

LANGUAGES

ACCEPTED

By PDAs

Recall: We formally defined pushdown automata as 6-tuples

$$M = (Q, \Sigma, \Gamma, \Delta, q_0, F)$$

Set of states
input alphabet
stack alphabet (contains \perp)

transition relation (nondeterministic!)

initial state $\in Q$

accept states $\subseteq Q$.

$$\Delta \subseteq (Q \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\})) \times (Q \times \Gamma^*)$$

A PDA which accepts $L = \{a^n b^n \mid n \geq 0\}$ by empty stack was

$$M = (Q_0, Q_1, \{a, b\}, \{\perp, c\}, \Delta, q_0, \emptyset), \text{ where}$$

$$\Delta = \{((q_0, a, \epsilon), (q_0, c)), ((q_0, b, c), (q_1, \epsilon)), ((q_1, b, c), (q_1, \epsilon))\}$$

A configuration is an operational snapshot of the PDA, capturing its current state, remaining input string, and the current contents of the stack (read top to bottom).

Today: How do we show that $L(M) = L = \{a^n b^n \mid n \geq 0\}$?

- ① For every $w \in L$, there is a sequence of moves which takes M from the initial configuration to a final configuration, and
- ② for every $w \in \{a, b\}^* \setminus L$, no sequence of moves can take M from the initial configuration to a final configuration.

If we show this, we are done!

$$\Delta = \{((q_0, a, \epsilon), (q_0, c)), ((q_0, b, c), (q_1, \epsilon)), ((q_1, b, c), (q_1, \epsilon))\}$$

Recall: Suppose $((q, a, A), (q', s)) \in \Delta$. Then,
 $(q, a\omega, AS) \xrightarrow{M^1} (q', \omega, sS)$, for any $\omega \in \Sigma^*$, $S \in \Gamma^*$.

What configuration changes are possible in one move of M ?

$$(q_0, a\omega, S) \xrightarrow{M^1} (q_0, \omega, CS) \quad (q_0, ab, \perp) \rightarrow (q_0, b, CL)$$

$$(q_0, b\omega, CS) \xrightarrow{M^1} (q_1, \omega, S) \quad (q_1, \epsilon, \perp)$$

$$(q_1, b\omega, CS) \xrightarrow{M^1} (q_1, \omega, S) \quad (q_0, aabb, \perp) \rightarrow (q_0, abb, CL)$$

$$(q_1, b\omega, CS) \xrightarrow{M^1} (q_1, \omega, S) \quad (q_1, b, CL) \leftarrow (q_0, bb, CC)$$

$$(q_1, \epsilon, \perp)$$

What does it mean for M to accept a string ω ?

$$(q_0, \omega, \perp) \xrightarrow[M]{*} (q, \epsilon, \perp) \text{ for some } q \in Q$$
$$\xrightarrow[M]{*} (f, \epsilon, s) \text{ for fef, } s \in \Gamma^*$$

Want to prove:

For any $\omega \in \Sigma^*$, $(q_0, \omega, \perp) \xrightarrow[M]{*} (q, \epsilon, \perp)$ for some $q \in Q$
iff $\omega \in L$.

① $\omega = \epsilon$: $(q_0, \epsilon, \perp) \xrightarrow[M]{\circ} (q_0, \epsilon, \perp)$

② $\omega = b\omega'$: $b\omega' \notin L$ for any ω' , no transitions out of $(q_0, b\omega', \perp)$.

③ $\omega = a\omega'$: What do we do in this case?

To prove: For any $m, n, k \geq 0$, where $m+n > 0$,

$(q_0, a^m b^n, C^k \perp) \xrightarrow[m]{*} (q_1, \epsilon, 1)$ iff $n = m+k$.

Proof: By induction on m .

① $m=0$: Since $m+n > 0$, $n \neq 0$. Two cases arise.

(a) $k=0$: There are no transitions out of (q_0, b^n, \perp) .

(b) $k > 0$: $(q_0, b^n, C^k \perp) \xrightarrow[m]{1} (q_1, b^{n-1}, C^{k-1} \perp)$

Two cases arise.

(i) $n=1$: $(q_1, b^{n-1}, C^{k-1} \perp) = (q_1, \epsilon, C^{k-1} \perp)$

There are no transitions out of $(q_1, \epsilon, -)$.

$\text{So } (q_0, b^1, C^k \perp) \xrightarrow{*m} (q_1, \epsilon, \perp)$ iff $k=1$, i.e. $n=m+k$.

(ii) $n > 1$: The only transition out of $(q_1, b^{n-1}, C^{k-1} \perp)$ is of the form $(q_1, b^{n-1}, C^{k-1} \perp) \xrightarrow{1} (q_1, b^{n-2}, C^{k-2} \perp)$

Claim: For $l, p > 0$, $(q_1, b^l, C^p \perp) \xrightarrow{*m} (q_1, \epsilon, \perp)$ iff $l=p$

Prove this by induction on l .

$\text{So } (q_0, b^n, C^k \perp) \xrightarrow{1} (q_1, b^{n-1}, C^{k-1} \perp) \xrightarrow{*m} (q_1, \epsilon, \perp)$ iff $n-1=k-1$

iff $n=k+0$ iff $n=k+m$.

This concludes the case when $m=0$.

② $m = l+1$ for some $l > 0$:

Suppose $(q_0, a^{l+1}b^n, c^k \perp) \xrightarrow[m]{*} (q_1, \epsilon, \perp)$.

$(q_0, a^{l+1}b^n, c^k \perp) \xrightarrow[m]{1} (q_0, ab^n, c^{k+1} \perp)$

this is the only possible move!

If $(q_0, ab^n, c^{k+1} \perp) \xrightarrow[m]{*} (q_1, \epsilon, \perp)$, then

by IH, $n = l+k+1 = l+1+k = m+k$.

On the other hand, suppose $n = m+k = l+1+k$. Then,

$(q_0, a^{l+1}b^{l+1+k}, c^k \perp) \xrightarrow[m]{1} (q_0, ab^{l+1+k}, c^{k+1} \perp) \xrightarrow[m]{*} (q_1, \epsilon, \perp)$. by IH \square

Going back to our main proof,

③ $\omega = a\omega'$: Two cases arise

(a) $\omega = a^m b^n$ for some $m > 0$. Then,

$(q_0, \omega, L) = (q_0, a^m b^n, C^0 L)$. By the above,

$(q_0, a^m b^n, C^0 L) \xrightarrow[m]{*} (q_1, \epsilon, L)$ iff $n = m + 0$ i.e. $n = m$.

(b) $\omega \neq a^m b^n$ for any m, n . Then, $\omega = a^m b^n \omega'$, where ω' starts with a.

Two cases arise.

$$\underline{\text{Claim}}: (q_0, a^m \omega, \perp) \xrightarrow[m]{m} (q_0, \omega, C^m \perp)$$

Prove this by induction on m

$$(q_0, b^n, C^k \perp) \xrightarrow[m]{*^n} (q_1, \epsilon, \perp) \text{ iff } n = k \quad \begin{matrix} \text{(from earlier,)} \\ \text{since } m = 0 \end{matrix}$$

If $n > m$, putting these together,

$$(q_0, a^m b^n \omega', \perp) \xrightarrow[m]{*} (q_1, b^{n-m} \omega', \perp) \quad \begin{matrix} \text{no transitions out of} \\ \text{this configuration!} \end{matrix}$$

If $n \leq m$,

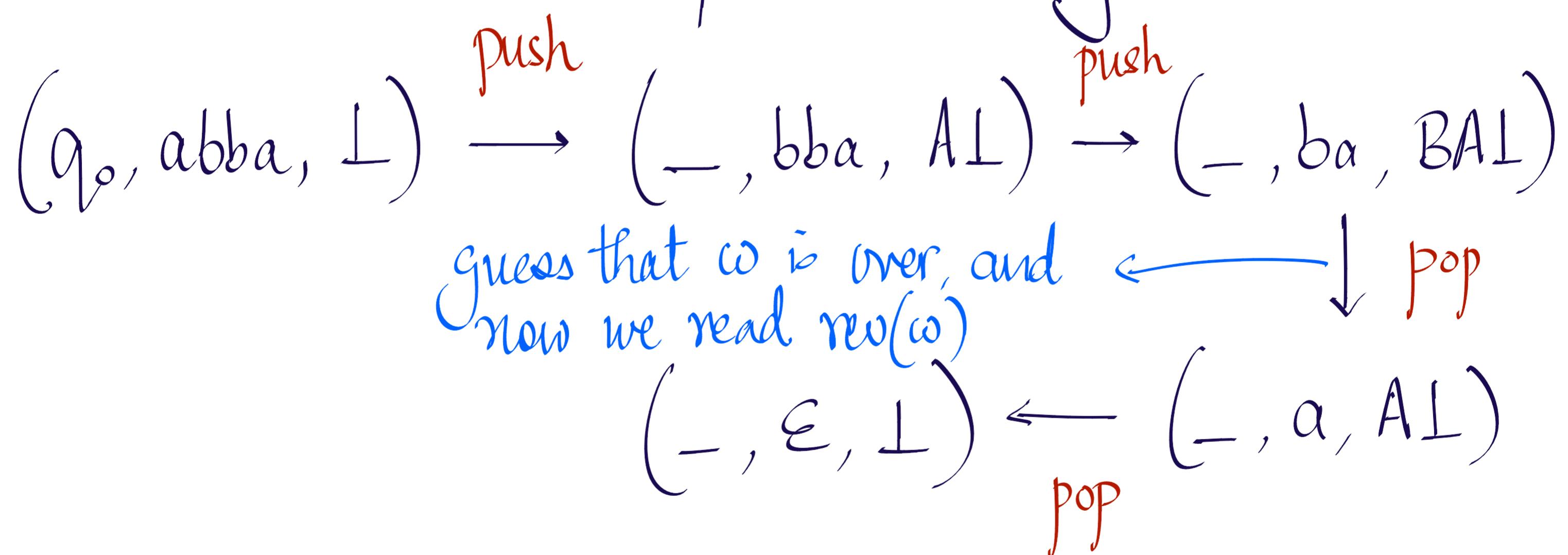
$$(q_0, a^m b^n \omega', \perp) \xrightarrow[m]{m} (q_0, b^n \omega', C^m \perp) \xrightarrow[m]{*} (q_1, \omega', C^{m-n} \perp)$$

ω' starts with a, so no transitions out of this configuration!

Design a PDA which recognizes by empty stack the language

$$L_{\text{pal}} = \{\omega \cdot \text{rev}(\omega) \mid \omega \in \{a, b\}^{2^*}\}$$

What behaviour do we expect on a string abba? $\text{abba} \in L$



What about on a string abb? $\text{abb} \notin L$.

$$\Delta = \{ ((q_0, a, \epsilon), (q_0, A)), ((q_0, b, \epsilon), (q_0, B)), \\ ((q_0, a, A), (q_1, \epsilon)), ((q_0, b, B), (q_1, \epsilon)), \\ ((q_1, a, A), (q_1, \epsilon)), ((q_1, b, B), (q_1, \epsilon)) \}$$

Claim: $M = (\{q_0, q_1\}, \{a, b\}, \{A, B, \perp\}, \Delta, q_0, \phi)$
 accepts L_{pal} by empty stack.

What 1-move changes in configuration are possible?

$$(q_0, a\omega, S) \xrightarrow{1_M} (q_0, \omega, AS)$$

$$(q_0, b\omega, S) \xrightarrow{1_M} (q_0, \omega, BS)$$

$$(q_0, a\omega, AS) \xrightarrow{1_M} (q_1, \omega, S)$$

$$(q_0, b\omega, BS) \xrightarrow{1_M} (q_1, \omega, S)$$

$$(q_1, a\omega, AS) \xrightarrow{1_M} (q_1, \omega, S)$$

$$(q_1, b\omega, BS) \xrightarrow{1_M} (q_1, \omega, S)$$

For which $\omega \in \Sigma^*$ is it true that
 $(q_0, \omega, L) \xrightarrow{*} (q, \epsilon, L)$ for some $q \in Q$?

$\omega = \epsilon$: trivially done

$\omega = a\omega'$:

$$(q_0, a\omega', L) \xrightarrow{1_M} (q_0, \omega', AL)$$

$\omega = b\omega'$:

$$(q_0, b\omega', L) \xrightarrow{1_M} (q_0, \omega', BL)$$

) and then what?

Try a few examples, figure out a general pattern.

$$(q_0, abba, \perp) \xrightarrow[\mathcal{M}]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, ba, BA\perp) \xrightarrow[\mathcal{M}]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, bba, A\perp) \xrightarrow[\mathcal{M}]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, aabbaa, \perp) \xrightarrow[\mathcal{M}]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, abbaa, A\perp) \xrightarrow[\mathcal{M}]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, baba, \perp) \not\xrightarrow[\mathcal{M}]{*} (q, \varepsilon, \perp) \text{ for any } q \in Q$$

$$(q_0, ba, AB\perp) \not\xrightarrow[\mathcal{M}]{*} (q, \varepsilon, \perp) \text{ for any } q \in Q$$