ENAE 441 - 0101
HW01: Orbits and 2BP

Due on September 25th. 2025 at 09:30 AM

Dr. Martin, 09:30 AM

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Problem 1: The Eccentricity Vector

1. Use the definition of the eccentricity vector **e** to prove that the total energy of the system,

$$\epsilon = \frac{\mathbf{v} \cdot \mathbf{v}}{2} - \frac{\mu}{r}$$

   is equal to

$$\epsilon = -\frac{\mu}{2a}.$$

2. Use the expression

$$\epsilon = \frac{\mathbf{v} \cdot \mathbf{v}}{2} - \frac{\mu}{r}$$

   to prove that

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0.$$

Solution

Part A

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = v^2$$

$$\mathbf{v} = \dot{\mathbf{r}}$$

$$|\mathbf{h}| = h$$

$$\mathbf{e} = \frac{1}{\mu}\left(\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{|\mathbf{r}|}\right)$$

$$\dot{\mathbf{r}} \times \mathbf{h} = vh\hat{r}$$

$$-\mu\frac{\mathbf{r}}{|\mathbf{r}|} = -\mu\hat{r}$$

$$\implies \mathbf{e} = \left(\frac{vh}{\mu} - 1\right)\hat{r}$$

at periapsis:

$$r_p v_p = h$$

$$r_p = a\left(1 - e\right)$$

$$\implies a = \frac{r}{1 - e}$$

$$\mathbf{e} = \left(\frac{rv^2}{\mu} - 1\right)\hat{r}$$

$$\implies v^2 = \frac{(e+1)\,\mu}{r}$$

$$\epsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

$$= \frac{v^2}{1} - \frac{v^2}{e+1}$$

$$= \frac{(e+1)\,\mu}{2r} - \frac{\mu}{r}$$

$$= \frac{(e+1)\,\mu - 2\mu}{2r}$$

$$= -\mu\frac{1 - e}{2r}$$

$$= -\frac{\mu}{2}\left(\frac{1-\cancel{e}}{r}\right)^{\cancel{\frac{1}{a}}}$$

$$\implies \epsilon = -\frac{\mu}{2a}.$$

Part B

$$\epsilon = \frac{\mathbf{v}-\mathbf{v}}{2} - \frac{\mu}{r}$$

$$= \frac{\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}}{2} - \frac{\mu}{r}$$

$$= \frac{(\dot{r})^2}{2} - \frac{\mu}{r}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \frac{2\dot{r}\ddot{r}}{2} + \frac{\mu\dot{r}}{r^2}$$

$$= \dot{r}\left(\ddot{r} + \left(\cancel{\frac{\mu}{r^2}}\right)^{-\ddot{r}}\right)$$

$$= \dot{r}\,\cancel{(\ddot{r} - \ddot{r})}^{\,0}$$

$$\implies \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0.$$

Problem 2: Polar Coordinates

Prove that

$$\ddot{\mathbf{r}} \times \mathbf{h} = \mu \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right).$$

Solution

$$
\begin{aligned}
\ddot{\mathbf{r}} \times \mathbf{h} &= \left(-\frac{\mu}{r^2}\hat{r}\right) \times \left(r^2\dot{\theta}\hat{h}\right) \\
&= -\mu\dot{\theta}\hat{r}\hat{h} \\
&= \mu\dot{\theta}\hat{\theta} \\
\implies \ddot{\mathbf{r}} \times \mathbf{h} &= \mu \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right).
\end{aligned}
$$

Problem 3: Orbital Element Calculations

Implement the following calculations in Python:

1. Takes state vector **X** and converts to orbital elements **œ**.

2. Takes vector of the orbital elements **œ** and converts to **X**.

3. Takes a set of orbital elements, and calculates the orbital period.

Code

See the Python code for this assignment.

Problem 4: Propagate and Visualize Orbits

Using SciPy's `solve_ivp` function that takes the cartesian state vector from Problem 3, $\mathbf{X}(t_0)$, and integrate the orbit for one period using the following differential equation:

$$^{\mathcal{N}}\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ -\mu x/r^3 \\ -\mu y/r^3 \\ -\mu z/r^3 \end{bmatrix} ; \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$
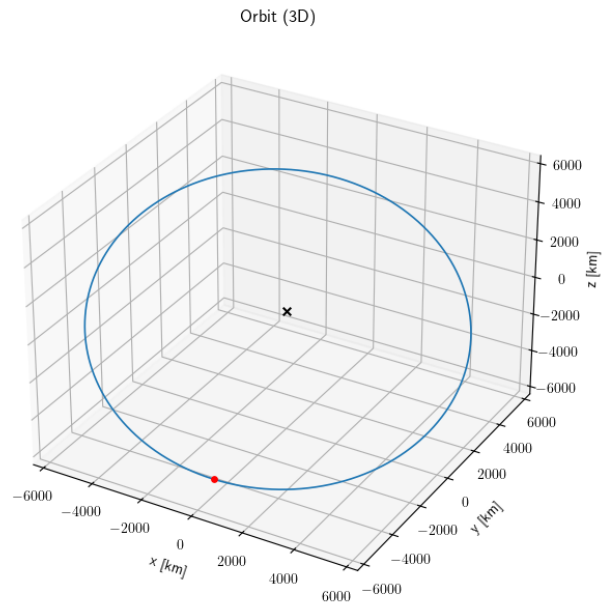
1. Plot the resulting orbit in 2D and 3D. Be sure that all axes have the same limits.

2. Calculate the orbital elements at each timestep, and plot them as a function of time.

3. In a few sentences, explain what you see and if it makes sense.

Solution

Part A
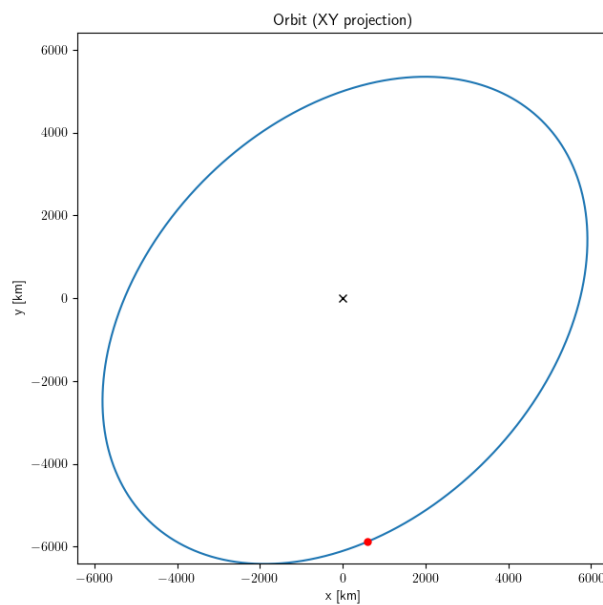
(a) Orbit Plot 2D

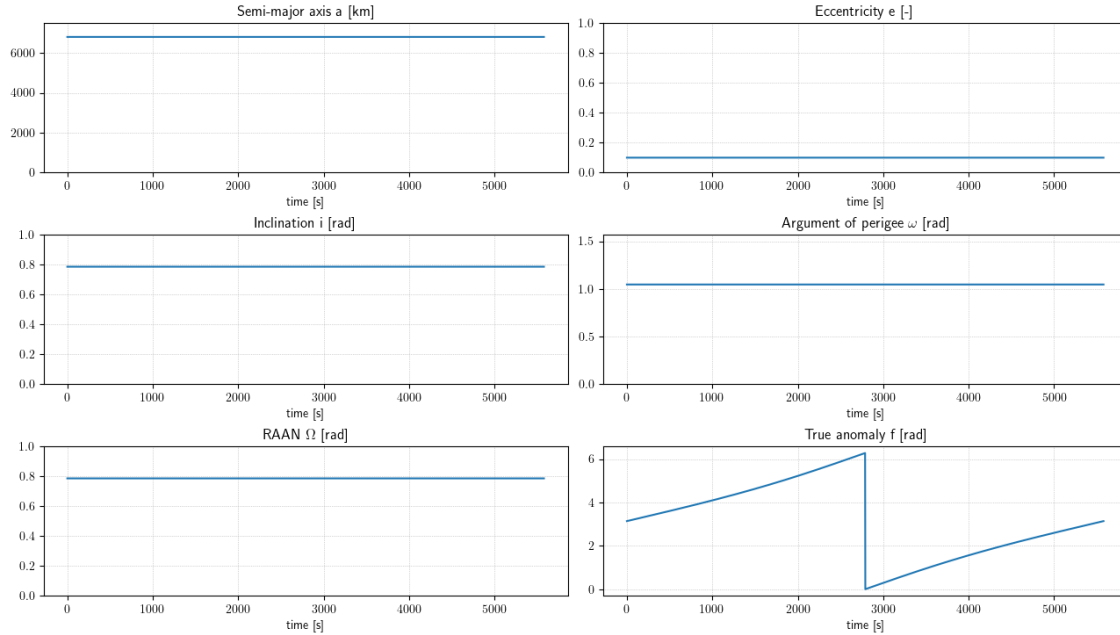(b) Orbit Plot 3D

Part B

Figure 2: Orbital Elements vs. Time

Part C

Due to being a pure two-body point-mass model (with no external forces nor third-body effects), the orbital plots depict a closed elliptical path, with perfectly repeated motion. A closed elliptical path should depict constant orbital elements a, e, i, ω, Ω — which we see in their respective orbital element plots. Such motion should also have steady procession in f wrt. time: beginning at $f = f_0$, wrapping at $f = 2\pi$, and continuing its procession until $f = f_0$ again — which we also see in its respective orbital element plot. Thus, all depicted behavior in the plots makes sense, as they line up with the expected behavior for a pure two-body system.

Code

See the Python code for this assignment.