Erasmus Mundus

Education and Culture

Vibot

Universitat de Girona

Visual Perception

Lecture 1B

Rigid Body Transformations

UNIVERSITÉ DE BOURGOGNE

UdG

HERIOT WATT UNIVERSITY

Contents

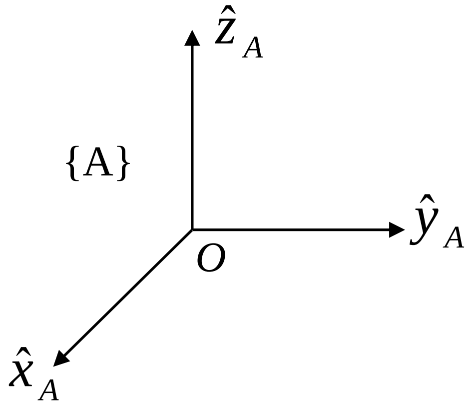
Contents

1.1 Cartesian coordinates, Points and Vectors

Coordinate System: Complete set of orthonormal vectors (perpendicular and unit) and coinciding in a point (origin).

Standard base vectors:

$$\hat{X}_A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{Y}_A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{Z}_A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

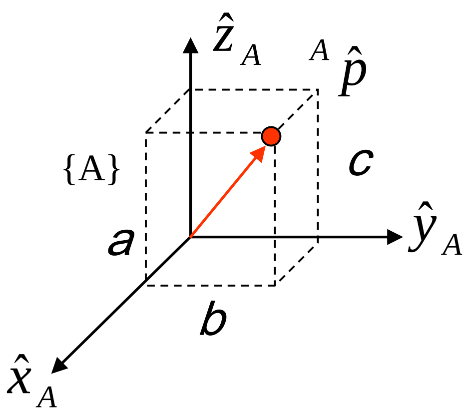


right-hand frame

Reference System: Unique/World coordinate system used for referencing points, vectors and other coordinate systems

1.1 Cartesian coordinates, Points and Vectors

Coordinates of a point p in space:



$${}^{A}\hat{p}_u$$

A → Referenced with respect to {A}

u → Corresponding to object/reference u

$$\hat{p} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\hat{x} + b\hat{y} + c\hat{z}$$

$$a = \hat{p} \cdot \hat{x} = p^T x$$

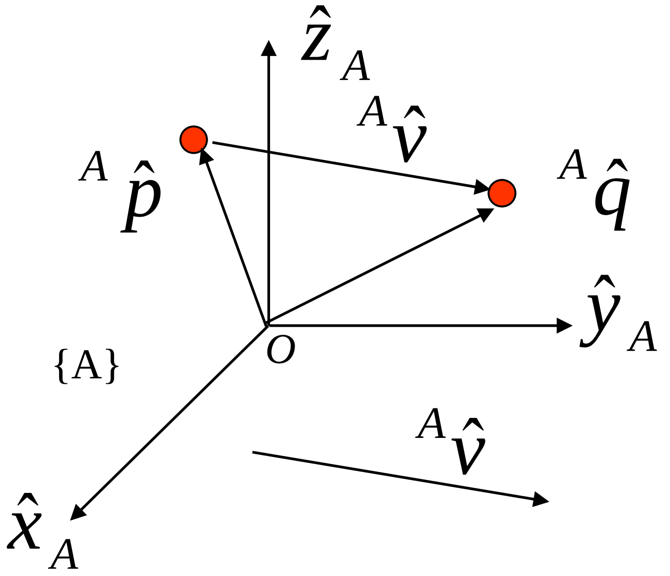
$$b = \hat{p} \cdot \hat{y} = p^T y$$

$$c = \hat{p} \cdot \hat{z} = p^T z$$

1.1 Cartesian coordinates, Points and Vectors

A "free" vector is defined by a pair of points (p, q):

$$^{A}\hat{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \qquad ^{A}\hat{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$



Coordinates of the vector v :

$$^{A}\hat{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix}$$

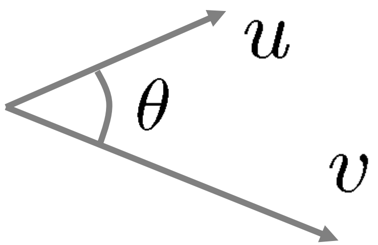
Contents

Contents

1.2 Inner product and Cross product

Inner product between two vectors:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



$$\langle u, v \rangle \doteq u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\cos(\theta) = \frac{\langle u,v \rangle}{||u|| \, ||v||}$$

$$||u|| \doteq \sqrt{u^T u} = \sqrt{u_1^2 + u_2^2 + u_3^3}$$

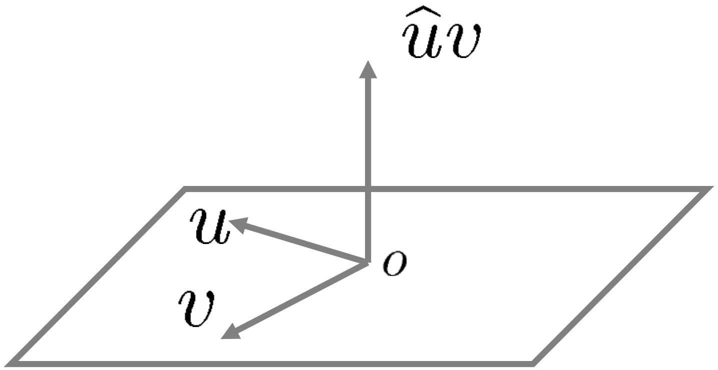
1.2 Inner product and Cross product

Cross product between two vectors:

$$u \times v \doteq \hat{u}v, \qquad u, v \in \mathbb{R}^3$$

$$\hat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Antisymmetric matrix



Contents

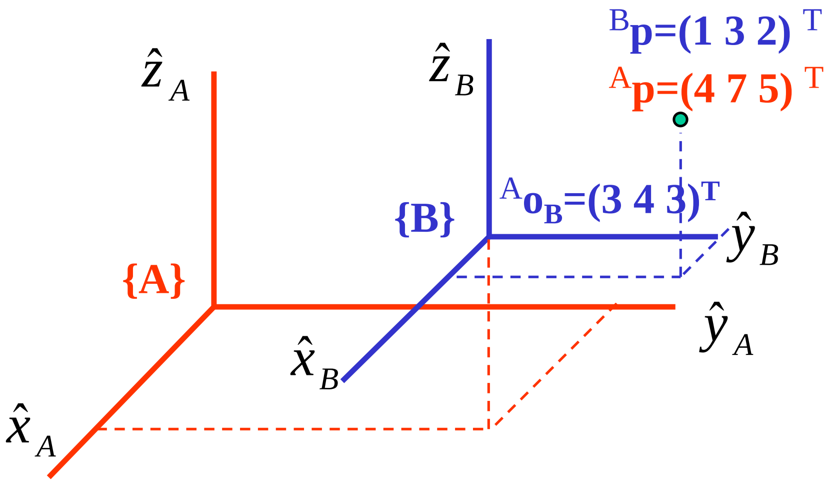
Contents

1.3 Translations

Translation vector:

- {B} is a translation of {A} to position ${}^{A}(3\ 4\ 3)^{T}$.
- Coodinate axes of {B} are paralel to coordinate axes of {A}.
- The translation can be representated by vectorial addition.

$$\begin{pmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \end{pmatrix} = \begin{pmatrix} {}^{B}p_x \\ {}^{B}p_y \\ {}^{B}p_z \end{pmatrix} + \begin{pmatrix} {}^{A}o_x \\ {}^{A}o_y \\ {}^{A}o_z \end{pmatrix} = \begin{pmatrix} {}^{B}p_x + {}^{A}o_x \\ {}^{B}p_y + {}^{A}o_y \\ {}^{B}p_z + {}^{A}o_z \end{pmatrix} \quad ; \quad \begin{pmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \end{pmatrix} = {}^{B}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + {}^{A}\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} = {}^{A}\begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix}$$



1.3 Translations

Translation matrix:

Coordinates are related by: $${}^{A}p = {}^{B}p + {}^{A}o_{B}$$

$${}^{A}p = T({}^{A}o_{B})\ {}^{B}p$$

$$T({}^{A}o_{B}) = {}^{A}\begin{pmatrix} 1 & 0 & 0 & {}^{A}o_{x} \\ 0 & 1 & 0 & {}^{A}o_{y} \\ 0 & 0 & 1 & {}^{A}o_{z} \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B} = {}^{A}\begin{pmatrix} I & t \\ 0 & 1 \end{pmatrix}_{B}$$

Contents

Contents

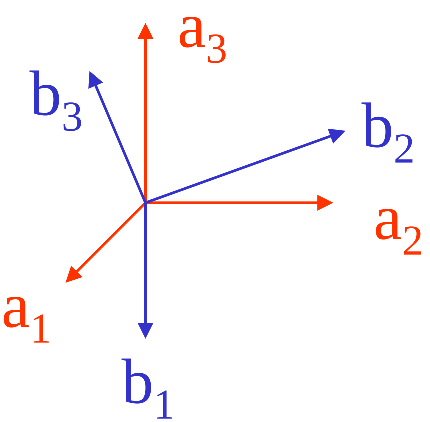
1.4 Rotations

Rotation matrix:

Let {A} and {B} two ortonormal coordinate systems with the same origin and unit vectors $\{a_1, a_2, a_3\}$ $\{b_1, b_2, b_3\}$.

Point p is represented by vector $\hat{p}$ in {A} and {B}

$$\begin{cases} {}^{A}\hat{p} = \left({}^{A}p_1 \quad {}^{A}p_2 \quad {}^{A}p_3\right)^T \\ {}^{B}\hat{p} = \left({}^{B}p_1 \quad {}^{B}p_2 \quad {}^{B}p_3\right)^T \end{cases}$$



$${}^{A}\hat{p} = {}^{A}p_1 \cdot {}^{A}\hat{a}_1 + {}^{A}p_2 \cdot {}^{A}\hat{a}_2 + {}^{A}p_3 \cdot {}^{A}\hat{a}_3$$

$${}^{A}\hat{p} = {}^{B}p_1 \cdot {}^{A}\hat{b}_1 + {}^{B}p_2 \cdot {}^{A}\hat{b}_2 + {}^{B}p_3 \cdot {}^{A}\hat{b}_3$$

Consider ${}^{A}p_k$

$${}^{A}p_k = {}^{A}\hat{p} \cdot {}^{A}\hat{a}_k = \left[\sum_{j=1}^{3} ({}^{B}p_j \cdot {}^{A}\hat{b}_j)\right] \cdot {}^{A}\hat{a}_k = \sum_{j=1}^{3} {}^{B}p_j \cdot \left({}^{A}\hat{b}_j \cdot {}^{A}\hat{a}_k\right) = \sum_{j=1}^{3} {}^{B}p_j \cdot r_{kj}$$

$$k = 1 \rightarrow {}^{A}p_1 = r_{11} \cdot {}^{B}p_1 + r_{12} \cdot {}^{B}p_2 + r_{13} \cdot {}^{B}p_3$$

$$k = 2 \rightarrow {}^{A}p_2 = r_{21} \cdot {}^{B}p_1 + r_{22} \cdot {}^{B}p_2 + r_{23} \cdot {}^{B}p_3$$

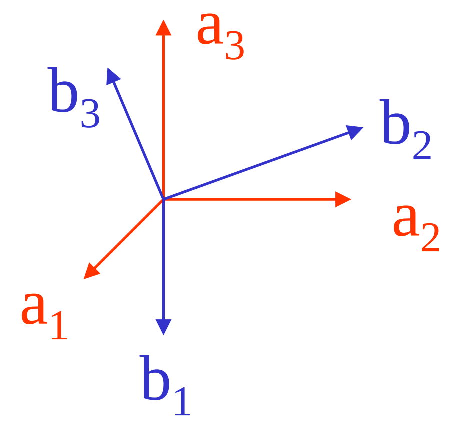
$$k = 3 \rightarrow {}^{A}p_3 = r_{31} \cdot {}^{B}p_1 + r_{32} \cdot {}^{B}p_2 + r_{33} \cdot {}^{B}p_3$$

$$\Rightarrow \begin{pmatrix} {}^{A}p_1 \\ {}^{A}p_2 \\ {}^{A}p_3 \end{pmatrix} = {}^{A}\begin{pmatrix} \hat{a}_1 \cdot \hat{b}_1 & \hat{a}_1 \cdot \hat{b}_2 & \hat{a}_1 \cdot \hat{b}_3 \\ \hat{a}_2 \cdot \hat{b}_1 & \hat{a}_2 \cdot \hat{b}_2 & \hat{a}_2 \cdot \hat{b}_3 \\ \hat{a}_3 \cdot \hat{b}_1 & \hat{a}_3 \cdot \hat{b}_2 & \hat{a}_3 \cdot \hat{b}_3 \end{pmatrix}_B \cdot \begin{pmatrix} {}^{B}p_1 \\ {}^{B}p_2 \\ {}^{B}p_3 \end{pmatrix}$$

1.4 Rotations

Rotation matrix:

$$
\begin{array}{cc}
{}^{A}R_{B} = & \begin{pmatrix} \hat{a}_1 \cdot \hat{b}_1 & \hat{a}_1 \cdot \hat{b}_2 & \hat{a}_1 \cdot \hat{b}_3 \\ \hat{a}_2 \cdot \hat{b}_1 & \hat{a}_2 \cdot \hat{b}_2 & \hat{a}_2 \cdot \hat{b}_3 \\ \hat{a}_3 \cdot \hat{b}_1 & \hat{a}_3 \cdot \hat{b}_2 & \hat{a}_3 \cdot \hat{b}_3 \end{pmatrix} \\
 & \begin{matrix} {}^{A}\hat{b}_1 & \quad {}^{A}\hat{b}_2 & \quad {}^{A}\hat{b}_3 \end{matrix}
\end{array}
$$



$$
\begin{array}{cc}
{}^{A}R_{B} = \begin{pmatrix} \hat{a}_1 \cdot \hat{b}_1 & \hat{a}_1 \cdot \hat{b}_2 & \hat{a}_1 \cdot \hat{b}_3 \\ \hat{a}_2 \cdot \hat{b}_1 & \hat{a}_2 \cdot \hat{b}_2 & \hat{a}_2 \cdot \hat{b}_3 \\ \hat{a}_3 \cdot \hat{b}_1 & \hat{a}_3 \cdot \hat{b}_2 & \hat{a}_3 \cdot \hat{b}_3 \end{pmatrix} & \begin{matrix} {}^{B}\hat{a}_1 \\ {}^{B}\hat{a}_2 \\ {}^{B}\hat{a}_3 \end{matrix}
\end{array}
$$

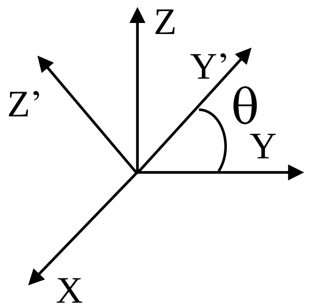
$$\left({}^{A}R_{B}\right)^{T} = {}^{B}R_{A}$$

$$R^{T}R = I,\ \det(R) = 1,\ \operatorname{rank}(R) = 3$$

1.4 Rotations

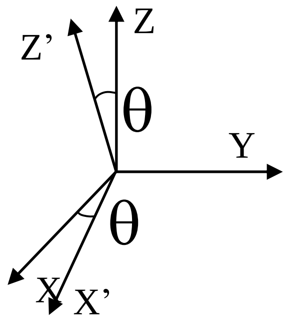
Basic Rotation matrices:

Rot(θ, x)



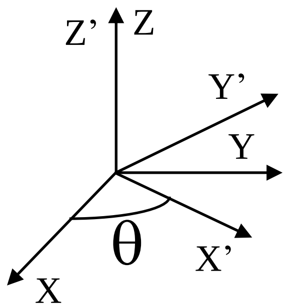
$$^{A}\begin{pmatrix} \hat{a}_1\cdot\hat{b}_1 & \hat{a}_1\cdot\hat{b}_2 & \hat{a}_1\cdot\hat{b}_3 \\ \hat{a}_2\cdot\hat{b}_1 & \hat{a}_2\cdot\hat{b}_2 & \hat{a}_2\cdot\hat{b}_3 \\ \hat{a}_3\cdot\hat{b}_1 & \hat{a}_3\cdot\hat{b}_2 & \hat{a}_3\cdot\hat{b}_3 \end{pmatrix}_{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix}$$

Rot(θ, y)



$$^{A}\begin{pmatrix} \hat{a}_1\cdot\hat{b}_1 & \hat{a}_1\cdot\hat{b}_2 & \hat{a}_1\cdot\hat{b}_3 \\ \hat{a}_2\cdot\hat{b}_1 & \hat{a}_2\cdot\hat{b}_2 & \hat{a}_2\cdot\hat{b}_3 \\ \hat{a}_3\cdot\hat{b}_1 & \hat{a}_3\cdot\hat{b}_2 & \hat{a}_3\cdot\hat{b}_3 \end{pmatrix}_{B} = \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}$$

Rot(θ, z)



$$^{A}\begin{pmatrix} \hat{a}_1\cdot\hat{b}_1 & \hat{a}_1\cdot\hat{b}_2 & \hat{a}_1\cdot\hat{b}_3 \\ \hat{a}_2\cdot\hat{b}_1 & \hat{a}_2\cdot\hat{b}_2 & \hat{a}_2\cdot\hat{b}_3 \\ \hat{a}_3\cdot\hat{b}_1 & \hat{a}_3\cdot\hat{b}_2 & \hat{a}_3\cdot\hat{b}_3 \end{pmatrix}_{B} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.4 Rotations

Rotation matrix:

Coordinates are related by: $^{A}p = {^{A}R_{B}}\ {^{B}p}$

$$^{A}p = T(^{A}R_{B})\ {^{B}p}$$

$$T(^{A}R_{B}) = {}^{A}\begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B} = {}^{A}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}_{B}$$

Contents

Contents

1.5 Homogeneous coordinates

Translation matrix:

Coordinates are related by: $^{A}p = {}^{B}p + {}^{A}o_{B}$

$$^{A}p = T({}^{A}o_{B})\ {}^{B}p \qquad T({}^{A}o_{B}) = {}^{A}\begin{pmatrix} 1 & 0 & 0 & {}^{A}o_{x} \\ 0 & 1 & 0 & {}^{A}o_{y} \\ 0 & 0 & 1 & {}^{A}o_{z} \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B} = {}^{A}\begin{pmatrix} I & t \\ 0 & 1 \end{pmatrix}_{B}$$

Rotation matrix:

Coordinates are related by: $^{A}p = {}^{A}R_{B}\ {}^{B}p$

$$^{A}p = T({}^{A}R_{B})\ {}^{B}p \qquad T({}^{A}R_{B}) = {}^{A}\begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B} = {}^{A}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}_{B}$$

1.5 Homogeneous coordinates

Composed matrix:

$$ {}^{A}p = {}^{A}T_{B}\ {}^{B}p $$

$$ {}^{A}T_{B} = {}^{A}\begin{pmatrix} r_{11} & r_{12} & r_{13} & {}^{A}o_{x} \\ r_{21} & r_{22} & r_{23} & {}^{A}o_{y} \\ r_{31} & r_{32} & r_{33} & {}^{A}o_{z} \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B} = {}^{A}\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}_{B} = \begin{bmatrix} R & t \end{bmatrix} $$

Point:

$$ {}^{A}p = \begin{bmatrix} {}^{A}p_{x} \\ {}^{A}p_{y} \\ {}^{A}p_{z} \\ 1 \end{bmatrix} $$

Vector:

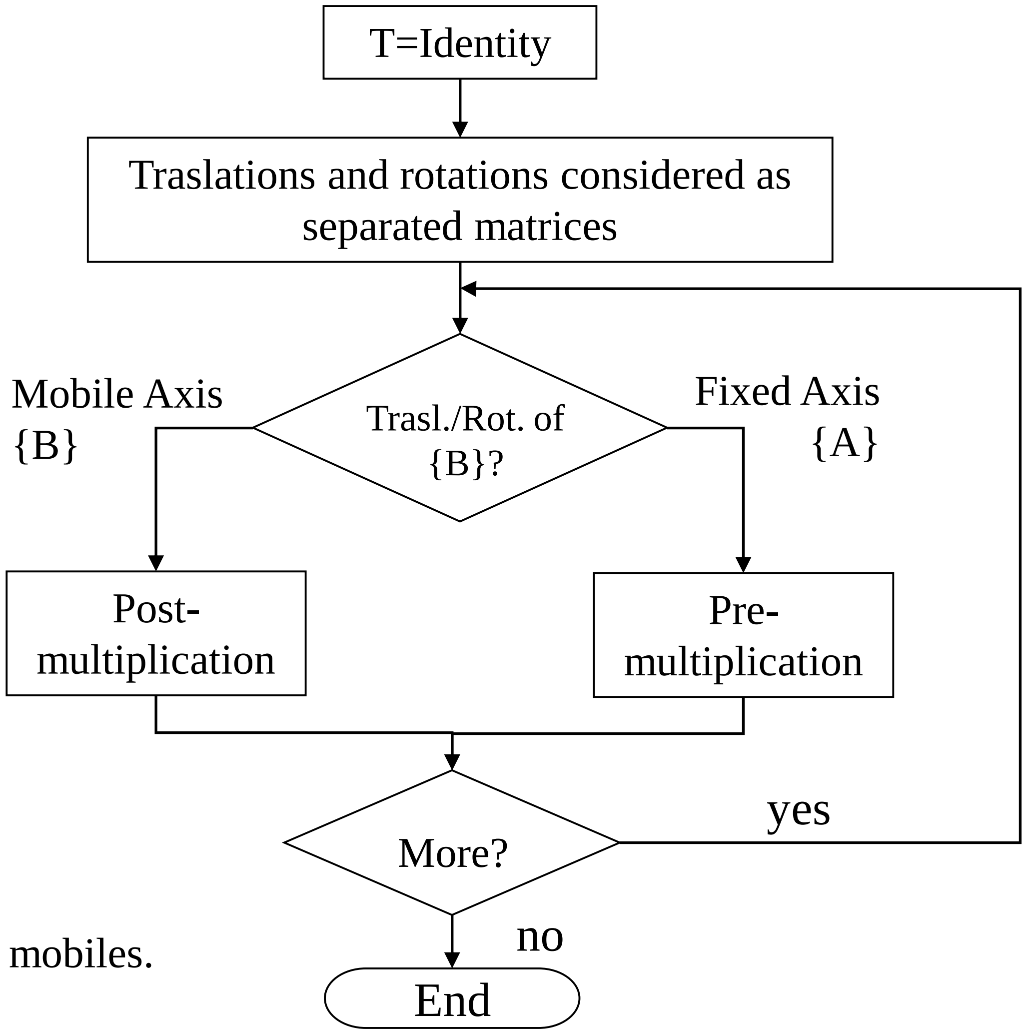
$$ {}^{A}v = \begin{bmatrix} {}^{A}v_{x} \\ {}^{A}v_{y} \\ {}^{A}v_{z} \\ 0 \end{bmatrix} $$

Contents

Contents

1.6 Composition of transformations

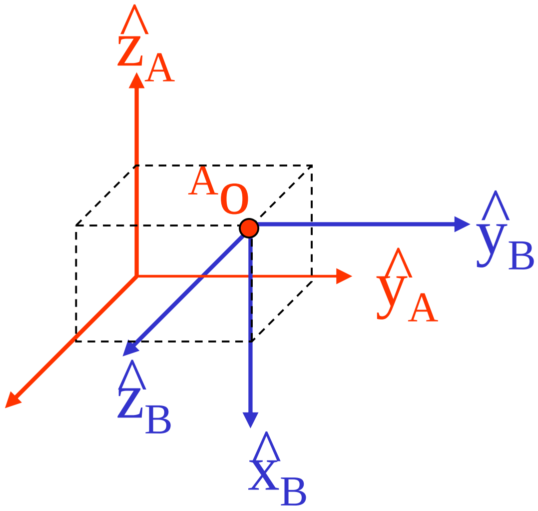
Algorithm

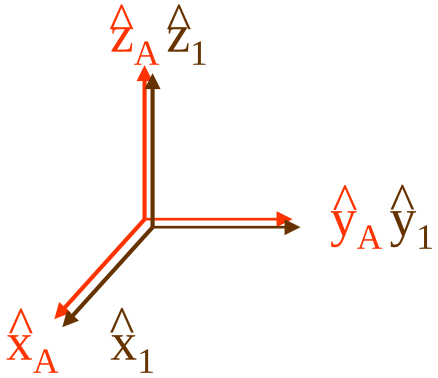


({*A*}) fixed and ({*B*}) mobiles.

1.6 Composition of transformations

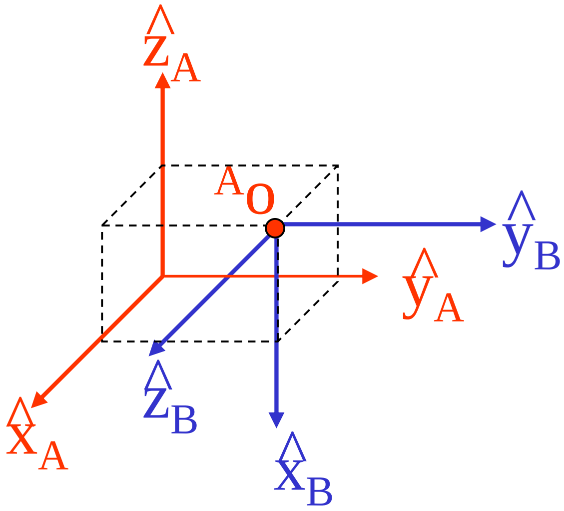
Example 1

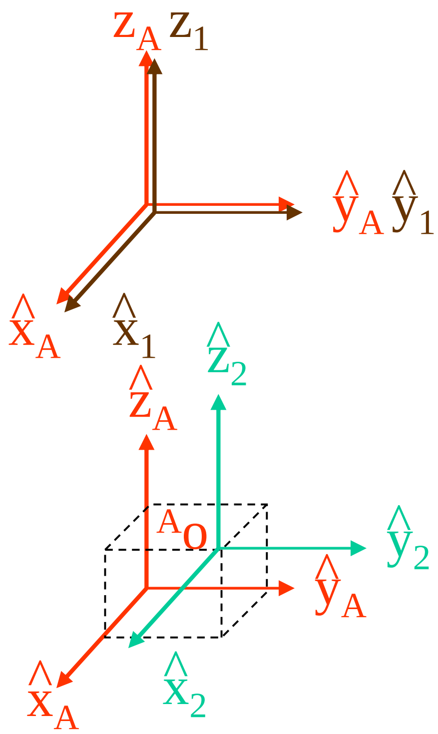




$$T = I$$

1.6 Composition of transformations





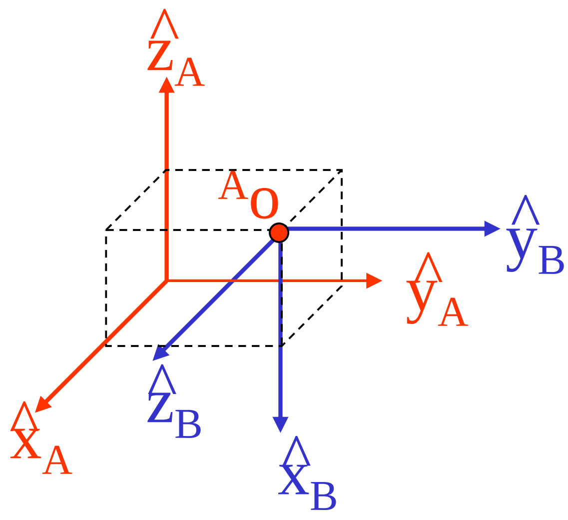
$$T = I$$

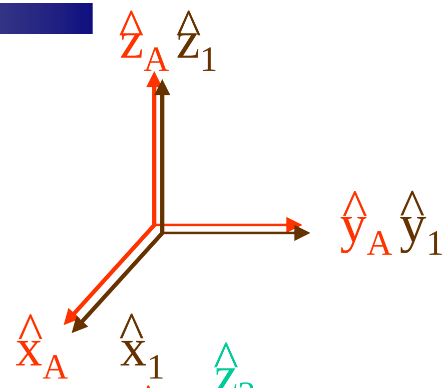
$$T = I \cdot Trans({}^{A}o)$$

o

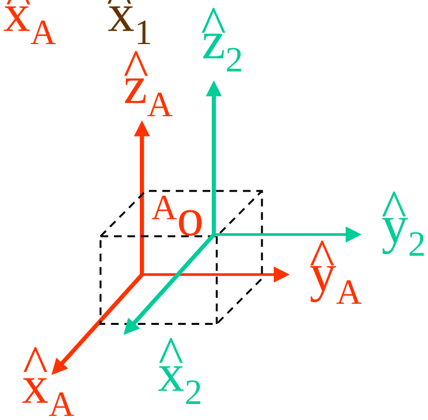
$$T = Trans({}^{A}o) \cdot I$$

1.6 Composition of transformations





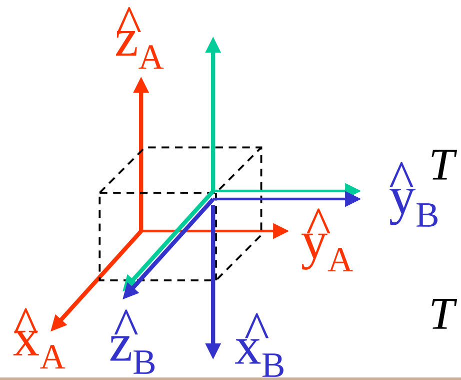
$$T = I$$



$$T = I \cdot Trans(^A o)$$

o

$$T = Trans(^A o) \cdot I$$

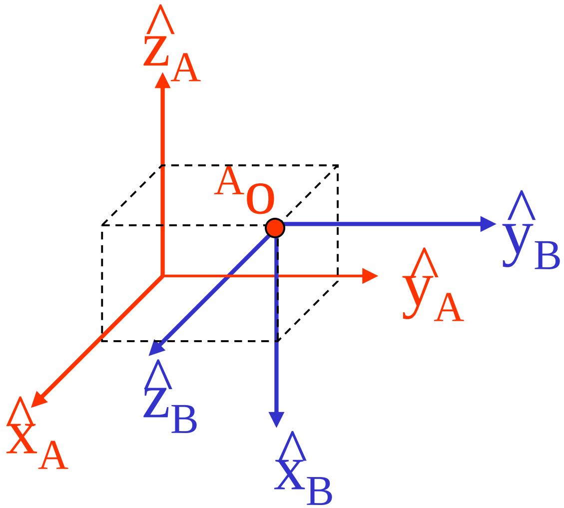


$$T = I \cdot Trans(^A o) \cdot Rot(\hat{y}_2, 90)$$

o

$$T = Trans(^A o) \cdot I \cdot Rot(\hat{y}_2, 90)$$

1.6 Composition of transformations



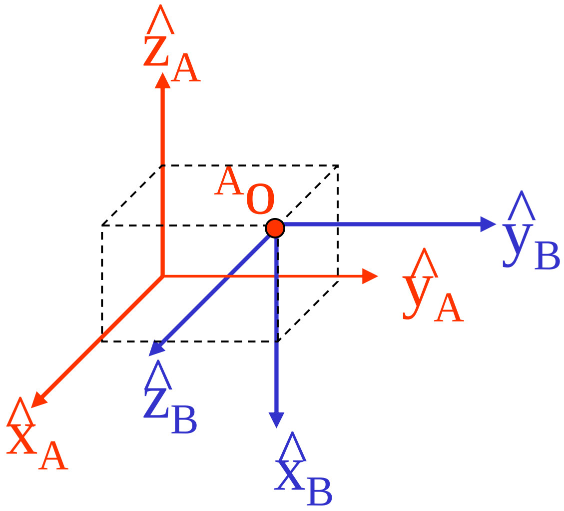
$$T = Trans(^A o) \cdot I \cdot Rot(\hat{y}_2, 90)$$

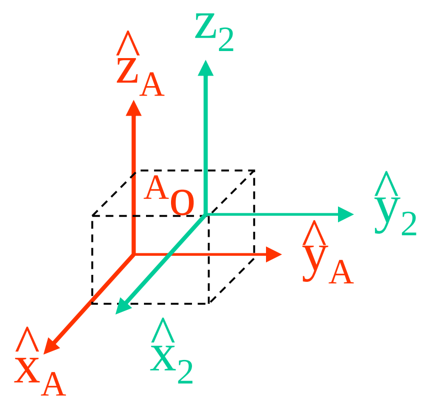
$$T = \begin{pmatrix} 1 & 0 & 0 & ^A o_x \\ 0 & 1 & 0 & ^A o_y \\ 0 & 0 & 1 & ^A o_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & ^A o_x \\ 0 & 1 & 0 & ^A o_y \\ -\sin\alpha & 0 & \cos\alpha & ^A o_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{matrix} \alpha = 90 \\ ^A o = \begin{pmatrix} 3 & 3 & 3 & 1 \end{pmatrix}^T \end{matrix}\right\} \Rightarrow T = {}^A\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 3 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}_B$$

1.6 Composition of transformations





$$T = I \cdot Trans(^A o)$$

o

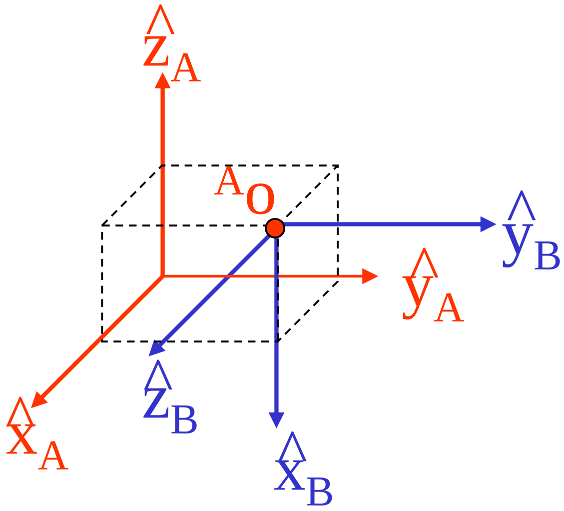
$$T = Trans(^A o) \cdot I$$

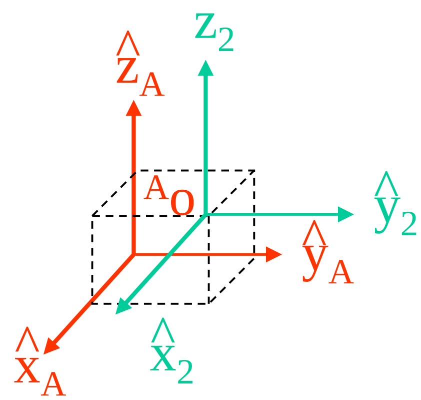
Be careful !!!

The result is different if the rotation is around the fixed axis (and not around the mobile axis)

1.6 Composition of transformations

Example 2

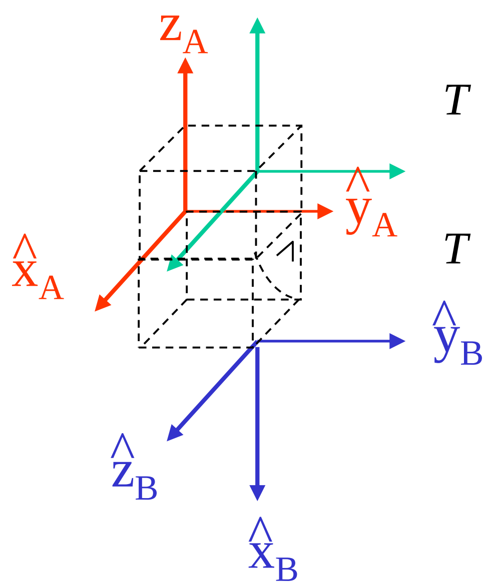




$$T = I \cdot Trans({}^{A}o)$$

o

$$T = Trans({}^{A}o) \cdot I$$



$$T = Rot(\hat{y}_2, 90) \cdot I \cdot Trans({}^{A}o)$$

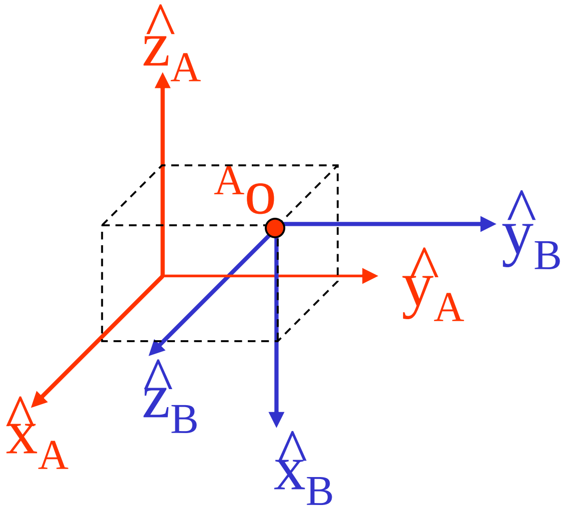
o

$$T = Rot(\hat{y}_2, 90) \cdot Trans({}^{A}o) \cdot I$$

Be careful !!!

The result is different if the rotation is about the fixed axis (and not about the mobile axis)

1.6 Composition of transformations



Be careful !!!
The result is different if the rotation is about the fixed axis (and not about the mobile axis)

$$T = Rot(\hat{y}_2, 90) \cdot Trans(^A o) \cdot I$$

$$T = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & ^A o_x \\ 0 & 1 & 0 & ^A o_y \\ 0 & 0 & 1 & ^A o_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & \cos\alpha \cdot {^A o_x} + \sin\alpha \cdot {^A o_z} \\ 0 & 1 & 0 & ^A o_y \\ -\sin\alpha & 0 & \cos\alpha & -\sin\alpha \cdot {^A o_x} + \cos\alpha \cdot {^A o_z} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \alpha = 90 \\ ^A o = \begin{pmatrix} 3 & 3 & 3 & 1 \end{pmatrix}^T \end{array}\right\} \Rightarrow T = {}^A\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 3 \\ -1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}_B$$

Contents

Contents

1.7 Inverse transformation

Homogeneous Inverse Transformation

$$T = \left(\begin{array}{ccc|c} & R & & p \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow T^{-1} = \left(\begin{array}{ccc|c} & R^T & & -R^T \cdot p \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

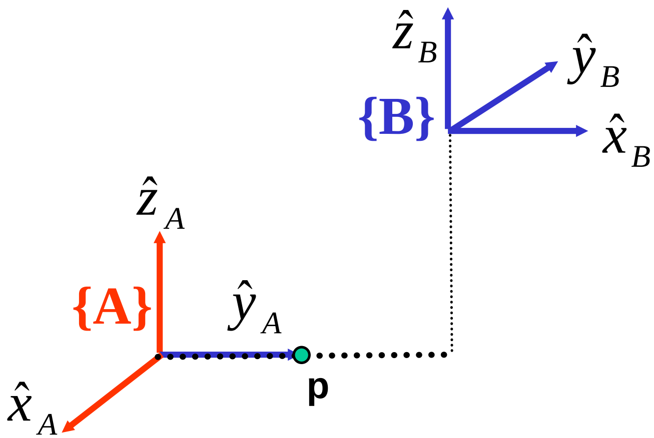
Example:

$$T = \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow T^{-1} = \left(\begin{array}{ccc|c} 0 & 1 & 0 & -2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$-R^T \ p = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$$

1.7 Inverse transformation

The homogeneous transformation that maps {B} with respect to {A}, is ${}^{A}T_{B}$. What are the coordinates of the point ${}^{A}p=(0,1,0)$ with respect the coordinate system {B} ?



$${}^{A}p = T\,{}^{B}p$$
$$T^{-1}\cdot{}^{A}p = T^{-1}\cdot T\,{}^{B}p$$
$$T^{-1}\cdot{}^{A}p = {}^{B}p$$

Then,

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} 0 & 1 & 0 & -2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T^{-1}\cdot{}^{A}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = {}^{B}\begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Contents

Contents

1.8 Parametrization of a Rotation matrix

RPY angles: Roll, Pitch and yaw

$$R({}^{A}Z,\beta_1)R({}^{A}Y,\beta_2)R({}^{A}X,\beta_3)=\begin{pmatrix}\cos\beta_1 & -\sin\beta_1 & 0\\ \sin\beta_1 & \cos\beta_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\beta_2 & 0 & \sin\beta_2\\ 0 & 1 & 0\\ -\sin\beta_2 & 0 & \cos\beta_2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\beta_3 & -\sin\beta_3\\ 0 & \sin\beta_3 & \cos\beta_3\end{pmatrix}$$

Euler angles: ZYZ, …

$$R({}^{B}Z,\beta_1)R({}^{B}Y,\beta_2)R({}^{B}Z,\beta_3)=\begin{pmatrix}\cos\beta_1 & -\sin\beta_1 & 0\\ \sin\beta_1 & \cos\beta_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\beta_2 & 0 & \sin\beta_2\\ 0 & 1 & 0\\ -\sin\beta_2 & 0 & \cos\beta_2\end{pmatrix}\begin{pmatrix}\cos\beta_3 & -\sin\beta_3 & 0\\ \sin\beta_3 & \cos\beta_3 & 0\\ 0 & 0 & 1\end{pmatrix}$$

1.8 Parametrization of a Rotation matrix

RPY angles:

$$R({}^{A}Z,\beta_1)R({}^{A}Y,\beta_2)R({}^{A}X,\beta_3)=\begin{pmatrix}\cos\beta_1 & -\sin\beta_1 & 0\\ \sin\beta_1 & \cos\beta_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\beta_2 & 0 & \sin\beta_2\\ 0 & 1 & 0\\ -\sin\beta_2 & 0 & \cos\beta_2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\beta_3 & -\sin\beta_3\\ 0 & \sin\beta_3 & \cos\beta_3\end{pmatrix}=$$

$$\begin{pmatrix}\cos\beta_1 & -\sin\beta_1 & 0\\ \sin\beta_1 & \cos\beta_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\beta_2 & \sin\beta_2\sin\beta_3 & -\sin\beta_2\cos\beta_3\\ 0 & \cos\beta_3 & -\sin\beta_3\\ -\sin\beta_2 & \cos\beta_2\sin\beta_3 & \cos\beta_2\cos\beta_3\end{pmatrix}=$$

$$\begin{pmatrix}\cos\beta_1\cos\beta_2 & \cos\beta_1\sin\beta_2\sin\beta_3-\sin\beta_1\cos\beta_3 & \sin\beta_1\sin\beta_3-\cos\beta_1\sin\beta_2\cos\beta_3\\ \sin\beta_1\cos\beta_2 & \sin\beta_1\sin\beta_2\sin\beta_3+\cos\beta_1\cos\beta_3 & -\sin\beta_1\sin\beta_2\cos\beta_3-\cos\beta_1\sin\beta_3\\ -\sin\beta_2 & \cos\beta_2\sin\beta_3 & \cos\beta_2\cos\beta_3\end{pmatrix}$$

Contents

Contents

1.9 Extracting the rotation angles of a rotation matrix

Given a Rotation Matrix R, we decide the parametrization

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} \cos\beta_1 \cos\beta_2 & \cos\beta_1 \sin\beta_2 \sin\beta_3 - \sin\beta_1 \cos\beta_3 & \sin\beta_1 \sin\beta_3 - \cos\beta_1 \sin\beta_2 \cos\beta_3 \\ \sin\beta_1 \cos\beta_2 & \sin\beta_1 \sin\beta_2 \sin\beta_3 + \cos\beta_1 \cos\beta_3 & -\sin\beta_1 \sin\beta_2 \cos\beta_3 - \cos\beta_1 \sin\beta_3 \\ -\sin\beta_2 & \cos\beta_2 \sin\beta_3 & \cos\beta_2 \cos\beta_3 \end{pmatrix}$$

And we can extract the angles from that parametrization

$$\beta_2 = \mathrm{asin}(r_{31})$$

$$\beta_3 = \mathrm{asin}(r_{32} / \cos\beta_2)$$

$$\beta_1 = \mathrm{atan}(r_{12} / r_{11})$$

1.9 Extracting the rotation angles of a rotation matrix

Given a Rotation Matrix R, we decide the parametrization

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} \cos\beta_1 \cos\beta_2 & \cos\beta_1 \sin\beta_2 \sin\beta_3 - \sin\beta_1 \cos\beta_3 & \sin\beta_1 \sin\beta_3 - \cos\beta_1 \sin\beta_2 \cos\beta_3 \\ \sin\beta_1 \cos\beta_2 & \sin\beta_1 \sin\beta_2 \sin\beta_3 + \cos\beta_1 \cos\beta_3 & -\sin\beta_1 \sin\beta_2 \cos\beta_3 - \cos\beta_1 \sin\beta_3 \\ -\sin\beta_2 & \cos\beta_2 \sin\beta_3 & \cos\beta_2 \cos\beta_3 \end{pmatrix}$$

And we can extract the angles from that parametrization

$$\beta_2 = \mathrm{asin}(r_{31})$$
$$\beta_3 = \mathrm{asin}(r_{32} / \cos\beta_2)$$
$$\beta_1 = \mathrm{atan}(r_{12} / r_{11})$$

$\cos(\beta) = \cos(-\beta)$

$\sin(\beta) = \sin(\beta + \pi/2)$

$\mathrm{atan}(\beta) = \mathrm{atan}(\beta + \pi)$

Angles are defined partially
Angles in the $(0, 2\pi)$ rang are desired

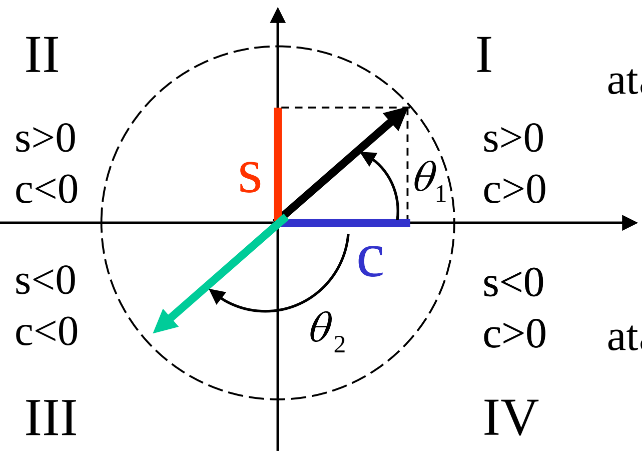
1.9 Extracting the rotation angles of a rotation matrix

atan2(s,c)

- **Problem of $\tan^{-1}$:** $-\frac{\pi}{2} \le \tan^{-1}(t) \le \frac{\pi}{2} \Rightarrow \exists \theta_1 \ne \theta_2 \,/\, \tan(\theta_1) = \tan(\theta_2)$

$$\mathrm{atan2}(s,c) = \tan^{-1}\left(\frac{s}{c}\right) + \pi$$

$$\mathrm{atan2}(s,c) = \tan^{-1}\left(\frac{s}{c}\right)$$



$$\mathrm{atan2}(s,c) = \tan^{-1}\left(\frac{s}{c}\right) - \pi$$

$$\mathrm{atan2}(s,c) = \tan^{-1}\left(\frac{s}{c}\right)$$

Solution:

$$\mathrm{atan2}(s,c) = \begin{cases} \left.\begin{matrix} c > 0 \\ Q-I, Q-IV \end{matrix}\right\} \Rightarrow \mathrm{atan2}(s,c) = \tan^{-1}\left(\frac{s}{c}\right) \\ \left.\begin{matrix} c = 0 \\ Q-I, Q-IV \end{matrix}\right\} \Rightarrow \mathrm{atan2}(s,c) = \mathrm{sgn}(s)\cdot\frac{\pi}{2} \\ \left.\begin{matrix} c < 0 \\ Q-II, Q-III \end{matrix}\right\} \Rightarrow \mathrm{atan2}(s,c) = \tan^{-1}\left(\frac{s}{c}\right) + \mathrm{sgn}(s)\cdot\pi \end{cases}$$

1.9 Extracting the rotation angles of a rotation matrix

Given a Rotation Matrix R, we decide the parametrization

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} \cos\beta_1 \cos\beta_2 & \cos\beta_1 \sin\beta_2 \sin\beta_3 - \sin\beta_1 \cos\beta_3 & \sin\beta_1 \sin\beta_3 - \cos\beta_1 \sin\beta_2 \cos\beta_3 \\ \sin\beta_1 \cos\beta_2 & \sin\beta_1 \sin\beta_2 \sin\beta_3 + \cos\beta_1 \cos\beta_3 & -\sin\beta_1 \sin\beta_2 \cos\beta_3 - \cos\beta_1 \sin\beta_3 \\ -\sin\beta_2 & \cos\beta_2 \sin\beta_3 & \cos\beta_2 \cos\beta_3 \end{pmatrix}$$

And we can extract the angles from that parametrization

$$\begin{array}{lcl} \beta_2 = \mathrm{asin}(r_{31}) & & \beta_3 = \mathrm{atan2}(r_{32}, r_{33}) \\ \beta_3 = \mathrm{asin}(r_{32} / \cos\beta_2) & \Longrightarrow & \beta_1 = \mathrm{atan2}(r_{21}, r_{11}) \\ \beta_1 = \mathrm{atan}(r_{12} / r_{11}) & & \beta_2 = \mathrm{atan2}(\text{-}r_{31}, r_{32} / \sin\beta_3) \end{array}$$

Contents

Contents

1.10 Computing the closest rotation matrix

Any *mxn* matrix *M* can be expressed in terms of its Singular Value Decomposition as:

$$M = UDV^T ; SVD(M) = UDV^T$$

where:

U is an *nxn* rotation matrix,
V is an *mxm* rotation matrix, and
D is an *mxn* diagonal matrix (i.e off-diagonals are all 0).

1.10 Computing the closest rotation matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

M is 3x3, the inverse of M is,

$$M^{-1} = \left(UDV^T\right)^{-1} = V\left(UD\right)^{-1} = VD^{-1}U^T$$

where:

$$D^{-1} = \begin{pmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{pmatrix}$$

U is an *3*x*3* rotation matrix,
V is an *3*x3 rotation matrix, and
D is an *3*x*3* diagonal matrix (i.e off-diagonals are all 0).

M is a Rotation Matrix if D is a Rotation matrix, i.e. its elements are unitarian.

$$SVD(M) = U\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}V^T \qquad R = U\begin{pmatrix} \mathrm{sgn}(\lambda_1) & 0 & 0 \\ 0 & \mathrm{sgn}(\lambda_2) & 0 \\ 0 & 0 & \mathrm{sgn}(\lambda_3) \end{pmatrix}V^T$$

sgn(x)=1 if x>0; sgn(x)=-1 if x<0

Contents