Lecture 8 - Propositional logic: Wrap up

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Hilbert system: Recap

$$
\begin{array}{ll}
\textbf{(H1)} & \varphi \supset (\psi \supset \varphi) \\
\textbf{(H2)} & (\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi)) \\
\textbf{(H3)} & (\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)
\end{array}
$$

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\ \mathrm{MP}$$

- **Theorem (Monotonicity)**: If $\Gamma \vdash_{\mathcal{H}} \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_{\mathcal{H}} \varphi$.
- **Theorem (Composing Proofs/Cut)**: If $\Gamma \vdash \alpha$ and $\Gamma, \alpha \vdash \beta$, then $\Gamma \vdash \beta$.
- **Deduction Theorem**: $\Gamma \cup \{\varphi\} \vdash_{\mathcal{H}} \chi$ iff $\Gamma \vdash_{\mathcal{H}} \varphi \supset \chi$.
- The Deduction Theorem allows us to prove things more easily than just using H1, H2, H3, and MP, so we use it as a proof rule DT.

Hilbert system: Recap

- The Hilbert system is **complete** for propositional logic.
- Show proof of a stronger statement, by contradiction.
- Assume that $X \nvdash_{\mathcal{H}} \varphi$ for some X and φ. Then show that $X \nvDash \varphi$.
- Need to build τ such that $\tau \vDash \psi$ for every $\psi \in X$, but $\tau \nvDash \varphi$.
- Strategy:
    - Build a (countable) set EP of expressions for which τ is a model
    - Expand this set as much as possible, while ensuring it does not derive φ
    - Extract τ from this set somehow; that's the required valuation!

Hilbert system: Recap

- *EP* should be as large as possible (**maximal**) and include X, but not φ.
- Systematically examine each PL expression, then add it to *EP* or not.
- PL is countable; assume an enumeration of expressions $\varphi_0, \varphi_1, \ldots$
- Examine each expression according to this sequence.
- When does an expression make it into *EP*?
- All of X goes into *EP*. φ does not.
- We should be able to construct a valuation τ s.t. $\tau \models EP$. So,
  - $\neg\psi \in EP$ iff $\psi \notin EP$
  - $\psi \supset \chi \in EP$ iff either $\psi \notin EP$ or $\chi \in EP$
- What about atomic propositions?

Hilbert system: Recap

- We know that $X \subseteq EP$ and $\varphi \notin EP$.
- Suppose $EP \vdash \varphi$. By soundness, $EP \vDash \varphi$. So if $\tau \vDash EP$, then $\tau \vDash \varphi$. **Bad!**
- Throw in an arbitrary formula as long as its addition does not prove φ.
- Build a sequence of sets $X_0, X_1, \ldots$ starting from $X_0 = X$.
- At the i^{th} step, examine φ_i. Check if $X_i, \varphi_i \vdash_{\mathcal{H}} \varphi$.
- If not, add φ_i to X_i to get $X_{i+1} = X_i \cup \{\varphi_i\}$.
- Otherwise $X_{i+1} = X_i$.
- Move on to the next index $i + 1$, and repeat.
- $EP = \bigcup_{i \geqslant 0} X_i$

Hilbert system: Recap

- If ψ omitted from *EP*, it is because $EP \cup \{\psi\} \vdash_{\mathscr{H}} \varphi$; so *EP* **maximal**
- *EP* does not derive φ
- Construct τ s.t. it assigns all atoms in *EP* to *T*, and all other atoms to *F*.
- Have to show that *EP* "agrees" with τ (as stated earlier).
- Have to show that $\tau \vDash \psi$ for all $\psi \in EP$.
- Showing these two statements concludes the proof!

An interesting fallout of our proof

- **Theorem (C1)**: $X \vDash \varphi$ iff $X_0 \vDash \varphi$ for some $X_0 \subset_{\text{fin}} X$.
- **Proof**: Easy if X is finite. Consider an infinite set X and an expression φ such that $X \cup \{\varphi\} \subseteq \mathsf{PL}$.
- If $X \vDash \varphi$, then (by strong completeness), $X \vdash_{\mathcal{H}} \varphi$.
- Since proofs are finite, $X_0 \vdash_{\mathcal{H}} \varphi$ for some $X_0 \subset_{\text{fin}} X$.
- By soundness, $X_0 \vDash \varphi$.
- Thus, if $X \vDash \varphi$, then $X_0 \vDash \varphi$ for some $X_0 \subset_{\text{fin}} X$.
- The other direction holds by monotonicity.

The Compactness Theorem

- The above is one way of formulating the **Compactness Theorem**.
- The more traditional form is as follows.
- **Theorem (C2)**: A set is satisfiable iff all its finite subsets are satisfiable.
- Often used in the contrapositive form
- "If a set of expressions is unsatisfiable, then some finite subset is unsatisfiable."
- We show that C1 is equivalent to C2.

The Compactness Theorem (Proof of C2 using C1)

Theorem (C2): A set is satisfiable iff all its finite subsets are satisfiable.

Proof for the $(\Rightarrow)$ direction is easy.

Proof $(\Leftarrow)$: Immediate if X is finite.

Consider an infinite set X whose all finite subsets are satisfiable. Assume, towards a contradiction, that X is not satisfiable.

Thus, any τ such that $\tau \models X$ will also be such that $\tau \models p \wedge \neg p$ for some $p \in AP$.

Thus, $X \models p \wedge \neg p$.

Using Theorem C1, there is an $X_0 \subset_{\text{fin}} X$ such that $X_0 \models p \wedge \neg p$.

This contradicts our assumption that all finite subsets of X are satisfiable.

The Compactness Theorem (Proof of C1 using C2)

Theorem (C1): $X \vDash \varphi$ iff $X_0 \vDash \varphi$ for some $X_0 \subset_{\text{fin}} X$.

Proof: Immediate if X is finite.

Consider an infinite X and a $\varphi \in \mathsf{PL}$ such that $X \vDash \varphi$.

Then, $X \cup \{\neg\varphi\}$ is unsatisfiable.

Thus, by Theorem C2, there is an $X_0 \subset_{\text{fin}} X$ such that $X_0 \cup \{\neg\varphi\}$ is unsatisfiable.

Thus, $X_0 \vDash \varphi$.

Application of compactness – Graph colouring

- A **colouring** of a graph $G = (V, E)$ is given by a function which maps vertices to colours, such that vertices along an edge get different colours.
- G is k-colourable if it has a colouring using k distinct colours $\{1, \dots, k\}$.
- We can encode k-colourability of a graph in propositional logic
- Assume we have $p_{v,i} \in AP$ for each vertex v and $i \in \{1, \dots, k\}$
- What are the statements we would like to make?
- We will have to write one PL expression for each vertex in the graph
- We will later (soon?) see a better way to do this!

Application of compactness – Graph colouring

- "Each vertex gets one of these k colours"
- For each $v \in V$, we write a PL expression α_v as follows.

$$\alpha_v \coloneqq \bigvee \left\{ p_{v,i} \mid 1 \leqslant i \leqslant k \right\}$$

- "If a vertex has colour i, it cannot simultaneously have colour $j \neq i$"
- Encoded as β_v per vertex v

$$\beta_v \coloneqq \bigwedge \left\{ p_{v,i} \supset \left(\bigwedge \left\{ \neg p_{v,j} \mid 1 \leqslant j \leqslant k, j \neq i \right\} \right) \mid 1 \leqslant i \leqslant k \right\}$$

Application of compactness – Graph colouring

- "If a vertex u shares an edge with vertex v, they get different colours"
- We write an expression $\gamma_{u,v}$ for all pairs of vertices $u, v \in V$, and later restrict it to only those which share an edge.

$$\gamma_{u,v} \coloneqq \bigwedge \left\{ p_{u,i} \supset \neg p_{v,i} \;\middle|\; 1 \leqslant i \leqslant k \right\}$$

- Let $S_G \coloneqq \left\{ \alpha_v, \beta_v \;\middle|\; v \in V \right\} \cup \left\{ \gamma_{u,v} \;\middle|\; (u, v) \in E \right\}$
- **Exercise**: Show that G is k-colourable iff S_G is satisfiable.

Application of compactness – Graph colouring

- Consider an infinite G where every finite subgraph is k-colourable
- Is G itself k-colourable?
- How do we prove this?
- Naïve: Take the "union" of all k-colourings for all finite subgraphs
- Does not work! The same vertex might be assigned different colours as part of different subgraphs.
- It is not obvious how to patch together the different k-colourings for all the finite subgraphs
- **Compactness to the rescue!**

Application of compactness – Graph colouring

- Consider an infinite G where every finite subgraph is k-colourable
- Every finite subgraph $G' = (W, F)$ of G is k-colourable
- By our earlier equivalence of colourability and satisfiability, the set

$$S_{G'} := \left\{\alpha_v, \beta_v \,\middle|\, v \in W\right\} \cup \left\{\gamma_{u,v} \,\middle|\, (u,v) \in F\right\} \text{ is satisfiable.}$$

- Every finite subset of S_G is a subset of $S_{G'}$ for some finite subgraph G'
  - Why a subset and not **equal** to $S_{G'}$ for some G'?
  - One could take arbitrary finite subsets of S_G
  - They might not contain all the right α_v, β_v, and γ_v to be equal to some $S_{G'}$.
- So every finite subset of S_G is satisfiable.
- By the Compactness Theorem (C2, $\Leftarrow$), S_G itself is satisfiable.
- So G is k-colourable.

Other applications of compactness

- Topology: Equivalent statement to C1 and C2 saying that a particular kind of topology is compact
- Some proofs of the Compactness Theorem closely follow proofs of related statements about compact sets in topology
- Not isolated to fields that use the word "compact"
- Orderings: Can use the Compactness Theorem to show that every partially ordered set may be totally ordered
- **Exercise**: Prove the above statement.

Propositional logic: Wrap up

- Each statement about the world a proposition
- Atomic propositions + connectives $\wedge, \vee, \neg, \supset$
- Each proposition has an associated truth value
- Invalidity shown via resolution algorithm
- Saw the Hilbert axiomatization system $\mathcal{H}$ for PL
- Hard to do proofs directly in $\mathcal{H}$; use Deduction Theorem
- Proved Soundness and Completeness for $\mathcal{H}$
- Proof of Completeness gave us a useful corollary of Compactness
- **What next?** More expressive logics!