ENAE 441 - 0101

HW01: Orbits and 2BP

Due on September 25th. 2025 at 09:30 AM

Dr. Martin, 09:30 AM

Vai Srivastava

October 27, 2025

Problem 1: The Eccentricity Vector

1. Use the definition of the eccentricity vector $\mathbf{e}$ to prove that the total energy of the system,
$$\epsilon = \frac{\mathbf{v}\cdot\mathbf{v}}{2} - \frac{\mu}{r}$$
is equal to
$$\epsilon = -\frac{\mu}{2a}.$$
2. Use the expression
$$\epsilon = \frac{\mathbf{v}\cdot\mathbf{v}}{2} - \frac{\mu}{r}$$
to prove that
$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0.$$

Solution

Part A

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{v} &= |\mathbf{v}|^2 = v^2 \\
\mathbf{v} &= \dot{\mathbf{r}} \\
|\mathbf{h}| &= h \\
\mathbf{e} &= \frac{1}{\mu}\left(\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{|\mathbf{r}|}\right) \\
\dot{\mathbf{r}} \times \mathbf{h} &= vh\hat{r} \\
-\mu\frac{\mathbf{r}}{|\mathbf{r}|} &= -\mu\hat{r} \\
\implies \mathbf{e} &= \left(\frac{vh}{\mu} - 1\right)\hat{r} \\
\text{at periapsis:} & \\
r_p v_p &= h \\
r_p &= a\left(1 - e\right) \\
\implies a &= \frac{r}{1-e} \\
\mathbf{e} &= \left(\frac{rv^2}{\mu} - 1\right)\hat{r} \\
\implies v^2 &= \frac{\left(e+1\right)\mu}{r} \\
\epsilon &= \frac{v^2}{2} - \frac{\mu}{r} \\
&= \frac{v^2}{1} - \frac{v^2}{e+1} \\
&= \frac{\left(e+1\right)\mu}{2r} - \frac{\mu}{r} \\
&= \frac{\left(e+1\right)\mu - 2\mu}{2r} \\
&= -\mu\frac{1-e}{2r}
\end{aligned}$$

Problem 1 continued on next page...

$$
\begin{aligned}
&= -\frac{\mu}{2}\left(\cancelto{\frac{1}{a}}{\frac{1-e}{r}}\right) \\
\implies \epsilon &= -\frac{\mu}{2a}.
\end{aligned}
$$

Part B

$$
\begin{aligned}
\epsilon &= \frac{\mathbf{v}-\mathbf{v}}{2} - \frac{\mu}{r} \\
&= \frac{\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}}{2} - \frac{\mu}{r} \\
&= \frac{\left(\dot{r}\right)^2}{2} - \frac{\mu}{r} \\
\frac{\mathrm{d}\epsilon}{\mathrm{d}t} &= \frac{2\dot{r}\ddot{r}}{2} + \frac{\mu\dot{r}}{r^2} \\
&= \dot{r}\left(\ddot{r} + \cancelto{-\ddot{r}}{\left(\frac{\mu}{r^2}\right)}\right) \\
&= \dot{r}\,\cancelto{0}{\left(\ddot{r}-\ddot{r}\right)} \\
\implies \frac{\mathrm{d}\epsilon}{\mathrm{d}t} &= 0.
\end{aligned}
$$

Problem 2: Polar Coordinates

Prove that

$$\ddot{\mathbf{r}} \times \mathbf{h} = \mu \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right).$$

Solution

$$\begin{aligned}
\ddot{\mathbf{r}} \times \mathbf{h} &= \left(-\frac{\mu}{r^2}\hat{r}\right) \times \left(r^2\dot{\theta}\hat{h}\right) \\
&= -\mu\dot{\theta}\hat{r}\hat{h} \\
&= \mu\dot{\theta}\hat{\theta} \\
\implies \ddot{\mathbf{r}} \times \mathbf{h} &= \mu \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right).
\end{aligned}$$

Problem 3: Orbital Element Calculations

Implement the following calculations in Python:

1. Takes state vector $\mathbf{X}$ and converts to orbital elements **œ**.
2. Takes vector of the orbital elements **œ** and converts to $\mathbf{X}$.
3. Takes a set of orbital elements, and calculates the orbital period.

Code

See the Python code for this assignment.

Problem 4: Propagate and Visualize Orbits

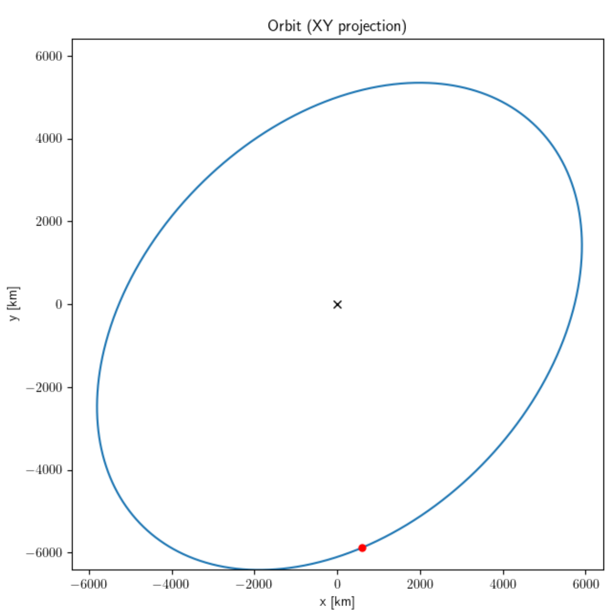
Using SciPy's `solve_ivp` function that takes the cartesian state vector from Problem 3, $\mathbf{X}(t_0)$, and integrate the orbit for one period using the following differential equation:

$$^{\mathcal{N}}\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ -\mu x/r^3 \\ -\mu y/r^3 \\ -\mu z/r^3 \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

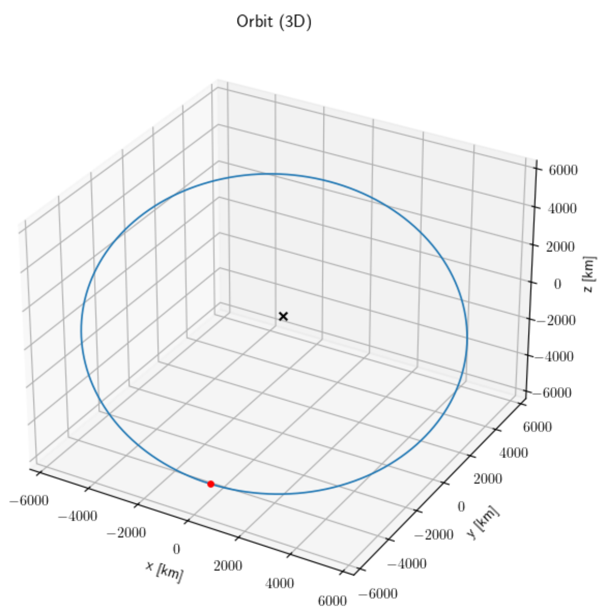
1. Plot the resulting orbit in 2D and 3D. Be sure that all axes have the same limits.
2. Calculate the orbital elements at each timestep, and plot them as a function of time.
3. In a few sentences, explain what you see and if it makes sense.

Solution

Part A



(a) Orbit Plot 2D



(b) Orbit Plot 3D

Part B

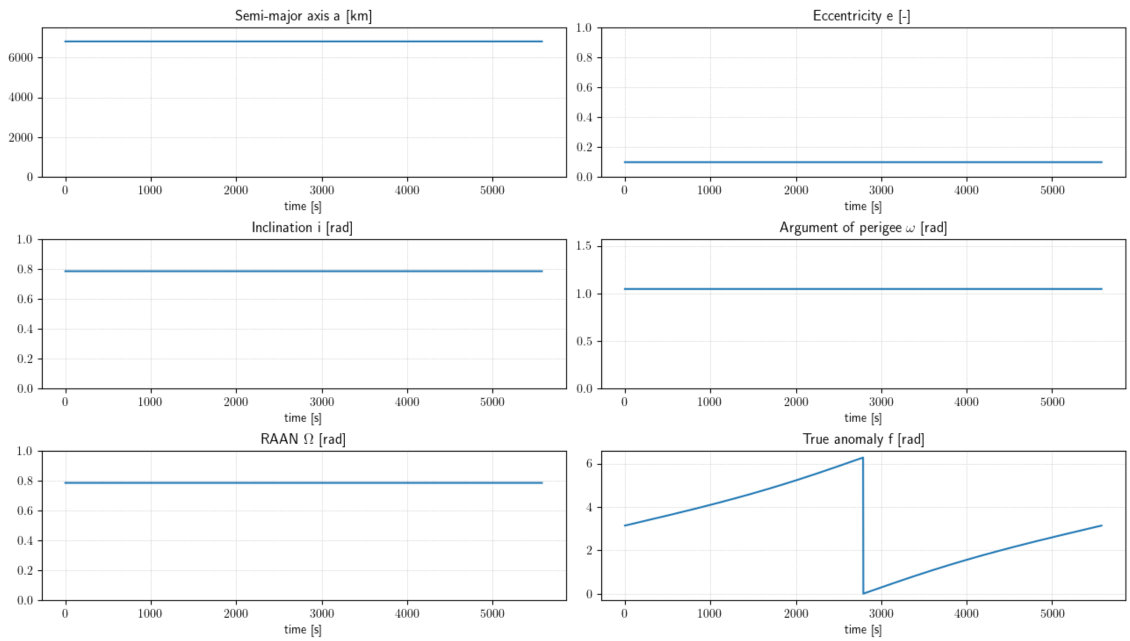


Figure 2: Orbital Elements vs. Time

Part C

Due to being a pure two-body point-mass model (with no external forces nor third-body effects), the orbital plots depict a closed elliptical path, with perfectly repeated motion. A closed elliptical path should depict constant orbital elements a, e, i, ω, Ω — which we see in their respective orbital element plots. Such motion should also have steady procession in f wrt. time: beginning at $f = f_0$, wrapping at $f = 2\pi$, and continuing its procession until $f = f_0$ again — which we also see in its respective orbital element plot. Thus, all depicted behavior in the plots makes sense, as they line up with the expected behavior for a pure two-body system.

Code

See the Python code for this assignment.