# Lecture 17 - Completeness

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# Recap: System $\vdash_{HK}$ for FO

All generalizations of the following, along with MP.

| | |
|---|---|
| **(H1a)** | $\varphi \supset (\psi \supset \varphi)$ |
| **(H1b)** | $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$ |
| **(H1c)** | $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$ |
| **(H2a)** | $x \equiv x$ |
| **(H2b)** | $x \equiv y \supset \big((\varphi(x) \supset \varphi(y)) \wedge (\varphi(y) \supset \varphi(x))\big)$ |
| **(H3a)** | $\forall x.\ [\varphi \supset \psi] \supset (\forall x.\ [\varphi] \supset \forall x.\ [\psi])$ |
| **(H3b)** | $\varphi \supset \forall x.\ [\varphi]$ where $x$ does not appear free in $\varphi$ |
| **(H3c)** | $\forall x.\ [\varphi] \supset \varphi\{t/x\}$ for any term $t$ |

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\ \text{MP}$$

We denote provability in this system with the symbol $\vdash_{HK}$.

# Recap: Completeness of $\vdash_{HK}$

- **Gödel's Completeness Theorem (1929)**: If $\Gamma \models \varphi$, then $\Gamma \vdash_{HK} \varphi$
- Equivalent statement: *Any consistent set of expressions is satisfiable*
- Start out with a consistent set of expressions $\Gamma$
- Assume $\mathscr{V} \setminus \mathsf{vars}(\Gamma)$ to be infinite. (Can always do this!)
- $\exists$-fulfilled $\Gamma$: for every expression of the form $\neg\forall x.\ [\alpha] \in \Gamma$, there exists some term $t$ such that $\neg\alpha\{t/x\} \in \Gamma$
- **Lindenbaum's Lemma**: Every consistent set can be extended to an $\exists$-fulfilled maximally consistent set (MCS).

# Recap: Proof of Lindenbaum's Lemma

- Similar structure to the completeness proof for $\mathsf{PL}$
- Fix enumerations, for each expression, put it into the set or not
- $\Gamma_0 \coloneqq \Gamma$, and for every $i \geqslant 0$,

$$\Gamma_{i+1} \coloneqq \begin{cases} \Gamma_i' & \text{if } \Gamma_i' \text{ consistent and } \varphi_i \text{ not of the form } \neg\forall x.\ [\alpha] \\ \Gamma_i' \cup \{\neg\alpha\{y/x\}\} & \text{if } \Gamma_i' \text{ consistent, } \varphi_i = \neg\forall x.\ [\alpha]\text{, and} \\ & \quad y \text{ the first variable not in } \mathsf{fv}(\Gamma_i) \cup \mathsf{vars}(\varphi_i) \\ \Gamma_i & \text{if } \Gamma_i' \text{ not consistent} \end{cases}$$

  where $\Gamma_i' = \Gamma_i \cup \{\varphi_i\}$.
- Finally, $\Gamma_{\mathrm{ext}} \coloneqq \bigcup_{i \geqslant 0} \Gamma_i$
- Showed: $\Gamma_{\mathrm{ext}}$ is a maximally consistent and $\exists$-fulfilled extension of $\Gamma$.

# A useful property of $\exists$-fulfilled MCSs

**Lemma**: Let $\Gamma_{ext}$ be any $\exists$-fulfilled MCS. Then, for all expressions $\alpha$ and $\beta$

1. $\neg\alpha \in \Gamma_{ext}$ iff $\alpha \notin \Gamma_{ext}$
2. $\alpha \supset \beta \in \Gamma_{ext}$ iff $\alpha \notin \Gamma_{ext}$ or $\beta \in \Gamma_{ext}$
3. $\Gamma_{ext} \vdash \alpha$ iff $\alpha \in \Gamma_{ext}$. In particular, all $\alpha \in \Gamma_{ext}$ such that $\vdash_{HK} \alpha$.
4. $\forall x.\ [\alpha] \in \Gamma_{ext}$ iff $\alpha\{t/x\} \in \Gamma_{ext}$, for all terms $t$.

**Proof**: Statements (1)–(3) follow as for PL. Consider (4). If $\forall x.\ [\alpha] \in \Gamma_{ext}$, then $\Gamma_{ext} \vdash \forall x.\ [\alpha]$, by (3). We also have $\Gamma_{ext} \vdash \forall x.\ [\alpha] \supset \alpha\{t/x\}$ for any term $t$, by **(H3c)**. Thus, by MP, $\Gamma_{ext} \vdash \alpha\{t/x\}$ for any $t$, and so $\alpha\{t/x\} \in \Gamma_{ext}$ by (3). Now suppose $\forall x.\ [\alpha] \notin \Gamma_{ext}$, then $\neg\forall x.\ [\alpha] \in \Gamma_{ext}$, by (1). Since $\Gamma_{ext}$ is $\exists$-fulfilled, we have $\neg\alpha\{y/x\} \in \Gamma_{ext}$ for some $y \in \mathscr{V}$. Thus, $\alpha\{y/x\} \notin \Gamma_{ext}$, and thus it is not the case that $\alpha\{t/x\} \in \Gamma$ for all terms $t$.

# From an ∃-fulfilled MCS to a model

- What did we want to show? *Any consistent set of expressions is satisfiable*
- So far: *Any consistent set of expressions can be extended to an* $\exists$*-fulfilled MCS*
- If I can produce a model for this $\exists$-fulfilled MCS, done!
- Suppose $\Gamma_{\text{ext}}$ is an $\exists$-fulfilled MCS corresponding to a consistent $\Gamma$.
- Need to build an interpretation $\mathcal{I} = ((M, \iota), \sigma)$ such that $\mathcal{I} \models \Gamma_{\text{ext}}$.
- We will, in fact, show that for every $\varphi$, $\mathcal{I} \models \varphi$ iff $\varphi \in \Gamma_{\text{ext}}$.
- We have a signature $\Sigma = (\mathcal{C}, \mathcal{F}, \mathcal{P})$. Need to
  - Define $M$
  - Fix interpretations via $\iota$ for every symbol in $\Sigma$ to get $\mathcal{M} = (M, \iota)$
  - Fix an assignment $\sigma$

# Some postulates about equality

For any $t, t_1, \dots, t_n, u_1, \dots, u_n \in \mathrm{T}(\Sigma)$, we have the following:

- $\vdash_{HK} t \equiv t$
- $\vdash_{HK} (t_1 \equiv t_2) \supset (t_2 \equiv t_1)$
- $\vdash_{HK} (t_1 \equiv t_2 \wedge t_2 \equiv t_3) \supset (t_1 \equiv t_3)$
- $\vdash_{HK} (\bigwedge_{1 \leqslant i \leqslant n} t_i \equiv u_i) \supset (f(t_1, \dots, t_n) \equiv f(u_1, \dots, u_n))$ for any $n$-ary $f \in \mathscr{F}$
- $\vdash_{HK} (\bigwedge_{1 \leqslant i \leqslant n} t_i \equiv u_i) \supset (P(t_1, \dots, t_n) \supset P(u_1, \dots, u_n))$ for any $n$-ary $P \in \mathscr{P}$

**Exercise**: Show that these proofs exist!

# Defining a domain $M$

- We need some domain $M$
- Every term symbol needs to be a name for some element of $M$
- What about terms where $t \equiv u$ belongs to $\Gamma_{ext}$?
- They should map to the same element.
- Define a binary relation $\simeq$ such that $t \simeq u$ iff $t \equiv u \in \Gamma_{ext}$
- Equality axioms guarantee that $\simeq$ is an equivalence relation
- **Exercise**: Verify this! (What does this require you to show?)
- For every $t \in \mathrm{T}(\Sigma)$, let $[t]$ denote the equivalence class containing $t$
- Define $M := \{[t] \mid t \in \mathrm{T}(\Sigma)\}$

# Defining $\iota$: Symbols in $\mathscr{P}$

- Let $P \in \mathscr{P}$ be an $n$-ary relation symbol. Define
  $\iota(P) = \mathsf{P}_{\mathscr{M}} := \left\{([t_1], \ldots, [t_n]) \,\middle|\, P(t_1, \ldots, t_n) \in \Gamma_{\text{ext}}\right\}$
- **Claim**: $\mathsf{P}_{\mathscr{M}}$ is well-defined

# Defining $\iota$: Symbols in $\mathscr{P}$

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- **Claim**: $\mathsf{P}_{\mathscr{M}}$ is well-defined
- If $t_i \simeq u_i$ for $1 \leqslant i \leqslant n$ and $P(t_1, \dots, t_n) \in \Gamma_{\text{ext}}$, then $P(u_1, \dots, u_n) \in \Gamma_{\text{ext}}$.
- From the final equality postulate, we get
$$\vdash_{HK} (t_1 \equiv u_1 \wedge \dots \wedge t_n \equiv u_n) \supset P(t_1, \dots, t_n) \supset P(u_1, \dots, u_n)$$
- Since $t_i \simeq u_i$ for $1 \leqslant i \leqslant n$, we have $t_i \equiv u_i \in \Gamma_{\text{ext}}$ for $1 \leqslant i \leqslant n$
- We also know that $P(t_1, \dots, t_n) \in \Gamma_{\text{ext}}$, so the claim follows.
- **Exercise**: How do we get *containment*? MP on these just gives us derivability, right?

# Defining $\iota$: Symbols in $\mathcal{F}$ and $\mathcal{C}$

- Let $f \in \mathcal{F}$ be an $n$-ary function symbol. Define $\iota(f) = \mathsf{f}_{\mathcal{M}}$ as follows: $\mathsf{f}_{\mathcal{M}}([t_1], \dots, [t_n]) := [f(t_1, \dots, t_n)]$.
- **Claim**: $\mathsf{f}_{\mathcal{M}}$ is well-defined

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- Let $f \in \mathscr{F}$ be an $n$-ary function symbol. Define $\iota(f) = \mathsf{f}_{\mathscr{M}}$ as follows: $\mathsf{f}_{\mathscr{M}}([t_1], \dots, [t_n]) \coloneqq [f(t_1, \dots, t_n)]$.
- **Claim**: $\mathsf{f}_{\mathscr{M}}$ is well-defined
- If $t_i \simeq u_i$ for $1 \leqslant i \leqslant n$, then $f(t_1, \dots, t_n) \simeq f(u_1, \dots, u_n)$.
- If $t_i \simeq u_i$ for $1 \leqslant i \leqslant n$, then $t_i \equiv u_i \in \Gamma_{\text{ext}}$ for $1 \leqslant i \leqslant n$.
- By the fourth equality postulate, $\vdash_{HK} f(t_1, \dots, t_n) \equiv f(u_1, \dots, u_n)$.
- So $f(t_1, \dots, t_n) \equiv f(u_1, \dots, u_n) \in \Gamma_{\text{ext}}$, and so $f(t_1, \dots, t_n) \simeq f(u_1, \dots, u_n)$.
- Let $c \in \mathscr{C}$ be a constant symbol. Define $\iota(c) = \mathsf{c}_{\mathscr{M}} \coloneqq [c]$
- $\mathscr{M} = (M, \iota)$ is the structure we will use to define our model.
- For $x \in \mathscr{V}$, define $\sigma(x) \coloneqq [x]$. $\mathscr{I} = (\mathscr{M}, \sigma)$ is our candidate model.

# The model $\mathcal{I}$

- Want to show $\mathcal{I} \vDash \varphi$ iff $\varphi \in \Gamma_{ext}$ for any $\varphi \in \mathsf{FO}_\Sigma$.
- Proof by induction on the structure of $\varphi$.
- Base case: $\mathcal{I} \vDash \varphi$ iff $\varphi \in \Gamma_{ext}$ for any atomic formula $\varphi$ (**Exercise!**)
- We omit the (straightforward) cases when $\varphi = \neg\psi$ and $\varphi = \psi \supset \xi$
- Consider the case when $\varphi = \forall x.\ [\psi]$.
    - $\forall x.\ [\psi] \in \Gamma_{ext}$ iff (from the useful properties of $\Gamma_{ext}$),
    - $\psi\{t/x\} \in \Gamma_{ext}$ for every $t \in \mathrm{T}(\Sigma)$ iff (by IH),
    - $(\mathcal{M}, \sigma) \vDash \psi\{t/x\}$ for every $t$ iff (by Substitution Lemma),
    - $(\mathcal{M}, \sigma[x \mapsto [t]]) \vDash \psi\{t/x\}$ for every $t$ iff (by the semantics of $\forall$)
    - $\mathcal{I} \vDash \forall x.\ [\psi]$
- Thus, every consistent $\Gamma$ (can be extended to an $\exists$-fulfilled MCS $\Gamma_{ext}$ which) is satisfiable.