Lecture 18 - Natural Deduction

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COL703/COL7203 - Logic for Computer Science

Proof rules for propositional fragment

Introduction rule	Elimination rule
$\dfrac{\Gamma \vdash \varphi_0 \qquad \Gamma \vdash \varphi_1}{\Gamma \vdash \varphi_0 \wedge \varphi_1} \wedge i$	$\dfrac{\Gamma \vdash \varphi_0 \wedge \varphi_1}{\Gamma \vdash \varphi_j} \wedge e_j$
$\dfrac{\Gamma \vdash \varphi_j}{\Gamma \vdash \varphi_0 \vee \varphi_1} \vee i_j$	$\dfrac{\Gamma \vdash \varphi_0 \vee \varphi_1 \qquad \Gamma, \varphi_0 \vdash \psi \qquad \Gamma, \varphi_1 \vdash \psi}{\Gamma \vdash \psi} \vee e$
$\dfrac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \supset \psi} \supset i$	$\dfrac{\Gamma \vdash \varphi \supset \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \supset e$
$\dfrac{\Gamma, \varphi \vdash \neg\psi \qquad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \neg\varphi} \neg i$	$\dfrac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \neg e$

Proof rules for ∃ and ∀

Introduction rule	Elimination rule
$\dfrac{\Gamma \vdash \varphi\{y/x\}}{\Gamma \vdash \forall x.\ [\varphi]}\ \forall\mathsf{i}$ (y fresh)	$\dfrac{\Gamma \vdash \forall x.\ [\varphi]}{\Gamma \vdash \varphi\{t/x\}}\ \forall\mathsf{e}$
$\dfrac{\Gamma \vdash \varphi\{t/x\}}{\Gamma \vdash \exists x.\ [\varphi]}\ \exists\mathsf{i}$	$\dfrac{\Gamma \vdash \exists x.\ [\varphi] \quad \Gamma, \varphi\{y/x\} \vdash \psi}{\Gamma \vdash \psi}\ \exists\mathsf{e}$ (y fresh)

where t is a term in the language, and $y \in \mathscr{V}$ is fresh if $y \notin \mathsf{vars}(\Gamma \cup \{\varphi, \psi\})$.

$$\dfrac{}{\Gamma \vdash \varphi}\ \mathsf{Ax} \quad (\varphi \in \Gamma)$$

We say that $\Gamma \vdash_{\mathscr{G}} \varphi$ if there is a proof of φ from assumptions Γ using Ax and the rules in both the above tables.

Example 0

Show that $\forall x.\ [P(x)] \vdash_{\mathcal{G}} \exists x.\ [P(x)]$

$$\dfrac{\dfrac{\dfrac{}{\forall x.\ [P(x)] \vdash \forall x.\ [P(x)]}\,\mathsf{Ax}}{\forall x.\ [P(x)] \vdash P(t)}\,\forall\mathsf{e}}{\forall x.\ [P(x)] \vdash \exists x.\ [P(x)]}\,\exists\mathsf{i}$$

Example 1

Let $\Gamma = \left\{\forall x.\ [L(x) \supset \neg U(x)], \forall x.\ [P(x) \supset \neg T(x)], L(a) \vee T(a)\right\}$. Show that $\Gamma \vdash_{\mathcal{G}} P(a) \supset \neg U(a)$.

Example 1

Let $\Gamma = \left\{\forall x.\ [L(x) \supset \neg U(x)], \forall x.\ [P(x) \supset \neg T(x)], L(a) \vee T(a)\right\}$. Show that $\Gamma \vdash_{\mathcal{G}} P(a) \supset \neg U(a)$.

Let $\Gamma' = \Gamma \cup \{P(a)\}$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\Gamma', T(a), \neg L(a) \vdash \forall x.\ [P(x) \supset \neg T(x)]$}
\RightLabel{$\forall$e}
\UnaryInfC{$\Gamma', T(a), \neg L(a) \vdash P(a) \supset \neg T(a)$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\Gamma', T(a), \neg L(a) \vdash P(a)$}
\RightLabel{$\supset$e}
\BinaryInfC{$\Gamma', T(a), \neg L(a) \vdash \neg T(a)$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\Gamma', T(a), \neg L(a) \vdash T(a)$}
\RightLabel{$\neg$i}
\BinaryInfC{$\Gamma', T(a) \vdash \neg\neg L(a)$}
\RightLabel{$\neg$e}
\UnaryInfC{$\Gamma', T(a) \vdash L(a)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\Gamma' \vdash \forall x.\ [L(x) \supset \neg U(x)]$}
\RightLabel{$\forall$e}
\UnaryInfC{$\Gamma' \vdash L(a) \supset \neg U(a)$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\Gamma' \vdash L(a) \vee T(a)$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\Gamma', L(a) \vdash L(a)$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma', T(a) \vdash L(a)$}
\RightLabel{$\vee$e}
\TrinaryInfC{$\Gamma' \vdash L(a)$}
\RightLabel{$\supset$e}
\BinaryInfC{$\Gamma' \vdash \neg U(a)$}
\RightLabel{$\supset$i}
\UnaryInfC{$\Gamma \vdash P(a) \supset \neg U(a)$}
\end{prooftree}
$$

Example 2

Show that $\vdash_{\mathscr{G}} (\neg\varphi \lor \neg\psi) \supset \neg(\varphi \land \psi)$.

Example 2

Show that $\vdash_{\mathscr{G}} (\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$.

Let $\Gamma = \{\neg\varphi \vee \neg\psi\}$ and $\Delta = \Gamma \cup \{\neg\varphi\}$.

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$.

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \wedge \psi)$.

$$\dfrac{\begin{array}{c} ??? \\ \vdots \\ \alpha \vdash \neg\varphi \vee \neg\psi \end{array}}{\vdash \alpha \supset (\neg\varphi \vee \neg\psi)} \supset\mathrm{i}$$

- **Tip:** If you have to prove $\Gamma \vdash_{\mathscr{G}} \alpha \vee \beta$, where $\Gamma \nvdash_{\mathscr{G}} \alpha$ and $\Gamma \nvdash_{\mathscr{G}} \beta$, use $\neg$e!
- Same for if you have to prove $\Gamma \vdash_{\mathscr{G}} \exists x.\ [\alpha]$, but $\Gamma \nvdash_{\mathscr{G}} \alpha(t)$ for any t.
- If all else fails, look to $\neg$e for help!

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \wedge \psi)$.

$$\dfrac{\begin{array}{c} ??? \\ \vdots \\ \alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi) \end{array}}{\alpha \vdash \neg\varphi \vee \neg\psi}\ \neg e$$

The only way we know to get a "brand new" expression headed by $\neg$ is $\neg i$!

Suppose we had a formula β such that the following held, then done.

$$\dfrac{\dfrac{\begin{array}{c} \vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \neg\beta \end{array} \quad \begin{array}{c} \vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \beta \end{array}}{\alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi)}\ \neg i}{\alpha \vdash \neg\varphi \vee \neg\psi}\ \neg e$$

But what is this β supposed to be?

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \wedge \psi)$.

$$\dfrac{\begin{matrix} ??? \\ \vdots \\ \alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi) \end{matrix}}{\alpha \vdash \neg\varphi \vee \neg\psi}\ \neg e$$

The only way we know to get a "brand new" expression headed by $\neg$ is $\neg i$!

Suppose we had a formula β such that the following held, then done.

$$\dfrac{\dfrac{\begin{matrix} \vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \neg\beta \end{matrix} \quad \begin{matrix} \vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \beta \end{matrix}}{\alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi)}\ \neg i}{\alpha \vdash \neg\varphi \vee \neg\psi}\ \neg e$$

But what is this β supposed to be? What can we prove from this context?

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \wedge \psi)$.

$$
\dfrac{\dfrac{\dfrac{}{\neg(\neg\varphi \vee \neg\psi), \neg\varphi \vdash \neg\varphi}\,\mathsf{Ax}}{\neg(\neg\varphi \vee \neg\psi), \neg\varphi \vdash \neg\varphi \vee \neg\psi}\,\vee\mathsf{i}_0 \qquad \dfrac{}{\neg(\neg\varphi \vee \neg\psi), \neg\varphi \vdash \neg(\neg\varphi \vee \neg\psi)}\,\mathsf{Ax}}{\dfrac{\neg(\neg\varphi \vee \neg\psi) \vdash \neg\neg\varphi}{\neg(\neg\varphi \vee \neg\psi) \vdash \varphi}\,\neg\mathsf{e}}\,\neg\mathsf{i}
$$

Similarly, $\neg(\neg\varphi \vee \neg\psi) \vdash \psi$. **Exercise**: Draw this proof tree.

Can use Monotonicity and $\wedge\mathsf{i}$ to get a proof π of

$$\alpha, \neg(\neg\varphi \vee \neg\psi) \vdash_{\mathscr{G}} \varphi \wedge \psi$$

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \wedge \psi)$.

$$\dfrac{\dfrac{}{\alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \alpha}\,\mathrm{Ax} \qquad \begin{array}{c}\pi \\ \vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash (\varphi \wedge \psi)\end{array}}{\dfrac{\alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi)}{\alpha \vdash \neg\varphi \vee \neg\psi}\,\neg\mathrm{e}}\,\neg\mathrm{i}$$

Exercise: Prove that $\forall x.\ [P(x) \vee \neg P(x)]$.

Example 4

Show that $\vdash_{\mathscr{G}} \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]$.

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$$\dfrac{\begin{array}{c} ??? \\ \vdots \\ \neg\forall x.\ [P(x)] \vdash \exists x.\ [\neg P(x)] \end{array}}{\vdash \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]}\ \supset\mathsf{i}$$

Example 4

Show that $\vdash_{\mathcal{G}} \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]$.

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}???\\ \vdots \\ \neg\forall x.\ [P(x)], \neg\exists x.\ [\neg P(x)] \vdash \neg\gamma\end{array} \quad \begin{array}{c}???\\ \vdots \\ \neg\forall x.\ [P(x)], \neg\exists x.\ [\neg P(x)] \vdash \gamma\end{array}}{\neg\forall x.\ [P(x)] \vdash \neg\neg\exists x.\ [\neg P(x)]}\ \neg\mathrm{i}}{\neg\forall x.\ [P(x)] \vdash \exists x.\ [\neg P(x)]}\ \neg\mathrm{e}}{\vdash \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]}\ \supset\mathrm{i}
$$

Why move to $\vdash_{\mathscr{G}}$?

- One main reason for moving to $\vdash_{\mathscr{G}}$ was intuitiveness
- Easier proofs, as we just saw
- Another reason is convenience for automation
- Proof search in $\vdash_{\mathscr{H}}$ is not syntactically decidable (even for PL)
  - Have to search through (infinitely many possible) instantiations of axiom schema which might appear in a proof
- Is $\vdash_{\mathscr{G}}$ better?
- We will see that $\vdash_{\mathscr{G}}$ enjoys some nice properties.
- Monotonicity and cut hold as usual.
- Is there anything that helps with proof search?

Unnecessary detours in proofs

Consider a proof of the following sort.

$$\dfrac{\dfrac{}{\varphi, \psi \vdash \varphi}\,\text{Ax} \qquad \dfrac{}{\varphi, \psi \vdash \psi}\,\text{Ax}}{\dfrac{\varphi, \psi \vdash \varphi \land \psi}{\varphi, \psi \vdash \varphi}\,\land e_0}\,\land i$$

Unnecessary detours in proofs

Consider a proof of the following sort.

$$\dfrac{\dfrac{}{\varphi, \psi \vdash \varphi}\,\mathrm{Ax} \qquad \dfrac{}{\varphi, \psi \vdash \psi}\,\mathrm{Ax}}{\dfrac{\varphi, \psi \vdash \varphi \land \psi}{\varphi, \psi \vdash \varphi}\,\land e_0}\,\land i$$

We first introduce an $\land$, and then immediately eliminate it.

Could have replaced this entire proof by the following, smaller proof without any such wasteful detours involving large expressions.

$$\dfrac{}{\varphi, \psi \vdash \varphi}\,\mathrm{Ax}$$

Unnecessary detours in proofs

- **Exercise**: What does a proof involving a detour on an $\vee$ or a $\supset$ look like?
- Detours on these operators involve the introduction rule for that operator, immediately followed by the elimination rule.
- What about $\neg$?

Unnecessary detours in proofs

- **Exercise**: What does a proof involving a detour on an $\vee$ or a $\supset$ look like?
- Detours on these operators involve the introduction rule for that operator, immediately followed by the elimination rule.
- What about $\neg$?
- Clearly $\neg$i followed by $\neg$e is **not** an unnecessary detour.
- We could not have done the earlier proofs without using this combo!
- However, the expressions we used for $\neg$i were informed by the context and the expected conclusion.
- Can we eliminate all unnecessary detours?
- Given Γ, α, is there some finite set to which every expression occurring in any proof of $\Gamma \vdash_{\mathcal{G}} \alpha$ belongs?