Lecture 7 - Completeness for the Hilbert system

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Hilbert system: Recap

$$\textbf{(H1)} \quad \varphi \supset (\psi \supset \varphi)$$

$$\textbf{(H2)} \quad (\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$$

$$\textbf{(H3)} \quad (\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$$

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\ \mathrm{MP}$$

- We denote provability in this system with the symbol $\vdash_{\mathscr{H}}$.
- $\Gamma \vdash_{\mathscr{H}} \varphi$ denotes that there is a proof of φ in System $\mathscr{H}$ using the expressions in Γ as assumptions
- **Theorem (Monotonicity)**: If $\Gamma \vdash_{\mathscr{H}} \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_{\mathscr{H}} \varphi$.

Composing proofs (Cut)

Theorem: If $\Gamma \vdash \alpha$ and $\Gamma, \alpha \vdash \beta$, then $\Gamma \vdash \beta$.

Proof: Suppose there is a proof π of $\Gamma \vdash \alpha$ and a proof ϖ of $\Gamma, \alpha \vdash \beta$.

Suppose α is never "used" in ϖ, i.e. no sequent $\Gamma, \alpha \vdash \alpha$ in ϖ.

In such a case, $\Gamma \vdash \beta$ (by ϖ itself).

Otherwise, consider the leaves of ϖ labelled by $\Gamma, \alpha \vdash \alpha$.

Replace each such leaf by π. This yields a valid proof of $\Gamma \vdash \beta$.

Deduction theorem

- Recall: Logical consequence corresponded directly to implication ($\Gamma \models \varphi$ iff $\left(\bigwedge_{\psi \in \Gamma} \psi\right) \supset \varphi$ is valid)
- A similar thing exists for proofs in $\vdash_{\mathscr{H}}$; can think of the assumptions as implying the conclusion
- Useful to assume something, get results, then **discharge** assumption
- **Deduction Theorem**: $\Gamma \cup \{\varphi\} \vdash_{\mathscr{H}} \chi$ iff $\Gamma \vdash_{\mathscr{H}} \varphi \supset \chi$.
- We will often use Γ, φ as notational shorthand for $\Gamma \cup \{\varphi\}$.

Deduction theorem: Proof ($\Leftarrow$)

Suppose $\Gamma \vdash_{\mathcal{H}} \varphi \supset \chi$ via a proof π.

Then, by monotonicity, $\Gamma, \varphi \vdash_{\mathcal{H}} \varphi \supset \chi$.

Also, $\Gamma, \varphi \vdash_{\mathcal{H}} \varphi$ since $\varphi \in \Gamma \cup \{\varphi\}$.

So we get the following proof tree:

$$\dfrac{\dfrac{\pi + \text{Monotonicity}}{\Gamma, \varphi \vdash \varphi \supset \chi} \qquad \dfrac{}{\Gamma, \varphi \vdash \varphi}\,\text{Ax}}{\Gamma, \varphi \vdash \chi}\,\text{MP}$$

Deduction theorem: Proof ($\Rightarrow$)

Suppose $\Gamma, \varphi \vdash_{\mathcal{H}} \chi$ via a proof tree π.

We show (by induction on the structure of π) that $\Gamma \vdash_{\mathcal{H}} \varphi \supset \psi$ for every sequent $\Gamma, \varphi \vdash \psi$ appearing in π.

Base case: Consider a leaf of π labelled by $\Gamma, \varphi \vdash \psi$.

- $\psi = \varphi$: $\Gamma \vdash_{\mathcal{H}} \varphi \supset \varphi$ (by the earlier proof and monotonicity).
- $\psi \in \Gamma$ or ψ is an instance of **(H1)**, **(H2)**, or **(H3)**: By **(H1)**, we know that $\psi \supset (\varphi \supset \psi)$ is valid. We can now build the following proof.

$$\dfrac{\dfrac{}{\Gamma \vdash \psi} \qquad \dfrac{}{\Gamma \vdash \psi \supset (\varphi \supset \psi)}\,\text{H1}}{\Gamma \vdash \varphi \supset \psi}\,\text{MP}$$

Deduction theorem: Proof $(\Rightarrow)$

Induction Hypothesis: $\Gamma \vdash_{\mathscr{H}} \varphi \supset \psi$ for every $\Gamma, \varphi \vdash \psi$ in π at height $< k$.

Induction Step: Consider a sequent of the form $\Gamma, \varphi \vdash \psi$ appearing in π at height $k \neq 0$. This must be obtained by a subproof as follows.

$$\dfrac{\Gamma, \varphi \vdash \xi \supset \psi \qquad \Gamma, \varphi \vdash \xi}{\Gamma, \varphi \vdash \psi}\ \text{MP}$$

By IH, we get proofs of $\Gamma \vdash \varphi \supset (\xi \supset \psi)$ and $\Gamma \vdash \varphi \supset \xi$. We can now build the following proof, appealing to **(H2)**.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (\varphi \supset (\xi \supset \psi)) \supset ((\varphi \supset \xi) \supset (\varphi \supset \psi))}\ \text{H2} \qquad \begin{array}{c}\text{IH}\\ \vdots \\ \Gamma \vdash \varphi \supset (\xi \supset \psi)\end{array}}{\Gamma \vdash (\varphi \supset \xi) \supset (\varphi \supset \psi)}\ \text{MP} \qquad \begin{array}{c}\text{IH}\\ \vdots \\ \Gamma \vdash \varphi \supset \xi\end{array}}{\Gamma \vdash \varphi \supset \psi}$$

Using the Deduction Theorem

- Using the Deduction Theorem (DT) simplifies proofs a lot.
- **Example**: Show that $\alpha \supset (\alpha \supset \beta) \supset \beta$.
- Difficult if we just have **(H1)**, **(H2)**, **(H3)**, and **MP**.

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- Difficult if we just have **(H1)**, **(H2)**, **(H3)**, and **MP**.
- Instead, use DT. Equivalent: **Show that** $\alpha, \alpha \supset \beta \vdash_{\mathscr{H}} \beta$.
- Let $\Gamma = \{\alpha, \alpha \supset \beta\}$.

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$$\dfrac{\dfrac{}{\Gamma \vdash \alpha \supset \beta}\,\text{Ax} \qquad \dfrac{}{\Gamma \vdash \alpha}\,\text{Ax}}{\Gamma \vdash \beta}\,\text{MP}$$

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- Proof rule DT to switch between equivalent formulations.

Using the Deduction Theorem

Show that $\neg\alpha \supset \beta, \alpha \supset \beta \vdash_{\mathcal{H}} \beta$.

- Suppose we had proofs of $\neg\beta \supset \neg\alpha$ and $\neg\beta \supset \alpha$
- Can get β from these using **(H3)** and applications of MP.

Using the Deduction Theorem

Show that $\neg\alpha \supset \beta, \alpha \supset \beta \vdash_{\mathcal{H}} \beta$.

- Suppose we had proofs of $\neg\beta \supset \neg\alpha$ and $\neg\beta \supset \alpha$
- Can get β from these using **(H3)** and applications of MP.

$$
\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (\neg\beta \supset \neg\alpha) \supset ((\neg\beta \supset \alpha) \supset \beta)}\,\text{H3} \qquad \begin{matrix}\text{???} \\ \vdots \\ \Gamma \vdash \neg\beta \supset \neg\alpha\end{matrix}}{\Gamma \vdash (\neg\beta \supset \alpha) \supset \beta}\,\text{MP} \qquad \begin{matrix}\text{???} \\ \vdots \\ \Gamma \vdash \neg\beta \supset \alpha\end{matrix}}{\Gamma \vdash \beta}\,\text{MP}
$$

- Suppose we show that $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$ for any φ and ψ.
- Then $\Gamma \vdash \neg\beta \supset \neg\alpha$ and $\Gamma \vdash \neg\beta \supset \neg\neg\alpha$.
- Suppose we also show that $\neg\neg\varphi \vdash \varphi$ for any φ. Then done!

Using the Deduction Theorem

- Show that $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$ for any φ and ψ.
- Equivalent to showing that $\varphi \supset \psi, \neg\psi \vdash \neg\varphi$. Let $\Gamma = \{\varphi \supset \psi, \neg\psi\}$.
- Suppose we proved $\neg\neg\varphi \supset \psi$ and $\neg\neg\varphi \supset \neg\psi$. Then use **(H3)** and MP.
- To show $\Gamma \vdash \neg\neg\varphi \supset \neg\psi$, equivalent: $\Gamma, \neg\neg\varphi \vdash \neg\psi$; use Ax.
- **Exercise**: Show that if $\Gamma, \varphi \vdash \psi$, then $\Gamma, \neg\neg\varphi \vdash \psi$ for all φ, ψ.

Using the Deduction Theorem

- Show that $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$ for any φ and ψ.
- Equivalent to showing that $\varphi \supset \psi, \neg\psi \vdash \neg\varphi$. Let $\Gamma = \{\varphi \supset \psi, \neg\psi\}$.
- Suppose we proved $\neg\neg\varphi \supset \psi$ and $\neg\neg\varphi \supset \neg\psi$. Then use **(H3)** and MP.
- To show $\Gamma \vdash \neg\neg\varphi \supset \neg\psi$, equivalent: $\Gamma, \neg\neg\varphi \vdash \neg\psi$; use Ax.
- **Exercise**: Show that if $\Gamma, \varphi \vdash \psi$, then $\Gamma, \neg\neg\varphi \vdash \psi$ for all φ, ψ.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{H3}
\UnaryInfC{$\Gamma \vdash (\neg\neg\varphi \supset \neg\psi) \supset (\neg\neg\varphi \supset \psi) \supset \neg\varphi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\Gamma, \neg\neg\varphi \vdash \neg\psi$}
\RightLabel{DT}
\UnaryInfC{$\Gamma \vdash (\neg\neg\varphi \supset \neg\psi)$}
\RightLabel{MP}
\BinaryInfC{$\Gamma \vdash (\neg\neg\varphi \supset \psi) \supset \neg\varphi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\Gamma \vdash \varphi \supset \psi$}
\RightLabel{DT}
\UnaryInfC{$\Gamma, \varphi \vdash \psi$}
\RightLabel{\textbf{Exercise!}}
\UnaryInfC{$\Gamma, \neg\neg\varphi \vdash \psi$}
\RightLabel{DT}
\UnaryInfC{$\Gamma \vdash \neg\neg\varphi \supset \psi$}
\RightLabel{MP}
\BinaryInfC{$\Gamma \vdash \neg\varphi$}
\end{prooftree}
$$

Maximal sets

- So far, proofs of an expression from some context (of the form $X \vdash_{\mathcal{H}} \psi$)
- What do we know if there is **no** proof of a ψ from some X?
- There is a "largest possible extension of X" which **does not** prove ψ, any "extension" of which **does** prove ψ.
- **Theorem**: If $X \nvdash_{\mathcal{H}} \psi$, then there is a maximal Y s.t. $X \subseteq Y$ and $Y \nvdash_{\mathcal{H}} \psi$.
- **Proof**: Consider some X and ψ such that $X \nvdash_{\mathcal{H}} \psi$.
- PL is countable, assume a fixed enumeration of expressions $\varphi_0, \varphi_1, \ldots$

Maximal sets: Proof

- Basic idea: Examine each expression in PL, choose whether to throw it in or not (depending on whether it derives ψ or not).
- Build a sequence of sets where $X_0 \coloneqq X$ and each X_{i+1} defined as below.

$$X_{i+1} \coloneqq \begin{cases} X_i & \text{if } X_i, \varphi_i \vdash_{\mathcal{H}} \psi \\ X_i \cup \{\varphi_i\} & \text{otherwise} \end{cases} \qquad \text{Set } Y \coloneqq \bigcup_{i \geqslant 0} X_i.$$

- For each expression φ_i, it either goes in at the X_{i+1} stage (and therefore into Y) or not at all.
- Y is a countable union of sets X_i.
- We will overload notation and use $Y \vdash_{\mathcal{H}} \alpha$ (for any α), even though Y is countable, to denote a proof of α from some finite subset of Y.
- $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq Y$, and $X_i \nvdash_{\mathcal{H}} \psi$ for every $i \geqslant 0$.

Maximal sets: Proof (contd.)

- **First, show $Y \nvdash_{\mathcal{H}} \psi$ by contradiction**. Suppose π is a proof of $Y \vdash \psi$.
- π is a finite tree, can only use finitely many assumptions (at the leaves).
- Consider any assumption of the form $Y \vdash \varphi_i$. Then, $\varphi_i \in Y$ (since this sequent was a leaf and φ_i is an assumption).
- Suppose the largest index of any such assumption in π is k.
- Since $\varphi_i \in Y$, it must be that $\varphi_i \in X_j$ for some j.
- Either $\varphi_i \in X$ and $j = 0$, or $j = i + 1$ since φ_i is, at the very latest, examined when we construct X_{i+1}.
- $X_{i+1} \subseteq X_{k+1}$, and since we chose i arbitrarily, **every** assumption in π belongs to X_{k+1}. Therefore, $X_{k+1} \vdash_{\mathcal{H}} \psi$.
- This contradicts our claim that $X_i \nvdash_{\mathcal{H}} \psi$ for every $i \geqslant 0$, so $Y \nvdash_{\mathcal{H}} \psi$.

Maximal sets: Proof (contd.)

- **Now show that Y is a maximal non-deriving context**. Consider any $\varphi_\ell \notin Y$.
- Then $\varphi_\ell \notin X_{\ell+1}$, which can only happen if $X_\ell, \varphi_\ell \vdash_{\mathcal{H}} \psi$.
- Since $X_\ell \subseteq Y$, by monotonicity, $Y, \varphi_\ell \vdash_{\mathcal{H}} \psi$.
- So for any Z s.t. $Y \subset Z$, it is the case that $Z \vdash_{\mathcal{H}} \psi$, i.e. Y is maximal.

Hilbert system: Completeness

- Usually we are only interested in proof search
- So why worry about contexts that **do not** prove an expression?
- To show that the Hilbert system is **complete**
- "Any valid expression in propositional logic is provable in this system"
- **Theorem (Completeness)**: For any PL expression φ, if $\models \varphi$, then $\vdash_{\mathcal{H}} \varphi$.
- We will prove by contradiction a stronger claim, unimaginatively called
- **Thm (Strong Completeness)**: For all $X \cup \{\varphi\}$, if $X \models \varphi$ then $X \vdash_{\mathcal{H}} \varphi$.

Hilbert system: Completeness (Intuition)

- Having assumed $X \nvdash_{\mathcal{H}} \varphi$, we will aim to show that $X \nvDash \varphi$.
- There is some valuation τ such that $\tau \vDash \psi$ for every $\psi \in X$, but $\tau \nvDash \varphi$.
- How do we demonstrate such a valuation?
- Unclear how to build such a τ directly
- Instead, we build the largest possible set *EP* of expressions for which τ is a model, and extract τ from this.
- What properties should such an *EP* satisfy?

Hilbert system: Completeness (Intuition)

- We want $\psi \in EP$ iff $\tau \vDash \psi$ for any $\psi \in \mathsf{PL}$.
- Since $\tau \vDash X$ and $\tau \nvDash \varphi$, we want $X \subseteq EP$, but $\varphi \notin EP$.
- EP should "agree" with τ
  - $\neg\psi \in EP$ iff $\psi \notin EP$
  - $\psi \supset \chi \in EP$ iff either $\psi \notin EP$ or $\chi \in EP$
- For each $\psi \in \mathsf{PL}$, we need to either add it or its negation to EP
- Need to do this in a systematic manner.
  - What if I throw in α and $\neg\beta$ but then add $\alpha \supset \beta$?
  - No valuation is a model for this set of expressions.
- We set up the notion of maximal non-deriving contexts just for this!
- We drop the $\mathscr{H}$ subscript in the proof that follows, as it is implicit.

Hilbert system: Proof of strong completeness

- We show that if $X \nvdash \varphi$, then $X \nvDash \varphi$
- There is a maximal Y s.t. $Y \nvdash \varphi$. So for every $\psi \notin Y$, $Y, \psi \vdash \varphi$.
- If $Y \vdash \psi$, by Cut, we would get $Y \vdash \varphi$, so we get $Y \vdash \psi$ iff $\psi \in Y$.
- We define a valuation τ as follows

$$\tau(p) = \begin{cases} T & \text{if } p \in Y \cap AP, \\ F & \text{otherwise} \end{cases}$$

- Suppose we show that $\psi \in Y$ iff $\tau \vDash \psi$ for all $\psi \in \mathsf{PL}$.
- Then, since $X \subseteq Y$, $\tau \vDash X$. Also, since $Y \nvdash \varphi$, $\varphi \notin Y$, and $\tau \nvDash \varphi$.
- So $X \nvDash \varphi$, and we have the required contradiction.

Hilbert system: Proof of strong completeness (contd.)

- So now we show that $\psi \in Y$ iff $\tau \vDash \psi$ for all $\psi \in \mathsf{PL}$.
- As usual, by induction on the structure of ψ.
- $\psi = p \in AP$: By definition of τ, $p \in Y$ iff $\tau \vDash p$
- $\psi = \neg\chi$: First show that $\neg\chi \in Y$ iff $\chi \notin Y$ for any χ.
  - Suppose $\{\neg\chi, \chi\} \subseteq Y$.
  - **Exercise**: Show that $\neg\chi, \chi \vdash \alpha$ in $\mathscr{H}$ for any χ, α.
  - So if $\{\neg\chi, \chi\} \subseteq Y$, then $Y \vdash \varphi$ (contradiction)
  - Suppose $\{\neg\chi, \chi\} \cap Y = \emptyset$.
  - By maximality, $Y, \neg\chi \vdash \varphi$ and $Y, \chi \vdash \varphi$.
  - But as we saw earlier, $\neg\alpha \supset \beta, \alpha \supset \beta \vdash \beta$, so $Y \vdash \varphi$ (contradiction)
- So $\neg\chi \in Y$ iff $\chi \notin Y$ iff (by IH) $\tau \nvDash \chi$ iff $\tau \vDash \neg\chi$.

Hilbert system: Proof of strong completeness (contd.)

- We show that $\psi \in Y$ iff $\tau \vDash \psi$ for all $\psi \in \mathsf{PL}$ (contd.)
- Recall that $Y \vdash \psi$ iff $\psi \in Y$ for any ψ.
- $\psi = \chi \supset \xi$: First show that $\chi \supset \xi \in Y$ iff $\chi \notin Y$ or $\xi \in Y$.
- First we show that if $\chi \supset \xi \in Y$, then $\chi \notin Y$ or $\xi \in Y$.
  - Suppose $\chi \supset \xi \in Y$. Then $Y \vdash \chi \supset \xi$.
  - Either $\neg\chi \in Y$ or $\chi \in Y$ (by maximality)
  - Now, if $\chi \in Y$, then $Y \vdash \chi$, and by MP, we get $Y \vdash \xi$, i.e. $\xi \in Y$.
  - Therefore, $\neg\chi \in Y$, or $\xi \in Y$.

Hilbert system: Proof of strong completeness (contd.)

- We show that $\psi \in Y$ iff $\tau \vDash \psi$ for all $\psi \in \mathsf{PL}$ (contd.)
- $\psi = \chi \supset \xi$: First show that $\chi \supset \xi \in Y$ iff $\chi \notin Y$ or $\xi \in Y$.
- Now we show that if $\chi \notin Y$ or $\xi \in Y$ then $\chi \supset \xi \in Y$.
- Suppose $\chi \notin Y$.
  - Then $\neg\chi \in Y$ and $Y \vdash \neg\chi$. By monotonicity, $Y, \chi \vdash \neg\chi$.
  - Since $Y, \chi \vdash \chi$, we can compose these proofs, and get $Y, \chi \vdash \alpha$ for any α.
  - In particular, $Y, \chi \vdash \xi$, and DT gives us $Y \vdash \chi \supset \xi$, i.e. $\chi \supset \xi \in Y$.
- Suppose $\xi \in Y$. Then $Y \vdash \xi$, and monotonicity gives us $Y, \chi \vdash \xi$. Using DT, we get $Y \vdash \chi \supset \xi$, i.e. $\chi \supset \xi \in Y$.
- So $\chi \supset \xi \in Y$ iff $\chi \notin Y$ or $\xi \in Y$ iff (by IH) $\tau \nvDash \chi$ or $\tau \vDash \xi$ iff $\tau \vDash \chi \supset \xi$.