

COL352 Problem Sheet 2

January 27, 2025

The problems marked as ** are relatively harder than the other problems

Problem 1. Given a language L over alphabet Σ , define the language $\text{cdr}(L) = \{y \in \Sigma^* \mid ay \in L \text{ for some } a \in \Sigma\}$. Show that the regular languages are closed under $\text{cdr}()$

Problem 2. Conclude that a star-free regex always generates a finite language.

Problem 3 (Efficiency of NFA). Let $L_k = \{x \in \{0, 1\}^* \mid |x| \geq k \text{ and the } k\text{'th character of } x \text{ from the end is a } 1\}$. Prove that every DFA that recognizes L_k has at least 2^k states. Also show that, on the other hand, there is an NFA with $k + 1$ states that recognizes L_k .

Problem 4. A coNFA is like an NFA, except it accepts an input w if and only if every possible state it could end up in when reading w is an accept state. (By contrast, an NFA accepts w iff there exists an accept state it could end up in when reading w .) Show that the class of languages recognized by coNFAs is exactly the regular languages.

Problem 5. Construct the minimal DFA D that recognizes the language

$$\{x \in \{0, 1\}^* \mid x \text{ is the binary representation of a number coprime with } 6\}.$$

Prove its minimality by giving a string $z_{q,q'}$ for each pair of distinct states q, q' such that exactly one of $\delta(q, z_{q,q'})$ and $\delta(q', z_{q,q'})$ is an accepting state of D .

Problem 6. Prove that for any infinite regular language L , there exist two infinite regular languages L_1, L_2 such that $L = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset^{**}$.

Problem 7. Construct a **minimal** DFA which accept the language $L = \{w \mid w \in \{a, b\}^* \text{ and } Na(w) \bmod 3 = Nb(w) \bmod 3\}$, where $Na(w)$ and $Nb(w)$ return the number of occurrences of a and b in w respectively.

Problem 8. Prove that the following languages are not regular.

1. $\{xx \mid x \in \{0, 1\}^*\}$
2. $\{x \in \{0, 1\}^* \mid x = \text{reverse}(x)\}$
3. $\{0^{n_1}10^{n_2}1 \cdots 0^{n_k}1 \mid k, n_1, n_2, \dots, n_k \in \mathbb{N} \cup \{0\} \text{ and } n_1, \dots, n_k \text{ are distinct}\}$
4. $\{xyx \mid x, y \in \{0, 1\}^* \text{ and } |x| > 0, |y| > 0\}$
5. $\{x \in \{0, 1\}^* \mid x \text{ is the binary representation of } 3^{n^2}, \text{ without leading } 0\text{'s, for some } n \in \mathbb{N}\}$
6. $\{0^m1^n \mid m \neq n\}$ (As a challenge, construct a clean proof using the pumping lemma only.)
7. $\{x \in \{a, b, c\}^* \mid x \text{ contains an equal number of occurrences of } ab \text{ and } ba \text{ as substrings}\}$
8. $\{x \in \{0, 1\}^* \mid x \text{ is the binary representation of } n!, \text{ without leading } 0\text{'s, for some } n \in \mathbb{N}\}$

Problem 9. Design a context free grammar for the language $\{x \in \{0, 1\}^* \mid \#0\text{'s in } x = \#1\text{'s in } x\}^{**}$.