Discrete-time Signal Processing

MsCV Vibot - UE4 Digital Signal Processing

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December 11, 2017

Contents

Preamble

- Main references:

References

[1] *Digital Signal Processing, Principles, Algorithms, and Applications*, John G. Proakis, Dimitris G. Manolakis.

[2] *Linear Processing for Discrete-Time Signal*, Frédéric Truchetet University of Burgundy

[3] *Discrete Time Signal Processing*, Benoît Champagne, Fabrice Labeau, Dpt of Electrical & Computer Engineering McGill University

- Course flow:
  - 8x2H Main course mix with tutorials
  - Assessment : closed book

    ⇒ *1 sheet of A4 paper allowed!*
    - double-sided
    - manuscript
    - no hard copy

1 Introduction

Introduction

Signal
Quantity that varies as function of *time* and/or *space* and has ability to convey information

- Signals are ubiquitous in science and engineering:
  - Electrical signals: currents and voltages in AC circuits, radio communications audio and video signals.
  - Mechanical signals: sound or pressure waves, vibrations in a structure, earthquakes.
  - Biomedical signals: electro-encephalogram, lung and heart monitoring, X-ray and other types of images.
  - Finance: time variations of a stock value or a market index.
- By extension, any series of measurements of a physical quantity can be considered a signal (temperature measurements for instance)

Types of signals and representations

Analog signal
$t \in \mathbb{R} \to x_a(t) \in \mathbb{R}$ or $\mathbb{C}$

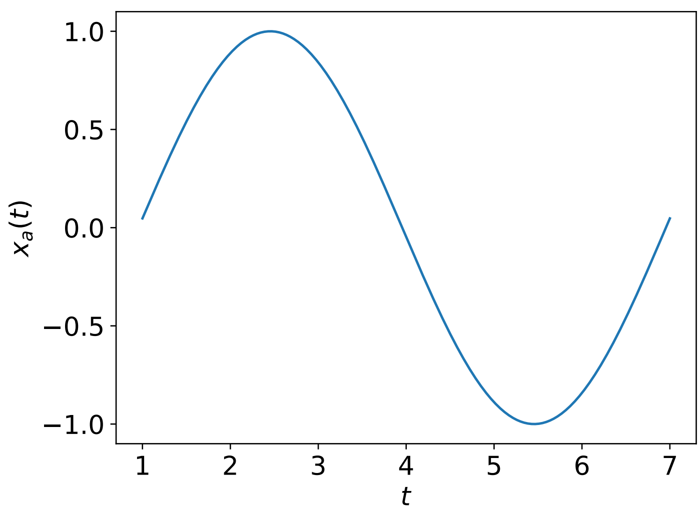
Discrete signal
$n \in \mathbb{Z} \to x[n] \in \mathbb{R}$ or $\mathbb{C}$

Digital signal
$n \in \mathbb{Z} \to x_d[n] \in A$, where A represents a finite set of signal levels.

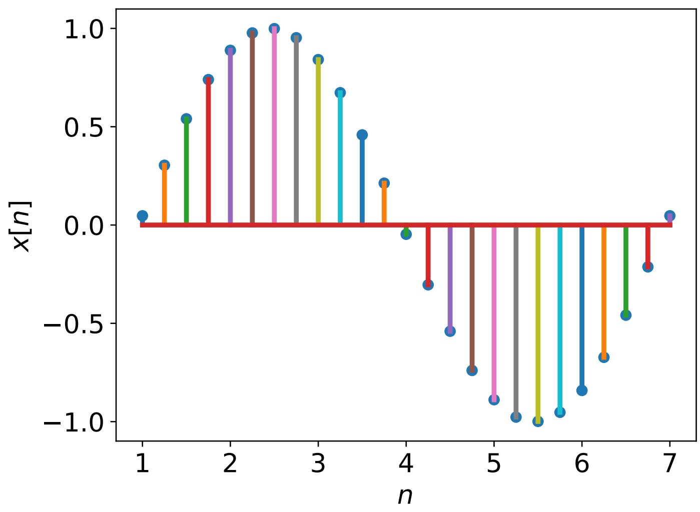
Multi-channel signal
$x(t) = (x_1(t), \ldots, x_N(t))$

Multi-dimensional signal
$x(t_1, \ldots, t_N)$

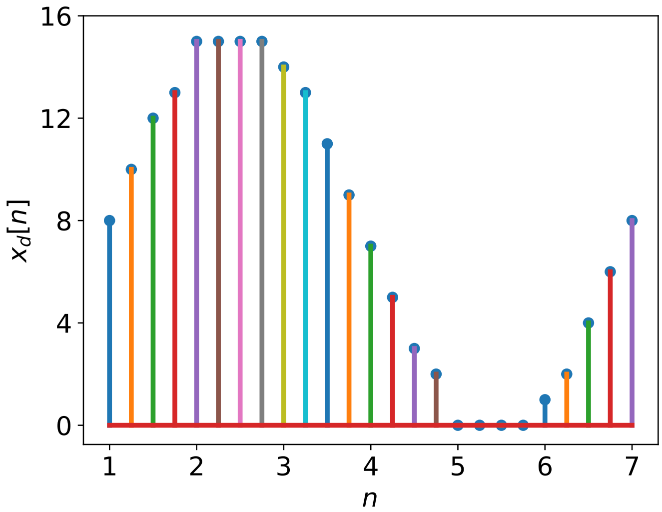
Example: analog signal



Example: discrete signal



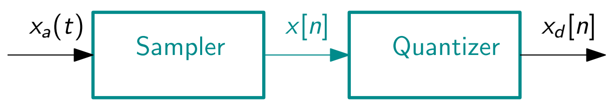
Example: digital signal



Generic structure of a DSP

1. The A/D (analog-to-digital) converter transforms the analog signal $x_a(t)$ into a digital signal (sampler + quantizer) $x_d[n]$
2. The DSP (Digital Signal Processing) performs the desired operations on the digital signal $x_d[n] \rightarrow y_d[n]$
3. The D/A (digital-to-analog) converter transformrs the digital output into an analog signal $y_a(t)$

A/D converter

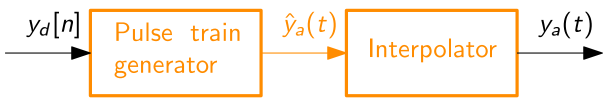


- Sampler: analog input is transformed into a DT signal $x[n] = x_a(nT_s)$ where T_s is the sampling period.
- Quantizer: the DT signal is approximated by a digital signal $x_d[n]$ with only a finite set of possible levels.
- The number of representation levels is generally equal to 2^b where b is the number of bits in a word
- In most systems, the set of discrete levels is uniformly spaced
- WAVE file example:
  - sampling rate: 44.1kHz (sampling rate used for audio CD's)
  - level resolution: 16 bits per sample (some systems use 24 bits)

Digital system

- Similar to a microprocessor: ability to perform mathematical operations and store intermediate results in internal memory
- Operations can be described of mean of an algorithm
- Important distinctions
  - *Real-time system*: computing associated to each sampling interval can be accomplished in a time $\leq$ the sampling interval
  - *Off-line system*: requires the use of external data storage units

D/A converter



- Pulse train generator: the digital signal $y_d[n]$ is transformed into a sequence of scaled, analog pulses
- Interpolator: the high frequency components of $\hat{y}_a(t)$ are removed via low-pass filtering to produce a smooth analog output $y_a(t)$
- One device can generally take care of both steps.

Pros and cons of DSP

Advantages

- Robustness (signal levels can be regenerated)
- Storage capability (can interfaced to low-cost devices for storage)
- Flexibility (software programmable)
- Structure (easy interconnection of DSP blocks)

Disadvantages

- Cost/complexity added by A/D and D/A conversion
- Input signal bandwidth is technology limited
- Quantization effects

2 Discrete-Time signals

Discrete-Time signals

Definition

sequence of real or complex numbers, that is, a mapping from the set of integers $\mathbb{Z}$ into $\mathbb{R}$ or $\mathbb{C}$, as in:

$$n \in \mathbb{Z} \rightarrow x[n] \in \mathbb{R}\, or\, \mathbb{C}$$

- n is called the discrete-time index
- x[n], the nth number in the sequence, is called a sample

Description

- Sequence notation:

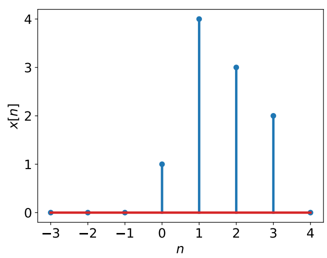
$$x = \{\cdots, 0, \underline{0}, 1, 4, 1, 0, 0, \cdots\},$$

  where underline indicates origin of time: $n = 0$

- Table:

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\cdots$
$x[n]$	$\cdots$	0	0	0	1	4	1	0	0	$\cdots$

- Graphical:



Description

- Explicit mathematical expression:

$$x[n] = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ 1/n & n > 0. \end{cases}$$

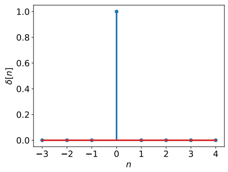
- Recursive approach:

$$x[n] = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ \frac{1}{2}x[n-1] & n > 0. \end{cases}$$

Basic Discrete-time signals

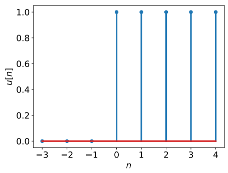
- Unit pulse:

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$



- Unit step:

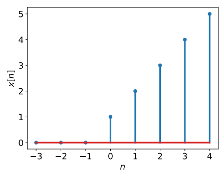
$$u[n] = \begin{cases} 1 & n \geqslant 0, \\ 0 & n < 0. \end{cases}$$



Basic Discrete-time signals

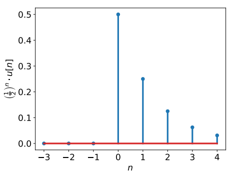
- Ramp function:

$$r[n] = n \cdot u[n].$$



- Exponential sequence:

$$x[n] = a^n \cdot u[n].$$



Uniform sampling

- DT signals are commonly generated via uniform (or periodic) sampling of an analog signal $x_a(t)$:

$$x[n] = x_a\left(nT_s\right), \; n \in \mathbb{Z},$$

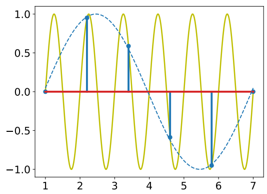
where $T_s > 0$: sampling period.

- Example of a sine wave with frequency F defined by $x_a(t) = \sin 2\pi F t$

$$x[n] = \sin(2\pi F n T_s) = \sin\left(2\pi \frac{F}{F_s} n\right) = \sin(\omega n),$$

  - $F_s = 1/T_s$: sampling frequency,
  - ω : normalized radian frequency of the DT signal.

Nyquist rate



Nyquist rate: $2 \cdot F$
The sampling frequency F_s must satisfy

$$F_s \gg 2 \cdot F$$

Basic operations on signal

In the set $\mathcal{S}$ of all DT signals the following operations can be defined:

scaling
$(\alpha x)[n] = \alpha \cdot x[n]$

addition
$(x + y)[n] = x[n] + y[n]$

multiplication
$(xy)[n] = x[n] \cdot y[n]$

Property
$\mathcal{S}$ equipped with addition and scaling is a vector space.

Classes of signals

Energy signals
all $x \in \mathcal{S}$ with finite energy:

$$\mathcal{E}_x \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Power signals
all $x \in \mathcal{S}$ with finite power:

$$\mathcal{P}_x \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty$$

Classes of signals

Bounded signals
all $x \in \mathcal{S}$ that can be bounded:

$$\exists B_x \in \mathbb{R}^+ / \forall n \in \mathbb{Z},\, |x[n]| \leq B_x$$

Absolutely summable
all $x \in \mathcal{S}$ such that:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Discrete convolution

Discrete convolution of x and y

$$(x * y)\,[n] = x[n] * y[n] \triangleq \sum_{k=-\infty}^{\infty} x[k] \cdot y[n-k]$$

Properties

- commutative law: $x * y = y * x$
- associative law: $(x * y) * z = x * (y * z)$
- convolution by unit pulse: $x * \delta = x$

Correlation of DT signals

- Signal correlation is an operation similar to signal convolution with different physical meaning
- Can be applied to *energy signals*
- Crosscorrelation: performed on two signals
  - can be considered as a measure of similarity of two signals
  - application when the signal is corrupted by noise
- Autocorrelation: performed on one signal
  - indicates how the signal energy (power) is distributed within the signal
  - applications of signal autocorrelation are in radar, sonar, satellite, and wireless communications systems

Crosscorrelation

Definition

$$R_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]y[k-n] = \sum_{k=-\infty}^{\infty} x[k+n]y[k]$$

Property

$$R_{xy}[n] = R_{yx}[-n]$$

Link with convolution

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

$$R_{xy}[n] = x[n] * y[-n]$$

Autocorrelation

Definition

$$R_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[k-n] = \sum_{k=-\infty}^{\infty} x[k+n]x[k]$$

Properties

- Even function:
$$R_{xx}[n] = R_{xx}[-n]$$
- Energy:
$$R_{xx}[0] = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \mathcal{E}_x$$
$$\forall n \in \mathbb{Z},\, R_{xx}[n] < R_{xx}[0]$$

Correlation coefficient

Definition

$$c_{xy} = \frac{R_{xy}[0]}{\sqrt{R_{xx}[0] \cdot R_{yy}[0]}}$$

Properties

- Similarity measurement of two signals
- $-1 \leq c_{xy} \leq 1$
- Geometrically represents angle between euclidean vectors x and y
$$C_{xy=}\frac{x \cdot y}{\sqrt{|x|^2\,|y|^2}} = \frac{x \cdot y}{|x|\,|y|} \triangleq \cos(x, y)$$

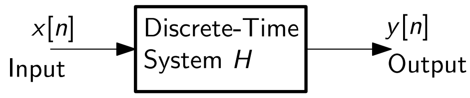
Correlation coefficient

- $c_{xy} \simeq 1 \Rightarrow x$ and y are very similar (almost overlap)
- $c_{xy} \simeq 0 \Rightarrow x$ and y are very different (orthogonal)
- $c_{xy} \simeq -1 \Rightarrow x$ and y are asimilar (opposite direction, but almost the same sample values)
- An also be defined in terms of paramter n:
$$-1 \leq c_{xy}[n] = \frac{R_{xy}[n]}{\sqrt{R_{xx}[0]R_{yy}[0]}} \leq 1$$

3 Discrete-Time systems

Definition

- A Discrete-Time system is a mapping H from $\mathcal{S}$ into itself :



- The system output $y[n]$ generally depends on $x[k]$ for all values of $k \in \mathbb{Z}$
- Notations:

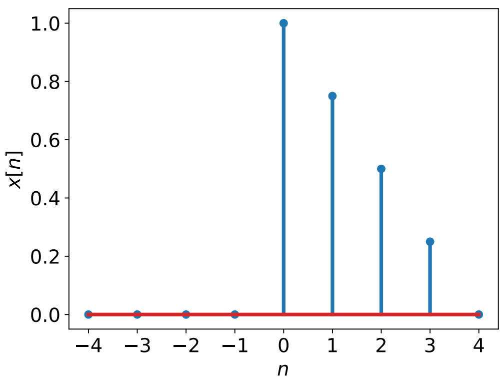
$$y[n] = H(x[n]) \triangleq (Hx)[n]$$

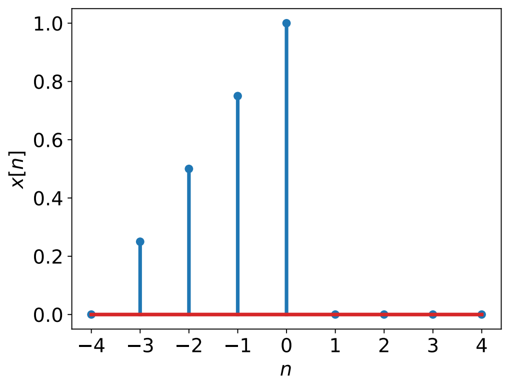
3.1 Basic systems

Basic systems

Time reversal
$y[n] = (Rx)[n] \triangleq x[-n]$

- Mirror image about origin:





Basic systems

Delay or shift by integer k

$$y[n] = (D_k x)[n] \triangleq x[n-k]$$

- Interpretation:
  - $k \geq 0 \Rightarrow$ graph of $x[n]$ shifted by k units to the right
  - $k \leq 0 \Rightarrow$ graph of $x[n]$ shifted by $|k|$ units to the left
- Application: any signal $x \in \mathcal{S}$ can be expressed as a linear combination of shifted impulses:

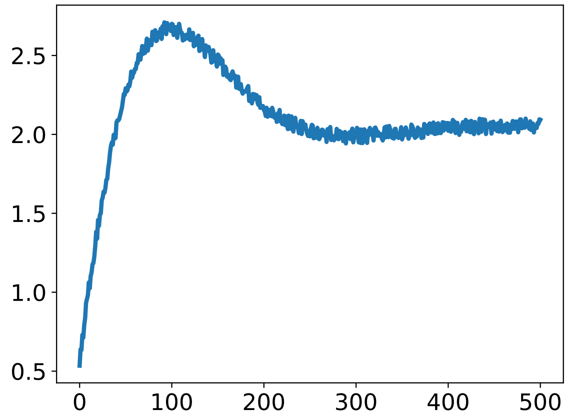
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

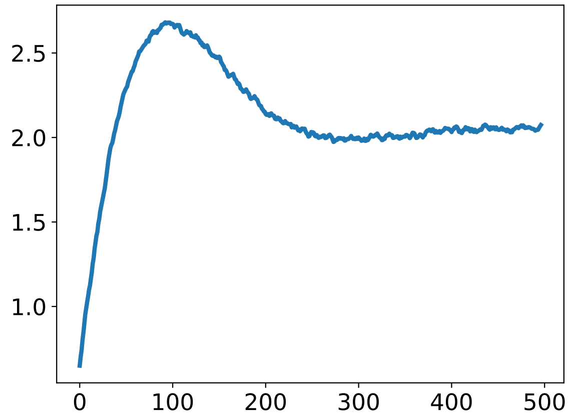
$$x[n] = (x * \delta)[n]$$

Basic systems

Moving average system

$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^{N} x[n-k]$$





Systems properties

1. Static or dynamic (memoryless or not)
2. Causal versus anti-causal
3. Linear or non-linear
4. Time invariant or not
5. Stable or not

Systems properties: static or dynamic ?

Static
$y[n] = (Hx)[n]$ is a function of $x[n]$ only.

- Static systems are memoryless:

$$y[n] = (x[n])^2$$

- Dynamic systems require memory:

$$y[n] = \frac{1}{2}(x[n-1] + x[n])$$

Systems properties: causal versus anti-causal

Causal
$y[n]$ only depends on values $x[k]$ for $k \leq n$.

- Present output depend only on past and present inputs
- Example:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Anti-causal
y[n] only depends on values x[k] for k>n.

Systems properties: linear or not ?

Linearity

$$\forall (\alpha, \beta) \in \mathbb{C}^2,\ \forall (x, y) \in \mathcal{S}^2,\ H(\alpha x + \beta y) = \alpha H(x) + \beta H(y)$$

- Example of a linear system:

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$

- Example of a non-linear system:

$$y[n] = (x[n])^2$$

Systems properties: time invariant or not ?

Time-invariant

$$\forall (n, k) \in \mathbb{Z}^2,\ (Hx)[n] = y[n] \Rightarrow (Hx)[n-k] = y[n-k]$$

- Example of a time invariant system: the moving average system.

$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^{N} x[n-k]$$

- Example of system not time invariant:

$$y[n] = x[2n]$$

Systems properties: stable or not ?

Stable

x bounded $\Rightarrow y = Hx$ bounded

Stable

if $\forall n \in \mathbb{Z}\, |x[n]| \leq B_x$ then $\exists B_y / \forall n \in \mathbb{Z},\ |y[n]| \leq B_y$

- A system is stable (Bounded Input Bounded Output) if every bounded input produces a bounded output.

3.2 Linear Time-Invariant (LTI) systems

Linear Time-Invariant (LTI) systems

- DT systems that are both *Linear* and *Time-Invariant* play a central role in digital signal processing:
  - Many physical systems are either LTI or approximately so
  - Many efficient tools are available for the analysis and design of LTI systems

Fundamental property

Let H a LTI system and $y = Hx$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x * h[n],$$

with $h \triangleq H\delta$ known as impulse response of H.

Proof of the fundamental property

First we have:

$$y[n] = (Hx)[n] = H(x[n]).$$

And for any DT signal, we can write:

$$x[n] = x * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Invoking linearity:

$$y[n] = H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) = \sum_{k=-\infty}^{\infty} x[k]H(\delta[n-k])$$

Invoking Time-Invariant property:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]HD_k(\delta[n]) = \sum_{k=-\infty}^{\infty} x[k]D_kH(\delta[n]) = \sum_{k=-\infty}^{\infty} x[k]D_kh[n]$$

Graphical interpretation

- to compute the sample values of $y[n]$ according to $y[n] = h * x[n]$:
  1. Time reverse sequence of $h[k]$:
     $\Rightarrow h[-k]$
  2. Shift $h[-k]$ by n samples:
     $\Rightarrow h[-(k-n)] = h[n-k]$
  3. Multiply sequences $x[k]$ and $h[n-k]$ and sum over k:
     $\Rightarrow y[n]$
- Example with $h = \{\cdots, 0, 1, \underline{0}, -1, 0, \cdots\}$ and $x = u$:

$x[n]$	0	0	0	0	1	*1*	1	1	1
mask			$\rightarrow$	-1	0	*1*	$\rightarrow$		
$y[n]$	0	0	0	1	**1**	0	0	0	0

Characterization of a LTI system

A LTI system is fully characterized by the knowledge of its impulse response $h = H\delta$. For any other input x we have:

$$Hx = x * h.$$

- Example of the accumulator system defined by $y[n] = \sum_{k=-\infty}^{n} x[k]$:
  - Let's define as input x the unit pulse δ:

$$h[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$h[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

$$h[n] = u[n]$$

Causality of a LTI system

Causality
A LTI system is causal if and only if:

$$\forall n < 0,\, h[n] = 0$$

- Proof:

$$y[n] = h * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[n] = \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] \cdots$$

$\Rightarrow$Clearly, $y[n]$ only depends on values $x[m]$ for $m \leq n$ if and only if $h[k] = 0$ for $k < 0$

Stability of a LTI system

Stability
A LTI system is stable if and only if the sequence $h[n]$ is absolutely summable, that is:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Example: consider LTI system with impulse response $h[n] = \alpha^n u[n]$:
- Causality: $h[n] = 0$ for $n < 0 \Rightarrow$ causal
- Stability:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |\alpha|^n$$

clearly the sum diverges if $|\alpha| \geq 1$ while it converges if $|\alpha| < 1$:

$$\sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-\alpha} < \infty$$

Thus the system is stable provided $|\alpha| < 1$

FIR and IIR

FIR system
An LTI system has a *Finite Impulse Response* (*FIR*) if we can find integers $N_1 \leq N_2$ such that:

$$h[n] = 0 \,\text{when}\; n < N_1 \;\textit{or}\; n > N_2$$

- Otherwise the LTI system has an *Infinite Impulse Response* (*IIR*).
- *FIR systems are necessarily stable*:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=N_1}^{N_2} |h[n]| < \infty$$

- The impulse response is often called a convolution mask.

FIR and IIR system examples

1. Let the LTI system described by:

$$h[n] = u[n] - u[n-N]$$

$$h[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

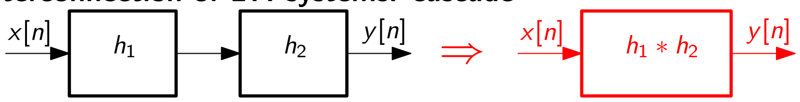
$\Rightarrow$ the system is FIR with $N_1 = 0$ and $N_2 = N-1$

2. Let the LTI system described by:

$$h[n] = \alpha^n u[n]$$

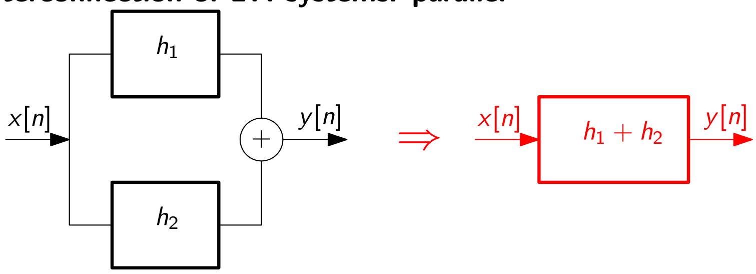
$\Rightarrow$ the system is IIR (cannot find any N_2)

Interconnection of LTI systems: cascade



$$y = h_2 * (h_1 * x) = (h_2 * h_1) * x = (h_1 * h_2) * x$$

Interconnection of LTI systems: parallel



$$y = (h_1 * x) + (h_2 * x) = (h_1 + h_2) * x$$

4 The Z-Transform (ZT)

4.1 Definition

Introduction

Definition

The ZT is a transformation that maps DT signal $x[n]$ into a function of the complex variable z, defined as:

$$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n},\ z \in \mathbb{C}$$

- The domain of $x(z)$ is the set of all $z \in \mathbb{C}$ such that the series converges absolutely, that is:

$$\text{Dom}(X) = \left\{ z \in \mathbb{C} / \sum_{n=-\infty}^{\infty} |x[n]z|^{-n} < \infty \right\}$$

- The domain of $X(z)$ is called the *Region Of Convergence* (ROC).
- The ROC *only depends on* $|z|$: if $z \in$ ROC, so is $ze^{i\phi}$ for any angle ϕ
- Within the ROC, $X(z)$ is an analytic function of complex variable z:
  - $X(z)$ is smooth,
  - derivative exists, etc.
- *Both* X(z) and the ROC *are needed* when specifying a ZT.

ZT of unit step

- Unit step: $x[n] = u[n]$
  1. ZT:

$$X(z) = \sum_{n=0}^{\infty} z^{-n},\ z \in \mathbb{C}$$

$$x(z) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \frac{1}{1-\frac{1}{z}}$$

  2. ROC:

$$\left|\frac{1}{z}\right| < 1$$

$$|z| > 1$$

ZT of a signal defined by a sequence

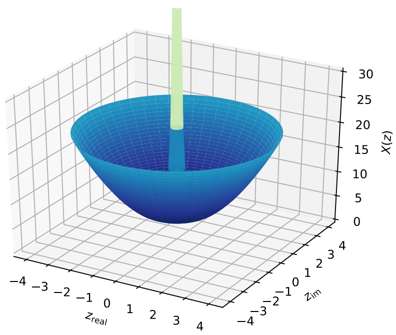
- Signal defined by: $x = \{\cdots, 1, 1, \underline{1}, 1, 1, \cdots\}$
  1. ZT:

$$X(z) = z^{-2} + z^{-1} + 1 + z + z^2$$

  2. ROC:

$$0 < |z| < \infty$$

  3. Modulus of ZT:



4.2 Study of the ROC

Signal with finite duration

Definition
A signal with finite duration is defined such that:

$$\exists\,(N_1, N_2) \in \mathbb{Z}^2,\ N_1 \leq N_2 / \forall n < N_1 \text{ and } \forall n > N_2,\ x[n] = 0$$

- ZT is defined by:

$$\begin{aligned} X(z) &= \sum_{n=N_1}^{N_2} x[n] z^{-n} \\ &= x[N_1] z^{-N_1} + x[N_1+1] z^{-N_1-1} + \cdots + x[N_2] z^{-N_2} \end{aligned}$$

- ZT exists $\forall z \in \mathbb{C}$, except possibly at $z = 0$ and $z = \infty$:
  - $N_2 > 0 \Rightarrow z = 0 \notin$ ROC
  - $N_1 < 0 \Rightarrow z = \infty \notin$ ROC

Theorem

Radius of convergence
To any power series $\sum_{n=0}^{\infty} c_n w^n$, we can associate a *radius of convergence*

$$R_w = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|,$$

such that:

- if $|w| < R_w \Rightarrow$the series converges absolutely
- if $|w| > R_w \Rightarrow$the series diverges

Causal signals

- Suppose $x[n] = 0$ for $n < 0$:

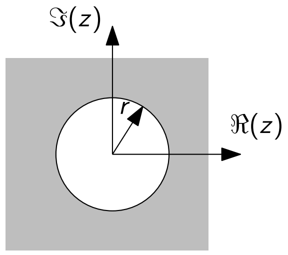
$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} c_n w^n$$

- The ROC is given by:

$$|w| < R_w = \lim_{n\to\infty}\left|\frac{x[n]}{x[n+1]}\right|$$

$$|z| > \frac{1}{R_w} \equiv r$$

- The ROC is the *exterior of a circle of radius* r:



Causal signals: example

- Consider the causal sequence:

$$x[n] = a^n u[n]$$

- ZT:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

$$X(z) = \frac{1}{1 - az^{-1}}, \text{ provided } \left|az^{-1}\right| < 1$$

- The ROC is the exterior of a circle of radius $r = |a|$

Anti-causal signals

- Suppose $x[n] = 0$ for $n > 0$:

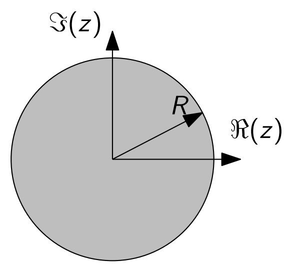
$$X(z) = \sum_{n=-\infty}^{0} x[n]z^{-n} = \sum_{n=0}^{\infty} x[-n]z^{n} = \sum_{n=0}^{\infty} c_n w^n$$

- The ROC is given by:

$$|w| < R_w = \lim_{n\to\infty}\left|\frac{x[-n]}{x[-n-1]}\right|$$

$$|z| < R_w \equiv R$$

- The ROC is the *interior of a circle of radius* R:



Anti-causal signals: example

- Consider the anti-causal sequence:

$$x[n] = -a^n u[-n-1]$$

- ZT:

$$X(z) = -\sum_{n=-\infty}^{1} a^n z^{-n} = -\sum_{n=1}^{\infty} \left(a^{-1}z\right)^n$$

$$\begin{aligned} X(z) &= -\frac{a^{-1}z}{1-a^{-1}z}, \text{ provided } \left|a^{-1}z\right| < 1 \\ &= \frac{1}{1-az^{-1}} \end{aligned}$$

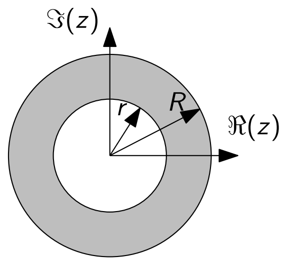
- The ROC is the interior of a circle of radius $r = |a|$

Arbitrary signals

- The series $X(z)$ can be decomposed as:

$$X(z) = \underbrace{\sum_{n=-\infty}^{-1} x[n]z^{-n}}_{\text{needs } |z|<R} + \underbrace{\sum_{n=0}^{\infty} x[n]z^{-n}}_{\text{needs } |z|>r}$$

- If $r < R$, the ZT exists and ROC: $r < |z| < R$:



- if $r > R$, the ZT does not exists.

Arbitrary signals: example 1

Consider the DT signal:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n}_{\text{needs } |z|>\frac{1}{2}} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n}_{\text{needs } |z|<2}$$

The two series will converge iff ROC: $\frac{1}{2} < |z| < 2$

$$X(z) = \frac{1}{1-\frac{1}{2z}} - 1 + \frac{1}{1-\frac{z}{2}} = \frac{2-\frac{5}{2}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}$$

Arbitrary signals: example 2

Consider the DT signal:

$$x[n] = 2^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} (2)^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n}$$

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n}_{\text{needs } |z|>2} + \underbrace{\sum_{n=1}^{\infty} (2z)^n}_{\text{needs } |z|<\frac{1}{2}}$$

Since $\{|z| > 2\} \cap \left\{|z| < \frac{1}{2}\right\} = \emptyset$, the ROC is empty and the ZT does not exist.

4.3 Properties of the ZT

Introductory remarks

- Notations for ZT pairs:

$$x[n] \overset{z}{\longleftrightarrow} X(z), z \in \mathcal{R}_x$$
$$y[n] \overset{z}{\longleftrightarrow} Y(z), z \in \mathcal{R}_y$$

  $\mathcal{R}_x$ and $\mathcal{R}_y$ denote the ROC of $X(z)$ and $Y(z)$ respectively.

- When stating a property, the corresponding ROC must also be specified
- In some cases, the true ROC may be larger than the one indicated

Basic symmetries

Basic symmetries

$$x[-n] \overset{z}{\longleftrightarrow} X(z^{-1}),\ z^{-1} \in \mathcal{R}_x$$
$$x^*[n] \overset{z}{\longleftrightarrow} X^*(z^*),\ z \in \mathcal{R}_x$$

- Proof:

  Let be $x_f[n] = x[-n]$:

$$X_f(z) = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$$

$$X_f(z) = \sum_{n=-\infty}^{\infty} x[n] z^{n} = X(z^{-1})$$

  Assuming that the ROC $\mathcal{R}_x$ was defined by: $r < |z| < R$, then the ROC of $X_f(z)$ is: $1/R < |z| < 1/r$

Linearity and Time shift

Linearity

$$\forall\,(a,b)\in\mathbb{C}^2,\; ax[n]+by[n] \overset{z}{\longleftrightarrow} aX(z)+bY(z),\; z\in\mathcal{R}_x\cap\mathcal{R}_y$$

Time shift

$$\forall d\in\mathbb{Z},\; x[n-d] \overset{z}{\longleftrightarrow} z^{-d}X(z),\; z\in\mathcal{R}_x$$

- Let's denote $X_d(z)$ the ZT of the DT x shifted (or delayed) by d:

$$X_d(z)=\sum_{n=-\infty}^{\infty} x[n-d]z^{-n}$$

$$X_d(z)=\sum_{l=-\infty}^{\infty} x[l]z^{-d}z^{-l},\text{ with } l=n-d$$

Exponential modulation

Exponential modulation (scaling)

$$a^n x[n] \overset{z}{\longleftrightarrow} X(z/a),\; z/a\in\mathcal{R}_x$$

- Proof:

$$\sum_{n=-\infty}^{\infty} a^n x[n]z^{-n}=\sum_{n=-\infty}^{\infty} x[n]\left(a^{-1}z\right)^{-n}$$

- Assuming that the ROC $\mathcal{R}_x$ was defined by: $r<|z|<R$, then:

$$r<\left|a^{-1}z\right|<R$$

Exponential modulation: example

$$x[n]=\cos\left(\omega_0 n\right)u[n]$$

$$x[n]=\frac{1}{2}e^{j\omega_0 n}u[n]+\frac{1}{2}e^{-j\omega_0 n}u[n]$$

$$X(z)=\frac{1}{2}ZT\left\{e^{j\omega_0 n}u[n]\right\}+\frac{1}{2}ZT\left\{e^{-j\omega_0 n}u[n]\right\}$$

$$X(z)=\underbrace{\frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}}}_{|z|>\left|e^{j\omega_0}\right|=1}+\underbrace{\frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}}}_{|z|>\left|e^{-j\omega_0}\right|=1}$$

$$X(z)=\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}},\;\text{ROC}:|z|>1$$

Differentiation

Differentiation

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz},\, z \in \mathcal{R}_x$$

- Proof:

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n]\frac{dz^{-n}}{dz}$$

$$\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

Differentiation: example

$$x[n] = na^n u[n]$$

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right),\, |z| > |a|$$

$$X(z) = -z\left(-az^{-2}\right)\left(\frac{1}{(1-az^{-1})^2}\right),\, \mathrm{ROC}: |z| > |a|$$

$$Z(x) = \frac{az^{-1}}{(1-az^{-1})^2},\, \mathrm{ROC}: |z| > |a|$$

Convolution

Convolution

$$x[n] * y[n] \overset{z}{\longleftrightarrow} X(z)Y(z),\, z \in \mathcal{R}_x \cap \mathcal{R}_y$$

- Let be $c[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$:

$$C(z) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]y[n-k]z^{-n}$$

$$C(z) = \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]y[l]z^{-l-k},\text{ with } l = n-k$$

$$C(z) = \sum_{l=-\infty}^{\infty} y[l]z^{-l}\sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

Initial value

Initial value (causal case)
For x[n] causal (i.e. $x[n] = 0\, for\, n < 0$), we have:

$$\lim_{z\to\infty} X(z) = x[0]$$

Initial value (anti-causal case)
For x[n] anti-causal (i.e. $x[n] = 0\, for\, n > 0$), we have:

$$\lim_{z^{-1}\to\infty} X(z) = x[0]$$

4.4 Rational ZTs

Rational function

Definition
$X(z)$ is a rational function in z (or z^{-1}) if:

$$X(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are polynomials in z (resp. z^{-1})

- Rational ZT plays a central role in DSP
- Essential for the realization of practical IIR filters
- Two important issues related to rational ZT are investigated:
  - Pole-Zero (PZ) characterization
  - Inversion via partial fraction expansion

Poles and zeros

Pole
$X(z)$ has a *pole* of order L at $z = p_0$ if:

$$X(z) = \frac{\psi(z)}{(z - p_0)^L},\ 0 < |\psi(p_0)| < \infty$$

Zero
$X(z)$ has a *zero* of order L at $z = z_0$ if:

$$X(z) = (z - z_0)^L\, \psi(z),\ 0 < |\psi(z_0)| < \infty$$

- The order L is sometimes referred as the multiplicity of the pole/zero.

Poles and zeros at ∞

Poles at ∞
$X(z)$ has a pole of order L at $z = \infty$ if:

$$X(z) = z^L \psi(z),\ 0 < |\psi(\infty)| < \infty$$

Zeros at ∞
X(z) has a zero of order L at $z = \infty$ if :

$$X(z) = \frac{\psi(z)}{z^L},\ 0 < |\psi(\infty)| < \infty$$

- Let be $X(z) = N(z)/D(z)$ (expressed in "simplified" form):
  - If order $(N(z))$ − order $(D(z)) = L > 0 \Rightarrow X(z)$ has a pole of order L at $z = \infty$
  - If order $(N(z))$ − order $(D(z)) = L < 0 \Rightarrow X(z)$ has a zero of order L at $z = \infty$

Poles and Zeros of a rational $X(z)$

Consider rational function $X(z) = N(z)/D(z)$:

- *Roots of $N(z) \Rightarrow$ zeros of $X(z)$*
- *Roots of $D(z) \Rightarrow$ poles of $X(z)$*
- Must take into account pole-zero cancellation:
  - common roots of $N(z)$ and $D(z)$ do not count as zeros and poles.
- Repeated roots in $N(z)$ (or $D(z)$) lead to multiple zeros (respectively poles).

Property
number of poles = number of zeros, if poles and zeros at 0 and ∞ are included.

Poles and Zeros of a rational: examples

- Example 1:

$$X(z) = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}} = \frac{z}{z^2 - 2z + 1} = \frac{z}{(z-1)^2}$$

  Corresponding poles and zeros:

  poles $p_1 = 1$, $L = 2$

  zeros $z_1 = 0$, $L = 1$ $z_2 = \infty$, $L = 1$

- Example 2:

$$X(z) = \frac{1 - z^{-4}}{1 + 3z^{-1}} = \frac{z^4 - 1}{z^3\,(z+3)}$$

  Corresponding poles and zeros:

  poles $p_1 = 0$, $L = 3$ $p_2 = -3$, $L = 1$

  zeros $z_{k\in[0,3]} = e^{jk\pi/2}0$, $L = 1$

Pole-zero and rational function link

Property
For rational functions $X(z) = N(Z)/D(z)$, knowledge of the poles and zeros (along with their order) completely specify $X(z)$, up to a scaling factor $G \in \mathbb{C}$.

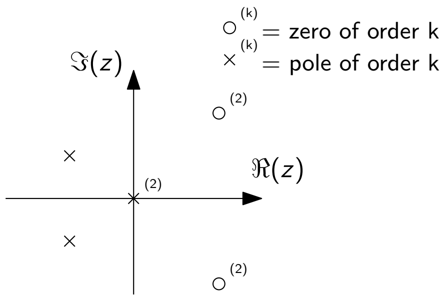
- Example:

  poles $p_1 = 2$, $L = 1$

  zeros $z_1 = 1$, $L = 1$

$$X(z) = G\frac{z-1}{z-2} = G\frac{1 - z^{-1}}{1 - 2z^{-1}}$$

Pole-zero (PZ) diagram



- The presence of poles or zeros at ∞ should be mentioned on the diagram
- It is useful to indicate ROC on the PZ-diagram

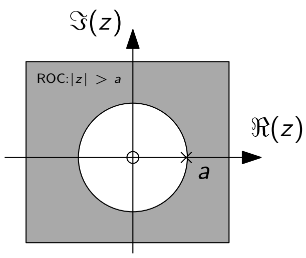
ROC and PZ diagram

- Consider $x[n] = a^n u[n]$, where $a > 0$:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \text{ ROC: } |z| > a$$

poles $p_1 = a$, $L = 1$

zeros $z_1 = 0$, $L = 1$



ROC for rational ZT

- ROC does not contain poles
  - because $X(z)$ does not converge at a pole
- ROC can always be extended to nearest pole
- ROC is delimited by poles
  - annular region between poles
- If we are given only $X(z)$, then several possible ROC:
  - any annular region between two poles of increasing magnitude
  - accordingly, several possible DT signals $x[n]$

4.5 Inverse ZT

Introduction

- The inverse Z-Transform consists in finding $x[n]$ given its ZT $X(z)$ and its corresponding ROC.
- Several methods exist:
  - Contour integration via residue theorem
  - Power series expansion
  - Partial fraction expansion
- *Partial fraction* is the most useful technique in the context of *rational ZTs*

Contour integration

Inverse Z-Transform

$$x[k] = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{k-1} dz$$

$$\begin{aligned}
\frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{k-1} dz &= \frac{1}{2\pi j} \oint_{\mathcal{C}} \sum_{n=-\infty}^{\infty} x[n] z^{-n} z^{k-1} dz \\
&= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi j} \oint_{\mathcal{C}} z^{k-1-n} dz
\end{aligned}$$

- Cauchy integral theorem:

$$\frac{1}{2\pi j} \oint_{\mathcal{C}} z^{k-1-n} dz = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

Inversion via Partial Fraction Expansion

- Let be a rational ZT defined according to:
  - $X(z) = \frac{N(z)}{D(z)}$
  - $N(z)$ and $D(z)$ are polynomials in z^{-1}
  - degree of $D(z) >$ degree of $N(z)$

- Under these conditions, $X(z)$ may be expressed as:

$$X(z) = \sum_{k=1}^{K} \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

  - $p_1, \cdots, p_K$ are the distinct poles of $X(z)$
  - $L_1, \cdots, L_K$ are the corresponding orders

Expression of the constants A_{kl}

$$X(z) = \sum_{k=1}^{K} \sum_{l=1}^{L_K} \frac{A_{kl}}{(1 - p_k z^{-1})^l}$$

Constants A_{kl} can be computed as follows:

- simple poles ($L_k = 1$):

$$A_{kl} \equiv \left(1 - p_k z^{-1}\right) X(z)\Big|_{z=p_k}$$

- multiple poles ($L_k > 1$):

$$A_{kl} \equiv \frac{1}{(L_k - l)!(-p_k)^{L_k - l}} \left\{ \frac{d^{L_k - l}}{(dz^{-1})^{L_k - l}} \left(1 - p_k z^{-1}\right)^{L_k} X(z) \right\}\Bigg|_{z=p_k}$$

Inversion method

Given $X(z)$ as above with ROC: $r < |z| < R$.

1. Determine the PFE of $X(z)$:
$$X(z) = \sum_{k=1}^{K}\sum_{l=1}^{L_K} \frac{A_{kl}}{(1-p_k z^{-1})^l}$$
2. Invoking linearity of the ZT, express $x[n]$ as:
$$x[n] = \sum_{k=1}^{K}\sum_{l=1}^{L_K} A_{kl}\mathcal{Z}^{-1}\left\{\frac{1}{(1-p_k z^{-1})^l}\right\}$$
3. Evaluate the elementary inverse ZTs:
   - simple poles ($L_k = 1$):
$$\frac{1}{1-p_k z^{-1}} \overset{\mathcal{Z}^{-1}}{\rightarrow} \begin{cases} p_k^n u[n] & \text{if } |p_k| \le r \\ -p_k^n u[-n-1] & \text{if } |p_k| \ge R \end{cases}$$
   - multiple poles ($L_k > 1$):
$$\frac{1}{(1-p_k z^{-1})^l} \overset{\mathcal{Z}^{-1}}{\rightarrow} \begin{cases} \binom{n+l-1}{l-1} p_k^n u[n] & \text{if } |p_k| \le r \\ -\binom{n+l-1}{l-1} p_k^n u[-n-1] & \text{if } |p_k| \ge R \end{cases}$$

Example

$$X(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})},\ |a| < |z| < |b|$$

- PFE can be written as:
$$X(z) = \frac{A_1}{(1-az^{-1})} + \frac{A_2}{(1-bz^{-1})},$$
  with :
  - $A_1 \equiv \left(1-az^{-1}\right) X(z)\big|_{z=a} = \frac{a}{a-b}$
  - $A_2 \equiv \left(1-bz^{-1}\right) X(z)\big|_{z=b} = \frac{b}{b-a}$
- Elementary inverse ZTs from 2 simple poles:
  - $\frac{1}{1-az^{-1}} \overset{\mathcal{Z}^{-1}}{\rightarrow} a^n u[n]$ since $|z| > |a|$ (causal)
  - $\frac{1}{1-bz^{-1}} \overset{\mathcal{Z}^{-1}}{\rightarrow} -b^n u[-n-1]$ since $|z| < |b|$ (anti-causal)
- Finally:
$$x[n] = \frac{a^{n+1}}{a-b}u[n] - \frac{b^{n+1}}{b-a}u[-n-1]$$

Putting $X(z)$ in a suitable form

- When applying the above PFE method to $X(z) = N(z)/D(z)$, it is essential that:
  - $N(z)$ and $D(z)$ be polynomials in z^{-1}
  - degree of $D(z) >$ degree of $N(z)$
- If either one of the above conditions are not satisfied, further algebraic manipulations must be applied to X(z)
- There are two common types of manipulations:
  - *polynomial division*
  - *use of shift property*

Polynomial division

- Find $Q(z)$ and $R(z)$, such that:
$$\frac{N(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)}$$
  - $Q(z)$ is a polynomial in z^{-1},
  - $R(z)$ is a polynomial in z^{-1} with degree less than that of $D(z)$.
- $Q(z)$ and $R(z)$ are determined using a division table:

$$\begin{array}{r|c} & Q(z) \\ \cline{2-2} D(z) & N(z) \\ & -Q(z)D(z) \\ \cline{2-2} & R(z) \end{array}$$

- To increase the z^{-1} power in $N(z)$, $D(z)$ and $N(z)$ are expressed in decreasing powers of z (e.g. $D(z) = 1 + 2z^{-1} + z^{-2}$)
- To decrease the z^{-1} power in $N(z)$, $D(z)$ and $N(z)$ are expressed in increasing powers of z (e.g. $D(z) = z^{-2} + 2z^{-1} + 1$)

Example

ZT of a causal signal

$$X(z) = \frac{-5 + 3z^{-1} + z^{-2}}{3 + 4z^{-1} + z^{-2}}$$

- Use long division to make the degree of numerator smaller than the degree of the denominator $\Rightarrow$ *decrease the z^{-1} power in $N(z)$*

$$\begin{array}{r|c} z^{-2} + 4z^{-1} + 3 & z^{-2} + 3z^{-1} - 5 \end{array}$$

$$\begin{array}{r|c} & 1 \\ \cline{2-2} z^{-2} + 4z^{-1} + 3 & z^{-2} + 3z^{-1} - 5 \\ & -\left(z^{-2} + 4z^{-1} + 3\right) \end{array}$$

$$\begin{array}{r|c} & 1 \\ \cline{2-2} z^{-2} + 4z^{-1} + 3 & z^{-2} + 3z^{-1} - 5 \\ & -\left(z^{-2} + 4z^{-1} + 3\right) \\ \cline{2-2} & -z^{-1} - 8 \end{array}$$

$X(z)$ rewrites $X(z) = 1 - \frac{z^{-1}+8}{z^{-2}+4z^{-1}+3}$

Example

$$X(z) = 1 - \frac{z^{-1} + 8}{z^{-2} + 4z^{-1} + 3}$$

- The denominator of the 2nd term has 2 roots, poles at $z = -1/3$ and $z = -1$, hence:

$$X(z) = 1 - \frac{z^{-1} + 8}{3\left(1 + \frac{1}{3}z^{-1}\right)\left(1 + z^{-1}\right)}$$

- The PFE gives:

$$X(z) = 1 - \frac{1}{3}\left(\frac{A_1}{1 + \frac{1}{3}z^{-1}} + \frac{A_2}{1 + z^{-1}}\right)$$

  with:
  - $A_1 = \left.\frac{z^{-1}+8}{1+z^{-1}}\right|_{z=-1/3} = -\frac{5}{2}$
  - $A_2 = \left.\frac{z^{-1}+8}{1+\frac{1}{3}z^{-1}}\right|_{z=-1} = \frac{21}{2}$

Example

$$X(z) = 1 + \frac{5}{6\left(1 + \frac{1}{3}z^{-1}\right)} - \frac{7}{2\left(1 + z^{-1}\right)}$$

- Causality of $x[n]$ determines the ROC of $X(z)$
  - ROC is supposed to be delimited by circles with radius 1/3 and/or 1
  - causal sequence $\Rightarrow$ ROC must extend outwards from the outermost pole $\Rightarrow$ ROC is $|z| > 1$
- The sequence $x[n]$ is then given by:

$$x[n] = \delta[n] + \frac{5}{6}\left(-\frac{1}{3}\right)^n u[n] - \frac{7}{2}\left(-1\right)^n u[n]$$

Use of shift property

- In some cases, a simple multiplication by z^k is sufficient to put $X(z)$ into a suitable format, that is:

$$Y(z) = z^k X(z) = \frac{N(z)}{D(z)}$$

  where $N(z)$ and $D(z)$ satisfy previous conditions
- The PFE method is then applied to $Y(z)$, yielding a DT signal $y[n]$
- Finally, the shift property is applied to recover $x[n]$:

$$x[n] = y[n - k]$$

Example

$$X(z) = \frac{1 - z^{-128}}{1 - z^{-2}},\ |z| > 1$$

- $X(z)$ can be rewritten according to:

$$X(z) = Y(z) - z^{-128}Y(z)$$

  where:

$$Y(z) = \frac{1}{1 - z^{-2}} = \frac{1}{\left(1 - z^{-1}\right)\left(1 + z^{-1}\right)}$$

- ZT of $Y(z)$ is:
$$y[n] = \frac{1}{2}\left(1 + (-1)^n\right) u[n]$$
- Using *linearity* and *shift property*:
$$x[n] = y[n] - y[n-128]$$
- Therefore:
$$x[n] = \frac{1}{2}\left(1 + (-1)^n\right)\left(u[n] - u[n-128]\right)$$

5 Fourier Transform of DT signals

5.1 Definition

Definition

DTFT (Discrete-Time Fourier Transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},\ \omega \in \mathbb{R}$$

- $X(\omega)$ is *continuous* and 2π *periodic*:
  - $X(\omega) = X(\omega + 2\pi)$
- Nyquist frequency is defined by: $\omega_N = \pi$
- $X(\omega)$ is called the spectrum of $x[n]$: $X(\omega) = |X(w)|\, e^{j\angle X(\omega)}$
  - $|X(w)|$: magnitude spectrum
  - $\angle X(\omega)$: phase spectrum
- The Fourier Transform is a specific case of the ZT taking $z = e^{j\omega}$ with $|z| = 1 \in$ ROC.

Fourier Transform of a sampled continuous signal

- Let $s_e(t)$ the sampled expression of the continuous signal $s(t)$ with sampling period T_s:

$$s_e(t) = s(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$$

- If we denote $s[k] = s(kTs)$, we have:

$$s_e(t) = \sum_{k=-\infty}^{\infty} s[k]\delta(t - kT_s)$$

- The Fourier transform gives:

$$TF\{s_e(t)\} = \hat{s}_e(\omega) = s[k] \sum_{k=-\infty}^{\infty} s[k]TF\{\delta(t - kT_s)\}$$

- Applying the delay theorem:

$$\hat{s}_e(\omega) = s[k] \sum_{k=-\infty}^{\infty} s[k]e^{-jkwT_s}TF\{\delta(t)\} = s[k] \sum_{k=-\infty}^{\infty} s[k]e^{-jkwT_s}$$

- Finally:

$$\hat{s}_e(\omega) = S(\omega T_s)$$

with the Nyquist frequency equal to:

$$\omega_N = \frac{\pi}{T_s}$$

Inverse DTFT

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega,\ n \in \mathbb{Z}$$

Proof: Note that $\int_{-\pi}^{\pi} e^{j\omega n}d\omega = 2\pi\delta[n]$:

$$\begin{aligned}
\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega &= \int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\right)e^{j\omega n}d\omega \\
&= \sum_{k=-\infty}^{\infty} x[k]\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \\
&= 2\pi\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \\
&= 2\pi x[n]
\end{aligned}$$

5.2 Convergence of the DTFT

Convergence of the DTFT

- For the DTFT to exist, the series $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ *must converge*
- That is, the partial sum

  $$X_M(\omega) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$$

  must converge to a limit $X(\omega)$ as $M \to \infty$
- Absolutely summable signals
  - $X_M(\omega)$ converges uniformly to $X(\omega)$
  - $X(\omega)$ is continuous
- Energy signals
  - $X_M(\omega)$ does not necessarily converge
  - $X(\omega)$ may be discontinuous at certain points
- Power signals
  - Most power signals do not have a DTFT
  - Exceptions including: Periodic signals, Unit step

5.3 Properties

Properties

- Linearity

  $$ax[n] + by[n] \overset{\mathcal{F}}{\leftrightarrow} aX(\omega) + bY(\omega)$$

- Time shift

  $$x[n-d] \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega d}X(\omega)$$

- Frequency modulation

  $$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\leftrightarrow} X(\omega - \omega_0)$$

- Differentiation

  $$nx[n] \overset{\mathcal{F}}{\leftrightarrow} j\frac{dX(\omega)}{d\omega}$$

Even and odd component definition

DT signal

$$
\begin{aligned}
x[n] &= x_e[n] + x_o[n] \\
x_e[n] &\triangleq \frac{1}{2}\left(x[n] + x^*[-n]\right) = x_e^*[-n] \\
x_o[n] &\triangleq \frac{1}{2}\left(x[n] - x^*[-n]\right) = -x_o^*[-n]
\end{aligned}
$$

DTFT

$$
\begin{aligned}
X(\omega) &= X_e(\omega) + X_o(\omega) \\
X_e(\omega) &\triangleq \frac{1}{2}\left(X(\omega) + X^*(w)\right) = X_e^*(-\omega) \\
X_o(\omega) &\triangleq \frac{1}{2}\left(X(\omega) - X^*(w)\right) = -X_o^*(-\omega)
\end{aligned}
$$

Basic symmetries

Real and imaginary parts decomposition

$$
\begin{aligned}
x[n] &= x_R[n] + jx_I[n] \\
X(\omega) &= X_R(\omega) + jX_I(\omega)
\end{aligned}
$$

- $x[-n] \overset{\mathcal{F}}{\leftrightarrow} X(-\omega)$
- $x^*[n] \overset{\mathcal{F}}{\leftrightarrow} X^*(-\omega)$
- $x_R[n] \overset{\mathcal{F}}{\leftrightarrow} X_e(\omega)$
- $jx_I[n] \overset{\mathcal{F}}{\leftrightarrow} X_o(\omega)$
- $x_e[n] \overset{\mathcal{F}}{\leftrightarrow} X_R(\omega)$
- $x_o[n] \overset{\mathcal{F}}{\leftrightarrow} jX_I(\omega)$

More properties

Convolution

$$
x[n]\text{*y[n]} \overset{\mathcal{F}}{\leftrightarrow} X(\omega)Y(\omega)
$$

Multiplication

$$
x[n]y[n] \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\phi)Y(\omega-\phi)d\phi
$$

Parseval's relation

$$
\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega
$$

Plancherel's relation

$$
\sum_{n=-\infty}^{\infty} x[n]y[n]^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)Y^*(\omega)d\omega
$$

6 Analysis of LTI systems in the z-Domain

6.1 LTI systems described by LCCDE

Linear Constant Coefficient Difference Equations

Definition

A DT system can be described by an LCCDE of order N if:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

where $a_0 \neq 0$ and $a_N \neq 0$.

- If we further assume initial *rest conditions*, i.e.:

$$\forall n < n_0,\, x[n] = 0 \Rightarrow \forall n < n_0,\, y[n] = 0$$

  LCCDE corresponds to unique causal LTI system.

Example: Accumulator system

$$x[n] \to y[n] = \sum_{k=-\infty}^{n} x[k]$$

This LTI system can be rewritten according to:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n-1} x[k] + x[n] \\ &= y[n-1] + x[n] \end{aligned}$$

$\Rightarrow$ LCCDE of order $N = 1$ ($M = 0$, $a_0 = 1$, $a_1 = -1$, $b_0 = 1$)
LCCDEs lead to efficient recursive implementation:

- Recursive because computation of $y[n]$ make use past output signal values ($y[n-1]$)
- Efficient: in the case of the accumulator it requires only 1 adder and 1 memory unit instead of an infinite number of adders and memory units.

6.2 One-sided Z-Transform

One-sided Z-Transform

- The two-sided ZT requires that the corresponding signals be specified for entire time range $-\infty < n < \infty$
  - Prevent evaluation of the output of non-relaxed systems
- The one-sided ZT can be used to solve difference equations with initial conditions

Definition

$$X^+(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Characteristics

1. Does not contain information about the signal $x[n]$ for negative values of time ($n < 0$)
2. It is *unique* only for *causal* signals
3. one-sided ZT of $x[n]$ is identical to the two-sided ZT of $x[n]u[n]$

Properties

Almost all properties for the two-sided ZT carry over to the one-sided ZT with exception of the *shifting property.*

Case 1: Time Delay

If

$$x[n] \overset{z^+}{\longleftrightarrow} X^+(z)$$

then

$$\forall k > 0,\; x[n-k] \overset{z^+}{\longleftrightarrow} z^{-k}\left(X^+(z) + \sum_{n=1}^{k} x[-n]z^n\right)$$

Case 2: Time advance

If

$$x[n] \overset{z^+}{\longleftrightarrow} X^+(z)$$

then

$$\forall k > 0,\; x[n+k] \overset{z^+}{\longleftrightarrow} z^{k}\left(X^+(z) - \sum_{n=0}^{k-1} x[n]z^{-n}\right)$$

Example

Determine the step response of the system with IC $y[-1] = 1$:

$$y[n] = \alpha y[n-1] + x[n], \text{ with } -1 < \alpha < 1$$

1. By taking the one-sided ZT of both sides:
$$Y^+(z) = \alpha z^{-1}\left(Y^+(z) + y[-1]z\right) + X^+(z).$$
2. Compute one-sided ZT of $x[n]$:
$$x[n] = u[n] \overset{z^+}{\longleftrightarrow} X^+(z) = \frac{1}{1-z^{-1}}$$
3. Solving for $Y^+[n]$:
$$Y^+(z) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{1}{(1-\alpha z^{-1})(1-z^{-1})}$$
4. Perform partial-fraction expansion:
$$Y^+(x) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{\frac{\alpha}{\alpha-1}}{1-\alpha z^{-1}} + \frac{\frac{1}{1-\alpha}}{1-z^{-1}}$$
5. Compute inverse ZT:
$$\begin{aligned} y[n] &= \alpha^{n+1}u[n] + \frac{1-\alpha^{n+1}}{1-\alpha}u[n] \\ &= \frac{1}{1-\alpha}\left(1-\alpha^{n+2}\right)u[n] \end{aligned}$$

Final Value Theorem

If

$$x[n] \overset{z^+}{\longleftrightarrow} X^+(z)$$

then

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X^+(z)$$

- The limit exists if the ROC of $(z-1)X^+(z)$ includes the unit circle
- Useful when the asymptotic behavior of a signal $x[n]$ is desired knowing its ZT

6.3 The system function

The system function

LTI system $\mathcal{H}$ (recall)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \text{ with } h[n] = \mathcal{H}\{\delta[n]\}$$

Definition
The system function of $\mathcal{H}$, denoted $H(z)$ is the ZT of $h[n]$:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n},\ z \in \mathcal{R}_H$$

where $\mathcal{R}_H$ denotes the corresponding ROC.

- If $H(z)$ and $\mathcal{R}_H$ are known, $h[n]$ can be recovered via inverse ZT
- if $z = e^{j\omega} \in \mathcal{R}_H$ (the ROC contains the unit circle) then

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \equiv H(\omega)$$

  $\Rightarrow$ the system function evaluated at $z = e^{j\omega}$ corresponds to the frequency response at angular frequency ω

Properties
Let $\mathcal{H}$ be LTI system with system function $H(z)$ and ROC $\mathcal{R}_H$.

- If $y[n]$ denotes the response of $\mathcal{H}$ to arbitrary input $x[n]$, then:

$$Y(z) = H(z)X(z)$$

- LTI system $\mathcal{H}$ is causal iff $\mathcal{R}_H$ is the exterior of a circle (including ∞)
- LTI system $\mathcal{H}$ is stable iff $\mathcal{R}_H$ contains the unit circle:

$$\begin{aligned} \mathcal{H}\,\text{stable} &\Leftrightarrow \sum_n |h[n]| < \infty \\ &\Leftrightarrow e^{j\omega} \in \mathcal{R}_H \end{aligned}$$

$$|H(z)| \leq \sum_n \left|h[n]z^{-n}\right|$$

  evaluated on the unit circle: $z = e^{i\omega}$:

$$|H(z)| \leq \sum_n |h[n]| < \infty$$

LCCDE system function

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Taking ZT on both sides:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Leading to a rational system function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Rational system with real coefficients

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- In many applications, coefficients a_k and b_k are real implying:

$$H^*(z) = H(z^*)$$

- Thus, if z_k is a zero of $H(z)$ then:

$$H(z_k^*) = (H(z_k))^* = 0^* = 0$$

  which shows that z_k^* is also a zero of $H(z)$

- More generally, it can be shown that complex poles (or zeros) occur in complex conjugate pairs:
  - if p_k is a pole of order l of $H(z)$, so is p_k^*
  - if z_k is a zero of order l of $H(z)$, so is z_k^*

6.4 Response of rational system Functions

Response of rational system functions

- Let be $H(z) = \frac{B(z)}{A(z)}$ the system function of a LCCDE system:
  - roots of $A(z)$ are the poles of $H(z)$
  - roots of $B(z)$ are the zeros of $H(z)$
- Let assume that the input signal $x[n]$ has a rational ZT of the form $X(z) = \frac{N(z)}{Q(z)}$
- If the system is initially relaxed the initial conditions ($y[-1] = y[-2] = \cdots = y[-N] = 0$) the output of the system has the form:

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

- The partial fraction expansion of $Y(z)$ yields

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

  if $\forall k \forall m$, $p_k \neq q_m$ and there is no pole-zero cancellation.

Response of rational system functions

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

- The inverse transform of $Y(z)$ can be written

$$y[n] = \underbrace{\sum_{k=1}^{N} A_k (p_k)^n u[n]}_{\text{natural response}} + \underbrace{\sum_{k=1}^{L} Q_k (q_k)^n u[n]}_{\text{forced response}}$$

  - Influence of the input signal on the *natural response* is through the scale factor $\{A_k\}$
  - Influence of the system on the *forced response* is through the scale factor $\{Q_k\}$

Response with nonzero initial conditions

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- $x[n]$ is assumed to be causal
- effects of all previous input signals are reflected in the initial conditions $y[-1]$, $y[-2]$, $, \cdots, y[-N]$
- To determine $y[n]$, $\forall n \geq 0$, the one-sided ZT can be used:

$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-k} \left(Y^+(z) + \sum_{n=1}^{k} y[-n] z^n \right) + \sum_{k=0}^{M} b_k z^{-k} X^+(z)$$

$$Y^+(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} X^+(z) - \frac{\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y[-n] z^n}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Response with nonzero initial conditions

Since $x[n]$ is causal $X^+(z) = X(z)$ and the expression can be written:

$$Y^+(z) = \underbrace{H(z)X(z)}_{Y_{zs(z)}} + \underbrace{\frac{N_0(z)}{A(z)}}_{Y_{zi}^+}$$

- $Y_{zs}(z) = H(z)X(z)$ is the *zero-state* response of the system
- $Y_{zi}^+(z) = \frac{N_0(z)}{A(z)}$ is the *zero-input* response of the system
- The zero-state response of the system remains the same and gives:

$$y_{zs}[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n] + \sum_{k=1}^{L} Q_k (q_k)^n u[n]$$

- Since$Y_{zi}^+(z) = \frac{N_0(z)}{A(z)}$ its poles are $p_1, \cdots, p_N$ and the zero-input response has the form:

$$y_{zi}[n] = \sum_{k=1}^{N} D_k (p_k)^n u[n]$$

Response with nonzero initial conditions

- The terms involving the poles $\{p_k\}$ can be combined:

$$y[n] = y_{zs}[n] + y_{zi}[n] = \sum_{k=1}^{N} A_k^{'}(p_k)^n u[n] + \sum_{k=1}^{L} Q_k(q_k)^n u[n]$$

  with $A_k^{'} = A_k + D_k$.

- Effect of initial conditions alter the natural response of the system through modification of the scale factors
- No new poles are introduced by the nonzero initial conditions
- There is no effect on the forced response of the system

6.5 Schur-Cohn Stability test

Schur-Cohn stability test

Reminder

- LTI system $\mathcal{H}$ is causal iff $\mathcal{R}_H$ is the exterior of a circle (including ∞)
- LTI system $\mathcal{H}$ is stable iff $\mathcal{R}_H$ contains the unit circle

$\Rightarrow$ A causal system described by its rational system function is stable iff its poles are strictly inside the unit circle.

Let be $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$ the denominator polynomial of $H(z)$. A polynomial of degree m is denoted by:

$$A_m(z) = \sum_{k=0}^{m} a_m[k] z^{-k} \quad a_m(0) = 1$$

The reverse polynomial $B_m(z)$ of degree m is defined as:

$$B_m(z) = z^{-m} A_m(z^{-1}) = \sum_{k=0}^{m} a_m[m-k] z^{-k}$$

Schur-Cohn stability test

The set of coefficients K_1, K_2, $\cdots$, K_N must satisfy the condition $|K_m| < 1$

1. First iteration:

$$A_N(z) = A(z)$$

   and $K_N = a_N(N)$

2. Compute the lower-degree polynomial:

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

   and $K_m = a_m(m)$

3. Loop to step 2 until it fails or $m = 1$

6.6 Frequency response of rational systems

Frequency response of rational systems

- Knowing poles and zeros of $H(z)$ can be expressed as:

$$H(z) = Gz^{-K} \frac{\prod_{k=0}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

- Frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = Ge^{-j\omega K} \frac{\prod_{k=0}^{M} (j\omega - z_k)}{\prod_{k=1}^{N} (j\omega - p_k)}$$

- Define:

$$V_k(\omega) = \left|e^{j\omega} - z_k\right| U_k(\omega) = \left|e^{j\omega} - p_k\right|$$
$$\theta_k(\omega) = \angle\left(e^{j\omega} - z_k\right) \phi_k(\omega) = \angle\left(e^{j\omega} - p_k\right)$$

- Magnitude response:

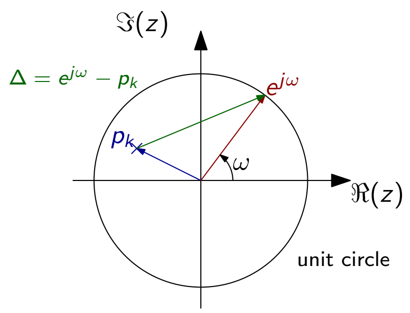
$$|H(\omega)| = |G| \frac{V_1(\omega) \cdots V_M(\omega)}{U_1(\omega) \cdots U_N(\omega)}$$

- Phase response:

$$\angle H(\omega) = \angle G - \omega K + \sum_{k=1}^{\infty} \theta_k(\omega) - \sum_{k=1}^{N} \phi_k(\omega)$$

Geometrical interpretation

- Consider pole p_k:



  - $\Delta = e^{j\omega} - p_k$: vector joining p_k to point $e^{j\omega}$ on unit circle
  - $U_k(\omega) = |\Delta|$: length of vector Δ
  - $\phi_k(\omega) = \angle\Delta$: angle between Δ and real axis

- A similar interpretation holds for the terms $V_k(\omega)$ and $\theta_k(\omega)$ associated to the zeros z_k

Some basic principles

- For stable and causal systems, the poles are located inside the unit circle; zeros can be anywhere.

- Poles near the unit circle at $p = re^{j\omega_0}$ $(r < 1)$ give rise to:
  - peak in $|H(\omega)|$ near ω_0
  - rapid phase variation near ω_0

- Zeros near the unit circle at $z = re^{j\omega_0}$ give rise to:
  - deep notch in $|H(\omega)|$ near ω_0
  - rapid phase variation near ω_0

6.7 Analysis of certain basic systems

First order LTI systems

The system function is given by:

$$H(z) = G\frac{1 - bz^{-1}}{1 - az^{-1}}$$

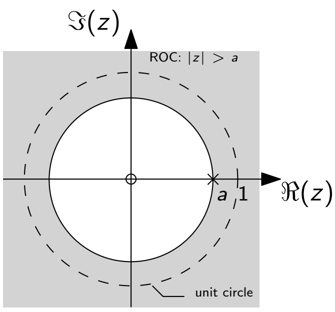
- Poles and zeros are:
  - pole: $z = a$ (simple)
  - zero: $z = b$ (simple)
- Practical requirements:
  - causality: ROC: $|z| > |a|$
  - stability: $|a| < 1$
- Impulse response (ROC: $|z| > |a|$):

$$h[n] = G\left(1 - \frac{b}{a}\right)a^n u[n] + G\frac{b}{a}\delta[n]$$

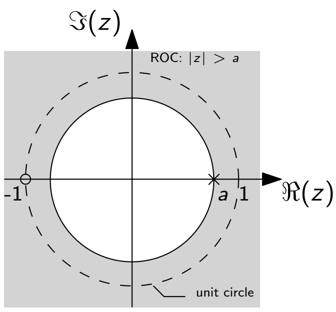
Low-pass case

$$H(z) = G\frac{1 - bz^{-1}}{1 - az^{-1}}$$

- To get a low-pass behavior: $a = 1 - \epsilon$, where $0 < \epsilon \ll 1$
- Additional attenuation of high-frequency is possible by proper placement of the zero $z = b$.

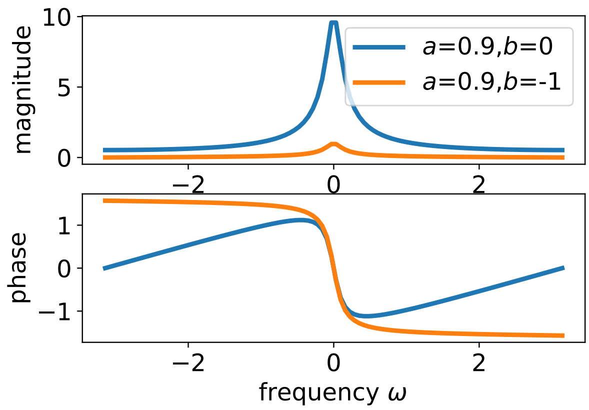


$H_1(z) = G_1 \frac{1}{1-az^{-1}}$ (zero: $b = 0$)
$G_1 = 1 - a \Rightarrow H_1(\omega = 0) = 1$



$H_2(z) = G_2 \frac{1+z^{-1}}{1-az^{-1}}$ (zero: $b = -1$)
$G_2 = \frac{1-a}{2} \Rightarrow H_2(\omega = 0) = 1$

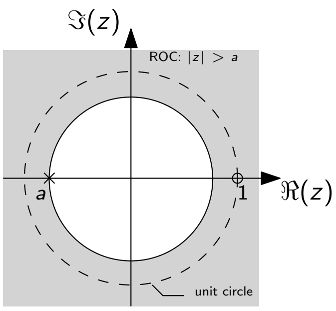
Frequency responses of the corresponding low-pass systems



High-pass case

$$H(z) = G\frac{1 - bz^{-1}}{1 - az^{-1}}$$

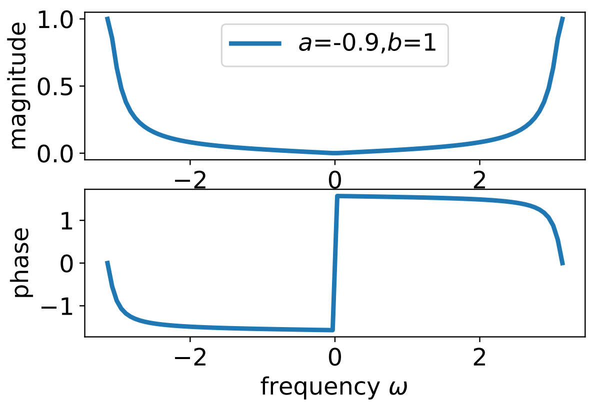
- To get a high-pass behavior: $a = -1 + \epsilon$, where $0 < \epsilon \ll 1$
- To get a high attenuation of the DC component, one has to locate the zero at or near $b = 1$



$H_3(z) = G_3\frac{1-z^{-1}}{1-az^{-1}}$ (zero: $b = 1$)
$G_3 = \frac{1+a}{2} \Rightarrow H_3(\omega = -\pi) = 1$

Frequency response of the corresponding high-pass system

- Frequency response:



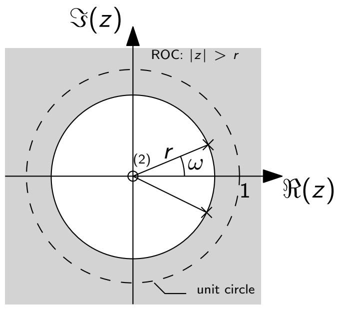
Second order systems

System function

$$H(z) = G\frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

- Poles
  - if $a_1^2 > 4a_2$: 2 distinct poles (real) at $p_{1,2} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_2}$
  - if $a_1^2 = 4a_2$: double pole (real) at $p_1 = -\frac{a_1}{2}$
  - if $a_1^2 < 4a_2$: 2 distinct poles (complex) at $p_{1,2} = -\frac{a_1}{2} \pm j\frac{1}{2}\sqrt{4a_2 - a_1^2}$
- Practical requirements:
  - causality: ROC: $|z| > \max\{|p_1|, |p_2|\}$
  - stability: $|p_1| < 1$ and $|p_2| < 1 \Leftrightarrow |a_2| < 1$ and $a_2 > |a_1| - 1$

Second order systems: resonator

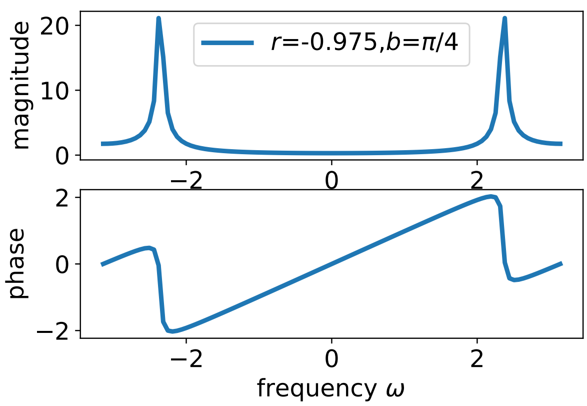


$p_1 = re^{j\omega_0}$
$p_2 = re^{-j\omega_0} = p_1^*$

$$\begin{aligned} H(z) &= G\frac{1}{\left(1 - re^{j\omega_0 z^{-1}}\right)\left(1 - re^{-j\omega_0 z^{-1}}\right)} \\ &= G\frac{1}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}} \end{aligned}$$

Second order systems: resonator

- Frequency response:



- The frequency response clearly shows peaks around $\pm\omega_0$.
- For r close to 1 (but < 1), $|H(\omega)|$ reaches a maximum at $\pm\omega_0$

FIR filters

System function

$$
\begin{aligned}
H(z) = B(z) &= b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \\
&= b_0 \left(1 - z_1 z^{-1}\right) \cdots \left(1 - z_M z^{-1}\right)
\end{aligned}
$$

- This is zeroth order rational system: $A(z) = 1$
  - The M zeros z_k can be anywhere in the complex plane
  - There is multiple pole of order M at $z = 0$
- Practical requirement: none
  - Above system is always causal and stable
- Impulse response:

$$
h[n] = \begin{cases} b_n & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}
$$

FIR filters: moving average system

- Difference equation:

$$
y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]
$$

- System function:

$$
H(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} = \frac{1}{M} \frac{1 - z^{-M}}{1 - z^{-1}}
$$

- PZ analysis: roots of the numerator

$$
z^M = 1 \Rightarrow z = e^{j2\pi k/M},\ k = 0, 1, \cdots, M-1
$$

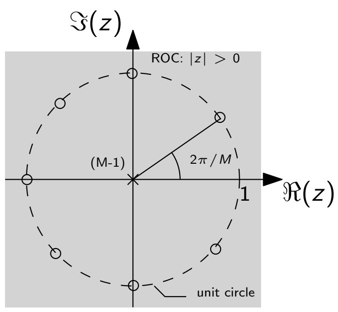
$\Rightarrow$ there is no pole at $z = 1$ because of PZ cancellation:

$$
H(z) = \frac{1}{M} \prod_{k=1}^{M-1} \left(1 - e^{j2\pi k/M} z^{-1}\right)
$$

FIR filters: moving average system

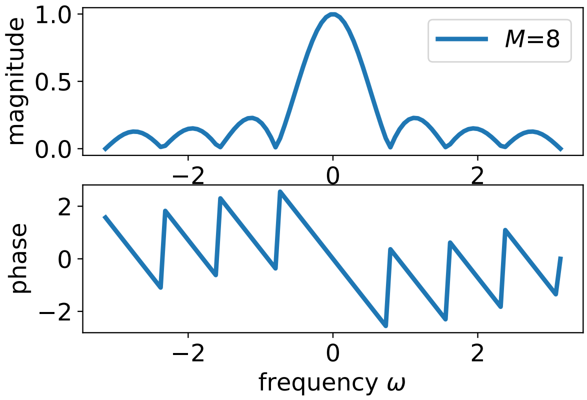
$$
H(z) = \frac{1}{M} \prod_{k=1}^{M-1} \left(1 - e^{j2\pi k/M} z^{-1}\right)
$$

- PZ diagram for $M = 8$



FIR filters: moving average system

- Frequency response:



7 Discrete Fourier Transform (DFT)

Introduction

*D*iscrete *T*ime *F*ourier *T*ransform $\neq$ *D*iscrete *F*ourier *T*ransform

Definition of the DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},\ \omega \in [-\pi, \pi]$$

Several drawbacks from a computational viewpoint:

- the summation over n is infinite
- the variable ω is continuous

In many situation, it is either not possible, or not necessary to implement the infinite summation:

- only the signal samples $x[n]$ from n to $N-1$ are available
- the signal is known to be zero outside this range; or
- the signal is periodic with period N

7.1 The DFT and its inverse

The DFT and its inverse

Definition of the DFT

$$\begin{aligned} X[k] = \mathrm{DFT}_N\left\{x[n]\right\} &\triangleq \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N},\ k \in \mathbb{Z} \\ &\triangleq \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n},\ \omega = 2\pi k/N \end{aligned}$$

- The N-point DFT is a transformation that maps DT signal samples $\{x[n]\}$ into a periodic sequence $\{X[k]\}$
- Only samples $x[0], \dots, x[N-1]$ are used in the computation
- The N-point DFT is *periodic*, with period N: $X[k+N] = X[k]$
- The "D" in DFT stands for discrete frequency (i.e. ω_k)

Examples

1. Consider

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1, \dots, N-1 \end{cases}$$

$$\forall k \in \mathbb{Z},\ X[k] = 1$$

2. Let

$$x[n] = a^n,\ n = 0, \ldots, N-1$$

$$\begin{aligned}\forall k \in \mathbb{Z},\ X[k] &= \sum_{n=0}^{N-1} \rho_k^n, \text{ where } \rho_k \triangleq a e^{-j2\pi k/N} \\ &= \begin{cases} N & \text{if } \rho_k = 1 \\ \frac{1-\rho_k^N}{1-\rho} & \text{otherwise} \end{cases}\end{aligned}$$

case $a = 1$
$\Rightarrow X[k] = \begin{cases} N & \text{if } k = 0 \\ 0 & \text{if } k = 1, \ldots, N-1 \end{cases}$
case $a = e^{j2\pi l/N}$
$\Rightarrow X[k] = \begin{cases} N & \text{if } k = l \text{ modulo } N \\ 0 & \text{otherwise} \end{cases}$

Inverse DFT (IDFT)

Definition of the IDFT

$$\begin{aligned}\tilde{x}[n] = \mathrm{IDFT}_N\left\{X[k]\right\} &= \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N},\ n \in \mathbb{Z} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\omega_k n},\ \omega = 2\pi k/N\end{aligned}$$

- In general, $\forall n \in \mathbb{Z},\ \tilde{x}[n] \neq x[n]$
- Only the samples $X[0], \ldots, X[N-1]$ are used in the computation
- The N-point IDFT is periodic, with period N: $\tilde{x}[n+N] = \tilde{x}[n]$

IDFT Theorem

IDFT Theorem

If X[k] is the N_point DFT of the samples $\{x[0], \ldots, x[N-1]\}$ then:

$$x[n] = \tilde{x}[n] = IDFT_N\left\{X[k]\right\},\ n = 0, \ldots, N-1$$

- Nothing is said about sample $x[n]$ out of the range $n = 0, \ldots, N-1$
- The IDFT $\tilde{x}[n]$ is periodic with period N whereas no such requirement is imposed on the original signal $x[n]$
- Values of $x[n]$ for $n < 0$ and for $n \geq N$ cannot in general be recovered from the DFT samples $X[k]$
- There are two important special cases when the complete signal $x[n]$ can be recovered from the DFT samples $X[k]$:
  - $x[n]$ is periodic with period N
  - $x[n]$ is known to be zero for $n < 0$ and for $n \geq N$

7.2 Relationship between the DFT and the DTFT

Introduction

The DFT may be viewed as a finite approximation to the DTFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} \approx X(\omega_k = \frac{2\pi k}{N}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_k n}$$

- It should not be possible to recover the DTFT exactly from the DFT
  - an arbitrary signal $x[n]$ cannot be recovered entirely from its N-point DFT
- However, in the following two special cases the DTFT can be evaluated exactly at any frequency $\omega \in [-\pi, \pi]$ if the DFT is known:
  - finite length signals
  - N-periodic signals

Finite length signals

Assumption
Suppose $x[n] = 0$ for $n < 0$ and for $n \geq N$

Inverse DFT
In this case $x[n]$ can be recovered entirely from its N-point DFT:

$$\tilde{x}[n] = \text{IDFT}\left\{X[k]\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N},\ n \in \mathbb{Z}$$

- For $n = 0, \ldots, N-1$ the IDFT theorem yields: $x[n] = \tilde{x}[n]$
- For $n < 0$ and for $n \geq N$, by assumption: $x[n] = 0$

$$x[n] = \begin{cases} \tilde{x}[n] & \text{if}\, 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

Finite length signals

Relationship between DFT and DTFT
In this case the DTFT $X(\omega = \omega_k = 2\pi k/N)$ can be completely reconstructed from the N-point DFT $X[k]$:

$$X(\omega_k) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = X[k]$$

In the general case, i.e. ω arbitrary, the following theorem can be applied.

Theorem
$X(\omega)$ and $X[k]$ respectively denote the DTFT and N-point DFT of signal $x[n]$ ($x[n] = 0$ for for $n < 0$ and for $n \geq N$:

$$X(\omega) = \sum_{k=0}^{N-1} X[k]P(\omega - \omega_k)$$

where

$$P(\omega) \triangleq \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n}$$

Finite length signals

$$X(\omega)=\sum_{k=0}^{N-1}X[k]P(\omega-\omega_k)\,\text{with}\,\mathrm{P}(\omega)\triangleq\frac{1}{N}\sum_{n=0}^{N-1}e^{-j\omega n}$$

Properties of $P(\omega)$:

- The theorem provides a kind of interpolation formula for evaluating $X(\omega)$ in between adjacent values of $X(\omega_k)=X[k]$
- Periodicity: $P(\omega+2\pi)=P(\omega)$
- If $\omega=2\pi l\,(l\in\mathbb{Z})$ then $e^{-j\omega n}=e^{-j2\pi ln}=1$ so that $P(\omega)=1$
- If $\omega\neq 2\pi l$

$$P(\omega)=\frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{j\omega}}=\frac{1}{N}e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

- Note that at frequency $\omega_k=2\pi/N$

$$P(\omega_k)=\begin{cases}1 & k=0\\ 0 & k=1,\ldots,N-1\end{cases}$$

Periodic signals

Assumption
Suppose $x[n]$ is N-periodic, i.e. $x[n+N]=x[n]$

Inverse DFT
In this case $x[n]$ can be recovered entirely from its N-point DFT:

$$\tilde{x}[n]=\mathrm{IDFT}\left\{X[k]\right\}=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N},\; n\in\mathbb{Z}$$

- For $n=0,\ldots,N-1$ the IDFT theorem yields: $x[n]=\tilde{x}[n]$
- Since both $x[n]$ and $\tilde{x}[n]$ are known to be N-periodic, it follows that $x[n]=\tilde{x}[n]$ must also be true for $n<0$ and for $n\geq N$:

$$x[n]=\tilde{x}[n],\;\forall n\in\mathbb{Z}$$

Periodic signals

Relationship between DFT and DTFT
Since the N-periodic signal $x[n]$ can be recovered completely from its N-point DFT $X[k]$, it should be possible to reconstruct the DTFT $X(\omega)$ from $X[k]$.

Theorem
$X(\omega)$ and $X[k]$ respectively denote the DTFT and N-point DFT of signal $x[n]$

$$X(\omega)=\frac{2\pi}{N}\sum_{-\infty}^{\infty}X[k]\delta_a(\omega-\omega_k)$$

where $\delta_a(\omega)$ denotes an analog delta function centered at $\omega=0$

- $X(\omega)\Leftrightarrow$ periodic train of infinite impulses in the ω domain
- When $x[n]$ is N-periodic, the DFT admits a Fourier series interpretation since the IDFT provides an expansion of $x[n]$ as a sum of harmonically related complex exponential signals $e^{j\omega_k n}$:

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\omega_k n},\; n\in\mathbb{Z}$$

Signal reconstruction via DTFT sampling

1. Let $X(\omega)$ be the DTFT of signal $x[n]$, $n \in \mathbb{Z}$, that is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},\ \omega \in \mathbb{R}.$$

2. Consider the sampled values of $X(\omega)$ at uniformly spaced frequencies $\omega_k = 2\pi k/N$ for $k = 0, \ldots, N-1$.

3. Suppose we compute the IDFT of the samples $X(\omega_k)$:

$$\hat{x}[n] = \text{IDFT}\{X(\omega_k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(\omega_k)e^{j\omega_k n}$$

What is the relationship between the original signal $x[n]$ and the reconstructed sequence $\hat{x}[n]$?

- $\hat{x}[n]$ is N-periodic, while $x[n]$ may not be
- Even for $n = 0, \ldots, N-1$ there is no reason for $\hat{x}[n]$ to be equal to $x[n]$

Signal reconstruction via DTFT sampling

Theorem

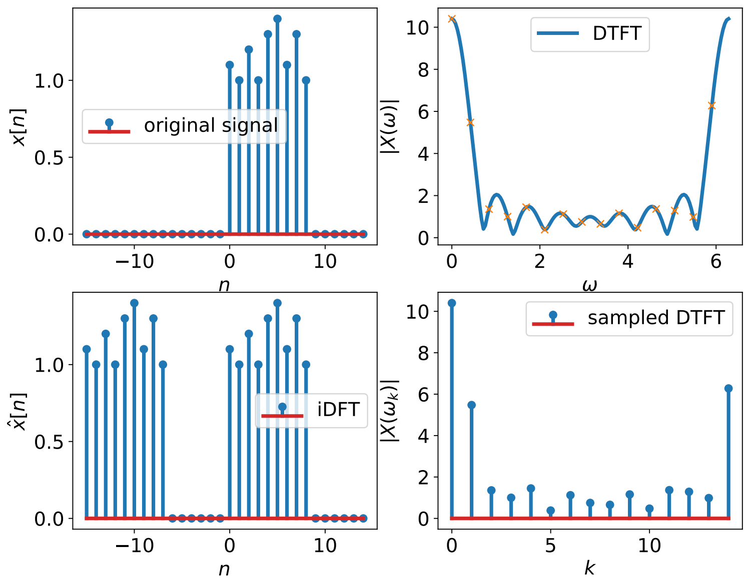
$$\hat{x} = \text{IDFT}\{X(\omega_k)\} = \sum_{r=-\infty}^{\infty} x[n-rN]$$

- $\hat{x}[n]$ is an infinite sum of the sequences $x[n-rN]$, $r \in \mathbb{Z}$:
- Each of these sequences $x[n-rN]$ is a shifted version of $x[n]$ by an integer multiple of N
- Depending on whether or not these shifted sequences overlap, we distinguish two important cases:
   1. Time limited signal: suppose $x[n] = 0$ for $n < 0$ and for $n \geq N$. Then there is no temporal overlap of the sequences $x[n-rN]$. We can recover $x[n]$ *exactly* from one period of $\hat{x}[n]$:

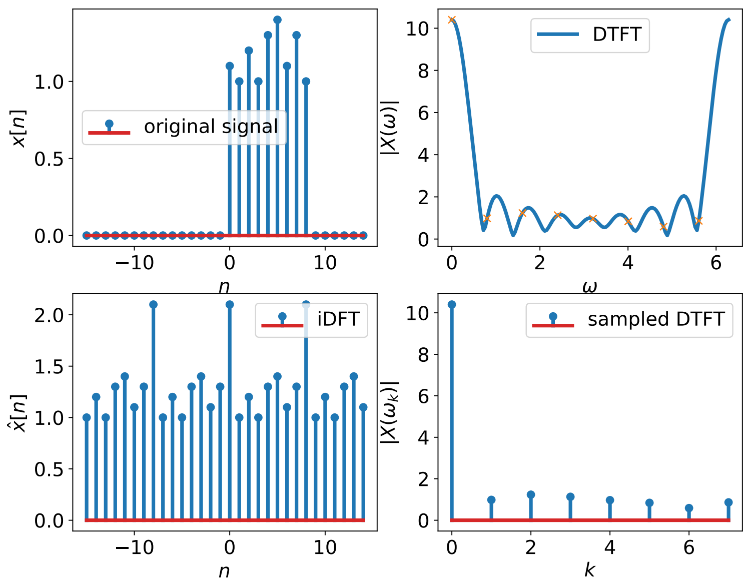
   $$x[n] = \begin{cases} \hat{x}[n] & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

   2. Non time-limited signal: suppose that $x[n] \neq 0$ for some $n < 0$ or $n \geq N$. Then, the sequences $x[n-rN]$ for different values of r will overlap in the time-domain. In this case, it is not true that $\hat{x}[n] = x[n]$ for all $0 \leq n \leq N-1 \Rightarrow$ *temporal aliasing*

Time limited signal



Non time-limited signal



7.3 Properties of the DFT

Properties of the DFT

Notations

$x[n]$ and $y[n]$ are defined over $0 \le n \le N-1$:

$$x[n] \overset{\mathrm{DFT}_N}{\leftrightarrow} X[k]$$
$$y[n] \overset{\mathrm{DFT}_N}{\leftrightarrow} Y[k]$$

$X[k]$ and $Y[k]$ are viewed as N-periodic sequences, defined for all $k \in \mathbb{Z}$.

Modulo N operation

any integer $n \in \mathbb{Z}$ can be expressed uniquely as $n = k + rN$ where $k \in \{0, \ldots, N-1\}$ and $r \in \mathbb{Z}$:

$$(n)_N = n \text{ modulo N} \triangleq k$$

Time reversal and complex conjugation

Circular time reversal

Given a sequence $x[n]$, $0 \leq n \leq N-1$, its circular reversal (CR) is defined as:

$$\mathrm{CR}\left\{x[n]\right\} = x\left[(-n)_N\right], \; 0 \leq n \leq N-1$$

Example: Let $x[n] = 6 - n$ for $n = 0, \ldots, 5$.

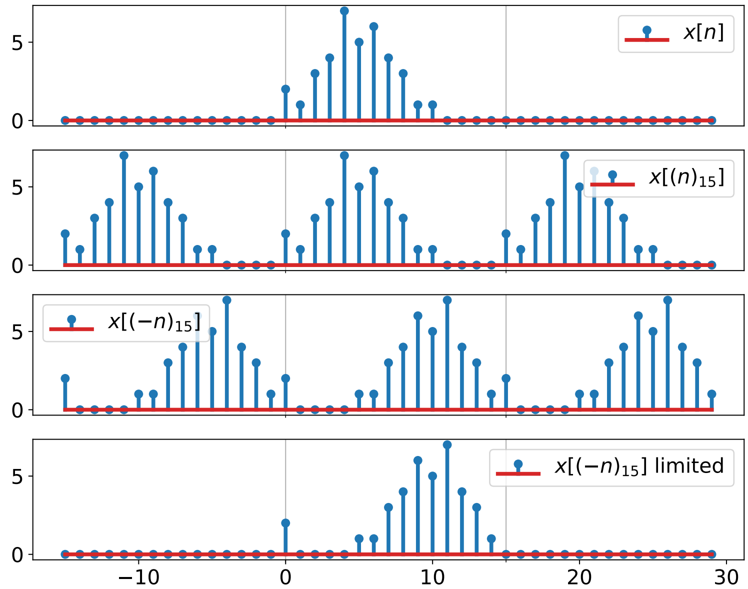
n	0	1	2	3	4	5
$x[n]$	6	5	4	3	2	1
$(-n)_6$	0	5	4	3	2	1
$x\left[(-n)_6\right]$	6	1	2	3	4	5

Time reversal and complex conjugation

Interpretation:

- Circular reversal can be seen as an operation on the set of samples $x[0], \ldots, x[N-1]$:
  - $x[0]$ is left unchanged
  - for $k = 1$ to $N-1$ samples $x[k]$ and $x[N-k]$ are exchanged
- One can also see this operation consisting in:
  1. periodizing the samples of $x[n]$, $0 \leq n \leq N-1$ with period N
  2. time-reversing the periodized sequence
  3. keeping only the samples between 0 and $N-1$

Time reversal and complex conjugation



Time reversal and complex conjugation

Property

$$x\left[(-n)_N\right] \overset{\mathrm{DFT}_N}{\leftrightarrow} X[-k]$$

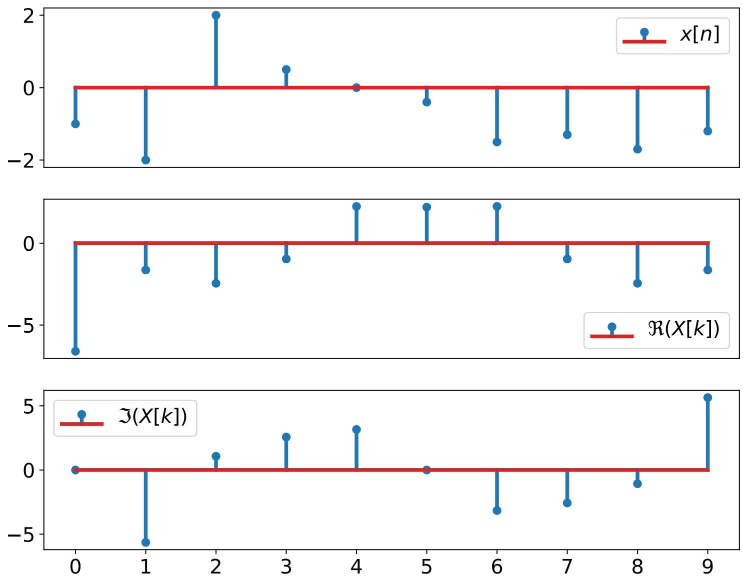
$$x^*[n] \overset{\mathrm{DFT}_N}{\leftrightarrow} X^*[-k]$$

$$x^*\left[(-n)_N\right] \overset{\mathrm{DFT}_N}{\leftrightarrow} X^*[k]$$

Remarks:

- Since $X[k]$ is periodic, $X[-k] = X\left[(-k)_N\right]$
- For real-valued signals:
  - $x[n] = x^*[n] \Leftrightarrow X[k] = X[-k]$
  - $X[0]$ is real
  - if N is even: $X[N/2]$ is real
  - $X[N-k] = X^*[k]$ for $1 \le k \le N-1$

Time reversal and complex conjugation



Linearity

Linearity

$$ax[n] + by[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} aX[k] + bY[k]$$

Even and odd decomposition

Conjugate symmetric components of finite sequences

$$x_{e,N}[n] \triangleq \frac{1}{2}\left(x[n] + x^*\left[(-n)_N\right]\right)$$
$$x_{e,N}[n] \triangleq \frac{1}{2}\left(x[n] - x^*\left[(-n)_N\right]\right)$$

In the case of a N-periodic sequence the modulo N operation can be omitted:

$$X_e[k] \triangleq \frac{1}{2}\left(X[k] + X^*[-k]\right)$$
$$X_o[k] \triangleq \frac{1}{2}\left(X[k] - X^*[-k]\right)$$

$$\Re\left\{x[n]\right\} \overset{DFT_N}{\longleftrightarrow} X_e[k]$$
$$j\Im\left\{x[n]\right\} \overset{DFT_N}{\longleftrightarrow} X_o[k]$$
$$x_{e,N}[n] \overset{DFT_N}{\longleftrightarrow} \Re\left\{X[k]\right\}$$
$$x_{o,N}[n] \overset{DFT_N}{\longleftrightarrow} j\Im\left\{X[k]\right\}$$

Circular shift

Definition

Given a sequence $x[n]$ defined over the interval $0 \leq n \leq N-1$, we define its circular shift by k as follows:

$$\mathrm{CS}_k\left\{x[n]\right\} = x\left[(n-k)_N\right], \, 0 \leq n \leq N-1$$

Example: Let $x[n] = 6 - n$ for $n = 0, \ldots, 5$.

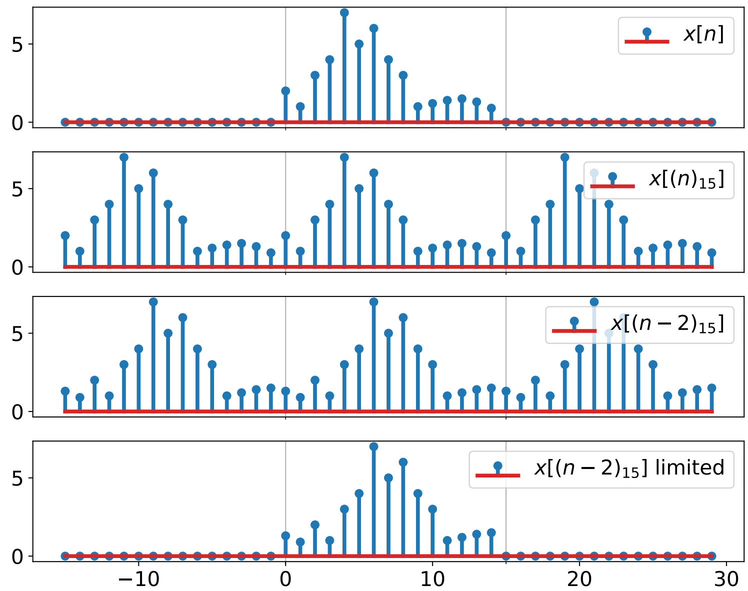
n	0	1	2	3	4	5
$x[n]$	6	5	4	3	2	1
$(n-2)_6$	4	5	0	1	2	3
$x\left[(n-2)_6\right]$	2	1	6	5	4	3

Circular shift

Interpretation:

- Can be seen as an operation on the set of signal samples $x[n]$ in which:
  - signal samples $x[n]$ are shifted as in a conventional shift
  - any signal sample leaving the interval $0 \leq n \leq N-1$ from one end reenters by the other end
- Alternatively, it may be interpreted as follows:
  1. periodizing the samples of $x[n]$, $0 \leq n \leq N-1$ with period N
  2. delaying the periodized sequence by k samples
  3. keeping only the samples between 0 and $N-1$

Circular shift



Circular shift

Circular shift property

$$x\left[(n-m)_N\right] \overset{\mathrm{DFT}_N}{\longleftrightarrow} e^{-j2\pi mk/N} X[k]$$

Frequency shift property

$$e^{j2\pi nm/N} x[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} X[k-m]$$

- Since the DFT $X[k]$ is already periodic, the modulo N operation is not needed here, that is: $X[(k-m)-N] = X[k-m]$.

Circular convolution

Definition
Let $x[n]$ and $y[n]$ be 2 sequences defined over $0 \leq n \leq N-1$:

$$x[n] \circledast y[n] \triangleq \sum_{m=0}^{N-1} x[m] y\left[(n-m)_N\right], \, 0 \leq n \leq N-1$$

Circular convolution property

$$x[n] \circledast y[n] \stackrel{\mathrm{DFT}_N}{\longleftrightarrow} X[k]Y[k]$$

Multiplication Property

$$x[n]y[n] \stackrel{\mathrm{DFT}_N}{\longleftrightarrow} \frac{1}{N} X[k] \circledast Y[k]$$

Other properties

Plancherel's relation

$$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]Y^*[k]$$

Parseval's relation

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

- Parseval's relation is a special case of Plancherel's relation: with $y[n] = x[n]$
- It allows the computation of the energy of the signal samples $x[n]\,(n = 0, \ldots, N-1)$ directly from the DFT samples $X[k]$

7.4 Relation between linear and circular convolutions

Introduction

Linear convolution
Time domain expression:

$$y_l[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k], \, n \in \mathbb{Z}$$

Frequency domain representation via DTFT:

$$Y_l(\omega) = X_1(\omega)X_2(\omega), \, \omega \in [0, 2\pi]$$

Circular convolution
Time domain expression:

$$y_c[n] = x_1[n] \circledast x_2[n] = \sum_{k=0}^{N-1} x_1[k]x_2\left[(n-k)_N\right], \, 0 \leq n \leq N-1$$

Frequency domain representation via N-point DFT

$$Y_c[k] = X_1[k]X_2[k], \, k \in \{0, \ldots N-1\}$$

A necessary condition...

Circular convolution and linear convolution are equivalent if:

$$y_l[n] = \begin{cases} y_c[n] & \text{if } 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$true if signals $x_1[n]$ and $x_2[n]$ have both finite length.

Finite length assumption

Suppose that $x_1[n]$ and $x_2[n]$ are time limited to $0 \le n < N_1$ and $0 \le n < N_2$ respectively then the linear convolution is *time limited to* $0 \le n < N_1 + N_2 - 1$

Example: Consider $x_1[n] = \{\underline{1}, 1, 1, 1\}$ and $x_2[n] = \{\underline{1}, 1/2, 1/2\}$

- $N_1 = 4$ and $N_2 = 3$
- $y_l[n] = \{1, 1.5, 2, 2, 1, .5\}$
- $\Rightarrow$ $N_3 = 6 = N_1 + N_2 - 1$

...proved to be a sufficient condition

Assuming $N \ge N_1 + N_2 - 1$:

1. Linear convolution gives:

$$y_l[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] = \sum_{k=0}^{n} x_1[k]x_2[n-k],\ 0 \le n < N$$

2. Circular convolution gives:

$$\begin{aligned} y_c[n] &= \sum_{k=0}^{N-1} x_1[k]x_2\left[(n-k)_N\right],\ 0 \le n < N \\ &= \sum_{k=0}^{n} x_1[k]x_2[n-k] + \underbrace{\sum_{k=n+1}^{N-1} x_1[k]x_2[N+n-k]}_{=0} \end{aligned}$$

Conclusion

The linear and circular convolution are equivalent if and only if:

$$N \ge N_1 + N_2 - 1$$

Relationship between $y_c[n]$ and $y_l[n]$

Assuming that $N \ge \max\{N_1, N_2\}$ the DFT of the 2 sequences $x_1[n]$ and $x_2[n]$ are samples of the corresponding DTFT:

$$\begin{aligned} N \ge N_1 &\Rightarrow X_1[k] = X_1(\omega_k),\ \omega_k = 2\pi k/N \\ N \ge N_2 &\Rightarrow X_2[k] = X_2(\omega_k) \end{aligned}$$

The DFT of the circular convolution is just the product of DFT:

$$\begin{aligned} Y_c[k] &= X_1[k]X_2[k] \\ &= X_1(\omega_k)X_2(\omega_k) = Y_l(\omega_k) \end{aligned}$$

$\Rightarrow$ $Y_c[k]$ is also made of uniformly samples of the DTFT $Y_l(\omega)$ of the linear convolution $y_l[n]$
$\Rightarrow$ The circular convolution $y_c[n]$ can be computed as the N-point IDFT of these frequency samples:

$$y_c[n] = \text{IDFT}_N\{Y_c[k]\} = IDFT_N\{Y_l(\omega_k)\}$$

Relationship between $y_c[n]$ and $y_l[n]$

Applying the "signal reconstruction via DTFT sampling" theorem we obtain:

$$y_c[n] = \sum_{r=-\infty}^{\infty} y_l[n - rN],\ 0 \leq n < N$$

The circular convolution is obtained by:

- a N-periodic repetition of the linear convolution $y_l[n]$
- and a windowing to limit the nonzero samples to the instants 0 to $N-1$

To get $y_c[n] = y_l[n]$ for $0 \leq n < N$, temporal aliasing must be avoided: the length of DFT$\geq$length of $y_l[n]$, i.e. :

$$N \geq N_1 + N_2 - 1$$

Linear convolution via DFT

Linear convolution via DFT can be summarized according to the following steps:

- Suppose that $x_1[n]$ and $x_2[n]$ are time limited to $0 \leq n < N_1$ and $0 \leq n < N_2$ respectively
- Select DFT size $N \geq N_1 + N_2 - 1$ (usually, $N = 2^k$)
- Compute the DFTs:

$$\begin{aligned} X_1[k] &= \mathrm{DFT}_N\left\{x_1[n]\right\},\ 0 \leq k < N \\ X_2[k] &= \mathrm{DFT}_N\left\{x_2[n]\right\},\ 0 \leq k < N \end{aligned}$$

- Compute the IDTF:

$$x_1[n] * x_2[n] = \begin{cases} \mathrm{IDFT}_N\left\{X_1[k]X_2[k]\right\} & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

7.5 The FFT

The FFT

Recall of the DFT

$$X[k] = \mathrm{DFT}_N\left\{x[n]\right\} \triangleq \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

For each coefficient:

- N complex multiplications
- $N-1$ complex additions

Then for the DFT$_N$:

- N^2 complex multiplications
- $N\left(N-1\right)$ complex additions

$\Rightarrow$ *algorithm complexity* $= N^2$

The FFT

Fast Fourier Transform

The FFT was developed in 1965 by Cooley and Tukey

- Assuming $N = 2^k$
- Considering even and odd part of the signals the DFT_N is split into 2 $\mathrm{DFT}_{N/2}$
- The FFT leads to:
  - $\frac{N}{2}\log_2 N$ complex multiplications
  - $N\log_2 N$ complex additions
- The algorithm complexity becomes $\frac{N}{2}\log_2 N$