Lecture 9 - First-order logic

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COL703/COL7203 - Logic for Computer Science

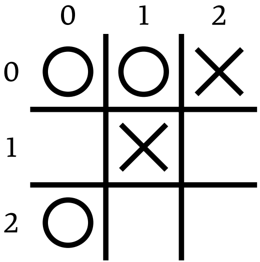
Back to Tic-Tac-Toe

- Every statement about the world was modelled as a proposition.
- But what if we want to make a statement about everyone in the world?
- Recall our Tic-Tac-Toe example.
- There was no link between $P_{ij}^{\bigcirc}$ and $P_{ij}^{\times}$.
- Could have valuations where they were simultaneously made true.
- We wanted to say "No square simultaneously contains a $\bigcirc$ and a $\times$"
- Had to write a long conjunctive expression, which talked about each square individually.
- What if I played on a $5 * 5$ grid? Or a $27 * 27$ grid? Infeasible to write such a formula!

Back to Tic-Tac-Toe: First-Order logic

- Would like $P^{\bigcirc}$ and $P^{\times}$ to be statements that can be made about *any* cell
- We call these **predicates**; fundamental building blocks now
- Can think of a proposition as a 0-ary predicate
- A cell is (i, j) where i is the row and j the column
- The following expression represents the grid below

$$\mathrm{circ}(0,0) \wedge \mathrm{circ}(0,1) \wedge \mathrm{cross}(0,2) \wedge \mathrm{cross}(1,1) \wedge \mathrm{circ}(2,0)$$



Back to Tic-Tac-Toe: First-Order logic

- Real power of predicates: **variables** and **quantification**
- We wanted to say "No square simultaneously contains a ○ and a ×"
- Move to **First-Order Logic** (FOL)!
- It allows us to talk about **all** and/or **some** elements in the universe
- **All** elements: $\forall$ (mnemonic: upside-down A, for "all")
- **Some** element: $\exists$ (mnemonic: backward E, for "exists")
- "For every cell, it is not the case that the cell contains a circle as well as that the cell contains a cross."
- Notation that captures "for every cell" gives us a small expression which talks about **all** cells in a grid **of any size**!

First-Order logic: Syntax

- We need to define the objects about which one can state a predicate.
- This requires us to define a notion of **terms**.
- Have variables (atomic) and *functions* to build bigger terms.
- The set of FOL expressions one ends up with depends on the chosen sets of functions and predicates.
- This is called a **signature**.

First-Order logic: Syntax

- We have a countable set of variables $x, y, z \ldots \in \mathscr{V}$
- We have a countable set of function symbols $f, g, h \ldots \in \mathscr{F}$, and a countable set of relation/predicate symbols $P, Q, R \ldots \in \mathscr{P}$
- 0-ary function symbols are constant symbols in $\mathscr{C}$
- $(\mathscr{C}, \mathscr{F}, \mathscr{P})$ is a signature Σ
- Grammar for FOL is as follows

$$\varphi, \psi \coloneqq t_1 = t_2 \big| P(t_1, \ldots, t_n) \big| \neg\varphi \big| \varphi \wedge \psi \big| \varphi \vee \psi \big| \varphi \supset \psi \big| \exists x.\ [\varphi]\ \big| \forall x.\ [\varphi]$$

where P is an n-ary predicate symbol in Σ, and the term syntax is

$$t \coloneqq x \in \mathscr{V} \mid c \in \mathscr{C} \mid f(t_1, \ldots, t_m)$$

where f is an m-ary function symbol in Σ.

A note on quantification

- $\forall$: "every", "all", "each", "any"
- $\exists$: "some", "many", "certain", "there exists", "at least one"
- Pay attention to the negations of these!
- $\neg\forall x.\ [\varphi]$ is equivalent to $\exists x.\ [\neg\varphi]$
- $\neg\exists x.\ [\varphi]$ is equivalent to $\forall x.\ [\neg\varphi]$
- We will see how to prove these later.

Tic-Tac-Toe

- What is the signature that we need for the Tic-Tac-Toe example?
- $\Sigma = \{\mathscr{C}, \mathscr{F}, \mathscr{P}\}$, where
- Constants: $\mathscr{C} = \{0, 1, 2\}$
- Functions: $\mathscr{F} = \emptyset$
- Predicates: $\mathscr{P} = \{\mathsf{circ}/2, \mathsf{cross}/2\}$
- For a bigger grid, add more constants
- $\mathscr{C} = \{0, \ldots, n-1\}$ for a grid of size $n \times n$.
- We might need more functions and relations too, depending on the expressions we want to write.

Tic-Tac-Toe: FOL expressions

Express the following in FOL.

- At least one cell has a circle

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- Win: circles line up horizontally

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- Win: circles line up along the main diagonal

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- Win: circles line up along the main diagonal: $\forall i.\ [\mathrm{circ}(i,i)]$

Tic-Tac-Toe: FOL Expressions

- What about the antidiagonal?
- Needs us to talk about cells of the form $(i, 2-i)$
- Constants: $\mathscr{C} = \{0, 1, 2\}$
- Functions: $\mathscr{F} = \{\mathsf{f}/1\}$, where $\mathsf{f}(x) = 2 - x$
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- Win: circles line up along the antidiagonal: $\forall i.\ [\mathsf{circ}(i, \mathsf{f}(i))]$
- A win for circle is a disjunction of these four FOL expressions

About quantification

- Win: circles line up horizontally: $\exists h.\ [\forall v.\ [\mathrm{circ}(h, v)]]$
- "There is a row such that for every column, a circle appears in the corresponding cell"
- What if we inverted the quantifiers?
- What does $\forall v.\ [\exists h.\ [\mathrm{circ}(h, v)]]$ capture?
- "For every column, there is a row such that a circle appears in the corresponding cell"
- Would it still capture the win condition where circles line up horizontally?

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- Pay attention to the order of quantifiers!

Graphs

- Consider a graph $G = (V, E)$
- Constants: $\mathscr{C} = \emptyset$
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- Undirected graph: $\forall u.\ [\forall v.\ [E(u, v) \supset E(v, u)]]$
- Transitive graph (a path from u to v implies an edge between u and v) : $\forall u.\ [\forall v.\ [\forall w.\ [(E(u, w) \wedge E(w, v)) \supset E(u, v)]]]$

Graph colouring

- $\Sigma = (\emptyset, \emptyset, \{E/2, C_1/1, \ldots, C_k/1\})$
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$$\forall v. \left[\bigwedge_{1 \leqslant i \leqslant k} \left\{ C_i(v) \supset \bigwedge_{\substack{1 \leqslant j \leqslant k \\ j \neq i}} \neg C_j(v) \right\} \right]$$

- Vertices along an edge do not get the same colour

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$$\forall u. \left[\forall v. \left[E(u, v) \supset \bigwedge_{1 \leqslant i \leqslant k} \neg\{C_i(u) \wedge C_i(v)\} \right] \right]$$