TCP Velocity Control

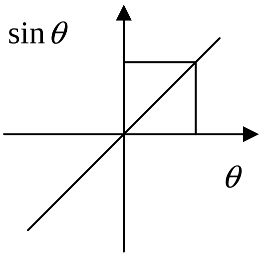
Differential Motions

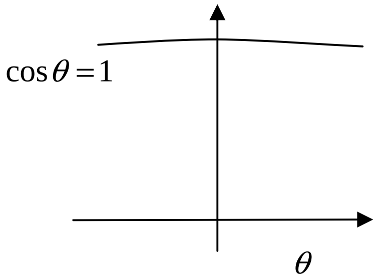
We start by considering differential (i.e. very small) rotational motions about the *x*, *y* and *z* axes.

Note that for **small angles**, θ

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1$$





Example

Take $\theta = 0.1$ radians (approximately *5.7* degrees)

$\sin\theta = 0.0998334$ small angle approximation = *0.1* error = *0.0001666*
$\cos\theta = 0.9950041$ small angle approximation = *1.0* error = *0.0049959*

The differential rotation matrices for rotations of δx, δy and δz about the *x*, *y* and *z* axes become -

$$x \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad y \approx \begin{pmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad z \approx \begin{pmatrix} 1 & -\delta z & 0 & 0 \\ \delta z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The general differential rotation matrix for rotations about all three axes is therefore -

$$\begin{pmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z + \delta x \cdot \delta y & 1 - \delta x \cdot \delta y \cdot \delta z & -\delta x & 0 \\ -\delta y + \delta x \cdot \delta z & \delta x + \delta y \cdot \delta z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can further approximate all higher order terms (two or more deltas multiplied together) to zero -

$$\begin{pmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem 1.

Differential rotations are commutative. The order in which the rotations are applied is irrelevant due to the assumption that only small angles are used.

Prove this theorem for yourself by multiplying differential rotation matrices in different orders and showing that they give the same result after higher order terms are set to zero.

Theorem 2.

A differential rotation, $\delta\theta$, about an arbitrary vector, k, where

$$k = \begin{pmatrix} k_x \\ k_y \\ k_z \\ 1 \end{pmatrix}$$

is equivalent to 3 differential rotations δx, δy and δz about the x, y and z axes, where

$$\delta x = k_x.\delta\theta \qquad \delta y = k_y.\delta\theta \qquad \delta z = k_z.\delta\theta$$

Now consider a homogeneous transform, T, which undergoes some small change, dT,

$$T + dT = [\text{translation } (dx, dy, dz), \text{ rotation } (k, \delta\theta)].T$$

$$\therefore dT = \left[translation\begin{pmatrix} dx & dy & dz \end{pmatrix}, rotation\begin{pmatrix} \delta x & \delta y & \delta z \end{pmatrix} - I\right] \cdot T$$

Where I is the identity matrix.

We can construct a **differential operator**, Δ, from the expression in the square brackets -

$$dT = \Delta \cdot T$$

where

$$\Delta = \begin{pmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

$$T = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{ll} \delta x = 0.1 & dx = 1 \\ \delta y = 0.0 & dy = 0 \\ \delta z = 0.0 & dz = 0.5 \end{array}$$

$$\Delta = \begin{pmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0.1 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$dT = \Delta T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0.7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now the new transform, *T'*, is given by -

$$T' = T + dT$$

$$\therefore T' = \begin{pmatrix} 0 & 0 & 1 & 6 \\ 1 & -0.1 & 0 & 2 \\ 0.1 & 1 & 0 & 0.7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Jacobians

Definition

$$D = \begin{pmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

Represents the changes to be made to a point in Cartesian space (small changes).

$$D_\theta = \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{pmatrix}$$

Represents the associated changes to the joints in the robotic arm.

We can relate these 2 vectors by a matrix called a **Jacobian** –

$$D = J \cdot D_\theta \qquad D_\theta = J^{-1} \cdot D$$

A six-jointed arm doesn't make a good example, as it is too messy. So we return to the θ-r manipulator.

For the θ-r manipulator we can define the Cartesian differential displacement vector thus -

$$D = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

If we arbitrarily set the interval of time, *dt*, over which these displacements occur to be *1* (by choosing a suitable unit for measuring time) then *D* can be turned into an instantaneous velocity vector -

$$D = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \qquad \dot{x} = \frac{dx}{dt} \qquad \dot{y} = \frac{dy}{dt}$$

For the two joints in our arm the same thing applies -

$$D_\theta = \begin{pmatrix} \dot{\theta} \\ \dot{r} \end{pmatrix} \begin{matrix} \leftarrow \text{Joint 1} \\ \leftarrow \text{Joint 2} \end{matrix}$$

The Jacobian is then defined by the matrix -

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{r} \end{pmatrix}$$

We can now work out the J_{ij} terms. We know that -

$$x = r \cdot \cos\theta$$

Remember that *x*, *y*, θ and *r* are all functions of time. They are *x(t)*, *y(t)*, $\theta(t)$ and *r(t)* really.

Taking derivatives yields -

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta + r\frac{d}{dt}(\cos\theta)$$

$$\frac{d}{dt}(\cos\theta(t)) = -\sin\theta(t)\frac{d\theta}{dt}$$

$$\therefore\ \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

We also know that -

$$y = r.\sin\theta$$

So, again, taking derivatives -

$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\frac{d}{dt}(\sin\theta)$$

$$\therefore\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

The Jacobian then becomes -

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{r} \end{pmatrix}$$

We can obtain the inverse Jacobian in the same way -

$$r^2 = x^2 + y^2$$

$$\therefore 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\therefore \dot{r} = \frac{x}{r}.\dot{x} + \frac{y}{r}.\dot{y}$$

Also,

$$\tan\theta = \frac{y}{x}$$

$$\therefore \dot{\theta}\sec^2\theta = \frac{-y}{x^2}\dot{x} + \frac{1}{x}\dot{y}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\therefore \sec^2\theta = \frac{r^2}{x^2}$$

$$\therefore \dot{\theta} = \frac{-y}{r^2}\dot{x} + \frac{x}{r^2}\dot{y}$$

And the inverse Jacobian is therefore -

$$\begin{pmatrix} \dot{\theta} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{-y}{r^2} & \frac{x}{r^2} \\ \frac{x}{r} & \frac{y}{r} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Note that there is a singularity at $r = 0$. The speed of the rotational joint approaches infinity as the length of the arm, governed by the prismatic joint, approaches zero.