

Lecture 7 - Completeness for the Hilbert system

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Hilbert system: Recap

$$(H1) \quad \varphi \supset (\psi \supset \varphi)$$

$$(H2) \quad (\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$$

$$(H3) \quad (\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$$

$$\frac{\varphi \supset \psi \quad \varphi}{\psi} \text{MP}$$

- We denote provability in this system with the symbol $\vdash_{\mathcal{H}}$.
- $\Gamma \vdash_{\mathcal{H}} \varphi$ denotes that there is a proof of φ in System \mathcal{H} using the expressions in Γ as assumptions
- **Theorem (Monotonicity):** If $\Gamma \vdash_{\mathcal{H}} \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_{\mathcal{H}} \varphi$.

Composing proofs (Cut)

Theorem: If $\Gamma \vdash \alpha$ and $\Gamma, \alpha \vdash \beta$, then $\Gamma \vdash \beta$.

Proof: Suppose there is a proof π of $\Gamma \vdash \alpha$ and a proof ω of $\Gamma, \alpha \vdash \beta$.
Suppose α is never “used” in ω , i.e. no sequent $\Gamma, \alpha \vdash \alpha$ in ω .
In such a case, $\Gamma \vdash \beta$ (by ω itself).
Otherwise, consider the leaves of ω labelled by $\Gamma, \alpha \vdash \alpha$.
Replace each such leaf by π . This yields a valid proof of $\Gamma \vdash \beta$.

Deduction theorem

- Recall: Logical consequence corresponded directly to implication
($\Gamma \models \varphi$ iff $(\bigwedge_{\psi \in \Gamma} \psi) \supset \varphi$ is valid)
- A similar thing exists for proofs in $\vdash_{\mathcal{H}}$; can think of the assumptions as implying the conclusion
- Useful to assume something, get results, then **discharge** assumption
- Deduction Theorem:** $\Gamma \cup \{\varphi\} \vdash_{\mathcal{H}} \chi$ iff $\Gamma \vdash_{\mathcal{H}} \varphi \supset \chi$.
- We will often use Γ, φ as notational shorthand for $\Gamma \cup \{\varphi\}$.

Deduction theorem: Proof (\Leftarrow)

Suppose $\Gamma \vdash_{\mathcal{H}} \varphi \supset \chi$ via a proof π .

Then, by monotonicity, $\Gamma, \varphi \vdash_{\mathcal{H}} \varphi \supset \chi$.

Also, $\Gamma, \varphi \vdash_{\mathcal{H}} \varphi$ since $\varphi \in \Gamma \cup \{\varphi\}$.

So we get the following proof tree:

$$\begin{array}{c} \pi + \text{Monotonicity} \\ \vdots \\ \Gamma, \varphi \vdash \varphi \supset \chi \end{array} \quad \begin{array}{c} \text{Ax} \\ \hline \Gamma, \varphi \vdash \varphi \\ \text{MP} \\ \hline \Gamma, \varphi \vdash \chi \end{array}$$

Deduction theorem: Proof (\Rightarrow)

Suppose $\Gamma, \varphi \vdash_{\mathcal{H}} \chi$ via a proof tree π .

We show (by induction on the structure of π) that $\Gamma \vdash_{\mathcal{H}} \varphi \supset \psi$ for every sequent $\Gamma, \varphi \vdash \psi$ appearing in π .

Base case: Consider a leaf of π labelled by $\Gamma, \varphi \vdash \psi$.

- $\psi = \varphi$: $\Gamma \vdash_{\mathcal{H}} \varphi \supset \varphi$ (by the earlier proof and monotonicity).
- $\psi \in \Gamma$ or ψ is an instance of **(H1)**, **(H2)**, or **(H3)**: By **(H1)**, we know that $\psi \supset (\varphi \supset \psi)$ is valid. We can now build the following proof.

$$\frac{\frac{}{\Gamma \vdash \psi} \quad \frac{}{\Gamma \vdash \psi \supset (\varphi \supset \psi)} \text{H1}}{\Gamma \vdash \varphi \supset \psi} \text{MP}$$

Deduction theorem: Proof (\Rightarrow)

Induction Hypothesis: $\Gamma \vdash_{\mathcal{H}} \varphi \supset \psi$ for every $\Gamma, \varphi \vdash \psi$ in π at height $< k$.

Induction Step: Consider a sequent of the form $\Gamma, \varphi \vdash \psi$ appearing in π at height $k \neq 0$. This must be obtained by a subproof as follows.

$$\frac{\Gamma, \varphi \vdash \xi \supset \psi \quad \Gamma, \varphi \vdash \xi}{\Gamma, \varphi \vdash \psi} \text{MP}$$

By IH, we get proofs of $\Gamma \vdash \varphi \supset (\xi \supset \psi)$ and $\Gamma \vdash \varphi \supset \xi$. We can now build the following proof, appealing to **(H2)**.

$$\frac{\frac{\Gamma \vdash (\varphi \supset (\xi \supset \psi)) \supset ((\varphi \supset \xi) \supset (\varphi \supset \psi)) \quad \text{H2} \quad \frac{\Gamma \vdash \varphi \supset (\xi \supset \psi)}{\Gamma \vdash (\varphi \supset \xi) \supset (\varphi \supset \psi)} \text{MP} \quad \frac{\Gamma \vdash \varphi \supset \xi}{\Gamma \vdash \varphi \supset \psi} \text{IH}}{\Gamma \vdash \varphi \supset \psi} \text{MP}$$

Using the Deduction Theorem

- Using the Deduction Theorem (DT) simplifies proofs a lot.
- **Example**: Show that $\alpha \supset (\alpha \supset \beta) \supset \beta$.
- Difficult if we just have **(H1)**, **(H2)**, **(H3)**, and **MP**.

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- **Example:** Show that $\alpha \supset (\alpha \supset \beta) \supset \beta$.
- Difficult if we just have **(H1)**, **(H2)**, **(H3)**, and **MP**.
- Instead, use DT. Equivalent: **Show that** $\alpha, \alpha \supset \beta \vdash_{\mathcal{H}} \beta$.
- Let $\Gamma = \{\alpha, \alpha \supset \beta\}$.

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- Let $\Gamma = \{\alpha, \alpha \supset \beta\}$.

$$\frac{\frac{}{\Gamma \vdash \alpha \supset \beta} \text{Ax} \quad \frac{}{\Gamma \vdash \alpha} \text{Ax}}{\Gamma \vdash \beta} \text{MP}$$

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- Proof rule **DT** to switch between equivalent formulations.

Using the Deduction Theorem

Show that $\neg\alpha \supset \beta, \alpha \supset \beta \vdash_{\mathcal{H}} \beta$.

- Suppose we had proofs of $\neg\beta \supset \neg\alpha$ and $\neg\beta \supset \alpha$
- Can get β from these using **(H3)** and applications of **MP**.

Using the Deduction Theorem

Show that $\neg\alpha \supset \beta, \alpha \supset \beta \vdash_{\mathcal{H}} \beta$.

- Suppose we had proofs of $\neg\beta \supset \neg\alpha$ and $\neg\beta \supset \alpha$
- Can get β from these using **(H3)** and applications of **MP**.

$$\begin{array}{c}
 \frac{\Gamma \vdash (\neg\beta \supset \neg\alpha) \supset ((\neg\beta \supset \alpha) \supset \beta)}{\Gamma \vdash (\neg\beta \supset \alpha) \supset \beta} \text{H3} \qquad \begin{array}{c} \text{???} \\ \vdots \\ \Gamma \vdash \neg\beta \supset \neg\alpha \end{array} \qquad \begin{array}{c} \text{???} \\ \vdots \\ \Gamma \vdash \neg\beta \supset \alpha \end{array} \\
 \hline
 \Gamma \vdash \beta \qquad \text{MP} \qquad \text{MP}
 \end{array}$$

- Suppose we show that $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$ for any φ and ψ .
- Then $\Gamma \vdash \neg\beta \supset \neg\alpha$ and $\Gamma \vdash \neg\beta \supset \neg\neg\alpha$.
- Suppose we also show that $\neg\neg\varphi \vdash \varphi$ for any φ . Then done!

Using the Deduction Theorem

- Show that $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$ for any φ and ψ .
- Equivalent to showing that $\varphi \supset \psi, \neg\psi \vdash \neg\varphi$. Let $\Gamma = \{\varphi \supset \psi, \neg\psi\}$.
- Suppose we proved $\neg\neg\varphi \supset \psi$ and $\neg\neg\varphi \supset \neg\psi$. Then use **(H3)** and **MP**.
- To show $\Gamma \vdash \neg\neg\varphi \supset \neg\psi$, equivalent: $\Gamma, \neg\neg\varphi \vdash \neg\psi$; use **Ax**.
- **Exercise:** Show that if $\Gamma, \varphi \vdash \psi$, then $\Gamma, \neg\neg\varphi \vdash \psi$ for all φ, ψ .

Using the Deduction Theorem

- Show that $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$ for any φ and ψ .
- Equivalent to showing that $\varphi \supset \psi, \neg\psi \vdash \neg\varphi$. Let $\Gamma = \{\varphi \supset \psi, \neg\psi\}$.
- Suppose we proved $\neg\neg\varphi \supset \psi$ and $\neg\neg\varphi \supset \neg\psi$. Then use **(H3)** and **MP**.
- To show $\Gamma \vdash \neg\neg\varphi \supset \neg\psi$, equivalent: $\Gamma, \neg\neg\varphi \vdash \neg\psi$; use **Ax**.
- **Exercise:** Show that if $\Gamma, \varphi \vdash \psi$, then $\Gamma, \neg\neg\varphi \vdash \psi$ for all φ, ψ .

		_____ Ax
		$\Gamma \vdash \varphi \supset \psi$
		_____ DT
		$\Gamma, \varphi \vdash \psi$
		_____ Exercise!
		$\Gamma, \neg\neg\varphi \vdash \psi$
		_____ DT
		$\Gamma \vdash \neg\neg\varphi \supset \psi$
		_____ MP
		$\Gamma \vdash \neg\varphi$
_____ H3	_____ Ax	
$\Gamma \vdash (\neg\neg\varphi \supset \neg\psi) \supset (\neg\neg\varphi \supset \psi) \supset \neg\varphi$	$\Gamma, \neg\neg\varphi \vdash \neg\psi$	
	_____ DT	
	$\Gamma \vdash (\neg\neg\varphi \supset \neg\psi)$	
	_____ MP	
	$\Gamma \vdash (\neg\neg\varphi \supset \psi) \supset \neg\varphi$	

Maximal sets

- So far, proofs of an expression from some context (of the form $X \vdash_{\mathcal{H}} \psi$)
- What do we know if there is **no** proof of a ψ from some X ?
- There is a “largest possible extension of X ” which **does not** prove ψ , any “extension” of which **does** prove ψ .
- **Theorem:** If $X \not\vdash_{\mathcal{H}} \psi$, then there is a maximal Y s.t. $X \subseteq Y$ and $Y \not\vdash_{\mathcal{H}} \psi$.
- **Proof:** Consider some X and ψ such that $X \not\vdash_{\mathcal{H}} \psi$.
- **PL** is countable, assume a fixed enumeration of expressions $\varphi_0, \varphi_1, \dots$

Maximal sets: Proof

- Basic idea: Examine each expression in **PL**, choose whether to throw it in or not (depending on whether it derives ψ or not).
- Build a sequence of sets where $X_0 := X$ and each X_{i+1} defined as below.

$$X_{i+1} := \begin{cases} X_i & \text{if } X_i, \varphi_i \vdash_{\mathcal{H}} \psi \\ X_i \cup \{\varphi_i\} & \text{otherwise} \end{cases} \quad \text{Set } Y := \bigcup_{i \geq 0} X_i.$$

- For each expression φ_i , it either goes in at the X_{i+1} stage (and therefore into Y) or not at all.
- Y is a countable union of sets X_i .
- We will overload notation and use $Y \vdash_{\mathcal{H}} \alpha$ (for any α), even though Y is countable, to denote a proof of α from some finite subset of Y .
- $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots \subseteq Y$, and $X_i \not\vdash_{\mathcal{H}} \psi$ for every $i \geq 0$.

Maximal sets: Proof (contd.)

- **First, show $Y \not\vdash_{\mathcal{H}} \psi$ by contradiction.** Suppose π is a proof of $Y \vdash \psi$.
- π is a finite tree, can only use finitely many assumptions (at the leaves).
- Consider any assumption of the form $Y \vdash \varphi_i$. Then, $\varphi_i \in Y$ (since this sequent was a leaf and φ_i is an assumption).
- Suppose the largest index of any such assumption in π is k .
- Since $\varphi_i \in Y$, it must be that $\varphi_i \in X_j$ for some j .
- Either $\varphi_i \in X$ and $j = 0$, or $j = i + 1$ since φ_i is, at the very latest, examined when we construct X_{i+1} .
- $X_{i+1} \subseteq X_{k+1}$, and since we chose i arbitrarily, **every** assumption in π belongs to X_{k+1} . Therefore, $X_{k+1} \vdash_{\mathcal{H}} \psi$.
- This contradicts our claim that $X_i \not\vdash_{\mathcal{H}} \psi$ for every $i \geq 0$, so $Y \not\vdash_{\mathcal{H}} \psi$.

Maximal sets: Proof (contd.)

- **Now show that Y is a maximal non-deriving context.** Consider any $\varphi_\ell \notin Y$.
- Then $\varphi_\ell \notin X_{\ell+1}$, which can only happen if $X_\ell, \varphi_\ell \vdash_{\mathcal{H}} \psi$.
- Since $X_\ell \subseteq Y$, by monotonicity, $Y, \varphi_\ell \vdash_{\mathcal{H}} \psi$.
- So for any Z s.t. $Y \subset Z$, it is the case that $Z \vdash_{\mathcal{H}} \psi$, i.e. Y is maximal.

Hilbert system: Completeness

- Usually we are only interested in proof search
- So why worry about contexts that **do not** prove an expression?
- To show that the Hilbert system is **complete**
- “Any valid expression in propositional logic is provable in this system”
- **Theorem (Completeness)**: For any PL expression φ , if $\models \varphi$, then $\vdash_{\mathcal{H}} \varphi$.
- We will prove by contradiction a stronger claim, unimaginatively called
- **Thm (Strong Completeness)**: For all $X \cup \{\varphi\}$, if $X \models \varphi$ then $X \vdash_{\mathcal{H}} \varphi$.

Hilbert system: Completeness (Intuition)

- Having assumed $X \not\vdash_{\mathcal{H}} \varphi$, we will aim to show that $X \not\models \varphi$.
- There is some valuation τ such that $\tau \models \psi$ for every $\psi \in X$, but $\tau \not\models \varphi$.
- How do we demonstrate such a valuation?
- Unclear how to build such a τ directly
- Instead, we build the largest possible set EP of expressions for which τ is a model, and extract τ from this.
- What properties should such an EP satisfy?

Hilbert system: Completeness (Intuition)

- We want $\psi \in EP$ iff $\tau \models \psi$ for any $\psi \in PL$.
- Since $\tau \models X$ and $\tau \not\models \phi$, we want $X \subseteq EP$, but $\phi \notin EP$.
- EP should “agree” with τ
 - $\neg\psi \in EP$ iff $\psi \notin EP$
 - $\psi \supset \chi \in EP$ iff either $\psi \notin EP$ or $\chi \in EP$
- For each $\psi \in PL$, we need to either add it or its negation to EP
- Need to do this in a systematic manner.
 - What if I throw in $\neg\alpha$ and $\neg\beta$ but then add $\alpha \supset \beta$?
 - No valuation is a model for this set of expressions.
- We set up the notion of maximal non-deriving contexts just for this!
- We drop the \mathcal{H} subscript in the proof that follows, as it is implicit.

Hilbert system: Proof of strong completeness

- We show that if $X \not\models \varphi$, then $X \not\models \varphi$
- There is a maximal Y s.t. $Y \not\models \varphi$. So for every $\psi \notin Y$, $Y, \psi \vdash \varphi$.
- If $Y \vdash \psi$, by Cut, we would get $Y \vdash \varphi$, so we get $Y \vdash \psi$ iff $\psi \in Y$.
- We define a valuation τ as follows

$$\tau(p) = \begin{cases} T & \text{if } p \in Y \cap AP, \\ F & \text{otherwise} \end{cases}$$

- Suppose we show that $\psi \in Y$ iff $\tau \models \psi$ for all $\psi \in PL$.
- Then, since $X \subseteq Y$, $\tau \models X$. Also, since $Y \not\models \varphi$, $\varphi \notin Y$, and $\tau \not\models \varphi$.
- So $X \not\models \varphi$, and we have the required contradiction.

Hilbert system: Proof of strong completeness (contd.)

- So now we show that $\psi \in Y$ iff $\tau \models \psi$ for all $\psi \in \text{PL}$.
- As usual, by induction on the structure of ψ .
- $\psi = p \in \text{AP}$: By definition of τ , $p \in Y$ iff $\tau \models p$
- $\psi = \neg\chi$: First show that $\neg\chi \in Y$ iff $\chi \notin Y$ for any χ .
 - Suppose $\{\neg\chi, \chi\} \subseteq Y$.
 - **Exercise**: Show that $\neg\chi, \chi \vdash \alpha$ in \mathcal{H} for any χ, α .
 - So if $\{\neg\chi, \chi\} \subseteq Y$, then $Y \vdash \varphi$ (contradiction)
 - Suppose $\{\neg\chi, \chi\} \cap Y = \emptyset$.
 - By maximality, $Y, \neg\chi \vdash \varphi$ and $Y, \chi \vdash \varphi$.
 - But as we saw earlier, $\neg\alpha \supset \beta, \alpha \supset \beta \vdash \beta$, so $Y \vdash \varphi$ (contradiction)
- So $\neg\chi \in Y$ iff $\chi \notin Y$ iff (by IH) $\tau \not\models \chi$ iff $\tau \models \neg\chi$.

Hilbert system: Proof of strong completeness (contd.)

- We show that $\psi \in Y$ iff $\tau \models \psi$ for all $\psi \in \text{PL}$ (contd.)
- Recall that $Y \vdash \psi$ iff $\psi \in Y$ for any ψ .
- $\psi = \chi \supset \xi$: First show that $\chi \supset \xi \in Y$ iff $\chi \notin Y$ or $\xi \in Y$.
- First we show that if $\chi \supset \xi \in Y$, then $\chi \notin Y$ or $\xi \in Y$.
 - Suppose $\chi \supset \xi \in Y$. Then $Y \vdash \chi \supset \xi$.
 - Either $\neg\chi \in Y$ or $\chi \in Y$ (by maximality)
 - Now, if $\chi \in Y$, then $Y \vdash \chi$, and by **MP**, we get $Y \vdash \xi$, i.e. $\xi \in Y$.
 - Therefore, $\neg\chi \in Y$, or $\xi \in Y$.

Hilbert system: Proof of strong completeness (contd.)

- We show that $\psi \in Y$ iff $\tau \models \psi$ for all $\psi \in PL$ (contd.)
- $\psi = \chi \supset \xi$: First show that $\chi \supset \xi \in Y$ iff $\chi \notin Y$ or $\xi \in Y$.
- Now we show that if $\chi \notin Y$ or $\xi \in Y$ then $\chi \supset \xi \in Y$.
- Suppose $\chi \notin Y$.
 - Then $\neg\chi \in Y$ and $Y \vdash \neg\chi$. By monotonicity, $Y, \chi \vdash \neg\chi$.
 - Since $Y, \chi \vdash \chi$, we can compose these proofs, and get $Y, \chi \vdash \alpha$ for any α .
 - In particular, $Y, \chi \vdash \xi$, and **DT** gives us $Y \vdash \chi \supset \xi$, i.e. $\chi \supset \xi \in Y$.
- Suppose $\xi \in Y$. Then $Y \vdash \xi$, and monotonicity gives us $Y, \chi \vdash \xi$. Using **DT**, we get $Y \vdash \chi \supset \xi$, i.e. $\chi \supset \xi \in Y$.
- So $\chi \supset \xi \in Y$ iff $\chi \notin Y$ or $\xi \in Y$ iff (by IH) $\tau \not\models \chi$ or $\tau \models \xi$ iff $\tau \models \chi \supset \xi$.