

Introduction to Mobile Robotics

Robot Motion Planning

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Slides by Kai Arras Last update July 2011

With material from S. LaValle, JC. Latombe, H. Choset et al., W. Burgard

Robot Motion Planning

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- A*, Any-Angle A*, D*/D* Lite
- Dynamic Window Approach (DWA)
- Markov Decision Processes (MDP)

Robot Motion Planning

J.-C. Latombe (1991):

“...eminently necessary since, by definition,
a robot accomplishes tasks by moving in
the real world.”

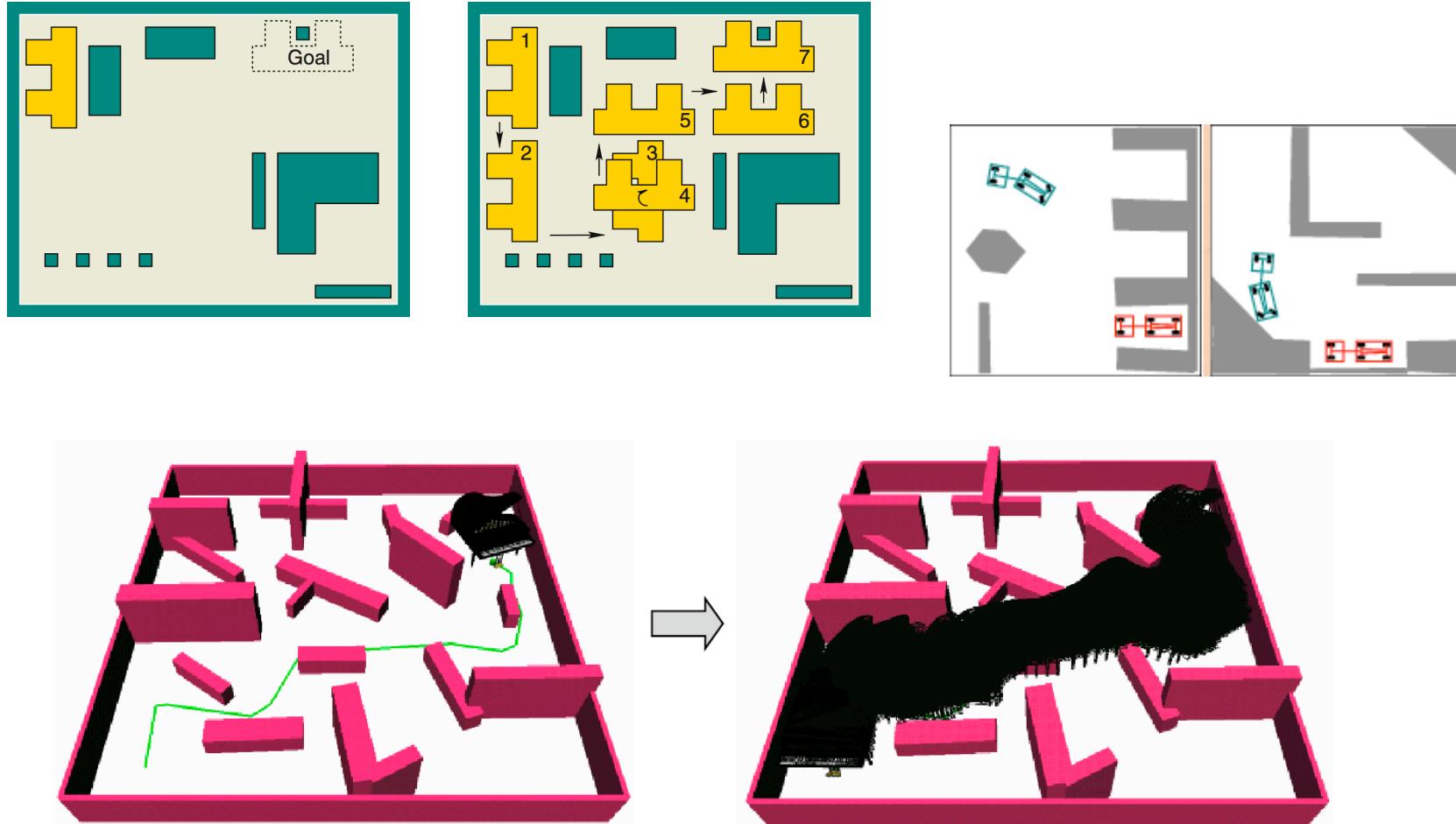
Goals

- Collision-free trajectories
- Robot should reach the goal location
as fast as possible

Problem Formulation

- The **problem of motion planning** can be stated as follows. Given:
 - A **start** pose of the robot
 - A desired **goal** pose
 - A geometric description of the **robot**
 - A geometric description of the **world**
- Find a path that moves the robot gradually from **start** to **goal** while **never touching** any obstacle

Problem Formulation



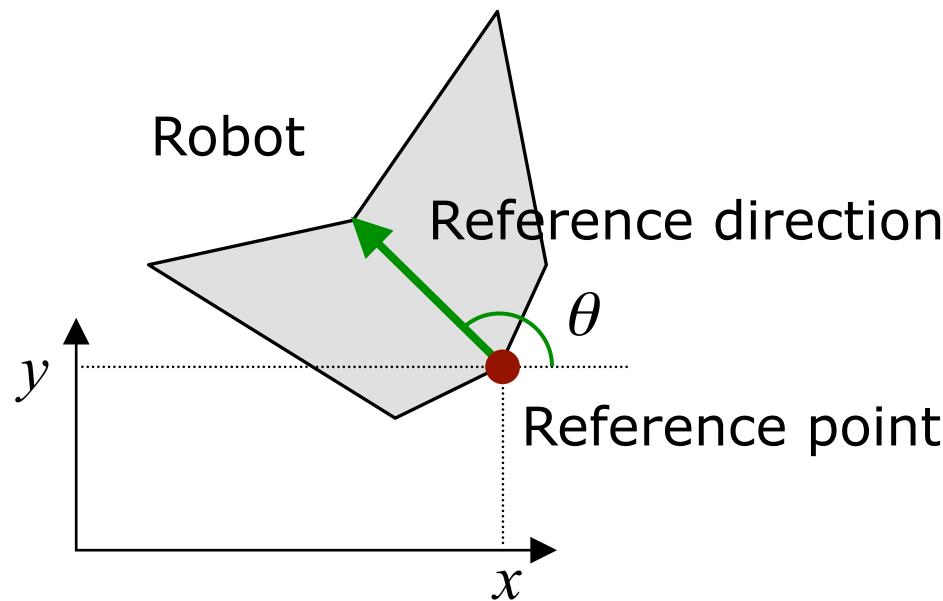
Motion planning is sometimes also called **piano mover's problem**

Configuration Space

- Although the motion planning problem is defined in the regular world, it lives in another space: the **configuration space**
- A robot configuration q is a specification of the positions of all robot points relative to a fixed coordinate system
- Usually a configuration is expressed as a **vector of positions** and **orientations**

Configuration Space

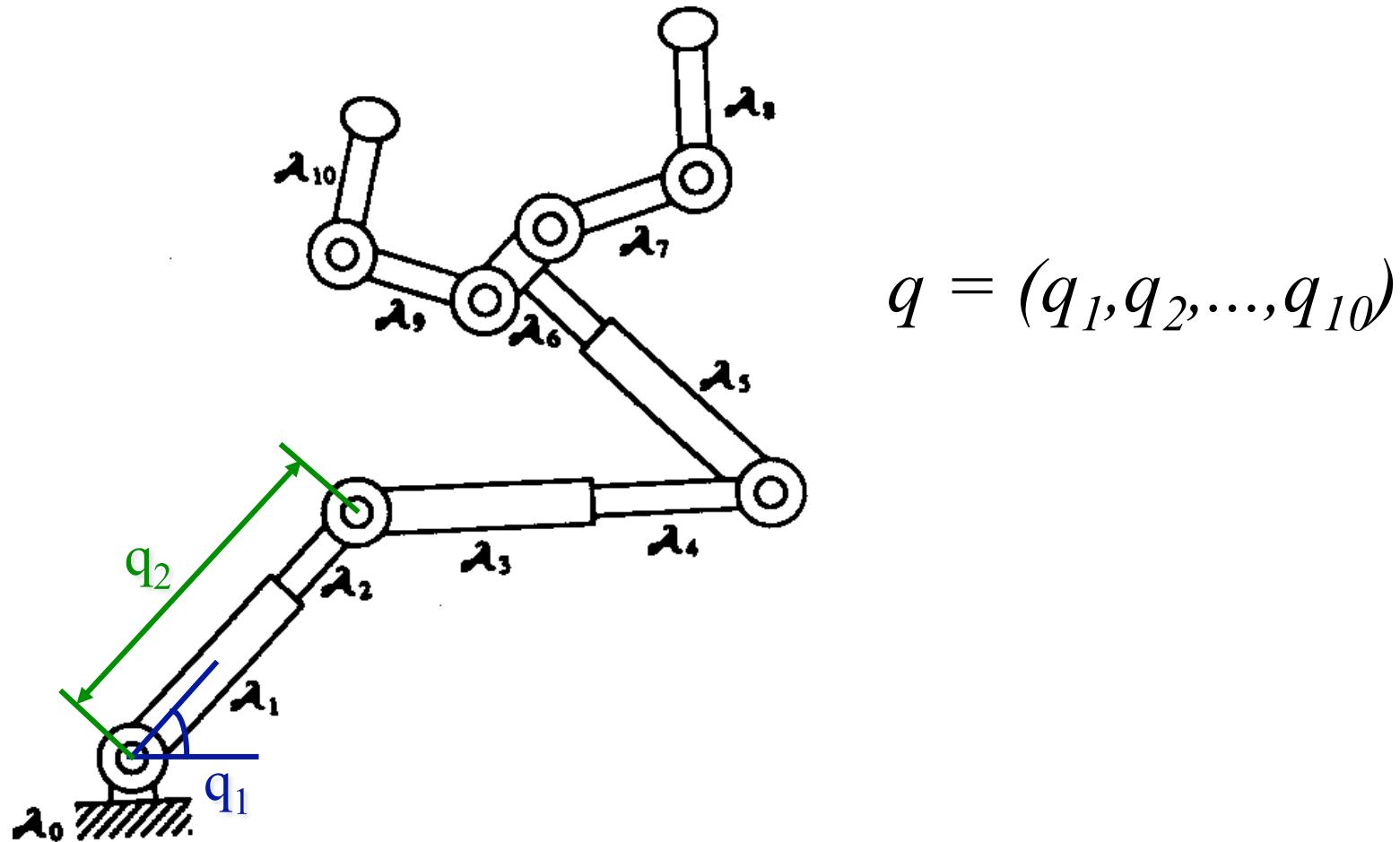
Rigid-body robot example



- 3-parameter representation: $q = (x, y, \theta)$
- In 3D, q would be of the form $(x, y, z, \alpha, \beta, \gamma)$

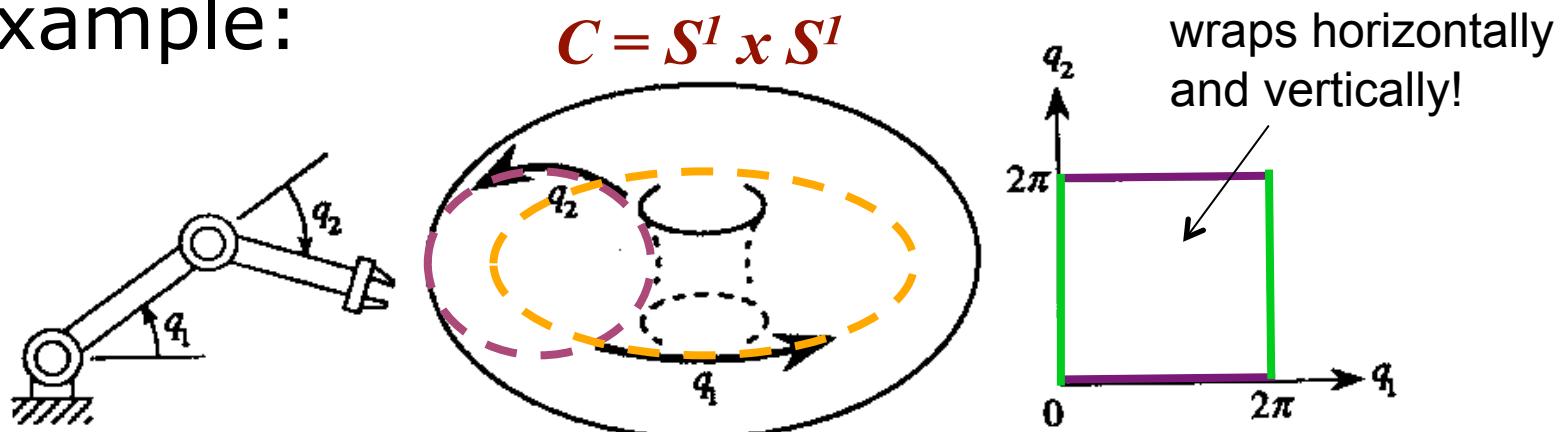
Configuration Space

Articulated robot example



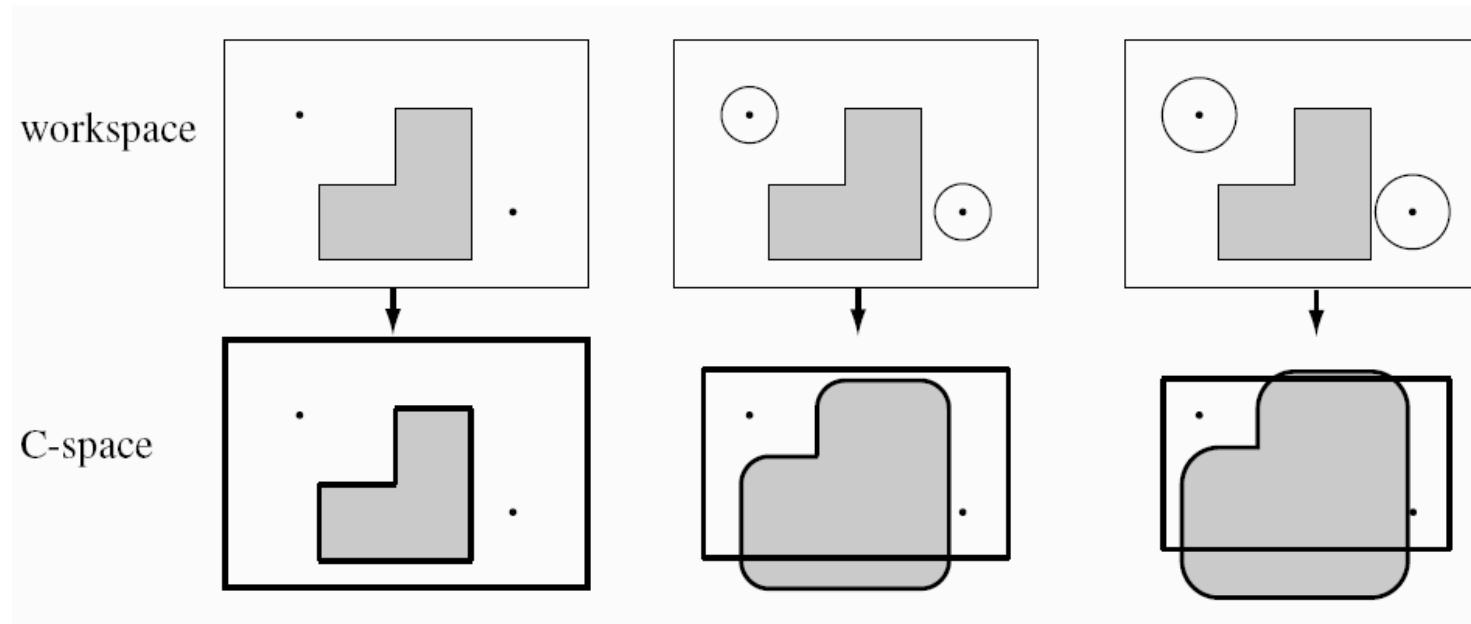
Configuration Space

- The configuration space (C-space) is the **space of all possible configurations**
- The topology of this space is usually **not** that of a Cartesian space
- The C-space is described as a **topological manifold**
- Example:



Configuration Space

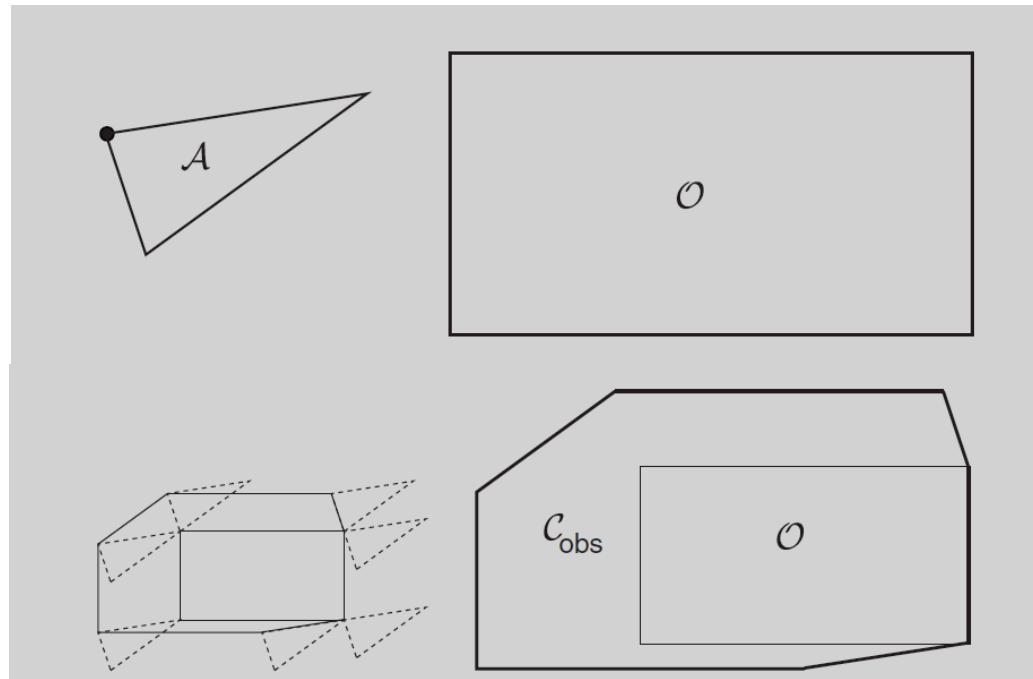
- Example: circular robot



- C-space is obtained by sliding the robot along the edge of the obstacle regions "blowing them up" by the robot radius

Configuration Space

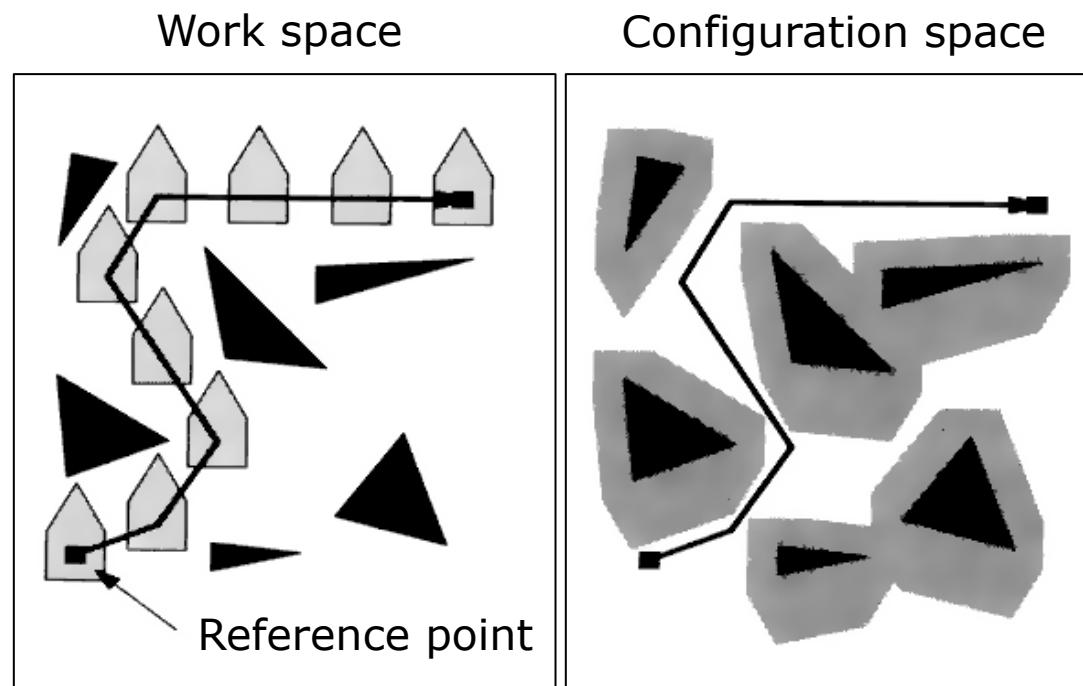
- Example: polygonal robot, translation only



- C-space is obtained by sliding the robot along the edge of the obstacle regions

Configuration Space

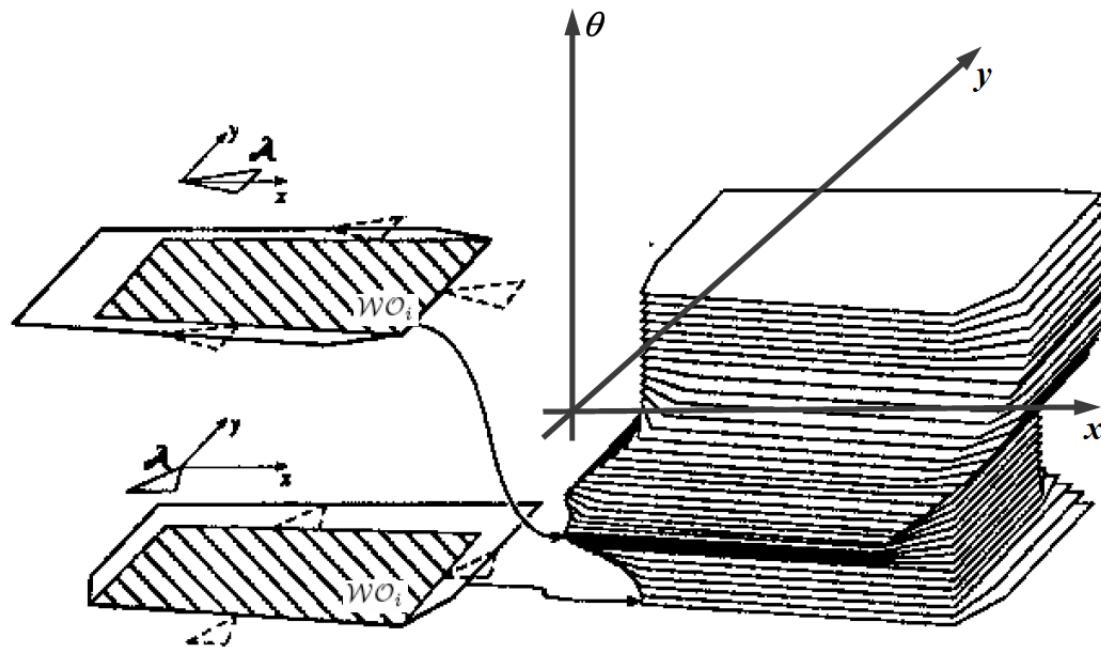
- Example: polygonal robot, translation only



- C-space is obtained by sliding the robot along the edge of the obstacle regions

Configuration Space

- Example: polygonal robot, trans+**rotation**



- C-space is obtained by sliding the robot along the edge of the obstacle regions in all orientations

Configuration Space

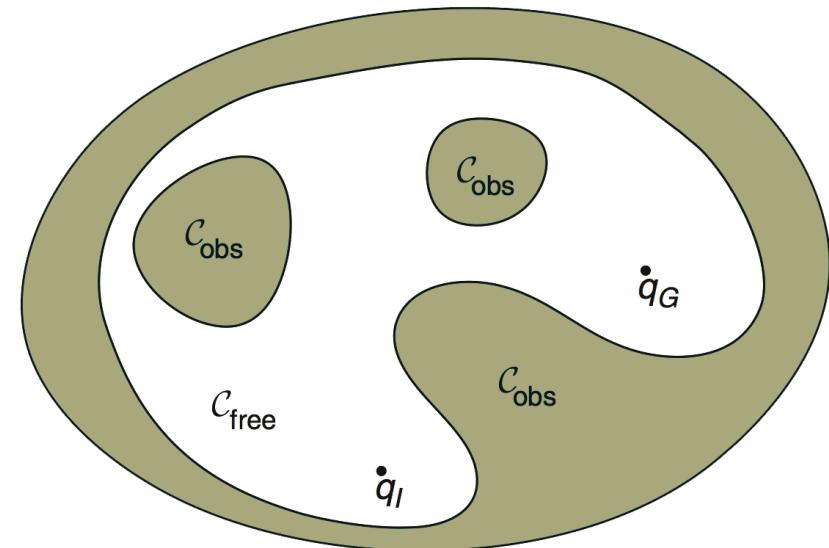
Free space and obstacle region

- With $\mathcal{W} = \mathbb{R}^m$ being the work space, $\mathcal{O} \in \mathcal{W}$ the set of obstacles, $\mathcal{A}(q)$ the robot in configuration $q \in \mathcal{C}$

$$\mathcal{C}_{free} = \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{O} = \emptyset\}$$

$$\mathcal{C}_{obs} = \mathcal{C}/\mathcal{C}_{free}$$

- We further define
 - q_I : start configuration
 - q_G : goal configuration



Configuration Space

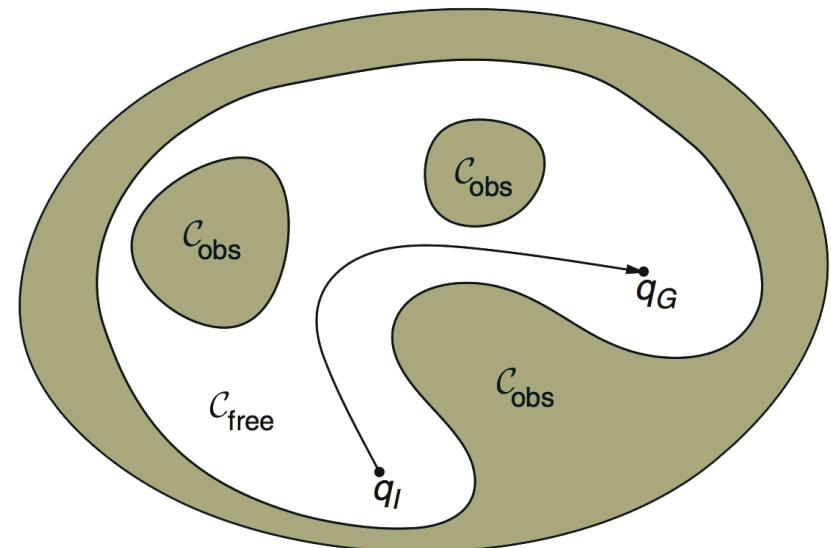
Then, motion planning amounts to

- Finding a continuous path

$$\tau : [0, 1] \rightarrow \mathcal{C}_{\text{free}}$$

with $\tau(0) = q_I$, $\tau(1) = q_G$

- Given this setting,
we can do planning
with the robot being
a point in C-space!



C-Space Discretizations

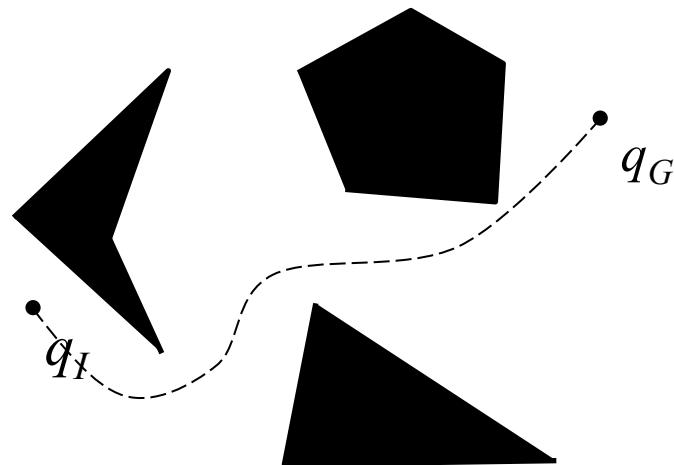
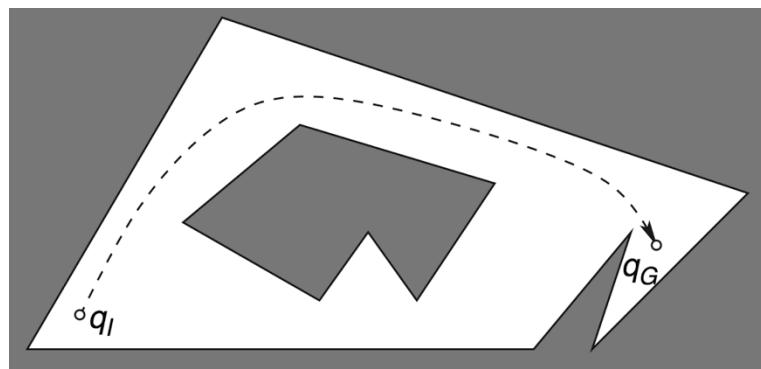
- Continuous terrain needs to be **discretized** for path planning
- There are **two general approaches** to discretize C-spaces:
 - **Combinatorial planning**
Characterizes C_{free} explicitly by capturing the connectivity of C_{free} into a graph and finds solutions using search
 - **Sampling-based planning**
Uses collision-detection to probe and incrementally search the C-space for solution

Combinatorial Planning

- We will look at four **combinatorial planning techniques**
 - Visibility graphs
 - Voronoi diagrams
 - Exact cell decomposition
 - Approximate cell decomposition
- They all produce a **road map**
 - A **road map** is a **graph** in C_{free} in which each vertex is a configuration in C_{free} and each edge is a collision-free path through C_{free}

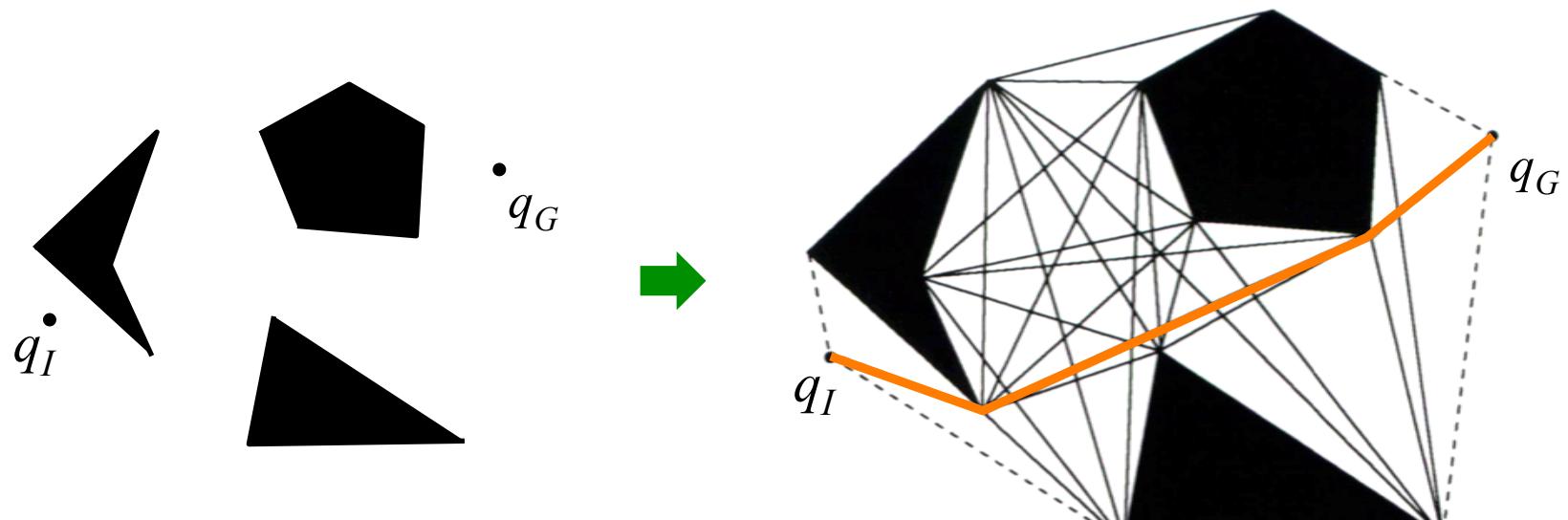
Combinatorial Planning

- Without loss of generality, we will consider a problem in $\mathcal{W} = \mathbb{R}^2$ with a **point robot** that cannot rotate. In this case: $\mathcal{C} = \mathbb{R}^2$
- We further assume a **polygonal** world



Visibility Graphs

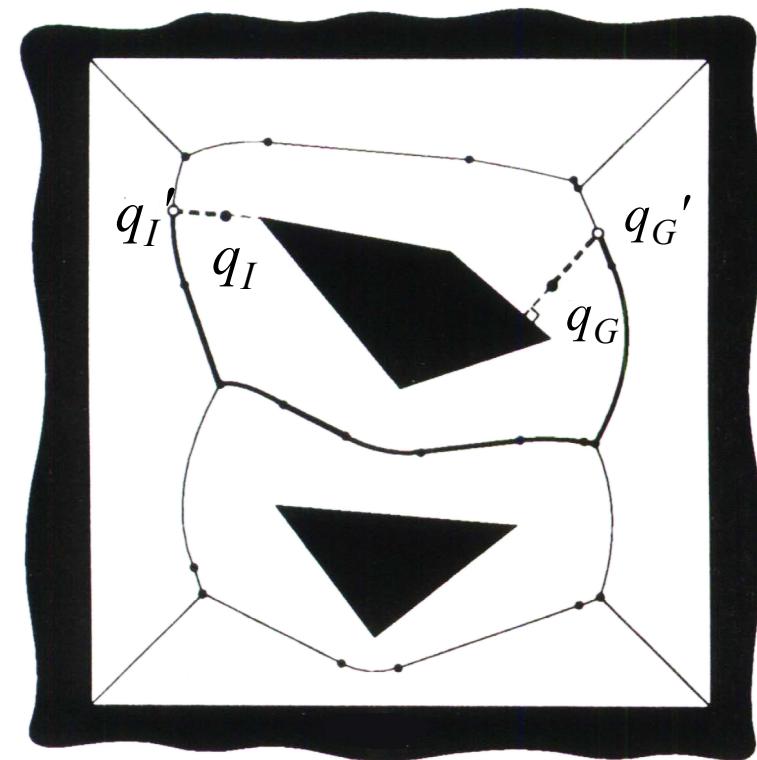
- **Idea:** construct a path as a polygonal line connecting q_I and q_G through vertices of C_{obs}
- Existence proof for such paths, **optimality**
- One of the earliest path planning methods



- Best algorithm: $O(n^2 \log n)$

Generalized Voronoi Diagram

- **Defined** to be the set of points q whose cardinality of the set of boundary points of C_{obs} with the same distance to q is greater than 1
- Let us decipher this definition...
- **Informally:** the place with the same **maximal clearance** from all nearest obstacles



Generalized Voronoi Diagram

- **Formally:**

Let $\beta = \partial C_{free}$ be the boundary of C_{free} , and $d(p,q)$ the Euclidian distance between p and q . Then, for all q in C_{free} , let

$$\text{clearance}(q) = \min_{p \in \beta} d(p, q)$$

be the *clearance* of q , and

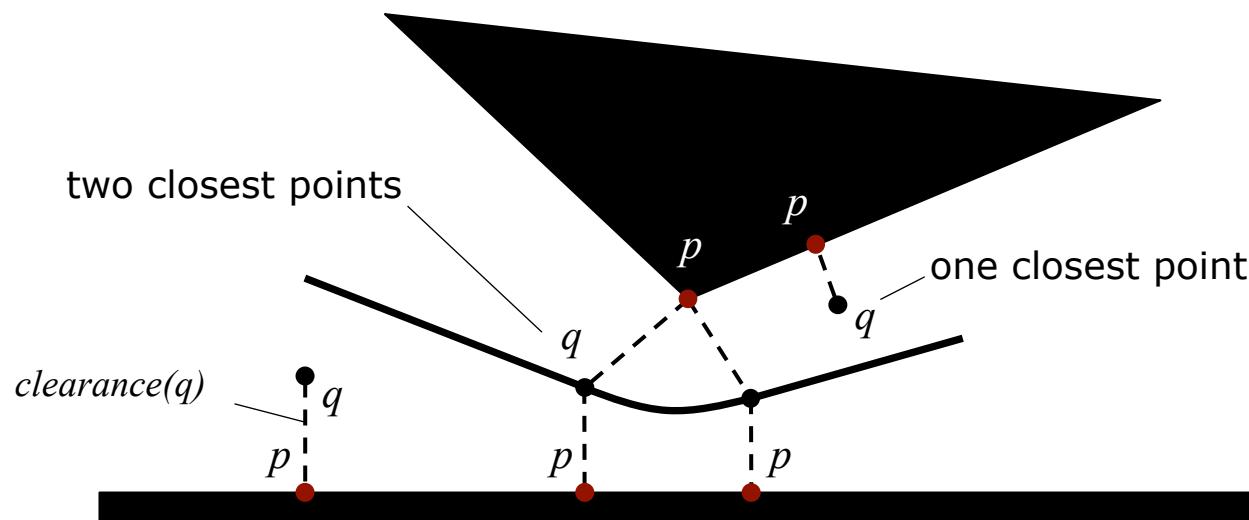
$$\text{near}(q) = \{p \in \beta \mid d(p, q) = \text{clearance}(q)\}$$

the set of "base" points on β with the same clearance to q . The **Voronoi diagram** is then the set of q 's with more than one base point p

$$V(C_{free}) = \{q \in C_{free} \mid |\text{near}(q)| > 1\}$$

Generalized Voronoi Diagram

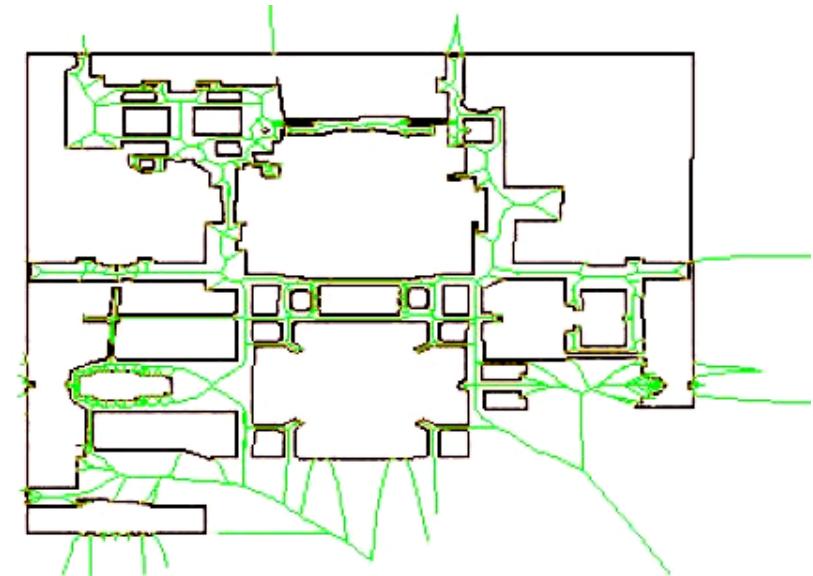
- Geometrically:



- For a polygonal C_{obs} , the Voronoi diagram consists of (n) lines and parabolic segments
- Naive algorithm: $O(n^4)$, best: $O(n \log n)$

Voronoi Diagram

- Voronoi diagrams have been well studied for (reactive) **mobile robot** path planning
- Fast methods exist to compute and update the diagram in real-time for low-dim. C's
 - **Pros:** maximize clearance is a good idea for an uncertain robot
 - **Cons:** unnatural attraction to open space, suboptimal paths
- Needs extensions

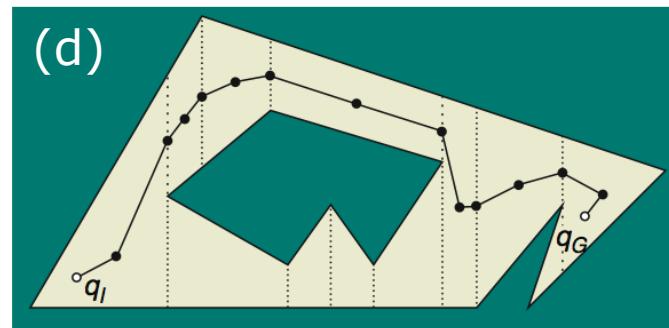
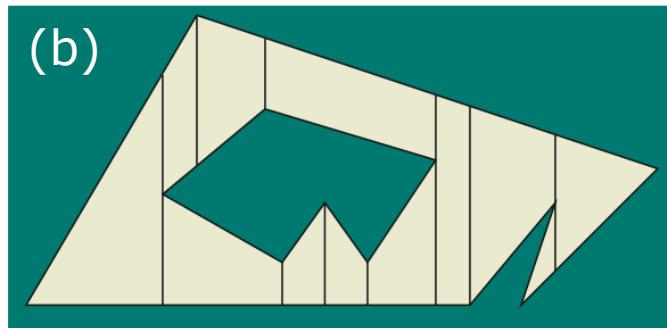
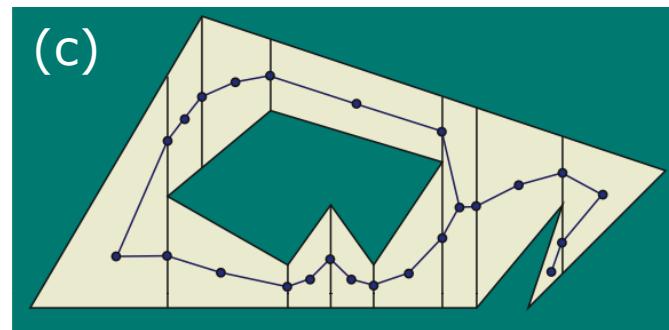
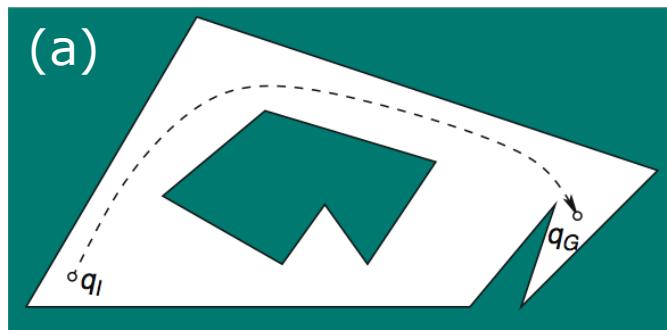


Exact Cell Decomposition

- **Idea:** decompose C_{free} into non-overlapping cells, construct connectivity graph to represent adjacencies, then search
- A popular implementation of this idea:
 1. Decompose C_{free} into **trapezoids** with vertical side segments by shooting rays upward and downward from each polygon vertex
 2. Place one **vertex** in the interior of every **trapezoid**, pick e.g. the centroid
 3. Place one **vertex** in every vertical **segment**
 4. Connect the vertices

Exact Cell Decomposition

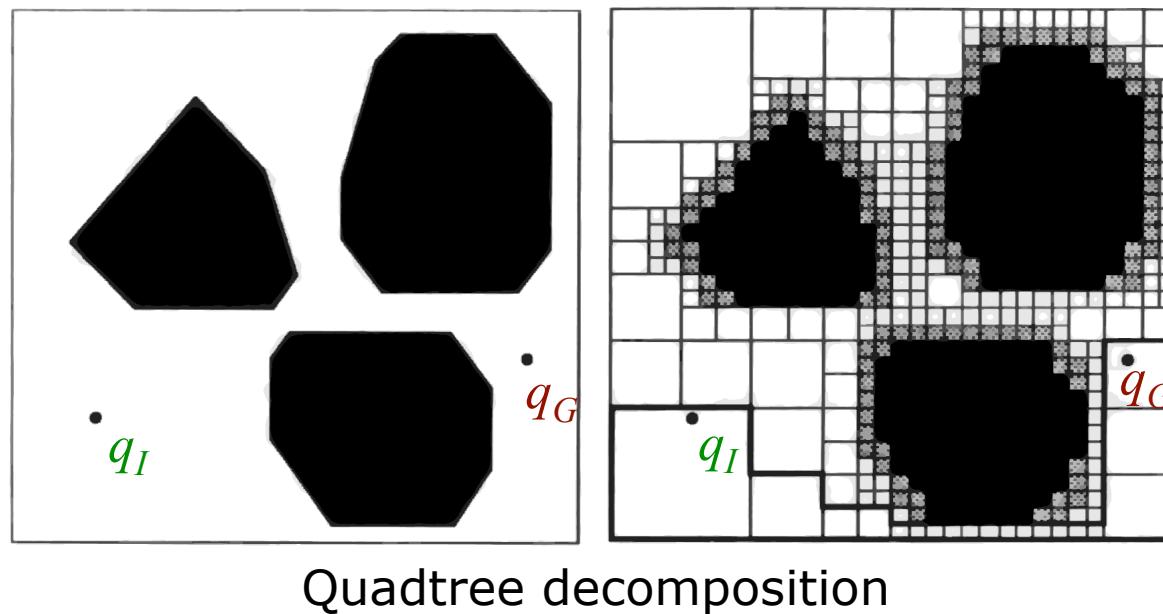
- Trapezoidal decomposition ($\mathcal{C} = \mathbb{R}^3$ max)



- Best known algorithm: $O(n \log n)$ where n is the number of vertices of C_{obs}

Approximate Cell Decomposition

- Exact decomposition methods can be involved and inefficient for complex problems
- Approximate decomposition uses cells with the **same simple predefined shape**



Approximate Cell Decomposition

- Exact decomposition methods can be involved and inefficient for complex problems
- Approximate decomposition uses cells with the **same simple predefined shape**
- **Pros:**
 - Iterating the **same** simple computations
 - Numerically more **stable**
 - **Simpler** to implement
 - Can be made **complete**

Combinatorial Planning

Wrap Up

- Combinatorial planning techniques are **elegant** and **complete** (they find a solution if it exists, report failure otherwise)
 - But: become **quickly intractable** when C-space dimensionality increases (or n resp.)
 - Combinatorial **explosion** in terms of **facets** to represent \mathcal{A} , \mathcal{O} , and \mathcal{C}_{obs} , especially when rotations bring in non-linearities and make C a nontrivial manifold
- Use **sampling-based planning**
Weaker guarantees but more efficient

Sampling-Based Planning

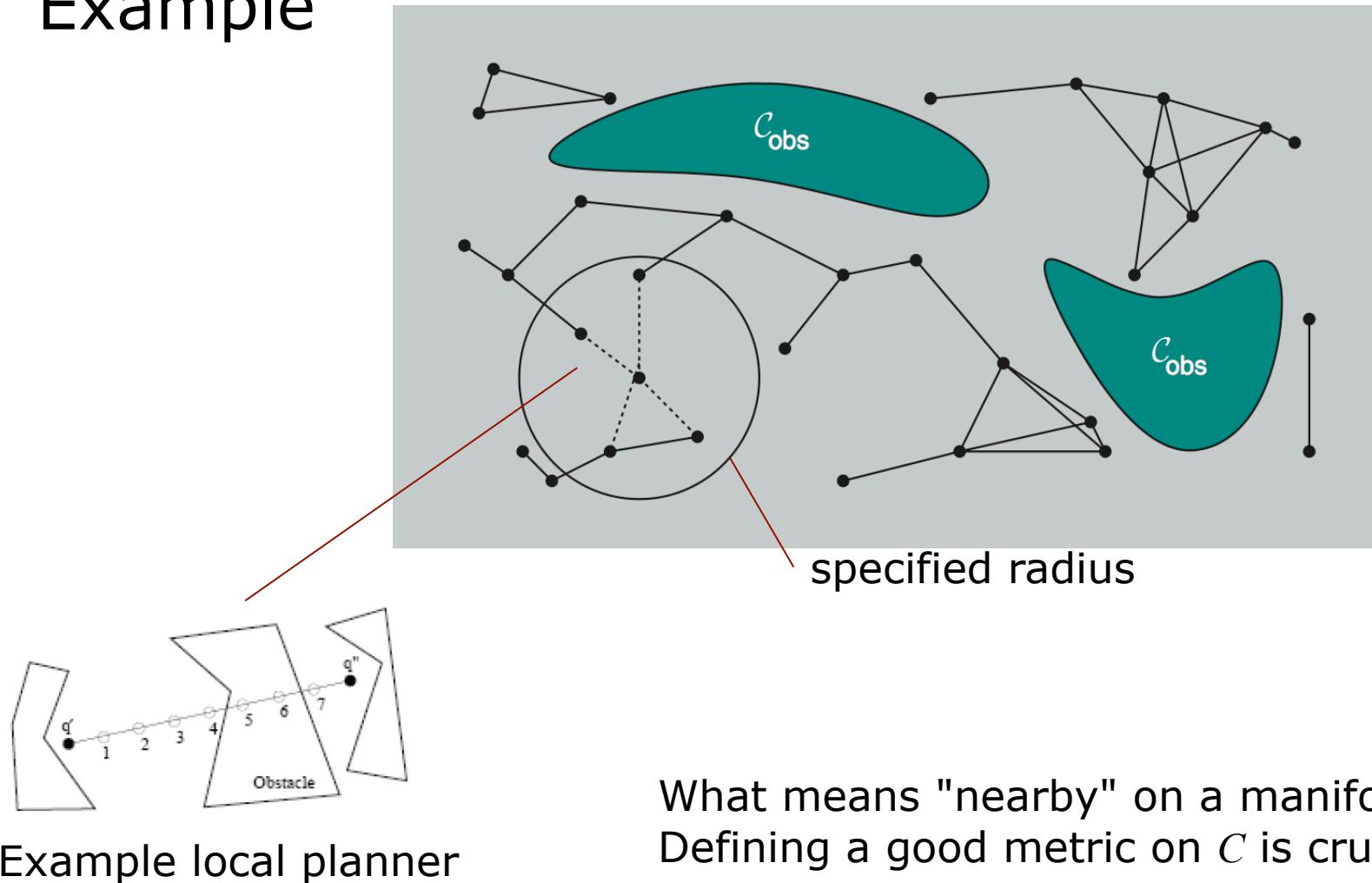
- Abandon the concept of explicitly characterizing C_{free} and C_{obs} and leave the algorithm **in the dark** when exploring C_{free}
- The only light is provided by a **collision-detection algorithm**, that probes C to see whether some configuration lies in C_{free}
- We will have a look at
 - **Probabilistic road maps** (PRM)
[Kavraki et al., 92]
 - **Rapidly exploring random trees** (RRT)
[Lavalle and Kuffner, 99]

Probabilistic Road Maps

- **Idea:** Take random samples from C , declare them as vertices if in C_{free} , try to connect nearby vertices with local planner
- The **local planner** checks if line-of-sight is collision-free (powerful or simple methods)
- Options for *nearby*: **k-nearest neighbors** or all neighbors within **specified radius**
- Configurations and connections are added to graph until roadmap is **dense enough**

Probabilistic Road Maps

- Example



Probabilistic Road Maps

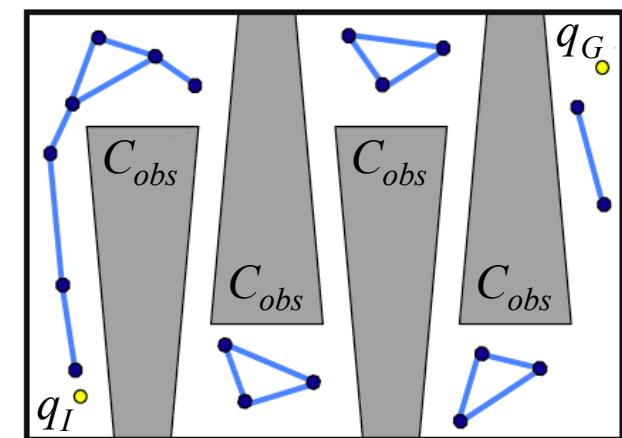
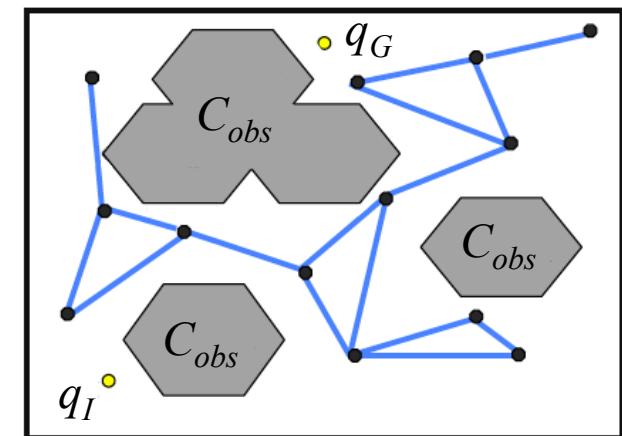
Good and bad news:

- **Pros:**

- *Probabilistically complete*
- Do not construct C-space
- Apply easily to high-dim. C's
- PRMs have solved previously unsolved problems

- **Cons:**

- Do not work well for some problems, narrow passages
- Not optimal, not complete

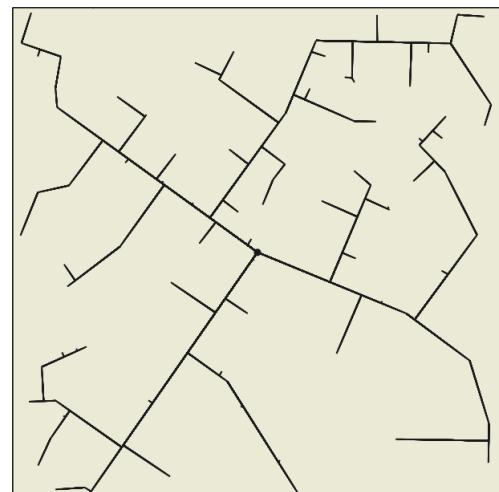


Probabilistic Road Maps

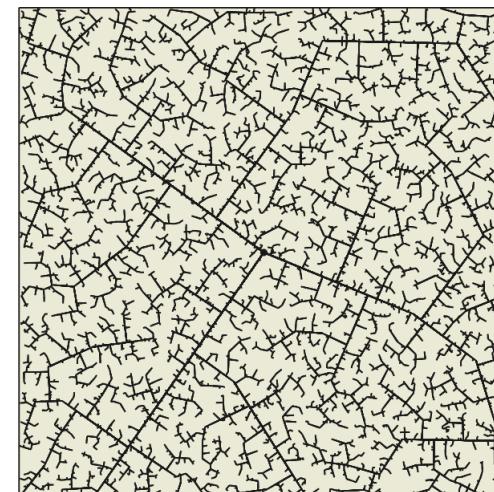
- How to **uniformly sample C** ? This is not at all trivial given its topology
- For example over spaces of rotations:
Sampling Euler angles gives samples near poles, not uniform over $SO(3)$. Use quaternions!
- However, PRMs are **powerful, popular** and **many extensions** exist: advanced sampling strategies (e.g. near obstacles), PRMs for deformable objects, closed-chain systems, etc.

Rapidly Exploring Random Trees

- **Idea:** aggressively probe and explore the C-space by **expanding incrementally** from an initial configuration q_0
- The explored territory is marked by a **tree rooted at q_0**



45 iterations



2345 iterations

RRTs

- The algorithm: Given C and q_0

Algorithm 1: RRT

```
1  $G.\text{init}(q_0)$ 
2 repeat
3    $q_{rand} \rightarrow \text{RANDOM\_CONFIG}(\mathcal{C})$ 
4    $q_{near} \leftarrow \text{NEAREST}(G, q_{rand})$ 
5    $G.\text{add\_edge}(q_{near}, q_{rand})$ 
6 until condition
```

← Sample from a **bounded region** centered around q_0

E.g. an axis-aligned relative random translation or random rotation

(but recall sampling over rotation spaces problem)



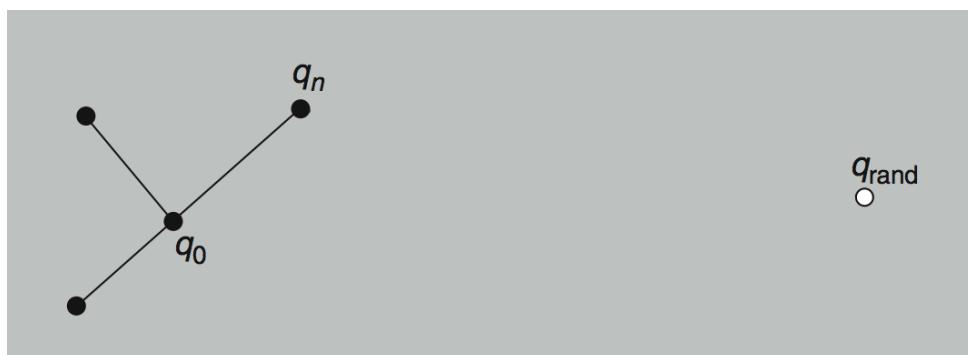
RRTs

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```

← Finds closest vertex in G
using a **distance function**
 $\rho : \mathcal{C} \times \mathcal{C} \rightarrow [0, \infty)$
formally a **metric**
defined on C

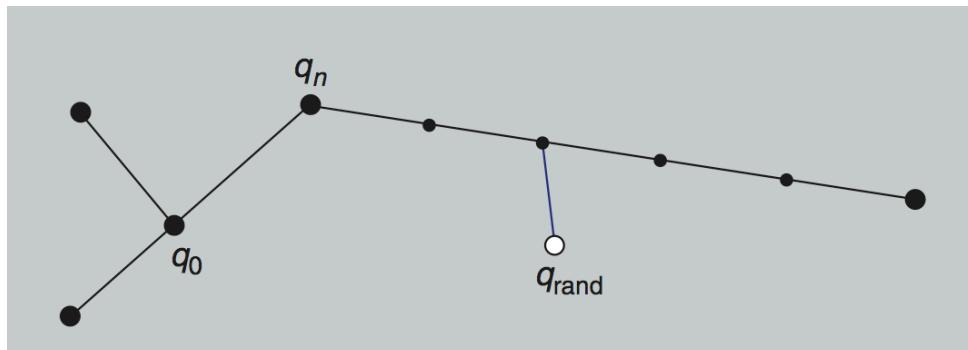


RRTs

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```



◀ Several strategies to find q_{near} given the closest vertex on G:

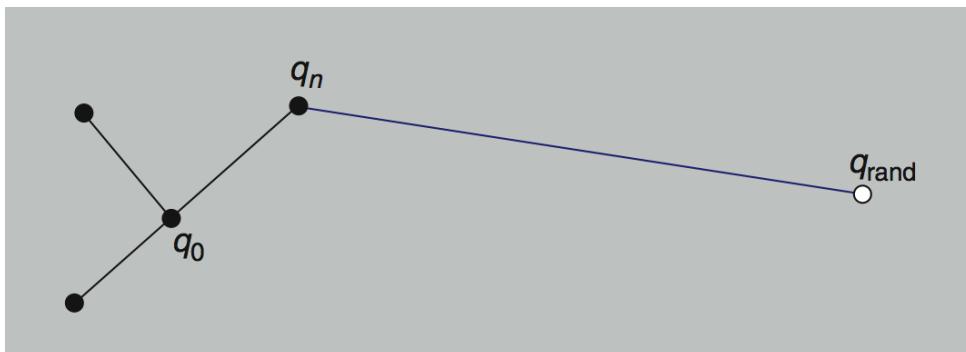
- Take closest vertex
- Check intermediate points at regular intervals and split edge at q_{near}

RRTs

- The algorithm

Algorithm 1: RRT

```
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```



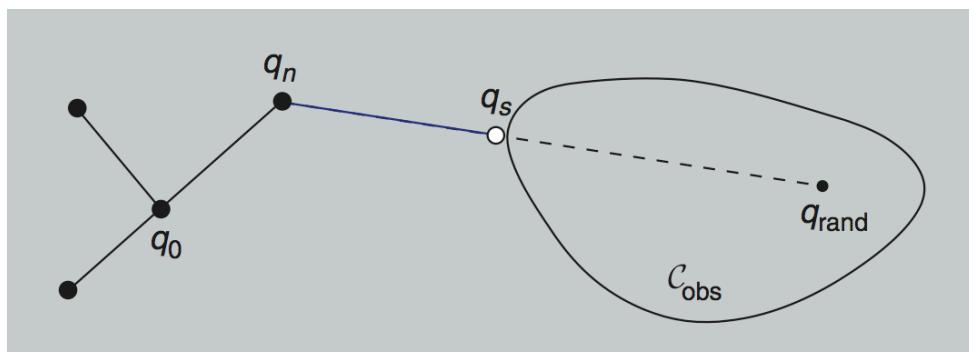
- ← Connect nearest point with random point using a **local planner** that travels from q_{near} to q_{rand}
- No collision: add edge
 - Collision: new vertex is q_i , as close as possible to C_{obs}

RRTs

- The algorithm

Algorithm 1: RRT

```
1  $G.\text{init}(q_0)$ 
2 repeat
3    $q_{rand} \rightarrow \text{RANDOM\_CONFIG}(\mathcal{C})$ 
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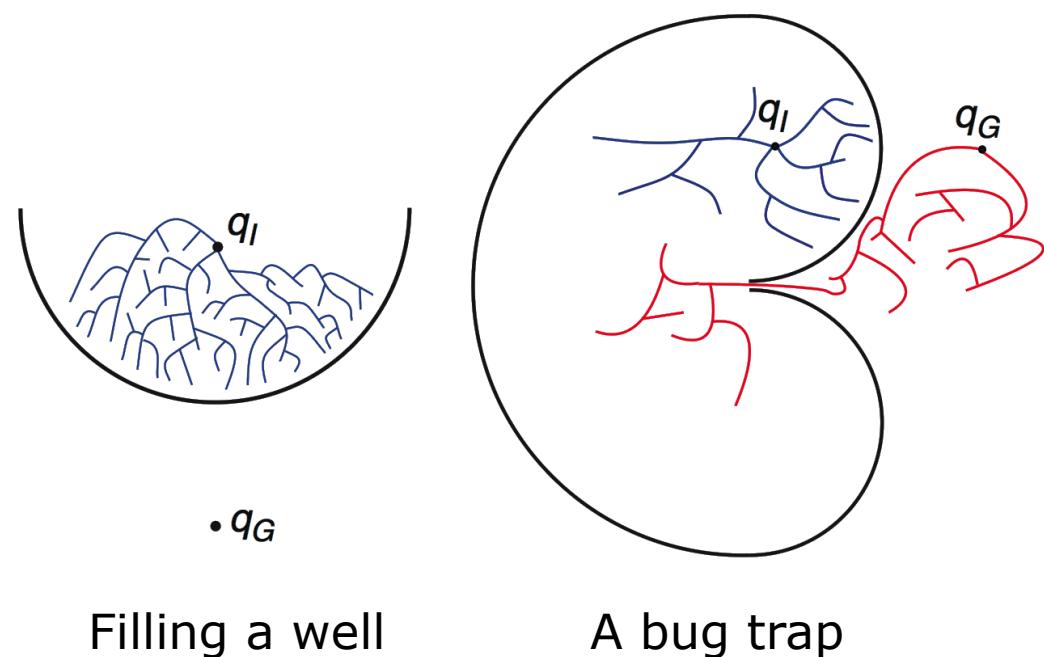
- ← Connect nearest point with random point using a **local planner** that travels from q_{near} to q_{rand}
- No collision: add edge
 - Collision: new vertex is q_i , as close as possible to C_{obs}

RRTs

- How to perform path planning with RRTs?
 1. Start RRT at q_I
 2. At every, say, 100th iteration, force $q_{rand} = q_G$
 3. If q_G is reached, problem is solved
- Why not picking q_G every time?
- This will fail and waste much effort in running into C_{Obs} instead of exploring the space

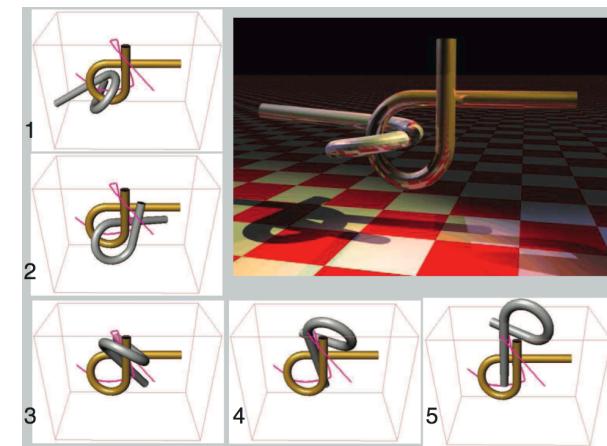
RRTs

- However, some problems require more effective methods: **bidirectional search**
- Grow **two** RRTs, one from q_I , one from q_G
- In every other step, try to extend each tree towards the newest vertex of the other tree



RRTs

- RRTs are popular, many extensions exist:
real-time RRTs, anytime
RRTs, for dynamic
environments etc.
- **Pros:**
 - Balance between greedy
search and exploration
 - Easy to implement
- **Cons:**
 - Metric sensitivity
 - Unknown rate of convergence



Alpha 1.0 puzzle.
Solved with
bidirectional RRT

From Road Maps to Paths

- All methods discussed so far **construct a road map** (without considering the query pair q_I and q_G)
- Once the investment is made, the **same road map** can be reused for **all** queries (provided world and robot do not change)
 1. **Find** the cell/vertex that contain/is close to q_I and q_G (not needed for visibility graphs)
 2. **Connect** q_I and q_G to the road map
 3. **Search** the road map for a path from q_I to q_G

Sampling-Based Planning

Wrap Up

- Sampling-based planners are **more efficient** in most **practical problems** but offer weaker guarantees
- They are **probabilistically complete**: the probability tends to 1 that a solution is found if one exists (otherwise it may still run forever)
- Performance degrades in problems with **narrow passages**. Subject of active research
- Widely used. Problems with high-dimensional and complex C-spaces are still computationally hard

Potential Field Methods

- All techniques discussed so far aim at capturing the connectivity of C_{free} into a graph
- **Potential Field methods** follow a different idea:

The robot, represented as a point in C , is modeled as a **particle** under the influence of a **artificial potential field** U

U superimposes

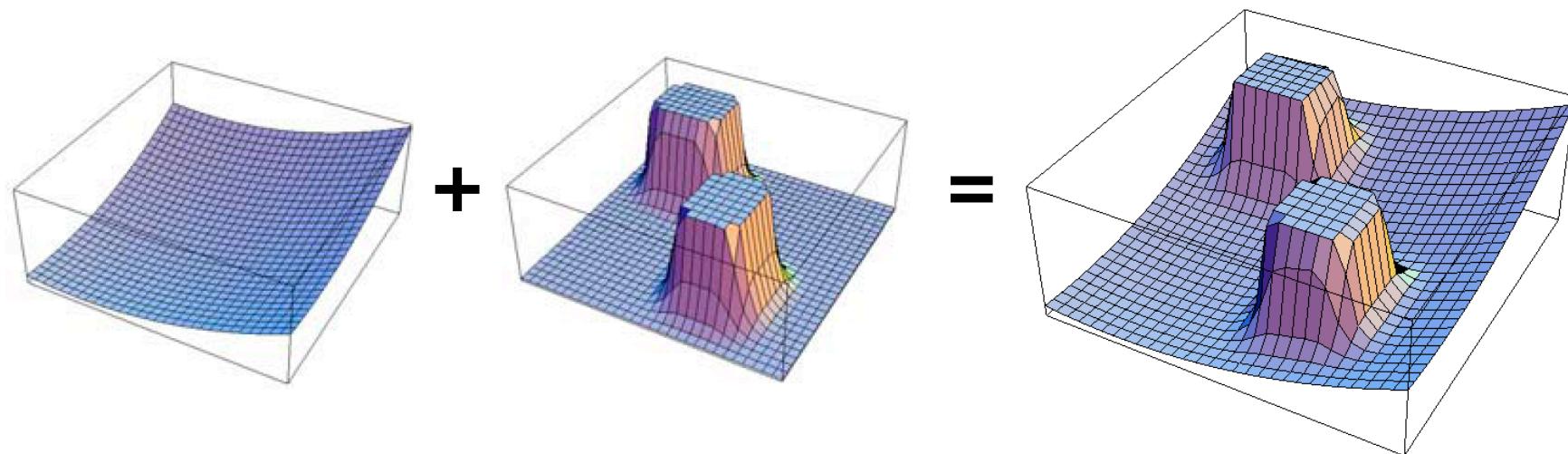
- **Repulsive forces** from obstacles
- **Attractive force** from goal

Potential Field Methods

- Potential function

$$\mathbf{U}(q) = \mathbf{U}_{att}(q) + \mathbf{U}_{rep}(q)$$

$$\vec{F}(q) = -\vec{\nabla}\mathbf{U}(q)$$



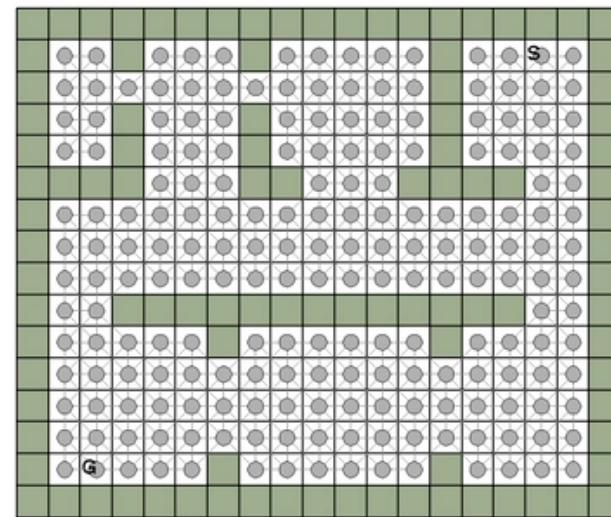
- Simply perform **gradient descent**
- Space typically discretized in a grid

Potential Field Methods

- Main problems: robot gets stuck in **local minima**
- Way out: Construct local-minima-free **navigation function** ("NF1"), then do gradient descent (e.g. bushfire from goal)
- The gradient of the potential function defines a **vector field** (similar to a policy) that can be used as **feedback control strategy**, relevant for an uncertain robot
- However, potential fields need to represent C_{free} **explicitely**. This can be too costly.

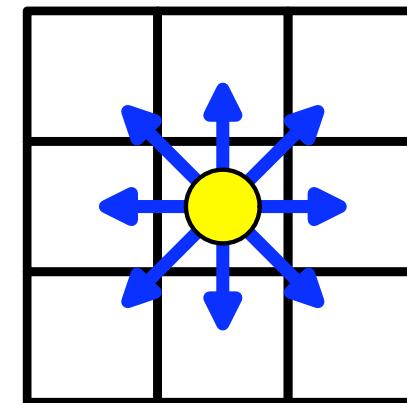
Robot Motion Planning

- Given a road map, let's do **search!**



A* Search

- A* is one of the most widely-known informed search algorithms with many applications in robotics
- *Where are we?*
A* is an instance of an **informed algorithm** for the general problem of **search**
- In robotics: planning on a 2D occupancy grid map is a common approach



Search

The problem of **search**: finding a sequence of actions (a *path*) that leads to desirable states (a *goal*)

- **Uninformed search:** besides the problem definition, no further information about the domain ("blind search")
- The only thing one can do is to expand nodes differently
- Example algorithms: breadth-first, uniform-cost, depth-first, bidirectional, etc.

Search

The problem of **search**: finding a sequence of actions (a *path*) that leads to desirable states (a *goal*)

- **Informed search:** further information about the domain through **heuristics**
- Capability to say that a node is "more promising" than another node
- Example algorithms: greedy best-first search, **A***, many variants of A*, D*, etc.

Search

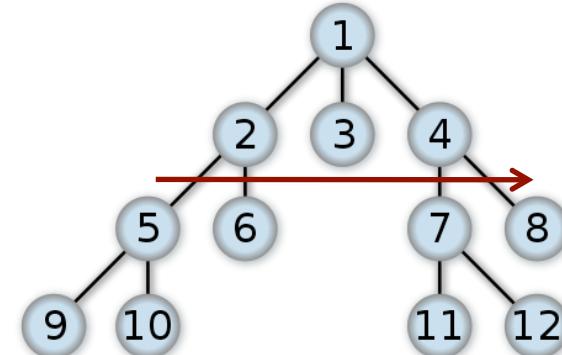
The performance of a search algorithm is measured in four ways:

- **Completeness:** does the algorithm find the solution when there is one?
- **Optimality:** is the solution the best one of all possible solutions in terms of path cost?
- **Time complexity:** how long does it take to find a solution?
- **Space complexity:** how much memory is needed to perform the search?

Uninformed Search

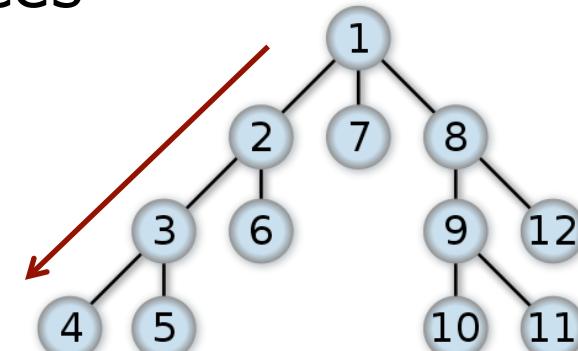
- **Breadth-first**

- Complete
- Optimal if action costs equal
- Time and space: $O(b^d)$



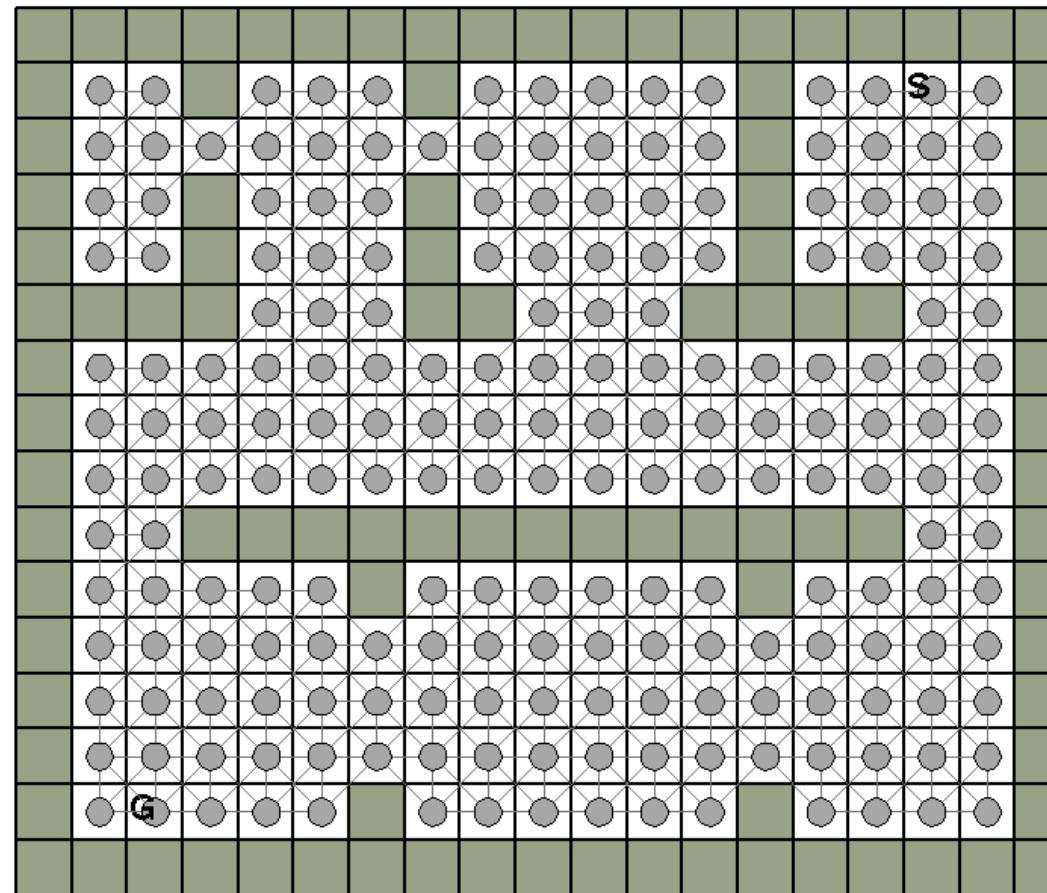
- **Depth-first**

- Not complete in infinite spaces
- Not optimal
- Time: $O(b^m)$
- Space: $O(bm)$ (can forget explored subtrees)



(b : branching factor, d : goal depth, m : max. tree depth)

Breadth-First Example



Informed Search

- Nodes are selected for expansion based on an **evaluation function** $f(n)$ from the set of generated but not yet explored nodes
- Then select node first with lowest $f(n)$ value
- Key component to every choice of $f(n)$:
Heuristic function $h(n)$
- Heuristics are most common way to inject domain knowledge and inform search
- Every $h(n)$ is a cost estimate of cheapest path from n to a goal node

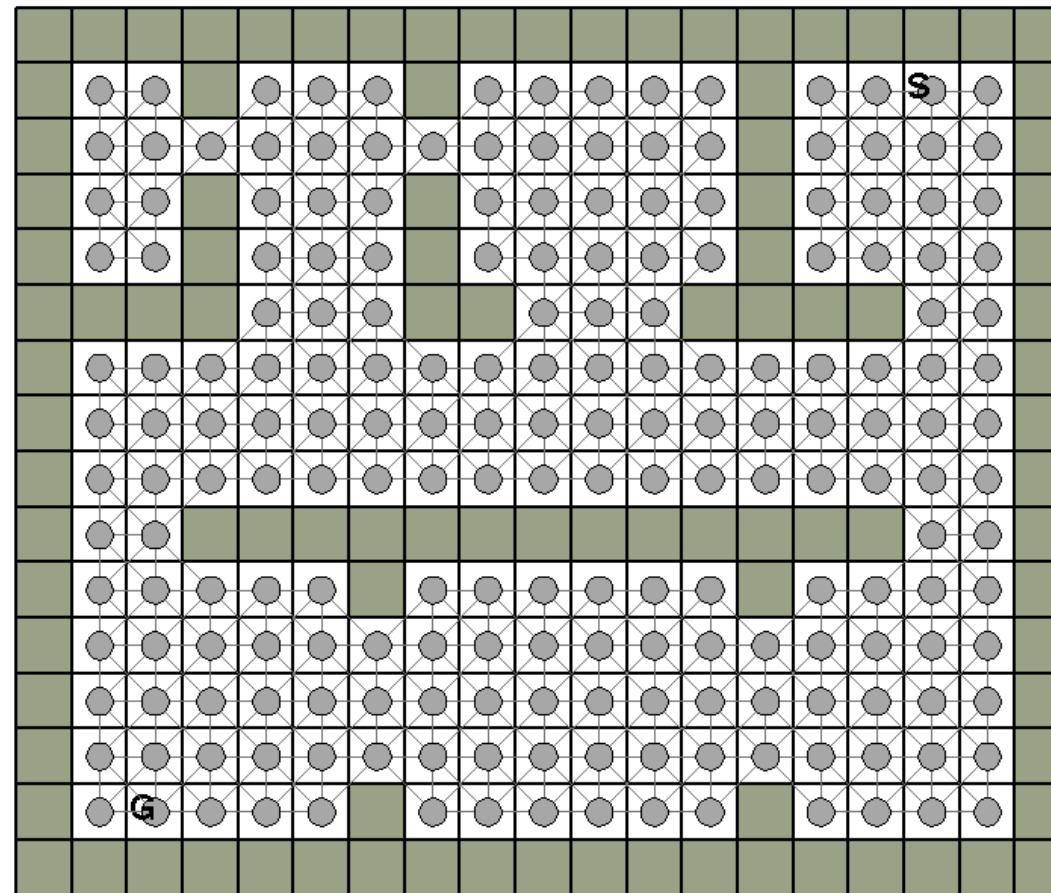
Informed Search

- **Greedy Best-First-Search**
 - Simply expands the node closest to the goal
$$f(n) = h(n)$$
 - Not optimal, not complete, complexity $O(b^m)$
- **A* Search**
 - Combines path cost to n , $g(n)$, and estimated goal distance from n , $h(n)$
$$f(n) = g(n) + h(n)$$
 - $f(n)$ estimates the cheapest path cost through n
 - If $h(n)$ is *admissible*: **complete** and **optimal!**

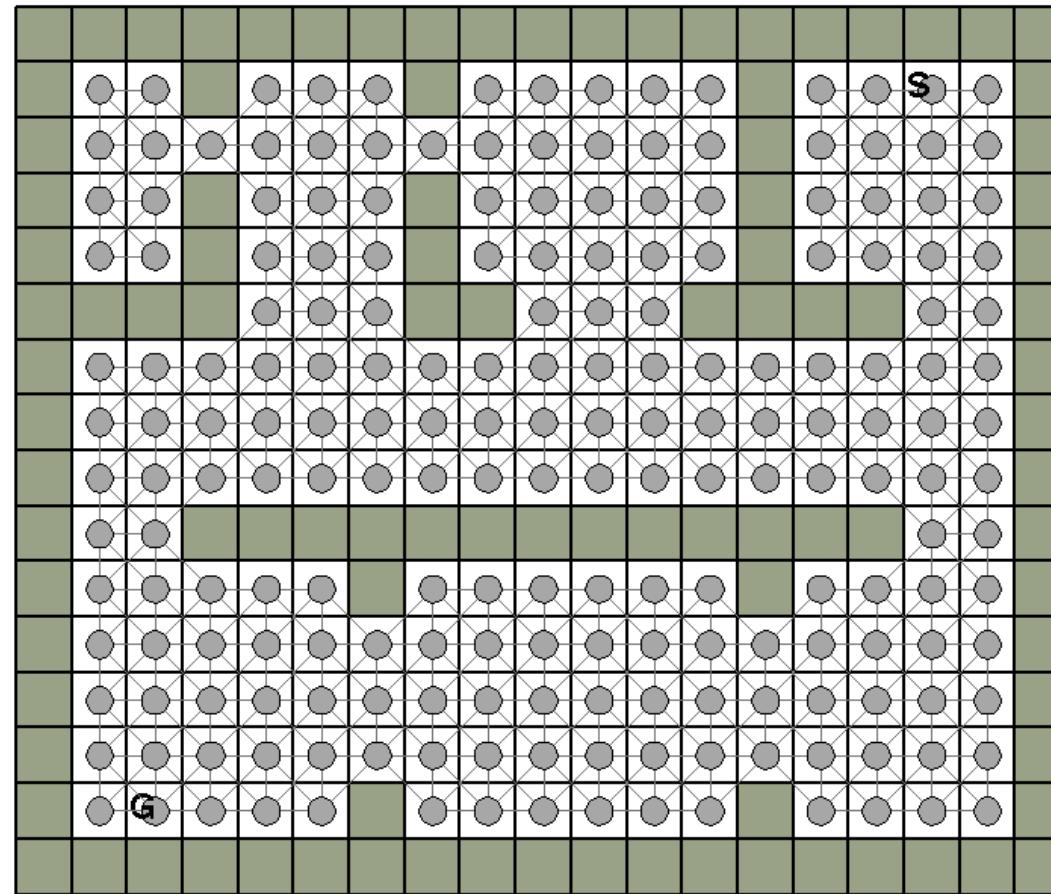
Heuristics

- **Admissible heuristic:**
 - Let $h^*(n)$ be the true cost of the optimal path from n to the goal. Then $h(\cdot)$ is admissible if the following holds for all n :
$$h(n) \leq h^*(n)$$
 ← be optimistic, never overestimate the cost
- Heuristics are problem-specific. Good ones (admissible, efficient) for **our task** are:
 - **Straight-line distance** $h_{SLD}(n)$
(as with any routing problem)
 - **Octile distance**: Manhattan distance extended to allow diagonal moves
 - Deterministic **Value Iteration**/Dijkstra $h_{VI}(n)$

Greedy Best-First Example



A* with h_{SLD} Example



Heuristics for A*

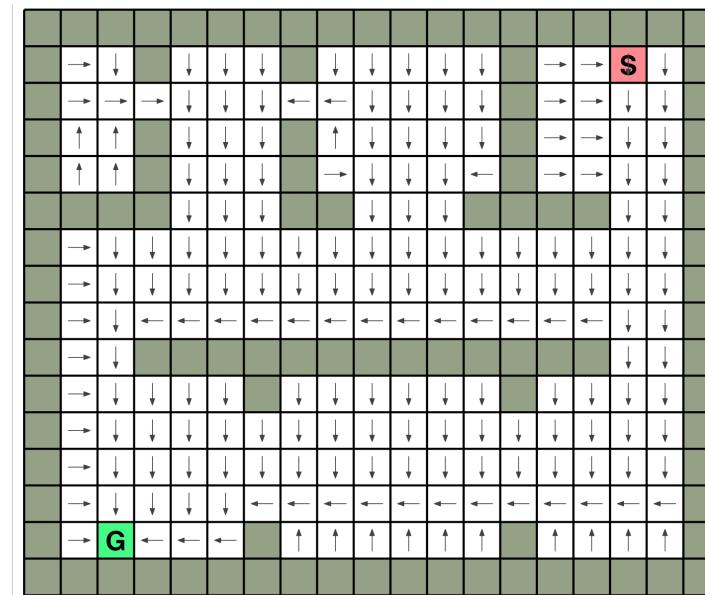
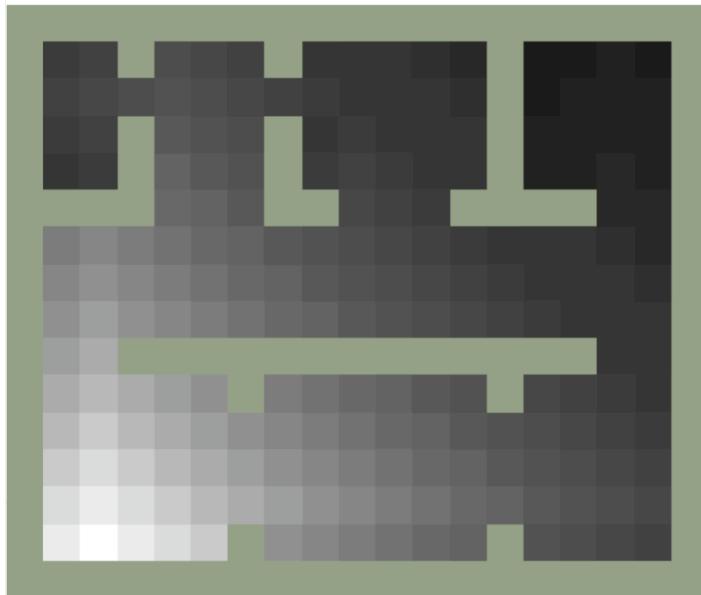
- Deterministic Value Iteration
 - Use Value Iteration for MDPs (later in this course) with rewards -1 and unit discounts
 - Like Dijkstra



- Precompute for dynamic or unknown environments where replanning is likely

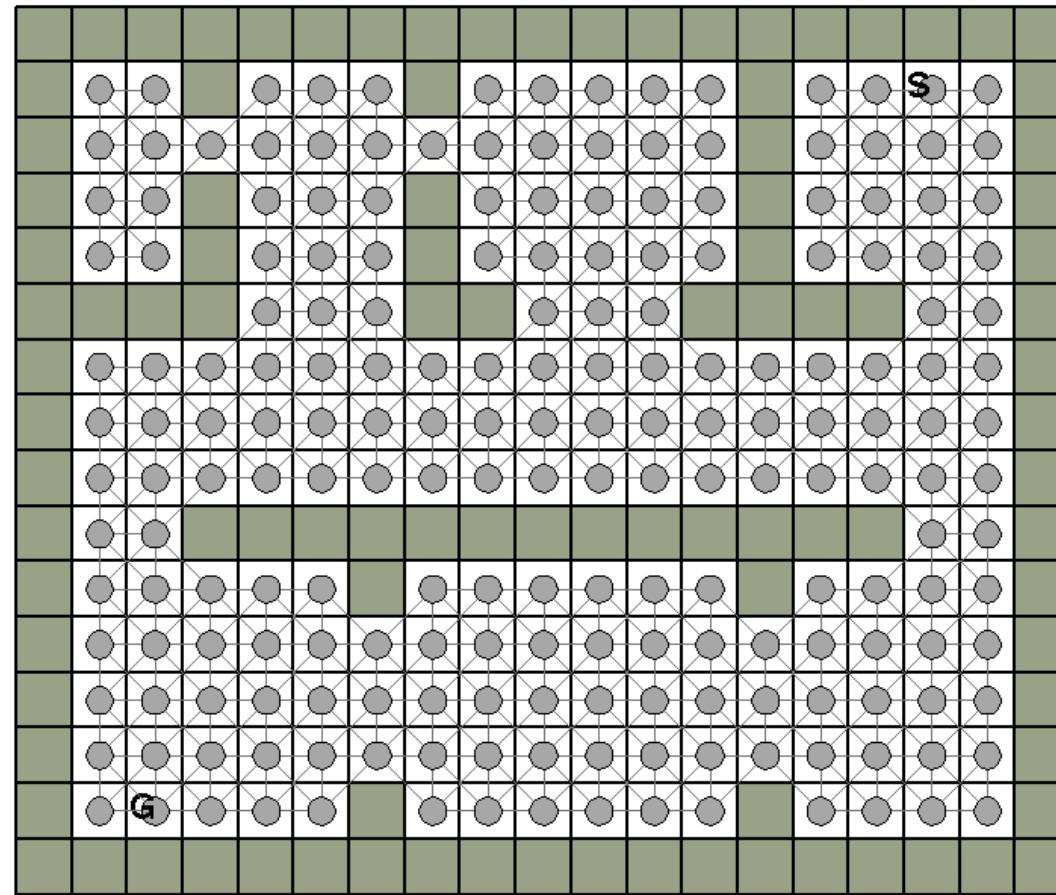
Heuristics for A*

- Deterministic Value Iteration



- Recall vector field from potential functions:
allows to implement a **feedback control strategy** for an uncertain robot

A* with h_{VI} Example



Problems with A* on Grids

1. The shortest path is often very **close to obstacles** (cutting corners)
 - Uncertain path execution increases the risk of collisions
 - Uncertainty can come from delocalized robot, imperfect map or poorly modeled dynamic constraints
2. Trajectories **aligned to the grid** structure
 - Path looks unnatural
 - Such paths are longer than the true shortest path in the continuous space

Problems with A* on Grids

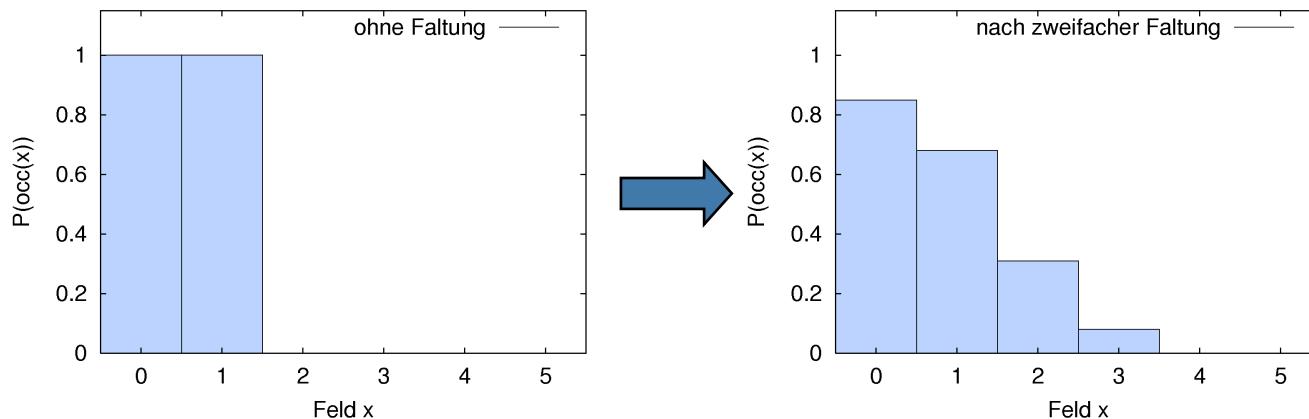
3. When the path turns out to be blocked during traversal, it needs to be **replanned from scratch**
 - In unknown or dynamic environments, this can occur very often
 - Replanning in large state spaces is costly
 - Can we reuse the initial plan?

Let us look at **solutions** to these problems...

Map Smoothing

- Given an occupancy grid map
- **Convolution** blurs the map M with kernel k (e.g. a Gaussian kernel)

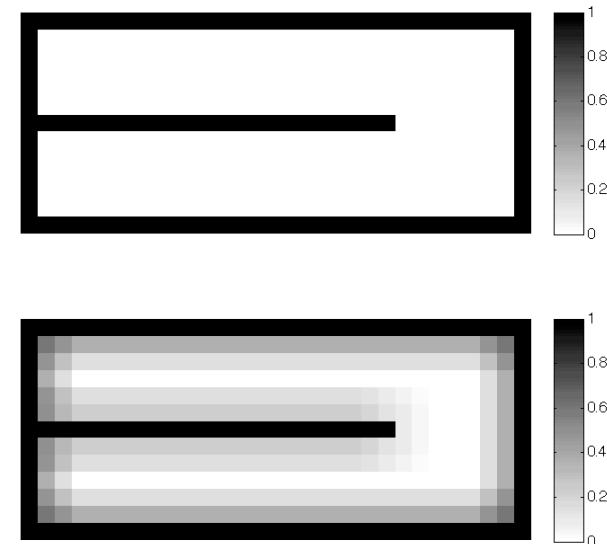
$$(M * k)[i] = \sum_{j=-\infty}^{\infty} M[j] k[i - j]$$



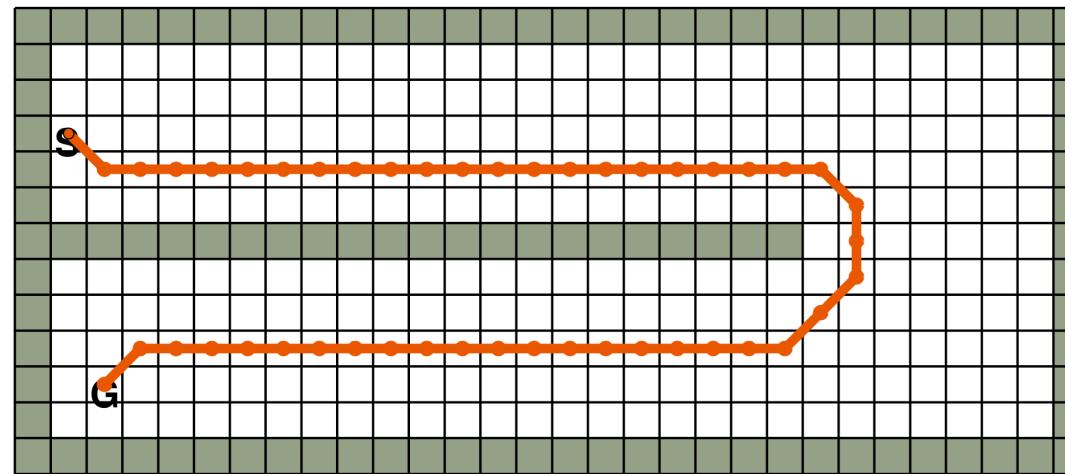
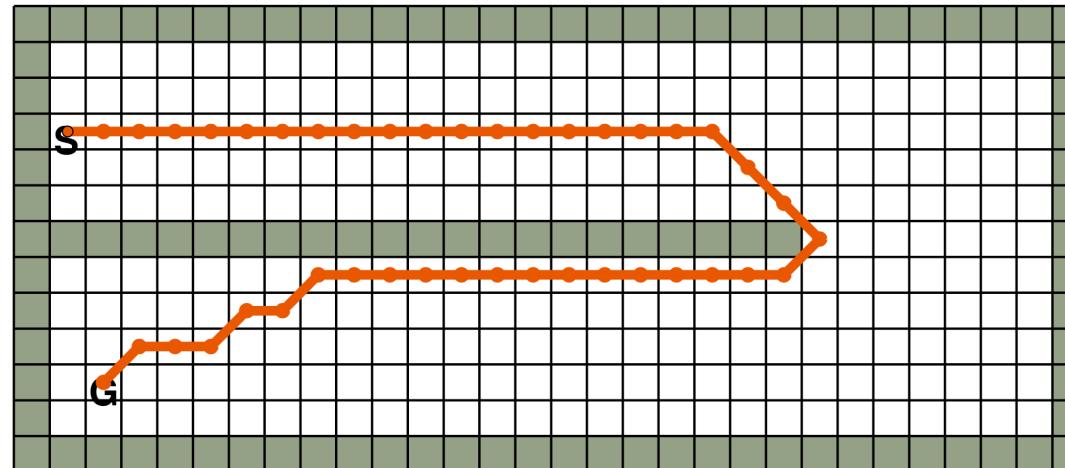
1D example: cells before and after two convolution runs

Map Smoothing

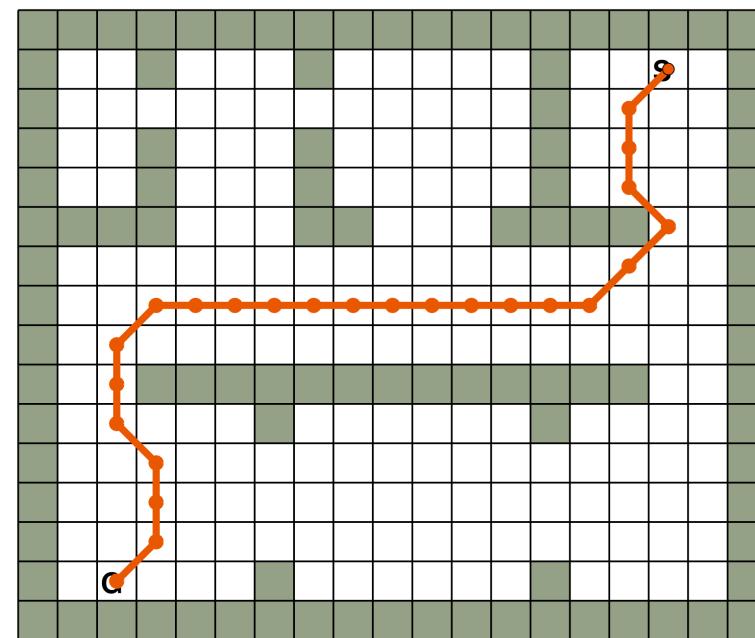
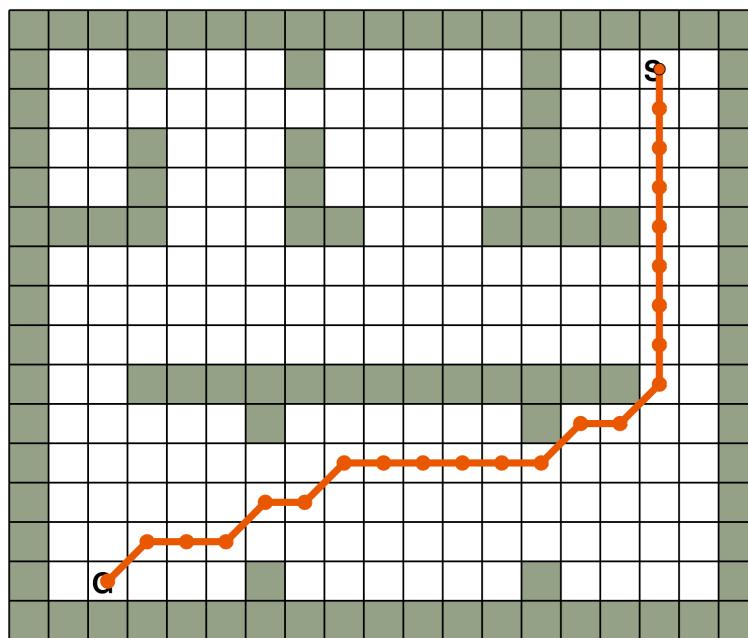
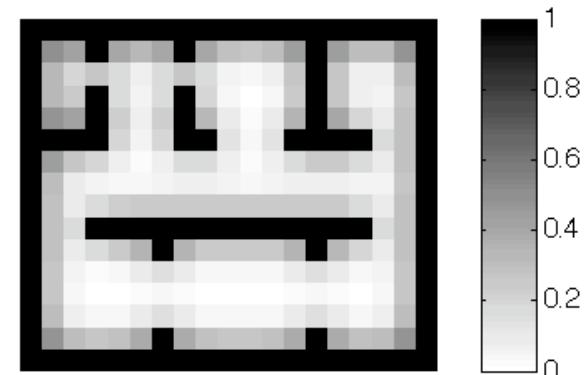
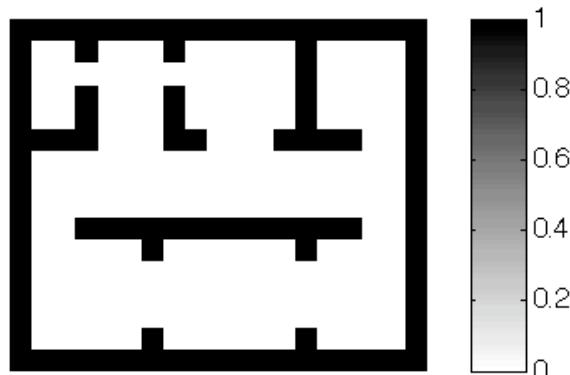
- Leads to above-zero probability areas around obstacles. Obstacles **appear bigger** than in reality
- Perform A* search in **convolved map** with evalution function
$$f(n) = g(n) \cdot p_{occ}(n) + h(n)$$
 $p_{occ}(n)$: occupancy probability of node/cell n
- Could also be a term for cell traversal cost



Map Smoothing

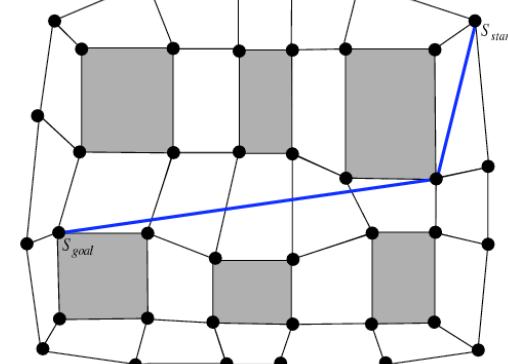
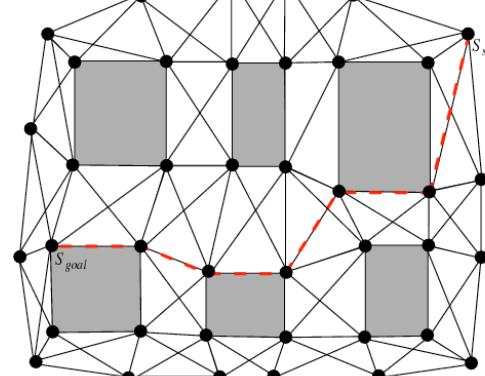
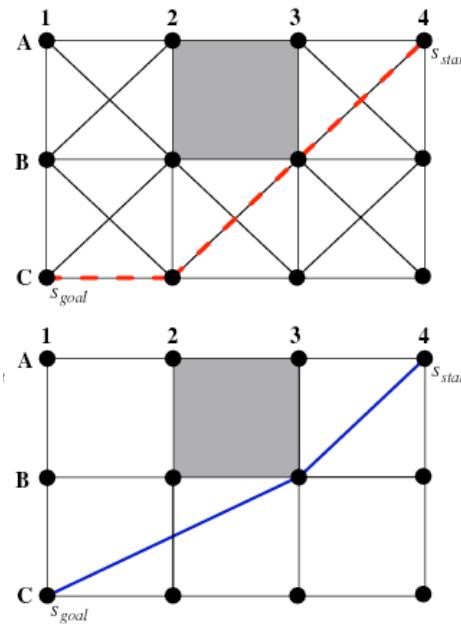


Map Smoothing



Any-Angle A*

- **Problem:** A* search only considers paths that are **constrained to graph edges**
- This can lead to **unnatural**, grid-aligned, and **suboptimal** paths



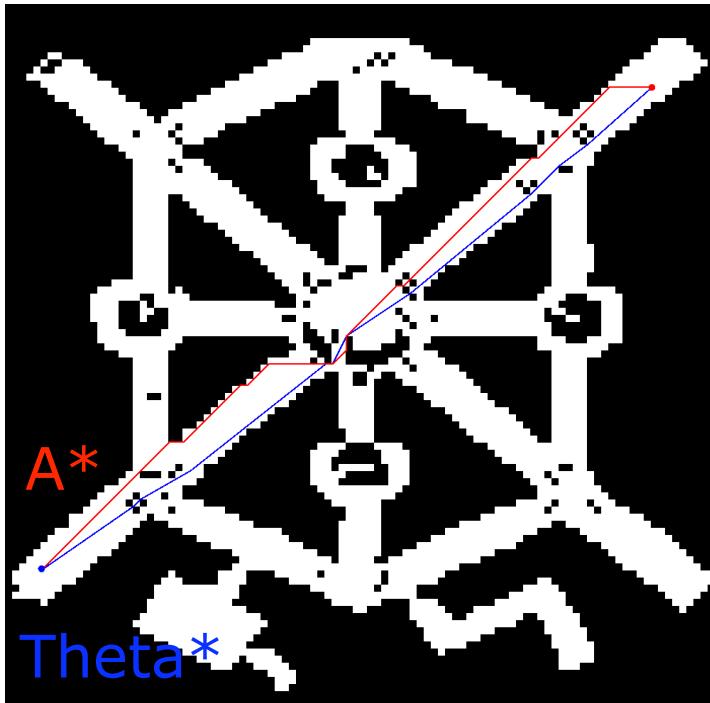
Pictures from [Nash et al. AAAI'07]

Any-Angle A*

- Different approaches:
 - **A* on Visibility Graphs**
Optimal solutions in terms of path length!
 - **A* with post-smoothing**
Traverse solution and find pairs of nodes with direct line of sight, replace by line segment
 - **Field D*** *[Ferguson and Stentz, JFR'06]*
Interpolates costs of points not in cell centers.
Builds upon D* family, able to efficiently replan
 - **Theta*** *[Nash et al. AAAI'07, AAAI'10]*
Extension of A*, nodes can have non-neighboring successors based on a line-of-sight test

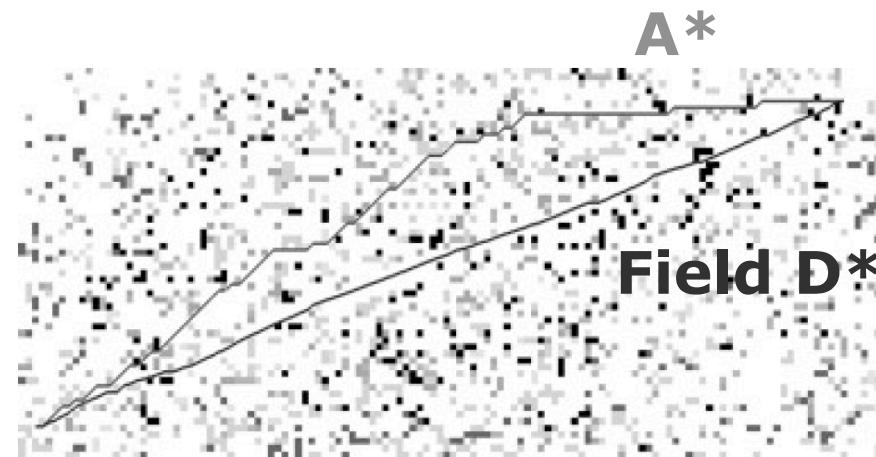
Any-Angle A* Examples

- Theta*



Game environment

- Field D*

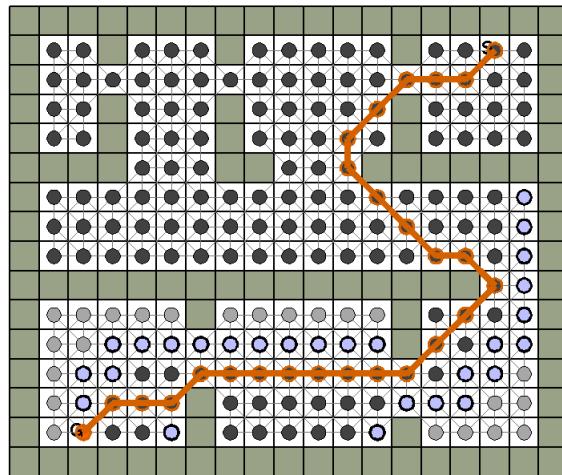


Outdoor environment.
Darker cells have larger
traversal costs

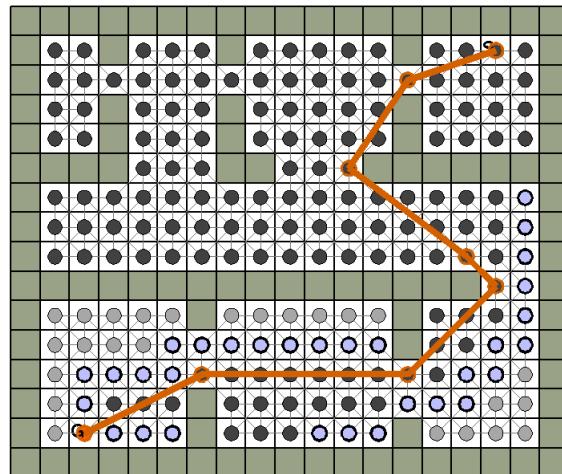
Any-Angle A* Examples

- ## ■ A* vs. Theta*

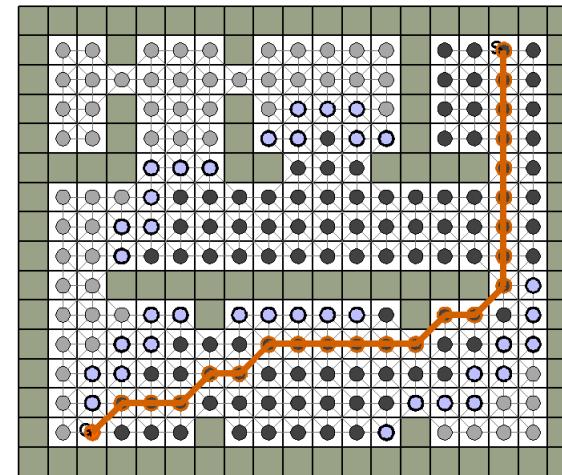
(*len*: path length, *nhead* = # heading changes)



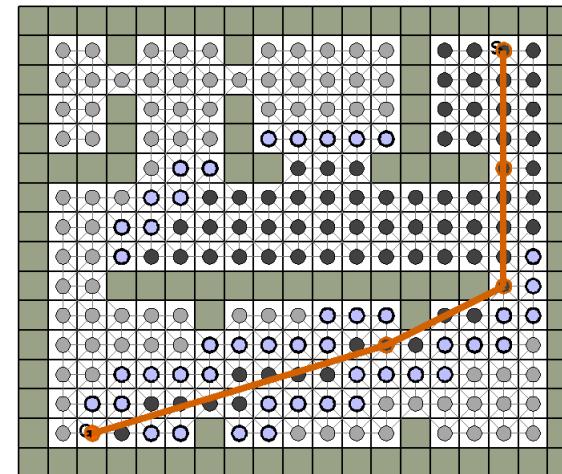
len: 30.0
nhead: 11



len: 28.9
nhead: 5

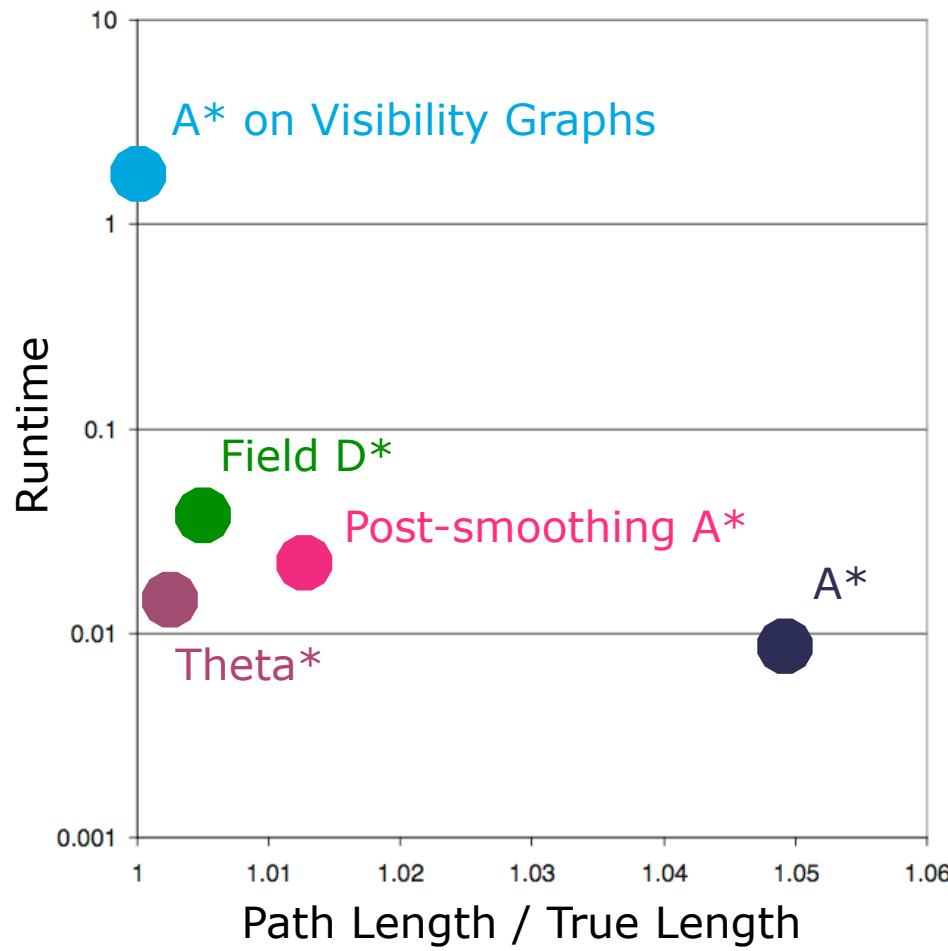


len: 24.1
nhead: 9



len: 22.9
nhead: 2

Any-Angle A* Comparison



- **A* PS** and **Theta*** provide the best trade off for the problem
- **A* on Visibility Graphs** scales poorly (but is optimal)
- **A* PS** does not always work in nonuniform cost environments. Shortcuts can end up in expensive areas

[Daniel et al. JAIR'10]

D* Search

- **Problem:** In unknown, partially known or dynamic environments, the planned path may be blocked and we need to **replan**
- Can this be done efficiently, avoiding to replan the **entire path?**
- **Idea:** Incrementally repair path keeping its modifications local around robot pose
- Several approaches implement this idea:
 - **D*** (Dynamic A*) [*Stentz, ICRA'94, IJCAI'95*]
 - **D* Lite** [*Koenig and Likhachev, AAAI'02*]
 - **Field D*** [*Ferguson and Stentz, JFR'06*]

D*/D* Lite

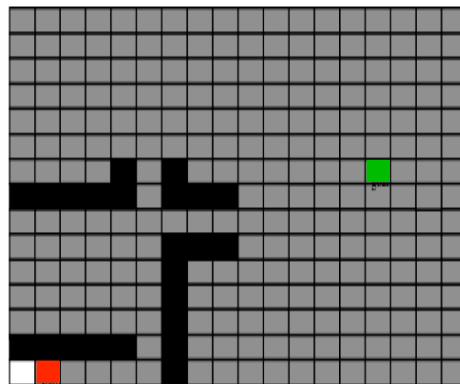
- **Main concepts**

- **Switched search direction:** search from goal to the current vertex. If a change in edge cost is detected during traversal (around the current robot pose), only few nodes near the goal (=start) need to be updated
- These nodes are nodes those **goal distances** have changed or not been calculated before AND are **relevant** to recalculate the new shortest path to the goal
- **Incremental heuristic search** algorithms: able to focus and build upon previous solutions

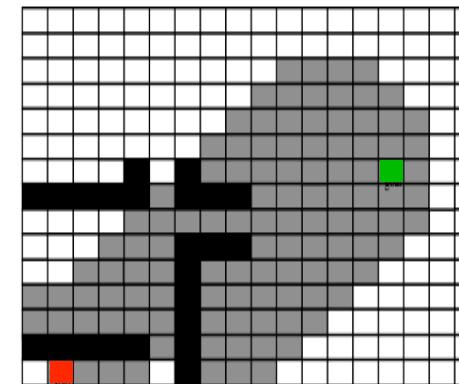
D* Lite Example

- Situation at start

Breadth-
First-
Search

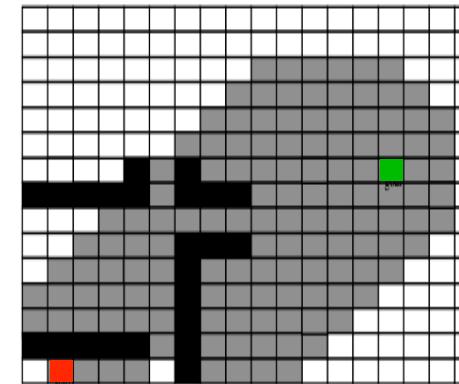


A*



Start
 Goal
 Expanded nodes
(goal distance
calculated)

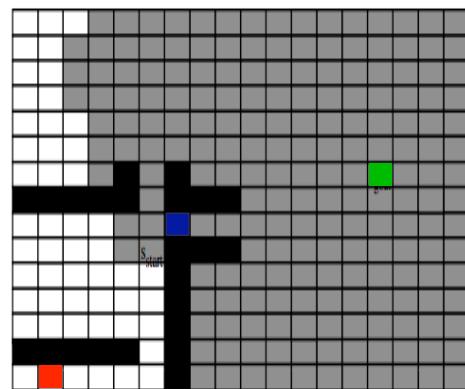
D* Lite



D* Lite Example

- After discovery of blocked cell

Breadth-
First-
Search

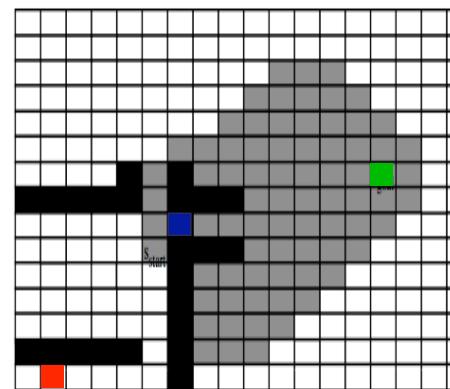


■ Blocked cell

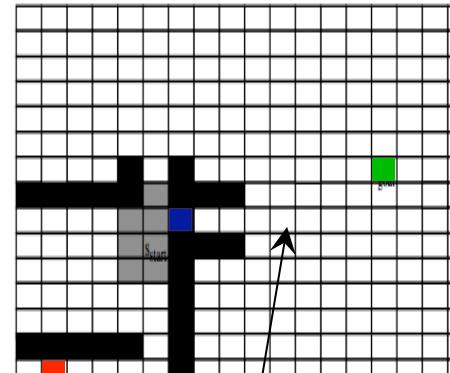
■ Updated nodes

All other nodes remain
unaltered, the shortest path
can reuse them.

A*

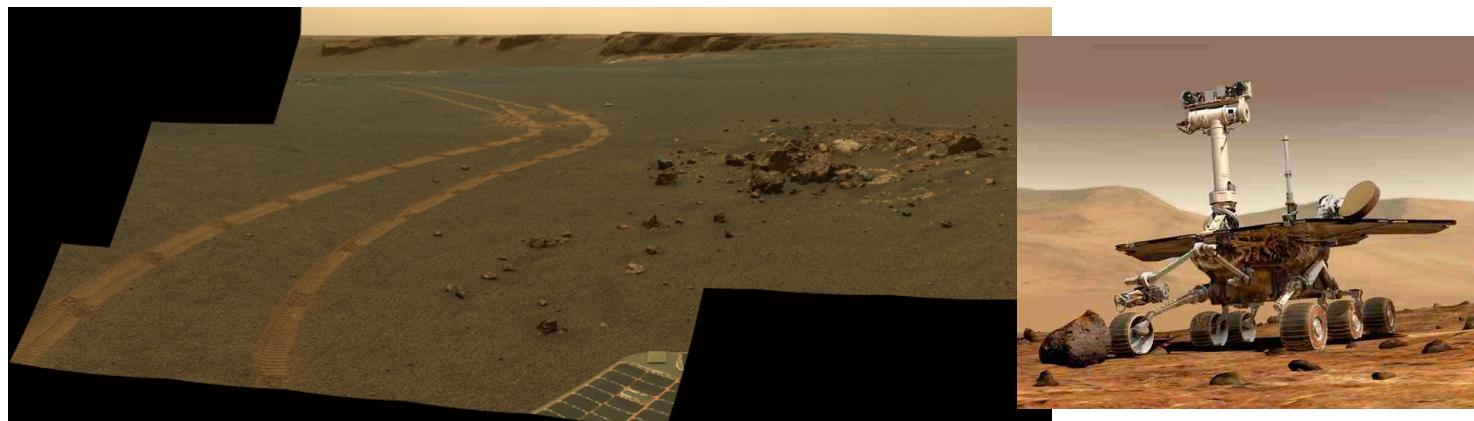


D* Lite



D* Family

- **D* Lite** produces the same paths than D* but is **simpler** and more **efficient**
- D*/D* Lite are **widely used**
- **Field D*** was running on Mars rovers Spirit and Opportunity (retrofitted in yr 3)



Tracks left by a drive executed with Field D*

Still in Dynamic Environments...

- Do we really need to replan the entire path for **each obstacle** on the way?
- What if the robot has to react **quickly** to unforeseen, fast moving obstacles?
 - Even D* Lite can be too slow in such a situation
- Accounting for the **robot shape** (it's not a point)
- Accounting for **kinematic** and **dynamic** vehicle **constraints**, e.g.
 - Decceleration limits,
 - Steering angle limits, etc.

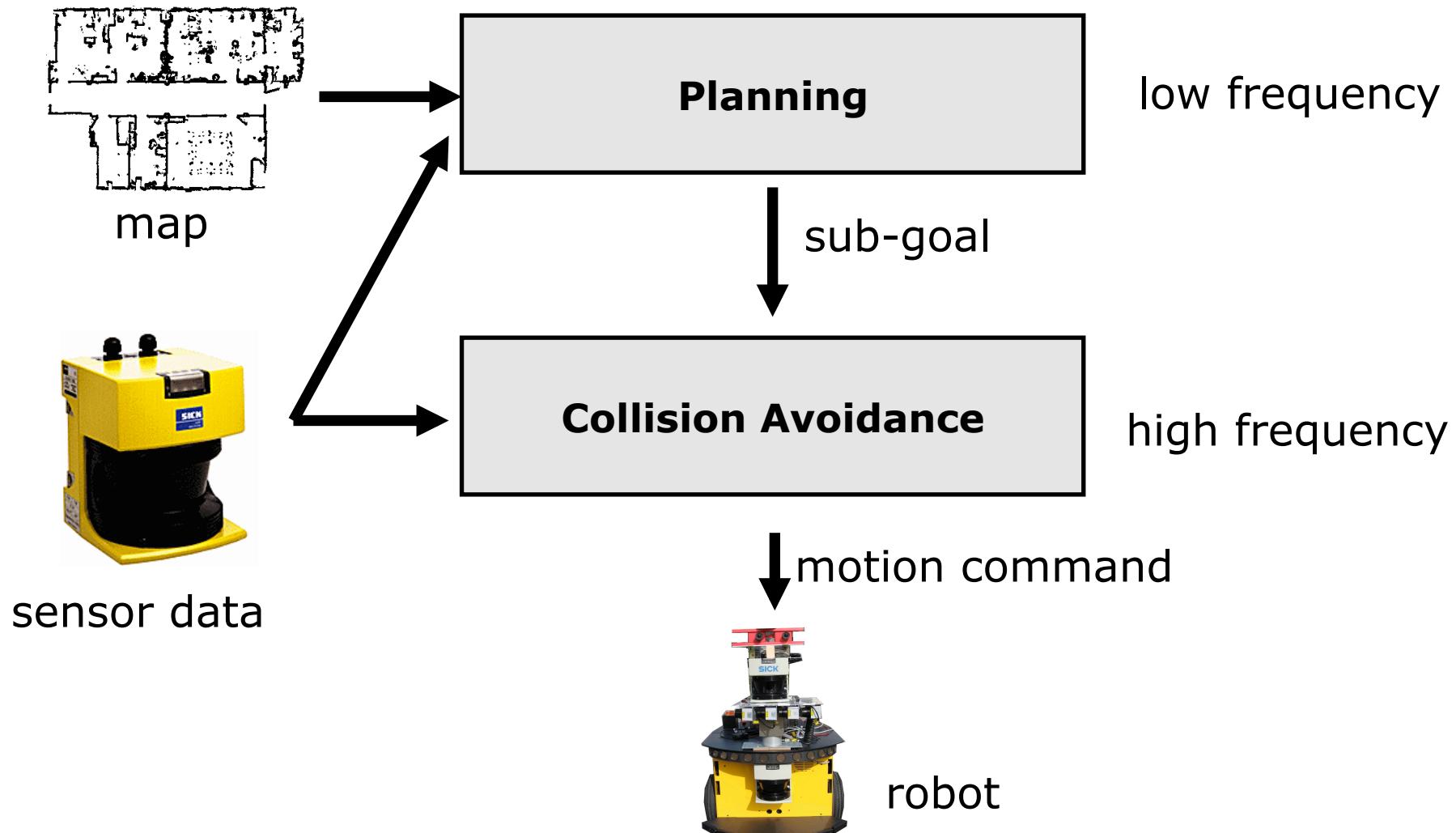
Collision Avoidance

- This can be handled by techniques called **collision avoidance** (obstacle avoidance)
- A well researched subject, different **approaches** exist:
 - Dynamic Window Approaches
[Simmons, 96], [Fox et al., 97], [Brock & Khatib, 99]
 - Nearness Diagram Navigation
[Minguez et al., 2001, 2002]
 - Vector-Field-Histogram+
[Ulrich & Borenstein, 98]
 - Extended Potential Fields
[Khatib & Chatila, 95]

Collision Avoidance

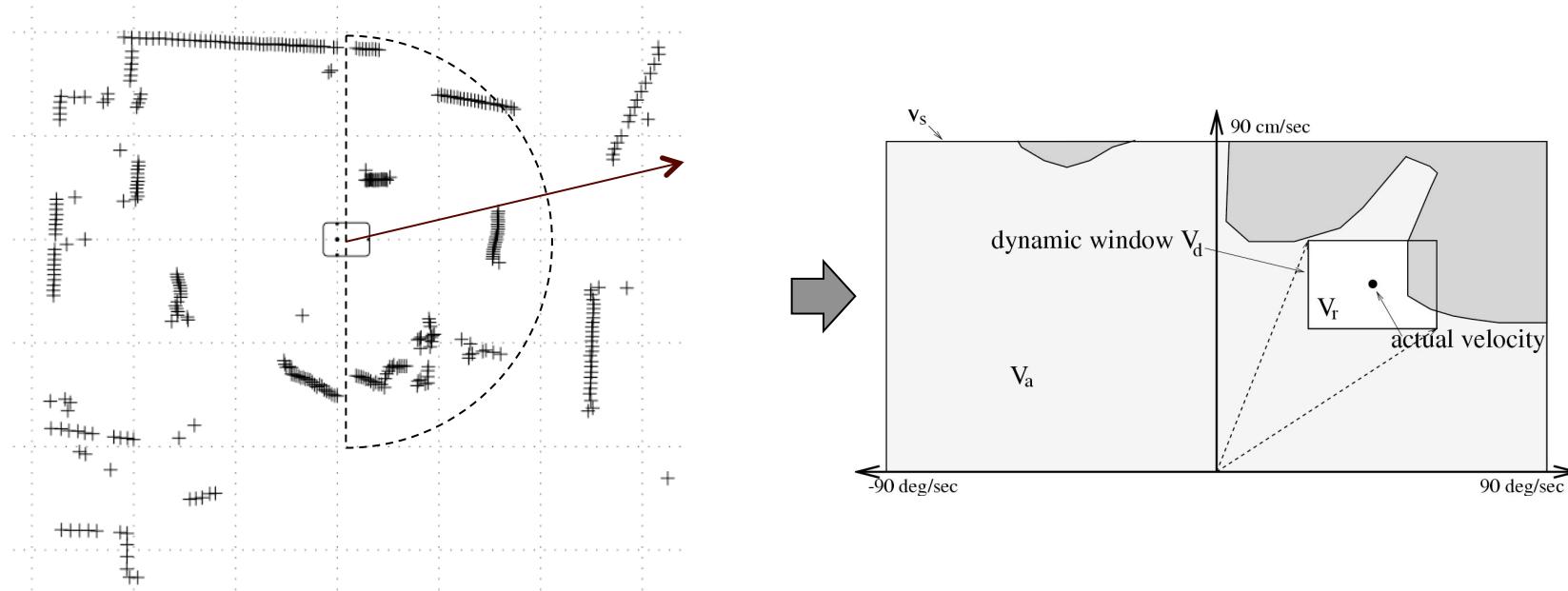
- Integration into general motion planning?
- It is common to subdivide the problem into a global and local planning task:
 - An approximate **global planner** computes paths ignoring the kinematic and dynamic vehicle constraints
 - An accurate **local planner** accounts for the constraints and generates (sets of) feasible local trajectories ("collision avoidance")
- What do we loose? What do we win?

Two-layered Architecture



Dynamic Window Approach

- **Given:** path to goal (a set of via points), range scan of the local vicinity, dynamic constraints
- **Wanted:** collision-free, safe, and fast motion towards the goal

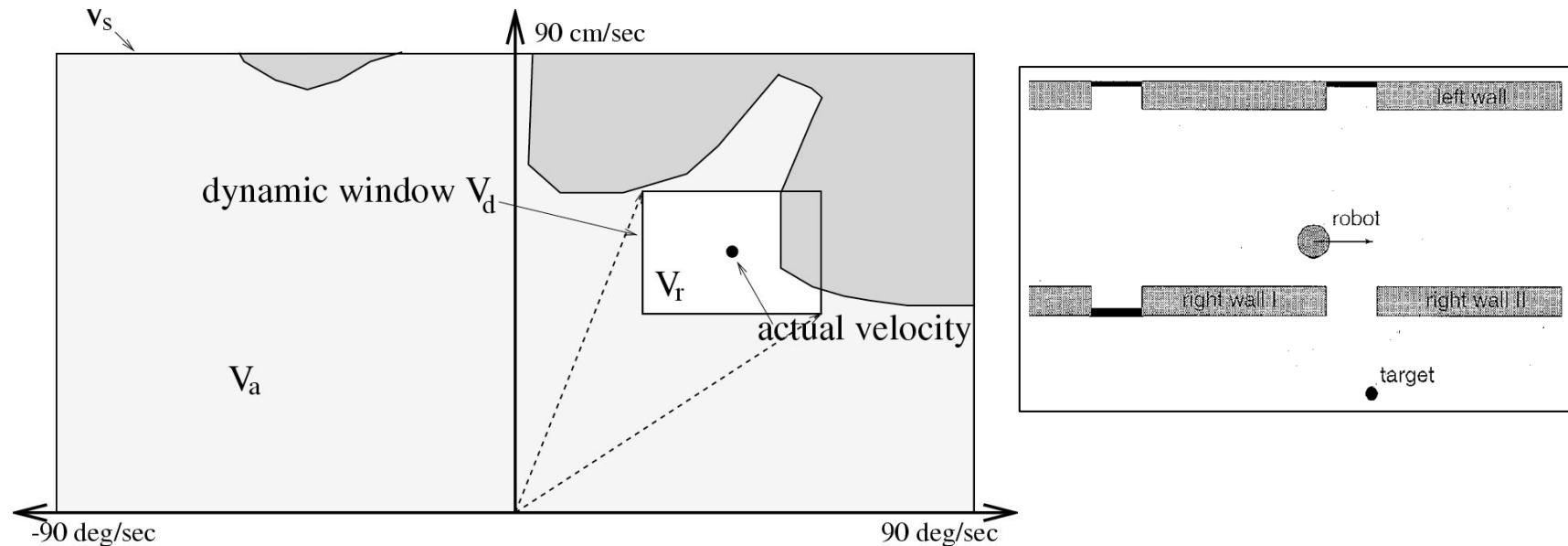


Dynamic Window Approach

- **Assumption:** robot takes motion commands of the form (v, ω)
- This is saying that the robot moves (instantaneously) on **circular arcs** with radius $r = v / \omega$
- **Question:** which (v, ω) 's are
 - **reasonable:** that bring us to the goal?
 - **admissible:** that are collision-free?
 - **reachable:** under the vehicle constraints?

DWA Search Space

- 2D velocity search space



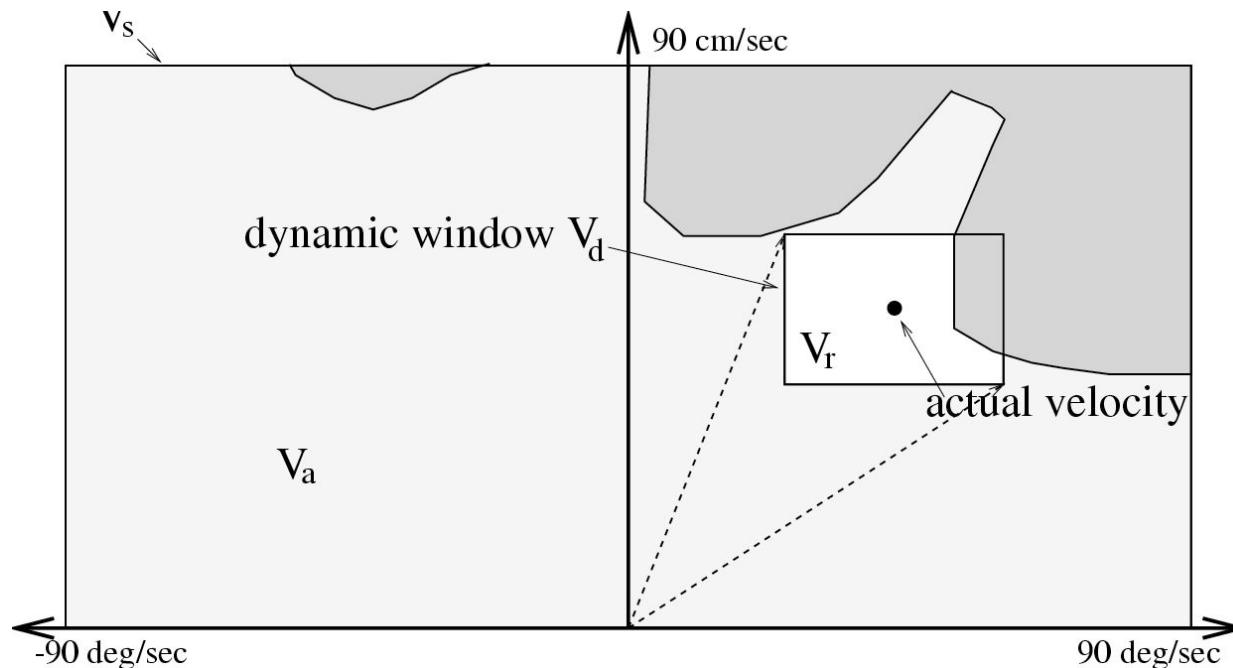
- V_s = all possible speeds of the robot
- V_a = obstacle free area
- V_d = speeds reachable within one time frame given acceleration constraints

$$Space = V_s \cap V_a \cap V_d$$

Reachable Velocities

- Speeds are reachable if

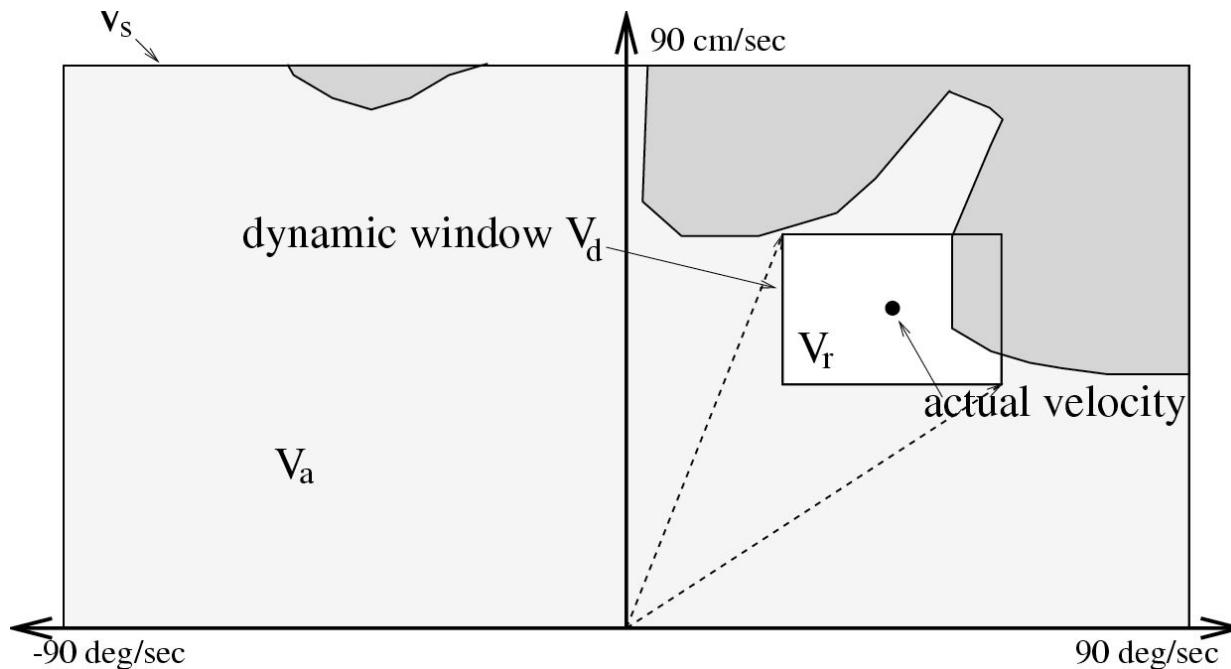
$$V_d = \{(v, \omega) \mid v \in [v - a_{trans}t, v + a_{trans}t] \wedge \omega \in [\omega - a_{rot}t, \omega + a_{rot}t]\}$$



Admissible Velocities

- Speeds are admissible if

$$V_a = \{(v, \omega) \mid v \leq \sqrt{2\text{dist}(v, \omega)a_{trans}} \wedge \\ \omega \leq \sqrt{2\text{dist}(v, \omega)a_{rot}}\}$$



Dynamic Window Approach

- How to choose (v, ω) ?
- Pose the problem as an **optimization problem** of an objective function within the dynamic window, search the maximum
- The objective function is a **heuristic navigation function**
- This function encodes the incentive to minimize the travel time by “driving **fast** and **safe** in the **right direction**”

Dynamic Window Approach

- Heuristic navigation function
- Planning restricted to (v, ω) -space
- Here: assume to have precomputed goal distances from NF1 algorithm

Navigation Function: [Brock & Khatib, 99]

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Dynamic Window Approach

- Heuristic navigation function
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Maximizes
velocity

Dynamic Window Approach

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Maximizes
velocity

Rewards alignment
to NF1/A* gradient

Dynamic Window Approach

- Heuristic navigation function
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$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Maximizes velocity

Rewards alignment to NF1/A* gradient

Rewards large advances on NF1/A* path

Dynamic Window Approach

- Heuristic navigation function
- Planning restricted to (v, ω) -space
- Here: assume to have precomputed goal distances from NF1 algorithm

Navigation Function: [Brock &

Comes in when goal region reached

$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

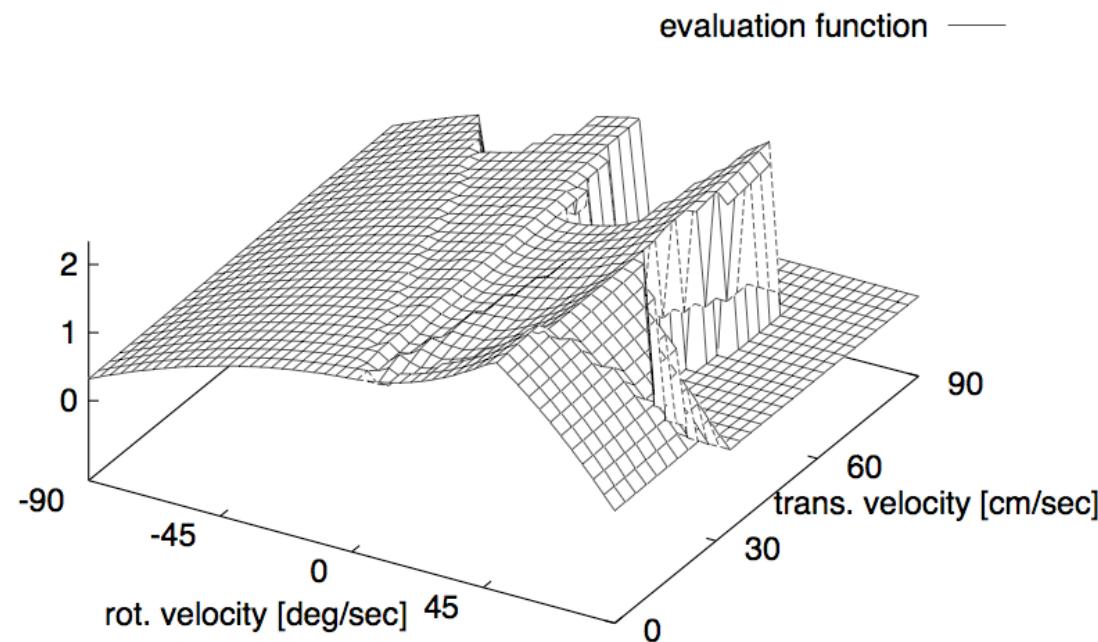
Maximizes velocity

Rewards alignment to NF1/A* gradient

Rewards large advances on NF1/A* path

Dynamic Window Approach

- Navigation function example

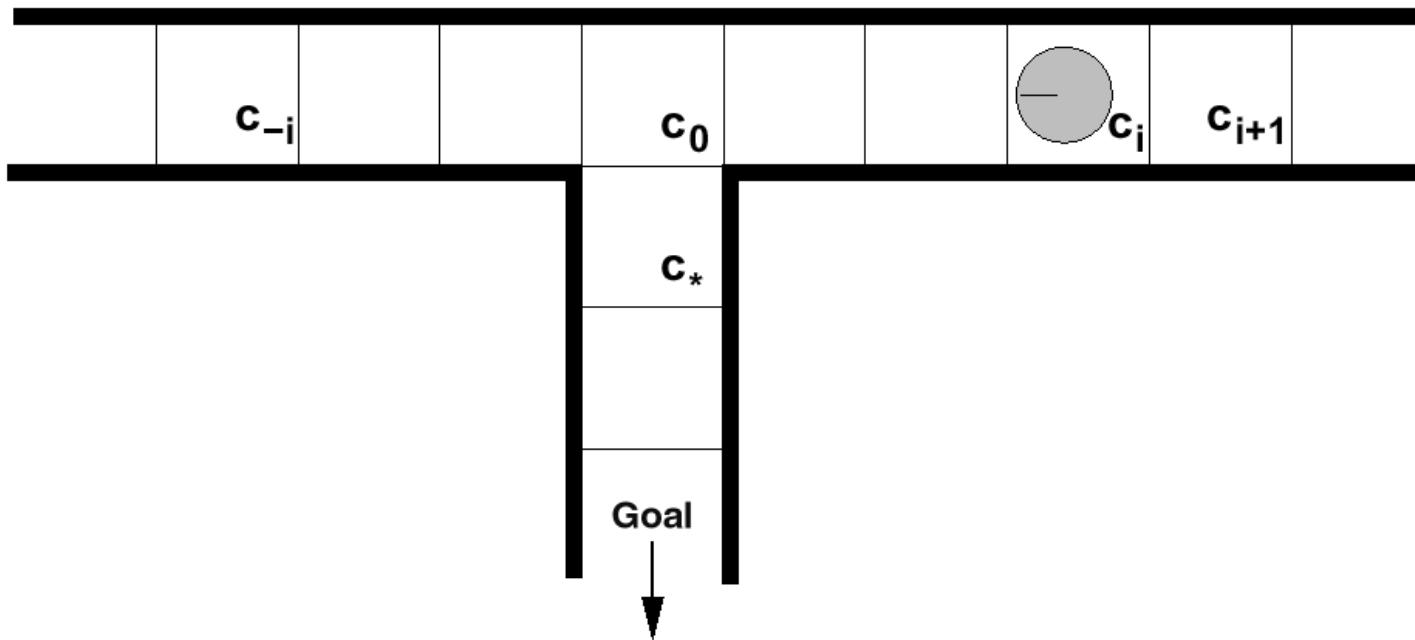


- Now perform search/optimization
- Find maximum

Dynamic Window Approach

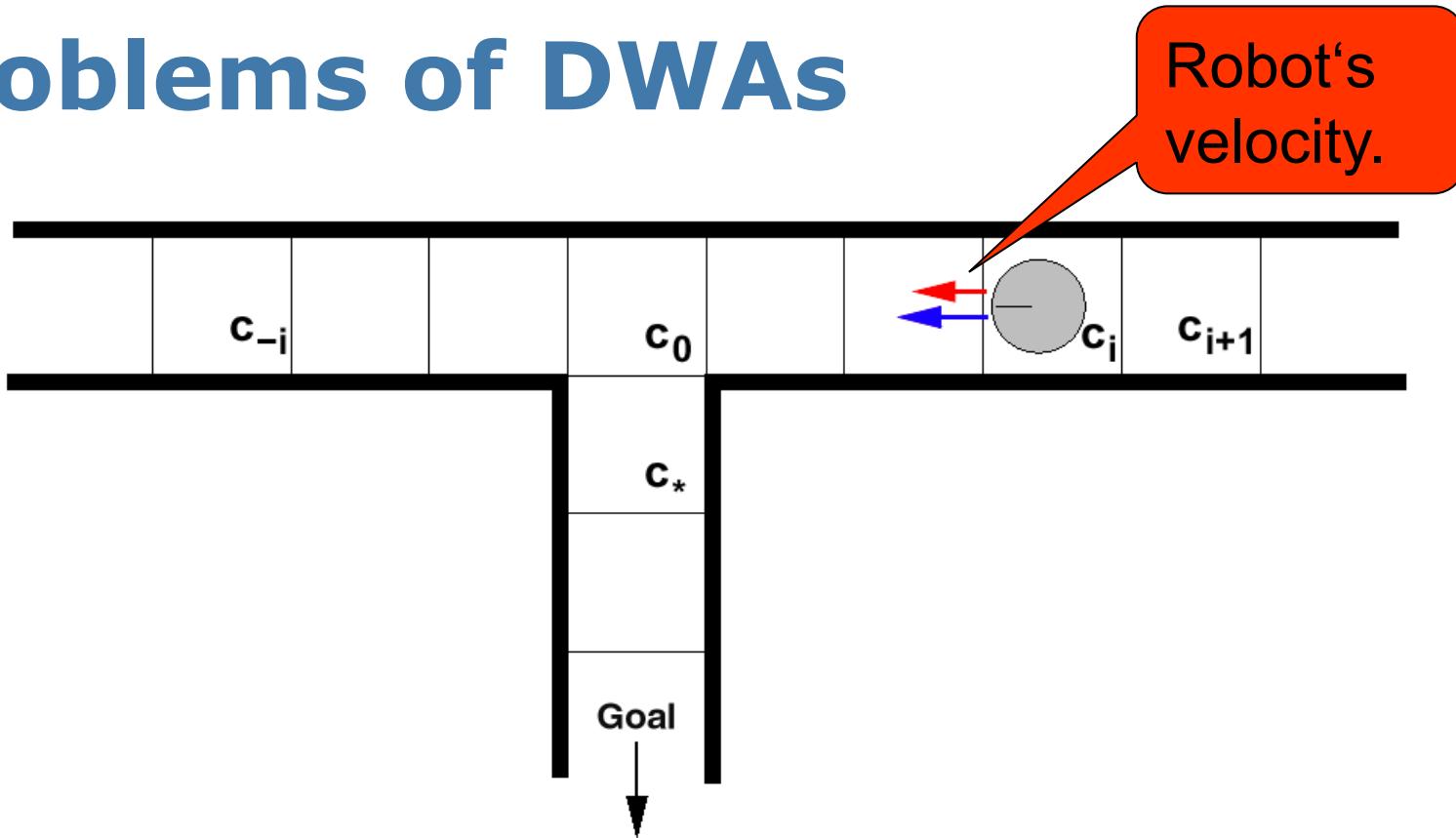
- Reacts quickly at low CPU requirements
- Guides a robot on a collision free path
- Successfully used in many real-world scenarios
- Resulting trajectories sometimes suboptimal
- Local minima might prevent the robot from reaching the goal location (regular DWA)
- Global DWA with NF1 overcomes this problem

Problems of DWAs



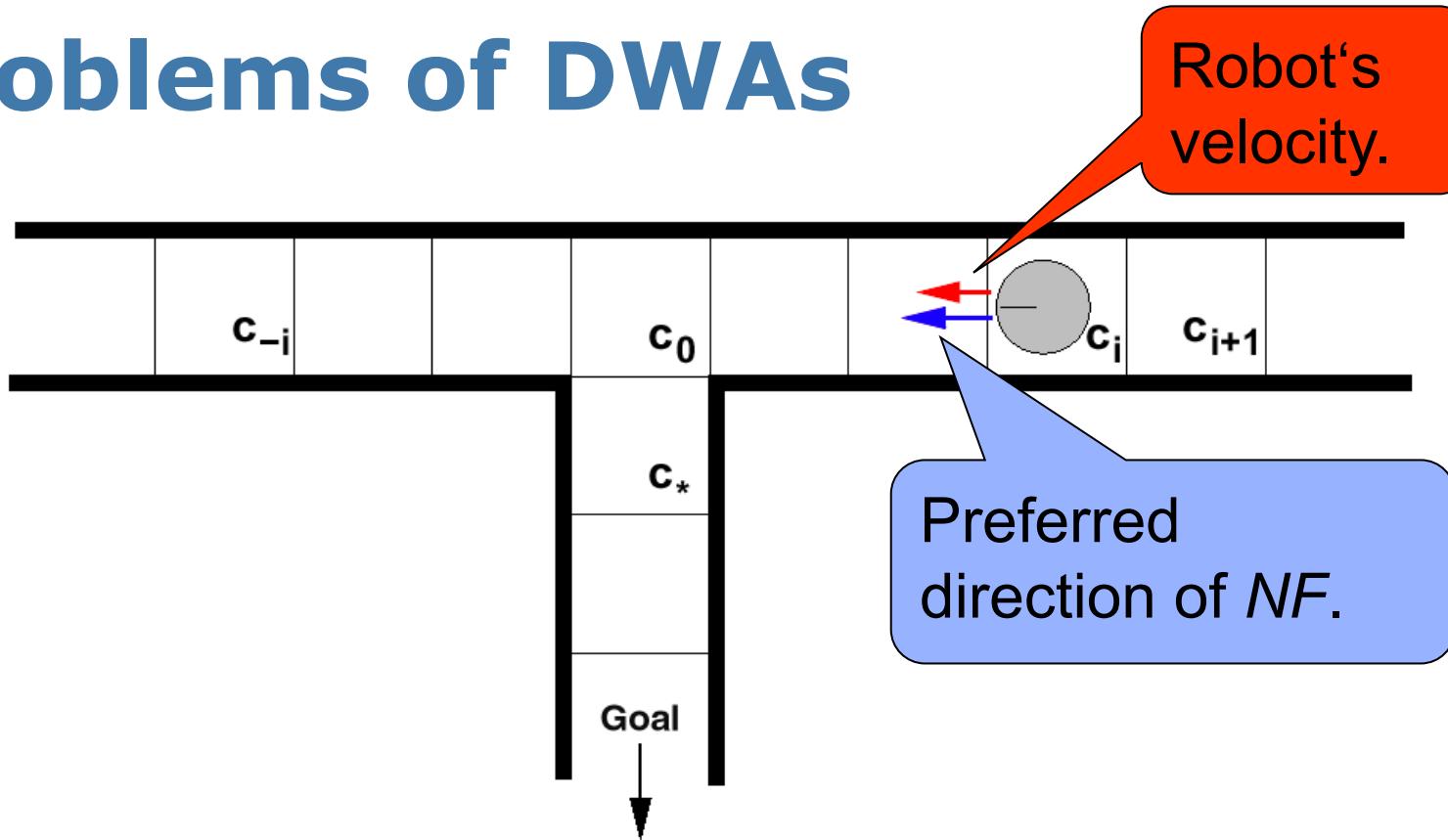
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



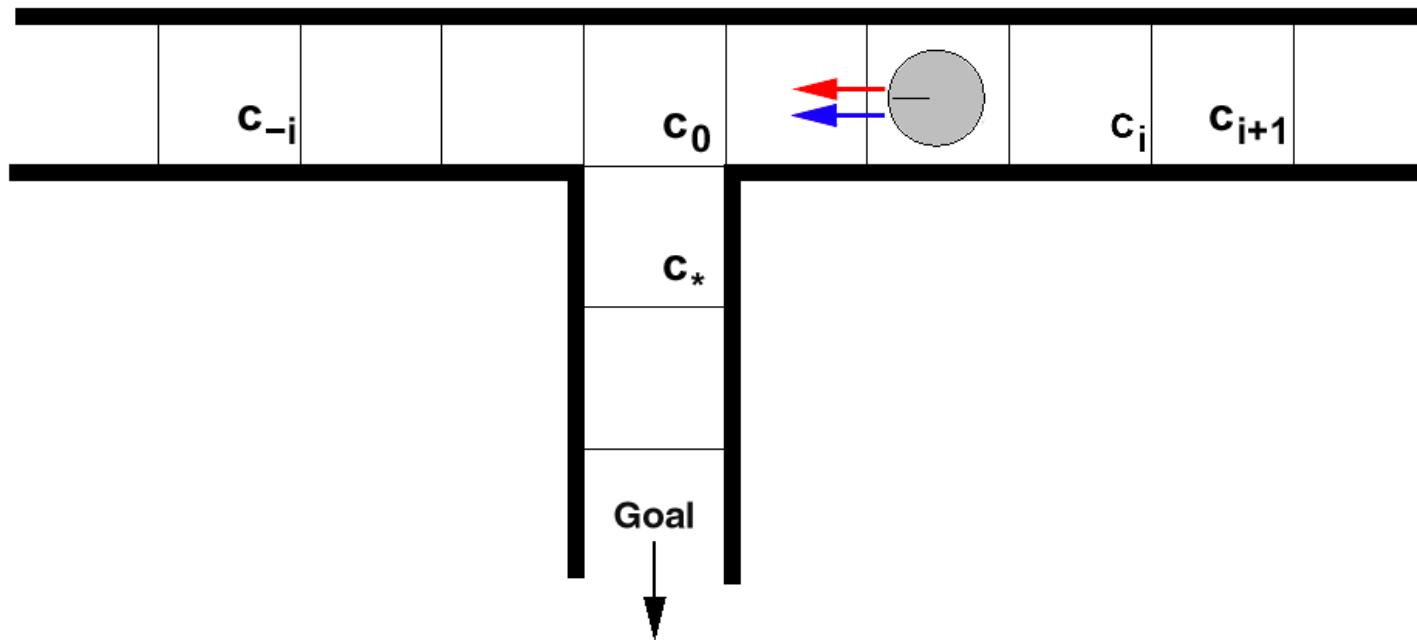
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



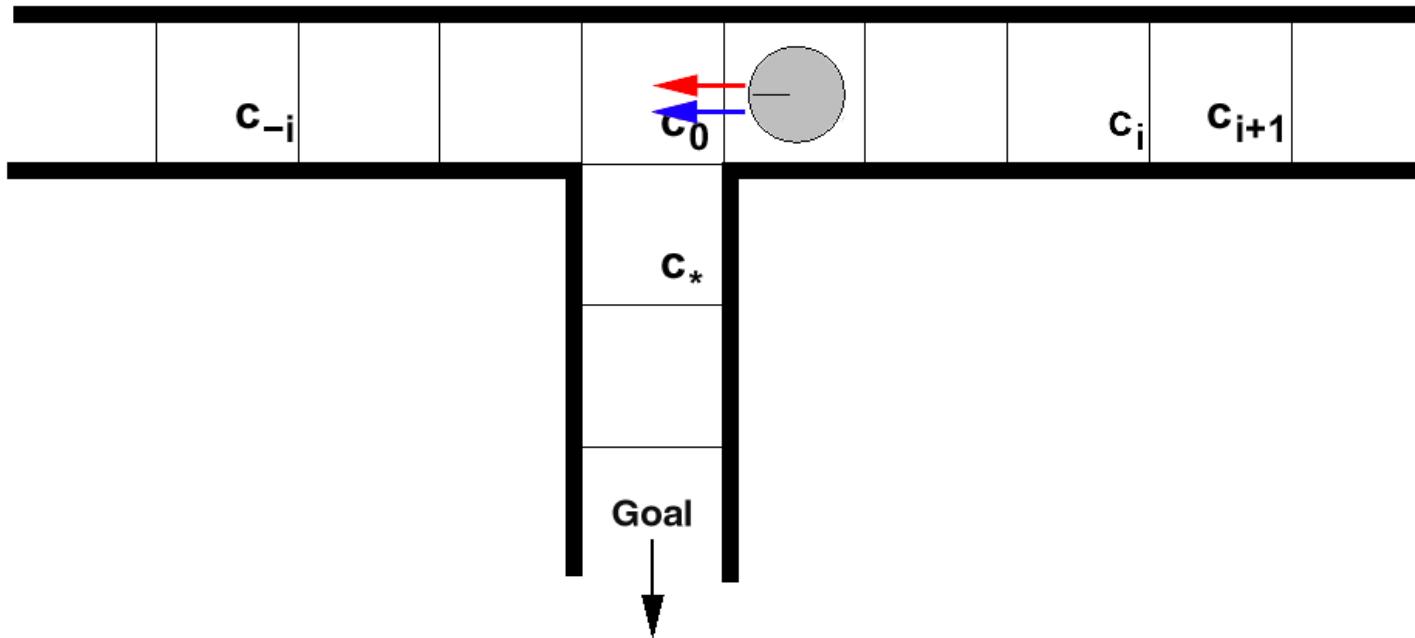
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



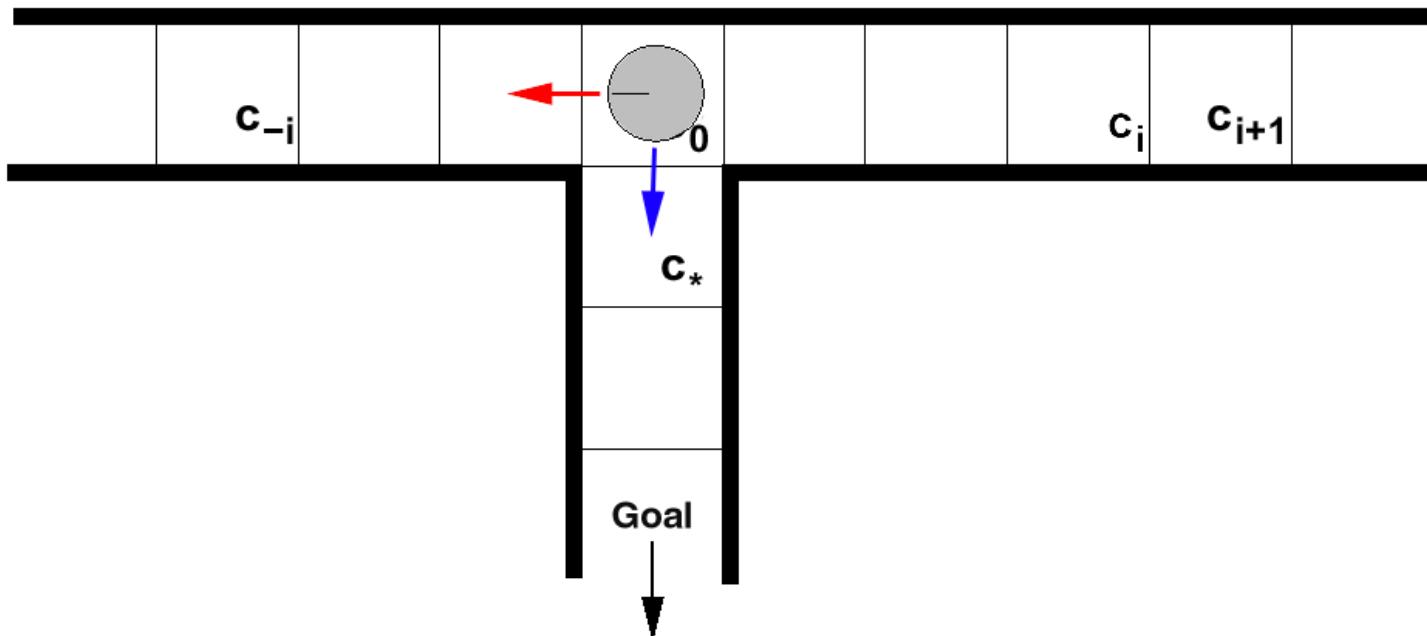
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

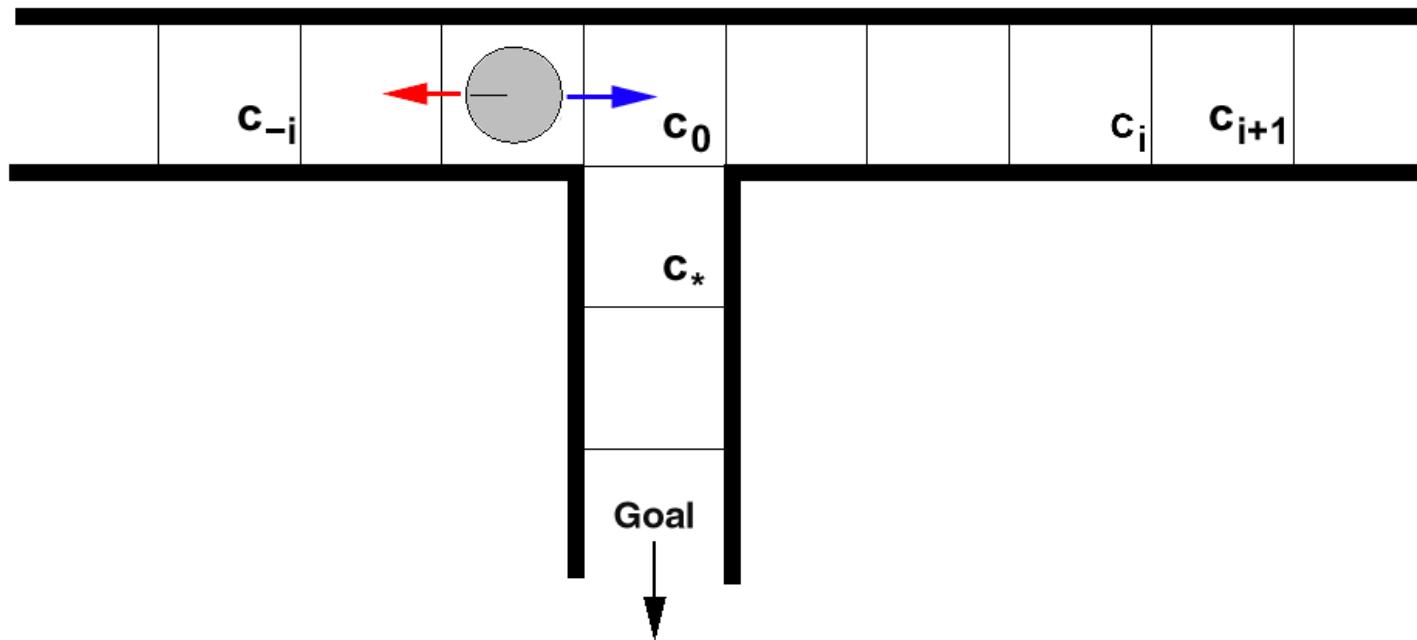
Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

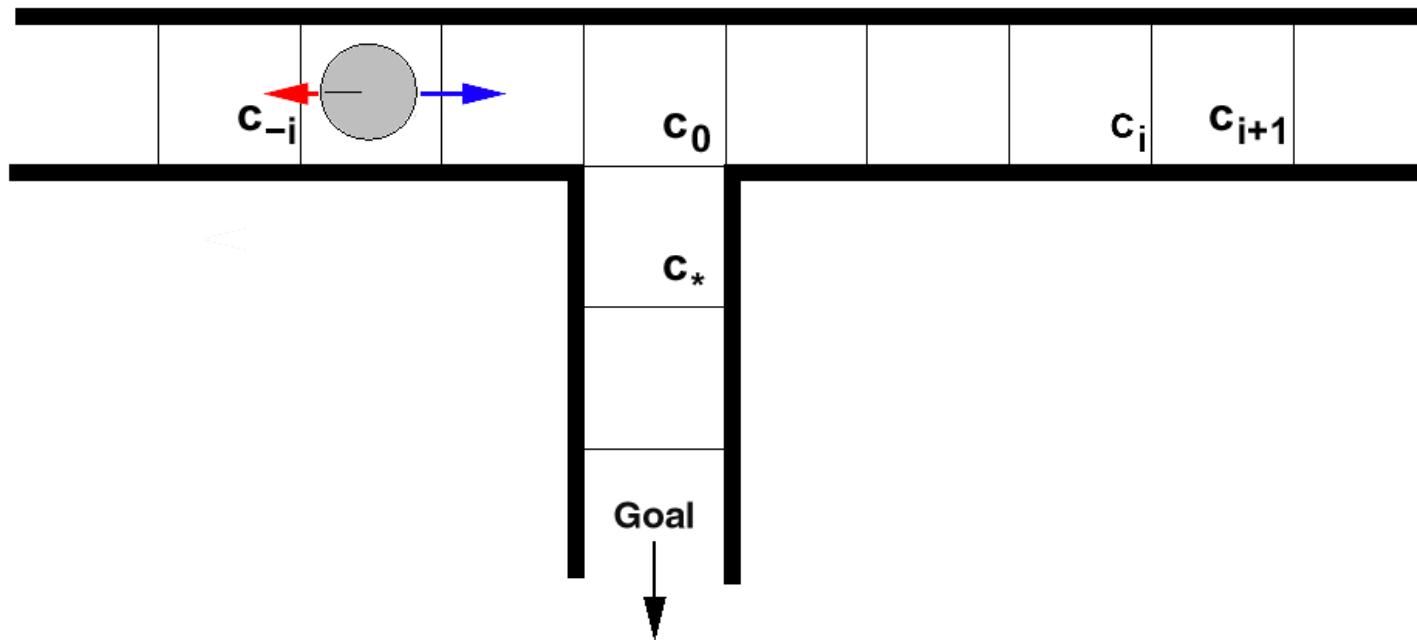
The robot drives too fast at c_0 to enter corridor facing south.

Problems of DWAs



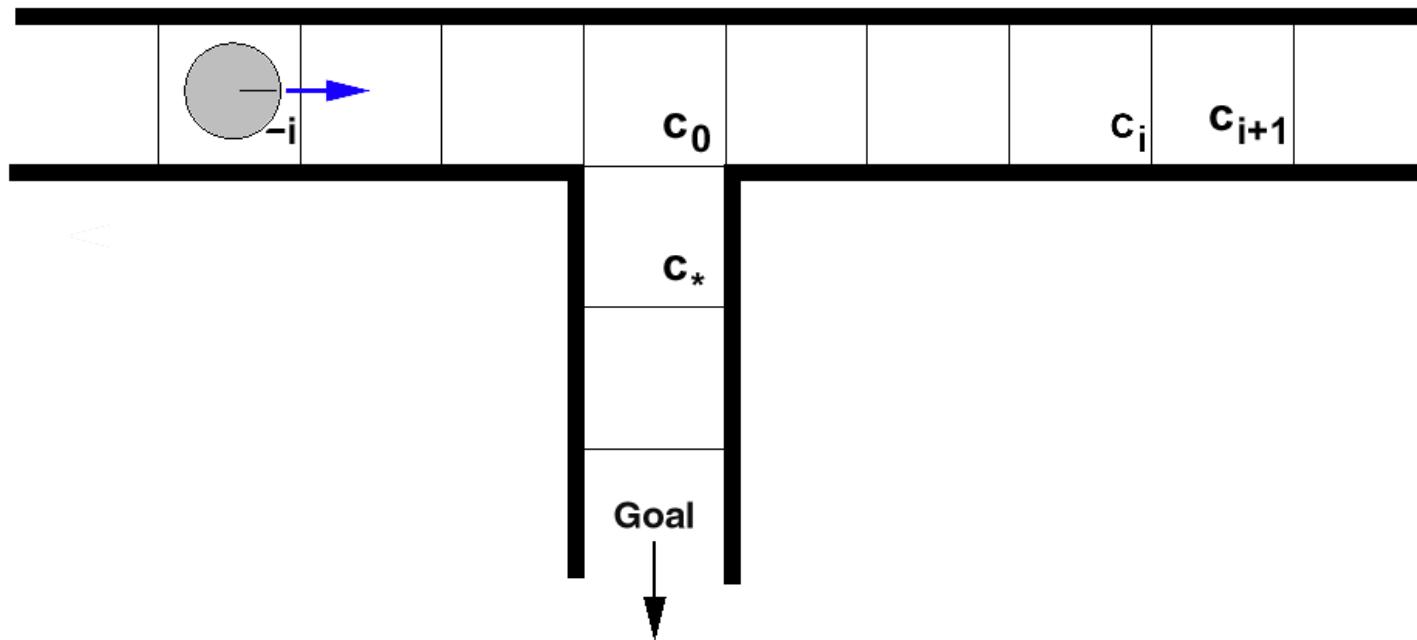
$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

Problems of DWAs



$$NF = \alpha \cdot vel + \beta \cdot nf + \gamma \cdot \Delta nf + \delta \cdot goal$$

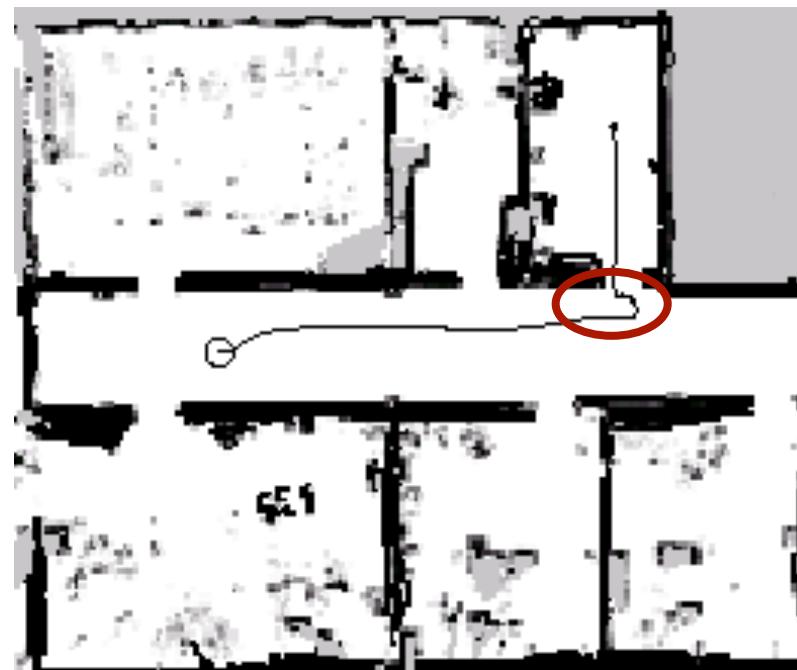
Problems of DWAs



- Same situation as in the beginning
- DWAs have problems to reach the goal

Problems of DWAs

- Typical problem in a real world situation:



- Robot does not slow down early enough to enter the doorway.

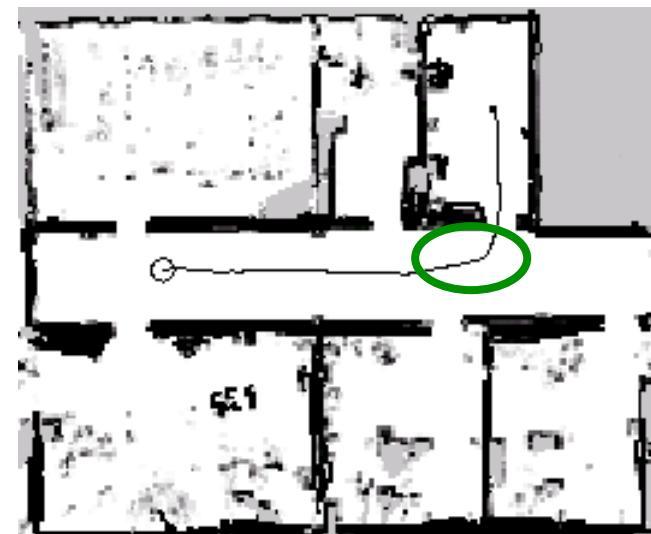
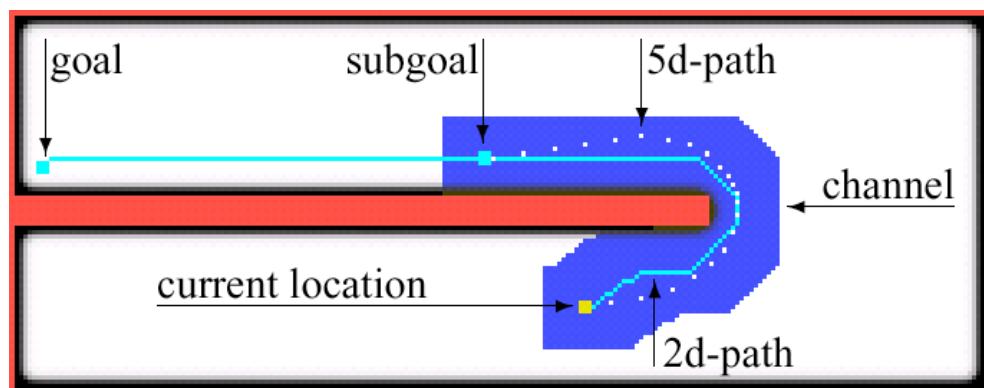
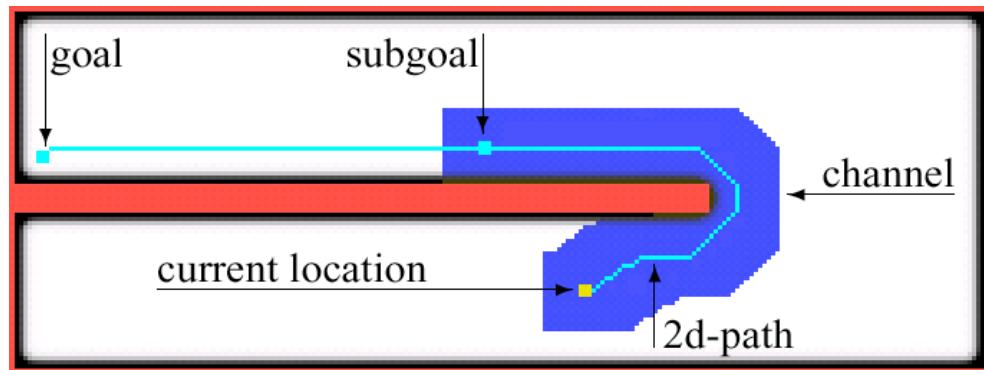
Alternative: 5D-Planning

- Plans in the **full** $\langle x, y, \theta, v, \omega \rangle$ -**configuration** space using A*
 - Considers the robot's kinematic constraints
- **Idea:** search in the discretized $\langle x, y, \theta, v, \omega \rangle$ -space
- **Problem:** search space too large to be explored in real-time
- **Solution:** restrict the full search space to "channels"

5D-Planning

- Use A* to find a trajectory in the **2D $\langle x, y \rangle$ -space**
- Choose a **subgoal** lying on the 2D-path within the channel
- Use A* in the "**channel**" **5D-space** to find a sequence of steering commands to reach the subgoal

5D-Planning Example



Summary (1 of 3)

- Motion planning lives in the **C-space**
- **Combinatorial** planning methods scale poorly with C-space dimension and non-linearity but are **complete** and **optimal**
- **Sampling-based** planning methods have weaker guarantees but are more efficient
- They all produce a **road map** that captures the connectivity of the C-space
- For planning on the road map, use **heuristic search** methods such as A*

Summary (2 of 3)

- Deterministic value iteration or Dijkstra yields the **optimal heuristic** for A*. Precompute if on-line replanning is likely
- A* in smoothed grid maps helps to keep the robot **away** from obstacles
- Any-angle A* methods produce **shorter** paths with **fewer** heading changes
- D*/D* Lite **avoids** replanning from **scratch** and finds the (usually few) nodes to be updated for on-line replanning

Summary (3 of 3)

- In highly dynamic environments, reactive **collision avoidance** methods that account for the **kinematic** and **dynamics** vehicle constraints become necessary
- Decoupling into an approximative **global** and an accurate **local** planning problem, integration using a layered architecture
- The Dynamic Window Approach optimizes a navigation function to trade off **feasible**, **reasonable**, and **admissible** motions

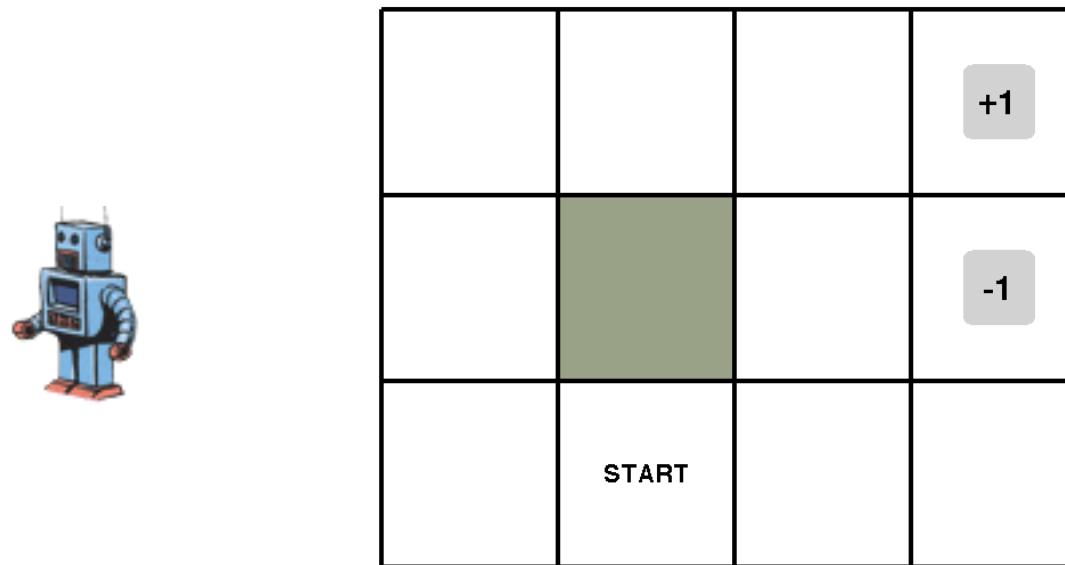
Uncertain Path Execution

- Have you ever **become lost** while trying to follow a **path** (e.g. printed out from Google maps)?
 - Problem: **path execution** is inherently **uncertain!**
 - Even the best **path** is worthless if the robot is unable to follow it
 - Reasons: Underlying trajectory controller, DWA, imperfect models of map/dynamics
- Instead of a plan, you need a **policy**



Markov Decision Process

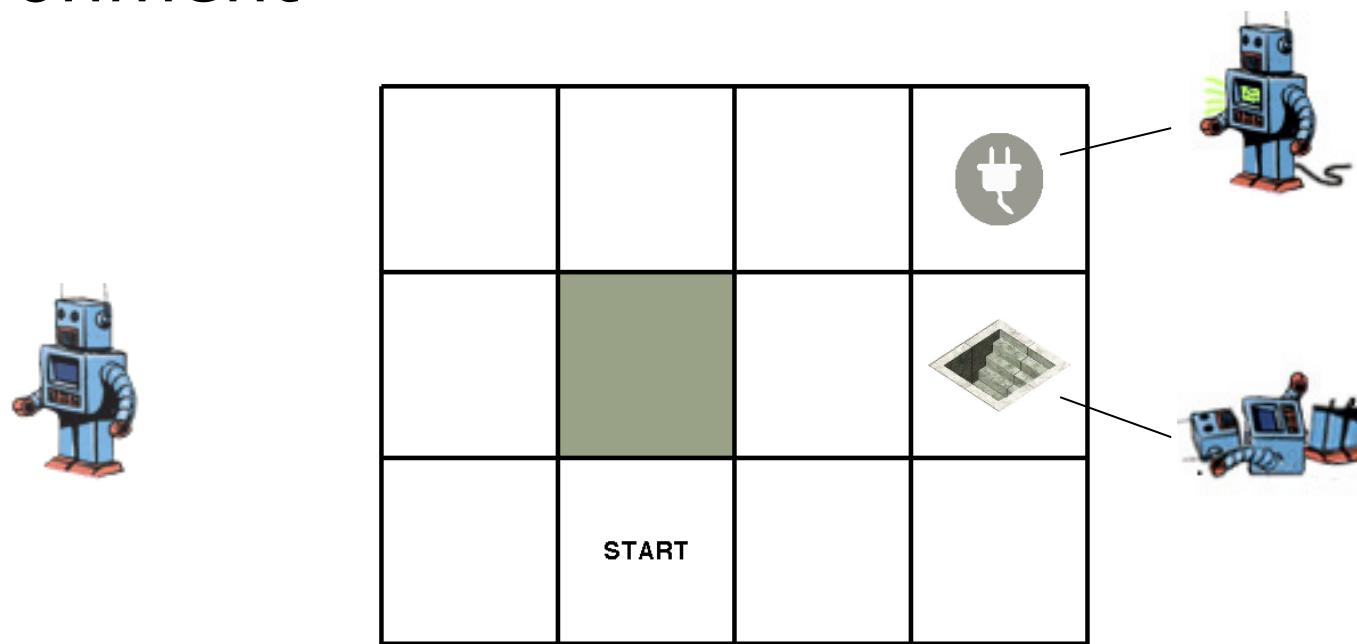
- Consider an agent acting in this environment



- Its mission is to reach the goal marked by +1 avoiding the cell labelled -1

Markov Decision Process

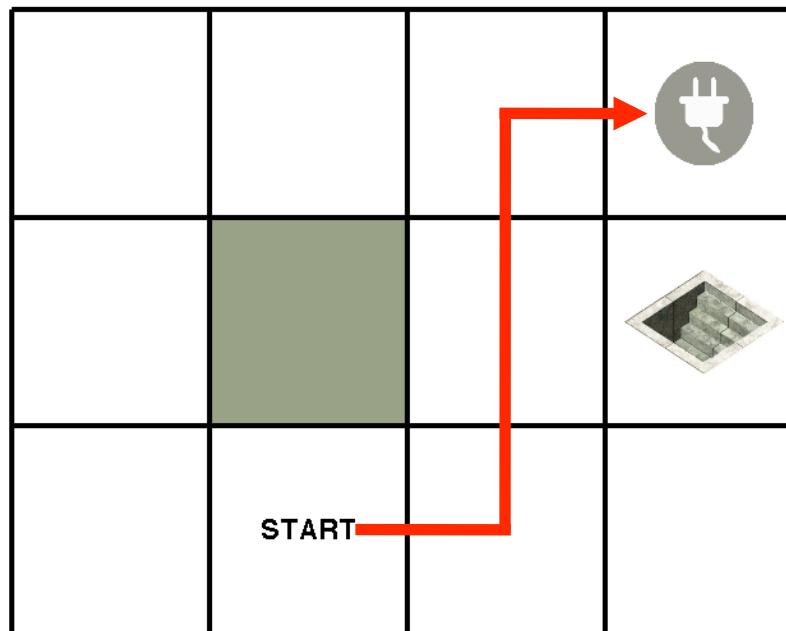
- Consider an agent acting in this environment



- Its mission is to reach the goal marked by +1 avoiding the cell labelled -1

Markov Decision Process

- Easy! Use a search algorithm such as A*



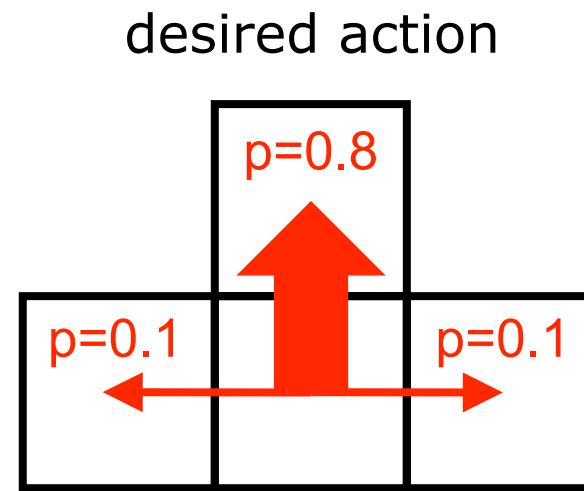
- Best solution (shortest path) is the action sequence [*Right, Up, Up, Right*]

What is the problem?

- Consider a non-perfect system in which actions are performed with a **probability less than 1**
 - What are the best actions for an agent under this constraint?
 - Example: a mobile robot does not *exactly* perform a desired motion
 - Example: human navigation
-  Uncertainty about performing actions!

MDP Example

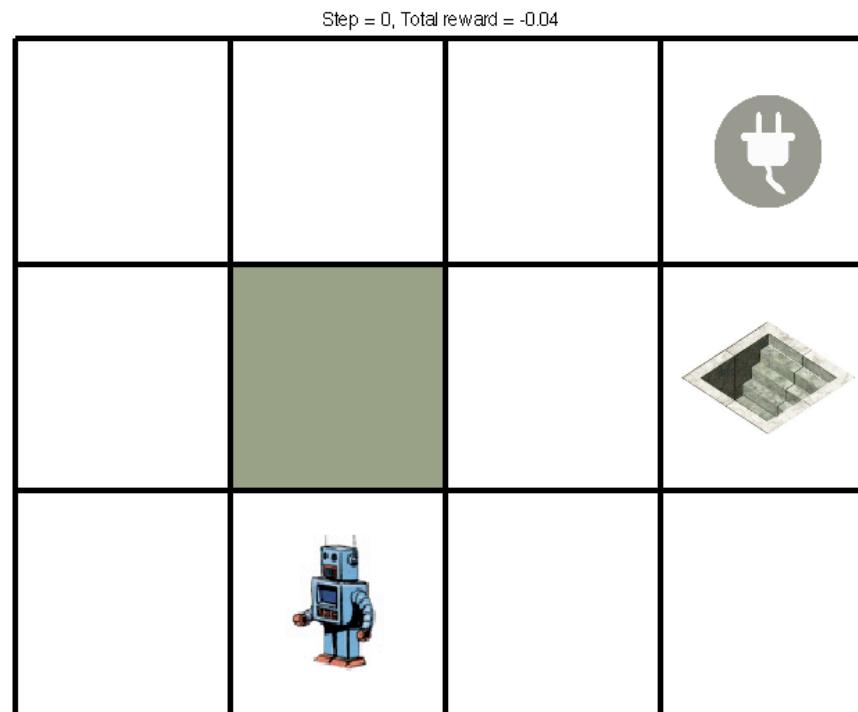
- Consider the **non-deterministic transition model** (N / E / S / W):



- Intended action is executed with $p=0.8$
- With $p=0.1$, the agent moves left or right
- Bumping into a wall "reflects" the robot

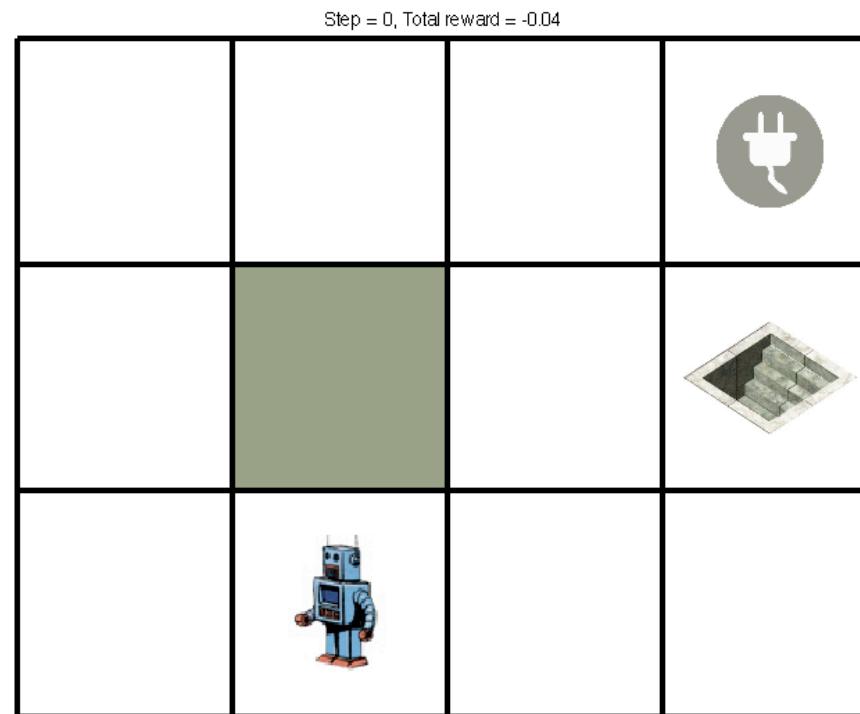
MDP Example

- Executing the **A* plan** in this environment



MDP Example

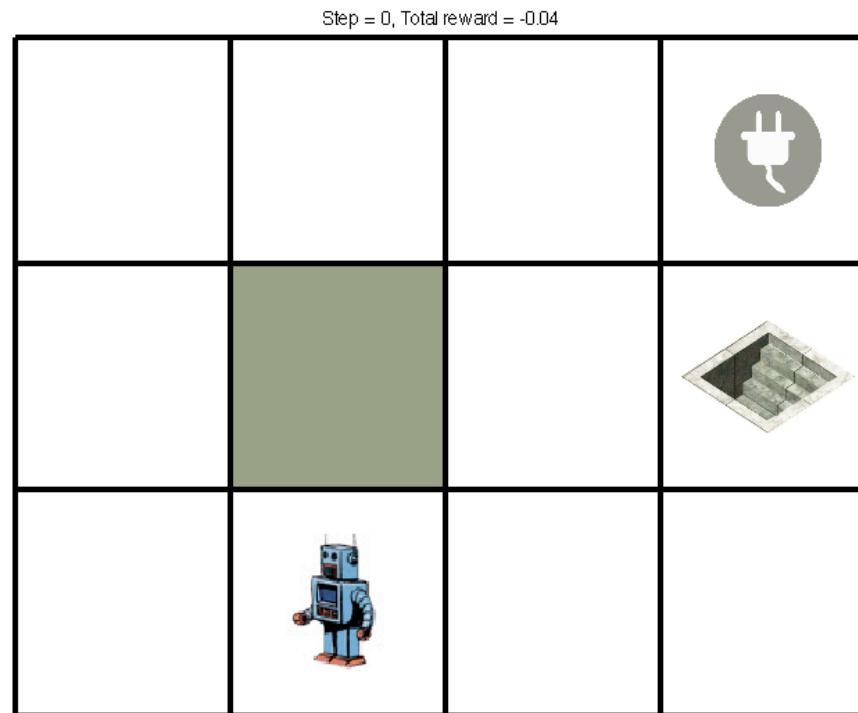
- Executing the **A* plan** in this environment



But: transitions are non-deterministic!

MDP Example

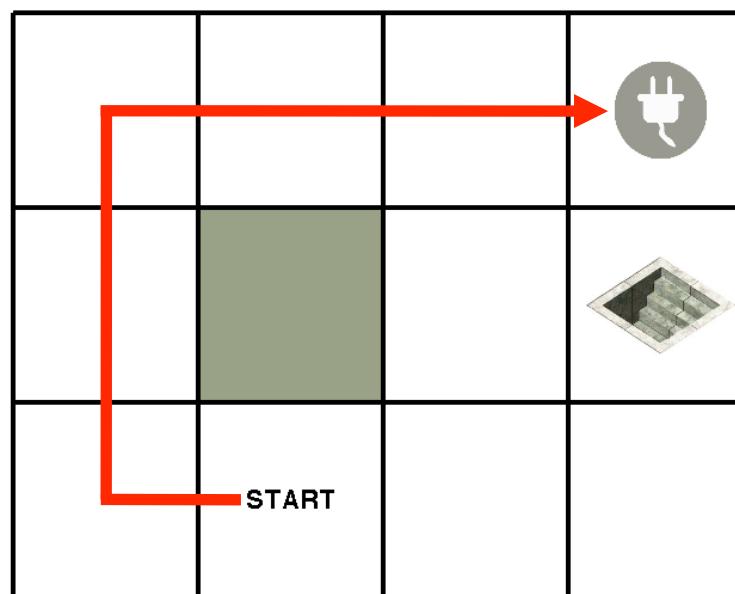
- Executing the **A* plan** in this environment



This will happen sooner or later...

MDP Example

- Use a **longer** path with **lower** probability to end up in cell labelled **-1**



- This path has the **highest overall utility**
- Probability $0.8^6 = 0.2621$

Transition Model

- The probability to reach the next state s' from state s by choosing action a

$$T(s, a, s')$$

is called **transition model**

Markov Property:

The transition probabilities from s to s' **depend only on the current state s** and not on the history of earlier states

Reward

- In each state s , the agent receives a **reward** $R(s)$
- The reward may be **positive** or **negative** but must be **bounded**
- This can be generalized to be a function $R(s,a,s')$. Here: consider only $R(s)$, does not change the problem

Reward

- In our example, the reward is **-0.04** in all states (e.g. the cost of motion) except the terminal states (that have rewards **+1/-1**)
- A negative reward gives agent an **incentive to reach the goal quickly**
- Or: "living in this environment is not enjoyable"

| | | | |
|-------|-------|-------|-------|
| -0.04 | -0.04 | -0.04 | +1 |
| -0.04 | | -0.04 | -1 |
| -0.04 | -0.04 | -0.04 | -0.04 |

MDP Definition

- Given a **sequential decision problem** in a fully observable, stochastic environment with a known Markovian transition model
- Then a **Markov Decision Process** is defined by the components
 - *Set of states:* S
 - *Set of actions:* A
 - *Initial state:* s_0
 - *Transition model:* $T(s, a, s')$
 - *Reward function:* $R(s)$

Policy

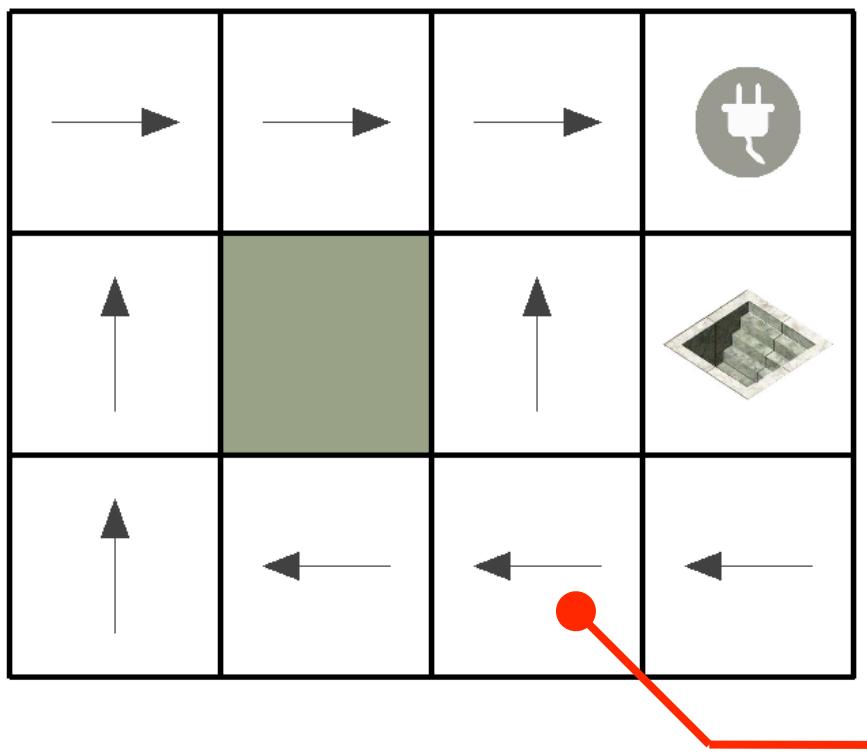
- An MDP solution is called **policy** π
- A policy is a mapping from states to actions

policy : *States* \mapsto *Actions*

- In each state, a policy tells the agent **what to do next**
- Let $\pi(s)$ be the *action* that π specifies for s
- Among the many policies that solve an MDP, the **optimal policy** π^* is what we seek. We'll see later what *optimal* means

Policy

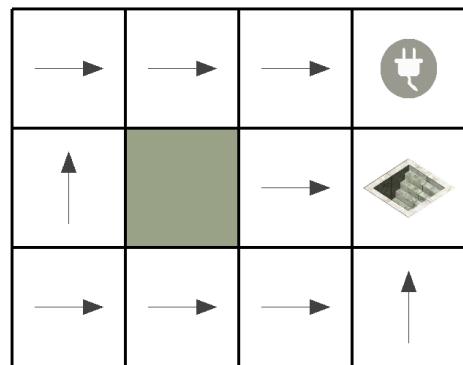
- The optimal policy for our example



Conservative choice
Take long way around
as the cost per step of
-0.04 is small compared
with the penalty to fall
down the stairs and
receive a **-1** reward

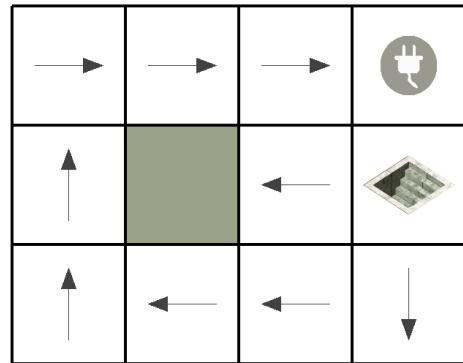
Policy

- When the balance of risk and reward changes, **other policies are optimal**



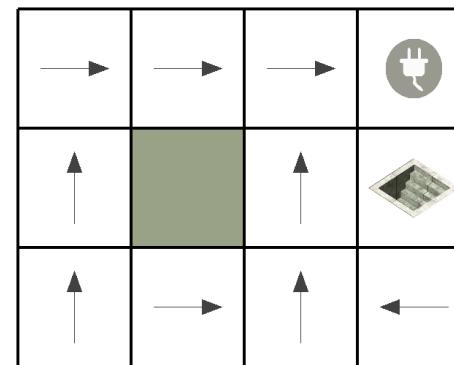
$$R < -1.63$$

Leave as soon as possible



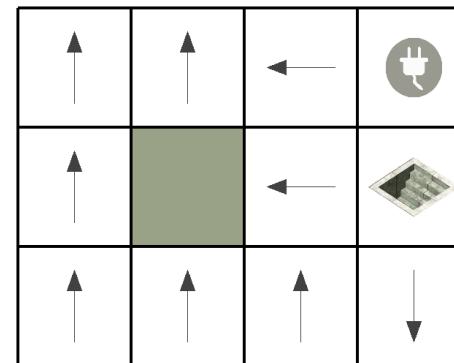
$$-0.02 < R < 0$$

No risks are taken



$$-0.43 < R < -0.09$$

Take shortcut, minor risks



$$R > 0$$

Never leave (inf. #policies)

Utility of a State

- The **utility of a state** $U(s)$ quantifies the **benefit** of a state for the **overall task**
- We first define $U^\pi(s)$ to be the **expected utility of all state sequences that start in s given π**

$$U^\pi(s) = E \left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s \right]$$

- $U(s)$ evaluates (and encapsulates) all possible futures **from s onwards**

Utility of a State

- With this definition, we can express $U^\pi(s)$ as a **function of its next state s'**

$$\begin{aligned} U^\pi(s) &= E \left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s \right] \\ &= E \left[R(s_0) + R(s_1) + R(s_2) + \dots \mid \pi, s_0 = s \right] \\ &= E \left[R(s_0) \mid s_0 = s \right] + E \left[R(s_1) + R(s_2) + \dots \mid \pi \right] \\ &= R(s) + E \left[\sum_{t=0}^{\infty} R(s_t) \mid \pi, s_0 = s' \right] \\ &= R(s) + U^\pi(s') \end{aligned}$$

Optimal Policy

- The utility of a state allows us to apply the **Maximum Expected Utility principle** to define the optimal policy π^*
- The **optimal policy** π^* in s chooses the action a that maximizes the expected utility of s (and of s')

$$\pi^*(s) = \operatorname{argmax}_a E[U^\pi(s)]$$

- Expectation taken over all policies

Optimal Policy

- Substituting $U^\pi(s)$

$$\begin{aligned}\pi^*(s) &= \operatorname{argmax}_a E\left[U^\pi(s)\right] \\ &= \operatorname{argmax}_a E\left[R(s) + U^\pi(s')\right] \\ &= \operatorname{argmax}_a E\left[R(s)\right] + E\left[U^\pi(s')\right] \\ &= \operatorname{argmax}_a E\left[U(s')\right] \\ &= \operatorname{argmax}_a \sum_{s'} T(s, a, s') U(s')\end{aligned}$$

- Recall that $E[X]$ is the weighted average of all possible values that X can take on

Utility of a State

- The **true utility of a state** $U(s)$ is then obtained by application of the optimal policy, i.e. $U^{\pi^*}(s) = U(s)$. We find

$$\begin{aligned} U(s) &= \max_a E \left[U^\pi(s) \right] \\ &= \max_a E \left[R(s) + U^\pi(s') \right] \\ &= \max_a E \left[R(s) \right] + E \left[U^\pi(s') \right] \\ &= R(s) + \max_a E \left[U(s') \right] \\ &= R(s) + \max_a \sum_{s'} T(s, a, s') U(s') \end{aligned}$$

Utility of a State

- This result is noteworthy:

$$U(s) = R(s) + \max_a \sum_{s'} T(s, a, s') U(s')$$

We have found a direct relationship between the **utility of a state** and the **utility of its neighbors**

- The utility of a state is the immediate reward for that state plus the expected utility of the next state, **provided** the agent chooses the **optimal** action

Bellman Equation

$$U(s) = R(s) + \max_a \sum_{s'} T(s, a, s') U(s')$$

- For each state there is a Bellman equation to compute its utility
- There are **n states** and **n unknowns**
- Solve the system using Linear Algebra?
- No! The max-operator that chooses the optimal action makes the system nonlinear
- We must go for an **iterative approach**

Discounting

We have made a **simplification** on the way:

- The utility of a state sequence is often defined as the sum of **discounted** rewards

$$U^\pi(s) = E \left[\sum_{t=0}^{\infty} \gamma^t R(s_t) \mid \pi, s_0 = s \right]$$

with $0 \leq \gamma \leq 1$ being the *discount factor*

- Discounting says that **future** rewards are **less significant** than **current** rewards.
This is a natural model for many domains
- The other expressions change accordingly

Separability

We have made an **assumption** on the way:

- Not all utility functions (for state sequences) can be used
- The utility function must have the **property of separability** (a.k.a. stationarity), e.g. additive utility functions:

$$U([s_0 + s_1 + \dots + s_n]) = R(s_0) + U([s_1 + \dots + s_n])$$

- Loosely speaking: the preference between two state sequences is unchanged over different start states

Utility of a State

- The **state utilities** for our example

| | | | |
|-------|-------|-------|-------|
| 0.812 | 0.868 | 0.918 | +1 |
| 0.762 | | 0.66 | -1 |
| 0.705 | 0.655 | 0.611 | 0.388 |

- Note that utilities are higher closer to the goal as fewer steps are needed to reach it

Iterative Computation

Idea:

- The utility is computed iteratively:

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s, a, s') U_i(s')$$

- Optimal utility: $U^* = \lim_{t \rightarrow \infty} U_t$
- Abort, if change in utility is below a threshold

Dynamic Programming

- The utility function is the basis for **Dynamic Programming**
- Fast solution to compute n -step decision problems
- Naive solution: $O(|A|^n)$
- Dynamic Programming: $O(n |A| |S|)$
- But: what is the correct value of n ?
- If the graph has loops: $n \rightarrow \infty$

The Value Iteration Algorithm

Algorithm 1: Value Iteration

In: An MDP with

- States and action sets S, A ,
- Transition model $T(s, a, s')$,
- Reward function $R(s)$,
- Discount factor γ

Out: The utility of all states U

$U' \leftarrow 0$

repeat

$U \leftarrow U'$

foreach state s in S **do**

$| \quad U(s) \leftarrow R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U(s')$

end

until *close-enough*(U, U')

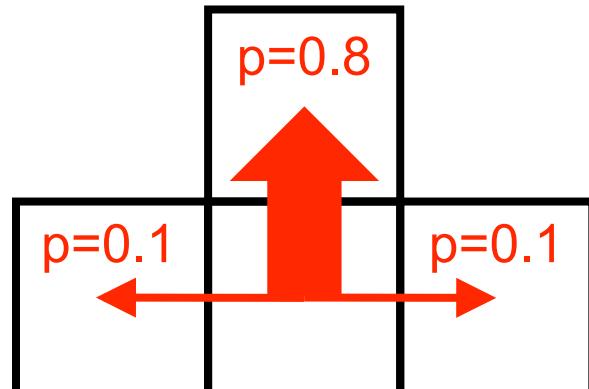
return U

Value Iteration Example

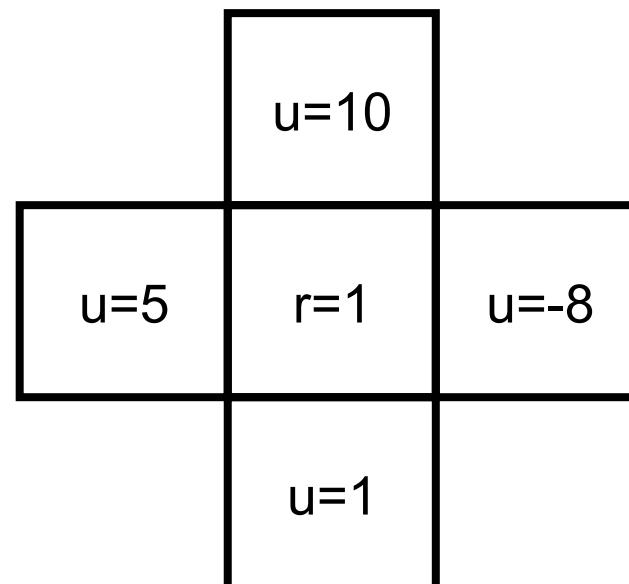
- Calculate utility of the center cell

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s, a, s') U_i(s')$$

desired action = Up



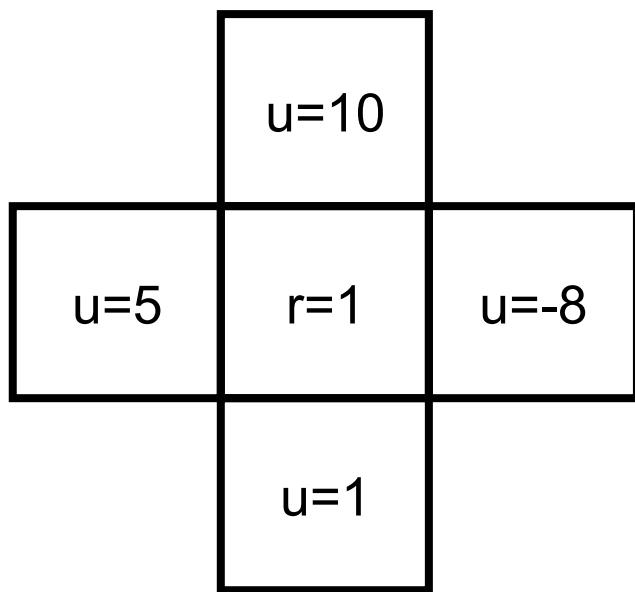
Transition Model



State space
(u=utility, r=reward)

Value Iteration Example

$$U_{i+1}(s) \leftarrow R(s) + \max_a \sum_{s'} T(s, a, s') U_i(s')$$



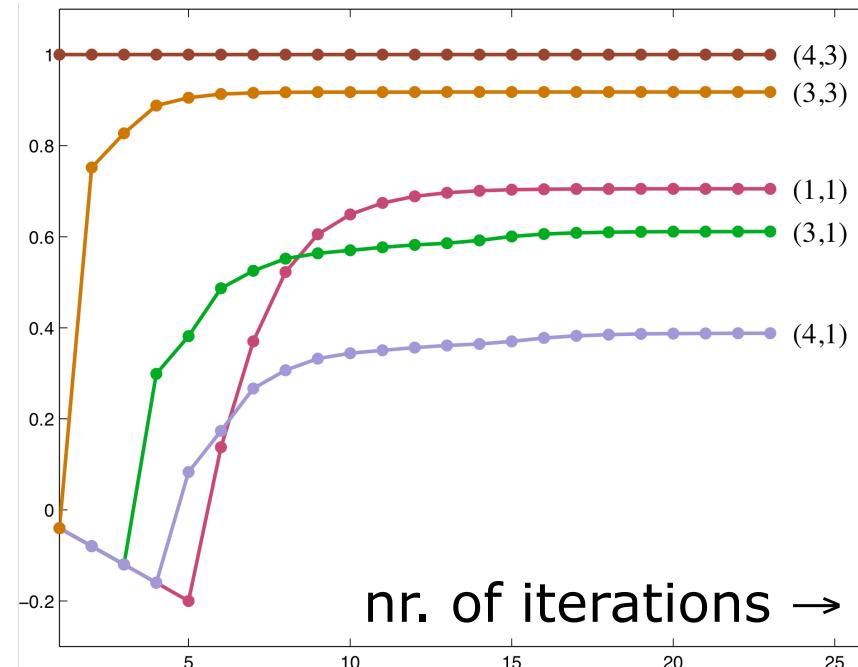
$$\begin{aligned} &= reward + \max\{ \\ &\quad 0.1 \cdot 1 + 0.8 \cdot 5 + 0.1 \cdot 10 \quad (\leftarrow), \\ &\quad 0.1 \cdot 5 + 0.8 \cdot 10 + 0.1 \cdot -8 \quad (\uparrow), \\ &\quad 0.1 \cdot 10 + 0.8 \cdot -8 + 0.1 \cdot 1 \quad (\rightarrow), \\ &\quad 0.1 \cdot -8 + 0.8 \cdot 1 + 0.1 \cdot 5 \quad (\downarrow)\} \\ &= 1 + \max\{5.1 (\leftarrow), 7.7 (\uparrow), \\ &\quad -5.3 (\rightarrow), 0.5 (\downarrow)\} \\ &= 1 + 7.7 \\ &= 8.7 \end{aligned}$$

Value Iteration Example

- In our example

| | | | |
|-------|-------|-------|-------|
| 0.812 | 0.868 | 0.918 | +1 |
| 0.762 | | 0.66 | -1 |
| 0.705 | 0.655 | 0.611 | 0.388 |

(1,1)



- States far from the goal first accumulate negative rewards until a path is found to the goal

Convergence

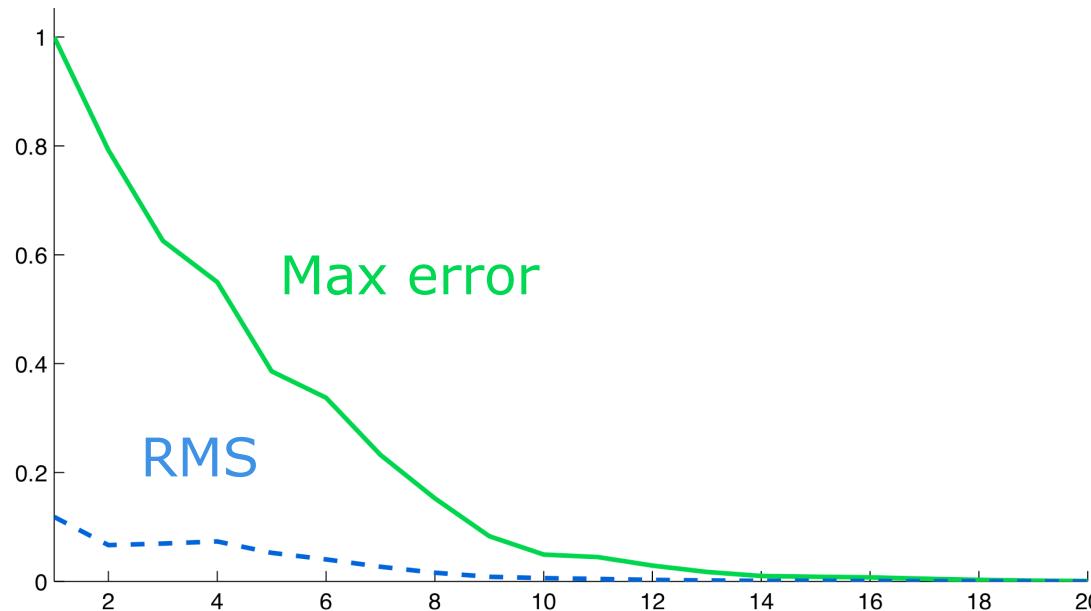
- The condition $\text{close-enough}(U, U')$ in the algorithm can be formulated by

$$RMS = \frac{1}{|S|} \sqrt{\sum_s (U(s) - U'(s))^2}$$

$$RMS(U, U') < \epsilon$$

- Different ways to detect convergence:
 - RMS error: root mean square error
 - Max error: $\|U - U'\| = \max_s |U(s) - U'(s)|$
 - Policy loss

Convergence Example



- What the agent cares about is **policy loss**: How well a policy based on $U_i(s)$ performs
- Policy loss converges much faster (because of the argmax)

Value Iteration

- Value Iteration finds the **optimal solution** to the Markov Decision Problem!
- **Converges** to the **unique solution** of the Bellman equation system for $\gamma < 1$
- Initial values for U' are arbitrary
- Proof involves the concept of *contraction*.
$$\|B U_i - B U'_i\| \leq \gamma \|U_i - U'_i\|$$
 with B being the Bellman operator (see textbook)
- VI propagates information through the state space by means of **local updates**

Optimal Policy

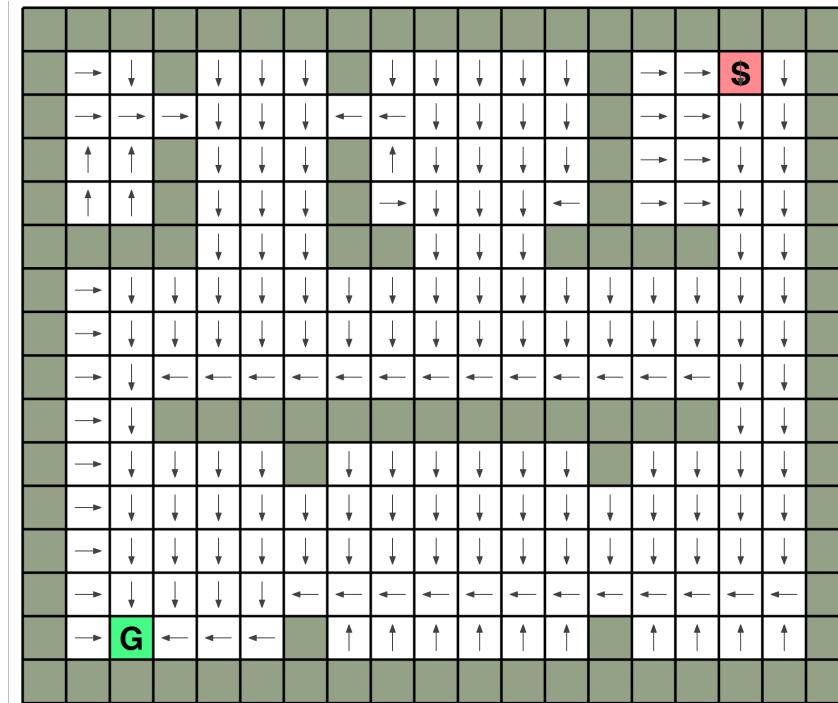
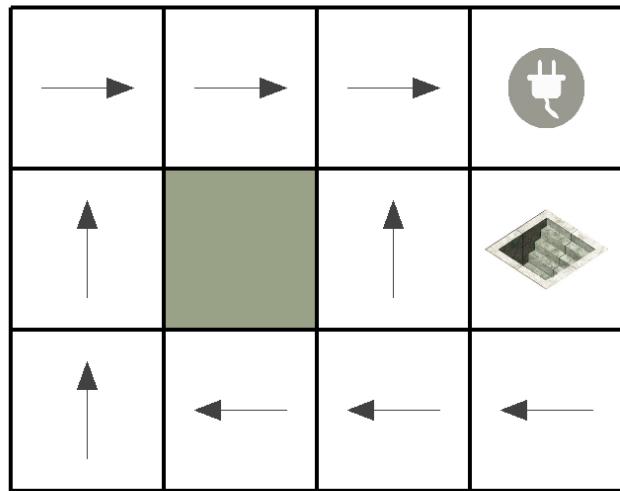
- How to finally compute the **optimal policy**? Can be easily extracted along the way by

$$\pi^*(s) = \operatorname{argmax}_a \sum_{s'} T(s, a, s') U(s')$$

- **Note:** $U(s)$ and $R(s)$ are quite different quantities. $R(s)$ is the **short-term** reward for being in s , whereas $U(s)$ is the **long-term** reward **from s onwards**

Optimal Policy

- Examples



Summary

- MDPs describe an uncertain agent with a **stochastic transition model**
- The solution is called **policy** that is a mapping from **states to actions**
- Value Iteration is a instance of dynamic programming, converges for lower-than-one discounts or finite horizons
- A policy allows to implement a **feedback control strategy**, the robot can never become lost anymore

What's missing...?

- Good solutions to **jointly plan the path under local constraints** that overcome the decoupling of global and local planning
- Good solutions to implement **feasible feedback** control strategies
- Problem: the **curse of dimensionality**
- AI/planning people and control theory people need to **talk more**
- Hence, the robot motion planning problem is not fully solved yet, but **good** solutions for many **practical** problems exist