(z) = \dfrac{1 + 2z^{-2}}{1 + z^{-2}}$

$$X(z) = 1 + 2z^{-2} \cdot \left(\frac{1}{1+z^{-2}}\right) = (1 + z^{-2} + z^{-2})\left(\frac{1}{1+z^{-2}}\right)$$

$$X(z) = \frac{1+z^{-2}}{1+z^{-2}} + \frac{z^{-2}}{1+z^{-2}} = 1 + \frac{z^{-2}}{1+z^{-2}}$$

$$\begin{array}{r|l} 2z^{-2}+1 & 1+z^{-2} \\ \hline -2z^{-2} & 2 \\ \hline -1 & \end{array}$$

or $X(z) = 2 - \dfrac{1}{1+z^{-2}} \leadsto Y(z) = \dfrac{1}{1+z^{-2}} = \dfrac{a}{1+jz^{-1}} + \dfrac{b}{1-jz^{-1}}$

$$= \frac{1}{(1+jz^{-1})(1-jz^{-1})}$$

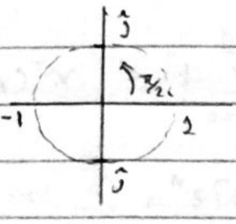
Then $a = Y(z) \cdot (1 + jz^{-1})\Big|_{z=-j} = \dfrac{1}{2}$ $\qquad \cos(\theta) = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

$b = Y(z) \cdot (1 - jz^{-1})\Big|_{z=j} = \dfrac{1}{2}$ $\qquad \sin(\theta) = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$Y(z) = \frac{1}{2}\left(\frac{1}{1+jz^{-1}} + \frac{1}{1-jz^{-1}}\right) \rightarrow y[n] = \frac{1}{2}\left((-j)^n u[n] + j^n u[n]\right)$$

$j = e^{j\pi/2}$

$-j = e^{-j\pi/2}$



$$y[n] = \tfrac{1}{2}\left(e^{j\pi/2 n} + e^{j\pi/2 n}\right) u[n]$$

$$y[n] = \cos(\pi/2\, n)\, u[n]$$

$$x[n] = 2\delta[n] - \cos(\pi/2\, n)\, u[n]$$

② Use the one-sided ZT to determine $y[n]$, $n>0$

a) $y[n] - \frac{3}{2}y[n-1] + \frac{1}{2}y[n-2] = 0$

$y[-1]=1, \quad y[-2]=0$

ZT^+

$$Y^+(z) - \frac{3}{2}z^{-1}\left(Y^+(z) + \cancel{y[-1]}^{\,1}z\right) + \frac{1}{2}z^{-2}\left(Y^+[z] + \cancel{y[-1]}^{\,1}z + \cancel{y[-2]}^{\,0}z^2\right) = 0$$

$$Y^+(z)\left[-\frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} + 1\right] - \frac{3}{2} + \frac{1}{2}z^{-1} = 0$$

$$Y^+(z) = \frac{\frac{3}{2} - \frac{1}{2}z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{\frac{3}{2} - \frac{1}{2}z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{a}{1-z^{-1}} + \frac{b}{1-\frac{1}{2}z^{-1}}$$

$$Y^+(z)(1-z^{-1})\Big|_{z=1} = a = 2$$

$$Y^+(z)(1-\tfrac{1}{2}z^{-1})\Big|_{z=\frac{1}{2}} = b = -\frac{1}{2} \qquad Y^+(z) = \frac{2}{1-z^{-1}} - \frac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}}$$

$$y[n] = 2u[n] - \frac{1}{2}\left(\frac{1}{2}\right)^n u[n]$$

③ $y[n] = \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + x[n]$

Determine the impulse response and the step response

$$Y(z) = \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + X(z)$$

$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = X(z)$$

$$Y(z) = \underbrace{\frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}}_{H(z)} X(z)$$

→ Find the poles and if they are in the unit circle is stable

- Schur-Cohn stability test $|K_m| < 1$
- First iteration $A_N(z) = A(z)$ and $K_N = a_N(N)$
- Secon $A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$ and $K_m = a_m(m)$

$$\Rightarrow A_2(z) = 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}$$

$$K_2 = \frac{1}{8} \qquad |K_2| < 1$$

$$\Rightarrow A_1(z) = \frac{1}{1-(\frac{1}{8})^2}\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} - \frac{1}{8}\left(z^{-2} - \frac{3}{4}z^{-1} + \frac{1}{8}\right)\right)$$

$K_1 = \frac{1}{1-(K)^2}\left(-3/4 + 3/4 - 1/8\right)$ $\quad |K_1| < 1$

$\Rightarrow$ stable

or $H(z) = \frac{2}{1 - 1/2\, z^{-1}} - \frac{1}{1 - 1/4\, z^{-1}}$

$\Rightarrow$ 2 poles: 1/2 and 1/4 are inside of the unit circle and because the system is assumed to be causal, it is stable

- impulse response $x[n] = \delta[n]$

$X(z) = 1$

$\Rightarrow$ compute the inverse ZT

$$h[n] = 2 \cdot (1/2)^n u[n] - (1/4)^n u[n]$$

- step response $x[n] = u[n]$

$$X(z) = \frac{1}{1 - z^{-1}}$$

$$Y(z) = H(z) \cdot X(z)$$

$$Y(z) = \frac{A}{1 - 1/2\, z^{-1}} + \frac{B}{1 - 1/4\, z^{-1}} + \frac{C}{1 - z^{-1}}$$

$A = -2$

$B = 1/3$

$C = 8/3$

$$y[n] = \left(-2\,(1/2)^n + 1/3\,(1/4)^n + 8/3\right) u[n]$$

(4) If $H(z) = \frac{z^{-1} + 1/2\, z^{-2}}{1 - 3/5\, z^{-1} + 2/25\, z^{-2}} = z^{-1}\left(\frac{1 + 1/2\, z^{-1}}{1 - 3/5\, z^{-1} + 2/25\, z^{-2}}\right)$

$$H(z) = z^{-1}\left(\frac{-7/2}{1 - 1/5\, z^{-1}} + \frac{9/2}{1 - 2/5\, z^{-1}}\right)$$

a) impulse response

$$h[n] = \left[-\frac{7}{2}\left(\frac{1}{5}\right)^{n-1} + \frac{9}{2}\left(\frac{2}{5}\right)^{n-1}\right] u[n-1]$$

b) zero-state step response

$\forall n > 0,\ y[-n] = 0$

$$Y(z) = H(z) * X(z) = H(z) * \frac{1}{1 - z^{-1}}$$

$$Y(z) = \left(\frac{7/8}{1 - 1/5\, z^{-1}} + \frac{-3}{1 - 2/5\, z^{-1}} + \frac{25/8}{1 - z^{-1}} \right)$$

$$y[n] = \underbrace{\left[7/8 \left(1/5\right)^n - 3\left(2/5\right)^n \right] u[n]}_{\text{natural response}} + \underbrace{25/8\, u[n]}_{\text{forced response}}$$

c) with non-null initial conditions

$y[-1] = 1 \ ;\ y[-2] = 2$