

Lecture 16 - FO Completeness

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Recap: FO Resolution

- **Substitution Lemma:** Given an interpretation $\mathcal{I} = ((M, \iota), \sigma)$, an expression $\varphi \in \text{FO}_{\Sigma}$, and a substitution $\{u/x\}$ such that $u^{\mathcal{I}} = m \in M$, $\mathcal{I} \models \varphi\{u/x\}$ iff $\mathcal{I}[x \mapsto m] \models \varphi$.
- Let δ_1, δ_2 be clauses s.t. $\text{fv}(\delta_1) \cap \text{fv}(\delta_2) = \emptyset$
- Let $P \in \mathcal{P}$ be a k -ary predicate symbol
- Let $L_1 = \{P(u_1, \dots, u_k) \in \delta_1 \mid u_1, \dots, u_k \in T(\Sigma)\}$ such that $\delta_1 = \delta'_1 \cup L_1$
- Let $L_2 = \{\neg P(v_1, \dots, v_k) \in \delta_2 \mid v_1, \dots, v_k \in T(\Sigma)\}$ such that $\delta_2 = \delta'_2 \cup L_2$
- Denote by \bar{L}_2 the set $\{P(v_1, \dots, v_k) \in \delta_2 \mid v_1, \dots, v_k \in T(\Sigma)\}$
- Let $L_1 \cup \bar{L}_2$ be unifiable, with θ an mgu

$$\frac{\delta'_1 \cup L_1 \quad \delta'_2 \cup L_2}{\theta(\delta'_1 \cup \delta'_2)} \theta$$

Towards a proof system

- The resolution procedure (linking unsatisfiability to the derivation of an empty clause) is sound and complete
- However, this rule does not provide a complete proof system for first-order logic (**Exercise**: Think about why!)
- Move to a less minimal proof system (which might be complete)
- Can extend $\vdash_{\mathcal{H}}$ to get \vdash_{HK} for FOL
- What rules do we keep? What do we add?

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- Can extend $\vdash_{\mathcal{H}}$ to get \vdash_{HK} for FOL
- What rules do we keep? What do we add?
- Keep all the existing axioms and **MP**
- All FO expression instances of **PL** tautologies are valid
- A **generalization** of an expression φ is any $\forall x_1 \dots x_n. \varphi$, where $n \geq 0$.

System \vdash_{HK} for FO

All generalizations of the following, along with **MP**.

$$\text{(H1a)} \quad \varphi \supset (\psi \supset \varphi)$$

$$\text{(H1b)} \quad (\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$$

$$\text{(H1c)} \quad (\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$$

$$\text{(H2a)} \quad x \equiv x$$

$$\text{(H2b)} \quad x \equiv y \supset ((\varphi(x) \supset \varphi(y)) \wedge (\varphi(y) \supset \varphi(x)))$$

$$\text{(H3a)} \quad \forall x. [\varphi \supset \psi] \supset (\forall x. [\varphi] \supset \forall x. [\psi])$$

$$\text{(H3b)} \quad \varphi \supset \forall x. [\varphi] \quad \text{where } x \text{ does not appear free in } \varphi$$

$$\text{(H3c)} \quad \forall x. [\varphi] \supset \varphi\{t/x\} \quad \text{for any term } t$$

$$\frac{\varphi \supset \psi \quad \varphi}{\psi} \text{MP}$$

We denote provability in this system with the symbol \vdash_{HK} .

Soundness of \vdash_{HK}

- **Theorem (Soundness):** If $\vdash_{HK} \varphi$, then $\models \varphi$
- Show that axioms are valid, and that **MP** preserves validity
- Might need the following lemma: For every φ , every $x \in \mathcal{V}$, and every $y \notin \text{vars}(\forall x. [\varphi])$,
 $\models (\forall x. [\varphi] \supset \forall y. [\varphi\{y/x\}]) \wedge (\forall y. [\varphi\{y/x\}] \supset \forall x. [\varphi])$.
- For all Γ, α, β , we have the following
- **Deduction Theorem:** $\Gamma, \alpha \vdash_{HK} \beta$ iff $\Gamma \vdash_{HK} \alpha \supset \beta$.
- **Cut** is admissible: If $\Gamma \vdash \alpha$ and $\Gamma, \alpha \vdash \beta$, then $\Gamma \vdash \beta$.
- **Lemma (Replacement by new variables):** Suppose $\Gamma \vdash_{HK} \varphi$, and $y \in \mathcal{V} \setminus (\text{vars}(\Gamma) \cup \text{vars}(\varphi))$. Then, $\Gamma\{y/x\} \vdash_{HK} \varphi\{y/x\}$.
- **Exercise:** Prove all these statements.

Substituting bound variables: Equivalence

Lemma: For every φ , every $x \in \mathcal{V}$, and every $y \notin \text{vars}(\forall x. [\varphi])$,

$$\vdash_{HK} (\forall x. [\varphi] \supset \forall y. [\varphi\{y/x\}]) \wedge (\forall y. [\varphi\{y/x\}] \supset \forall x. [\varphi]).$$

Proof: Enough to show (\Rightarrow) , i.e. $\vdash_{HK} \forall x. [\varphi] \supset \forall y. [\varphi\{y/x\}]$.

(\Leftarrow) follows since $x \notin \text{vars}(\forall y. [\varphi\{y/x\}])$, and $\varphi\{y/x\}\{x/y\} = \varphi$.

$\forall y. [\forall x. [\varphi] \supset \varphi\{y/x\}] \supset (\forall y. [\forall x. [\varphi]] \supset \forall y. [\varphi\{y/x\}])$	H3a	$\forall y. [\forall x. [\varphi] \supset \varphi\{y/x\}]$	H3c+G	π
$\forall y. [\forall x. [\varphi]] \supset \forall y. [\varphi\{y/x\}]$		$\forall y. [\forall x. [\varphi]] \supset \forall y. [\varphi\{y/x\}]$		

$\forall x. [\varphi] \supset \forall y. [\forall x. [\varphi]]$	H3b	$\forall y. [\forall x. [\varphi]] \supset \forall y. [\varphi\{y/x\}]$	π \vdots	Cut
$\forall x. [\varphi] \supset \forall y. [\varphi\{y/x\}]$				

Universal generalization lemma

Lemma (Universal generalization): Suppose $\Gamma \vdash_{HK} \varphi\{y/x\}$, where $y \notin \text{fv}(\Gamma) \cup \text{fv}(\varphi)$. Then, $\Gamma \vdash_{HK} \forall x. [\varphi]$.

Proof: Suppose $\Gamma \vdash_{HK} \varphi\{y/x\}$ via a proof π . We first show the following:

For any sequent $\Gamma \vdash_{HK} \alpha_i$ appearing in the proof π , $\Gamma \vdash_{HK} \forall y. [\alpha_i]$.

(Then, $\Gamma \vdash_{HK} \forall y. [\varphi\{y/x\}]$, and by the previous lemma, $\Gamma \vdash_{HK} \forall x. [\varphi]$.)

The proof is by induction on the structure of π .

Base case(s): Suppose α_i is an instance of an axiom. Then, $\forall x. [\alpha_i]$ is a generalization of an axiom, and hence, also an axiom. Otherwise, suppose $\alpha_i \in \Gamma$. Then, $y \notin \text{fv}(\alpha_i)$. Thus, **(H3b)** gives us $\alpha_i \supset \forall y. [\alpha_i]$.

$$\frac{\frac{}{\Gamma \vdash \alpha_i \supset \forall y. [\alpha_i]} \text{H3b} \quad \frac{}{\Gamma \vdash \alpha_i} \text{Ax}}{\Gamma \vdash \forall y. [\alpha_i]} \text{MP}$$

Universal generalization lemma: Proof

Induction case: α_i is obtained by applying **MP** to some $\alpha_j \supset \alpha_i$ and α_j , both appearing in shorter subtrees. By IH, $\Gamma \vdash \forall y. [\alpha_j \supset \alpha_i]$ and $\Gamma \vdash \forall y. [\alpha_j]$.

$$\begin{array}{c}
 \frac{\Gamma \vdash \forall y. [\alpha_j \supset \alpha_i] \supset (\forall y. [\alpha_j] \supset \forall y. [\alpha_i])}{\Gamma \vdash \forall y. [\alpha_j] \supset \forall y. [\alpha_i]} \text{H3a} \qquad \frac{\Gamma \vdash \forall y. [\alpha_j \supset \alpha_i] \quad \Gamma \vdash \forall y. [\alpha_j]}{\Gamma \vdash \forall y. [\alpha_i]} \text{MP} \\
 \text{IH} \qquad \qquad \qquad \text{IH} \\
 \vdots \qquad \qquad \qquad \vdots
 \end{array}$$

Completeness of \vdash_{HK}

- **Gödel's Completeness Theorem (1929):** If $\Gamma \models \varphi$, then $\Gamma \vdash_{HK} \varphi$
- Want a slightly different, equivalent formulation of this statement
- Introduce a notion of **consistency**
- An expression φ is said to be **consistent** if $\not\vdash_{HK} \neg\varphi$
- A finite set $\{\varphi_1, \dots, \varphi_n\}$ is consistent if $\bigwedge_{1 \leq i \leq n} \varphi_i$ is consistent
- An arbitrary set Γ is consistent if each of its finite subsets is consistent.
- Equivalent statement: *Any consistent set of expressions is satisfiable*
- **Exercise:** Show that this is equivalent to the Completeness statement.

Completeness of \vdash_{HK}

- Suppose we start out with a consistent set of expressions Γ
- The proof becomes easier if we can assume $\mathcal{V} \setminus \text{vars}(\Gamma)$ to be infinite.
- We achieve this as follows. Let $\mathcal{V} = \{x_0, x_1, x_2, \dots\}$
- Partition this set into $\mathcal{V}_e = \{x_0, x_2, x_4, \dots\}$ and $\mathcal{V}_o = \{x_1, x_3, x_5, \dots\}$
- Given a Γ , form Δ by systematically replacing each occurrence (free or bound) of x_i in Γ by x_{2i} for all $i \geq 0$.
- $\text{vars}(\Delta) \subseteq \mathcal{V}_e$, so $\mathcal{V} \setminus \text{vars}(\Delta)$ is infinite.
- We now need to prove the following:
 - If Γ is consistent, then Δ is consistent
 - If Δ is satisfiable, then Γ is satisfiable
- Once we prove these, we can assume $\mathcal{V} \setminus \text{vars}(\Gamma)$ to be infinite in the rest of the presentation.

Γ consistent $\Rightarrow \Delta$ consistent

- Proof by contradiction. Suppose Δ is inconsistent.
- Then, there is a $\{\delta_1, \dots, \delta_k\} \subseteq_{\text{fin}} \Delta$ such that $\vdash_{HK} \neg(\delta_1 \wedge \dots \wedge \delta_k)$
- Let n be such that $i < 2n$ for every i where $x_i \in \text{fv}(\bigcup_{1 \leq j \leq k} \delta_j)$.
- Replace every $x_{2j} \in \text{vars}(\bigcup_{1 \leq j \leq k} \delta_j)$ by x_{2n+j} to get $\{\rho_1, \dots, \rho_k\}$
- **Claim:** $\vdash_{HK} \neg(\rho_1 \wedge \dots \wedge \rho_k)$ **Exercise:** Prove this claim.
- Replace every x_{2n+j} by x_j to get $\{\gamma_1, \dots, \gamma_k\} \subseteq_{\text{fin}} \Gamma$
- $\vdash_{HK} \neg(\gamma_1 \wedge \dots \wedge \gamma_k)$
- Thus, Γ is inconsistent.

Δ satisfiable $\Rightarrow \Gamma$ satisfiable

- Suppose $(\mathcal{M}, \sigma) \models \Delta$.
- Only variables from \mathcal{V}_e appear in Δ
- We replace every occurrence of x_{2i} by x_i to get Γ
- $(\mathcal{M}, \sigma') \models \Gamma$, where $\sigma'(x_i) = \sigma(x_{2i})$
- Thus, if Δ is satisfiable, then so is Γ

Lindenbaum's Lemma

- A set Γ is maximally consistent if Γ is consistent, and $\Gamma \cup \{\varphi\}$ is inconsistent for any FO expression $\varphi \notin \Gamma$.
- A set Γ is said to be \exists -fulfilled iff for every expression of the form $\neg\forall x. [\alpha] \in \Gamma$, there exists some term t such that $\neg\alpha\{t/x\} \in \Gamma$.
- **Lindenbaum's Lemma:** Every consistent set can be extended to an \exists -fulfilled MCS.
- Given a consistent Γ , we build an \exists -fulfilled MCS which extends Γ .
- As earlier, fix an enumeration of expressions, and examine each.

Lindenbaum's Lemma: Proof

- Fix an enumeration $\varphi_0, \varphi_1, \varphi_2, \dots$ of the expressions in \mathbf{FO}_Σ
- Also fix an enumeration x_0, x_1, x_2, \dots of the variables in \mathcal{V}
- Now, we build the following sequence $\Gamma_0, \Gamma_1, \dots$ of sets of formulas.
- $\Gamma_0 := \Gamma$, and for every $i \geq 0$,

$$\Gamma_{i+1} := \begin{cases} \Gamma'_i & \text{if } \Gamma'_i \text{ consistent and } \varphi_i \text{ not of the form } \neg\forall x. [\alpha] \\ \Gamma'_i \cup \{\neg\alpha\{y/x\}\} & \text{if } \Gamma'_i \text{ consistent, } \varphi_i = \neg\forall x. [\alpha], \text{ and} \\ & y \text{ the first variable not in } \mathbf{fv}(\Gamma_i) \cup \mathbf{vars}(\varphi_i)^1 \\ \Gamma_i & \text{if } \Gamma'_i \text{ not consistent} \end{cases}$$

where $\Gamma'_i = \Gamma_i \cup \{\varphi_i\}$.

- Finally, $\Gamma_{\text{ext}} := \bigcup_{i \geq 0} \Gamma_i$

¹We can get away with only requiring that y is the first variable not in $\mathbf{fv}(\Gamma_i) \cup \mathbf{fv}(\alpha)$ as long as we somehow ensure that $y \notin \mathbf{bv}(\alpha)$

Lindenbaum's Lemma: Proof

- **Claim:** Γ_{ext} is maximally consistent and \exists -fulfilled.
- We first show that each Γ_i is consistent (by induction on i)
- **Base case:** $\Gamma_0 = \Gamma$, consistent by assumption.
- **Induction step:** Suppose Γ_i is consistent. Two cases arise: Either $\Gamma'_i = \Gamma_i \cup \{\varphi_i\}$ is consistent or not.
- In the latter case, $\Gamma_{i+1} = \Gamma_i$, and Γ_{i+1} is also consistent.
- If $\Gamma'_i = \Gamma_i \cup \{\varphi_i\}$ is consistent, and if φ_i is not of the form $\neg\forall x. [\alpha]$, then $\Gamma_{i+1} = \Gamma'_i$, so consistent by construction.

Lindenbaum's Lemma: Proof

- If $\varphi_i = \neg\forall x. [\alpha]$ for some α , and $\Gamma_i \cup \{\neg\forall x. [\alpha]\}$ is consistent, we set $\Gamma_{i+1} = \Gamma_i \cup \{\neg\forall x. [\alpha], \neg\alpha\{y/x\}\}$, where y is the first variable not in $\text{fv}(\Gamma_i) \cup \text{vars}(\varphi_i)$
- Suppose towards a contradiction that Γ_{i+1} is not consistent
- There is $\{\gamma_1, \dots, \gamma_k\} \subseteq_{\text{fin}} \Gamma_i$ such that $\neg\forall x. [\alpha], \gamma_1, \dots, \gamma_k \vdash \alpha\{y/x\}$. **Why?**
- Since $y \notin \text{fv}(\Gamma_i) \cup \text{vars}(\varphi_i)$, we can use Universal Generalization to get $\neg\forall x. [\alpha], \gamma_1, \dots, \gamma_k \vdash \forall x. [\alpha]$.
- One can avoid using $\neg\varphi$ as an assumption to prove φ for any φ . So $\gamma_1, \dots, \gamma_k \vdash \forall x. [\alpha]$
- But this contradicts the fact that $\Gamma_i \cup \{\neg\forall x. [\alpha]\}$ is consistent!
- So Γ_{i+1} is consistent for every i .

Lindenbaum's Lemma: Proof

- Γ_{ext} is consistent, since each finite subset of Γ_{ext} is also a finite subset of Γ_i for some $i \geq 0$. **Exercise:** Why only one Γ_i and not multiple?
- For every φ_ℓ such that $\Gamma_{\text{ext}} \cup \{\varphi_\ell\}$ is consistent, $\Gamma_\ell \cup \varphi_\ell$ is also consistent (reasoning as above), so $\varphi_\ell \in \Gamma_{\ell+1} \subseteq \Gamma_{\text{ext}}$. Therefore, Γ_{ext} is maximally consistent.
- Consider $\varphi_\ell = \neg \forall x. [\alpha] \in \Gamma_{\text{ext}}$. Note that $\Gamma_\ell \cup \{\varphi_\ell\}$ is consistent (as above). So $\neg \alpha\{y/x\} \in \Gamma_{\ell+1} \subseteq \Gamma_{\text{ext}}$ for some y , by construction. Therefore, Γ_{ext} is also \exists -fulfilled.
- Thus, we have shown that every consistent set Γ can be extended to an \exists -fulfilled MCS Γ_{ext} .