ENAE 441 - 0101
HW04: Random Variables

Due on November 11th. 2025 at 09:30 AM

Dr. Martin, 09:30 AM

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November 10, 2025

Problem 1: Standard Deviation

Consider two zero-mean uncorrelated random variables W and V with standard deviations σ_w and σ_v respectively. What is the standard deviation of the random variable $X = W + V$? Note: Do not assume gaussian distribution.

Solution

If W, V are zero-mean and uncorrelated, then

$$\text{Var}(W + V) = \text{Var}(W) + \text{Var}(V) + 2\,\text{Cov}(W, V) = \sigma_w^2 + \sigma_v^2$$

Hence

$$\boxed{\sigma_X = \sqrt{\sigma_w^2 + \sigma_v^2}}$$

Problem 2: Correlation Coefficient

Consider two scalar RVs X and Y.

   (a) Prove that if X and Y are independent, then their correlation coefficient $\rho = 0$.

   (b) Find an example of two RVs that are not independent but that have a correlation coefficient of zero.

   (c) Prove that if Y is a linear function of X then $\rho = \pm 1$.

Solution

Part A

If X, Y are independent and have finite second moments, then

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0,$$

so

$$\boxed{\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = 0}$$

(provided $\sigma_X, \sigma_Y > 0$)

Part B

Let $X \sim \text{Unif}(-1, 1)$ and $Y = X^2$. Then

$$\mathbb{E}[X] = 0, \quad \mathbb{E}[Y] = \frac{1}{3}, \quad \mathbb{E}[XY] = \mathbb{E}\left[X^3\right] = 0,$$

so $\text{Cov}(X, Y) = 0$ and $\rho = 0$, yet Y is a deterministic function of $X \implies$ not independent.

Part C

If $Y = aX + b$ with $a \neq 0$,

$$\text{Cov}(X, Y) = \text{Cov}(X, aX + b) = a\,\text{Var}(X), \quad \sigma_Y = |a|\sigma_X$$

so

$$\boxed{\rho = \frac{a\,\text{Var}(X)}{\sigma_X |a|\sigma_X} = \text{sgn}(a) \in \{\pm 1\}}$$

(If $a = 0$, then $\sigma_Y = 0$ and ρ is undefined.)

Problem 3: Probability Density

Consider the following function

$$f_{XY}(x,y) = \begin{cases} ae^{-2x}e^{-3y} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

and answer the following questions.

(a) Find the value of a so that $f_{XY}(x,y)$ is a valid joint probability density function.

(b) Calculate $\bar{x}$ and $\bar{y}$.

(c) Calculate $\mathbb{E}\left[X^2\right]$, $\mathbb{E}\left[Y^2\right]$, and $\mathbb{E}[XY]$.

(d) Calculate the autocorrelation matrix of the random vector $Z = [X,Y]^T$.

(e) Calculate the variance σ_x^2, variance σ_y^2, and the covariance C_{XY}.

(f) Calculate the autocovariance matrix of the random vector $Z = [X,Y]^T$.

(g) Calculate the correlation coefficient between X and Y.

Solution

Part A

$$1 = \iint f_{XY} = a\left(\int_0^\infty e^{-2x}dx\right)\left(\int_0^\infty e^{-3y}dy\right) = a\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{a}{6} \Rightarrow a = 6.$$

Thus $f_{XY}(x,y) = 6e^{-2x}e^{-3y}$ and factorizes:

$$\boxed{X \sim \text{Exp}(2), Y \sim \text{Exp}(3)} \qquad \text{independent}$$

Part B

$$\boxed{\bar{x} = \mathbb{E}[X] = \frac{1}{2}, \quad \bar{y} = \mathbb{E}[Y] = \frac{1}{3}}$$

Part C

For $\text{Exp}(\lambda)\mathbb{E}\left[X^2\right] = \frac{2}{\lambda^2}$

$$\mathbb{E}\left[X^2\right] = \frac{2}{2^2} = \frac{1}{2}$$
$$\mathbb{E}\left[Y^2\right] = \frac{2}{3^2} = \frac{2}{9}$$
$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{2}\cdot\frac{1}{3}$$

$$\boxed{\mathbb{E}[XY] = \frac{1}{6}}$$

Part D

$$R_Z = \begin{bmatrix} \mathbb{E}\left[X^2\right] & \mathbb{E}[XY] \\ \mathbb{E}[YX] & \mathbb{E}\left[Y^2\right] \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{9} \end{bmatrix}$$

Part E

$$\sigma_x^2 = \mathbb{E}\left[X^2\right] - \bar{x}^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\sigma_y^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$$C_{XY} = \mathbb{E}[XY] - \bar{x}\bar{y} = \frac{1}{6} - \frac{1}{6} = 0$$

Part F

$$K_Z = \begin{bmatrix} \sigma_x^2 & C_{XY} \\ C_{XY} & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$$

Part G

$$\rho = \frac{C_{XY}}{\sigma_x \sigma_y} = 0$$

Problem 4: Covariance and Variance

Prove the following two results from lecture where $x \sim \mathcal{N}\left(\bar{x}, \sigma_x^2\right)$ and $e \sim \mathcal{N}\left(0, \sigma_e^2\right)$ and $y = cx + de$.

  (a) $\operatorname{Cov}\left(X, Y\right) = \mathbb{E}[(x - \bar{x})(y - \bar{y})] = \mathbb{E}[XY] - \bar{x}\bar{y}$

  (b) $\operatorname{Var}\left(Y\right) = \mathbb{E}\left[(y - \bar{y})^2\right] = c^2\sigma_x^2 + d^2\sigma_e^2$

Solution

Part A

Let $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$.

$$
\begin{aligned}
\mathbb{E}\left[\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right] &= \mathbb{E}\left[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y\right] \\
&= \mathbb{E}[XY] - \mu_X \mathbb{E}[Y] - \mu_Y \mathbb{E}[X] + \mu_X \mu_Y \\
&= \mathbb{E}[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y \\
&= \mathbb{E}[XY] - \mu_X \mu_Y
\end{aligned}
$$

Thus

$$
\operatorname{Cov}\left(X, Y\right) = \mathbb{E}\left[\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]
$$

$$
\implies \boxed{\operatorname{Cov}\left(X, Y\right) = \mathbb{E}[XY] - \bar{x}\bar{y}}
$$

Part B

$$
\bar{y} = \mathbb{E}[Y] = c\mathbb{E}[x] + d\mathbb{E}[e] = c\bar{x}
$$

Then

$$
Y - \bar{y} = c(x - \bar{x}) + de
$$

Hence

$$
\operatorname{Var}\left(Y\right) = \mathbb{E}\left[(c(x - \bar{x}) + de)^2\right] = c^2\sigma_x^2 + d^2\sigma_e^2 + 2cd\mathbb{E}[(x - \bar{x})e]
$$

If x and e are independent (or just uncorrelated), $\mathbb{E}[(x - \bar{x})e] = 0$, giving

$$
\boxed{\operatorname{Var}\left(Y\right) = c^2\sigma_x^2 + d^2\sigma_e^2}
$$

Problem 5: Central Limit Theorem + Mappings

(a) In python, generate a random variable x_1 distributed by a uniform distribution $x_i \sim \mathcal{U}[-1, 1]$ using `np.random.uniform` function. Sample $N = 10$ points from this distribution, plot those points as a histogram.

(b) Compute the sample mean, $\hat{\mu}$, and sample variance, $\hat{\sigma}^2$, of a sample set using functions `np.mean` and `np.var` functions, and determine if the reported values match the analytic mean and variance, $\mathbb{E}[x_1]$ and $\mathbb{E}\left[(x_1 - \mu_{x_1})^2\right]$ respectively? Repeat using $N = 10^i$ samples where $1 \leq i \leq 6$, reporting the sample mean. Report the values and explain what you observe.

(c) Create three new random variables x_2, x_3, x_4 in addition to x_1, each also distributed from a uniform distribution $\mathcal{U}[-1, 1]$. Sample $N = 100,000$ values from $x_1 - x_4$, and use these independent variables to compute a new set of random variables y_1, y_2, and y_3 defined as

$$y_1 = \frac{x_1 + x_2}{2}$$
$$y_2 = \frac{x_1 + x_2 + x_3}{3}$$
$$y_3 = \frac{x_1 + x_2 + x_3 + x_4}{4}$$

Using the sampled values of x_1 through x_4, plot a histogram $p(x_1), p(y_1), p(y_2), p(y_3)$.

(d) Explain what you see.

(e) Transform y_3 into two new random variables z and q and plot the resulting distributions of $p(z)$ and $p(q)$ next to the original distribution $p(y_3)$ where

$$z = g(y) = 2y + 3$$
$$q = f(y) = e^y$$

(f) Explain how the distribution changes with both transformations, and if it makes sense.
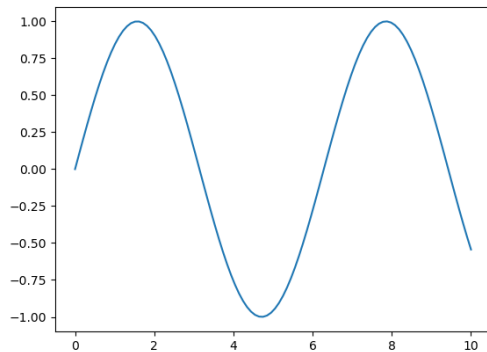
Solution

Part A

Figure 1: Example

Part B

```
1  0.7853981633974481
```

Part C

```
1  0.7853981633974481
```

Part D
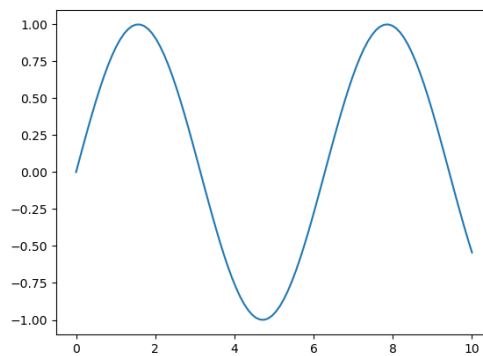
```
1  0.7853981633974481
```

Part E

Figure 2: Example

Part F

```
1  0.7853981633974481
```

Code

See the Python code for this assignment.