F21RO INTELLIGENT ROBOTICS TUTORIAL 3 EULER ANGLES

QUESTIONS

1. Derive robust formulae for the three Z-Y-Z Euler angles which describe the orientation contained in a homogeneous transformation of the form –

$$\begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. In what sense are the formulae you have derived robust?

3. Calculate the Z-Y-Z Euler angle set that would produce a re-orientation of a manipulator's tool centre point of 180° about its local *x*-axis followed by -90° about its local *y*-axis.

SOLUTIONS

1. We first require to generate the general Euler angle matrix for a Z-Y-Z set of rotations ϕ, θ, ψ.

   The general Euler matrix is formed by multiplying together the standard rotation matrices for a *z* rotation of ϕ, a *y* rotation of θ and a further *z* rotation of ψ (don't be confused by the subscripts; I only include them to remind us which axis each angle is rotating about) -

$$Rot(z,\phi_z)Rot(y,\theta_y)Rot(z,\psi_z)$$

$$\begin{bmatrix} \cos\phi_z & -\sin\phi_z & 0 & 0 \\ \sin\phi_z & \cos\phi_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Yielding (as in my slides) –

$$\begin{bmatrix} c\phi_z c\theta_y c\psi_z - s\phi_z s\psi_z & -c\phi_z c\theta_y s\psi_z - s\phi_z c\psi_z & c\phi_z s\theta_y & 0 \\ s\phi_z c\theta_y c\psi_z + c\phi_z s\psi_z & -s\phi_z c\theta_y s\psi_z + c\phi_z c\psi_z & s\phi_z s\theta_y & 0 \\ -s\theta_y c\psi_z & s\theta_y s\psi_z & c\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Next we need to set the general Euler angle matrix equal to the homogeneous transformation matrix (*Th*) and then equating terms will enable formulae to be derived for the three Euler angles –

$$\begin{bmatrix} c\phi_z c\theta_y c\psi_z - s\phi_z s\psi_z & -c\phi_z c\theta_y s\psi_z - s\phi_z c\psi_z & c\phi_z s\theta_y & 0 \\ s\phi_z c\theta_y c\psi_z + c\phi_z s\psi_z & -s\phi_z c\theta_y s\psi_z + c\phi_z c\psi_z & s\phi_z s\theta_y & 0 \\ -s\theta_y c\psi_z & s\theta_y s\psi_z & c\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   We inspect the general Euler angle matrix looking for terms that will generate tangents of the Euler angles so that we can use the *atan2* function to find the angles. There are no actual tangents in the matrix that we can use directly, of course, so we look for pairs of terms incorporating sines and cosines of a common angle so that we can divide the sine by the cosine and generate a tangent for that angle.

Here is a pair of likely candidates –

$$\sin(\phi).\sin(\theta) = a_y \quad \text{and} \quad \cos(\phi).\sin(\theta) = a_x$$

Dividing the first equation by the second, the *sin(θ)* on the left hand side cancels out, leaving us with a *sin(ϕ)* divided by a *cos(ϕ)* –

$$\sin(\phi)/\cos(\phi) = a_y/a_x$$
$$\therefore \tan(\phi) = a_y/a_x$$
$$\therefore \boldsymbol{\phi = \mathrm{atan2}(a_y,a_x)}$$

That gives us our first Euler angle (the first *z* rotation).

We now look for other pairs in the general Euler matrix that might deliver an equation for *θ* or for *ψ*.

These two look like they should gives us an equation for *ψ* in the same was as we got *ϕ* above.

$$-\sin(\theta).\cos(\psi) = n_z \text{ and } \sin(\theta).\sin(\psi) = o_z$$

$$\therefore \sin(\psi)/\cos(\psi) = o_z/-n_z$$
$$\therefore \tan(\psi) = o_z/-n_z$$
$$\therefore \boldsymbol{\psi = \mathrm{atan2}(o_z,-n_z)}$$

Finally, now we have equations for both *ϕ* and *ψ*, we look for terms that will give us an equation for *θ*.

$-\sin(\theta).\cos(\psi) = n_z$ and $\cos(\theta) = a_z$ look like good candidates –

$$\therefore \sin(\theta)/\cos(\theta) = -n_z/\cos(\psi).a_z$$
$$\therefore \tan(\theta) = -n_z/\cos(\psi).a_z$$
$$\therefore \boldsymbol{\theta = \mathrm{atan2}(-n_z, a_z.\cos(\psi))}$$

2. The above formulae are robust because they use the **atan2(*,*) function** which gives **greater accuracy than arcsin and arccos**, returns a result in the **correct quadrant** based on the signs of the two parameters and also handles the case where **the second argument is 0** sensibly.

3. We construct the homogeneous transform representing the required rotations (an *x* rotation of 180° and a *y* rotation of -90°). This will be our *Th* matrix in this case –

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(180) & -\sin(180) & 0 \\ 0 & \sin(180) & \cos(180) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{X} \quad \begin{pmatrix} \cos(-90) & 0 & \sin(-90) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-90) & 0 & \cos(-90) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Which is –

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we substitute the values in this matrix for the appropriate variables in the formulae derived in question 2 –

$\phi = \mathrm{atan2}(a_y,a_x)$	=	atan2(0,-1)	=	180°
$\psi = \mathrm{atan2}(o_z,-n_z)$	=	atan2(0,1)	=	0°
$\theta = \mathrm{atan2}(-n_z,\cos(\psi).a_z)$	=	atan2(1,0)	=	90°

So the Z-Y-Z Euler set (ϕ, θ, ψ) is (180°, 90°, 0°)