

# Lecture 21 - More About First-Order Theories

Vaishnavi Sundararajan

COL703/COL7203 - Logic for Computer Science

# Recap

- Looked at a few theories of “common” constructions
- Groups, fields, orders...
- Saw that one can define formulae which characterize subclasses of these constructions (groups with no elements of order 2 etc)
- Captured the  $<$  relation as a formula with two free variables in  $(\mathbb{R}, +, \times, 0)$
- Saw that  $<$  **cannot** be captured in  $(\mathbb{R}, +, 0)$
- Used an automorphism to show this

# Elementary classes

- We showed that  $<$  is not definable using the signature without  $\times$
- But we also defined entire classes of groups, fields etc via FO formulas
- How to show if an entire class of models is characterizable using FO?
- For a set  $X$  of  $\Sigma$ -sentences, we define

$$\text{Mod } X := \{\mathcal{M} \mid \mathcal{M} \text{ is a } \Sigma\text{-structure, and } \mathcal{M} \models X\}$$

- Let  $\mathcal{C}$  be a class of  $\Sigma$ -structures.  $\mathcal{C}$  is said to be
  - **elementary** if there is a  $\varphi \in \text{FO}_\Sigma$  such that  $\mathcal{C} = \text{Mod } \{\varphi\}$
  - **$\Delta$ -elementary** if there is a set  $X \subseteq \text{FO}_\Sigma$  such that  $\mathcal{C} = \text{Mod } X$ .
- **Elementary:** An FO sentence  $\varphi$  captures the exact class of models

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- Classes of equivalence relations, orders, and fields also elementary
- Let  $p$  be a prime. A field  $F$  has **characteristic  $p$**  if  $\underbrace{1 + \cdots + 1}_{p \text{ times}} = 0$ . If there is no such  $p$ , then  $F$  has characteristic  $0$ .
- The field  $\mathbb{R}$  of real numbers has characteristic  $0$ .
- Let  $\chi_p := \underbrace{1 + \cdots + 1}_{p \text{ times}} \equiv 0$
- The class of fields of characteristic  $p$  is  $\text{Mod } (\gamma_{\text{flds}} \wedge \chi_p)$
- The class  $\mathcal{C}$  of fields of characteristic  $0$  is  $\Delta$ -elementary
- $\mathcal{C} = \text{Mod } \{\gamma_{\text{flds}}\} \cup \{\neg \chi_p \mid p \text{ is a prime}\}$
- Is  $\mathcal{C}$  elementary?

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- $\mathcal{C} = \text{Mod } \{\gamma_{\text{flds}}\} \cup \{\neg \chi_p \mid p \text{ is a prime}\}$
- Is  $\mathcal{C}$  elementary? We can use Compactness to show that it is **not**.

# Compactness theorem

- **Compactness Theorem:** A set  $\Gamma$  of FO sentences is satisfiable iff every finite subset of  $\Gamma$  is satisfiable.
- **Proof:** Suppose  $\Gamma$  satisfiable. Then all finite subsets of  $\Gamma$  also satisfiable.
- Now suppose that  $\Gamma$  is not satisfiable. We know that every consistent set is satisfiable. So  $\Gamma$  is not consistent.
- So there is some  $\{\varphi_1, \dots, \varphi_n\} \subseteq_{\text{fin}} \Gamma$  such that  $\vdash \neg(\varphi_1 \wedge \dots \wedge \varphi_n)$
- But by Soundness,  $\models \neg(\varphi_1 \wedge \dots \wedge \varphi_n)$
- So there is a finite subset of  $\Gamma$  that is unsatisfiable.

## Compactness: Application

- Let  $\varphi$  be a sentence which holds in all fields of characteristic  $0$
- So  $\{\gamma_{\text{flds}}\} \cup \{\neg \chi_p \mid p \text{ is a prime}\} \models \varphi$
- Compactness tells us that there is some  $n_0$  such that  
 $\{\gamma_{\text{flds}}\} \cup \{\neg \chi_p \mid p \text{ is a prime}, p < n_0\} \models \varphi$
- Hence,  $\varphi$  is valid in all fields of characteristic  $\geq n_0$ !
- So, a sentence which is valid in all fields of characteristic  $0$  is also valid in all fields with a “sufficiently large” characteristic.
- So the class of fields with characteristic  $0$  is **not** elementary

# Elementary equivalence

- So far, we saw some classes of structures that FOL can characterize
- What about which classes can be distinguished via FOL sentences?
- When are two structures **not** distinguishable?
- When they satisfy the same sentences
- Two  $\Sigma$ -structures  $\mathcal{M}$  and  $\mathcal{M}'$  are said to be **elementarily-equivalent** (denoted  $\mathcal{M} \bowtie \mathcal{M}'$ ) if for every sentence  $\varphi \in \text{FO}_\Sigma$ ,  $\mathcal{M} \models \varphi$  iff  $\mathcal{M}' \models \varphi$
- For a  $\Sigma$ -structure  $\mathcal{M}$ , the **theory** of  $\mathcal{M}$  is the set of sentences it satisfies:  
 $\text{Th}(\mathcal{M}) = \{\varphi \mid \mathcal{M} \models \varphi\}.$
- **Theorem:** For two  $\Sigma$ -structures  $\mathcal{M}$  and  $\mathcal{M}'$ ,  $\mathcal{M} \bowtie \mathcal{M}'$  iff  $\mathcal{M}' \models \text{Th}(\mathcal{M})$ .
- **Exercise:** Prove this statement.

# Elementary equivalence

- Clear that any two isomorphic structures are elementarily-equivalent.
- Are any two elementarily-equivalent structures also isomorphic?
- Can we say something about the class of structures that are all elementarily-equivalent to a particular  $\mathcal{M}$ ?
- **Theorem:** For any  $\mathcal{M}$ ,  $\mathcal{C} = \{\mathcal{M}' \mid \mathcal{M} \models \mathcal{M}'\} = \text{Mod Th}(\mathcal{M})$  is  $\Delta$ -elementary.  $\mathcal{C}$  is the smallest  $\Delta$ -elementary class which contains  $\mathcal{M}$ .
- **Exercise:** Prove this!
- Is the class of all structures isomorphic to  $\mathcal{M}$  also  $\Delta$ -elementary?
- Does the cardinality of  $\mathcal{M}$  influence the answer?
- Suppose I have an uncountable  $\mathcal{M}$  (over a countable  $\Sigma$ )
- What can I say about all structures elementarily-equivalent to  $\mathcal{M}$ ?

## Downward Löwenheim-Skolem theorem

- **(Downward) Löwenheim-Skolem Theorem:** If a set  $\Gamma$  of sentences over a countable  $\Sigma$  is satisfiable, it is satisfied by a countable model
- **Proof:** Consider a satisfiable set  $\Gamma$  of FO sentences over a countable  $\Sigma$ .
- $\Gamma$  is consistent. **Exercise:** Prove this!
- For Completeness, we built a model whose elements were equivalence classes of terms of the language
- What is the cardinality of  $T(\Sigma)$  for a countable  $\Sigma$ ?

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- $\Gamma$  is consistent. **Exercise:** Prove this!
- For Completeness, we built a model whose elements were equivalence classes of terms of the language
- What is the cardinality of  $T(\Sigma)$  for a countable  $\Sigma$ ?
- How many equivalence classes can there be over a countable set?
- At most countably many
- Thus, every satisfiable set of sentences (over a countable signature) is consistent, and satisfiable in a countable model!

## Downward L-S: Application

**Theorem:** Let  $\Sigma$  be countable, and  $X \in \text{FO}_\Sigma$  be a set of sentences which has arbitrarily large finite models (i.e. for every  $n \in \mathbb{N}$  there is a model for  $X$  with cardinality at least  $n$ ). Then,  $X$  is satisfied by a countably infinite model.

**Proof:** Recall  $\varphi_{\geq n} = \exists x_1. \left[ \exists x_2. \left[ \dots \exists x_n. \left[ \bigwedge_{1 \leq i < j \leq n} \neg(x_i \equiv x_j) \right] \dots \right] \right]$ .

Define  $Y := X \cup \{\varphi_{\geq m} \mid m \geq 2\}$

Every model of  $Y$  is infinite, and also a model of  $X$ .

**Exercise:** Is there even one such?

So  $Y$  (and consequently  $X$ ) is satisfied by a countably infinite model, by downward L-S.

# Upward Löwenheim-Skolem theorem

- **(Upward) Löwenheim-Skolem Theorem:** If a set  $\Gamma \subseteq \text{FO}_\Sigma$  is satisfied by an infinite model, then, for any set  $A$ , there is a model for  $\Gamma$  which has at least as many elements as  $A$ .
- **Proof:** For each  $a \in A$ , let  $c_a \notin \mathcal{C}$  be a new constant such that  $c_a \neq c_b$  for distinct  $a, b \in A$ . Let  $\Sigma' = (\mathcal{C} \cup \{c_a \mid a \in A\}, \mathcal{F}, \mathcal{P})$ .
- Let  $G := \Gamma \cup \{\neg(c_a \equiv c_b) \mid a, b \in A, a \neq b\} \subseteq \text{FO}_{\Sigma'}$ . Suppose  $\mathcal{J} \models G$ .
- $\mathcal{J}$  is also a model for  $\Gamma$
- Clear that  $\mathcal{J}(c_a) \neq \mathcal{J}(c_b)$  (since  $\mathcal{J} \models \neg(c_a \equiv c_b)$  for distinct  $a, b \in A$ )
- Then,  $\{(a, \mathcal{J}(a)) \mid a \in A\}$  is an injective map from  $A$  to the domain of  $\mathcal{J}$ , and so the model  $\mathcal{J}$  for  $G$  has at least as many elements as  $A$ .
- **Exercise:** Show that  $G$  is satisfiable.

# Löwenheim-Skolem theorem

- By Downward L-S, an **uncountable**  $\mathcal{M}$  has an elementarily-equivalent **countable**  $\mathcal{M}'$
- $\mathcal{M}$  and  $\mathcal{M}'$  are clearly not isomorphic
- So elementary equivalence and isomorphism do not coincide
- The class of all structures isomorphic to  $\mathcal{M}$  is **not**  $\Delta$ -elementary
- By Upward L-S, a **countable**  $\mathcal{M}$  has an elementarily-equivalent **uncountable**  $\mathcal{M}'$

## So why did we do all this?

- Recall that we wanted to reduce truth to provability in our proof system
- Common enough setting: natural numbers
- What if I wanted to obtain every fact that is true about  $\mathbb{N}$ ?
- Consider all sentences true of the natural numbers:  $\text{Th}(\mathbb{N})$
- There is also an **uncountable** model which satisfies these sentences!
- So  $\text{Th}(\mathbb{N})$  is satisfied by multiple models of various cardinalities.
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- But do we need  $\text{Th}(\mathbb{N})$  to have exactly one model? Not necessarily
- All we want is a “nice” set of axioms  $\Gamma_{\mathbb{N}}$  such that  $\varphi \in \text{Th}(\mathbb{N})$  iff  $\Gamma_{\mathbb{N}} \vdash_{\mathcal{G}} \varphi$
- But Gödel’s incompleteness theorem says that **no such  $\Gamma_{\mathbb{N}}$  exists**.