MSc VIBOT/Computer Vision/MaIA

Applied Mathematics

Probability

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Problem Set

Problem 1

Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Problem 2

The birthday problem!

1. What is the probability that at least two people in a class of N students share the same birthday? We make the following assumptions:
   - 365 days in a year
   - Each day of the year has an equal chance of being somebody's birthday.
2. How large must the class be to make the probability of finding two people with the same birthday at least 50%?

Problem 3

- a) Let A and B be independent events. Show that $\overline{A}$ and B are also independent.
- b) Let A and B be two events. If the occurence of event B makes A more likely, then does the occurence of A make B more likely? Justify your answer.
- c) If event A is independent of itself, show that $P(A)$ is 1 or 0.
- d) If $P(A)$ is 1 or 0, show that A is independent of all events B.

Problem 4

Let a and b be two positive integers and X a discrete random variable such that

$$P(X = x) = \begin{cases} \frac{1}{a} - \frac{1}{b} & \text{if } 0 \leq x \leq ab \\ 0 & \text{if } x > ab \end{cases}$$

1. Find a condition on a and b, such that $p(x) = P(X = x)$ is the pmf of X.
2. Find the cdf $F_X(x)$ of X.
3. Calculate $\mathbb{E}(X)$. What values of a and b give $\mathbb{E}(X) = \frac{7}{2}$?

Problem 5

Let X and Y be two independent random variables with common distribution function F and density function f.

1. Show that $V = \max\{X, Y\}$ has distribution function $F_V(v) = F(v)^2$ and density function $f_V(v) = 2f(v)F(v)$.
2. Find the density function of $U = \min\{X, Y\}$.
3. Suppose X and Y are uniformly distributed on $[0, 1]$. Find $\mathbb{E}(U)$ and $\mathbb{E}(V)$.

Problem 6

We select a point X according to some probability density function f_X.
Find the value of a that minimizes the average value of the square distance between the point $(a, 1)$ and the random point $(X, 0)$ in the plane.

Problem 7

Buses arrive at a specified stop at 15-minute intervals starting at 7 a.m. (7:00, 7:15, 7:30, 7:45, ...). If the passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he/she waits

- (a) less than 5 minutes for the bus.
- (b) more than 10 minutes for the bus.

Problem 8

1. Let X be a Poisson random variable with parameter λ. Find $\mathbb{E}[X]$ and $\mathbb{V}\text{ar}\,(X)$.
2. A typesetter, on the average makes one error in every 500 words typset. Considering that a typical page contains 300 words, what is the probability that there will be no more than two errors in five pages?
3. Suppose that the length of a phone call in minutes is an exponential random variable with $\lambda = 1/10$. If somebody arrives immediately ahead of you at the telephone box, find the probability you have to wait
   - (a) more than 10 minutes
   - (b) between 10 and 20 minutes

4. Suppose that the lifetime of a light bulb is exponetially distributed with $\lambda = 1/1000$. If the light survives 500 hours, what is the probability that it will last another 1000 hours?

Problem 9

We study the traffic flow at the connexion of two highways. Let X and Y be respectively the number of vehicles arriving from the first and second branch. Hence, $S = X + Y$ is the number of vehicles on the highway after the connexion. Let us assume that X and Y are independent Poisson random variables with parameters $\lambda > 0$ and $\mu > 0$ respectively.

1. Show that S is also a Poisson random variable. Find $\mathbb{E}[S]$ and $\mathbb{V}\mathrm{ar}\,(S)$. (Hint: One can write S as union of disjoint events).
2. Show that the conditional distribution of X given $S = n (n \geq 0)$ is a Bionomial distribution with paremeters n and $p = \frac{\lambda}{\lambda+\mu}$.
3. Calculate $\mathbb{E}[X|S = n]$ and $\mathbb{V}\mathrm{ar}\,(X|S = n)$ and verify that one has the following relation

$$\mathbb{V}\mathrm{ar}\,(X) = \mathbb{E}[\mathbb{V}\mathrm{ar}\,(X|S)] + \mathbb{V}\mathrm{ar}\,(\mathbb{E}[X|S]).$$

Problem 10

Let X and Y be independent random variables, each uniformly distributed on the interval $[0, 2]$.

1. What is the pdf of X? What are the mean and variance of X?
2. Find the mean and variance of XY.
3. Find the probability that $XY \geq 1$, i.e. $P(XY \geq 1)$.

Problem 11

Suppose you are a witness to a night-time hit-and-run accident involving a taxi in Athens. All taxi cars in Athens are blue or green. You swear that the taxi was blue. Extensive testing shows that, under the dim lighting conditions, discrimination between blue and green is 75% reliable.

1. Is it possible to calculate the most likely color for the taxi?
2. What is your resulting estimate, given that 9 out of 10 Athenian taxis are green?

Problem 12

The inhabitants of an island tell the truth one third of the time. They lie with probability 2/3. On an occasion, after one of them made a statement, you ask another “was that statement true?" and he says “yes".
What is the probability that the statement was indeed true?

Problem 13

Three prisoners are informed by their jailer that one of them has been chosen to be executed at random with equal probability, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free.
The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from 1/3 to 1/2, since he would then be one of two prisoners.
What do you think of the jailer's reasoning?

Problem 14

Let x and y be independent random variables with means μ_x and μ_y, and variances σ_x^2 and σ_y^2 respectively. Show that

$$\text{Var}[xy] = \sigma_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2.$$

Problem 15

Let $X_1, X_2, ..., X_n$ be independent random variables uniformly distributed over the interval $[0, 1]$. Define the random variable $R_n = \min(X_1, X_2, \ldots, X_n)$.

1. Find the pdf $f_{R_n}(r)$ of R_n.
2. Compute the expected value $E[R_n]$. What is $\lim_{n\to\infty} E[R_n]$.

Problem 16

Each laptop has a lifetime that is exponentially distributed with parameter λ. The lifetime of laptops are independent of each other. Suppose you have two laptops which you begin using at the same time. Define T_1 as the time of the first failure (of one of the laptops) and T_2 as the time of the second failure.

1. Find the pdf of T_1, $f_{T_1}(t)$.
   HINT: how can you define T_1 in terms of the lifetimes of laptop one and laptop two?
2. Let $X = T_2 - T_1$. What is $f_{X|T_1}(x|t_1)$, i.e. the pdf of X conditioned on T_1?
3. Is X independent of T_1? Give a mathematical justification for your answer.
4. Find $f_{T_2}(t)$ and calculate $\mathbb{E}[T_2]$.
5. Now suppose you have 100 laptops, and let Y be the time of the first failure.
   Find $P(Y < 0.01)$.

Problem 17

A game in Le Creusot casino involves a device that produces independent samples of a continuous random variable X with probability density function

$$f_X(x) = \begin{cases} 3/x^4 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

The game is rather simple: the price to play is 1€, and the payoff in dollars is X.

1. What is your expected profit (profit = X - 1.00€) from one play of the game? Give a numerical answer with proper units.

2. What is the standard deviation of your profit from one play of the game?

3. You decide to play the game repeatedly until you profit at least 9.00€ on a single game (i.e. until the first $X \geq 10$) and then you will stop. Find the expected number of games you will play.

Problem 18

A software company, Creusot Soft, sells a software package first released in year 0. Every year after year 0, Creusot Soft launches either a major or a minor software development project, which may or may not result in the release of a new version.
Specifically, for every year $t = 1, \dots,$ there is probability α that Creusot Soft launches a major development project; and a probability $1 - \alpha$ that it launches a minor development project.
Given a major development project, there is a new release with probability p. Given a minor development project, there is a new release with probability r.
We assume that events in distinct years are independent.

1. What is the probability of a new release in year t.

2. Let M_t be the event that there was a major development project in year t. Let R_t be the event that there was a new release in year t. Let C be the event that 'either {there was a minor development project effort in year 1} or {there was a major development project in year 2 which resulted in a new release in year 2} or both'.

   - Show that event C can be written as $C = \overline{M_1} \cup (M_2 \cap R_2)$.
   - Find the probability of event C.

3. Are the events M_t and R_t independent events? Prove that they are independent or give a counterexample.

4. What is the probability of 4 new releases in the first 6 years?

5. Given that there were 5 major development projects in the first 8 years, what is the probability that there were 3 major development projects in the first 4 years?

6. Let T be the first year in which there is a major development project. Find $E[T^2]$.

Problem 19

Consider m data points (x_i, y_i), where the y_is are generated from the x_is according to a linear Gaussian model as follows:

$$P(y|x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-(\theta_1 x+\theta_2))^2}{2\sigma^2}}.$$

Find the values of θ_1, θ_2 and σ that maximizes the conditional log likelihood of the data.