

ENAE 441 - 0101
HW01: Orbits and 2BP

Due on September 25th. 2025 at 09:30 AM

Dr. Martin, 09:30 AM

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Problem 1: The Eccentricity Vector

1. Use the definition of the eccentricity vector \mathbf{e} to prove that the total energy of the system,

$$\epsilon = \frac{\mathbf{v} \cdot \mathbf{v}}{2} - \frac{\mu}{r}$$

is equal to

$$\epsilon = -\frac{\mu}{2a}.$$

2. Use the expression

$$\epsilon = \frac{\mathbf{v} \cdot \mathbf{v}}{2} - \frac{\mu}{r}$$

to prove that

$$\frac{d\epsilon}{dt} = 0.$$

Solution

Part A

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = v^2$$

$$\mathbf{v} = \dot{\mathbf{r}}$$

$$|\mathbf{h}| = h$$

$$\mathbf{e} = \frac{1}{\mu} \left(\dot{\mathbf{r}} \times \mathbf{h} - \mu \frac{\mathbf{r}}{|\mathbf{r}|} \right)$$

$$\dot{\mathbf{r}} \times \mathbf{h} = v h \hat{r}$$

$$-\mu \frac{\mathbf{r}}{|\mathbf{r}|} = -\mu \hat{r}$$

$$\Rightarrow \mathbf{e} = \left(\frac{v h}{\mu} - 1 \right) \hat{r}$$

at periapsis:

$$r_p v_p = h$$

$$r_p = a(1 - e)$$

$$\Rightarrow a = \frac{r}{1 - e}$$

$$\mathbf{e} = \left(\frac{r v^2}{\mu} - 1 \right) \hat{r}$$

$$\Rightarrow v^2 = \frac{(e + 1)\mu}{r}$$

$$\epsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

$$= \frac{v^2}{1} - \frac{v^2}{e + 1}$$

$$= \frac{(e + 1)\mu}{2r} - \frac{\mu}{r}$$

$$= \frac{(e + 1)\mu - 2\mu}{2r}$$

$$= -\mu \frac{1 - e}{2r}$$

$$\begin{aligned}
 &= -\frac{\mu}{2} \left(\frac{1-e}{r} \right) \\
 \Rightarrow \epsilon &= -\frac{\mu}{2a}.
 \end{aligned}$$

Part B

$$\begin{aligned}
 \epsilon &= \frac{\mathbf{v} \cdot \mathbf{v}}{2} - \frac{\mu}{r} \\
 &= \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{2} - \frac{\mu}{r} \\
 &= \frac{(\dot{r})^2}{2} - \frac{\mu}{r} \\
 \frac{d\epsilon}{dt} &= \frac{2\dot{r}\ddot{r}}{2} + \frac{\mu\dot{r}}{r^2} \\
 &= \dot{r} \left(\ddot{r} + \left(\frac{\mu}{r^2} \right) \right) \\
 &= \dot{r} (\ddot{r} - \ddot{r}) = 0 \\
 \Rightarrow \frac{d\epsilon}{dt} &= 0.
 \end{aligned}$$

Problem 2: Polar Coordinates

Prove that

$$\ddot{\mathbf{r}} \times \mathbf{h} = \mu \frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right).$$

Solution

$$\begin{aligned} \ddot{\mathbf{r}} \times \mathbf{h} &= \left(-\frac{\mu}{r^2} \hat{r} \right) \times \left(r^2 \dot{\theta} \hat{h} \right) \\ &= -\mu \dot{\theta} \hat{r} \hat{h} \\ &= \mu \dot{\theta} \hat{\theta} \\ \Rightarrow \ddot{\mathbf{r}} \times \mathbf{h} &= \mu \frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right). \end{aligned}$$

Problem 3: Orbital Element Calculations

Implement the following calculations in Python:

1. Takes state vector \mathbf{X} and converts to orbital elements \mathbf{oe} .
2. Takes vector of the orbital elements \mathbf{oe} and converts to \mathbf{X} .
3. Takes a set of orbital elements, and calculates the orbital period.

Code

See the [Python code](#) for this assignment.

Problem 4: Propagate and Visualize Orbits

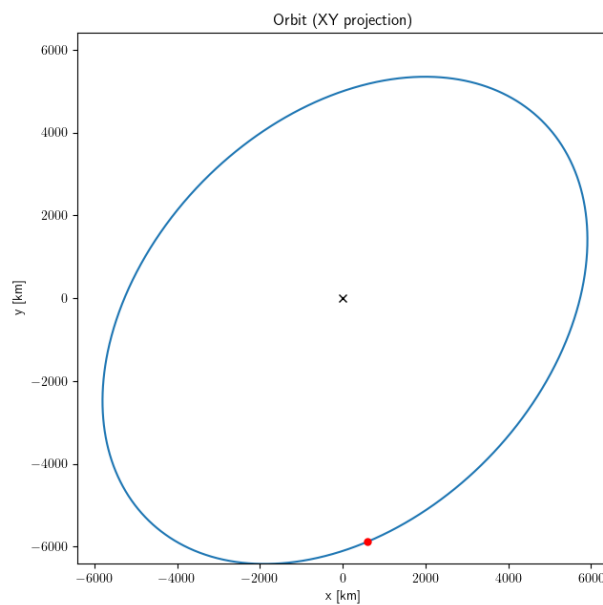
Using SciPy's `solve_ivp` function that takes the cartesian state vector from [Problem 3](#), $\mathbf{X}(t_0)$, and integrate the orbit for one period using the following differential equation:

$$\mathcal{N} \dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ -\mu x/r^3 \\ -\mu y/r^3 \\ -\mu z/r^3 \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

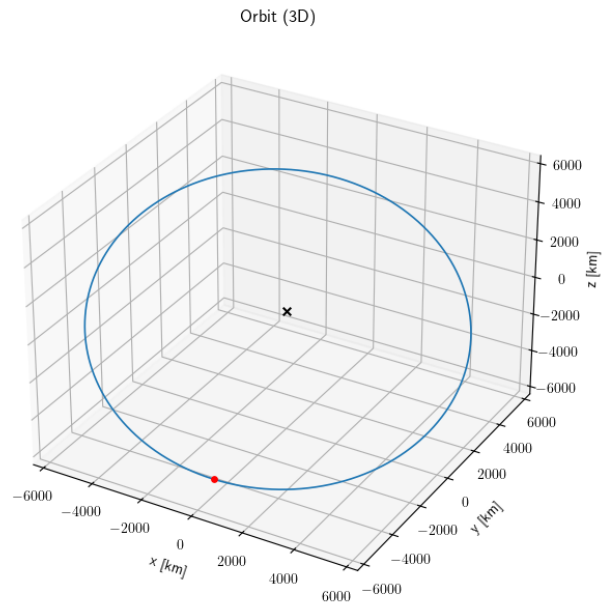
1. Plot the resulting orbit in 2D and 3D. Be sure that all axes have the same limits.
2. Calculate the orbital elements at each timestep, and plot them as a function of time.
3. In a few sentences, explain what you see and if it makes sense.

Solution

Part A



(a) Orbit Plot 2D



(b) Orbit Plot 3D

Part B

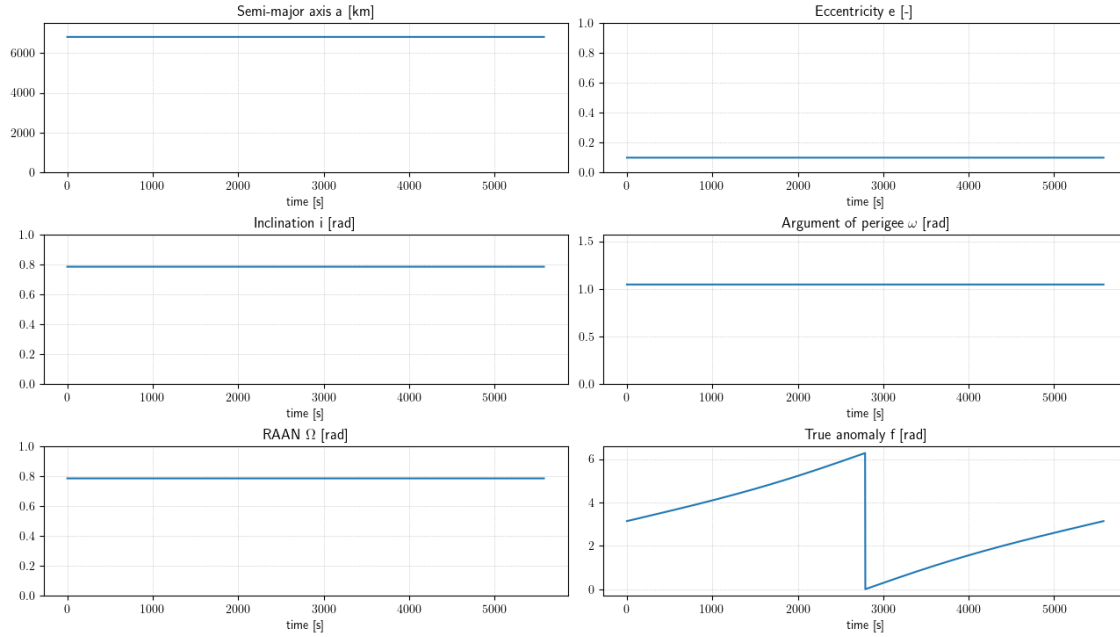


Figure 2: Orbital Elements vs. Time

Part C

Due to being a pure two-body point-mass model (with no external forces nor third-body effects), the orbital plots depict a closed elliptical path, with perfectly repeated motion. A closed elliptical path should depict constant orbital elements a, e, i, ω, Ω — which we see in their respective orbital element plots. Such motion should also have steady procession in f wrt. time: beginning at $f = f_0$, wrapping at $f = 2\pi$, and continuing its procession until $f = f_0$ again — which we also see in its respective orbital element plot. Thus, all depicted behavior in the plots makes sense, as they line up with the expected behavior for a pure two-body system.

Code

See the [Python code](#) for this assignment.