Homework # 5

ENGR 441

Instructions

This homework assignment introduces concepts related to stochastic linear systems and state estimation using batch least squares, recursive least squares, and Kalman Filters.

Please submit your answers using the following naming conventions to Gradescope in two files:

1. `{your_name}.pdf` — derivations and / or written explanations of results
2. `submission.py` — the python script for the autograder (must be named `submission.py`!)

Note the type of question is indicated by its color:

- **Blue** questions are solved via pen and paper and uploaded in your **PDF file**.
- **Red** questions correspond to **autograded functions** that must be populated in `submission.py`.
- **Black** questions solutions are to be answered in `submission.py` but are **not autograded** (e.g. plots, discussions, etc)

Programming Questions

Each assignment has starter code available on ELMS. Your grade depends exclusively on the output of functions labeled `#REQUIRED`. Additional helper functions are included in the starter code to assist with debugging.

IMPORTANT: Be sure to include the necessary `.npy` data files in your submission to ensure the autograder will work properly!

Questions

Imagine there is a spacecraft flying in a circular geosynchronous orbit around the Earth. To an observer on the rotating Earth, the spacecraft location appears static. You do not know the exact semi-major axis, but you can estimate it using a range measurements taken from a radio telescope on the ground

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Assume you are positioned directly beneath the satellite such that range measurement can be reduced to the following form

$$\rho = \sqrt{z^2} = z$$

For 50 consecutive nights, you go outside and point your radio antenna towards the spacecraft and collect 200 noisy range measurements. The radio antenna is not particularly precise, so it has continuous measurement noise properties $\tilde{\boldsymbol{v}}(t) \sim \mathcal{N}(0, V)$ where $V = 100\ \text{m}^2$. Use this information, and the measurements provided in `Files/HW5Measurements.npy`[1] to answer the following questions

Question 1. (30 pts) Batch Least Squares Estimation

a) Express the system in continuous time state-space form assuming a state of

$$\boldsymbol{X} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$$

and convert to a discrete time state-space model assuming $\Delta t = 10\text{s}$.

b) Use batch least squares algorithm[2] to estimate the spacecraft state from the first trial in `HW5Measurements.npy`. Plot the estimated position as a function of $k = \{1, \ldots, 200\}$ measurements.

c) Repeat the batch estimation process for all 50 trials included in `HW5Measurements.npy` and plot each trial's state estimates as a function of k. In addition, use the 50 trials to compute the sample mean ($\hat{\mu}_k$) and sample 3-sigma covariance ($\hat{\sigma}_k$) centered about that mean. Overlay sample mean and covariance atop the 50 trials. [3][4].

d) Using all 50 trials, plot histograms for each state estimate in $\hat{\boldsymbol{x}}$ after $k = 10, 50, 200$ measurements. Include the sample mean and error covariance for each distribution as annotations[5]. Do these values make sense? Why?

e) Measure the average amount of time it takes to compute an estimate as a function of k. Use Python's `time` library.

[1] The measurements provided are formatted as a $N \times M$ array where N corresponds to the trial and M corresponds to the number of measurements per trial and the units are meters.

[2] Hint: When you need to take a matrix inverse for BLLS and subsequent algorithms, use `np.linalg.pinv` rather than `np.linalg.inv` to avoid singularities

[3] Recall that the diagonal elements of the state error covariance matrix yield σ_i^2

[4] The matplotlib function `plt.fill_between` can be useful for visualization of the $\pm 3\sigma$ bounds that center on the mean

[5] Use `np.nanmean` and `np.nanstd` over all values for a particular k

Question 2. (30 pts) Recursive Least Squares Estimation

a) Using the same dataset, implement the recursive least squares algorithm to estimate $\hat{\boldsymbol{x}}$. Use an initial estimate of

$$\hat{\boldsymbol{x}}_0 = [0, 0, 42000 \text{ km}, 0, 0, 0]$$

and covariance of

$$P_0 = \begin{bmatrix} 50 \cdot \mathbb{I}_{3\times 3} \text{ km}^2 & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & 1 \cdot \mathbb{I}_{3\times 3} \text{ km}^2/\text{s}^2 \end{bmatrix}$$

Plot the state estimate for the 50 trials as a function of k.

b) Plot the sample mean state estimate μ_k across all trials and the $\pm$3-sigma error bounds around the estimate. Does this mirror the same plot in 1d? Why or why not?

c) Measure / plot the average amount of time it takes to perform each update from $k = \{1, \ldots, 200\}$ and compare against the batch least squares estimator. Explain any differences.

Question 3. (40 pts) Kalman Filtering

Imagine you are floating in interplanetary space and observing a spacecraft traveling at some undetermined position away from you. The spacecraft has the state $\boldsymbol{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$. The spacecraft is not under the influence of gravity, rather its motion is only driven by its current velocity vector, and some gaussian white noise acting on the acceleration terms — i.e.

$$\ddot{x}(t) = w_1(t) \tag{1}$$
$$\ddot{y}(t) = w_2(t) \tag{2}$$
$$\ddot{z}(t) = w_3(t) \tag{3}$$

where

$$\boldsymbol{w}(t) \sim \mathcal{N}(\boldsymbol{0}, W); \quad W = \begin{bmatrix} 10^{-5} & 0 & 0 \\ 0 & 10^{-5} & 0 \\ 0 & 0 & 10^{-5} \end{bmatrix} (\text{km/s}^2)^2$$

You have a magical sensor that can produce noisy measurements of the spacecraft's current position, but it is unable to resolve any information about it's velocity – i.e.

$$\boldsymbol{Y}(t) = [x, y, z]^T + \boldsymbol{v}(t) \quad \boldsymbol{v}(t) \sim \mathcal{N}(0, V)$$

where the noise associated with the sensor is characterized by the continuous time process noise matrix

$$V = \begin{bmatrix} 10^3 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 10^3 \end{bmatrix} (\text{km}^2)$$

Using your expert insight, you have a rough order of magnitude estimate for the spacecraft's initial state and covariance defined as:

$$\hat{\boldsymbol{x}}_0 = [0 \text{ km}, 0 \text{ km}, 500 \text{ km}, 0.01 \text{ km/s}, 0 \text{ km/s}, 0.01 \text{ km/s}]$$

km and covariance of

$$P_0 = \begin{bmatrix} 50 \cdot \mathbb{I}_{3\times 3} \text{ km}^2 & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & 0.1 \cdot \mathbb{I}_{3\times 3} \text{ km}^2/\text{s}^2 \end{bmatrix}$$

Using this information, answer the following questions:

a) Express this system in a discrete time state-space model. Specify A, C, F, H, Q, and R.

b) Implement a Kalman filter to estimate the position and velocity of the spacecraft using measurements provided in `HW5Measurements-P3.npy`. Report your best estimate of the state at the final time-step and it's associated error covariance matrix.

c) Plot the pure prediction estimates of the state and $\pm 3\sigma$ error bounds over time — i.e. $\hat{\boldsymbol{\mu}}_k^-$ and P_k^-.

d) Plot the measurement corrected estimates of the state and $\pm 3\sigma$ error bounds as a function of time — i.e. $\hat{\boldsymbol{\mu}}_k^+$ and P_k^+.

e) Explain the differences between the plots of $(\hat{\boldsymbol{\mu}}_k^-, P_k^-)$ and $(\hat{\boldsymbol{\mu}}_k^+, P_k^+)$

f) Using the history of $\hat{\boldsymbol{\mu}}_k^+$, plot the measurement residuals over time

$$\delta \boldsymbol{y} = \boldsymbol{y}_k - H_k \hat{\boldsymbol{\mu}}_k^+$$

Do these match expectation?