(1)

$$\left(\frac{\dot a}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\left[\underset{\Omega_{m,0}\,\rho_{crit,0}}{\rho_{m,0}}\left(\frac{a_0}{a}\right)^3 + \underset{\Omega_{r,0}}{\rho_{r,0}}\left(\frac{a_0}{a}\right)^4 + \underset{\Omega_{\Lambda 0}}{\rho_{\Lambda,0}}\right] - \frac{Kc^2}{a^2}$$

$K = 0$

a) $\Omega_{r0} = 1 \quad \Omega_{m0} = \Omega_\Lambda = 0$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{a_0}{a}\right)^{4}\rho_{crit\,0} \qquad a_0 \to 1$$

$$\frac{\dot a}{\cancel{a}} = \left(\frac{8\pi G\rho_{crit,0}}{3}\right)^{1/2}\frac{1}{a^{\cancel{2}}} \Rightarrow a\frac{da}{dt} = \left(\frac{8\pi G\rho_{crit\,0}}{3}\right)^{1/2}$$

$$\Rightarrow \left.\frac{a^2}{2}\right|_{a(t)}^{1} = \left(\frac{8\pi G\rho_{crit\,0}}{3}\right)^{1/2} t\Big|_t^{t_0}$$

$$\Rightarrow 1 - a(t)^2 = 2\left(\frac{8\pi G\rho_{crit\,0}}{3}\right)^{1/2}(t_0 - t)$$

$$H_0 = \left(\frac{8\pi G\rho_{crit\,0}}{3}\right)^{1/2}$$

$$\Rightarrow \boxed{a(t) = \sqrt{1 - 2H_0(t_0 - t)}}$$

b) $\Omega_{m0} = 1 \quad \Omega_{r0} = \Omega_{\Lambda 0} = 0$

$$\left(\frac{\dot a}{a}\right)^2 = \left(\frac{8\pi G\rho_{c0}}{3}\right)\frac{1}{a^3} \Rightarrow \frac{da}{dt} = \left(\frac{8\pi G\rho_{c0}}{3}\right)^{1/2}\frac{1}{a^{1/2}}$$

$$\Rightarrow \frac{2}{3}\left(1 - a^{3/2}\right) = \left(\frac{8\pi G\rho_{c0}}{3}\right)^{1/2}(t_0 - t)$$

$$\Rightarrow \boxed{a = \left(1 - \frac{3}{2} \cdot H_0 \cdot (t_0 - t)\right)^{2/3}}$$

c) $\Omega_\Lambda = 1$, $\Omega_{r0} = \Omega_{m0} = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G/3 \cdot \rho_{c0} \Rightarrow \frac{da}{a} = \left(8\pi G/3 \cdot \rho_{c0}\right)^{1/2} dt$$

$$\Rightarrow \ln\left(1/a(t)\right) = \left(8\pi G/3 \, \rho_{c0}\right)^{1/2} (t_0 - t)$$

$$\Rightarrow \boxed{a(t) = \exp\left(- H_0 (t_0 - t)\right)}$$

(2) $K = 0$ $\quad \Omega_{m0} = 1 \quad \Omega_{r0}, \Omega_{\Lambda 0} = 0$

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t)}$$

$$a(t) = \left(1 - \frac{3}{2} H_0 (t_0 - t)\right)^{2/3}$$

Big Bang $\rightarrow$ $z \rightarrow \infty$ $\quad a \rightarrow 0$

From Friedmann equation:

$$H(t)^2 = \frac{8\pi G}{3}\left[1 \cdot \left(a_0/a\right)^3\right]$$

$t = t_0$ $\quad a_0 = 1$ $\quad H = H_0$

$$H_0^2 = 8\pi G/3 \cdot \rho_{c0} \rightarrow \boxed{H_0 = \left(\frac{8\pi G \rho_{c,0}}{3}\right)^{1/2}}$$

Now, $a = 0 \Rightarrow t_0 - t_{BB} = 2/3H_0$ $\quad$ — (1)

time of BB

$$\therefore \quad 1+z = \frac{1}{a(t)} = \frac{1}{\left(1-\tfrac{3}{2}H_0(t_0-t)\right)^{2/3}}$$

$$\Rightarrow \quad 1-\tfrac{3}{2}H_0\left(t_0-t\right) = \left(\tfrac{1}{1+z}\right)^{3/2}$$

$$\Rightarrow \quad 1-\left(\tfrac{1}{1+z}\right)^{3/2} = \tfrac{3}{2}H_0\left(t_0-t\right)$$

$$\Rightarrow \quad t_0-t = \frac{2}{3H_0}\left[1-\frac{1}{(1+z)^{3/2}}\right] \quad -2$$

1 − 2

$$\Rightarrow \quad t-t_{BB} = \frac{2}{3H_0\,(1+z)^{3/2}}$$

$$\Rightarrow \quad \boxed{t_H(z) = \frac{2}{3H_0(1+z)^{3/2}}}$$

b) Lookback time : $t_H(0) - t_H(z)$

$$\boxed{t_{lb}(z) = \frac{2}{3H_0}\left[1-\frac{1}{(1+z)^{3/2}}\right]}$$

3) a) $\theta = D/d_A \quad$ size D

angle subtended by object.

Robertson-Walker metric : $dl^2 = a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$

$dr = 0 \qquad d\phi = 0$

go along transverse direction:

$$dl = a(t_e)\, r\, d\theta$$

$$D = \int a(t_e)\, r\, d\theta = a(t_e)\, r_e \int d\theta$$

Now, $1+z = \frac{a(t_0)}{a(t_e)}$ 1 at present time

$$\therefore \quad D = \frac{r_e}{1+z}\,\theta$$

$$\therefore \quad d_A = \frac{r_e}{1+z}$$

r_e Comoving distance to point

EDS universe:

Friedmann eq: $H(t)^2 = \underbrace{8\pi G/3 \,\big[\, \rho_{crit}}_{H_0} \cdot \Omega_{m,0} \left(a_0/a(t)\right)^3 \big]$

$a_0/a(t) \to 1+z$

$$\Rightarrow \quad H(z) = H_0 (1+z)^{3/2}$$

Light travelling along null geodesic: $ds = 0 \Rightarrow \frac{c\,dt}{a(t)} = \frac{dr}{(1-kr^2)^{1/2}} = dr$

Robertson-Walker

Now, $H(z) = \frac{1}{a}\, da/dt = \frac{da}{dz}\, dz/dt\, (1+z)$

$$a = \frac{1}{1+z} \Rightarrow da = -\frac{1}{(1+z)^2}\, dz$$

$$\Rightarrow H(z) = \frac{-1}{(1+z)^{\cancel{2}}} \, {}^{dz}/_{dt} \, (\cancel{1+z})$$

$$\Rightarrow dt = \frac{-dz}{H(z)(1+z)}$$

$(K=0)$

$$\Rightarrow dr = \frac{c\,dt}{a} = \frac{-c\,dz}{H(z)(\cancel{1+z})} \times (\cancel{1+z})$$

$$\therefore \int_0^{r_e} dr = \int_z^0 \frac{-c\,dz}{H(z)} \Rightarrow r_e = \int_0^z \frac{c\,dz}{H_0(1+z)^{3/2}}$$

$1+z = u$

$dz = du$

$$\int_1^{n+1} \frac{c\,du}{H_0\, u^{3/2}}$$

$$-2\,{}^{c}/_{H_0}\; u^{-1/2}\Big|_1^{n+1}$$

$$\Rightarrow r_e = \frac{-2c}{H_0} (1+z)^{-1/2}\Big|_0^z$$

$$\Rightarrow r_e = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

$$\therefore \boxed{d_A = \frac{2c}{H_0(1+z)}\left[1 - \frac{1}{\sqrt{1+z}}\right]}$$

b) $F_{obs} = \frac{L}{4\pi d_L^2}$ → total power emitted

observed flux

Photons emitted with wavelength λ_e at time intervals δt_e

observed with wave. λ_0 at time intervals δt_0

$$\frac{\lambda_e}{\lambda_0} = \frac{\delta t_e}{\delta t_0} = \frac{a_e}{a_0}$$

$(\text{as } \nu \propto 1/\lambda)$

For a single photon: $P_{em} = h\nu_e/\delta t_e$

$$P_{obs} = h\nu_0/\delta t_0 = h\nu_e/\delta t_e \cdot a_e^2/a_0^2$$

If the emitted light is at a comoving distance r_e,

It is currently at a distance of $a_0 r_e$

i.e light emitted at t_e has spread

into a sphere of radius $a_0 r_e$

& Surface area $= 4\pi a_0^2 r_e^2$

$$\therefore F_{obs} = \frac{P_{obs}}{4\pi a_0^2 r_e^2} = \frac{\overset{\to L \text{ for many photons}}{P_{em}}}{4\pi a_0^2 r_e^2} \cdot \left(a_e^2/a_0^2\right)$$

$$= \frac{L}{4\pi a_0^2 r_e^2}\left(a_e^2/a_0^2\right)$$

comparing to $\dfrac{L}{4\pi d_L^2}$

$$\& \quad d_L = a_0 r_e \cdot \left(a_0/a_e\right) = r_e(1+z) \qquad (a_0 = 1)$$

From prev. derivation,

$$\boxed{d_L = \frac{2c}{H_0}(1+z)\left[1 - \frac{1}{\sqrt{1+z}}\right]}$$

c) Friedmann eqn:

$$H(t)^2 = H_0^2 \left[\Omega_{m0} \left(\frac{a_0}{a}\right)^3 + \Omega_{r0} \left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda 0} \right]; K=0 \text{ flat Universe}$$

$$\Omega_0 = \Omega_{m0} + \Omega_{R0} + \Omega_{\Lambda 0} = 1$$

$$H(z) = H_0 \underbrace{\left[\Omega_{\Lambda 0} + \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 \right]^{1/2}}_{E(z)}$$

$$\int_{t_{BB}}^{t(z)} dt = \int_{\infty}^{z} \frac{-dz}{H(z)(1+z)}$$

Given $\Omega_{m0}, \Omega_\Lambda \rightarrow$ can get Ω_{r0}

$$\therefore \boxed{t_H(z) = \int_z^{\infty} \frac{dz}{H(z)(1+z)}}$$

$$\boxed{d_A(z) = \frac{r_e}{1+z} = \frac{1}{1+z} \int_z^0 \frac{-c\,dz}{H(z)}}$$

General expressions for $K=0$ Universe

d) $\rho(a)$

$$\rho_m = \rho_{m,0} \left(\frac{a_0}{a}\right)^3$$

$$\rho_{rad} = \rho_{rad,0} \left(\frac{a_0}{a}\right)^4$$

$$\rho_{vac} = \rho_{\Lambda,0}$$

$$z_{eq} = 3387$$

$H_0 = 67.66$ km/s/Mpc

$\Omega_{m,0} = 0.3111$

$\Omega_{\Lambda,0} = 0.6889$

$\rho_{crit,0} = 3H_0^2/8\pi G$

$\rho_{m,0} = \Omega_{m0}\, \rho_{crit,0}$

$\rho_{\Lambda,0} = \Omega_{\Lambda 0}\, \rho_{crit,0}$

$$\Omega_\Lambda(z) = \Omega_{\Lambda 0}/E^2(z), \quad \Omega_m(z) = \frac{\Omega_{m,0}(1+z)^3}{E^2(z)}, \quad \Omega_r(z) = \frac{\Omega_{r0}(1+z)^4}{E^2(z)}$$

Both equal at z: $\Omega_{m,0} = \Omega_{r,0}(1+z)$

$$z_{eq} = \frac{\Omega_{m,0}}{\Omega_{r,0}} - 1$$

$$\rightarrow \boxed{\Omega_{r0} = \Omega_{m0}/1+z}$$