# LIE GROUPS & GROUP REPRESENTATIONS

**MA419-TERM PAPER**

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## INTRODUCTION

The usual way to deal with symmetry in mathematics is by the use of the notion of a transformation group (like the dihedral group seen in class). The wonderful thing for us is that the groups that arise in the study of geometric symmetries are often themselves smooth manifolds. Such "group manifolds" are called Lie groups. In physics, Lie groups play a big role in connection with physical symmetries and conservation laws (Noether's theorem). Within physics, perhaps the most celebrated role played by Lie groups is in particle physics and gauge theory, and hence familiarity with the various representations of Lie groups is of paramount importance in any fundamental study of these topics. In mathematics, Lie groups play a prominent role in harmonic analysis (generalized Fourier theory), group representations, differential equations, and in virtually every branch of geometry including Riemannian geometry, Cartan geometry, algebraic geometry, Kahler geometry, and symplectic geometry.

# 1 GROUP REPRESENTATIONS

## 1.1 Basic definitions

Given an abstract group G and a vector space V, it is often possible to regard the elements of the group as invertible linear operators on the vector space. Once this is done, every element of G can be associated with the matrix of the operator in a fixed basis for V.

**Definition 1.** Let $\Gamma$ be a function that maps an element $g$ of G to the matrix $\Gamma(g)$. Then the totality of matrices $\Gamma(g)$ is called a **representation** of G if $\Gamma$ is a homomorphism i.e

$$\Gamma(g_1 * g_2) = \Gamma(g_1)\Gamma(g_2) \tag{1}$$

Notice that the product on the right hand side is a matrix product, while that on the left hand side is the product defined in the group. If the dimension of V is $n$ then the representation $\Gamma$ is said to be of degree $n$. For a given group element $g$, the entries of $\Gamma(g)$ are easily ascertained from the fact that $g$ acts as an invertible linear operator on V. Thus the $j^{th}$ column of $\Gamma(g)$ in a basis of V is picked out by the basis vector $x_j$ through $gx_j \to \Gamma(g)x_j$.

**Definition 2.** A representation $\Gamma$ of G is said to be faithful if $\Gamma(g_1) \neq \Gamma(g_2)$ for $g_1 \neq g_2$

For example, consider the subgroup of $\pi/2$ rotations of $D_8$, $R = \{R_{\pi/2}, R_\pi, R_{3\pi/2}, R_0\}$. Let the vector space V be $\mathbb{R}^2$ with the usual basis. Then, a representation of this rotation group would be

$$\Gamma(R_{\pi/2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \Gamma(R_\pi) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

$$\Gamma(R_{3\pi/2}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Gamma(R_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

**Definition 3.** Let $\Gamma$ be a representation of degree $n$ of a group G. The **character** of an element $g$ of G under the representation $\Gamma$ is the trace of the matrix $\Gamma(g)$.

$$\chi^\Gamma(g) = \mathrm{tr}\Gamma(g) = \sum_i^n [\Gamma(g)]_{ii} \tag{4}$$

A consequence of this definition is that if $\Gamma$ and $\Theta$ are two different representations of G related by a similarity transformation, then the character of any element of G has the same value in both the representations. Such representations are **equivalent representations** and are fundamentally not distinguished from each other. However, when two representations map to two different vector spaces, they remain no longer equivalent. To illustrate this, consider the following representation, $\Theta$, of the group of $\pi/2$ rotations acting on the vector space $\mathbb{R}^4$

$$\Theta(R_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Theta(R_{\pi/2}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

$$\Theta(R_\pi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \Theta(R_0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

### 1.2 Some important concepts

**Definition 4.** An **algebra** over a field is a vector space (over the same field) equipped with a bilinear product $f : V \times V \to V$

**Theorem 1.** *Every representation of a finite group* $G$ *is equivalent to a representation that maps all elements of* $G$ *to a unitary operator in some Hilbert space* $V$ *(a complete vector space equipped with a scalar product).*

*Proof.* Suppose $D(g)$ is a representation of a finite group $G$. Let

$$S = \sum_{g \in G} D(g)^\dagger D(g)$$

Note that $D(g)^\dagger D(g)$ is Hermitian. Moreover, it is a positive operator (has only real non-negative eigenvalues). It follows that $S$ is also a positive operator and we can diagonalize $S$ as

$$S = U^{-1} * d * U$$

where $d = \mathrm{diag}(d_1, d_2, ...)$ and $d_n \geqslant 0$. To show that $d$ is invertible, we assume the contrary. Let $d_i = 0$ for some $i \in \mathbb{N}$. Then, there exists an eigenvector such that $Sv = 0$. It would follow that

$$v^\dagger S v = \sum_{g \in G} \|D(g)v\|^2 = 0 \Rightarrow \|D(g)v\|^2 = 0 \quad \forall g \in G$$

which is impossible for $D(e_G) = \mathbb{I}$. Thus, $S$ must be invertible. We then define

$$S^{1/2} = U^{-1} \mathrm{diag}(\sqrt{d_1}, \sqrt{d_2}, ...) U$$

and let

$$D'(g) = S^{1/2} D(g) (S^{1/2})^{-1}$$

Note that $S^{1/2}$ is invertible as $S$ is invertible. We now only need to show that $D(g)$ is a unitary operator

$$\begin{aligned}
D'(g)^\dagger D'(g) =& (S^{1/2})^{-1} D(g)^\dagger S^{1/2} S^{1/2} D(g) (S^{1/2})^{-1} \\
=& (S^{1/2})^{-1} D(g)^\dagger S D(g) (S^{1/2})^{-1} \\
=& (S^{1/2})^{-1} D(g)^\dagger \left( \sum_{x \in G} D(x)^\dagger D(x) \right) D(g) (S^{1/2})^{-1} \\
=& (S^{1/2})^{-1} \sum_{x \in G} D(gx)^\dagger D(gx) (S^{1/2})^{-1} \\
=& (S^{1/2})^{-1} S (S^{1/2})^{-1} = \mathbb{I}
\end{aligned}$$

□

## 2 LIE GROUPS & LIE ALGEBRA

### 2.1 Lie Groups

**Definition 5.** A smooth manifold $G$ is called a **Lie Group** if it is a group (abstract group) such that the multiplication map $\mu : G \times G \to G$ and the inverse map inv $: G \to G$, given respectively by $\mu(g, h) = gh$ and $\mathrm{inv}(g) = g^{-1}$, are $C^\infty$ maps.

Consider the smooth 1D manifold $S^1$ (which is isomorphic to the group of complex numbers of modulus 1 under multiplication- $U(1)$). The symmetry of a unit circle is that its shape is invariant under unit rotations of any degree (corresponding to multiplications by $e^{i\theta}$ in $U(1)$) . The notion of any degree implies that there is continuously an infinite number of unit rotations one can perform such that the unit circle's shape is unchanged. To do so we need some continuous variable denoted $\theta$ to represent every possible rotation. Now we call these unit rotations $R$, and consider the representation of these unit rotations in $\mathbb{R}^2$:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{7}$$

Note that $R(0) = \mathbb{I}$. Now, consider some neighbourhood of the identity operator. We can perform a Taylor's expansion and write up to fist order in $d\theta << 1$,

$$R(d\theta) \approx \mathbb{I} + R'(0)d\theta \tag{8}$$

Note,

$$R'(\theta) = \begin{pmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix} \Rightarrow R'(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

As physicists, we want our operators to be Hermitian, since in physics want to make use of the eigenvalues to express conserved quantities in the symmetry. Thus, we define the **generator** of the unit rotation Lie group as following:

$$X = iR'(0) \tag{10}$$

From this, it would follow that $R(d\theta) \approx \mathbb{I} - id\theta X$. Now, finite rotations can be expressed as infinite successive infinitesimal rotations (which makes sense due to the closure property of the group) as follows:

$$R(\theta) = \lim_{k\to\infty} (\mathbb{I} - id\theta X)^k = \lim_{k\to\infty} (\mathbb{I} - i\theta X/k)^k = e^{-iX\theta} \tag{11}$$

The group of all rotations in 2D, represented in this way is also called the $SO(2)$ group.

### 2.2 Lie Algebra

**Definition 6.** A **Lie algebra** is a vector space $\mathfrak{g}$ over some field $F$ together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket satisfying the following axioms:

- The operation is bilinear
- The operation is alternative, $[x,x] = 0$ for all $x \in \mathfrak{g}$ and is hence skew symmetric, $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$
- The operation satisfies the Jacobi identity, $[x,[y,z]] + [y,[z,x]] + [z,[x,y] = 0$ for all $x, y, z \in \mathfrak{g}$

Any Lie group gives rise to a canonically determined Lie algebra (concretely, the tangent space at the identity). Conversely, for any finite-dimensional Lie algebra $\mathfrak{g}$, there exists a corresponding connected Lie group $G$ with Lie algebra $\mathfrak{g}$. Concretely, consider the group $SO(N)$ which acts on the $n-$dimensional vector space $\mathbb{R}^n$ and keeps the dot-product of two vectors invariant. The defining condition for such a group in the representation of $\mathbb{R}^n$ is

$$R^T R = \mathbb{I} \tag{12}$$

Consider a path in $SO(N)$ so that $R = R(t)$ and require $R(0) = \mathbb{I}$. Taking the derivative w.r.t t yields

$$\left(\frac{dR^T}{dt}\right)_{t=0} + \left(\frac{dR}{dt}\right)_{t=0} = 0 = \omega^T + \omega \tag{13}$$

The set of all $\omega$'s satisfying this condition forms a linear vector space (with basis element $R'(0)$), the tangent space at the identity also known as the Lie algebra of the group. Recall that in the case of $SO(2)$, we were able to generate the group representation by exponentiating $R'(0)$. Motivated by this, we define, in general, the generators of the Lie group using the basis elements of the Lie algebra at identity.

**Definition 7.** Let $\alpha_j$ be a set of parameters that parametrize the Lie Group, and let $D$ be a representation of the Lie group such that $D(0) = \mathbb{I}$. Then, the **generators of the Lie Group** are

$$X_j = i\frac{\partial D}{\partial \alpha_j}(0) \tag{14}$$

In such a case,

$$D(\alpha) = e^{-iX_j\alpha_j} \tag{15}$$

where the Einstein summation convention is implied.

For example, in the case of $U(1)$, a Lie algebra element has the form $i\alpha$, which would make it isomorphic to $\mathbb{R}$. This agrees with our intuitive notion that tangent to a circle at a point would be a line.

## 2.3 SO(3)

Consider the group of unit rotations in 3-D called $SO(3)$ (as a smooth manifold, is diffeomorphic to the real projective space $\mathbb{P}^3(\mathbb{R})$). Since we know that all unit rotations axes can be spanned by the $x, y, z$ axes, we can consider the following basis of $SO(3)$ in the representation of matrices acting on the $\mathbb{R}^3$ vector space.

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \quad R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, the generators of the Lie group in this representation and basis are

$$X_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad X_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad X_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

It can be checked that the generators of $SO(3)$ form the basis to the Lie algebra with

$$[X_i, X_j] = 2i\epsilon_{ijk}X_k \tag{17}$$

where $\epsilon_{ijk}$ is the completely antisymmetric Levi-Civita symbol. At this point, it is worth noting that $J_a = \hbar X_a$ become the angular momentum operators for spin-1 particles in Quantum Mechanics. Exactly as seen here, they are the generators of rotation in 3-D for wavefunctions with spin-1.

## 2.4 SU(2)

**Definition 8.** The special unitary group $SU(2)$ is defined as

$$SU(2) \equiv \{g \in GL(2, \mathbb{C}) | gg^\dagger = \mathbb{I}, \det(g) = 1\} \tag{18}$$

Based on the unitary property and the determinant 1 condition, we can write $SU(2)$ as

$$SU(2) = \left\{ \begin{pmatrix} \alpha^* & -\beta^* \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\} \tag{19}$$

A convenient parametrization of unitary $2 \times 2$ matrices is by means of the Cayley-Klein parameters $\xi_{0,1,2,3} \in \mathbb{R}$. These parameters are related to $\alpha$ and $\beta$ as follows:

$$\alpha = \xi_0 + i\xi_3 \quad \beta = \xi_2 - i\xi_1 \tag{20}$$

With this, the unitary matrices can be written as

$$U(\xi_0, \vec{\xi}) = \xi_0 \mathbb{I} - i\vec{\xi} \cdot \vec{\sigma} \tag{21}$$

with $\sigma_i$ being the 3 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

The determinant condition gives $\det(U) = \xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$. Since there are only three free parameters, $SU(2)$ can be realised as diffeomorphic to the 3-sphere $S^3$ embedded in 4-D. It is also worth noting that the Pauli matrices satisfy the following commutation relation giving rise to Lie algebra on $SU(2)$ called $\mathfrak{su}(2)$.

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{23}$$

The following choice of generators of $\mathfrak{su}(2)$ (basis for the algebra) is found to be more convenient

$$J_1 = \frac{\sigma_1}{2}, \quad J_2 = \frac{\sigma_2}{2}, \quad J_3 = \frac{\sigma_3}{2} \tag{24}$$

The commutation relations then become

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{25}$$

If we append a factor of $\hbar$, these just become the angular momentum operators for spin-1/2 particles in Quantum mechanics

# 3 LORENTZ GROUP

The Lorentz group is the set of all transformations that preserve the inner product of Minkowski space. Formally

**Definition 9.** The Lorentz group is defined to be

$$O(1,3) = \{\Lambda \in GL(4, \mathbb{R}) | \Lambda^T \eta \Lambda = \eta\} \tag{26}$$

where $\eta$ is the metric of Minkowski space

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \tag{27}$$

Note this is just one representation of the Lorentz group acting on a 4 dimensional vector space. $SO(1,3)$ is defined to be the subgroup of $O(1,3)$ such that $\det(\Lambda) = 1$. As before, if we consider a path in $SO(1,3)$ (a parameterization) such that $\Lambda(t = 0) = \mathbb{I}$, and differentiate the defining condition of the Lorentz group, we get

$$\omega^T \eta + \eta \omega = 0 \tag{28}$$

where the elements satisfying the above condition form an algebra. We can thus choose the Hermitian generators of this algebra (basis) to be

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

Where we call the first row to be generators of "boosts" along the 3 axes and the second row to be the generators of rotations about the 3 axes. Using the explicit form of the generators for $SO(1,3)$ we can derive the corresponding Lie algebra, $\mathfrak{so}(1,3)$ to be

$$[L'_i, L'_j] = i\epsilon_{ijk}L'_k, \quad [L'_i, K'_j] = i\epsilon_{ijk}K'_k, \quad [K'_i, K'_j] = -i\epsilon_{ijk}L'_k \tag{31}$$

where we have taken $L'_i = iL_i$ and $K'_i = iK_i$. With these generators, $SO(1,3)$ can be represented compactly as

$$\text{Boost: } \Lambda_\phi = e^{i\vec{\phi}\cdot\vec{K'}} \quad \text{Rotation: } R_\theta = e^{i\vec{\theta}\cdot\vec{L'}} \tag{32}$$

The two type of generators do not commute with each other. While the rotation generators close under commutation (giving $SO(3)$ as a subgroup of $SO(1,3)$), the boost generators do not. We can however define two sets of operators from the old ones that do close under commutation and commute with each other

$$N_i^{\pm} = \frac{L'_i \pm iK'_i}{2} \tag{33}$$

which lead to the commutation relations

$$[N_i^+, N_j^+] = i\epsilon_{ijk}N_k^+, \quad [N_i^-, N_j^-] = i\epsilon_{ijk}N_k^-, \quad [N_i^+, N_i^-] = 0 \tag{34}$$

We have, therefore, discovered that the Lie algebra $\mathfrak{so}(1,3)$ consists of two copies of the Lie algebra $\mathfrak{su}(2)$

### 3.1 The $(0,0)$ representation

The lowest order representation is as for $SU(2)$ trivial, because the vector space is 1D for both copies of the $\mathfrak{su}(2)$ Lie algebra. Our generators must therefore be $1 \times 1$ and the only $1 \times 1$ matrices satisfying the commutation relations are the trivial $0$

$$N_i^+ = N_i^- = 0 \quad \rightarrow \quad e^{-i\alpha_i N_i^+} = e^{-i\beta_i N_i^-} = \mathbb{I} \tag{35}$$

Therefore, the $(0,0)$ representation of the Lorentz group acts on objects that do not change under Lorentz transformations. This is called the Lorentz scalar representation. This is exactly how the the scalar fields of the Klein-Gordon equation transform under a Lorentz transformation.

### 3.2 The $(\frac{1}{2},0)$ representation

In this representation, we use the 2D representation for one copy of the $\mathfrak{su}(2)$ algebra $N_i^+$ i.e $N_i^+ = \sigma_i/2$ and the 1D representation for the other $N_i^-$, i.e $N_i^- = 0$

$$N_i^- = 0 \Rightarrow L'_i = iK'_i$$

Therefore,

$$N_i^+ = \frac{\sigma_i}{2} \quad \Rightarrow \quad K'_i = \frac{-i}{2}\sigma_i, \quad L'_i = \frac{1}{2}\sigma_i$$

Thus, the Lorentz group representations are given by

$$\text{Boost: } \Lambda_\phi = e^{\vec{\phi}\cdot\vec{\sigma}/2} \quad \text{Rotation: } R_\theta = e^{i\vec{\theta}\cdot\vec{\sigma}/2} \tag{36}$$

The important thing to notice is that here we have complex $2 \times 2$ matrices, representing the Lorentz transformations. The two component objects this representation acts on are called **left-chiral spinors**

$$\chi_L = \begin{pmatrix} (\chi_L)_1 \\ (\chi_L)_2 \end{pmatrix} \tag{37}$$

### 3.3 The $(0, \frac{1}{2})$ representation

This representation can be constructed analogous to the $(\frac{1}{2}, 0)$ case. Here we have

$$N_i^+ = 0, \quad N_i^- = \frac{\sigma_i}{2} \quad \Rightarrow \quad K_i' = \frac{i}{2}\sigma_i, \quad L_i' = \frac{1}{2}\sigma_i$$

We then get

$$\text{Boost: } \Lambda_\phi = e^{-\vec{\phi}\cdot\vec{\sigma}/2} \quad \text{Rotation: } R_\theta = e^{i\vec{\theta}\cdot\vec{\sigma}/2} \tag{38}$$

Rotations are the same as in the $(\frac{1}{2}, 0)$ representation, but boosts differ by a minus sign in the exponent. Therefore, both representations act on objects that are similar but not the same. We call these objects **right-chiral spinors**.

$$\chi_R = \begin{pmatrix} (\chi_R)^1 \\ (\chi_R)^2 \end{pmatrix} \tag{39}$$

### 3.4 Weyl Spinors

The left-chiral spinors and right-chiral spinors are together called **Weyl spinors** or **Weyl fermions** (once a tensor product with the vector space of space-time dependent wavefunctions has been performed). In Quantum Field Theory and relativistic Quantum Mechanics, these spinors describe massless spin-1/2 particles and satisfy the Weyl equation which is a simplified form of the Dirac equation.

$$\sigma^\mu \partial_\mu \chi_R = 0 \tag{40}$$

$$\bar{\sigma}^\mu \partial_\mu \chi_L = 0 \tag{41}$$

where $\sigma^\mu = (\mathbb{I}, \sigma_1, \sigma_2, \sigma_3)$ and $\bar{\sigma}^\mu = (\mathbb{I}, -\sigma_1, -\sigma_2, -\sigma_3)$ are duals to each other. As these equations violate parity individually, Weyl spinors have profound applications in the theory of weak interactions where parity is violated. These spinors also find their way into the theory of $\beta$-decay, initially proposed by Wolfgang Pauli, where they are used to describe essentially massless spin-1/2 neutrinos.

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