MSc VIBOT/Computer Vision/MaIA

Applied Mathematics

Linear Algebra

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Problem Set

Problem 1

Suppose that x, y and z are linearly independent vectors in $\mathbb{R}^n$. Prove that $x, x+y$ and $x+y+z$ are also independent.

Problem 2

Prove that the three vectors bellow are linearly independent

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}; \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}; \begin{bmatrix} -2 \\ 5 \\ 7 \end{bmatrix}.$$

Problem 3

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix}$.

Find a solvability condition on b, to the equation $Ax = b$.

Problem 4

Find the inverses of the following matrices using Gauss-Jordan elimination

$$A = \begin{bmatrix} 2 & 2 \\ 6 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5

Consider the matrix $F = \begin{bmatrix} 3 & 1 & -1 \\ -3 & 1 & 3 \\ 6 & 4 & 1 \end{bmatrix}$

1. Find the LU decomposition of F.

2. Calculate $\det(F)$.

3. Solve the linear system: $Fx = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$.

Problem 6

Let $A = \begin{bmatrix} 5 & -2 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
Find all solutions to the equation $Ax = 3x$.

Problem 7

Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$

1. Find the rref of A;

2. Find a basis for the column space $\mathrm{C}(A)$ of A;

3. Find a basis for the nullspace $\mathrm{N}(A)$ of A;

4. Find all solutions x to the system $Ax = b$, where $b = \begin{bmatrix} 7 \\ 10 \\ 2 \\ 8 \end{bmatrix}$.

Problem 8

Suppose we have 3 matrices A, B and C such that

1. A is 4×4; B is 4×3 and C is 3×4;

2. $Av = B(Cv)$ for all $v \in \mathbb{R}^4$.

Show that $\mathrm{N}(A)$ contains at least a nonzero vector, and that there exists a vector in $\mathbb{R}^4$ which is not in $\mathrm{C}(A)$.

Problem 9

Let A and B be two $n \times n$ matrices such that $AB = A + B$. Show that $AB = BA$. Such two matrices are called commutative matrices.

Problem 10

Let A and B be two projection matrices.

1. Show that $A + B$ is a projection matrix, if and only if $AB = BA = 0$.
2. Assuming $A + B$ is a projection matrix, show that

$$\ker(A + B) = \ker(A) \cap \ker(B)$$

Problem 11

Find the eigenvalues and eigenvectors of A, A^{-1}, A^2 and $A + 4I$, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Problem 12

Let $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

1. Find the eigenvalues and eigenvectors of A.
2. Explain how you find A^{100}?

Problem 13

Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Is A invertible? If yes, find A^{-1}.
Hint: you are given that $\det(A - \lambda I) = -(\lambda^3 - 3\lambda^2 + 4)$.

Problem 14

Prove the following results:

- $\ker(A) \subseteq \ker(A^2)$ for every square matrix A.
- If A is nonsingular, then $\ker(A) = \ker(A^2)$.

Problem 15

Find the parabola $A + Dt + Ct^2$ that comes closest to the values $b = (0; 0; 1; 0; 0)$ at the times $t = (-2; -1; 0; 1; 2)$.

Problem 16

We have measured some data (t_i, y_i) and we want to find the best weighted linear fit $y = \alpha + \beta t$, using some weights c_i given bellow:

t_i	0	1	2	-1
y_i	10	-4	-2	0
c_i	2	1	1	3

Find the equation of this line.

Problem 17

In a town called *Computer Vision Village*, the local newspaper *The Computer Visionist* has determined that a citizen who purchases a copy of their paper one day has 70% chance of buying the following day's edition. They have also determined that a person who does not purchase a copy of *The Computer Visionist* one day has 20% chance of purchasing it the next day. Records show that of the 1000 citizens of *Computer Vision Village*, exactly 750 purchased a copy of the newspaper on Day 0. To determine the appropriate amount of papers to press each day, the owner of *The Computer Visionist*, Mr Marr Rosenfeld, is interested the following types of questions:

1. If a person purchased a paper today, how likely is he to purchase a paper on Day 2? Day 3? Day n?
2. What sales figures can *The Computer Visionist* expect on Day 2? Day 3? Day n?
3. Will the sales figures fluctuate a great deal from day to day, or are they likely to become stable eventually?

MATLAB practicing

Problem 18

Matlab's function *rref* allows the user to compute the reduced row echelon form of a matrix.

1. Try *rref* with the following matrix $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$

   What's the number of pivots of A? What's the rank of A?.
2. Write a function which computes bases of the four fundamental subspaces of a matrix: column space, nullspace, row space and left nullspace (you are allowed to use only Matlab built-in function *rref*).
3. Using the function *rref*, write a function to compute the inverse of a matrix.

   Hint: Think about Gauss-Jordan elimination!

Problem 19

- First, show that the system of linear equations $Ax = b$ can be solved iff rank(A) = rank$([A\ b])$. (We have already showed that in class).

- Write a function which takes as input a vector v and an unspecified numbers of vectors $\{u_1, u_2, \ldots\}$. All input vectors are the same size. Your function should, as output, display a message on the screen saying whether $v \in \text{span}(u_1, u_2, \ldots)$ or $v \notin \text{span}(u_1, u_2, \ldots)$.

- Given $a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$, determine whether or not $c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ is in the span of vectors a and b?

Problem 20

You are given two function *mcode* and *mdecode*, which you must download form the course webpage!

1. Try the function *mdecode*(B,A) with the following matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

and

$$B = [339;\ 851;\ 1722;\ 3061;\ 432;\ 1101;\ 2219;\ 3901;\ 346;\ 955;\ 1956;\ 3446;\ 418;\ 1066;\ 2157;\ 3805;\ 284;\ 715;\ 1357;\ 2242;\ 415;\ 1022;\ 2021;\ 3509;\ 149;\ 182;\ 215;\ 248]^T$$

2. Explain how the message is coded and decoded using the nonsingular matrix A.

Problem 21

1. Use the least squares technique to find the line which best fits the following collection of points

$$(-3, 3), (-2, 3.5), (-1, 2.5), (0, -0.5), (1, -1.5), (2, -0.5), (3, -1), (4, -1.5).$$

Plot the data points and the approximation on the same figure.

2. Find the line that best approximates the points $(-3, 8.5)$, $(-2, 4.3)$, $(-1, 0.8)$, $(0, 0.1)$, $(1, 1)$, $(2, 3.8)$ and $(3, 9.2)$. What can you say about this approximation?

   Using MATLAB function *polyfit*, find the parabola that best approximates these points.

   Try to fit a cubic to these points. What can you say about this approximation? It is better than the quadratic approximation? (Expand your view of the graph!).

3. Find the polynomial of smallest degree which best approximates the following points:

$$(-2.7, 18), (-1.7, 0), (-0.8, 5.5), (0.2, 8.7), (1, 4), (1.6, 0.2), (2.6, 14).$$

Problem 22

A linear system of equations $Ax = b$ can be solved either by linear least squares (LLS) or by singular value decomposition (SVD).
Say we want to solve the system $Ax = b$, with $A = [c_1\ c_2\ c_3]$ and

$$c_1 = [1\ \ 2\ \ 4\ \ 8]^T,$$

$$c_2 = [3\ \ 6\ \ 9\ \ 12]^T,$$

$$c_3 = c_1 - 4c_2 + 10^{-6}.$$

The vector b is defined by

$$b = c_1 - 4c_2 + 10^{-3}.$$

1. Solve $Ax = b$ using the normal equation (LLS).
2. Solve $Ax = b$ using SVD.

For each method, compute the norm of the error $e = b - A\hat{x}$, where $\hat{x}$ is the obtained solution. What is the difference between both methods? Can you give an explanation?

Problem 23

SVD and Image Compression

The SVD of an $m \times n$ matrix A of rank k is given by $A = U\Sigma V^T$, which can also be written as $A = \sum_{i=1}^{k} \sigma_i u_i v_i^T$.
So, the matrix A is a sum of rank one matrices that are orthogonal with respect to the matrix inner product. Truncating the sum at p terms defines a rank p matrix $A_p = \sum_{i=1}^{p} \sigma_i u_i v_i^T$. If we approximate A with A_p, then we make an error equals to $E_p = A - A_p = \sum_{i=p+1}^{k} \sigma_i u_i v_i^T$. It can be shown that A_p is the best rank p approximation to A.
SVD can be used for image compression. In digital image processing, the quality of the compression on an $m \times n$ image I is often measured in term of Peak Signal to Noise Ratio (PSNR) which is defined as

$$PSNR = 10\, log_{10} \frac{(\text{max range})^2}{\sqrt{MSE}},$$

where *max range* is the maximum pixel value in the image and *MSE* is the mean square error defined by

$$MSE = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \|I(i,j) - I_c(i,j)\|^2 .$$

1. Select an image I of your choice (a grayscale image not too big in size, 256×256 pixels for instance).
2. Using SVD, display a rank 1 approximation of I. What can you say about the 'columns' in the rank 1 approximation of the image in terms of linear dependence/independence.
3. Reconstruct your image using different ranks approximation (4, 16, 32, 64, 128). For each approximation, show the absolute difference between the original image and the re-constructed images. Give also the PSNR values.

4. Explain how the reconstruction using rank approximation, provides a compression of the original image. What is the compression ratio?

5. Create a 256×256 random matrix I_2 (use the command *rand*). Convert the random matrix into an intensity image using the command *mat2gray*.

   For the images I and I_2, make a plot which shows how the singular values decay along the diagonal of Σ in the SVD. Explain the difference in the decay for the two images.

6. Another way of performing SVD compression is achieve by subtracting the mean of the original image before performing the SVD. The mean is then added back to the SVD construction to obtain the re-constructed image. This process is called the 'Rank-1 Update'.

   Apply 'Rank-1 Update' to I and I_2. What are the differences with the previous results?

7. What is, from your point of view, the reason why SVD in not a popular image compression tool?