Lecture 13 - Unification

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COL703/COL7203 - Logic for Computer Science

Recap: Substitutions

- A **substitution** θ is a partial map from $\mathscr{V}$ to $\mathrm{T}(\Sigma)$, with a finite domain
- Read $\theta = \{t/x\}$ as "x is replaced by t under θ"
- **Substitution Lemma**: Given an interpretation $\mathcal{I} = ((M, \iota), \sigma)$ for some Σ, a term $t \in \mathrm{T}(\Sigma)$, an expression $\varphi \in \mathsf{FO}_\Sigma$, and a substitution $\{u/x\}$ such that $u^{\mathcal{I}} = m \in M$, the following hold:
  - $(t\{u/x\})^{\mathcal{I}} = t^{\mathcal{I}[x \mapsto m]}$
  - $\mathcal{I} \vDash \varphi\{u/x\}$ iff $\mathcal{I}[x \mapsto m] \vDash \varphi$.
- Only consider "admissible" substitutions θ for terms/expressions; range of θ does not contain any variables that appear in the term/expression

Recap: Normal forms

- Prenex Normal Form (PNF): FO expression where all quantifiers "appear at the front"
- $Q_1 x_1 \dots Q_n x_n. [\varphi]$ is in PNF if φ is **quantifier-free (qf)**.
- For any FO expression φ, there exists a logically equivalent ψ in PNF.
- Choice of witness for $\exists$ might depend on value chosen for $\forall$ if $\exists$ appears "deeper" than $\forall$
- Move to **Skolem Normal Form**
- PNF expression $Q_1 x_1 \dots Q_n x_n. [\varphi]$ is in SNF if $Q_i = \forall$ for every $1 \leqslant i \leqslant n$.
- Intuition: Replace every $\exists y$ by a "Skolem function" which computes y using all the (other) variables y depends on.
- For any FO sentence φ, there exists an equisatisfiable ψ in SNF.

Recap: Herbrand models

- Universe is $T^g(\Sigma)$, the set of all ground terms over the signature Σ
- Map each symbol in the syntax to itself
- Assignments map variables to ground terms
- A sentence $\varphi \in FO_\Sigma$ is satisfiable iff its SNF form φ_{snf} is satisfiable iff Γ^g, the set of all ground instances of the qf subexpression in φ_{snf}, is satisfied by a Herbrand model.
- A sentence is unsatisfiable iff some finite set of ground instances of its qf subexpressions is unsatisfiable.

Unification

- Consider a signature $\Sigma = (\{m, n\}, \{f/2\}, \emptyset)$
- Now consider two terms $t_1 = f(m, y)$ and $t_2 = f(x, n)$
- What if I applied the substitution $\theta = \{m/x, n/y\}$ to t_1 and t_2?

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- What if I applied the substitution $\theta = \{m/x, n/y\}$ to t_1 and t_2?
- t_1 and t_2 **unify** to the same term $f(m, n)$ under θ
- **Unification** is the problem of finding a substitution θ so as to make some terms identical.
- One basically solves an equation of the form $t_1\theta = t_2\theta$ to see if there is some θ which assigns the right meanings to the variables in t_1 and t_2 and renders them the same.

Unifiability

- A finite set of terms $T = \{t_i \mid 1 \leqslant i \leqslant n\}$ is said to be **unifiable** if there exists a θ such that $t_i\theta = t_j\theta$ for all $1 \leqslant i, j \leqslant n$.
- θ is called a **unifier** of T
- So for our earlier example, consider $T = \{t_1, t_2\} = \{f(m, y), f(x, n)\}$
- T is unifiable, and $\theta = \{m/x, n/y\}$ is a unifier for T
- What about $T' = \{f(x, y), f(y, x)\}$?

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- What about $\theta'' = \{x/y, y/x\}$? Does θ'' cause T' to unify?
- **No!** $f(x, y)\theta'' = f(y, x)$ and $f(y, x)\theta'' = f(x, y)$.

Unifiability: More examples

- Consider a signature Σ with two distinct unary functions f and g
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- Is $T = \{f(x), g(y)\}$ unifiable? No! This is called a **clash**.
- The arity of f and g is immaterial; holds for any two distinct symbols
- If two terms are unifiable, then
  - Either they are headed by the same function symbol[1], or
  - They are both variables, or
  - One is headed by some function symbol and the other is a variable.
- Consider $T = \{x, y\}$ and $\theta = \{f(z)/x, f(z)/y\}$
- Is θ a unifier of T?

[1]The symbol that marks the root nodes of their ASTs

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- Consider $T = \{x, y\}$ and $\theta = \{f(z)/x, f(z)/y\}$
- Is θ a unifier of T? Yes
- Is $\theta' = \{x/y\}$ a unifier of T? Also yes!
- Can we compare θ and θ' using some ordering relation?

[1]The symbol that marks the root nodes of their ASTs

Generality of unifiers

- θ assigns a specific term to x and to y; θ' just replaces y by x
- θ' "less constrained" than θ
- Can apply $\tau = \{f(z)/x\}$ to the result of θ' to obtain the result of θ
- A substitution θ' is **at least as general as** another substitution θ (denoted $\theta' \succcurlyeq \theta$) if there exists a substitution τ such that $\theta = \tau \circ \theta'$ (where $\circ$ denotes function composition)
- $\theta' \sim \theta$ if $\theta' \succcurlyeq \theta$ and $\theta \succcurlyeq \theta'$.
- θ' is **strictly more general than** θ (denoted $\theta' \succ \theta$) if $\theta' \succcurlyeq \theta$ and $\theta \not\succcurlyeq \theta'$.
- **Exercises**: Show that, on the set of all substitutions from $\mathscr{V}$ to $T(\Sigma)$,
    - $\succcurlyeq$ is a reflexive transitive relation
    - $\succ$ is an irreflexive transitive relation
    - $\sim$ is an equivalence relation
    - If $\theta \sim \theta'$ and $\tau \circ \theta = \theta'$, then $\mathrm{rng}(\tau) \subseteq \mathscr{V}$.

Most general unifiers

- Let T be a unifiable set of terms
- θ' is called **a most general unifier (mgu)** of T if for each unifier θ of T, there is a τ such that $\theta = \tau \circ \theta'$.
- If a set of terms is unifiable, then it has an mgu
- Can a set have multiple mgus?

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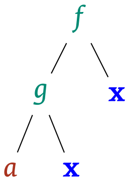
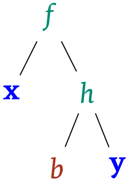
- If a set of terms is unifiable, then it has an mgu
- Can a set have multiple mgus? Yes!
- $T = \{x, y\}$ and $\theta = \{x/y\}$ and $\theta' = \{y/x\}$; both are mgus of T
- **Exercise**: If θ and θ' are both mgus of T, then $\theta \sim \theta'$.

More about unifiability: Example

Suppose $T = \{f(g(a,x),x), f(x,h(b,y))\}$ where $x, y \in \mathscr{V}$ and $a, b \in \mathscr{C}$.

Is T unifiable?





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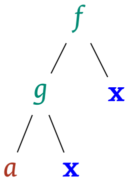

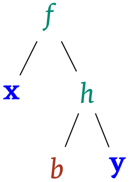

- Need to make x, $g(a,x)$, and $h(b,y)$ identical
- Two problems with this
  - $g \neq h$, so we have a **clash**, and $g(a,x)$ and $h(b,y)$ do not unify
  - x and $g(a,x)$ can never unify (this is called an **occurs check**)
- Obstacles of the above two sorts are the **only** roadblocks to unifiability
- If they do not feature, the set is unifiable!

A unification algorithm

- Start with a system of equations

$$
\begin{aligned}
l_1 &= r_1 \\
l_2 &= r_2 \\
&\vdots \\
l_n &= r_n
\end{aligned}
$$

- Perform a series of transformations till you cannot anymore.
- What sort of terms can occur in l_i and r_i? How do we handle them?
- What combinations already rule out unification?

A unification algorithm: Transformations

- $l_i = t \notin \mathscr{V}$ and $r_i = x$: Replace $l_i = r_i$ by $x = t$
- $l_i = x$ and $r_i = x$: Remove the equation
- $l_i = f(\ldots)$ and $r_i = g(\ldots)$: The following cases arise.
  - $f \neq g$: Clash; no unification possible. Terminate.
  - $f = g$: Then $l_i = f(t_1, \ldots, t_k)$ and $r_i = f(u_1, \ldots, u_k)$. Replace $l_i = r_i$ by k new equations, each of the form $t_j = u_j$, for $1 \leqslant j \leqslant k$.
- $l_i = x$ and $r_i = t \notin \mathscr{V}$ such that $x \in \mathsf{vars}(t)$: Occurs check; no unification possible. Terminate.
- $l_i = x$ and $r_i = t$ and $x \notin \mathsf{vars}(t)$: Replace every occurrence of x in $\{l_j \cup r_j \mid 1 \leqslant j \leqslant n, j \neq i\}$ by t.

Example

1

$$g(Y) = X$$

$$f(X, h(X), Y) = f(g(Z), W, Z)$$

Example

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$$g(Y) = X$$

$$f(X, h(X), Y) = f(g(Z), W, Z)$$

2

$$X = g(Y)$$

$$f(X, h(X), Y) = f(g(Z), W, Z)$$

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$$f(X, h(X), Y) = f(g(Z), W, Z)$$

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$$f(X, h(X), Y) = f(g(Z), W, Z)$$

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$$X = g(Y)$$
$$X = g(Z)$$
$$h(X) = W$$
$$Y = Z$$

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