ENAE 301:

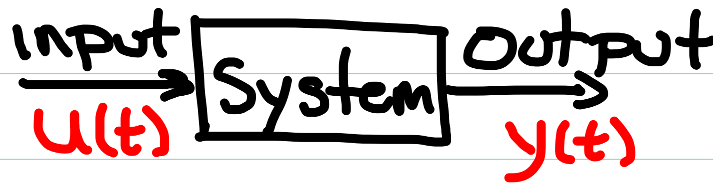
$$\xrightarrow{\text{force}} \boxed{\text{Dynamics}} \xrightarrow{\text{motion}}$$

More generally:

$$\xrightarrow{\text{input}} \boxed{\text{System}} \xrightarrow{\text{output}}$$

ENAE 301:

More generally:



A (dynamic) system "transforms" inputs $u(t)$ into outputs $y(t)$.

We must first understand as completely as possible this "transformation".

Simple Hovercraft Example

$$\frac{d}{dt}(mv) = f \quad \text{(dynamics)}$$

$$\frac{d}{dt}(y) = v \quad \text{(Kinematics)}$$

Where:

m = mass (assume constant)
v = velocity
y = position
f = applied force

Thus:

$$\left.\begin{aligned} m\frac{dv}{dt} &= f \\ \frac{dy}{dt} &= v \end{aligned}\right\} \text{Governing DE}$$

Equivalently: $m\dfrac{d^2y}{dt^2} = f$

Driving Force

Force driving system is due to fan:

$$f \approx K_f \, \omega$$

where K_f constant, ω is rotation rate of fan

Similarly: $\omega \approx K_m V_m$

where K_m constant, V_m is voltage applied to motor

Then $\boxed{\ddot{y}(t) = K\,u(t)}$, $K = \left[\dfrac{K_f K_m}{m}\right]$

treating $V_m(t) = u(t)$ as the input to the system

$$\xrightarrow{u(t)} \boxed{\text{System}} \xrightarrow{y(t)}$$

Analysis:

Given:

$$\boxed{\begin{aligned}\dot{v}(t) &= K u(t) \\ \dot{y}(t) &= v(t)\end{aligned}}$$

($v(t)$ velocity)

Then:

$$v(t) = v_0 + K\int_{\phi}^{t} u(\tau)\, d\tau$$

$$y(t) = y_0 + \int_{\phi}^{t} v(\sigma)\, d\sigma$$

Take $v_0 = y_0 = \phi$ for simplicity now, then

$$y(t) = K\int_{\phi}^{t}\left[\int_{\phi}^{\sigma} u(\tau)\, d\tau\right] d\sigma$$

So: $$y(t) = K\int_{\phi}^{t}\int_{\phi}^{\sigma} u(\tau)\, d\tau\, d\sigma$$ (Double integral!)

Or: $$y(t) = K\int_{\phi}^{t} (t-\tau)\, u(\tau)\, d\tau$$ (How...?)

Example Control Problem

Find $u(t)$ so that, for a specified t_f, y_f

$$V(t_f) = \emptyset \Rightarrow \boxed{\begin{aligned} \emptyset &= K\int_{\emptyset}^{t_f} u(\tau)\,d\tau \\ y_f &= K\int_{\emptyset}^{t_f} (t_f - \tau)\,u(\tau)\,d\tau \end{aligned}}$$

$$y(t_f) = y_f \Rightarrow$$

Solve for $u(\tau)$

Here, we are assuming vehicle starts at rest $(V(\emptyset) = \emptyset)$ on the "start line" $(y(\emptyset) = \emptyset)$.

Want the vehicle to move to position y_f in t_f seconds and stop there.

Many sol'ns $u(t)$ possible! Typically would also constrain:

1.) $|u(t)| \leq u_{max}$

2.) Behavior of $y(t)$, $t \in [\emptyset, t_f]$

Issues

1.) m, K_f, K_m not known precisely:

Hovercraft will not stop exactly where we want.

2.) Requires an accurate clock:

Must use correct $u(t)$ at exactly right times t.

3.) Cannot handle an external ("disturbance") force:

Headwind or cross-breeze will drive hovercraft off the track.

Mathematically sound, but not practical!

Do you drive like that? I hope not!

Mathematically sound, but not practical!

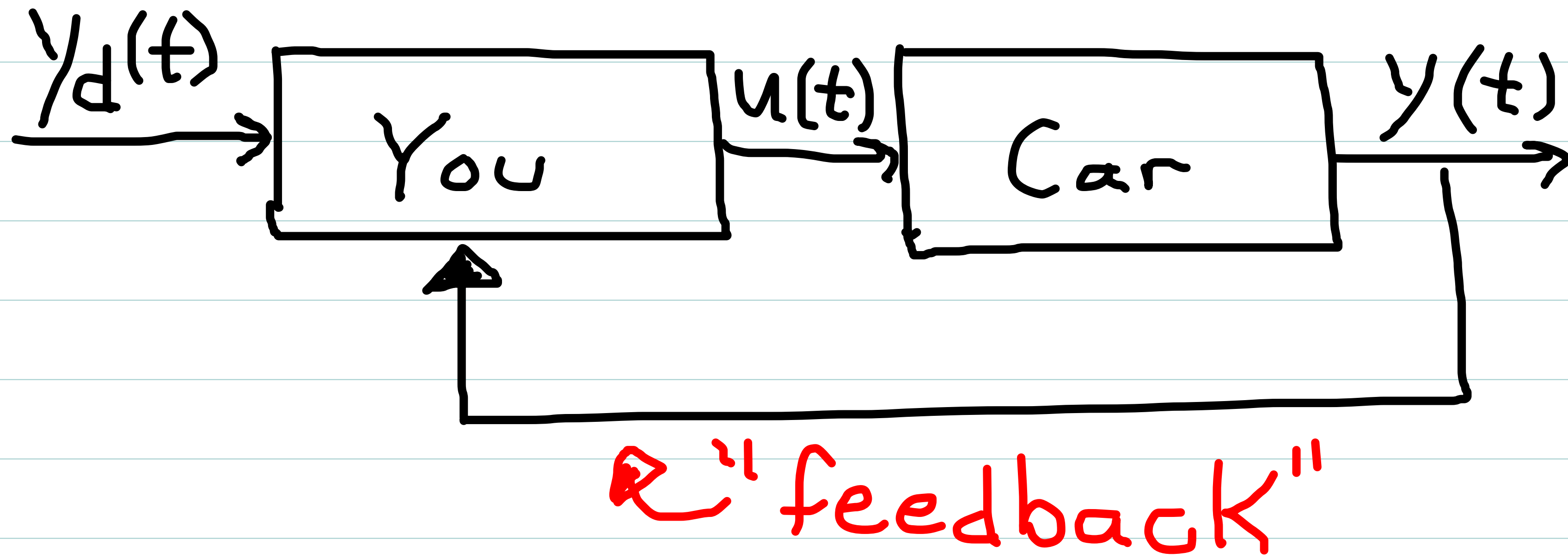
Do you drive like that? I hope not!

Instead you continually **compare** where you are ($y(t)$) with where you want to be ($y_d(t)$) and continually adjust actions ($u(t)$) based on difference.

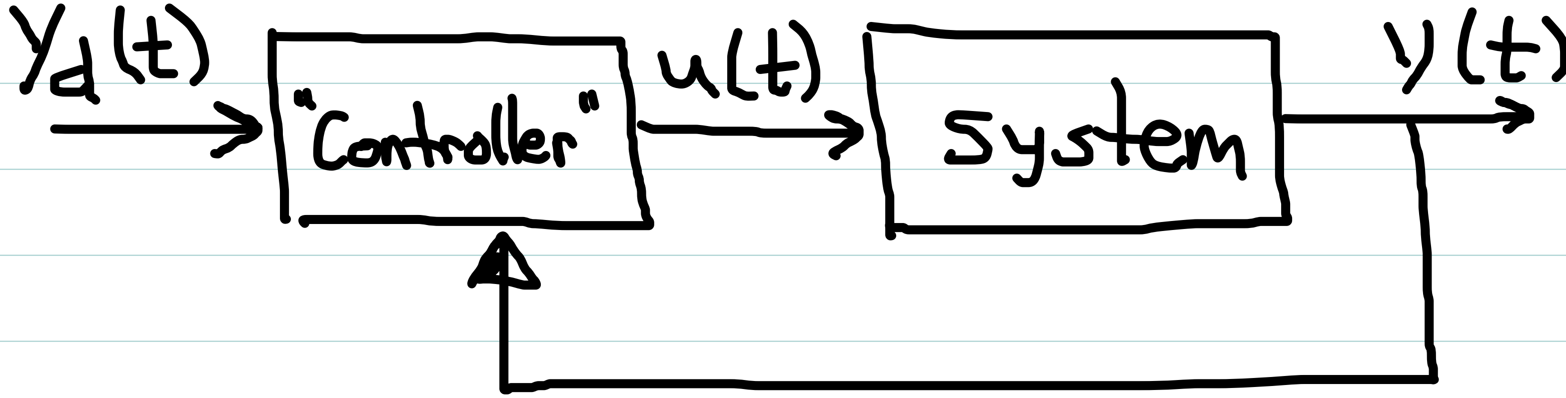
Mathematically sound, but not practical!

Do you drive like that? I hope not!

Instead you continually <u>compare</u> where you are ($y(t)$) with where you want to be ($y_d(t)$) and continually adjust actions ($u(t)$) based on difference.



Feedback Control



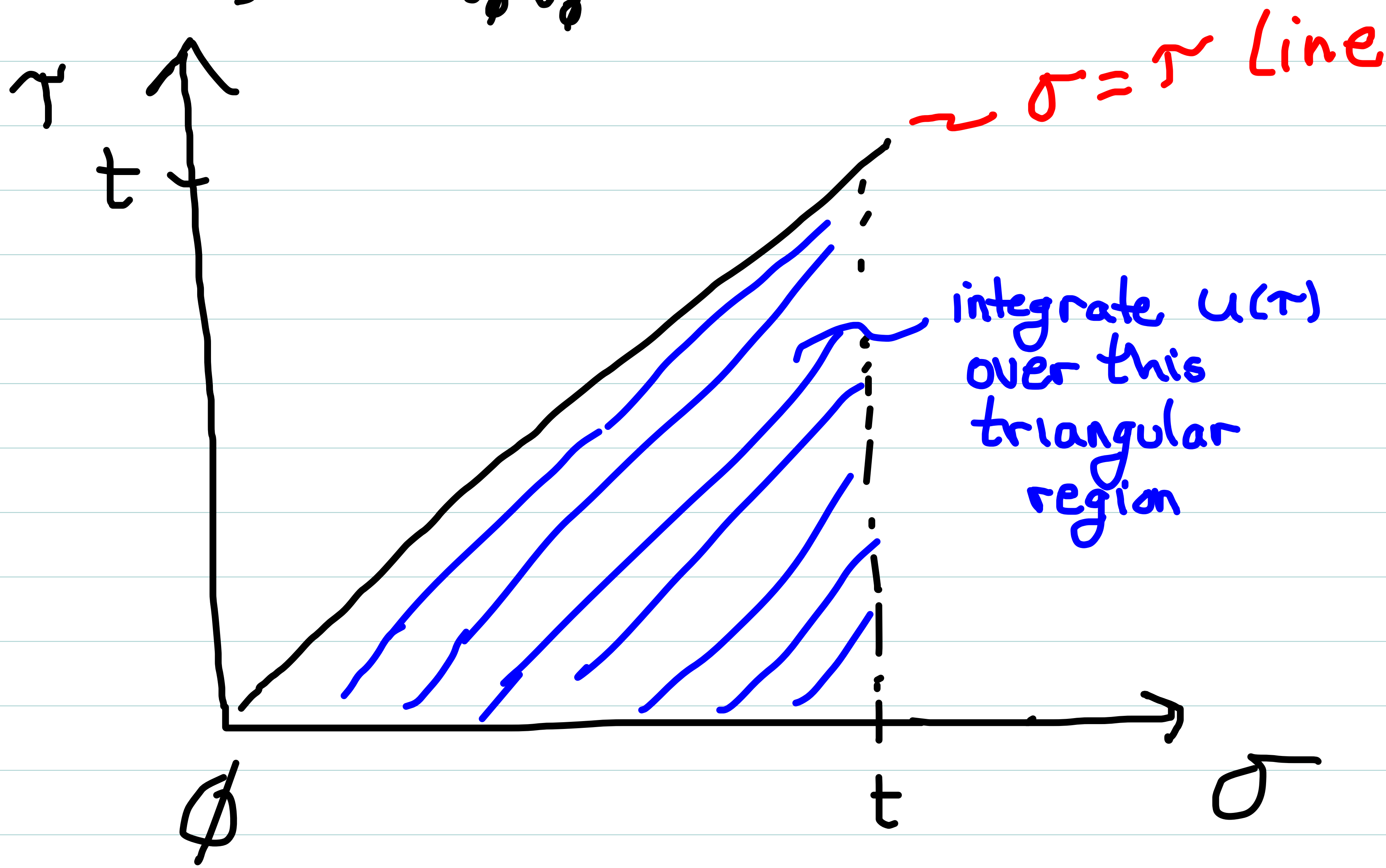
The controller is a mathematical algorithm (implemented as a computer program) which calculates required $u(t)$ from $y(t)$ and $y_d(t)$.

Addresses all 3 issues: uncertainty, disturbance, clocking

This course is about the derivation + implementation of suitable feedback control algorithms based on governing dynamics of system.

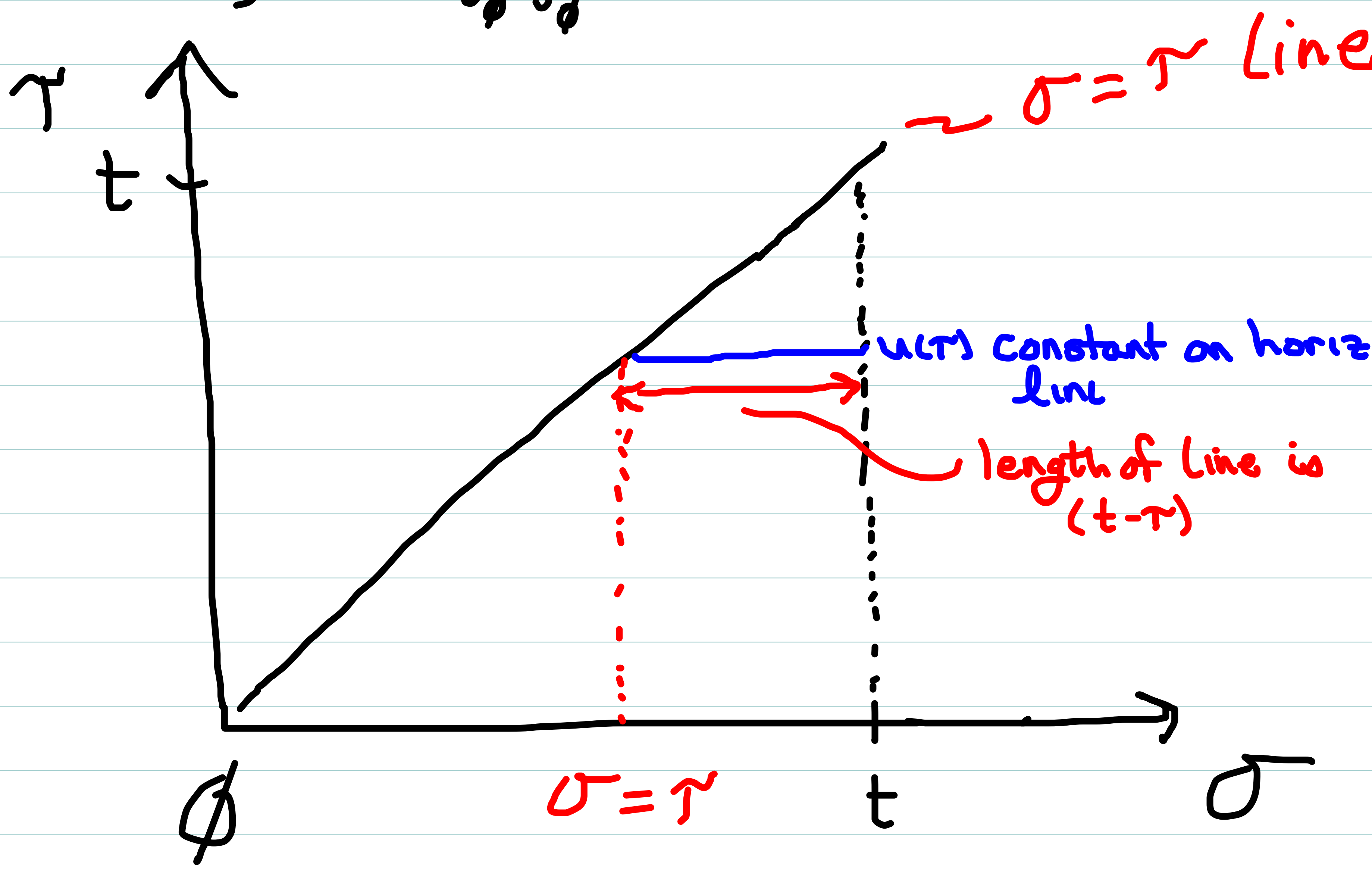
Reduction of the double integral

$$y(t) = K \int_0^t \int_0^\sigma u(\tau)\, d\tau\, d\sigma$$



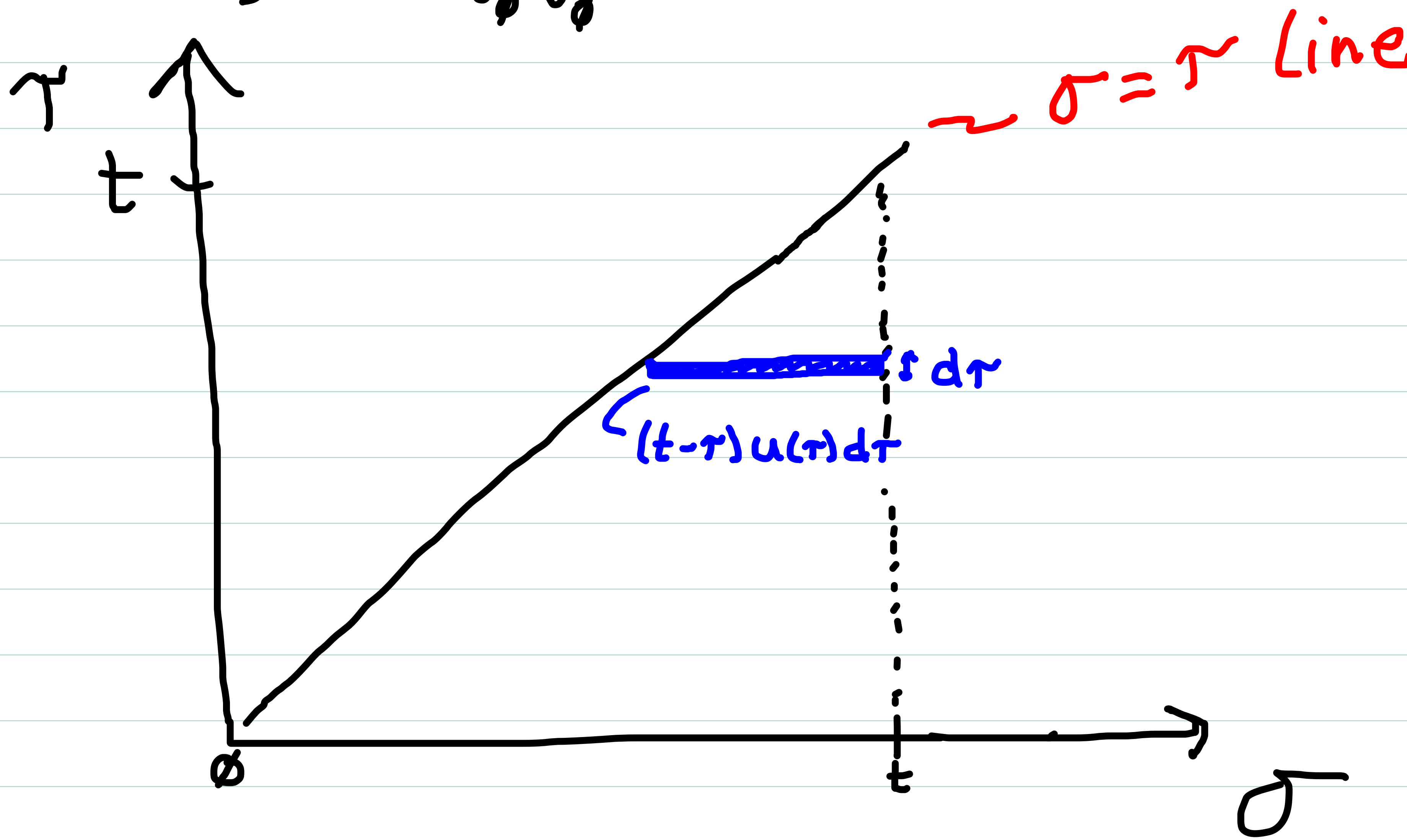
Reduction of this double integral

$$y(t) = K\int_{\phi}^{t}\int_{\phi}^{\sigma} u(\tau)\,d\tau\,d\sigma$$



Reduction of this double integral

$$y(t) = K\int_{\phi}^{t}\int_{\phi}^{\sigma} u(\tau)\, d\tau\, d\sigma$$



Integrate over all strips $\Rightarrow$ $y(t) = K\int_0^t (t-\tau)\, u(\tau)\, d\tau$
+ multiply by K :

An alternate form

Our sol'n has the general form:

$$\boxed{y(t) = \int_{\phi}^{t} g(t-\tau)\, u(\tau)\, d\tau}$$

where here $g(t) = Kt$ $\quad [\text{so } g(t-\tau) = K(t-\tau)]$

We will (indirectly) show that for any system, no matter how complex the dynamics, this relationship between $u(t)$ and $y(t)$ holds.

Different systems are characterized by different functions $g(t)$.

The characteristic function $g(t)$ is called the

Impulse response

Implication

Suppose:

$$y_1(t) = \int_0^t g(t-\tau)\,u_1(\tau)\,d\tau$$

$$y_2(t) = \int_0^t g(t-\tau)\,u_2(\tau)\,d\tau$$

are two known input-output pairs.

Suppose that $u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$; α_1, α_2 constant

Then:

$$y(t) = \int_0^t g(t-\tau)\left[\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau)\right] d\tau$$

$$= \alpha_1 \int_0^t g(t-\tau)\,u_1(\tau)\,d\tau + \alpha_2 \int_0^t g(t-\tau)\,u_2(\tau)\,d\tau$$

hence

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$ Principle of Linearity.

This suggests an approach:

1. Identify a "family" of functions $u_K(t)$ for which it is EASY to calculate response $y_K(t)$:

$$u_K(t) \mapsto y_K(t) \quad \text{(easy)}$$

2. "Break down" an arbitrarily complicated $u(t)$ into a linear combination of the $u_K(t)$:

$$u(t) = \sum \alpha_K u_K(t) \qquad \text{(easy?)}$$

3. Use Linearity:

$$y(t) = \sum \alpha_K y_K(t) \qquad \text{(easy)}$$

Time varying complex numbers

$$z(t) = a(t) + b(t)\,j$$
$$= r(t)\,e^{j\theta(t)}$$

Important example:

$$z(t) = e^{st} \text{ with } s \in \mathbb{C}$$

"complex exponential functions"

Let $s = \sigma + j\omega \quad \sigma, \omega \in \mathbb{R}$

So $\mathrm{Re}\{s\} = \sigma$, $\mathrm{Im}\{s\} = \omega$

① If $\omega = 0$, then

$$e^{st} = e^{\sigma t} \text{ (\underline{real} exponential)}$$

② If $\sigma = 0$ then

$$e^{st} = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Note: $\mathrm{Im}\{s\}$ gives *frequency* of the oscillations

(3.) Most general case

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}$$

$$= e^{\sigma t}[\cos\omega t + j\sin\omega t]$$

$$\left.\begin{aligned} Re\{e^{st}\} &= e^{\sigma t}\cos(\omega t) \\ Im\{e^{st}\} &= e^{\sigma t}\sin(\omega t) \end{aligned}\right]$$

$\sigma \rightarrow$ amplitude envelope

$\omega \rightarrow$ oscillation frequency

$s = \sigma + j\omega$ is the "Complex frequency"

Direct Solution

Our sol'n has the general pattern:

$$\boxed{y(t) = \int_{\phi}^{t} g(t-\tau)\, u(\tau)\, d\tau}$$

where here $g(t) = Kt$ $\left[\text{so } g(t-\tau) = K(t-\tau)\right]$

We will (indirectly) show that for <u>any</u> system, no matter how complex the dynamics, this relationship between $u(t)$ and $y(t)$ holds.

Different systems are characterized by different functions $g(t)$.

The characteristic function $g(t)$ is called the
<u>Impulse response</u>

Implication

Suppose:

$$y_1(t) = \int_0^t g(t-\tau)\, u_1(\tau)\, d\tau$$

$$y_2(t) = \int_0^t g(t-\tau)\, u_2(\tau)\, d\tau$$

are two **known** input-output pairs.

Suppose that $u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$; α_1, α_2 **constant**

Then:

$$y(t) = \int_0^t g(t-\tau)\left[\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau)\right] d\tau$$

$$= \alpha_1 \int_0^t g(t-\tau)\, u_1(\tau)\, d\tau + \alpha_2 \int_0^t g(t-\tau)\, u_2(\tau)\, d\tau$$

hence

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Principle of Linearity.

This suggests an approach:

① Identify a "family" of functions $u_k(t)$ for which it is EASY to calculate response $y_k(t)$:

$$u_k(t) \mapsto y_k(t) \quad \text{(easy)}$$

② "Break down" an arbitrarily complicated $u(t)$ into a linear combination of the $u_k(t)$:

$$u(t) = \sum \alpha_k u_k(t) \qquad \text{(easy?)}$$

③ Use Linearity:

$$y(t) = \sum \alpha_k y_k(t) \qquad \text{(easy)}$$

Time varying complex numbers

$$z(t) = a(t) + b(t)\,j$$
$$= r(t)\,e^{j\theta(t)}$$

Important example:

$$z(t) = e^{st} \text{ with } s \in \mathbb{C}$$

"Complex exponential functions"

Let $s = \sigma + j\omega \quad \sigma, \omega \in \mathbb{R}$

So $\mathrm{Re}\{s\} = \sigma$, $\mathrm{Im}\{s\} = \omega$

① If $\omega = \emptyset$, then

$$e^{st} = e^{\sigma t} \text{ (\underline{real} exponential)}$$

② If $\sigma = \emptyset$ then

$$e^{st} = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Note: $\mathrm{Im}\{s\}$ gives *frequency* of the oscillations

3. Most general case

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}$$

$$= e^{\sigma t}\left[\cos\omega t + j\sin\omega t\right]$$

$$\boxed{\begin{aligned} \mathrm{Re}\{e^{st}\} &= e^{\sigma t}\cos(\omega t) \\ \mathrm{Im}\{e^{st}\} &= e^{\sigma t}\sin(\omega t) \end{aligned}}$$

$\sigma \rightarrow$ amplitude envelope

$\omega \rightarrow$ oscillation frequency

$s = \sigma + j\omega$ is the "Complex frequency"

Utility of e^{st}

For different values of s, e^{st} is:

- a constant
- a real exponential
- a pure sine/cosine wave
- an exponentially decaying or increasing sine/cosine

Covers 90% of cases needed to solve linear diff'l eq'ns

Complex Amplitudes

Now consider $z(t) = Ae^{st}$
with both $A, s \in \mathbb{C}$.

$s = \sigma + j\omega$, $A = re^{j\varphi}$ (polar)

$$
\begin{aligned}
Ae^{st} &= \left(re^{j\varphi}\right)\left(e^{(\sigma + j\omega)t}\right) \\
&= \left(re^{\sigma t}\right)\left(e^{j(\omega t + \varphi)}\right) \\
&= re^{\sigma t}\left[\cos(\omega t + \varphi) + j\sin(\omega t + \varphi)\right]
\end{aligned}
$$

so $\boxed{\begin{aligned} \mathrm{Re}\{Ae^{st}\} &= re^{\sigma t}\cos(\omega t+\varphi) \\ \mathrm{Im}\{Ae^{st}\} &= re^{\sigma t}\sin(\omega t+\varphi) \end{aligned}}$

$r = |A|$ is initial amplitude of oscillations

$\varphi = \measuredangle A$ is phase shift of oscillations

$\varphi > 0$ called "phase lead"

$\varphi < 0$ called "phase lag"

Property of e^{st}:

Let $f(t) = e^{st}$ for any $s \in \mathbb{C}$

Then $\dot{f}(t) = \frac{d}{dt} f(t) = \frac{d}{dt}(e^{st})$

$= s e^{st}$

or $\dot{f}(t) = s f(t)$

Similarly: $\ddot{f}(t) = s^2 f(t)$

$f^{(3)}(t) = s^3 f(t)$, etc

Linear constant coefficient (time invariant) Diff Eq'n

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 \dot{y} + \alpha_0 y$$
$$= \beta_m u^{(m)} + \cdots + \beta_1 \dot{u} + \beta_0 u$$

where $\alpha_n, \ldots \alpha_0$ and $\beta_m \ldots \beta_0$ are
<u>real</u> and <u>constant</u>

Suppose $u(t) = U e^{st}$ with $s, U \in \mathbb{C}$

Is $y(t) = Y e^{st}$ a sol'n for some $Y \in \mathbb{C}$?

Substitute into DE

GIVES

$$r(s)Ye^{st} = q(s)Ue^{st}$$

With:

$$r(s) = \alpha_n s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

$$q(s) = \beta_m s^m + \cdots + \beta_1 s + \beta_0$$

So Assumption is consistent with

$$Y = \left[\frac{q(s)}{r(s)}\right]U = G(s)U$$

$$Ue^{st} = u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t) = Ye^{st}$$

system

$$Y = G(s)U$$

If $u(t) = Ue^{st}$ for some $U, s \in \mathbb{C}$
then $y(t) = Ye^{st}$, with $Y = G(s)U$

This is one possible sol'n of the DE, the forced sol'n, $\boxed{y_f(t)}$.

Other sol'ns are possible.

Other possible sol'ns

Now, suppose $u(t) = \emptyset$. Clearly here $y_f(t) = \emptyset$. But is $y(t) = \emptyset$ necessarily? Or can we still have sol'ns of the form $y(t) = Ce^{st}$? Substitute into DE:

$$r(s)Ce^{st} = \emptyset$$

which <u>can</u> be true for any s where

$$\boxed{r(s) = \emptyset}$$

$$r(s) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0$$

There are n values of s for which $r(s) = \emptyset$. We denote these

$$p_1, p_2, \cdots p_n$$

So $r(s)$ can be factored as

$$r(s) = \alpha_n (s - p_1)(s - p_2) \cdots (s - p_n)$$

$$= \alpha_n \prod_{k=1}^{n} (s - p_k)$$

For any p_K with $r(p_K) = \phi$,
$y(t) = e^{p_K t}$ is a sol'n of the DE
when $u(t) = \phi$. So is $y(t) = C_K e^{p_K t}$
for any constant C_K. So is any
<u>sum</u> of these terms:

$$y(t) = \sum_{K=1}^{n} C_K e^{p_K t} = y_h(t)$$

The "homogeneous" sol'n.

__Proof:__

Substitute $y(t) = \sum_{k=1}^{n} c_k e^{p_k t}$

into diff eq'n:

GIVES:

$$r(p_1)c_1 e^{p_1 t} + r(p_2)c_2 e^{p_2 t} + \cdots + r(p_n)c_n e^{p_n t} = 0$$

Which is true if $r(p_1) = r(p_2) = \cdots = r(p_n) = 0$

i.e. the p_k are zeros of polynomial $r(s)$

Since any $y_h(t)$ yields $\emptyset$ exactly when substituted into DE, we can add it to any other sol'n and still have a valid sol'n. Generally:

$$y(t) = y_h(t) + y_f(t)$$

where $y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t}$

and if $u(t) = U e^{st}$, then

$$y_f(t) = G(s) U e^{st}$$

Both complex! (Generally)

But $y_f(t)$ is complex generally...?

$\Longrightarrow$ because $u(t)$ is complex here

Suppose $u(t) = B\sin(\omega t + \varphi)$ (real)

$= \text{Im}\{Ue^{st}\}$ Take matching Im part

with $U = Be^{j\varphi}$

and $s = j\omega$

Then $y_f(t) = \text{Im}\{G(s)Ue^{st}\}$

and similarly for cosine inputs, taking real part

What about $y_h(t)$?

Contains terms e^{pt}, where $r(p) = \emptyset$.

If p is complex, $p = \sigma + j\omega$, $\omega \neq \emptyset$

then e^{pt} is complex

However: in this case $r(p) = \emptyset \Longrightarrow r(\bar{p}) = \emptyset$

i.e. $\bar{p}$ is also a zero of $r(s)$.

$\Longrightarrow$ Complex roots of polynomials occur in "conjugate pairs".

Hence, with complex roots, $y_h(t)$ will contain

$$C_1 e^{pt} + C_2 e^{\bar{p}t}$$

<u>FACT:</u>

$$C_2 = \overline{C_1}$$

i.e. coef of $e^{\bar{p}t}$ will <u>always</u> be the conjugate of the coef of e^{pt}.

Thus, if $r(s)$ has a complex root p, $y_h(t)$ will contain

$$ce^{pt} + \bar{c}e^{\bar{p}t} = ce^{pt} + \overline{ce^{pt}}$$

Recap (DE Review)

Any constant coef linear diff'l eq'n has sol'n:

$$y(t) = y_h(t) + y_f(t)$$

where

$$y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t}$$] homogeneous response

$$r(p_k) = 0, \quad k = 1, \ldots, n$$] p_k roots of char. poly $r(s)$

and $y_f(t)$ depends on specific forcing function (input)

("forced response")

For the specific case that $u(t) = Ue^{st}$, $U, s \in \mathbb{C}$

then

$$y_f(t) = G(s)u(t), \quad G(s) = \frac{q(s)}{r(s)}$$

"transfer function"

$$= Ye^{st}$$

with $Y = G(s)U$ (ordinary complex number multiplication!)

Complex math yields real sol'ns

Note that both $y_h(t)$ and $y_f(t)$ are complex-valued functions as we have written them

But physical systems will have only real-valued inputs and outputs.

For $y_f(t)$, note that we can express a real input as the real or imag part of a complex input:

$$u(t) = Ae^{\sigma t}\sin(\omega t+\varphi) = \mathrm{Im}\{\bar{U}e^{st}\}$$

$$\text{with } \bar{U} = Ae^{j\varphi} \text{ and } s = \sigma + j\omega$$

The corresponding real $y_f(t) = \mathrm{Im}\{G(s)\bar{U}e^{st}\}$

and similarly if input is cosinoidal we use the real part of the complex number calculation

Complex => real, cont

for $y_h(t)$:

Sol'n contains terms e^{pt}, $r(p)=0$

This will be complex if root p is complex, i.e.

$p = \sigma + j\omega$, $\omega \neq 0$.

However, if this is true then $\bar{p} = \sigma - j\omega$ will

<u>also</u> be a root of $r(s)$, i.e. $r(\bar{p}) = r(p) = 0$

Complex roots of polynomials always occur in "conjugate pairs"

So Sol'n for $y_h(t)$ will really look like

$$y_h(t) = c_1 e^{pt} + c_2 e^{\bar{p}t} + (\cdots \text{other terms})$$

Real-valued homogeneous response

So

$$y_h(t) = \underbrace{c_1 e^{pt} + c_2 e^{\bar{p}t}} + (\cdots \text{other terms})$$

Fact: It is always the case that $c_1 = \bar{c_2}$,

i.e. the coefs. of conjugate terms are themselves conjugates

(This is b/c boundary cond'ns on DE are also real)

$$\Rightarrow \text{so, } y_h(t) = ce^{pt} + \bar{c}e^{\bar{p}t} + (\cdots)$$

$$= \underbrace{ce^{pt} + \overline{ce^{pt}}}_{\text{sum of conjugate terms}} + (\cdots)$$

$$= \underbrace{2\,\mathrm{Re}\{ce^{pt}\}} + (\cdots)$$

$\Rightarrow$ The two complex terms from conjugate roots $p, \bar{p}$ combine to form a real function of time!

Conclusion

$\Rightarrow$ When $p = \sigma + j\omega$, $\omega \neq 0$ is a root of char poly $r(s)$, the homog. sol'n will contain the real-valued function

$$2\,\mathrm{Re}\{c e^{pt}\} = A e^{\sigma t} \cos(\omega t + \varphi)$$

where $A = 2|c|$, $\varphi = \angle c$ are determined by ICs.

$\Rightarrow$ $\left\{ \begin{array}{l} \text{Complex roots of } r(s) \text{ correspond to real} \\ \text{oscillations in homog. response.} \end{array} \right\}$

(Note, there may be several pairs of conjugate roots in $r(s)$, resulting in multiple oscillations w/ different frequencies + damping)

Recap (DE Review)

Any constant coef diff'l eq'n has sol'n

$$y(t) = y_h(t) + y_f(t)$$

where

$$y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t}$$ homogeneous response

$$r(p_k) = 0, \quad k=1,\dots,n$$ p_k roots of char. poly $r(s)$

and $y_f(t)$ depends on specific forcing function (input)
("forced response")

For the specific case that $u(t) = Ue^{st}$, $U, s \in \mathbb{C}$

then

$$y_f(t) = G(s)u(t), \quad G(s) = \frac{q(s)}{r(s)}$$ "transfer function"

$$= Ye^{st}$$

with $Y = G(s)U$ (ordinary complex number multiplication!)

Boundary/initial conditions

Undetermined coefs c_K in $y_h(t)$ determined by boundary conditions $y(0), \dot{y}(0), \dots, y^{(n-1)}(0)$

For simple problems, can often solve for c_K by substituting general form $y(t) = y_h(t) + y_f(t)$, differentiating, and matching stated B/Cs.

$\Rightarrow$ Results in a system of n equations in the n coefs c_K which can be solved (lin. algebra)

Warning: There are situations where this approach to compute c_K will not work.

Will cover this situation shortly.

Particularly if $u(t)$ is discontinuous at $t=0$ and one or more derivs of $u(t)$ appear in DE

Example

$$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = 2\dot{u}(t) + u(t)$$

where $y(0) = \dot{y}(0) = 0, \quad u(t) = 3\cos(2t - \pi/2)$

By inspection:

$$y(t) = \overbrace{c_1 e^{-t} + c_2 e^{-4t}}^{y_h(t)} + \overbrace{A\cos(2t + \varphi)}^{y_f(t)}$$

Only remaing problem is to calculate c_1, c_2, A, φ

Note: A, φ in $y_f(t)$ determined by $u(t)$, and are independent of c_1, c_2

With a little more calculation:

$$y(t) = \frac{1}{10}\left[7e^{-4t} - 4e^{-t}\right] + \underbrace{\left(\frac{3\sqrt{17}}{10}\right)}_{A}\cos\left(2t \underbrace{- \pi/2 - \tan^{-1} 1/4}_{\varphi}\right)$$

Forced response

(zero)

Here $u(t) = 3\cos(2t - \pi/2) = 3e^{0t}\cos(2t - \pi/2)$

$= \mathrm{Re}\{Ue^{st}\}$ with $U = 3e^{-\frac{\pi}{2}j}$

$s = 0 + 2j$

and $y_f(t) = \mathrm{Re}\{G(s)Ue^{st}\}$

$= \mathrm{Re}\{Ye^{st}\}$, $Y = G(s)U$

$= |Y|e^{0t}\cos(2t + \measuredangle Y)$ ($s = 0 + 2j$ from input)

All we need to do is the complex number multiplication $Y = G(s)U$ and convert to polar form $|Y|, \measuredangle Y$

$\Rightarrow$ We have U, still need transfer f'n $G(s)$

Diff'l eq'n is

$$\dddot{y} + 5\dot{y} + 4y = 2\dot{u} + u$$

$$G(s) = \frac{q(s)}{r(s)}$$

$q(s) =$

$r(s) =$

so finally $G(s) =$

Note that $r(s) = (s+1)(s+4)$

which also gives us the general form for $y_h(t)$

$$c_1 e^{-t} + c_2 e^{-4t}$$

Evaluate $G(s)$ at complex freq of input, $s = 2j$ here

$$G(2j) = \left. \frac{2s+1}{s^2+5s+4} \right|_{s=2j}$$

$$= \frac{1+4j}{10j}$$

$$= \frac{1}{10}(1-4j)$$

How...?

Here $u(t) = 3\cos(2t - \pi/2) = \mathrm{Re}\{U e^{st}\}$

with $s = 2j$ and $U = 3e^{-\pi/2 j}$

So $y_f(t) = \mathrm{Re}\{G(2j)(3e^{-\pi/2 j})(e^{2jt})\}$

$Y = G(2j)U$

with here: $G(s) = \dfrac{2s+1}{s^2+5s+4}$

$\Rightarrow G(2j) = \dfrac{1+4j}{(2j)^2+10j+4} = \frac{1}{10}(4-j) = \left(\frac{\sqrt{17}}{10} \measuredangle -\tan^{-1}\left(\frac{1}{4}\right)\right)$

Hence:

$$y_f(t) = \overbrace{\frac{3\sqrt{17}}{10}}^{|Y|} \cos(2t \overbrace{- \pi/2 - \tan^{-1}(1/4)}^{\measuredangle Y})$$

So we know $y_f(t)$ exactly at this point.

Homogeneous Sol'n

We have $r(s) = s^2 + 5s + 4$ (denom poly of $G(s)$)

Or: $r(s) = (s+1)(s+4)$

So $p_1 = -1$, $p_2 = -4$ and $y_h(t) = c_1 e^{-t} + c_2 e^{-4t}$

Then $y(t) = y_f(t) + y_h(t)$

$$= \frac{3\sqrt{17}}{10}\cos\left(2t - \tfrac{\pi}{2} - \tan^{-1}(1/4)\right) + c_1 e^{-t} + c_2 e^{-4t}$$

So $y(0) = c_1 + c_2 - \frac{3}{10} = 0$ (specified)

and $\dot{y}(0) = -c_1 - 4c_2 + 12/5 = 0$ (specified)

Impose Boundary Cond'ns

Equivalently

$$\begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/10 \\ -12/5 \end{bmatrix}$$

$\Rightarrow$ (linear algebra):

$$c_1 = -4/10, \quad c_2 = 7/10$$

So that:

$$y(t) = \frac{1}{10}\left[7e^{-4t} - 4e^{-t}\right] + \frac{3\sqrt{17}}{10}\cos\left(2t - \frac{\pi}{2} - \tan^{-1}\frac{1}{4}\right)$$

as claimed

final result

Recap

General sol'n of LTI DE is:

$$y(t) = y_h(t) + y_f(t)$$

Forced response $y_f(t)$ depends on $u(t)$

Homogeneous response is <u>independent</u> of $u(t)$:

$$\left. y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t} \quad \text{where } r(p_k) = \emptyset \right\} \text{ for } \underline{\text{any}}\ u(t)$$

Specific coefs c_k depend on initial conditions and $u(t)$.

Repeated roots of $r(s)$

Above formula for $r(s)$ assumes the roots p_k are <u>non-repeated</u>

Suppose instead that there are repeated roots, for example:

$$r(s) = (s-p_1)^{\ell}(s-p_{\ell+1})\cdots(s-p_n)$$

i.e. p_1 is repeated ℓ times. Then:

$$y_h(t) = (c_1 + c_2 t + c_3 t^2 + \cdots + c_\ell t^{\ell-1})e^{p_1 t} + \sum_{k=\ell+1}^{n} c_k e^{p_k t}$$

(will prove later)

(Natural) Modes

$r(p_k)=0$

$y_h(t)$ is a linear combination of $e^{p_k t}$ (or $t^i e^{p_k t}$). These describe solutions which are possible without any input

They are "natural" motions which are intrinsic to the dynamics of the system.

We call them the "MODES".

$\boxed{\text{MODES:}}$ Terms in sol'n for $y(t)$ of form e^{pt}, where $r(p) = \emptyset$

Two cases (non-repeated, to start)

① p real : e^{pt} is a real exponential function

"1st order mode"

(2) p complex: e^{pt} and $e^{\bar{p}t}$ both present in solution, and will combine to form the "2^{nd} order mode"

$$A e^{\sigma t} \cos(\omega t + \varphi)$$

where $\sigma = Re\{p\}$, $\omega = Im\{p\}$

and A, φ depend on the initial conditions

Stability

A mode e^{pt} is stable if

$$|e^{pt}| \to 0 \text{ as } t \to \infty$$

A system is stable if

$$|e^{p_k t}| \to 0 \text{ for all } k = 1, \ldots, n$$

i.e. if every mode is stable

Note: if true then $y_h(t) \to 0$ for any set of initial conditions.

Stability Condition

As usual, let $p = \sigma + j\omega$. Then:

$$|e^{pt}| = |e^{(\sigma + j\omega)t}|$$
$$= |e^{\sigma t} e^{j\omega t}| = |e^{\sigma t}|\,|e^{j\omega t}|$$
$$= |e^{\sigma t}|$$

So $|e^{pt}| \to \emptyset$ only if $\sigma < \emptyset$. Hence:

$$\boxed{\text{A mode is stable if } \sigma = \mathrm{Re}\{p\} < \emptyset}$$

System Stability

The system is stable if:

$$\mathrm{Re}\{p_k\} < 0 \text{ for all } k = 1, \ldots, n$$

$\Rightarrow$ all roots of $r(s)$ have negative real parts

$\Rightarrow$ all roots of $r(s)$ lie to the left of imaginary axis in the complex plane.

$\Rightarrow$ all roots of $r(s)$ lie in left half of complex plane (LHP)

Instability

A **mode** e^{pt} is **unstable** if $\text{Re}\{p\} > 0$

$\Rightarrow$ root p lies to right of imag Axis

$\Rightarrow$ p is in "right half plane" (RHP)

A **system** is unstable if:

$$\text{Re}\{p_k\} > 0 \text{ for \underline{any} } k = 1, \ldots, n$$

i.e. if **any** roots of $r(s)$ are in RHP.

What about repeated modes?

Repeated real modes have terms like:

$$t^i e^{pt} \quad \text{(powers of } t \text{ multiplying } e^{pt}\text{)}$$

Fact: for any $i > 0$, if $Re\{p\} < \emptyset$ then

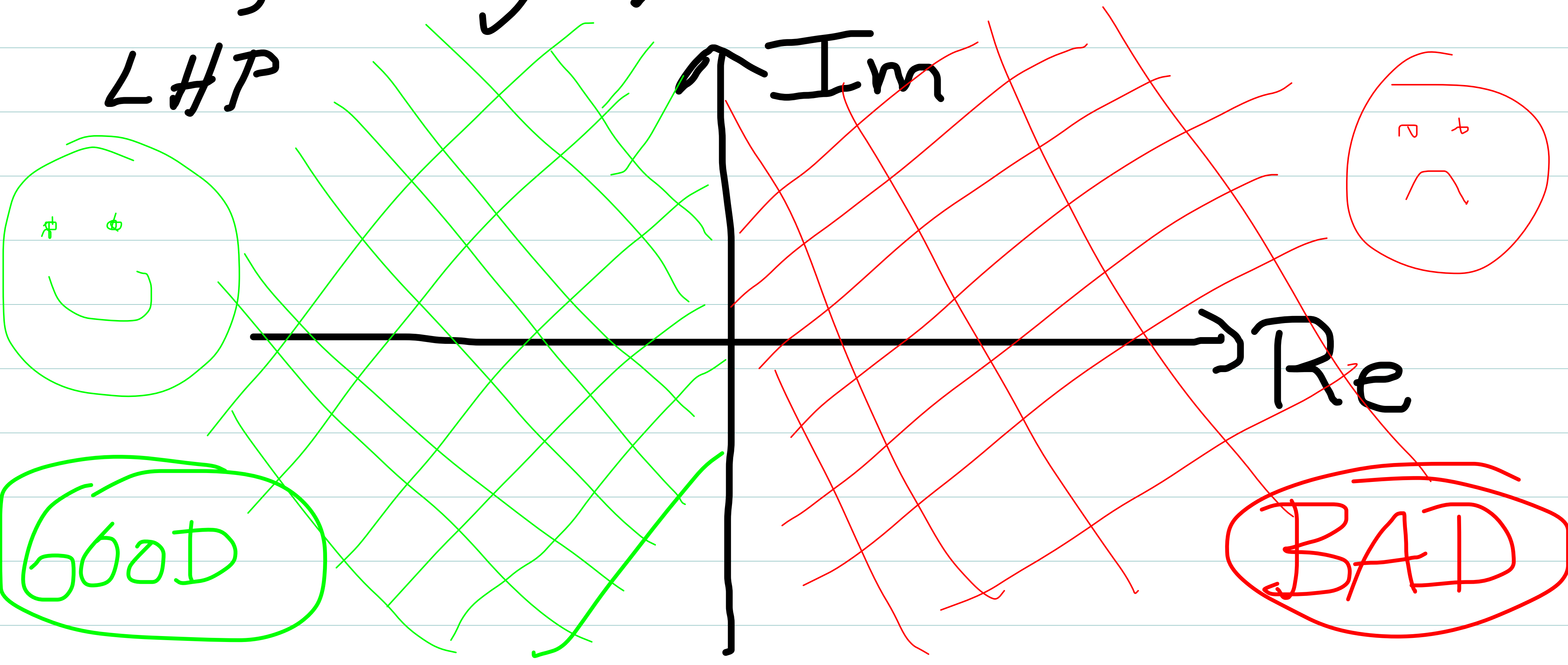
$$\lim_{t \to \infty} |t^i e^{pt}| \to \emptyset$$

Thus a repeated mode is stable as long as the repeated roots are in LHP.

Conversely, a repeated mode is unstable if repeated roots in RHP.

Hence:

A system is stable if all roots of $r(s)$, including repeated roots, lie in LHP



Recap

Stable mode: $|e^{pt}| \to 0$ as $t \to \infty$

$$\iff \operatorname{Re}\{p\} < 0$$

Unstable mode: $|e^{pt}| \to \infty$ as $t \to \infty$

$$\iff \operatorname{Re}\{p\} > 0$$

Stable **system**: $\operatorname{Re}\{p_k\} < 0$ for **all** $k = 1, \ldots, n$

Unstable **system**: $\operatorname{Re}\{p_k\} > 0$ for **any** $k = 1, \ldots, n$

What happens if $\operatorname{Re}\{p\} = 0$?

Marginally stable MODES

$$\mathrm{Re}\{p\} = \emptyset \Rightarrow |e^{pt}| = |e^{j\omega t}| = 1 \quad \forall t \geq 0$$

i.e. the magnitude is constant

$\Rightarrow$ neither increasing nor decreasing with time

$\Rightarrow$ neither stable nor unstable

"Marginally stable"

Repeated MODES with $\mathrm{Re}\{p\} = \emptyset$ will increase in magnitude polynomially in t

$\Rightarrow$ Not as "bad" as exponential growth

An alternate decomposition of $y(t)$

$$y(t) = y_h(t) + y_f(t)$$
$$= y_{tr}(t) + y_{ss}(t) \quad \} \text{(regroup terms)}$$

$y_{tr}(t)$ is the "transient response", which satisfies:

$$\lim_{t \to \infty} |y_{tr}(t)| \to 0$$

$y_{ss}(t)$ is the "steady-state" response, which is all remaining terms in $y(t)$.

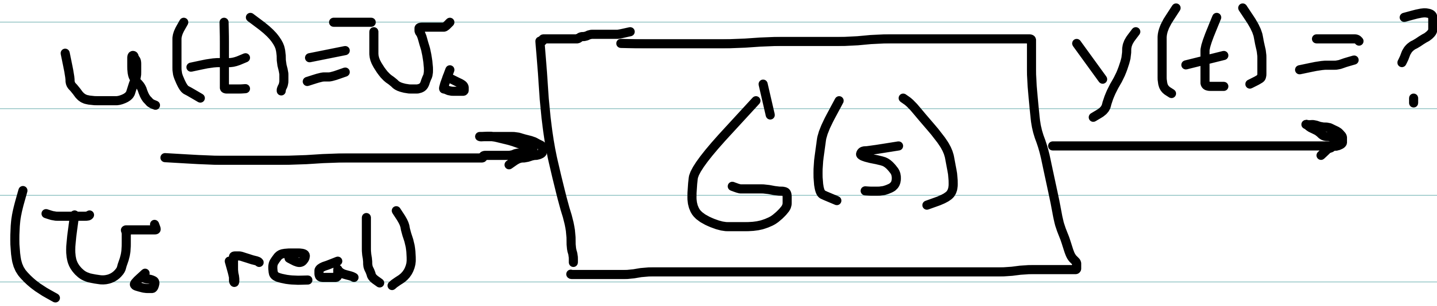
Notes:

1. If system is stable, $y_{tr}(t)$ contains $y_h(t)$ but $y_{tr}(t)$ would also contain decaying terms in $y_f(t)$ (if any).

2. Conversely, marginally stable terms in $y_h(t)$ (if any) would be part of $y_{ss}(t)$.

3. "steady-state" is not a useful concept if system is unstable.

Example: Stable system with constant input



constant!

$$y(t) = y_h(t) + y_f(t) = y_h(t) + \boxed{G(0)U_0}$$

Since system is stable, $y_h(t) \to 0$ as $t \to \infty$

So here: $y_{tr}(t) = y_h(t)$

$y_{ss}(t) = G(0)\,U_0$ (constant)

Very common and important case!

A Different Example

$$G(s) = \frac{s+2}{s(s+1)}, \quad u(t) = e^{-3t}$$

$$y_h(t) = c_1 + c_2 e^{-t}$$

$$y_f(t) = G(-3)e^{-3t} = -\tfrac{1}{6}e^{-3t}$$

$$y_{tr}(t) = c_2 e^{-t} - \tfrac{1}{6}e^{-3t}$$

$$y_{ss}(t) = c_1$$

Note: system is <u>not</u> stable here.

Convergence metrics

Useful to quantify how quickly stable modes decay to $\emptyset$.

"2% criterion": Defines the settling time t_s for a mode to be such that

$$|e^{pt}| \leq .02 \quad \forall t \geq t_s$$

For a 1st order mode ($p = \sigma$, real)

$$t_s = \frac{\ln(.02)}{\sigma} \approx \frac{4}{|\sigma|} = \frac{4}{|Re\{p\}|}$$

2^{nd} order settling time

For a 2^{nd} order mode $e^{pt}, e^{\bar{p}t}$ with $p = \sigma + j\omega$, $\omega \neq 0$, the calculation is more complicated due to the oscillations.

However:

$$t_s \approx \frac{4}{|\sigma|} = \frac{4}{|Re\{p\}|}$$

is still a useful approximation in these cases also.

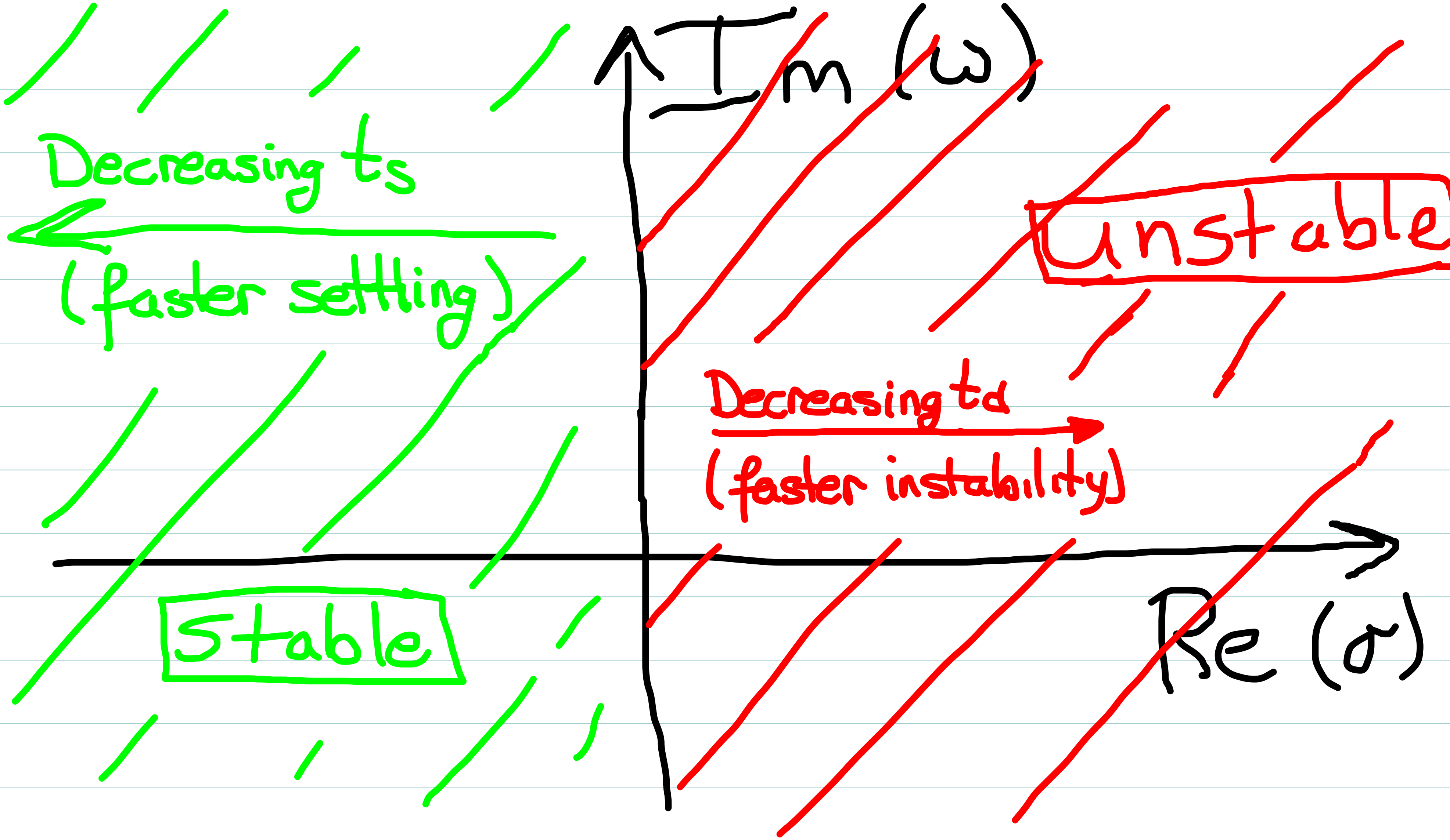
"Doubling time" of unstable modes

When $\sigma > 0$, the doubling time t_d is such that

$$|e^{\sigma t_d}| = 2 \quad \Rightarrow \quad \boxed{t_d \approx \frac{0.7}{\sigma}}$$

Smaller $t_d \iff$ "more unstable" system

$\Rightarrow$ Faster rate of increase for amplitude



Settling times decrease the further left of the imag axis the root p is.

To a <u>first approximation</u>, the settling time of a <u>system</u> is the settling time of its <u>slowest</u> mode

$\Rightarrow$ Mode <u>closest to imag Axis</u> determines settling time

$\Rightarrow$ Called the "<u>dominant mode</u>"

$\Rightarrow$ Only a "first cut." <u>Will refine later</u>

2nd Order "Damping ratio"

For 2nd order modes we also define the

<u>damping ratio</u>

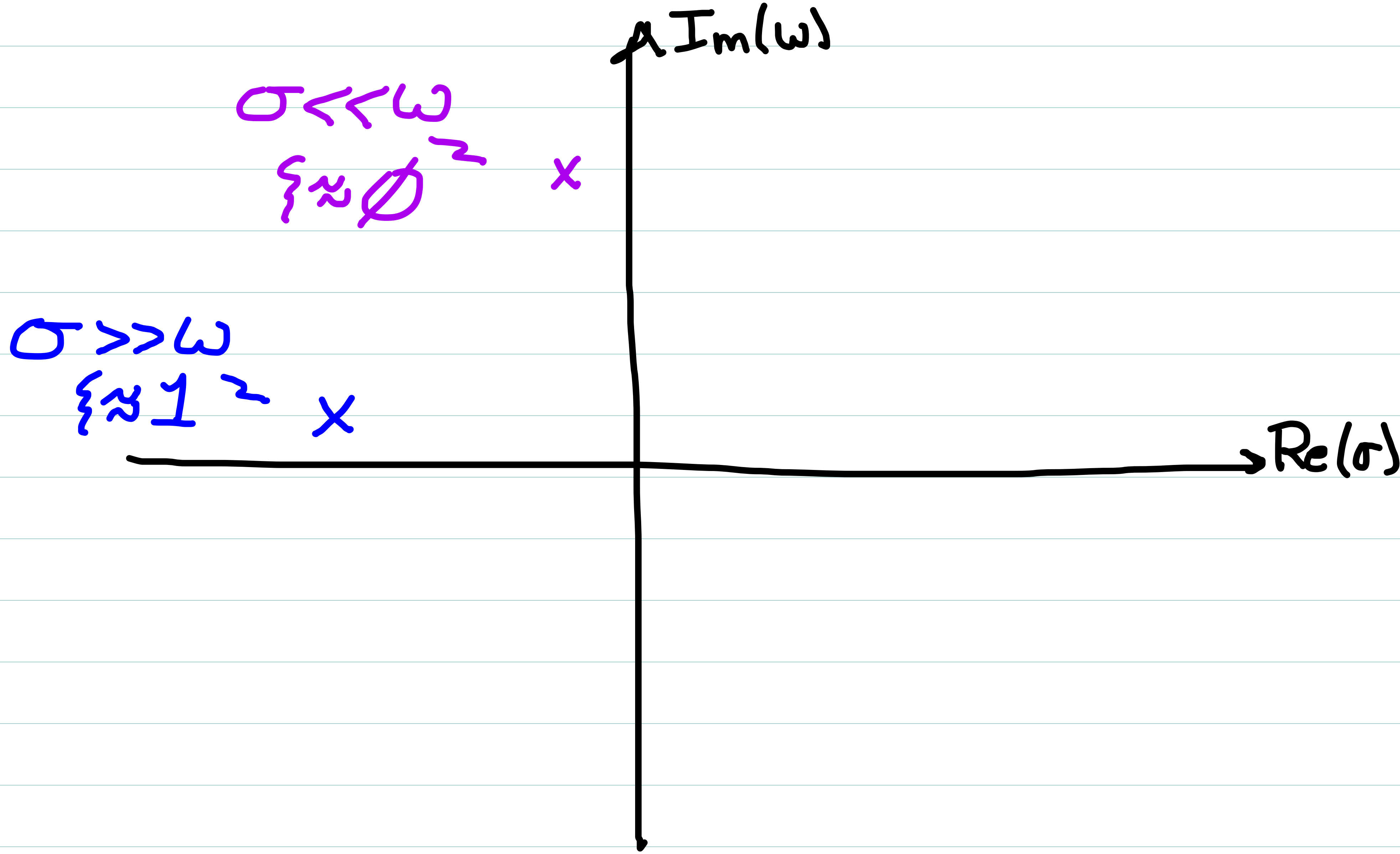
$$\boxed{\zeta = \left|\frac{\sigma}{p}\right| = \frac{|\sigma|}{\sqrt{\sigma^2+\omega^2}}}$$

A non-dimensional comparison of convergence rate to oscillation frequency

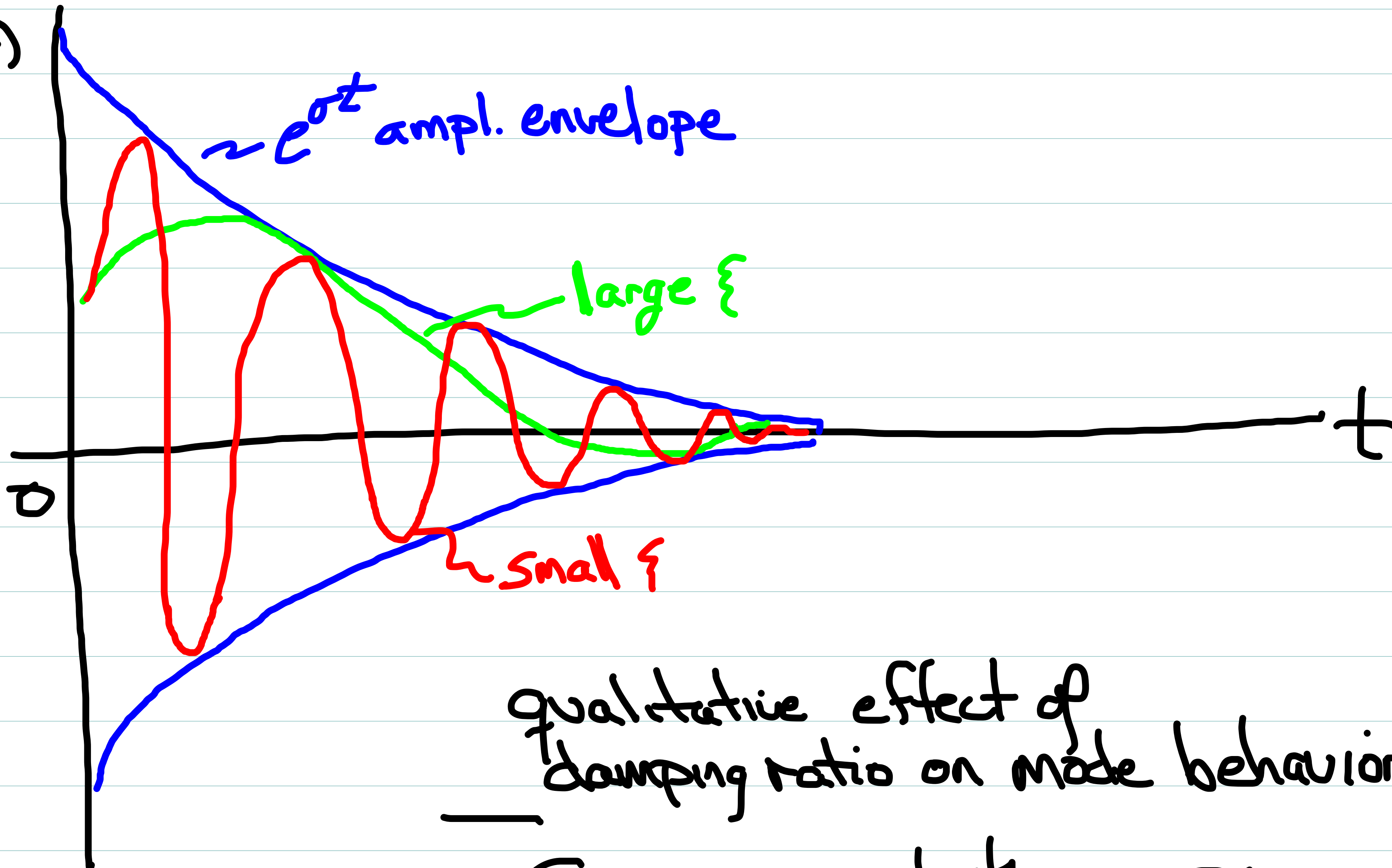
$0 \leq \zeta \leq 1$ for a stable mode

$\zeta \approx 0 \iff$ many oscillations before 2% criterion reached

$\zeta \approx 1 \iff$ less than one complete oscillation before 2% criterion reached.



Will explore in greater detail later



qualitative effect of damping ratio on mode behavior

Same σ in both CASES, different ω

Transfer functions

$$G(s) = \frac{q(s)}{r(s)}$$

Compactly gives us all information we need to predict major features of system response

- $y_h(t)$, modes, stability : all from $r(s)$ the denominator polynomial of $G(s)$

$$r(s) = \alpha_n \prod_{k=1}^{n} (s - p_k)$$

- forced response : Evaluate $G(s)$ at specific complex values of s.

Numerator Terms

Can also factor $q(s)$:

$$q(s) = \beta_m (s - z_1)(s - z_2) \cdots (s - z_m)$$

where $q(z_i) = \emptyset$ for $i = 1, \ldots, m$

The values z_i are called the zeros of $G(s)$

since $G(z_i) = \dfrac{\cancel{q(z_i)} \to \emptyset}{r(z_i)} = \emptyset$

The values p_k are called the poles of $G(s)$

since $G(p_k) = \dfrac{q(p_k)}{\emptyset \leftarrow \cancel{r(p_k)}} = \infty$

Zero/Pole/Gain (ZPK) form

$$G(s) = K\left[\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{k=1}^{n}(s - p_k)}\right]$$

Poles p_k satisfy $r(p_k) = \emptyset$

Zeros z_i satisfy $q(z_i) = \emptyset$

Gain: $K = \dfrac{\beta_m}{\alpha_n}$ (always real)

Alternate ZPK form:

When $G(s)$ has complex poles and/or zeros, we commonly combine the conjugate roots of $r(s)$ or $q(s)$ into 2^{nd} order polynomials.

For example, if $p = \sigma + j\omega$ and $\bar{p} = \sigma - j\omega$ are complex roots of $r(s)$:

$$(s-p)(s-\bar{p}) = \underline{s^2 - 2\sigma s + (\sigma^2 + \omega^2)}$$

replace with in $G(s)$

Transfer functions

$$G(s) = \frac{q(s)}{r(s)}$$

Compactly gives us all information we need to predict major features of system response

- $y_h(t)$, modes, stability : all from $r(s)$ the denominator polynomial of $G(s)$

$$r(s) = \alpha_n \prod_{k=1}^{n} (s - p_k)$$

- forced response : Evaluate $G(s)$ at specific complex values of s.

Numerator Terms

Can also factor $q(s)$:

$$q(s) = \beta_m (s - z_1)(s - z_2)\cdots(s - z_m)$$

where $q(z_i) = \emptyset$ for $i = 1, \ldots, m$

The values z_i are called the zeros of $G(s)$

$$\text{since } G(z_i) = \frac{\cancelto{\emptyset}{q(z_i)}}{r(z_i)} = \emptyset$$

The values p_K are called the poles of $G(s)$

$$\text{since } G(p_K) = \frac{q(p_K)}{\cancelto{\emptyset}{r(p_K)}} = \infty$$

Zero/Pole/Gain (ZPK) form

$$G(s) = K\left[\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{k=1}^{n}(s-p_k)}\right]$$

Poles p_k satisfy $r(p_k)=\varnothing$

Zeros z_i satisfy $q(z_i)=\varnothing$

Gain: $K = \frac{\beta_m}{\alpha_n}$ (always real)

Alternate ZPK form:

When $G(s)$ has complex poles and/or zeros, we commonly combine the conjugate roots of $r(s)$ or $q(s)$ into 2^{nd} order polynomials.

For example, if $p = \sigma + j\omega$ and $\bar{p} = \sigma - j\omega$ are complex roots of $r(s)$:

$$(s-p)(s-\bar{p}) = \underline{s^2 - 2\sigma s + (\sigma^2 + \omega^2)}$$

replace with in $G(s)$

Stability and $G(s)$

→ $G(s)$ is stable if all its poles are in LHP.

→ $G(s)$ is unstable if any of its poles are in RHP.

→ What role do zeros of $G(s)$ have in stability?

⟹ ABSOLUTELY NONE!

→ OK, so what role do zeros play?

Effect of zeros in $G(s)$

→ Certainly zeros influence the coefficients c_K of homogeneous response.

→ They also influence calculation of $y_f(t)$.

→ Special example: Suppose $u(t) = e^{z_i t}$

then:

$$y_f(t) = G(z_i)e^{z_i t} = \emptyset$$

The forced response is exactly zero here!

"Input absorbing" property of zeros

More complicated $u(t)$

$$u(t) = Ue^{st} \Rightarrow y_f(t) = G(s)Ue^{st}$$

Suppose $u(t) = U_1 e^{s_1 t} + U_2 e^{s_2 t}$

Substitute into DE, can show

$$y_f(t) = G(s_1)U_1 e^{s_1 t} + G(s_2)U_2 e^{s_2 t}$$

More complicated $u(t)$

$$u(t) = Ue^{st} \Rightarrow y_f(t) = G(s)Ue^{st}$$

Suppose $u(t) = \boxed{U_1 e^{s_1 t}} + \boxed{U_2 e^{s_2 t}}$

Substitute into DE, can show

$$y_f(t) = \boxed{G(s_1)U_1 e^{s_1 t}} + \boxed{G(s_2)U_2 e^{s_2 t}}$$

The sum of the responses to the individual parts of the input.

Linearity of Systems

If $y_1(t)$ is a possible sol'n of DE with input $u_1(t)$

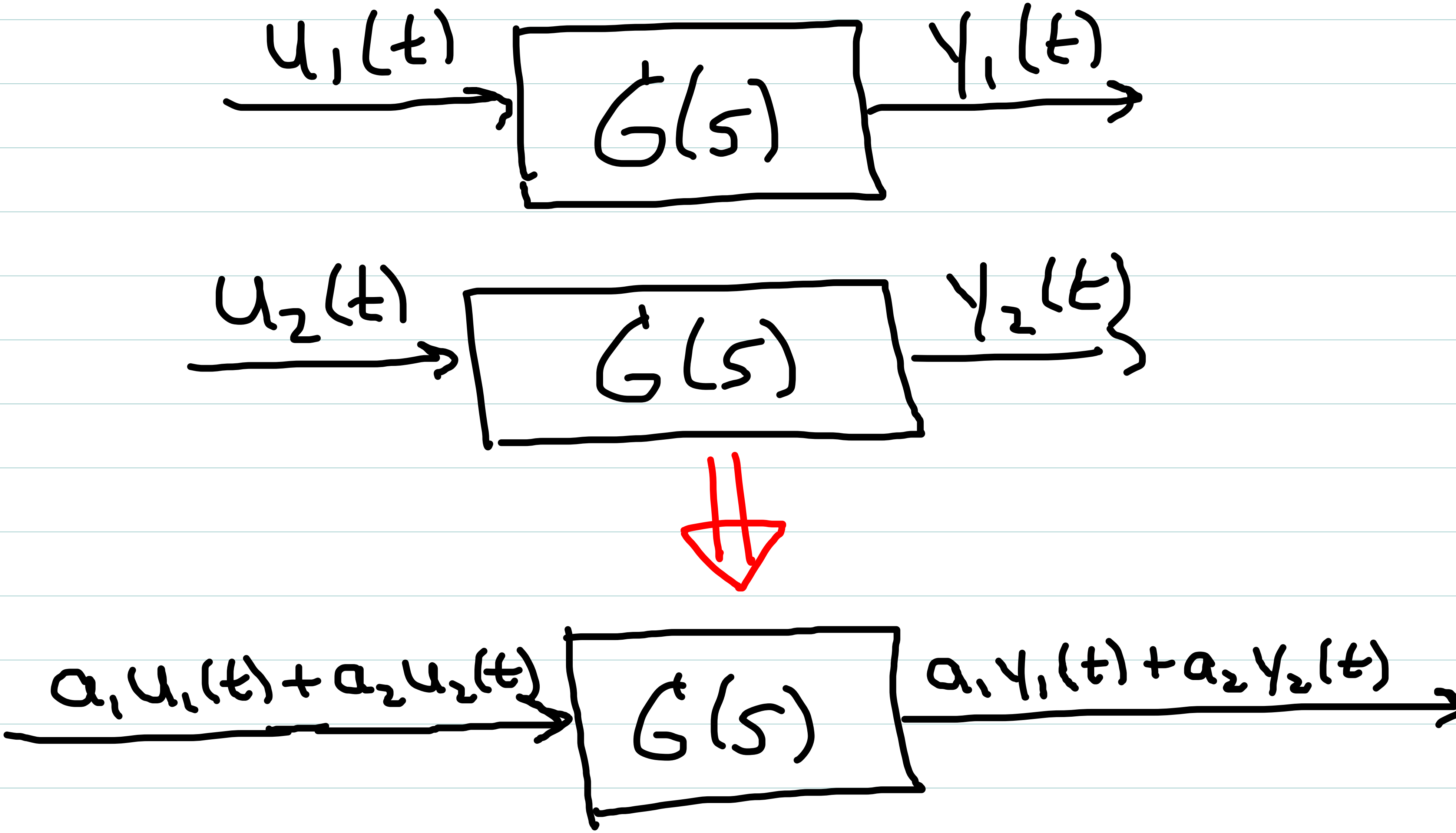
and similarly $y_2(t)$ is a sol'n for input $u_2(t)$

Then:

$$y(t) = a_1 y_1(t) + a_2 y_2(t)$$

is a sol'n for input $u(t) = a_1 u_1(t) + a_2 u_2(t)$

For **any** constants a_1, a_2 **and** any inputs $u_1(t), u_2(t)$



$u_1(t)$
$G(s)$
$y_1(t)$
$u_2(t)$
$G(s)$
$y_2(t)$
$a_1u_1(t)+a_2u_2(t)$
$G(s)$
$a_1y_1(t)+a_2y_2(t)$

We've already seen an example

$$u_1(t) = e^{st} \longrightarrow y_1(t) = G(s)e^{st}$$

$$u_2(t) = \emptyset \longrightarrow y_2(t) = y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t}$$

$$u(t) = U e^{st} = U e^{st} + \emptyset$$

$$= U u_1(t) + u_2(t)$$

$$\Rightarrow \quad y(t) = U y_1(t) + y_2(t)$$

$$= U G(s) e^{st} + y_h(t)$$

$$= y_f(t) + y_h(t)$$

Linearity can be used multiple times

$$u(t) = \sum_{i=1}^{N} a_i u_i(t) \implies y(t) = \sum_{i=1}^{N} a_i y_i(t)$$

$y_i(t)$ sol'n for $u_i(t)$

$\Rightarrow$ Holds for any number N

In particular,

$$u(t) = \sum_{i=1}^{N} U_i e^{s_i t} \implies y(t) = \sum_{i=1}^{N} G(s_i) U_i e^{s_i t}$$

Even for infinite sum, $N = \infty$.

Is this enough to make <u>any</u> u(t)?

Not quite, need to go to differential limit

$$\sum_{i=1}^{N} U_i e^{s_i t} \longrightarrow \int U(s) e^{st}\, ds$$

integral over all complex freqs

i.e. $u(t) = \int U(s) e^{st}\, ds$

$U(s)$ is the "amount" (complex amplitude) of e^{st} present in $u(t)$, for <u>each</u> $s \in \mathbb{C}$.

Similarly $y(t) = \sum_{i=1}^{N} G(s_i) U_i e^{s_i t} \rightarrow \int G(s) U(s) e^{st}\, ds$

OR: $y(t) = \int Y(s) e^{st}\, ds$ with $\boxed{Y(s) = G(s) U(s)}$

Laplace Transform

More formally, for any $f(t)$ define:

$$(1) \qquad f(t) = \frac{1}{2\pi j} \int F(s) e^{st} \, ds$$

Normalizing constant

where:

$$(2) \qquad F(s) = \int_0^{\infty} f(t) e^{-st} \, dt$$

Notation:

$F(s) = \mathcal{L}\{f(t)\}$ (transform)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$ (inverse transform)

Limitations of Laplace Transform

Only defined for $f(t)$ where the integral (2) converges.

Requires: $\quad e^{-\sigma_0 t}|f(t)| \to \emptyset$

for some finite $\sigma_0 \in \mathbb{R}$

The transform $F(s)$ is then defined for any

$s = \sigma + j\omega \quad$ with $\sigma \geq \sigma_0$

and the integral (1) is over all values of s which satisfy this condition. "region of convergence"

Examples

$f(t) = e^{pt}$ can be transformed for any finite $p \in \mathbb{C}$

However, $f(t) = e^{t^2}$ cannot be transformed since $e^{-\sigma_0 t} f(t) = e^{(t^2 - \sigma_0 t)} \to \infty$

for any finite σ_0.

Note:

When working with Laplace transforms we assume we are using values of s in the region of convergence. (ROC)

By above def'n of ROC,

$$\lim_{t \to \infty} e^{-st} f(t) = \phi$$

for these values of s.

Laplace Transform

More formally, for any $f(t)$ define:

(1) $$f(t) = \frac{1}{2\pi j} \int F(s) e^{st} \, ds$$

↳ Normalizing constant

Where:

(2) $$F(s) = \int_0^\infty f(t) e^{-st} \, dt$$

Notation: $F(s) = \mathcal{L}\{f(t)\}$ (transform)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$ (inverse transform)

Limitations of Laplace Transform

Only defined for $f(t)$ where the integral (2) converges.

<u>Requires:</u> $$e^{-\sigma_0 t}|f(t)| \to \emptyset$$

for some <u>finite</u> $\sigma_0 \in \mathbb{R}$

The transform $F(s)$ is then defined for any

$s = \sigma + j\omega$ with $\sigma \geq \sigma_0$

and the integral (1) is over all values of s which satisfy this condition. "region of convergence"

Examples

$f(t) = e^{pt}$ can be transformed for any finite $p \in \mathbb{C}$

However, $f(t) = e^{t^2}$ cannot be transformed since $e^{-\sigma_0 t} f(t) = e^{(t^2 - \sigma_0 t)} \to \infty$ for any finite σ_0.

Note:

When working with Laplace transforms we assume we are using values of s in the region of convergence. (ROC)

By above def'n of ROC,

$$\lim_{t \to \infty} e^{-st} f(t) = 0$$

for these values of s.

Fundamental Transform

(only one you need!)

$$\boxed{\mathcal{L}\{e^{pt}\} = \frac{1}{s-p}} \qquad \forall p \in \mathbb{C}$$

$$\mathcal{L}\{e^{pt}\} = \int_0^\infty e^{pt} e^{-st}\, dt = \int_0^\infty e^{(p-s)t}\, dt$$

$$= \left[\left(\frac{1}{p-s}\right) e^{(p-s)t}\right]_{t=0}^{t=\infty}$$

$$= \left(\frac{1}{p-s}\right)\left[\cancel{e^{(p-s)\infty}} - 1\right]$$

$$\emptyset$$

for any s in ROC

Property #1: Linearity

$$\boxed{\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)}$$

for any transformable functions $f_1(t)$, $f_2(t)$
any (complex) constants a_1, a_2

$$\int_0^\infty \{a_1 f_1(t) + a_2 f_2(t)\} e^{-st} dt$$

$$= a_1 \underbrace{\boxed{\int_0^\infty f_1(t) e^{-st} dt}}_{F_1(s)} + a_2 \underbrace{\boxed{\int_0^\infty f_2(t) e^{-st} dt}}_{F_2(s)}$$

And generally:

$$\mathcal{L}\left\{\sum^N a_i f_i(t)\right\} = \sum^N a_i F_i(s)$$

Linearity lets us build more complex transforms:

Consider: $f(t) = Ae^{at}\cos(bt+\psi)$

$$= ce^{pt} + \bar{c}e^{\bar{p}t}$$

with $p = a+bj$, $c = \left(\frac{A}{2}\right)e^{j\psi}$ (polar form)

Then by linearity

$$\boxed{\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{c}{s-p} + \frac{\bar{c}}{(s-\bar{p})}}$$

We can combine the two terms:

$$\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{c}{s-p} + \frac{\bar{c}}{s-\bar{p}}$$

$$= \frac{A[(s-a)\cos\psi - b\sin\psi]}{s^2 - 2as + (a^2+b^2)}$$

so

$$\boxed{\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{A[(s-a)\cos\psi - b\sin\psi]}{(s-a)^2 + b^2}}$$

But we will see it is often easier to keep the two terms separate when solving problems.

Fundamental Transforms

$$\mathcal{L}\{e^{pt}\} = \frac{1}{s-p} \quad \text{for any } p \in \mathbb{C}$$

$$\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{c}{s-p} + \frac{\bar{c}}{s-\bar{p}}$$

with $p = a+bj$ and $c = \left(\frac{A}{2}\right)e^{j\psi}$

What is $\mathcal{L}\{c\}$ for an arbitrary constant c?

i.e. $\mathcal{L}\{f(t)\}$ with $f(t) = c$ for all $t \geq 0$

Fundamental Transforms

$$\mathcal{L}\{e^{pt}\} = \frac{1}{s-p} \quad \text{for any } p \in \mathbb{C}$$

$$\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{c}{s-p} + \frac{\bar{c}}{s-\bar{p}}$$

with $p = a+bj$ and $c = \left(\frac{A}{2}\right)e^{j\psi}$

What is $\mathcal{L}\{c\}$ for an arbitrary constant c?

i.e. $\mathcal{L}\{f(t)\}$ with $f(t) = c$ for all $t \geq 0$

$$f(t) = c = c\,e^{\emptyset t} \Longrightarrow F(s) = \left.\frac{c}{s-p}\right]_{p=\emptyset}$$

Fundamental Transforms

$$\mathcal{L}\{e^{pt}\} = \frac{1}{s-p} \quad \text{for any } p \in \mathbb{C}$$

$$\mathcal{L}\{Ae^{at}\cos(bt+\varphi)\} = \frac{c}{s-p} + \frac{\bar{c}}{s-\bar{p}}$$

with $p = a + bj$ and $c = \left(\frac{A}{2}\right)e^{j\varphi}$

What is $\mathcal{L}\{c\}$ for an arbitrary constant c?

i.e. $\mathcal{L}\{f(t)\}$ with $f(t) = c$ for all $t \geq 0$

$$f(t) = c = c\,e^{\emptyset t} \implies F(s) = \left.\frac{c}{s-p}\right]_{p=\emptyset}$$

Hence $\boxed{\mathcal{L}\{c\} = \frac{c}{s}}$ for any $c \in \mathbb{C}$

Common Mistakes

$$\mathcal{L}\{c\} \neq c \qquad \left(\mathcal{L}\{c\} = \frac{c}{s} \right)$$

$$\mathcal{L}\{f_1(t)f_2(t)\} \neq F_1(s)F_2(s)$$

$$\mathcal{L}\{f_1(t)f_2(t)\} = \langle \text{unspeakably ugly} \rangle$$

Property #2: Diff'n rule

$$\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$$

$$= \int_0^\infty \frac{df}{dt} e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,df$$

(by parts) $$= \left[e^{-st} f(t)\right]_{t=0}^{t=\infty} + s \underbrace{\boxed{\int_0^\infty f(t) e^{-st}\,dt}}_{F(s)}$$

$$= sF(s) - f(0)$$

Higher Derivatives

$$\mathcal{L}\{\ddot{f}(t)\} = \mathcal{L}\{\dot{f}_1(t)\} \quad \text{with } f_1(t) = \dot{f}(t)$$

$$= sF_1(s) - f_1(0)$$

but $F_1(s) = \mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$

So $\boxed{\mathcal{L}\{\ddot{f}(t)\} = s^2F(s) - \dot{f}(0) - sf(0)}$

and generally

$$\boxed{\mathcal{L}\{f^{(k)}(t)\} = s^kF(s) - f^{(k-1)}(0) - sf^{(k-2)}(0) - \cdots - s^{k-1}f(0)}$$

Note: Laplace will allow us to directly account for IC effects (No Linear algebra!)

Property #3: "t-mult" rule

$$\boxed{\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)}$$

$$\begin{aligned}
\mathcal{L}\{tf(t)\} &= \int_0^\infty t f(t) e^{-st}\,dt \\
&= \int_0^\infty f(t)\left[t e^{-st}\right]dt \\
&= \int_0^\infty f(t)\left[-\frac{d}{ds} e^{-st}\right]dt \\
&= \int_0^\infty -\frac{d}{ds}\left[f(t) e^{-st}\right]dt \\
&= -\frac{d}{ds}\int_0^\infty f(t) e^{-st}\,dt
\end{aligned}$$

Use of t-mult rule

$$\mathcal{L}\{te^{pt}\} = -\frac{d}{ds}\left[\frac{1}{s-p}\right]$$

$$= \frac{1}{(s-p)^2}$$

Similarly $\mathcal{L}\{t^2 e^{pt}\} = \mathcal{L}\{t f_1(t)\}$, $f_1(t) = te^{pt}$

$$= -\frac{d}{ds} F_1(s) = -\frac{d}{ds}\left[\frac{1}{(s-p)^2}\right]$$

So $\mathcal{L}\{t^2 e^{pt}\} = \dfrac{2}{(s-p)^3}$

Generally:

$$\boxed{\mathcal{L}\{t^k e^{pt}\} = \frac{k!}{(s-p)^{k+1}}}$$

Recap: Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int F(s)e^{st}\,ds$$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt$$

Properties:

1.) Linearity: $\mathcal{L}\left\{\sum_{i=1}^{N} a_i f_i(t)\right\} = \sum_{i=1}^{N} a_i F_i(s)$

2.) Diff. rule: $\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$

$$\mathcal{L}\{\ddot{f}(t)\} = s^2F(s) - \dot{f}(0) - sf(0)$$

$$\vdots$$

$$\mathcal{L}\{f^{(k)}(t)\} = s^kF(s) - f^{(k-1)}(0) - \cdots - s^{k-1}f(0)$$

Use of $\mathcal{L}T$ for Diff'l Eqn

$$\mathcal{L}\{\alpha_n y^{(n)}(t) + \cdots + \alpha_1 \dot{y}(t) + \alpha_0 y(t)\}$$

$$= \mathcal{L}\{\beta_m u^{(m)} + \cdots + \beta_1 \dot{u}(t) + \beta_0 u(t)\}$$

Gives:

$$\alpha_n \left[s^n Y(s) - y^{(n-1)}(0) - s y^{(n-2)}(0) - \cdots - s^{n-1} y(0) \right]$$

$$+ \cdots + \alpha_1 \left[s Y(s) - y(0) \right] + \alpha_0 Y(s)$$

$$= \beta_m \left[s^m U(s) - u^{(m-1)}(0) - s u^{(m-2)}(0) - \cdots - s^{m-1} u(0) \right]$$

$$+ \cdots + \beta_1 \left[s U(s) - u(0) \right] + \beta_0 U(s)$$

Collect Terms

$$\boxed{r(s)Y(s) - c(s) = q(s)U(s) - b(s)}$$

$$r(s) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0$$

$$q(s) = \beta_m s^m + \cdots + \beta_1 s + \beta_0$$

$c(s)$ = $n-1$ order polynomial in s from IC terms on $y(t)$

$b(s)$ = $m-1$ order polynomial in s from IC terms on $u(t)$.

Re-arrange for $Y(s)$

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

IC terms

Or:

$$Y(s) = G(s)U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

Alternate def'n of TF:

$$G(s) = \left[\frac{Y(s)}{U(s)}\right]_{ICs=0} = \left[\frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}}\right]_{\substack{\text{zero}\\ ICs}}$$

Example

$$2y^{(3)} + 8\ddot{y} + 14\dot{y} + 10y = 3\ddot{u} + 15\dot{u} + 18u$$

$$2\left[s^3Y(s) - \ddot{y}_0 - s\dot{y}_0 - s^2y_0\right] + 8\left[s^2Y(s) - \dot{y}_0 - sy_0\right]$$

$$+ 14\left[sY(s) - y_0\right] + 10Y(s)$$

$$= 3\left[s^2U(s) - \dot{u}_0 - su_0\right] + 15\left[sU(s) - u_0\right] + 18U(s)$$

OR:

$$(2s^3 + 8s^2 + 14s + 10)Y(s)$$
$$- \left[2s^2y_0 + s(2\dot{y}_0 + 8y_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0)\right]$$
$$= (3s^2 + 15s + 18)U(s) - \left[3su_0 + (3\dot{u}_0 + 15u_0)\right]$$

Example

$$2y^{(3)} + 8\ddot{y} + 14\dot{y} + 10y = 3\ddot{u} + 15\dot{u} + 18u$$

$$2\left[s^3Y(s) - \ddot{y}_0 - s\dot{y}_0 - s^2y_0\right] + 8\left[s^2Y(s) - \dot{y}_0 - sy_0\right]$$

$$+\ 14\left[sY(s) - y_0\right] + 10Y(s)$$

$$= 3\left[s^2U(s) - \dot{u}_0 - su_0\right] + 15\left[sU(s) - u_0\right] + 18U(s)$$

OR:

$$\overbrace{(2s^3 + 8s^2 + 14s + 10)}^{r(s)}Y(s)$$

$$-\left[2s^2y_0 + s(2\dot{y}_0 + 8y_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0)\right]$$

$$= \underbrace{(3s^2 + 15s + 18)}_{q(s)}U(s) - \left[3su_0 + (3\dot{u}_0 + 15u_0)\right]$$

Example

$$2y^{(3)} + 8\ddot{y} + 14\dot{y} + 10y = 3\ddot{u} + 15\dot{u} + 18u$$

$$2\left[s^3Y(s) - \ddot{y}_0 - s\dot{y}_0 - s^2y_0\right] + 8\left[s^2Y(s) - \dot{y}_0 - sy_0\right]$$

$$+\ 14\left[sY(s) - y_0\right] + 10Y(s)$$

$$= 3\left[s^2U(s) - \dot{u}_0 - su_0\right] + 15\left[sU(s) - u_0\right] + 18U(s)$$

OR:

$$(2s^3 + 8s^2 + 14s + 10)Y(s) - \underbrace{\left[2s^2y_0 + s(2\dot{y}_0 + 8y_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0)\right]}_{c(s)}$$

$$= (3s^2 + 15s + 18)U(s) - \underbrace{\left[3su_0 + (3\dot{u}_0 + 15u_0)\right]}_{b(s)}$$

Thus:

$$Y(s) = \underbrace{\boxed{\left[\frac{3s^2+15s+18}{2s^3+8s^2+14s+10}\right]}}_{G(s)} U(s)$$

$$+ \left[\frac{2s^2y_0 + s(2\dot{y}_0 + 8y_0 - 3u_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0 - 3\dot{u}_0 - 15u_0)}{2s^3+8s^2+14s+10}\right]$$

$\Rightarrow$ We assume all ICs on $y(t)$ Known; and $u(t)$ Known so $U(s)$ can be computed and ICs on $u(t)$

$\Rightarrow$ All terms on RHS are Known, so we Know $Y(s)$

$\Rightarrow$ "Simply" invert transform to get $y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Inverse Transform

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= \frac{1}{2\pi j}\int Y(s)e^{st}\,ds$$

$\Rightarrow$ contour integral over ROC in complex plane

$\Rightarrow$ ugly! Math 463

$\Rightarrow$ We can sidestep this in many cases

General form of Y(s)

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

all polynomials

Suppose $U(s)$ is rational in s
(ratio of polynomials)

i.e. $U(s) = \dfrac{a(s)}{h(s)}$, $a(s)$ $h(s)$ polys

Note: ① Not true for every $u(t)$
② True for many "useful" $u(t)$

Then...

$$U(s)$$

$$Y(s) = \left[\frac{q(s)}{r(s)}\right]\left(\frac{a(s)}{h(s)}\right) + \frac{c(s)-b(s)}{r(s)}$$

$$= \frac{q(s)a(s) + h(s)[c(s)-b(s)]}{r(s)h(s)}$$

or

$$Y(s) = \frac{N(s)}{D(s)}$$

where both $N(s)$ and $D(s)$ are polynomials

(i.e. $Y(s)$ is rational)

$$Y(s) = \frac{N(s)}{D(s)}$$

Suppose $\deg\{N(s)\} < \deg\{D(s)\} = L$

Let d_ℓ be the roots of $D(s)$: $D(d_\ell) = \phi$

Then:

$$Y(s) = \frac{A_1}{s-d_1} + \frac{A_2}{s-d_2} + \cdots + \frac{A_L}{s-d_L}$$

$$= \sum_{\ell=1}^{L} \frac{A_\ell}{s-d_\ell}$$

"Partial fraction expansion"

and

$$y(t) = \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

How to find expansion coefficients

"Residue formula":

$$A_\ell = \left[(s-d_\ell)\,Y(s)\right]_{s=d_\ell}$$

(also called "cover up" rule).

Example:

$$Y(s) = \frac{2s+3}{(s+2)(s+3)}$$

$$Y(s) = \frac{A_1}{s+2} + \frac{A_2}{s+3}$$

$$A_1 = \left[\frac{2s+3}{s+3}\right]_{s=-2} = -1, \quad A_2 = \left[\frac{2s+3}{s+2}\right]_{s=-3} = 3$$

$$y(t) = 3e^{-3t} - e^{-2t}$$

Complex d_ℓ

Note if d_ℓ is a complex root of $D(s)$, then its conjugate $\bar{d}_\ell$ will also be a root.

The residue formula then tells us that

for d_ℓ: $A_\ell = \left[(s-d_\ell)Y(s)\right]_{s=d_\ell}$

and for $\bar{d}_\ell$ we instead have

$$\left[(s-\bar{d}_\ell)Y(s)\right]_{s=\bar{d}_\ell} = \bar{A}_\ell$$

i.e. the PFE coefficients are also conjugates

Complex d_ℓ (cont)

Thus, the expression for $y(t)$ will contain

$$A_\ell e^{d_\ell t} + \bar{A}_\ell e^{\bar{d}_\ell t}$$

$$= 2|A_\ell| e^{\sigma t} \cos(\omega t + \measuredangle A_\ell)$$

where $\sigma = \mathrm{Re}\{d_\ell\}$ $\omega = \mathrm{Im}\{d_\ell\}$

Example:

$$Y(s) = \frac{4(s^2+2s+6)}{(s+1)(s^2+4s+13)}$$

$$d_1 = -1; \quad d_2 = -2+3j; \quad d_3 = -2-3j = \bar{d}_2$$

Then:

$$A_1 = \left[(s+1)Y(s)\right]_{s=-1} = 2$$

$$A_2 = \left[(s+2-3j)Y(s)\right]_{s=-2+3j} = 1+j = \boxed{\sqrt{2}\,\angle^{\pi/4} = A_2}$$

$$A_3 = \left[(s+2+3j)Y(s)\right]_{s=-2-3j} = 1-j = \bar{A}_2$$

Hence:

$$y(t) = 2e^{-t} + \boxed{(1+j)e^{(-2+3j)t} + (1-j)e^{(-2-3j)t}}$$

or:

$$y(t) = 2e^{-t} + \boxed{2\sqrt{2}\,e^{-2t}\cos(3t + \pi/4)}$$

Recap

$$G(s) \qquad\qquad \text{IC terms}$$

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s)-b(s)}{r(s)}\right]$$

If $U(s)$ rational, $U(s) = \dfrac{a(s)}{h(s)}$

Then $\quad Y(s) = \dfrac{N(s)}{D(s)} \quad$ (also rational)

$$= \sum_{\ell=1}^{L} \frac{A_\ell}{(s-d_\ell)} \quad \text{where} \quad D(d_\ell) = \emptyset$$

and $\quad A_\ell = \big[(s-d_\ell)Y(s)\big]_{s=d_\ell}$

Inverse transform:

$$y(t) = \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

Assumptions

Above assumes:

1. $\deg\{N(s)\} < \deg\{D(s)\}$
2. No repeated roots of $D(s)$

} Simplest, most common case

Both can be relaxed:

1. Suppose $\text{Deg}\{N(s)\} = \text{Deg}\{D(s)\}$

Then do polynomial long division:

$$Y(s) = \frac{N(s)}{D(s)} = A_0 + \frac{N_1(s)}{D(s)}, \quad \text{Deg}\{N_1(s)\} < \text{Deg}\{D(s)\}$$

and $\dfrac{N_1(s)}{D(s)}$ can be expanded using above

So:

$$Y(s) = \frac{N(s)}{D(s)} = A_0 + \frac{N_1(s)}{D(s)}$$

$$= A_0 + \sum_{\ell=1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

PFE

Where:

$$A_\ell = \left[(s-d_\ell)\frac{N_1(s)}{D(s)}\right]_{s=d_\ell}$$

<u>Inverse transforming</u>:

$$y(t) = \mathcal{L}^{-1}\{A_0\} + \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

What is this?? We'll see later...

<u>Note</u>: $\text{Deg}\{N(s)\} > \text{Deg}\{D(s)\}$
nonphysical + won't be seen

Repeated Roots

Now suppose:

$$D(s) = (s-d_1)^K (s-d_{K+1}) \cdots (s-d_L)$$

i.e. d_1 is repeated K times, then:

$$Y(s) = \sum_{\ell=1}^{K} \frac{A_\ell}{(s-d_1)^\ell} + \sum_{\ell=K+1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

<u>for $\ell = K+1, \ldots, L$</u> :

$$A_\ell = \left[(s-d_\ell)Y(s)\right]_{s=d_\ell}$$ (unchanged)

<u>for $\ell = 1, \ldots, K$</u> :

(ugh!)

$$A_\ell = \frac{1}{(K-\ell)!}\left\{\frac{d^{K-\ell}}{ds^{K-\ell}}\left[(s-d_1)^K Y(s)\right]\right\}_{s=d_1}$$

Inverse Transform (Repeated Roots)

$$Y(s) = \sum_{\ell=1}^{K} \frac{A_\ell}{(s-d_1)^\ell} + \sum_{\ell=K+1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

$$\Rightarrow \quad y(t) = \sum_{\ell=1}^{K} \left(\frac{A_\ell t^{\ell-1}}{(\ell-1)!} \right) e^{d_1 t} + \sum_{\ell=K+1}^{L} A_\ell e^{d_\ell t}$$

Example:

$$Y(s) = \frac{2s+1}{(s+1)^3(s+2)} \qquad d_1 = -1, \; K=3 \qquad d_4 = -2$$

$$\Rightarrow y(t) = \left[A_1 + A_2 t + \frac{A_3}{2} t^2\right] e^{-t} + A_4 e^{-2t}$$

$$A_3 = \left[(s+1)^3 Y(s)\right]_{s=-1} = -1$$

$$A_2 = \left(\frac{1}{1}\right)\left\{\frac{d}{ds}\left[(s+1)^3 Y(s)\right]\right\}_{s=-1} = \left[\frac{3}{(s+2)^2}\right]_{s=-1} = 3$$

$$A_1 = \left(\frac{1}{2}\right)\left\{\frac{d^2}{ds^2}\left[(s+1)^3 Y(s)\right]\right\}_{s=-1}$$

$$= \left(\frac{1}{2}\right)\left\{\frac{d}{ds}\left[\frac{3}{(s+2)^2}\right]\right\}_{s=-1} = -3$$

And

$$A_4 = \left[(s+2)Y(s)\right]_{s=-2} = 3$$

So finally:

$$y(t) = \left[-3 + 3t - \frac{1}{2}t^2\right]e^{-t} + 3e^{-2t}$$

<u>Note</u>: You aren't responsible for repeated root residue formula. <u>However</u> you should know the general pattern for repeated root solutions.

Alternate System Models

- A dynamical analysis does not always result in a high-order DE Directly connecting $u(t)$ and $y(t)$
- Sometimes the analysis (initially) results in a system of 1st order DEs describing the evolution of the Dynamics
- Each first order equation describes the rate of change of a single physical variable (like airspeed, pitch angle, and angle of attack)
- Generically label these $x_K(t)$ $(K=1 \ldots n)$ Known as the $\boxed{\text{state variables}}$ for the system.

"State variable" form of Dynamics

System of 1^{st} order DEs describing how rate of change in each state depends on other states and forcing input

rate of change of each state — Linear combination of states — effect of input

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1 u(t) \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2 u(t) \\ \vdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_n u(t) \end{cases}$$

$\longrightarrow$ **n 1^{st} order DEs**

$\Rightarrow$ easier to represent in Matrix/Vector form

"State-space" Dynamical model

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\,u(t)$$ ← "state equation"

with:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (n \times 1)$$

"state vector"

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (n \times n)$$

What about output?

Output $y(t)$ can be any $\underline{1}$ of the states, or any weighted combination of states (and input) as appropriate.

i.e. $y(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t) + D u(t)$

or $y(t) = C \underline{x}(t) + D u(t)$ "output equation"

where $C = [C_1 \; C_2 \; \ldots \; C_n]$ $(1 \times n)$

So complete model is

Standard "state-space" model of dynamics

$$\boxed{\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B u(t) \\ y(t) &= C\underline{x}(t) + D u(t) \end{aligned}}$$

Example from HW #1

fan: $I\dot{\omega} = K_m i_m - D\omega$

motor: $L\dfrac{di_m}{dt} = V_m - R i_m - E\omega$

vehicle: $m\ddot{y} = K_f\omega \Rightarrow m\dot{v} = K_f\omega, \quad \dot{y} = \overset{\text{velocity}}{v}$

So:

$$\frac{d}{dt}\overbrace{\begin{bmatrix} \omega \\ i_m \\ v \\ y \end{bmatrix}}^{\underline{x}(t)} = \overbrace{\begin{bmatrix} -D/I & K_m/I & 0 & 0 \\ -E/L & -R/L & 0 & 0 \\ K_f/m & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}^{A}\begin{bmatrix} \omega \\ i_m \\ v \\ y \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1/L \\ 0 \\ 0 \end{bmatrix}}^{B}\overbrace{V_m}^{u}$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}}_{C}\underline{x}(t) + \underbrace{0}_{D}u(t)$$

Where is G(s) for this model?

Not as easy to see transfer function by inspection.

But, we can still use Laplace.

Laplace can be applied to vectors too, just apply it to each component of the vector

$$\mathcal{L}\{\underline{x}(t)\} = \underline{X}(s) = \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix} = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$

Linearazity:

$$\mathcal{L}\{A\underline{x}_1(t) + B\underline{x}_2(t)\} = A\underline{x}_1(s) + B\underline{x}_2(s)$$

Derivative rule

$$\mathcal{L}\{\dot{\underline{x}}(t)\} = \begin{bmatrix} sX_1(s) - X_1(0) \\ sX_2(s) - X_2(0) \\ \vdots \\ sX_n(s) - X_n(0) \end{bmatrix} = s\underline{X}(s) - \underline{X}_0$$

Apply Laplace to State space model

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$y(t) = C\underline{x}(t) + Du(t)$$

$$\underline{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

Initial state values

$$\Rightarrow \quad s\underline{x}(s) - \underline{x}_0 = A\underline{x}(s) + Bu(s)$$
$$y(s) = C\underline{x}(s) + Du(s)$$

1^{st} eq'n is equivalent to:

$$(sI - A)\underline{x}(s) = \underline{x}_0 + Bu(s) \qquad (I = n \times n \text{ identity})$$

$$\Rightarrow \quad \underline{x}(s) = [sI - A]^{-1}[\underline{x}_0 + Bu(s)]$$

Substitute into 2^{nd} eq'n:

$$\boxed{y(s) = C[sI - A]^{-1}\underline{x}_0 + [C(sI - A)^{-1}B + D]U(s)}$$

$$y(s) = \overbrace{\underbrace{C[sI-A]^{-1}\underline{x}_0}_{\text{effect of ICs}}}^{} + \overbrace{\underbrace{[C(sI-A)^{-1}B+D]U(s)}_{\text{effect of input}}}^{}$$

Recall: TF derived assuming ICs $=0 \Rightarrow \underline{x}_0 = \underline{0}$

Then

$$y(s) = \overbrace{\boxed{[C(sI-A)^{-1}B+D]}}^{G(s)}U(s)$$

Hence, for any (A,B,C,D) state space representation the corresponding transfer function is:

$$G(s) = [C\overbrace{(sI-A)^{-1}}^{n\times n \text{ matrix inverse}}B+D]$$

Now recall for arbitrary matrix M

$$M^{-1} = \frac{\mathrm{Adj}(M)}{\mathrm{Det}(M)}$$

Adj = $n\times n$ Matrix of cofactors

Det = Scalar Determinant

Thus

$$(s\mathbb{I}-A)^{-1} = \frac{Q(s)}{r(s)}$$

where

$$\left[\begin{array}{l} Q(s) = \mathrm{Adj}(s\mathbb{I}-A) \quad (n\times n \ \text{matrix}) \\ r(s) = \mathrm{Det}(s\mathbb{I}-A) \quad \underline{\text{polynomial}} \text{ in } s. \end{array}\right.$$

and

$$G(s) = \frac{CQ(s)B}{r(s)} + D = \frac{CQ(s)B + Dr(s)}{r(s)}$$

where both $CQ(s)B$ and $r(s)$ are polynomials

$\Rightarrow$ zeros where $\boxed{CQ(s)B + Dr(s)} = 0$ $\quad q(s)$

$\Rightarrow$ poles where $r(s) = 0$.

So the poles of $G(s)$ will satisfy

$$r(s) = \emptyset = \mathrm{Det}(s\mathbb{I} - A)$$

$\Rightarrow$ $(s\mathbb{I} - A)$ is <u>singular</u>, at these values of s, i.e. there exists nonzero $\underline{v}$

so that

$$(s\mathbb{I} - A)\underline{v} = \underline{0}$$

(singular matrices have nontrivial nullspace)

or:

$$A\underline{v} = s\underline{v} \quad \text{for any } s \text{ with } r(s) = 0$$

$$\boxed{\Rightarrow \text{poles of } G(s) \text{ are } \boxed{\underline{\text{eigenvalues}}} \text{ of } A!!}$$

Converting from $G(s)$ to state space

- Sometimes it is useful to reverse process described above, i.e.

  Given $G(s)$, find $[A, B, C, D]$

- In fact, for a given $G(s)$ there are

  infinitely many equivalent $[A, B, C, D]$

  $\Rightarrow$ many more DoF in $A(n^2)$, $B(n)$, $C(n)$

  than in polynomials $r(s)$ $(n+1)$ and $q(s)$ $(m \leq n)$

- One "canonical" conversion is easy to obtain where the coefs. of polys $r(s)$ and $q(s)$

  appear as rows and/or cols of $[A, B, C]$

- Known as "companion forms"

Companion form (one possibility)

Given $$G(s) = \frac{q(s)}{r(s)} = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

Take
$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix} \quad D = 0$$

(Note: $A_1 = A^T$, $B_1 = C^T$, $C_1 = B^T$ works too!)

Example

$$G(s) = \frac{3s^2 - 4s + 5}{s^3 + 2s^2 - s + 7}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & -4 & 3 \end{bmatrix} \qquad D = 0$$

One possible state space model for this TF

Philosophical Question: What is $t=0$?

$\Rightarrow$ The instant we start acting on the system with external input.

$\Rightarrow$ In control Theory, we assume these inputs are completely "off" for $t<0$.

$\Rightarrow$ $u(t), \dot{u}(t), \ddot{u}(t)$, etc all <u>zero</u> for $t<0$

$\Rightarrow$ <u>Discontinuities</u> exist when $u(0) \neq 0$

$$u(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$u(t) = \begin{cases} e^{pt}, & t \geq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

$$\Rightarrow \boxed{u(t) = e^{pt}\,\mathbb{1}(t)}$$

where

$$\boxed{\mathbb{1}(t) = \begin{cases} 1 & t \geq \emptyset \\ 0 & \text{otherwise} \end{cases}}$$

"Unit step function" (Very important!)

Now, Laplace is concerned about behavior of functions only for $t \geq \emptyset$.

For all intents and purposes, functions in Laplace are considered $\emptyset$ for $t < \emptyset$

Implication

Formally:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = e^{pt}\mathbb{1}(t)$$

$$= \begin{cases} e^{pt}, & t \geq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

Now generally, our diff'l eq'ns will involve <u>derivatives</u> of these discontinuous functions

$\Rightarrow$ creates <u>singularities</u> in analysis at $t=0$

$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \emptyset & t \neq \emptyset \\ & \end{cases}$$

Implication

Formally:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = e^{pt}\,\mathbb{1}(t)$$

$$= \begin{cases} e^{pt}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now generally, our diff'l eq'ns will involve derivatives of these discontinuous functions

$\Rightarrow$ creates singularities in analysis at $t=0$

$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \quad (???) \end{cases}$$

Theoretical problems in integrals when discontinuities or singularities at one of the end points.

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$\rightarrow$ possible problem here

Resolve these by taking lower limit at $t = \phi^-$ (the instant <u>before</u> $t = \phi$).

$\Rightarrow$ integral "sees" effect of singularities at $t = \phi$.

Starting the integral at 0^- instead of 0

- Avoids singularities at endpoints
- Causes transform to "see" singularities and discontinuities at $t=0$, so their effects will be reflected in the solutions for $y(t)$.

Hence:

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} \, dt$$

__Implications:__

Assumed ICs for $y(t)$: just __before__ $t=0$

$$\mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0^-)$$

$$\mathcal{L}\{\ddot{y}(t)\} = s^2Y(s) - \dot{y}(0^-) - sy(0^-)$$

etc.

$$\mathcal{L}\{\dot{u}(t)\} = sU(s) - u(0^-)$$

$$\mathcal{L}\{\ddot{u}(t)\} = s^2U(s) - \dot{u}(0^-) - su(0^-)$$

etc.

Always $=0$ in our analysis!

Thus:

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - \cancel{b(s)}^{\,\to\,\emptyset}}{r(s)}\right]$$

$$= \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s)}{r(s)}\right]$$

$\Rightarrow$ IC polynomial $b(s)$ for input vanishes

$\Rightarrow$ specific to controls convention for $u(t)$

$\Rightarrow$ Not a common assumption in regular math classes.

$\Rightarrow$ In controls, want to know effect of discontinuities

Common, discontinuous "test functions"

$u(t) = \mathbb{1}(t)$ (unit step) $\quad \mathcal{L}\{\mathbb{1}(t)\} = \frac{1}{s}$

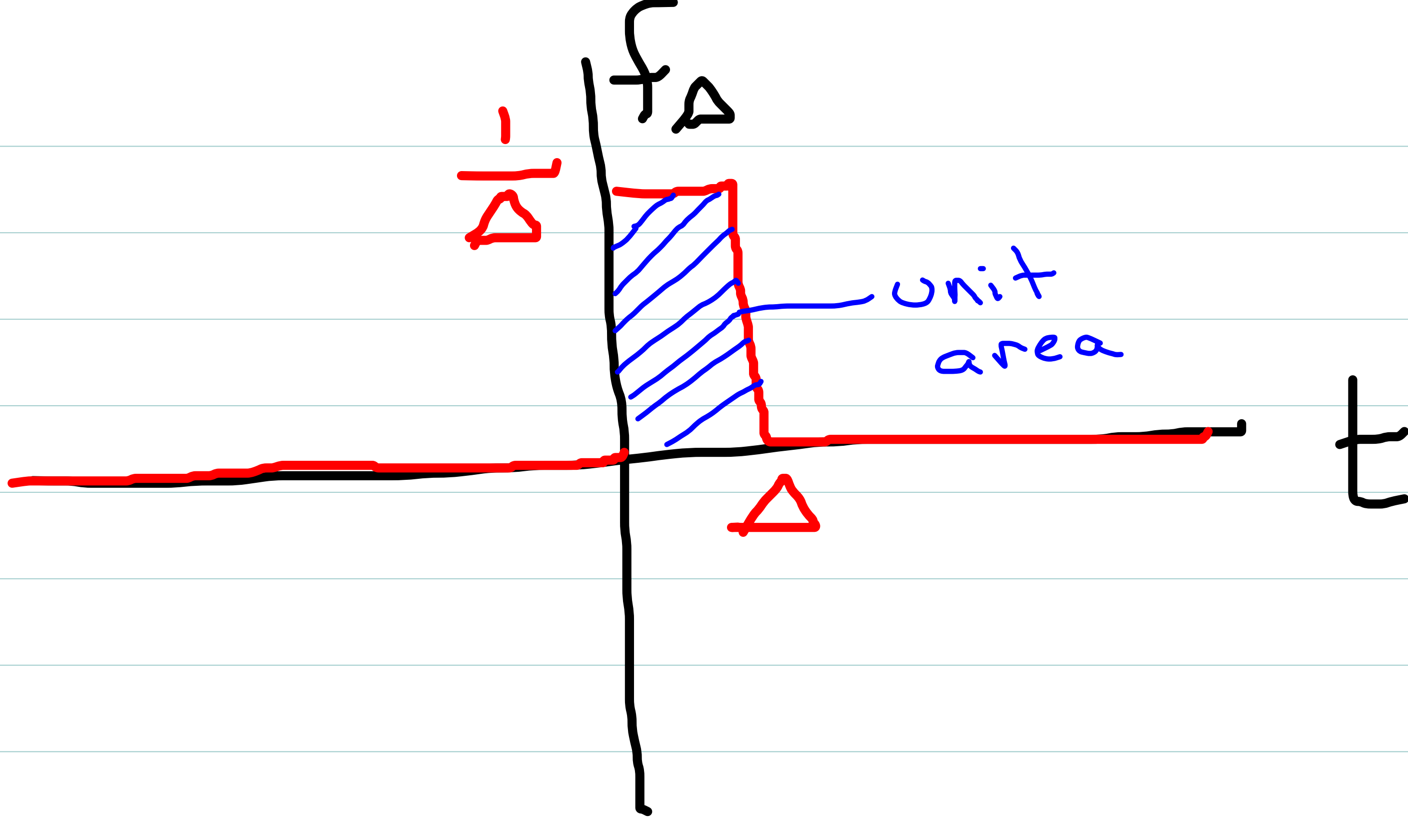
$$u(t) = \cos(\omega t)\,\mathbb{1}(t)$$

$$= \begin{cases} \cos(\omega t) & t \geq \emptyset \\ \emptyset & t < \emptyset \end{cases}$$

$$\Longrightarrow \quad u(t) = f_{\Delta}(t)$$

$$= \begin{cases} \frac{1}{\Delta} & \emptyset \leq t \leq \Delta \\ \emptyset & \text{otherwise} \end{cases}$$

"Unit pulse function"



Note: for any $\Delta > 0$

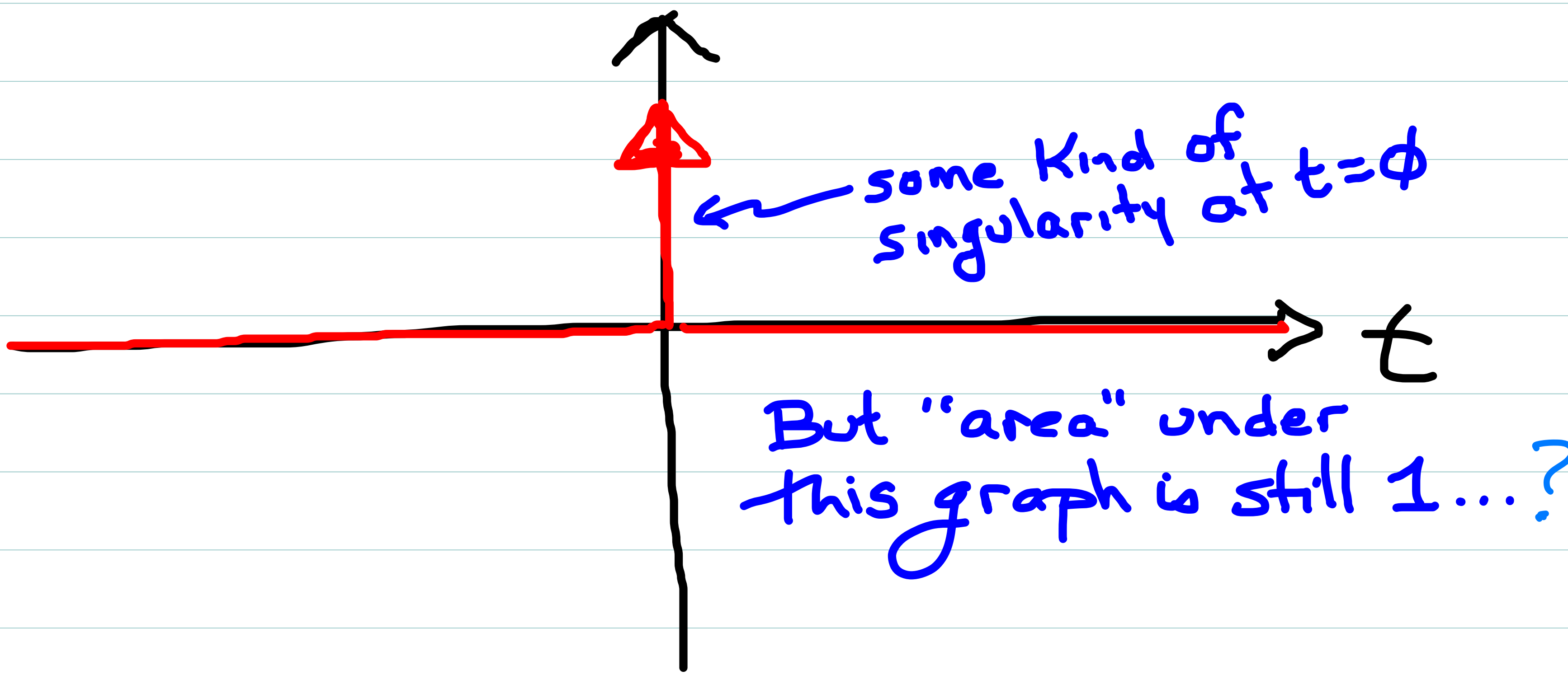
$$\int_{0^-}^{\infty} f_\Delta(t)\,dt = \int_{0^-}^{\Delta} \left(\frac{1}{\Delta}\right) dt$$

$$= 1$$

What is $\lim\limits_{\Delta \to 0} f_\Delta(t)$?

$$= \lim_{\Delta \to 0} \begin{cases} \frac{1}{\Delta} & \phi \le t \le \Delta \\ \phi & \text{otherwise} \end{cases}$$

$$= \begin{cases} \infty & t = \phi \\ \phi & \text{otherwise} \end{cases}$$



Define:

$$\boxed{\delta(t) = \lim_{\Delta \to 0} f_\Delta(t)}$$

"ideal impulse": models delivering a unit of input energy over negligibly small time.
(sharp "kick")

Alternate names:

"delta function"
"impulse function"
"Dirac delta"

Note: Not really a meaningful function at all!

More formally, belongs to a class of mathematical objects called
"distributions" or "generalized functions"

Suppose $\delta(t)$ appears in an integral

$$\int_{-\infty}^{\infty} \delta(t)\, h(t)\, dt \quad , \quad h(t) \text{ arbitrary f'n}$$

$$= \int_{-\infty}^{\infty} \left[\lim_{\Delta \to 0} f_\Delta(t) \right] h(t)\, dt$$

$$= \lim_{\Delta \to 0} \int_{-\infty}^{\infty} f_\Delta(t)\, h(t)\, dt$$

$$= \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \int_{0^-}^{\Delta} h(t)\, dt \right\}$$

$$\cong \left(\frac{1}{\Delta}\right) \left(\Delta\, h(\emptyset)\right)$$

$$= h(\emptyset)$$

Note: with $h(t) = 1$ for all t, we get

$$\int_{0^-}^{\infty} \delta(t)\, dt = 1$$

Defining Property of $\delta(t)$

$$\int_a^b \delta(t)h(t)\,dt = \begin{cases} h(\emptyset) & \text{if } \emptyset\in(a,b) \\ \emptyset & \text{otherwise} \end{cases}$$

"sifting property"

$\Rightarrow$ $\delta(t)$ is defined by what it does in an integral

Not as an ordinary function

Now we can compute:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt$$

$$= \left[e^{-st}\right]_{t=\emptyset} = 1$$

Thus:

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall
$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t=\phi \\ \phi & \text{otherwise} \end{cases}$$

which looks like $\frac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\left\{\frac{d}{dt}\mathbb{1}(t)\right\} = s\,\mathcal{L}\{\mathbb{1}(t)\} - \mathbb{1}(\phi^-)$$

Thus:

$$\mathcal{L}\{\delta(t)\} = 1$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall $\quad \dfrac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases}$

which looks like $\dfrac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\left\{\frac{d}{dt}\mathbb{1}(t)\right\} = s\,\overset{1/s}{\mathcal{L}\{\mathbb{1}(t)\}} - \cancelto{0}{\mathbb{1}(0^-)}$$

$$= 1 = \mathcal{L}\{\delta(t)\} \qquad \text{YES}$$

Recap: Unit Impulse

$$\delta(t) = \lim_{\Delta \to 0} f_\Delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

"sifting property":

$$\int_a^b \delta(t) h(t)\, dt = \begin{cases} h(0) & \text{if } 0 \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Laplace Transform:

$$\mathcal{L}\{\delta(t)\} = 1$$

Useful property:

$$\frac{d}{dt} \mathbb{1}(t) = \delta(t)$$

Impulse Response

The impulse response of a system is the output $y(t)$ when $u(t) = \delta(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[c(s) - b(s)]}{r(s)}$$

$\Rightarrow u(t) = \delta(t) \Rightarrow b(s) = \emptyset$ and $U(s) = 1$

$\Rightarrow$ all ICs on $y(t)$ zero $\Rightarrow c(s) = \emptyset$

So:

$$Y(s) = G(s)$$

and thus

$$y(t) = \mathcal{L}^{-1}\{G(s)\} \triangleq g(t)$$

$$\boxed{\text{The } \underline{\text{impulse response}}\ g(t) \text{ is the inverse transform of the transfer function } G(s)}$$

Conversely, Knowledge (or <u>measurement</u>) of $g(t)$ tells us what the transfer function is, and hence the governing diff'l eq'ns.

$\Longrightarrow$ Foundation of "system identification" theory.

Theoretical problems in integrals when discontinuities or singularities at one of the end points.

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt$$

possible problem here

Resolve these by taking lower limit at $t = 0^-$ (the instant before $t = 0$).

$\Rightarrow$ integral "sees" effect of singularities at $t = 0$.

Starting the integral at 0^- instead of 0

- Avoids singularities at endpoints
- Causes transform to "see" singularities and discontinuities at $t=0$, so their effects will be reflected in the solutions for $y(t)$.

Hence:

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st}\, dt$$

<u>Implications</u>:

Assumed ICs for $y(t)$: just <u>before</u> $t=0$

$$\mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0^-)$$

$$\mathcal{L}\{\ddot{y}(t)\} = s^2 Y(s) - \dot{y}(0^-) - s y(0^-)$$

etc.

$$\mathcal{L}\{\dot{u}(t)\} = sU(s) - u(0^-)$$

$$\mathcal{L}\{\ddot{u}(t)\} = s^2 U(s) - \dot{u}(0^-) - s u(0^-)$$

etc.

Always $= 0$ in our analysis!

$\Rightarrow b(s) = 0$ always

Thus:

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - \cancel{b(s)}^{\,\emptyset}}{r(s)}\right]$$

$$\boxed{= \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s)}{r(s)}\right]}$$

$\Rightarrow$ IC polynomial $b(s)$ for input vanishes

$\Rightarrow$ specific to controls convention for $u(t)$

$\Rightarrow$ Not a common assumption in regular math classes.

$\Rightarrow$ In controls, want to know effect of discontinuities

Common, discontinuous "test functions"

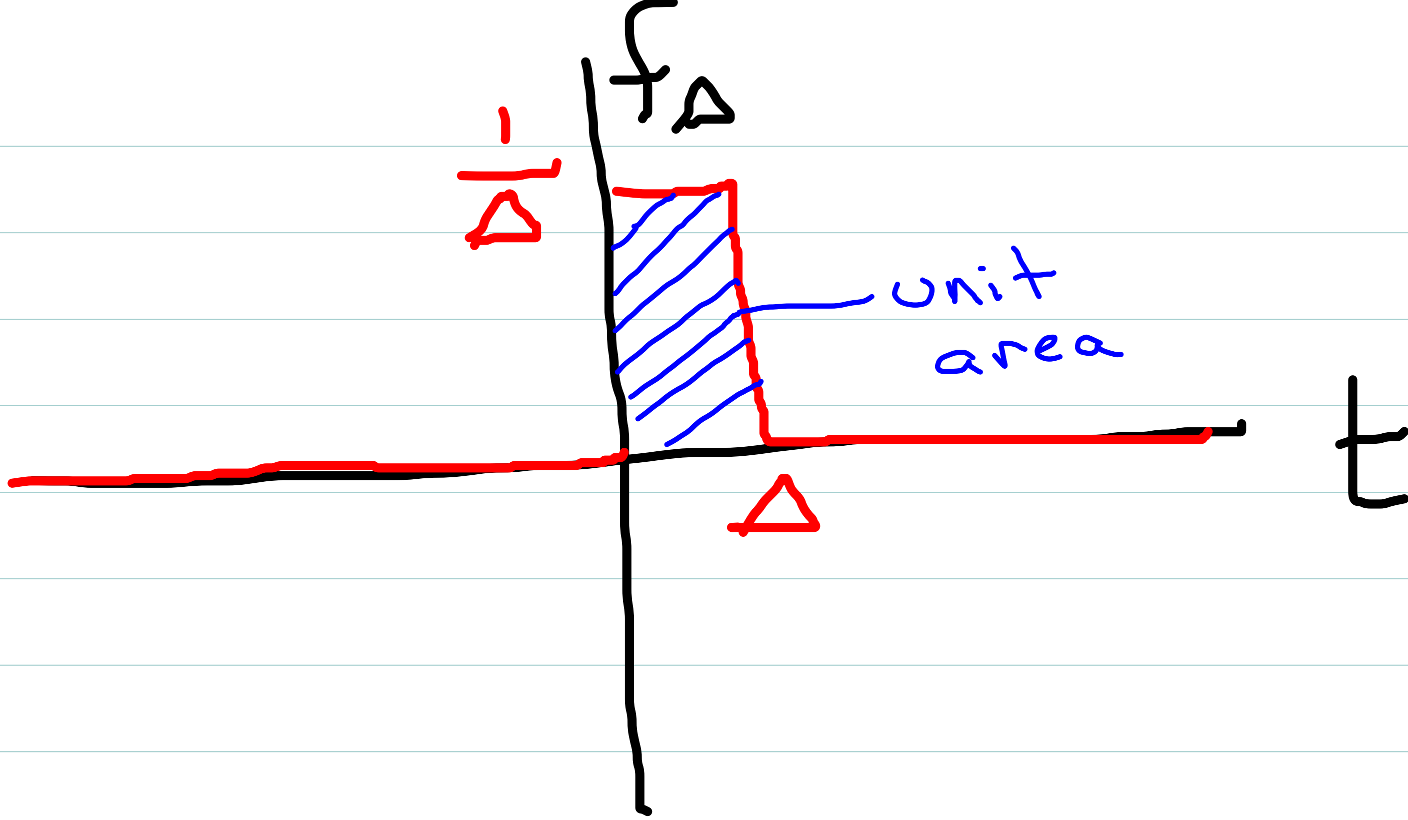
$u(t) = \mathbb{1}(t)$ (unit step) $\quad \mathcal{L}\{\mathbb{1}(t)\} = \frac{1}{s}$

$$u(t) = \cos(\omega t)\,\mathbb{1}(t)$$
$$= \begin{cases} \cos(\omega t) & t \geq \emptyset \\ \emptyset & t < \emptyset \end{cases}$$

$\Longrightarrow$ $u(t) = f_\Delta(t)$

$$= \begin{cases} \frac{1}{\Delta} & \emptyset \leq t \leq \Delta \\ \emptyset & \text{otherwise} \end{cases}$$

"Unit pulse function"



Note: for any $\Delta > 0$

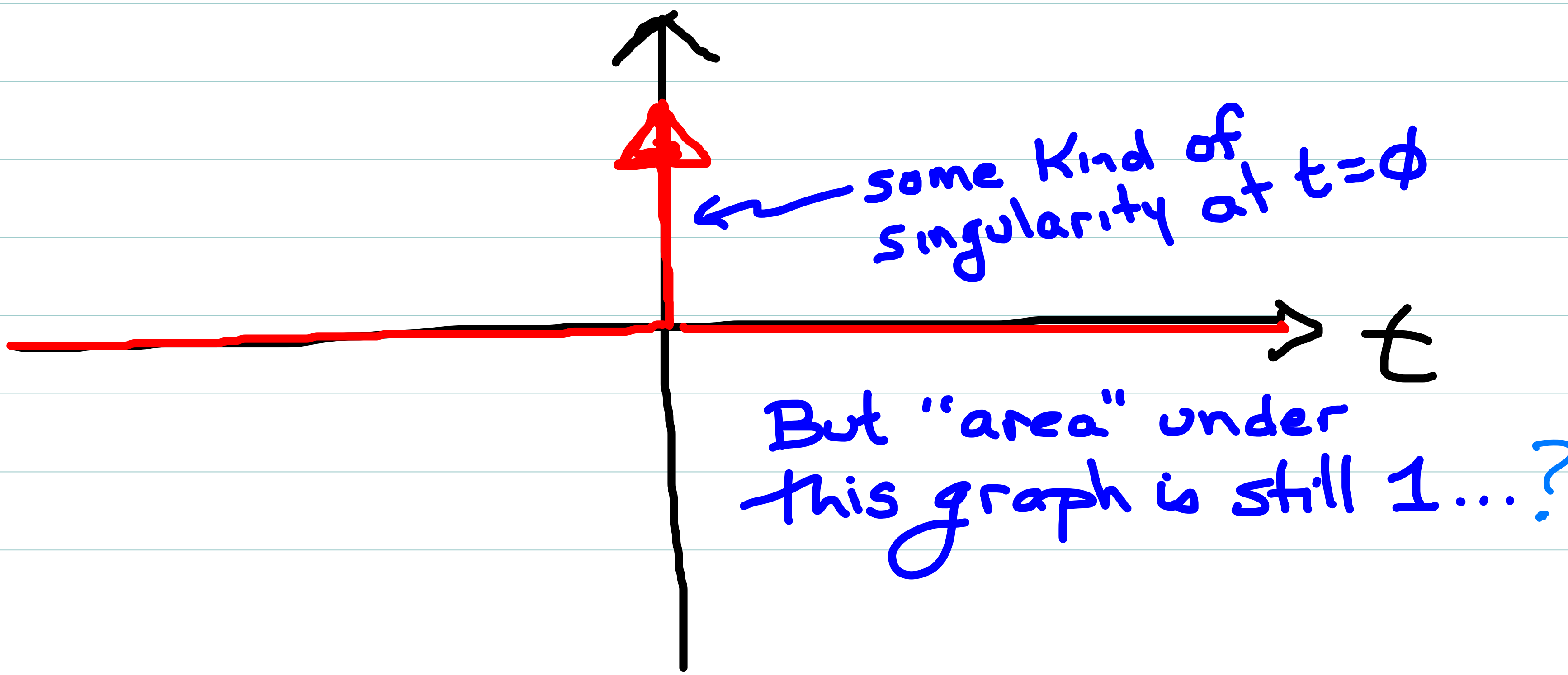
$$\int_{0^-}^{\infty} f_\Delta(t)\,dt = \int_{0^-}^{\Delta} \left(\frac{1}{\Delta}\right) dt$$

$$= 1$$

What is $\lim_{\Delta \to 0} f_\Delta(t)$?

$$= \lim_{\Delta \to 0} \begin{cases} \frac{1}{\Delta} & \phi \le t \le \Delta \\ \phi & \text{otherwise} \end{cases}$$

$$= \begin{cases} \infty & t = \phi \\ \phi & \text{otherwise} \end{cases}$$



Define:

$$\boxed{\delta(t) = \lim_{\Delta \to 0} f_\Delta(t)}$$

"ideal impulse": models delivering a unit of input energy over negligibly small time.

(sharp "kick")

Alternate names:

"delta function"
"impulse function"
"Dirac delta"

Note: *Not really a meaningful function at all!*

More formally, belongs to a class of mathematical objects called "distributions" or "generalized functions"

Suppose $\delta(t)$ appears in an integral

$$\int_{-\infty}^{\infty} \delta(t)\, h(t)\, dt \quad , \quad h(t) \text{ arbitrary f'n}$$

$$= \int_{-\infty}^{\infty} \left[\lim_{\Delta \to 0} f_\Delta(t)\right] h(t)\, dt$$

$$= \lim_{\Delta \to 0} \int_{-\infty}^{\infty} f_\Delta(t)\, h(t)\, dt$$

$$= \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \int_{0^-}^{\Delta} h(t)\, dt \right\}$$

$$\cong \left(\frac{1}{\Delta}\right) \left(\Delta\, h(0)\right)$$

$$= h(0)$$

Note: with $h(t) = 1$ for all t, we get

$$\int_{0^-}^{\infty} \delta(t)\, dt = 1$$

Defining Property of $\delta(t)$

$$\int_a^b \delta(t)h(t)\,dt = \begin{cases} h(0) & \text{if } 0 \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

"sifting property"

$\Rightarrow$ $\delta(t)$ is defined by what it does in an integral

Not as an ordinary function

Now we can compute:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt$$

$$= \left[e^{-st}\right]_{t=0} = 1$$

Thus:

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

$$\Rightarrow \mathcal{L}^{-1}\{c\} = c\,\delta(t)$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall $\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t=\emptyset \\ \emptyset & \text{otherwise} \end{cases}$

which looks like $\frac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\{\tfrac{d}{dt}\mathbb{1}(t)\} = s\,\mathcal{L}\{\mathbb{1}(t)\} - \mathbb{1}(\emptyset^-)$$

Thus:

$$\mathcal{L}\{\delta(t)\} = 1$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall

$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t=\emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

which looks like $\frac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\left\{\frac{d}{dt}\mathbb{1}(t)\right\} = s\,\underbrace{\mathcal{L}\{\mathbb{1}(t)\}}_{1/s} - \cancel{\mathbb{1}(\emptyset^-)} \to \emptyset$$

$$= 1 = \mathcal{L}\{\delta(t)\}$$

YES

Recap: Unit Impulse

$$\delta(t) = \lim_{\Delta \to 0} f_\Delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

"sifting property":

$$\int_a^b \delta(t) h(t)\, dt = \begin{cases} h(0) & \text{if } 0 \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Laplace Transform:

$$\mathcal{L}\{\delta(t)\} = 1$$

Useful property:

$$\frac{d}{dt} \mathbb{1}(t) = \delta(t)$$

Impulse Response

The *impulse response* of a system is the output $y(t)$ when $u(t) = \delta(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[c(s) - b(s)]}{r(s)}$$

$\Rightarrow u(t) = \delta(t) \Rightarrow b(s) = \emptyset$ and $U(s) = 1$

$\Rightarrow$ all ICs on $y(t)$ zero $\Rightarrow c(s) = \emptyset$

So:

$$Y(s) = G(s)$$

and thus

$$y(t) = \mathcal{L}^{-1}\{G(s)\} \triangleq g(t)$$

The <u>impulse response</u> $g(t)$ is the inverse transform of the transfer function $G(s)$

Conversely, knowledge (or <u>measurement</u>) of $g(t)$ tells us what the transfer function is, and hence the governing diff'l eq'ns.

$\Rightarrow$ Foundation of "system identification" theory.

Additional Laplace Property

For any two functions $f_1(t), f_2(t)$ with transforms $F_1(s), F_2(s)$

$$\mathcal{L}^{-1}\{F_1(s)F_2(s)\} = \int_{0^-}^{\infty} f_1(t-\tau)f_2(\tau)\,d\tau$$

"convolution"

Implication: $\mathcal{L}^{-1}\{G(s)U(s)\} = \int_{0^-}^{\infty} g(t-\tau)u(\tau)\,d\tau$

proving generally what we showed specifically for the hovercraft problem.

There we had $\ddot{y}(t) = Ku(t)$

$$\Rightarrow G(s) = \frac{K}{s^2} \Rightarrow g(t) = Kt \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

and thus $g(t-\tau) = K(t-\tau)$.

Note:

Laplace actually let's us "divide out" the effect of any Known input to recover the transfer function (impulse response)

$$Y(s) = G(s)\,U(s) \qquad \text{(assuming } \emptyset \text{ ICs)}$$

$$\left[Y(s) = G(s)\,U(s)\right] \times \left(\frac{1}{U(s)}\right)$$

$$\left[\frac{Y(s)}{U(s)}\right] = G(s)\left[\frac{U(s)}{U(s)}\right]$$

$$= G(s)\cdot 1$$

response to ideal impulse.

Structure of Impulse Response

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{q(s)}{r(s)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\sum_{K=1}^{n} \frac{\gamma_K}{(s-p_K)}\right\} \quad p_K \text{ poles of } G(s)$$

or

$$\boxed{\begin{aligned} g(t) &= \sum_{K=1}^{n} \gamma_K e^{p_K t} \\ \gamma_K &= \left[(s-p_K)\,G(s)\right]_{s=p_K} \end{aligned}}$$

(assuming non-repeated modes for simplicity)

$$g(t) = \sum_{K=1}^{n} \gamma_K e^{p_K t}$$

Note:

$\Longrightarrow$ $g(t)$ is a specific linear combination of the modes.

$\Longrightarrow$ Like a special homogeneous response

Alternate characterization of system stability

$$\boxed{\lim_{t \to \infty} |g(t)| \to \emptyset}$$

(if system is stable)

Impulse response in state-space

For a state-space model recall: $G(s) = C(sI-A)^{-1}B+D$

Assume $D=0$ for simplicity (most common case)

then $$g(t) = \mathcal{L}^{-1}\{C(sI-A)^{-1}B\}$$
$$= C\mathcal{L}^{-1}\{(sI-A)^{-1}\}B \quad \text{(linearity)}$$

Let $\Phi(s) \triangleq (sI-A)^{-1}$ (nxn matrix)

and $\phi(t) \triangleq \mathcal{L}^{-1}\{\Phi(s)\}$ (nxn matrix)

Then

$$g(t) = C\phi(t)B$$

$\Rightarrow$ what is this matrix $\phi(t)$??

Matrix Exponential Function

Note for scalar a,

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = \mathcal{L}\{(s-a)^{-1}\} = e^{at}$$

By analogy

$$\mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}^{-1}\{(s\mathbb{I}-A)^{-1}\} \triangleq e^{At}$$

The "matrix exponential function"

How to calculate it?

Laplace (and its inverse) works on matrices just like it does on vectors:

Apply inverse transform to each entry of $\Phi(s) = (s\mathbb{I}-A)^{-1}$ (so n^2 inverse transforms ugh!)

General Observations

Recall that $\Phi(s) = (sI - A)^{-1} = \dfrac{Q(s)}{r(s)}$

$Q(s) = \text{Adj}(sI - A)$ nxn matrix of polynomials in s

$r(s) = \text{Det}(sI - A)$ ordinary polynomial in s

Hence:

1.) Each entry of $\Phi(s)$ is rational
$\Rightarrow$ can use PFE tricks for inverse xform

2.) Each entry of $\Phi(s)$ has same denom, hence same poles (roots of $r(s)$)

Recall: roots of $r(s)$ same as eigenvalues of A

Thus: Each entry of $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$ will be a linear combination of $e^{\lambda_k t}$ where λ_k are eigenvalues of A

Example

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \Rightarrow \text{companion form. By inspection}$$

$$r(s) = s^2 + 5s + 6 = (s+2)(s+3)$$

$\Rightarrow$ each entry of e^{At} is a linear comb. of e^{-2t}, e^{-3t}

$$(sI - A) = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix} \qquad Q(s) = \mathrm{Adj}(sI - A) = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\[2ex] \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix}$$

inverse xform each entry separately using PFE

Example, cont

(1,1) $$\frac{s+5}{(s+2)(s+3)} = \frac{A_1}{s+2} + \frac{A_2}{s+3} \qquad A_1 = \frac{-2+5}{1} = 3 \qquad A_2 = \frac{-3+5}{-1} = -2$$

(1,2) $$\frac{1}{(s+2)(s+3)} = \frac{A_1}{s+2} + \frac{A_2}{s+3} \qquad A_1 = \frac{1}{1} = 1 \qquad A_2 = \frac{1}{-1} = -1$$

(2,1) $$\frac{-6}{(s+2)(s+3)} \text{ is just } -6 \times \left(\frac{1}{(s+2)(s+3)}\right)$$

(2,2) $$\frac{s}{(s+2)(s+3)} = \frac{A_1}{s+2} + \frac{A_2}{s+3} \qquad A_1 = \frac{-2}{1} = -2 \qquad A_2 = \frac{-3}{-1} = 3$$

Thus here:

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ 6(e^{-3t} - e^{-2t}) & 3e^{-3t} - 2e^{-2t} \end{bmatrix}$$

Recap

So with a state-space model we equivalently have

$$g(t) = Ce^{At}B$$

where:

- e^{At} is an $n \times n$ matrix
- each entry of e^{At} is a linear combination of MODES
- MODES determined by poles (p_k) $\Longleftrightarrow$ eigenvalues (λ_k) of A

and thus:

$$g(t) = Ce^{At}B = \sum_{k=1}^{n} \gamma_k e^{p_k t}$$

a linear combination of MODES, just like before.

Step Responses

The (unit) step response of a system is the output $y(t)$ when $u(t) = \mathbb{1}(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[\overset{\emptyset}{\cancel{c}}(s) - \overset{\emptyset}{\cancel{b}}(s)]}{r(s)}$$

$U(s) = \frac{1}{s}$ here, so

$$Y(s) = \left(\frac{1}{s}\right)G(s) = \frac{q(s)}{s\,r(s)}$$

Intermediate Case 3 Situations

If $1.1 < \frac{|P_2|}{|P_1|} < 5$ (or 8 or 10)

$$\frac{4}{|P_1|} < t_s < \frac{6}{|P_1|}$$

Unfortunately, there is no simple formula for interpolating between the two limits based on the exact ratio.

Step Responses

The (unit) <u>step response</u> of a system is the output $y(t)$ when $u(t) = \mathbb{1}(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[\cancelto{\emptyset}{c(s)} - \cancelto{\emptyset}{b(s)}]}{r(s)}$$

$U(s) = \frac{1}{s}$ here, so

$$Y(s) = \left(\frac{1}{s}\right)G(s) = \frac{q(s)}{s\,r(s)}$$

General Thoughts about step responses

① Every system has a unit step response:

$$Y(s) = \left[\left(\tfrac{1}{s}\right) G(s)\right]$$

$$y(t) = \mathcal{L}^{-1}\left\{\tfrac{1}{s} G(s)\right\} \triangleq y_{us}(t)$$

Find $y_{us}(t)$ as usual by partial fraction expansion and inverse transform of each term

However, we want to be able to predict main features of $y_{us}(t)$ by inspection for 1^{st} and 2^{nd} order systems

⇒ Very common special cases

⇒ "Building blocks" for more complex systems

② (Use of linearity, I)

$$u(t) = c\,\mathbb{1}(t) \implies y(t) = c\,y_{us}(t)$$

All $y(t)$ values are the unit step values multiplied by c.

Equivalent to "rescaling" vertical Axis on plot of $y(t)$, however horizontal (time) Axis is unaffected

$\implies$ Characteristic times (t_s, t_r, t_p) are unaffected

We'll encounter these shortly.

$\implies$ Corresponding $y(t)$ values scaled by c:

$$y_{ss} = c\,G(0), \quad y_p = c\,G(0)[1 + M_p]$$

$\implies$ True for any c, positive or negative

③ (Use of Linearity, II)

By definition, unit step response assumes all ICs are zero.

However, can easily "Add on" effects of nonzero ICs.

Nonzero Now

$$Y(s) = \left[\frac{1}{s}G(s)\right] + \left[\frac{C(s)}{r(s)}\right]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{(\tfrac{1}{s})G(s)\} + \mathcal{L}^{-1}\left\{\frac{C(s)}{r(s)}\right\}$$

$$= y_{us}(t) + \mathcal{L}^{-1}\left\{\frac{C(s)}{r(s)}\right\}$$

Added terms from ICs

Solve for last term by PFE

Effect of added terms on t_s, t_p, y_p etc depends on specific ICs. No simple formulae to quantify their effects.

"1st Order" Responses

$$\dot{y}(t) + \alpha_0 y(t) = \beta_0 u(t) \implies G(s) = \frac{\beta_0}{s+\alpha_0}$$

Single real pole at $p_1 = -\alpha_0$ (stable if $\alpha_0 > 0$)

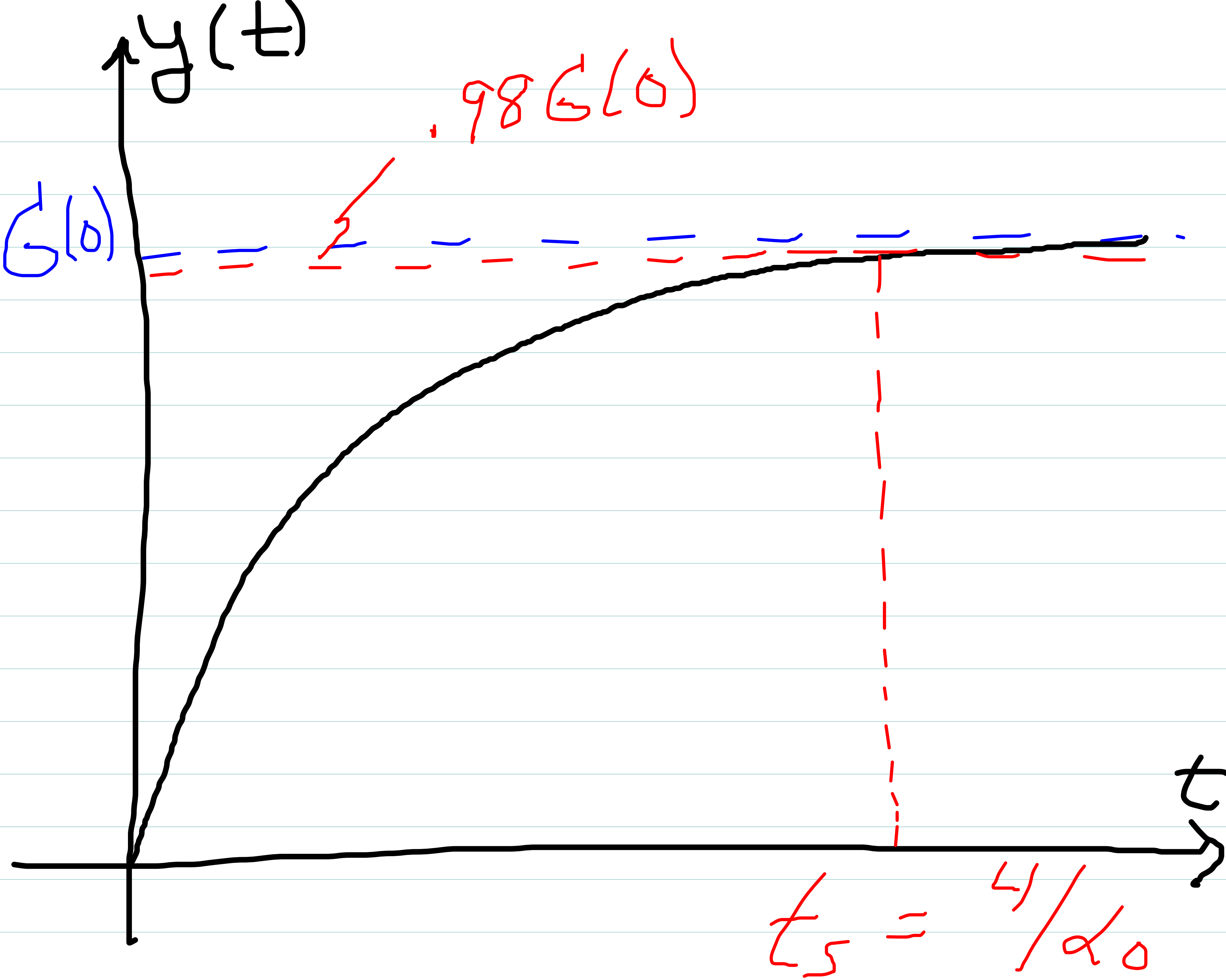
$$Y(s) = \frac{\beta_0}{s(s+\alpha_0)} = \frac{A_1}{s} + \frac{A_2}{s+\alpha_0}$$

$$A_1 = [sY(s)]_{s=0} = \frac{\beta_0}{\alpha_0} = G(0)$$

$$A_2 = [(s+\alpha_0)Y(s)]_{s=-\alpha_0} = \frac{-\beta_0}{\alpha_0} = -G(0)$$

Thus:

$$\boxed{y(t) = G(0)\left[1 - e^{-\alpha_0 t}\right]}$$


y(t)
.98G(0)
G(0)
t
$t_s = 4/\alpha_0$

Notes

1. Response asymptotically approaches steady-state

$$y_{ss}(t) = G(0)$$ (as expected)

2. Response **never** crosses its steady-state

3. Response **settles** within 2% of its steady-state in

$$t_s = \frac{4}{|Re\{p\}|} = \frac{4}{\alpha_0}$$

4. "Shape" of graph is same for any 1st order system

Responses only differ by:

- steady-state level, $G(0)$
- settling time, t_s

"2nd Order" Step Responses

$\ddot{y}(t) + \alpha_1 \dot{y}(t) + \alpha_0 y(t) = \beta_0 u(t) \implies G(s) = \dfrac{\beta_0}{s^2 + \alpha_1 s + \alpha_0}$

2 poles, both stable if $\alpha_1 > 0, \alpha_0 > 0$.

3 possibilities for poles:

1. $\alpha_1^2 < 4\alpha_0 \implies P_1, P_2$ complex conjugates
2. $\alpha_1^2 = 4\alpha_0 \implies P_1 = P_2$ repeated real
3. $\alpha_1^2 > 4\alpha_0 \implies P_1, P_2$ real, non-repeated

Case 1 is most interesting (and complicated)
tackle this after the other two

2^{nd} order response, Case 2

$$G(s) = \frac{\beta_0}{s^2+\alpha_1 s+\alpha_0} \qquad \alpha_1^2 = 4\alpha_0 \quad (\xi = 1)$$

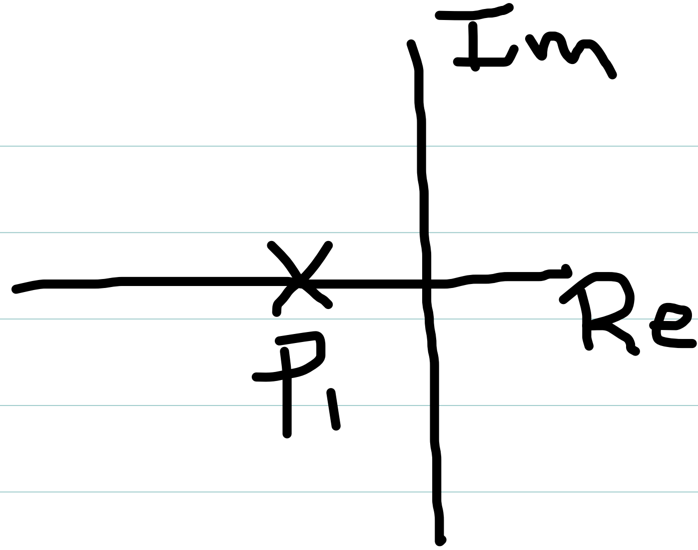
$$= \frac{\beta_0}{(s-p_1)^2} \qquad \text{repeated real pole}$$

$$Y(s) = \left(\frac{1}{s}\right)G(s) = \frac{A_1}{s} + \frac{A_2}{(s-p_1)} + \frac{A_3}{(s-p_1)^2}$$

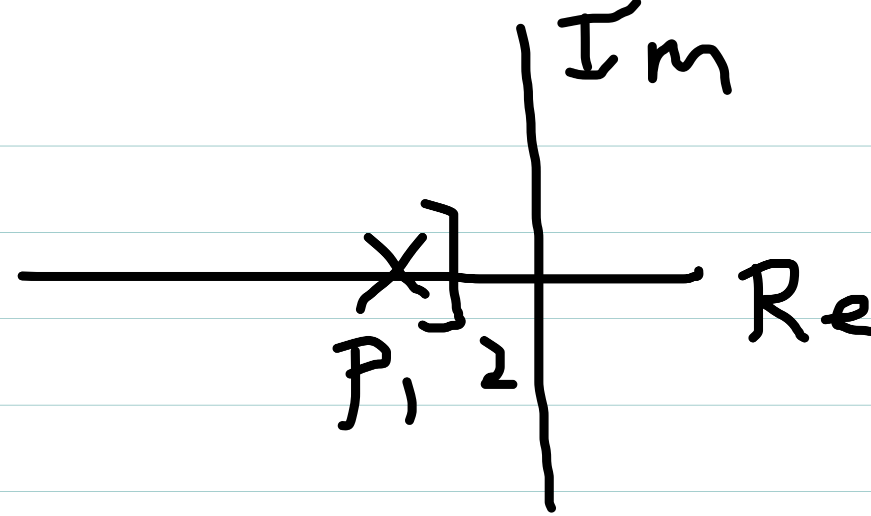
$$y(t) = G(\phi) + [A_2 + A_3 t]e^{p_1 t}$$

Non-oscillatory, since poles are real

features resemble 1^{st} order response
(No overshoot, $y_{ss} = G(0)$ approached asymptotically from below), but $\boxed{t_s \text{ 50\% longer } \left(\frac{6}{|p_1|}\right)}$

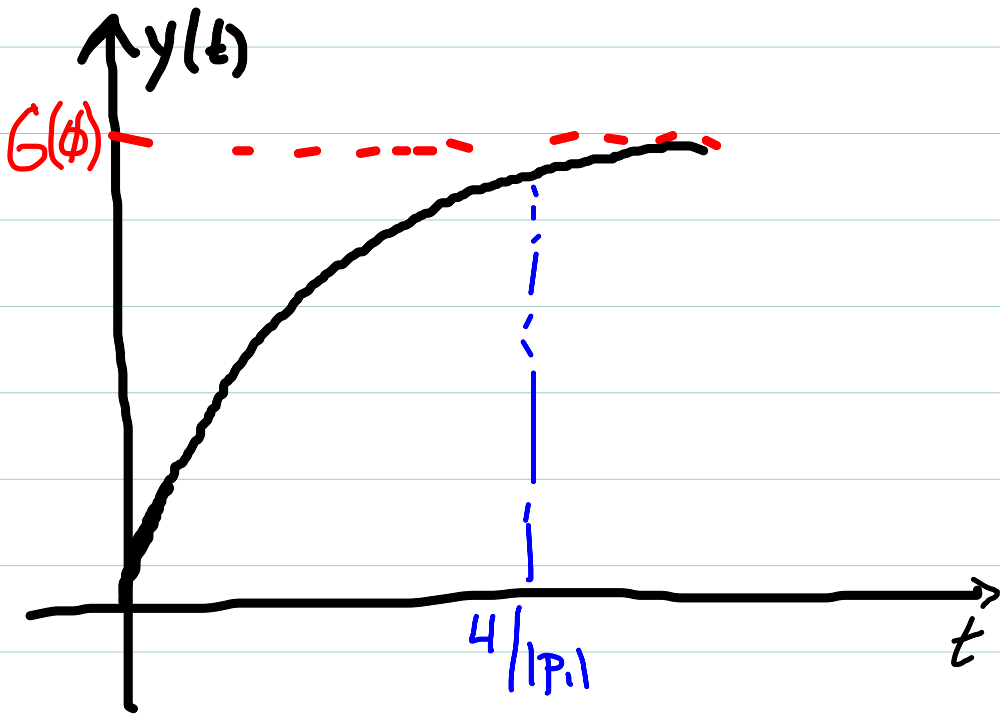


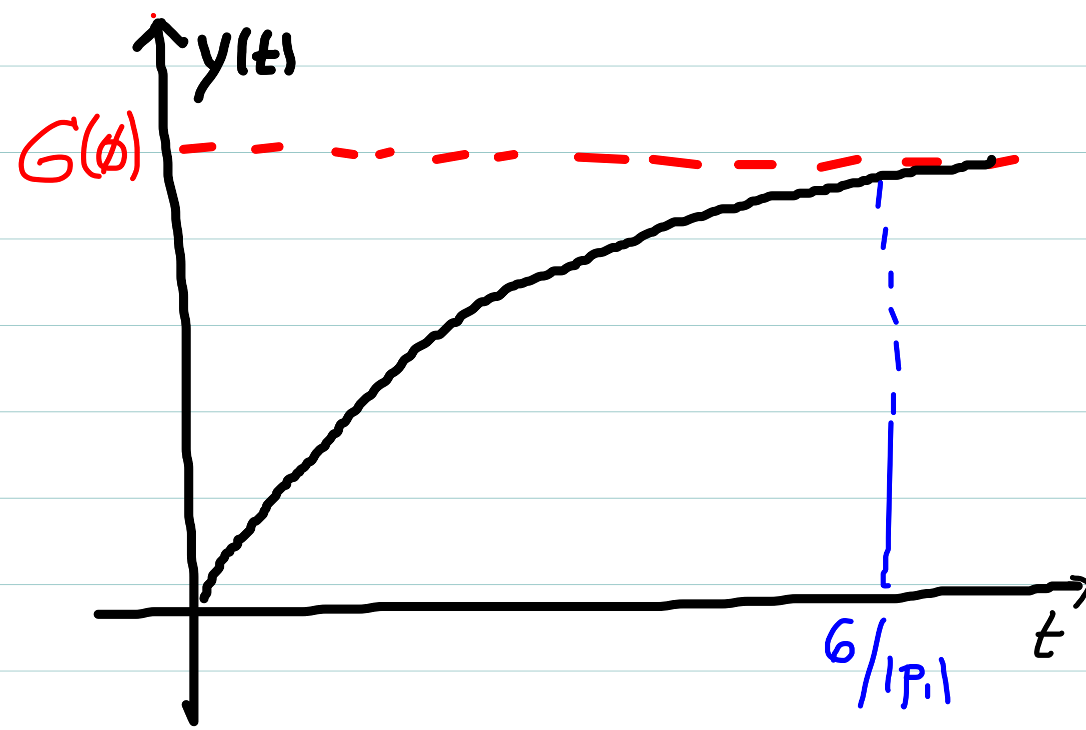
1^{st} order



2^{nd} order, repeated real

Add'l $te^{p_1 t}$ term "slows down" response.





2^{nd} order Response, Case 3

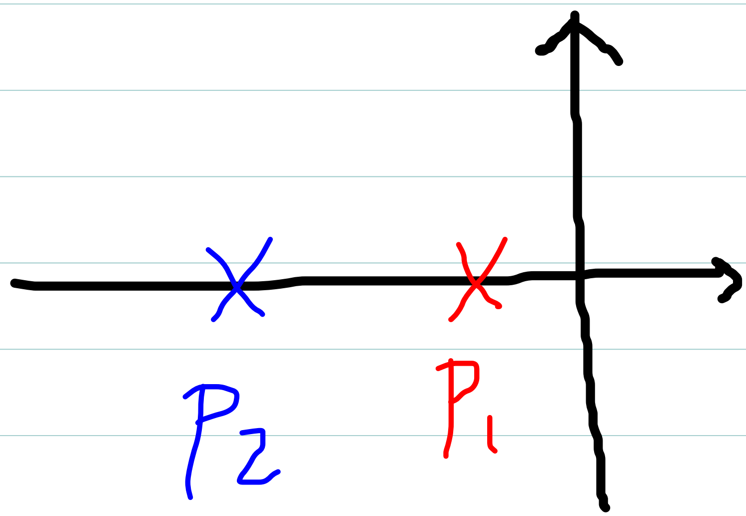
$\alpha_1^2 > 4\alpha_0$

$$Y(s) = \frac{\beta_0}{s(s-p_1)(s-p_2)} \qquad p_1 \neq p_2 .$$

$$\Rightarrow y(t) = G(0) + A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

Assume for notation sake that poles are numbered so that

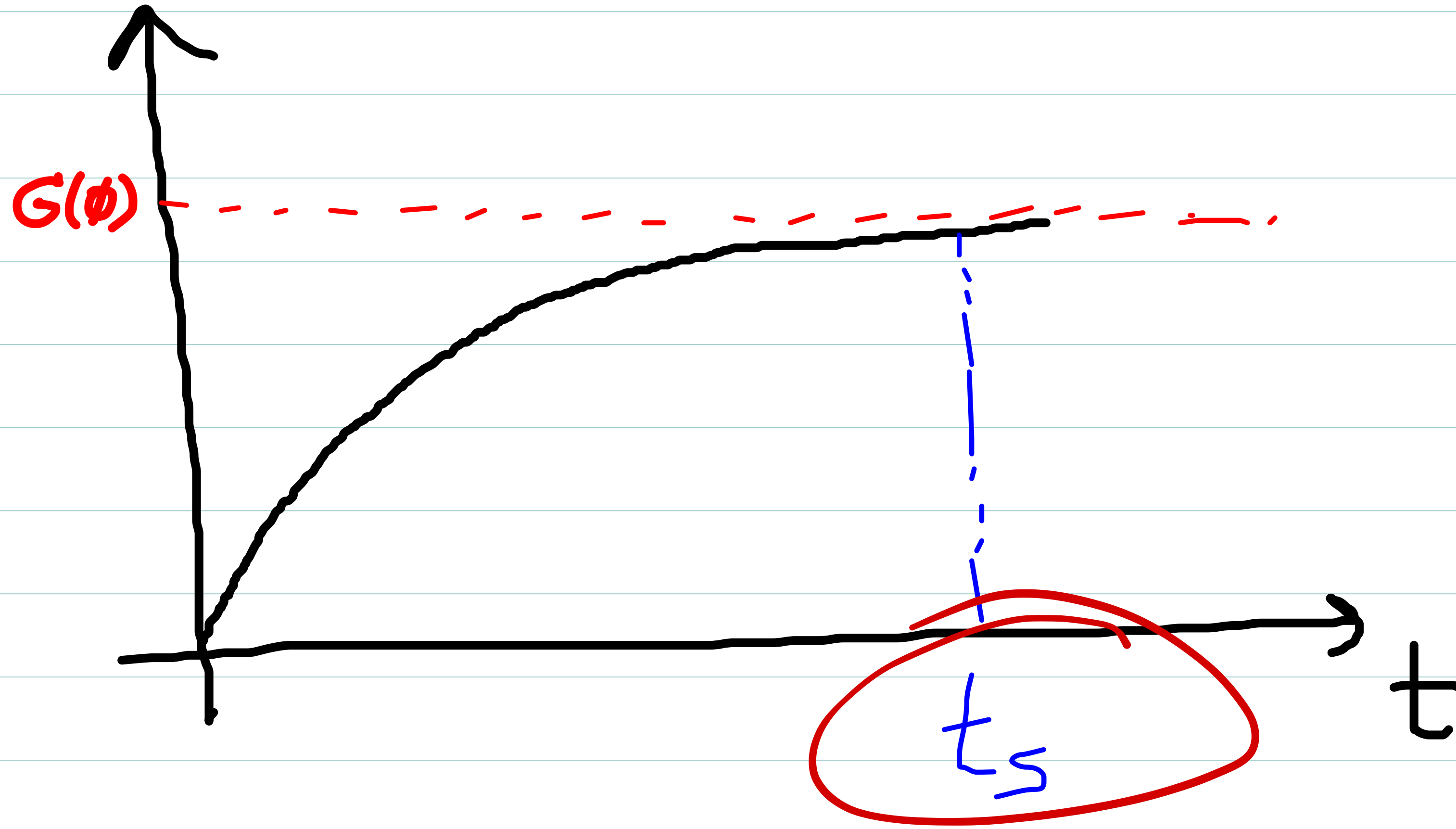
$$p_2 < p_1 \qquad (\Rightarrow |p_2| > |p_1| \text{ since } p_1, p_2 \text{ assumed negative})$$



p_1 is the "slow pole"

p_2 is the "fast pole"

General sol'n again resembles 1st order response



t_s difficult to quantify precisely for arbitrary p_1, p_2

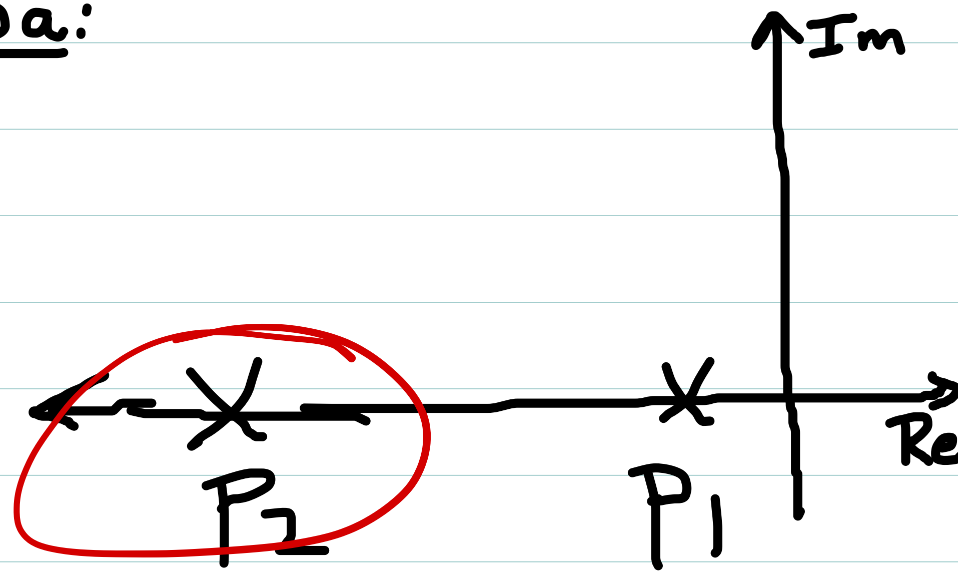
Two Limiting cases:

<u>Case 3a:</u> $|p_2| \gg |p_1|$

<u>Case 3b:</u> $|p_2| \approx |p_1|$

Case 3a:

$$y(t) = G(\phi) + A_1 e^{p_1 t} + A_2 e^{p_2 t}$$



$|p_2| >> |p_1|$

$\Rightarrow$ p_2 much further into LHP than p_1

$\Rightarrow$ $e^{p_2 t} \to \phi$ much faster than $e^{p_1 t}$

$\Rightarrow$ $e^{p_1 t}$ controls settling time ("slow pole")

So $\quad t_s \approx \dfrac{4}{|p_1|}$ in this case

$\Rightarrow$ Corresponds with previous "1st cut" of approximating system settling time with settling time of slowest mode.

Dominant Modes

When $|p_2| \gg |p_1|$ we say that mode $e^{p_1 t}$ "dominates" transient response, or that $e^{p_1 t}$ ("slow mode") is the

Dominant mode

What is a sufficient separation for a mode to be dominant

Generally, if $\boxed{|p_2| > 5|p_1| \quad \text{or} \quad |p_2| > 10|p_1|}$

i.e. if p_2 is 5-10 times further into LHP

$\Rightarrow$ setting time of $e^{p_2 t}$ 5-10 times faster than that of $e^{p_1 t}$

(5 is usually sufficient. Some authors use 8 or even 10)

Case 3b

$|p_2| \approx |p_1| \Rightarrow p_2 \approx p_1$ Poles are "nearly" repeated

Here it is best to approximate the settling time as though the poles were actually repeated

$$t_s \approx \frac{6}{|p_1|}$$

Simple rule of thumb for this:

$$1 \leq \frac{|p_2|}{|p_1|} \leq 1.1$$

Intermediate Case 3 Situations

If $1.1 < \dfrac{|P_2|}{|P_1|} < 5$ (or 8 or 10)

$$\frac{4}{|P_1|} < t_s < \frac{6}{|P_1|}$$

Unfortunately, there is no simple formula for interpolating between the two limits based on the exact ratio.

"2nd Order" Step Responses

$\ddot{y}(t)+\alpha_1\dot{y}(t)+\alpha_0 y(t)=\beta_0 u(t) \implies G(s) = \dfrac{\beta_0}{s^2+\alpha_1 s+\alpha_0}$

2 poles, both stable if $\alpha_1>0, \alpha_0>0$.

3 possibilities for poles:

① $\alpha_1^2<4\alpha_0 \implies p_1, p_2$ complex conjugates

② $\alpha_1^2=4\alpha_0 \implies p_1=p_2$ repeated real

③ $\alpha_1^2>4\alpha_0 \implies p_1, p_2$ real, non-repeated

Case ① is most interesting (and complicated)
tackle this after the other two

Useful Observation (Case 1)

$p_1 = \sigma + j\omega_d \qquad \omega_d = \mathrm{Im}\{p_1\}$

Note slight change of notation! $\omega \to \omega_d$

$$s^2 + \alpha_1 s + \alpha_0 = (s - p_1)(s - \bar{p}_1)$$

$$= s^2 - (p_1 + \bar{p}_1)s + p_1\bar{p}_1$$

$$= s^2 - 2\mathrm{Re}\{p_1\}s + |p_1|^2$$

$$= s^2 - 2\sigma s + (\sigma^2 + \omega_d^2)$$

Hence:

$$\boxed{\begin{aligned} \alpha_1 &= -2\sigma = -2\mathrm{Re}\{p_1\} \\ \alpha_0 &= \sigma^2 + \omega_d^2 = |p_1|^2 \end{aligned}}$$

Rapidly identify pole location from coefs.

2^{nd} Order Response, Case 1:

$$Y(s) = \frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)} = \frac{A_1}{s} + \frac{A_2}{(s-p_1)} + \frac{\bar{A}_2}{(s-\bar{p}_1)}$$

$$A_1 = [sY(s)]_{s=0} = \frac{\beta_0}{p_1\bar{p}_1} = \frac{\beta_0}{\alpha_0} = G(0)$$

$$A_2 = [(s-p_1)Y(s)]_{s=p_1} = \frac{\beta_0}{p_1(p_1-\bar{p}_1)} = \frac{\beta_0}{(\sigma+j\omega_d)(2j\omega_d)}$$

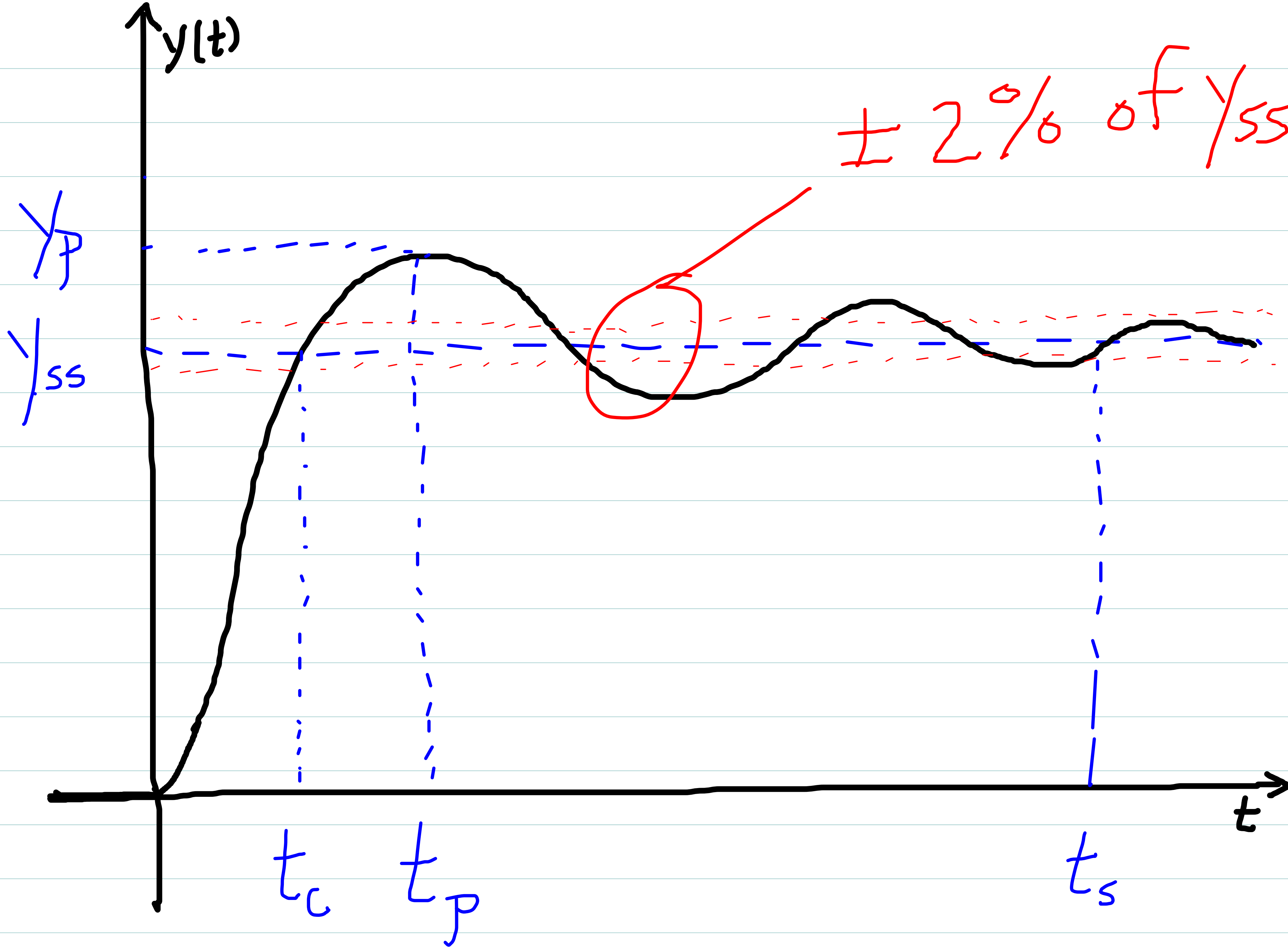
$$= \underbrace{\left(\frac{\beta_0}{2\alpha_0}\right)}_{\frac{1}{2}G(0)} \underbrace{\left(\frac{\alpha_0}{(\sigma+j\omega_d)(j\omega_d)}\right)}_{B}$$

SO:

$$y(t) = G(0) + 2|A_2|e^{\sigma t}\cos(\omega_d t + \measuredangle A_2)$$

OR:

$$\boxed{y(t) = G(0)\left[1 + |B|e^{\sigma t}\cos(\omega_d t + \measuredangle B)\right]}$$



y(t)
± 2% of y_ss
y_p
y_ss
t
t_c
t_p
t_s

General Observations

1. $y(t)$ continually oscillates about its steady-state value
$y_{ss} = G(\phi)$
2. t_c = time steady-state is first crossed
3. 1^{st} oscillation is largest, and creates an initial overshoot past the steady-state.
4. This initial overshoot has peak value y_p, and occurs at time t_p
5. Settling time t_s defined where response enters ±2% tolerance band and remains within it for times thereafter

Must learn to rapidly quantify these!!

$$y(t) = G(0)\left[1 + |B| e^{\sigma t}\cos(\omega_d t + \measuredangle B)\right]$$

where:

$$B = \frac{\alpha_0}{(j\omega_d)(\sigma + j\omega_d)} = \frac{|p_1|^2}{(j\omega_d)\, p_1}$$

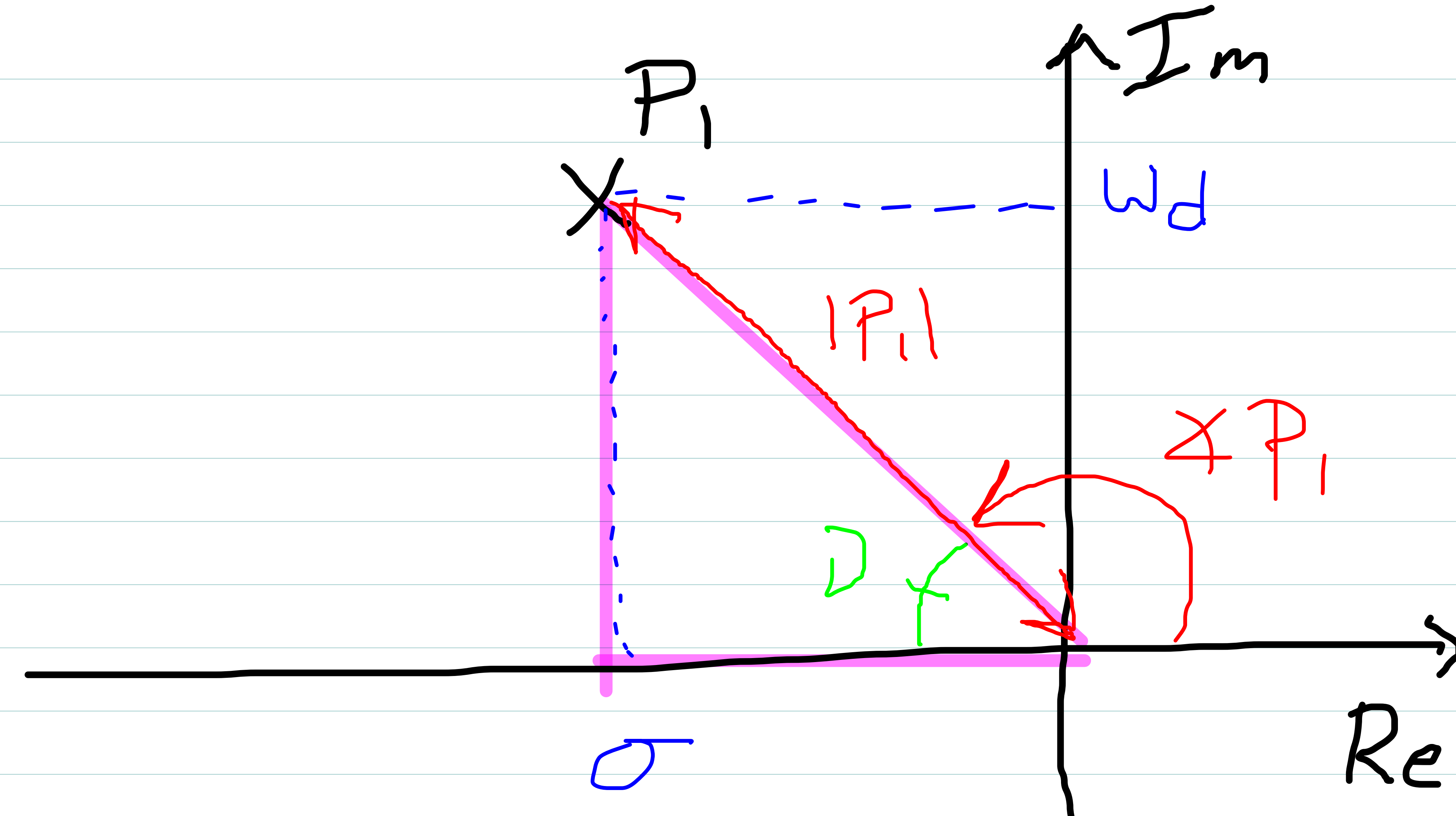
$\Rightarrow$ Transient features completely determined by location of pole $p_1 = \sigma + j\omega_d$ in complex plane

$$|B| = \frac{|p_1|^2}{|j\omega_d| \cdot |p_1|} = \frac{|p_1|}{\omega_d}$$

$$\measuredangle B = \cancelto{\phi}{\measuredangle |p_1|^2} - \left(\measuredangle(j\omega_d) + \measuredangle p_1\right)$$

$$= -\left(\frac{\pi}{2} + \measuredangle p_1\right)$$

must quantify this!



$$\measuredangle P_1 = \pi - \nu$$

<u>Note</u>: $\nu > 0$ is supplement of $\measuredangle P_1$

So:

$$\angle B = -\left(\frac{\pi}{2} + \angle p_1\right) = -\left(\frac{\pi}{2} + (\pi - \nu)\right)$$

$$= -\frac{3\pi}{2} + \nu$$

and thus:

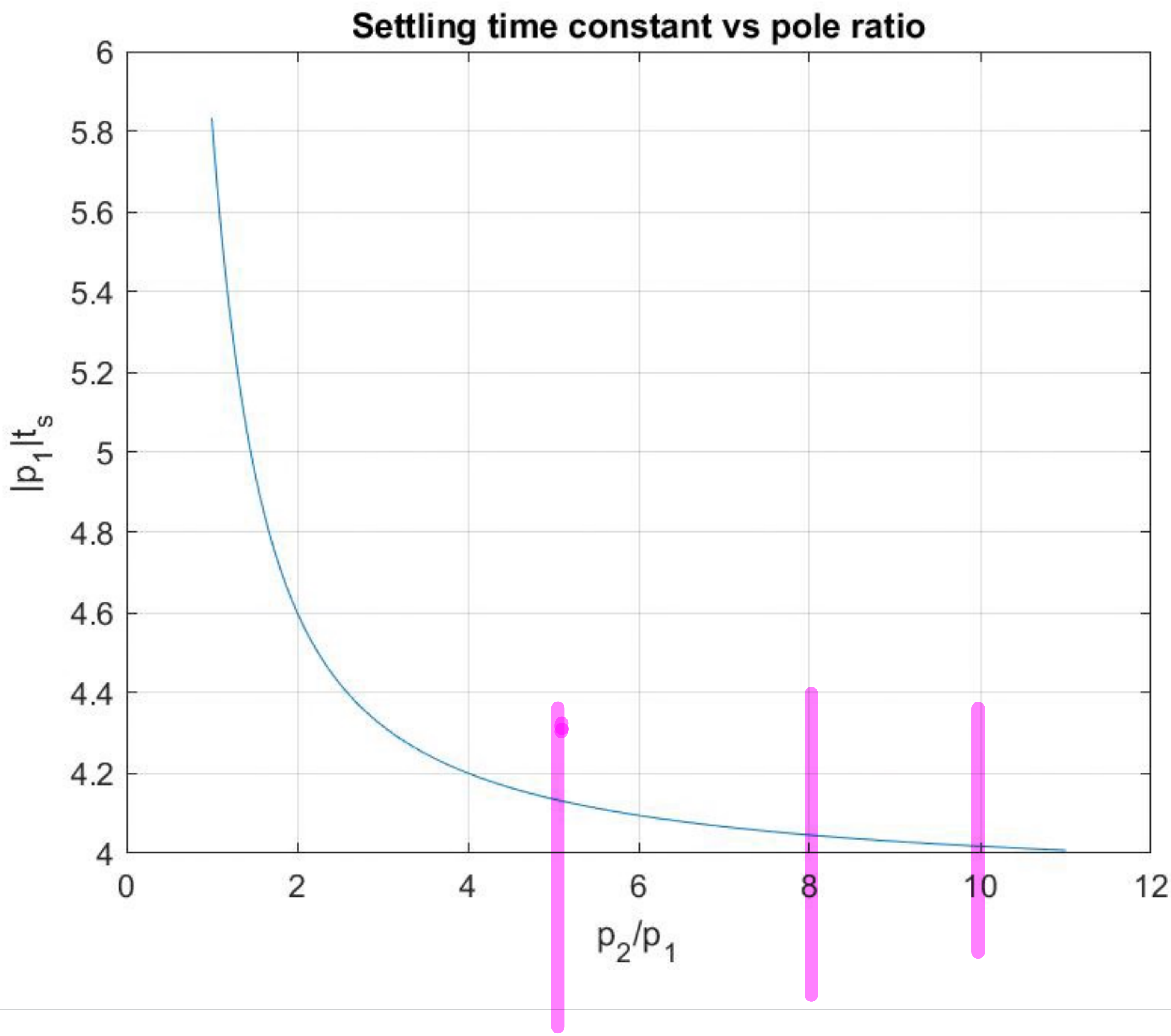
$$y(t) = G(0)\left[1 + \left(\frac{|p_1|}{\omega_d}\right) e^{\sigma t} \cos\left(\omega_d t - \frac{3\pi}{2} + \nu\right)\right]$$

so:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{|p_1|}{\omega_d}\right) e^{\sigma t} \sin(\omega_d t + \nu)\right]}$$

Need to understand how ν depends on p_1

2 real poles, $C\left(\frac{p_2}{p_1}\right)$ $\quad t_s = \dfrac{C(p_2/p_1)}{|p_1|}$



"2^{nd} Order" Step Responses

$\ddot{y}(t)+\alpha_1\dot{y}(t)+\alpha_0 y(t)=\beta_0 u(t) \implies G(s) = \dfrac{\beta_0}{s^2+\alpha_1 s+\alpha_0}$

2 poles, both stable if $\alpha_1>0, \alpha_0>0$.

3 possibilities for poles:

1. $\alpha_1^2<4\alpha_0 \implies p_1, p_2$ complex conjugates
2. $\alpha_1^2=4\alpha_0 \implies p_1=p_2$ repeated real
3. $\alpha_1^2>4\alpha_0 \implies p_1, p_2$ real, non-repeated

Case 1. is most interesting (and complicated)
tackle this after the other two

Useful Observation (Case 1)

$p_1 = \sigma + j\omega_d \qquad \omega_d = \mathrm{Im}\{p_1\}$

Note slight change of notation! $\omega \to \omega_d$

$$s^2 + \alpha_1 s + \alpha_0 = (s - p_1)(s - \bar{p}_1)$$

$$= s^2 - (p_1 + \bar{p}_1)s + p_1\bar{p}_1$$

$$= s^2 - 2\,\mathrm{Re}\{p_1\}s + |p_1|^2$$

$$= s^2 - 2\sigma s + (\sigma^2 + \omega_d^2)$$

Hence:

$$\boxed{\begin{aligned} \alpha_1 &= -2\sigma = -2\,\mathrm{Re}\{p_1\} \\ \alpha_0 &= \sigma^2 + \omega_d^2 = |p_1|^2 \end{aligned}}$$

Rapidly identify pole location from coefs.

2^{nd} Order Response, Case 1:

$$Y(s) = \frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)} = \frac{A_1}{s} + \frac{A_2}{(s-p_1)} + \frac{\bar{A}_2}{(s-\bar{p}_1)}$$

$$A_1 = [sY(s)]_{s=0} = \frac{\beta_0}{p_1 \bar{p}_1} = \frac{\beta_0}{\alpha_0} = G(0)$$

$$A_2 = [(s-p_1)Y(s)]_{s=p_1} = \frac{\beta_0}{p_1(p_1-\bar{p}_1)} = \frac{\beta_0}{(\sigma+j\omega_d)(2j\omega_d)}$$

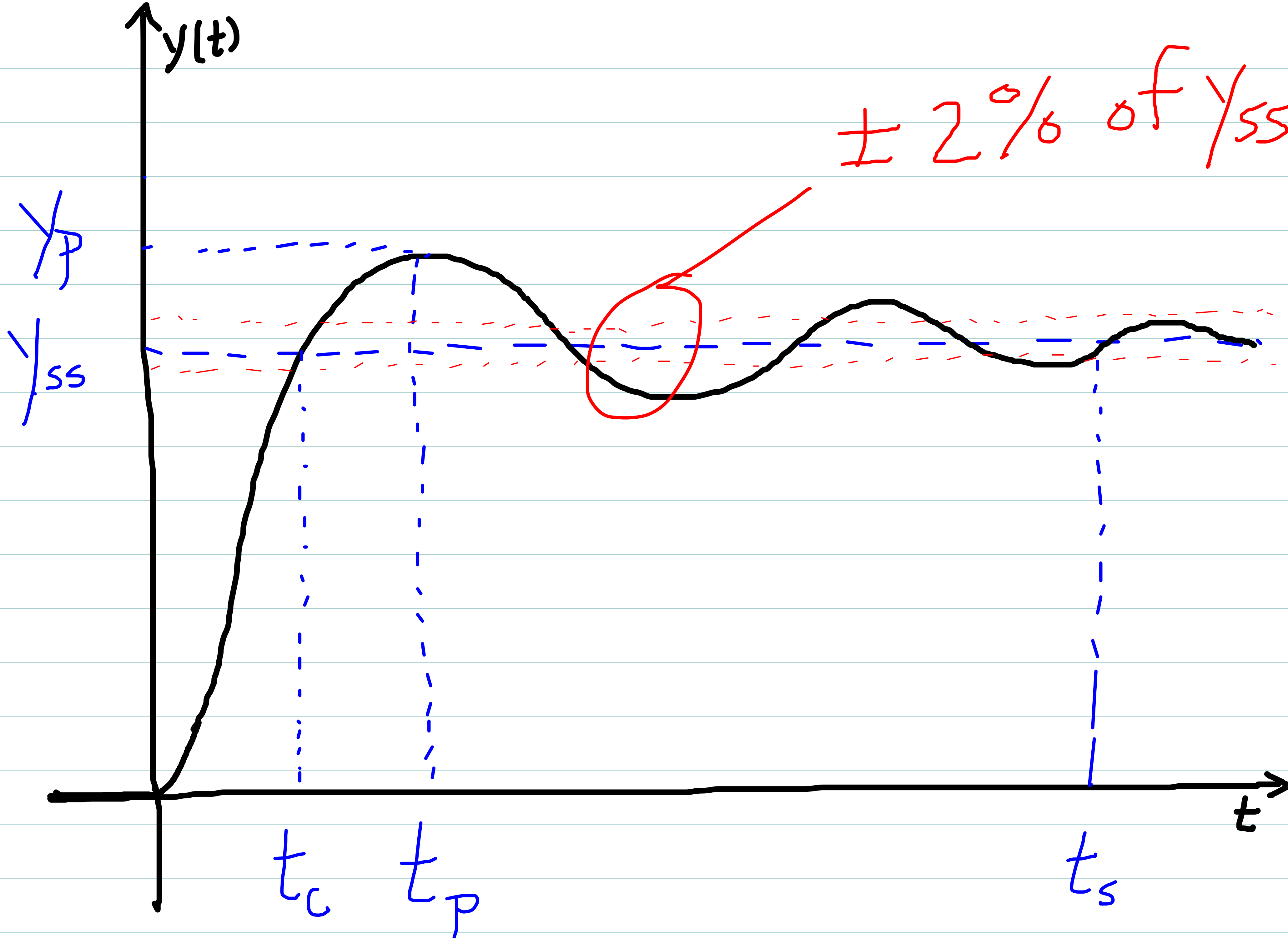
$$= \underbrace{\left(\frac{\beta_0}{2\alpha_0}\right)}_{\frac{1}{2}G(0)} \underbrace{\left(\frac{\alpha_0}{(\sigma+j\omega_d)(j\omega_d)}\right)}_{B}$$

SO:

$$y(t) = G(0) + 2|A_2| e^{\sigma t} \cos(\omega_d t + \angle A_2)$$

OR:

$$\boxed{y(t) = G(0)\left[1 + |B| e^{\sigma t} \cos(\omega_d t + \angle B)\right]}$$



y(t)
y_p
y_{ss}
± 2% of y_{ss}
t_c
t_p
t_s
t

General Observations

1. $y(t)$ continually oscillates about its steady-state value $y_{ss} = G(\phi)$
2. t_c = time steady-state is first crossed
3. 1^{st} oscillation is largest, and creates an initial overshoot past the steady-state.
4. This initial overshoot has peak value y_p, and occurs at time t_p
5. Settling time t_s defined where response enters $\pm 2\%$ tolerance band and remains within it for times thereafter

Must learn to rapidly quantify these!!

$$y(t) = G(0)\left[1 + |B| e^{\sigma t}\cos(\omega_d t + \measuredangle B)\right]$$

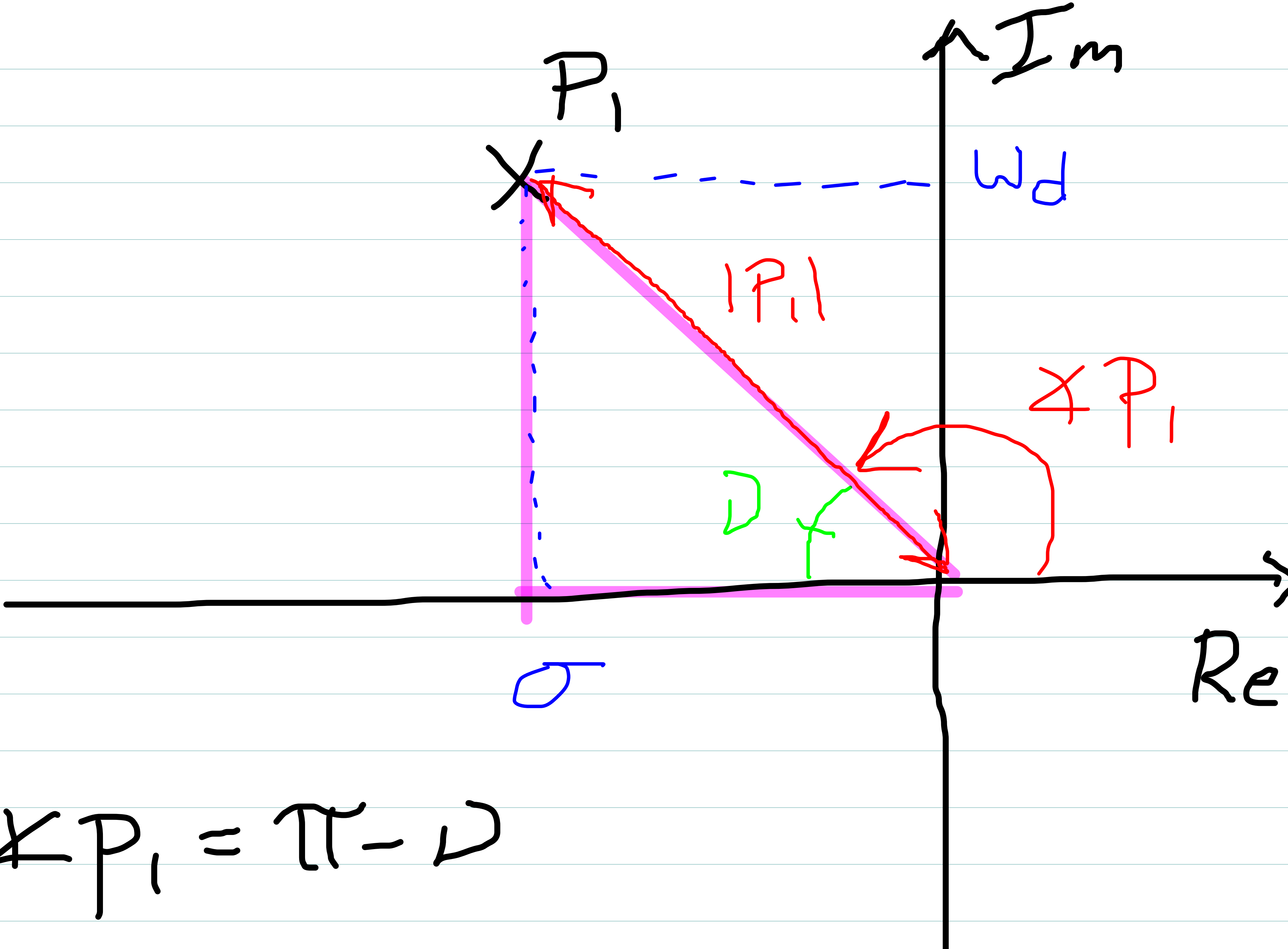
where:
$$B = \frac{\alpha_0}{(j\omega_d)(\sigma + j\omega_d)} = \frac{|p_1|^2}{(j\omega_d)\, p_1}$$

$\Rightarrow$ Transient features completely determined by location of pole $p_1 = \sigma + j\omega_d$ in complex plane

$$|B| = \frac{|p_1|^2}{|j\omega_d| \cdot |p_1|} = \frac{|p_1|}{\omega_d}$$

$$\measuredangle B = \cancel{\measuredangle |p_1|^2}^{\,\phi} - \left(\measuredangle (j\omega_d) + \measuredangle p_1\right)$$

$$= -\left(\frac{\pi}{2} + \measuredangle p_1\right)$$ must quantify this!



$$\measuredangle P_1 = \pi - \nu$$

Note: $\nu > \alpha$ is supplement of $\measuredangle P_1$

So:

$$\measuredangle B = -\left(\frac{\pi}{2} + \measuredangle p_1\right) = -\left(\frac{\pi}{2} + (\pi - \nu)\right)$$

$$= -\frac{3\pi}{2} + \nu$$

and thus:

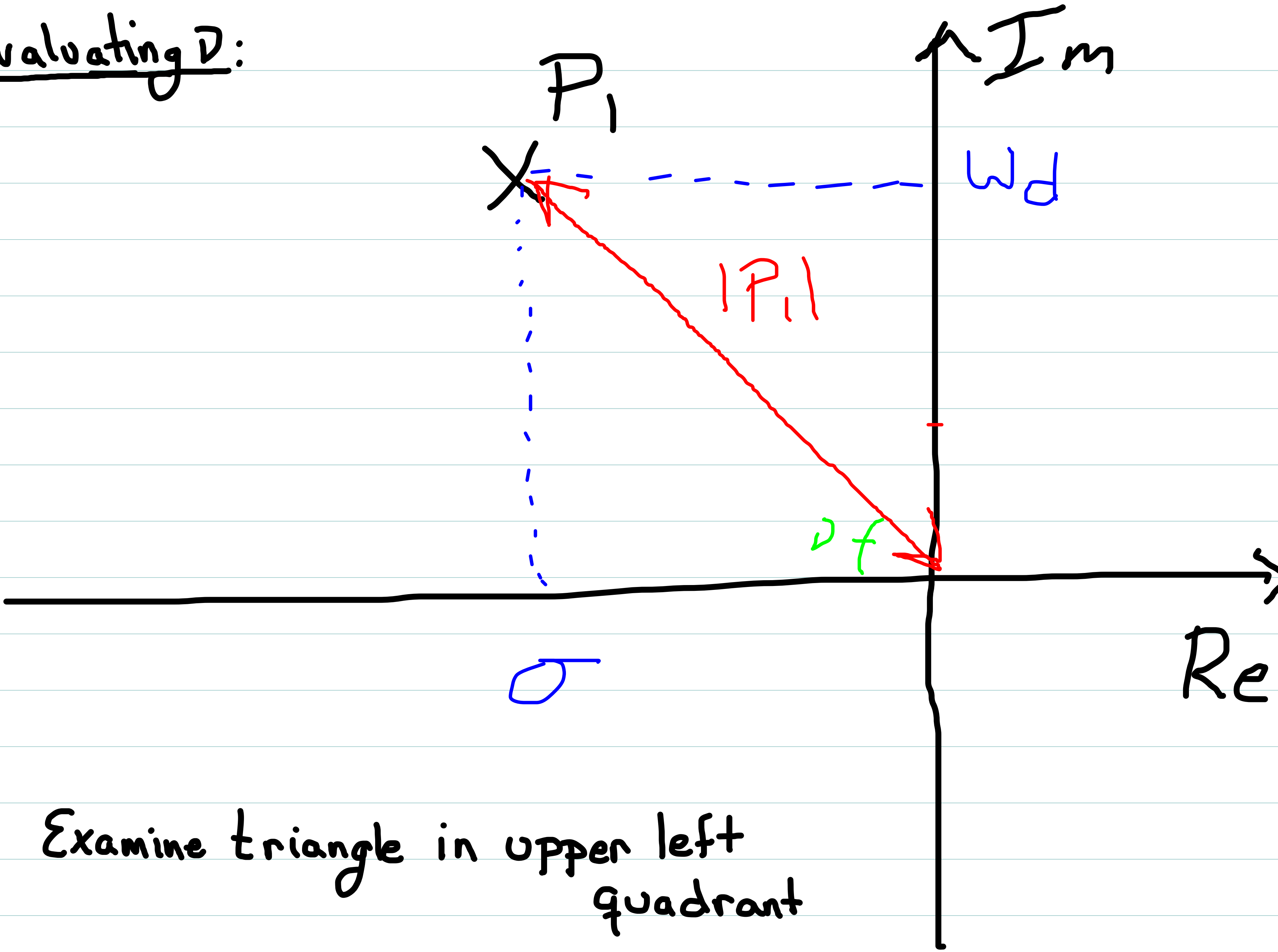
$$y(t) = G(0)\left[1 + \left(\frac{|p_1|}{\omega_d}\right)e^{\sigma t}\cos\left(\omega_d t - \frac{3\pi}{2} + \nu\right)\right]$$

so:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{|p_1|}{\omega_d}\right)e^{\sigma t}\sin(\omega_d t + \nu)\right]}$$

Need to understand how ν depends on p_1

Evaluating D:



Examine triangle in upper left quadrant

Two Useful Parameters

Define: $\omega_n = |p_1| = \sqrt{\sigma^2 + \omega_d^2}$ "natural" frequency

$\Rightarrow$ purely theoretical! ω_d is physical frequency of transient oscillations

Define: $$\xi = \frac{|\sigma|}{\omega_n} = \frac{|\sigma|}{\sqrt{\sigma^2 + \omega_d^2}}$$

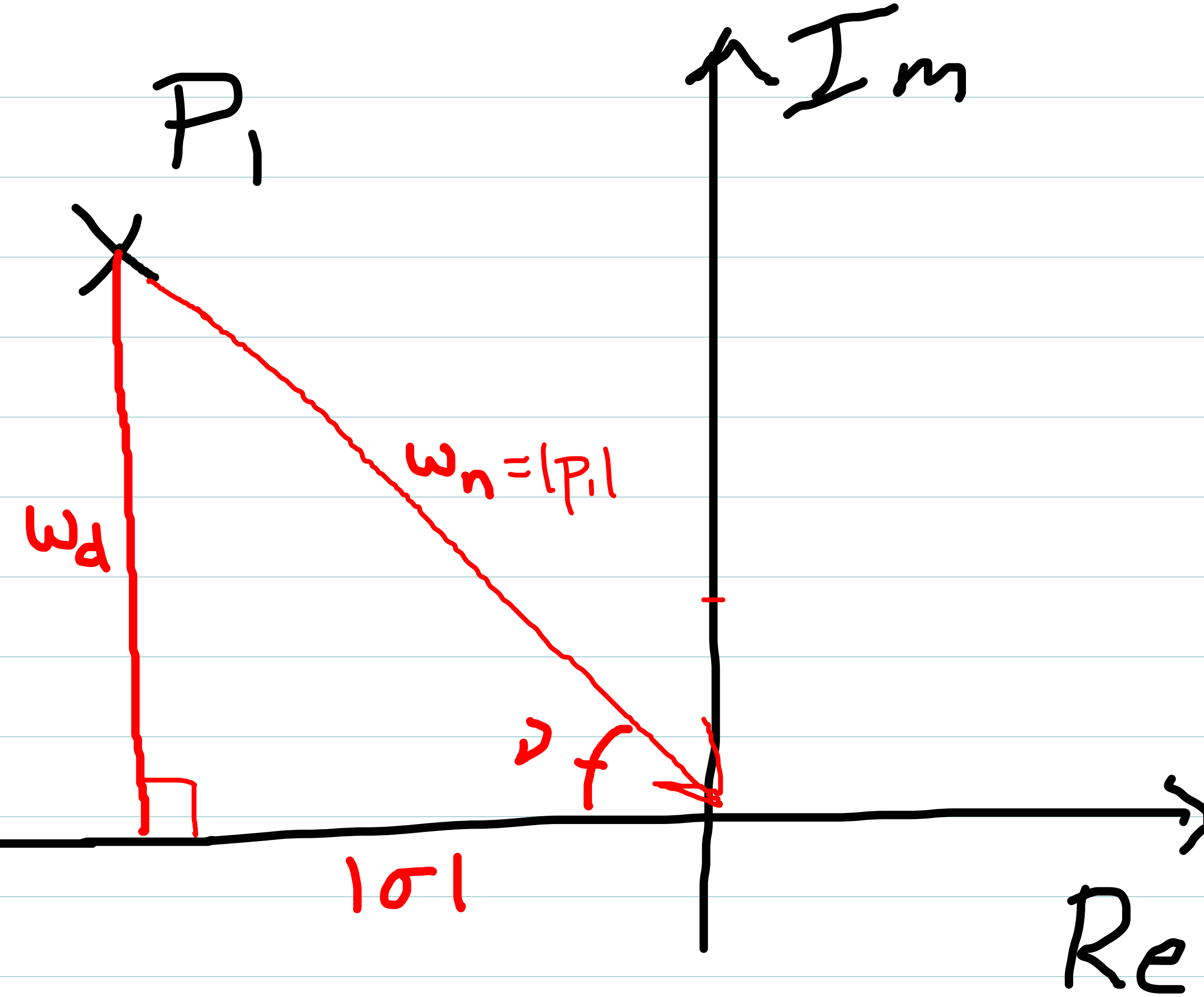
"Damping ratio"

$\Rightarrow$ A normalized measure of the number of transient oscillations observed before amplitude becomes negligible

Note:

$\phi \le \nu \le \pi/2$

So: $\phi \le \zeta \le 1$



$$\nu = \tan^{-1}\left(\frac{\omega_d}{|\sigma|}\right)$$

$$\nu = \sin^{-1}\left(\frac{\omega_d}{\omega_n}\right)$$

$$\nu = \cos^{-1}\left(\frac{|\sigma|}{\omega_n}\right) = \cos^{-1}\zeta$$ $\Leftarrow$ <u>very useful!</u>

Thus finally, the Case 1 step response is:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{\omega_n}{\omega_d}\right) e^{\sigma t} \sin(\omega_d t + \cos^{-1}\zeta)\right]}$$

We can now solve for important transient parameters

$\Rightarrow$ t_c: Solve for first $t>0$ such that

$$y(t) = y_{ss}(t) = G(0)$$

$$\Rightarrow \sin(\omega_d t + \cos^{-1}\zeta) = 0$$

$$\Rightarrow t_c = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$$

or:

$$\boxed{t_c = \frac{\pi - \rho}{\omega_d}}$$

$\Rightarrow$ For t_p, y_p

Solve for first $t>0$ such that

$$\dot{y}(t) = 0$$

$$\Rightarrow \boxed{t_p = \frac{\pi}{\omega_d}}$$

Substituting:

$$y_p = y(t_p) = G(0)\left[1 + e^{(\sigma\pi/\omega_d)}\right]$$

<u>Define</u>:

$$\boxed{\begin{aligned} M_p &= e^{(\sigma\pi/\omega_d)} \\ y_p &= G(0)\left[1 + M_p\right] \end{aligned}}$$

then:

Peak Overshoot

⇒ M_p is the Normalized peak overshoot

$$y_p = G(0)[1+M_p] \Longrightarrow M_p = \frac{y_p - G(0)}{G(0)} = \frac{y_p - y_{ss}}{y_{ss}}$$

⇒ M_p is entirely determined by damping ratio ξ

$$M_p = \exp\left[\frac{\sigma\pi}{\omega_d}\right]$$

$$= \exp\left[\frac{(-\xi\omega_n)\pi}{\omega_n\sqrt{1-\xi^2}}\right]$$

OR

$$\boxed{M_p = \exp\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right]}$$

$$\%OS = 100 \times M_p$$

Summary: Case I step response; $p_1 = \sigma + j\omega_d$

"Natural" frequency: $\omega_n = \sqrt{\sigma^2 + \omega_d^2} = |p_1|$

Damping ratio: $\xi = \dfrac{|\sigma|}{\omega_n}$

1st crossing: $t_c = \dfrac{\pi - \cos^{-1}\xi}{\omega_d} = \dfrac{\pi - \nu}{\omega_d}$, $\xi = \cos\nu$

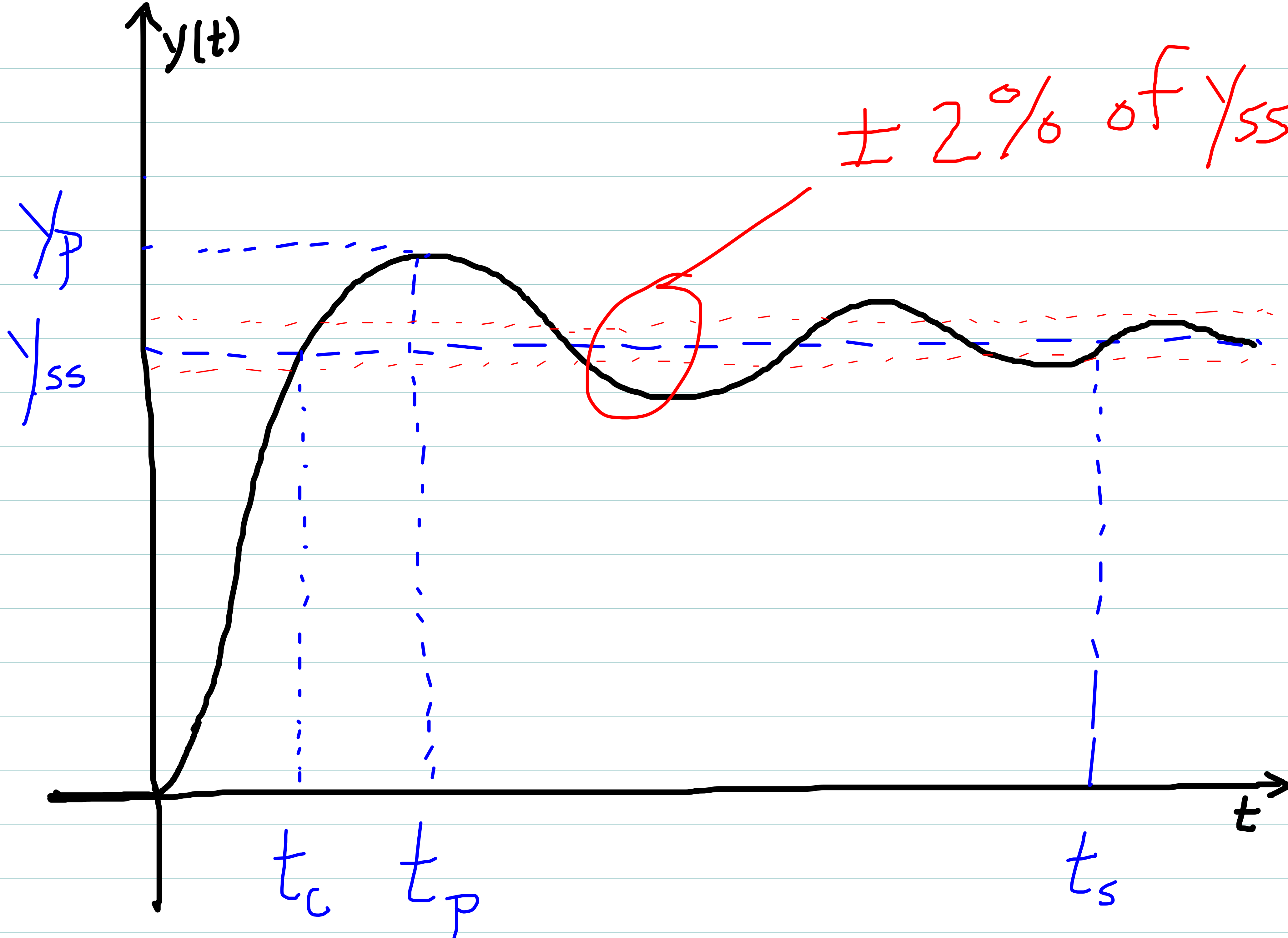
1st peak: $t_p = \dfrac{\pi}{\omega_d}$

Normalized overshoot: $M_p = \exp\left[\dfrac{\sigma\pi}{\omega_d}\right] = \exp\left[\dfrac{-\xi\pi}{\sqrt{1-\xi^2}}\right]$

$$M_p = \left[\frac{y_p - y_{ss}}{y_{ss}}\right]$$

Peak response: $y_p = y_{ss}[1 + M_p]$

$y_{ss} = G(\emptyset)$ for unit step



y(t)
y_p
y_{ss}
± 2% of y_{ss}
t
t_c
t_p
t_s

Settling Time

As usual, we can use the approximation

$$\boxed{t_s \approx \frac{4}{|Re\{p_1\}|} = \frac{4}{|\sigma|}}$$

But t_s is actually a function of ξ also here:

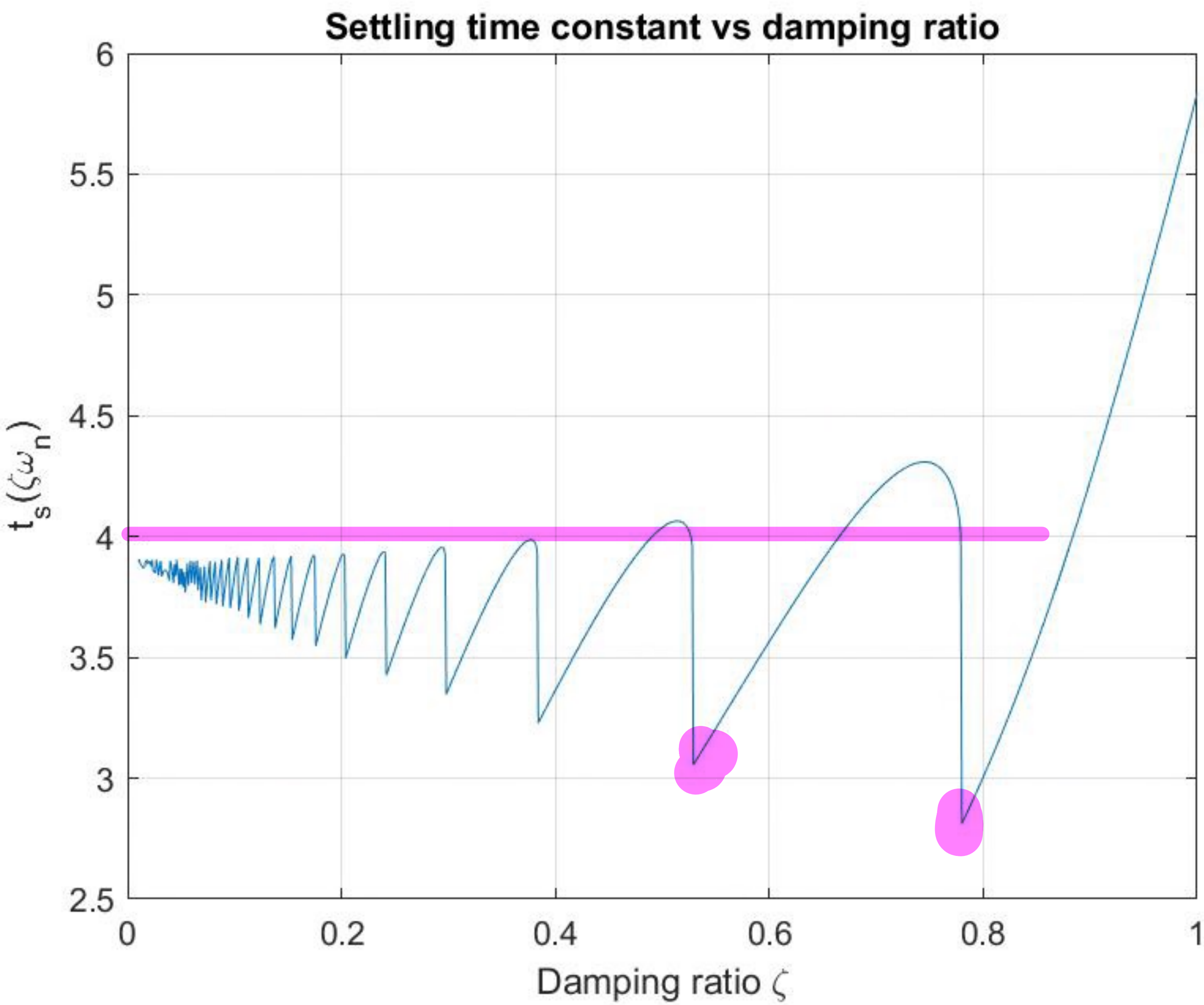
$$t_s = \frac{C(\xi)}{|\sigma|}$$

with $3 \leq C(\xi) \leq 5$ for most $0 \leq \xi < 0.9$

so 4 is an "average" value for $C(\xi)$

However for $0.95 \leq \xi \leq 1$ a better approximation is: $t_s \approx \dfrac{6}{|\sigma|}$

Complex Poles - $C(\xi)$ $\quad t_s = \frac{C(\xi)}{\xi \omega_n}$



A few more observations

$$\zeta = \frac{|\sigma|}{\omega_n} \implies \boxed{\sigma = -\zeta\omega_n}$$ (Stable system Assumed)

$$\omega_n = \sqrt{\sigma^2 + \omega_d^2}$$

$$\implies \omega_d^2 = \omega_n^2 - \sigma^2 = \omega_n^2 - (-\zeta\omega_n)^2 = \omega_n^2(1-\zeta^2)$$

so: $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$

Then note:

$$\boxed{\begin{aligned} s^2 + \alpha_1 s + \alpha_0 &= (s-p)(s-\bar{p}) \\ &= s^2 - 2\sigma s + (\sigma^2 + \omega_d^2) \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \end{aligned}}$$

all equivalent

Note:

The three possible cases for 2^{nd} order step responses can be categorized by ξ:

Case 1 (complex poles): $0 \leq \xi < 1$

$$\alpha_1^2 < 4\alpha_0 \Rightarrow (2\xi\omega_n)^2 < 4\omega_n^2 \quad ✓$$

Case 2 (repeated real poles): $\xi = 1$

$$\alpha_1^2 = 4\alpha_0 \Rightarrow (2\cancelto{1}{\xi}\omega_n)^2 = 4\omega_n^2 \quad ✓$$

Case 3 (distinct real poles): $\xi > 1$

$$\alpha_1^2 > 4\alpha_0 \Rightarrow 4\xi^2\omega_n^2 > 4\omega_n^2 \quad ✓$$

Note:

The three possible cases for 2nd order step responses can be categorized by ξ:

Case 1 (complex poles): $0 \le \xi < 1$ "underdamped"

$$\alpha_1^2 < 4\alpha_0 \Rightarrow (2\xi\omega_n)^2 < 4\omega_n^2 \checkmark$$

Case 2 (repeated real poles): $\xi = 1$ "critically damped"

$$\alpha_1^2 = 4\alpha_0 \Rightarrow (2\cancel{\xi}^{\,1}\omega_n)^2 = 4\omega_n^2 \checkmark$$

Case 3 (distinct real poles): $\xi > 1$ "overdamped"

$$\alpha_1^2 > 4\alpha_0 \Rightarrow 4\xi^2\omega_n^2 > 4\omega_n^2 \checkmark$$

Limiting case: $\xi \to \emptyset$

$\xi \to \emptyset \Rightarrow \sigma = -\xi\omega_n \to \emptyset \Rightarrow P_1 = j\omega_d$ (pure imaginary)

Overshoot $M_p = e^{(\sigma\pi/\omega_d)} \to 1$ (100% OS)

Peak: $y_p = G(\emptyset)[1 + M_p]$

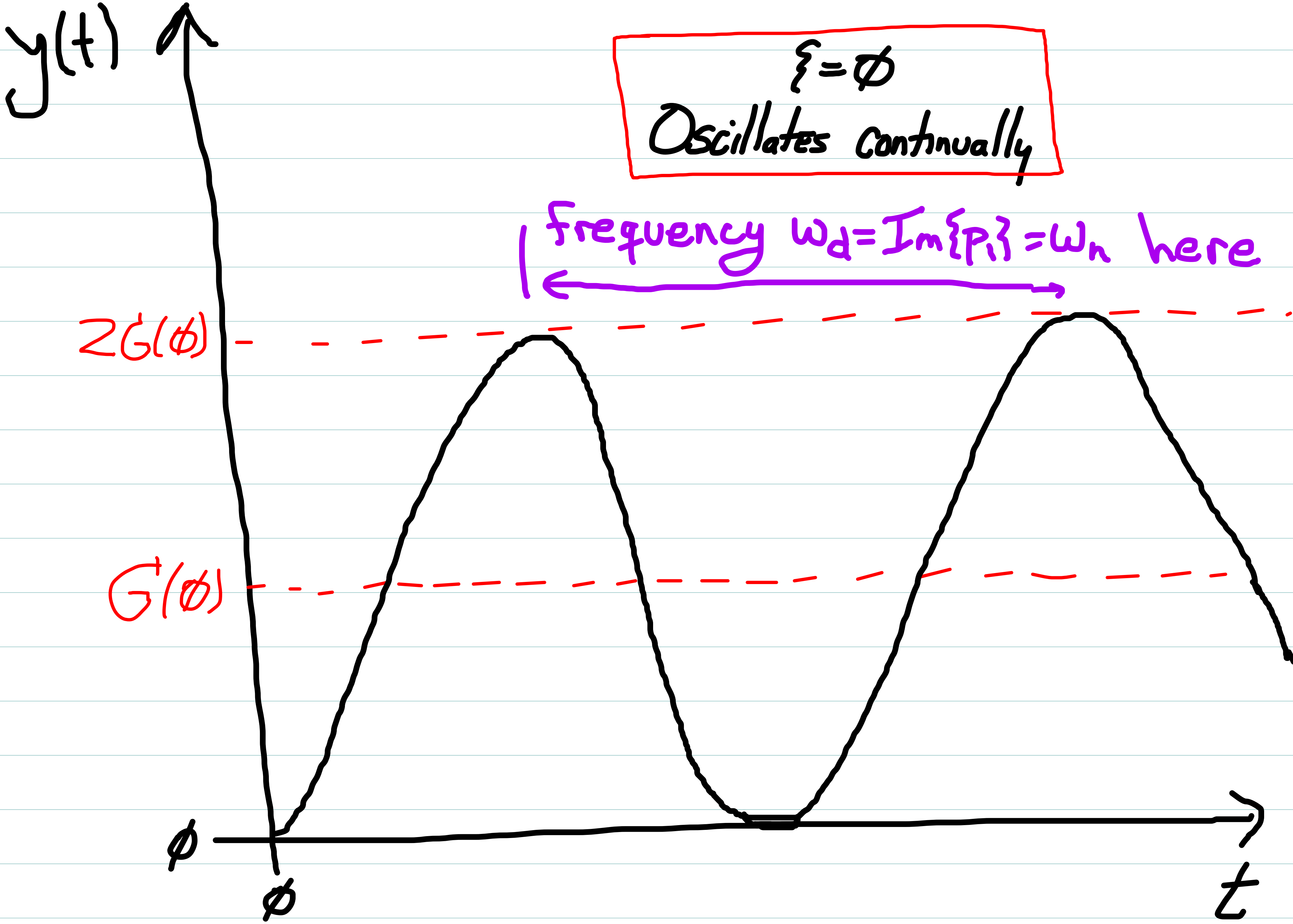
or $\quad y_p = 2y_{ss}$

Settling time: $\quad t_s \approx \dfrac{4}{|\sigma|} = \infty$

Never settles!

Response oscillates infinitely between $\emptyset$ and $2G(\emptyset)$ with frequency $\omega_d = \omega_n\sqrt{1-\xi^2} = \omega_n$

"Undamped"



y(t)
ξ = ∅
Oscillates continually
frequency ω_d = Im{p_i} = ω_n here
2G(∅)
G(∅)
∅
∅
t

Limiting Case, $\zeta \to 1$

$\zeta \to 1 \Rightarrow \quad \sigma = -\zeta\omega_n \to -\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \to \phi$$

Response does not oscillate!

Overshoot:

$$M_p = e^{(\sigma\pi/\omega_d)} = e^{-\omega_n\pi/\phi} = \phi$$

No overshoot

1st crossing:

$$t_c = \frac{\pi - \cos^{-1}\zeta}{\omega_d} = {}^{\pi/2}/_{\phi} = \infty$$

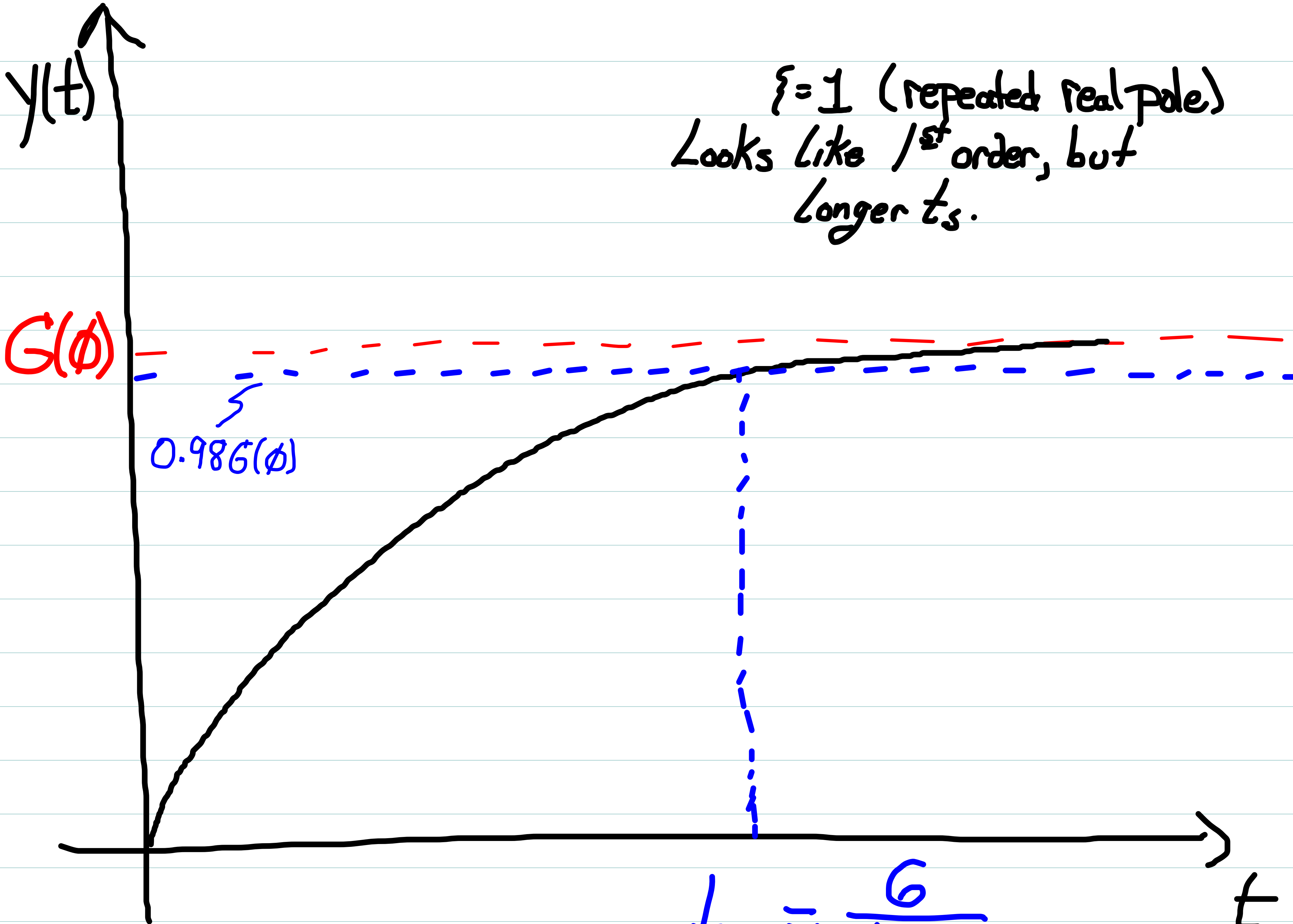
$\Rightarrow$ response asymptotes to y_{ss} from below

Settling:

$$t_s \approx \frac{6}{|\sigma|}$$

use 6 here

$\zeta = 1$ (repeated real pole)
Looks like 1^{st} order, but longer t_s.



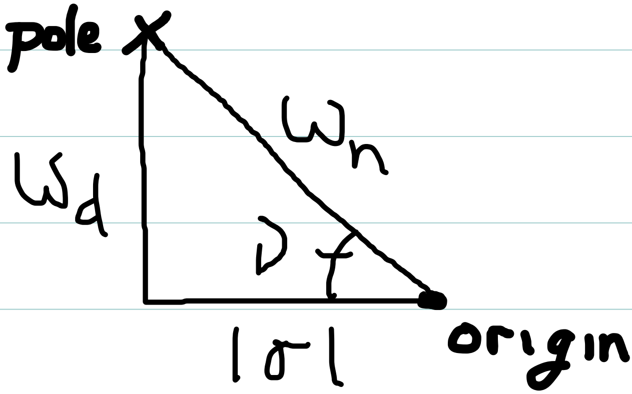
$$t_s = \frac{6}{|\sigma|}$$

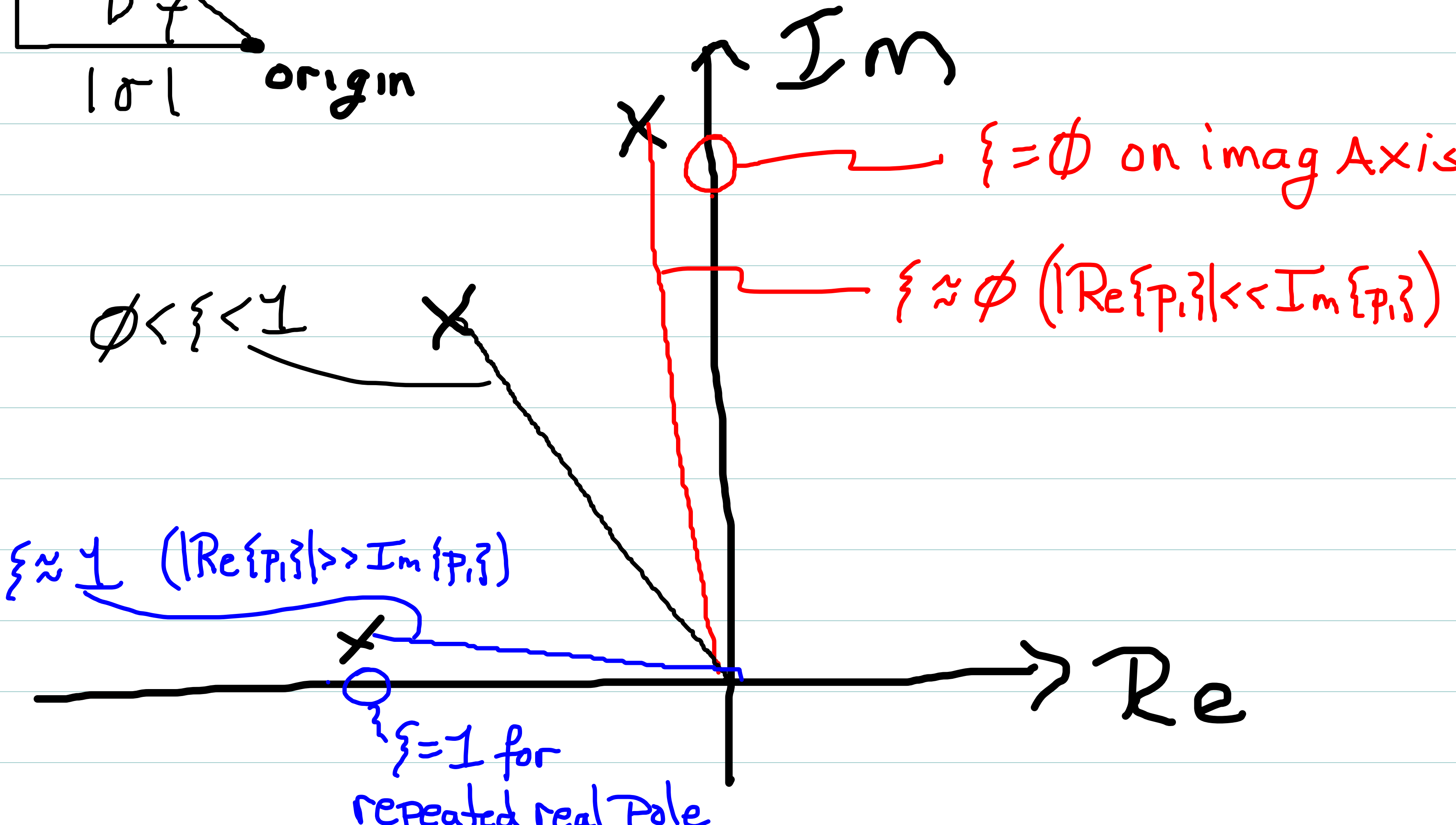
Graphical Interpretation of ζ:

$\boxed{\zeta = \cos\nu}$:

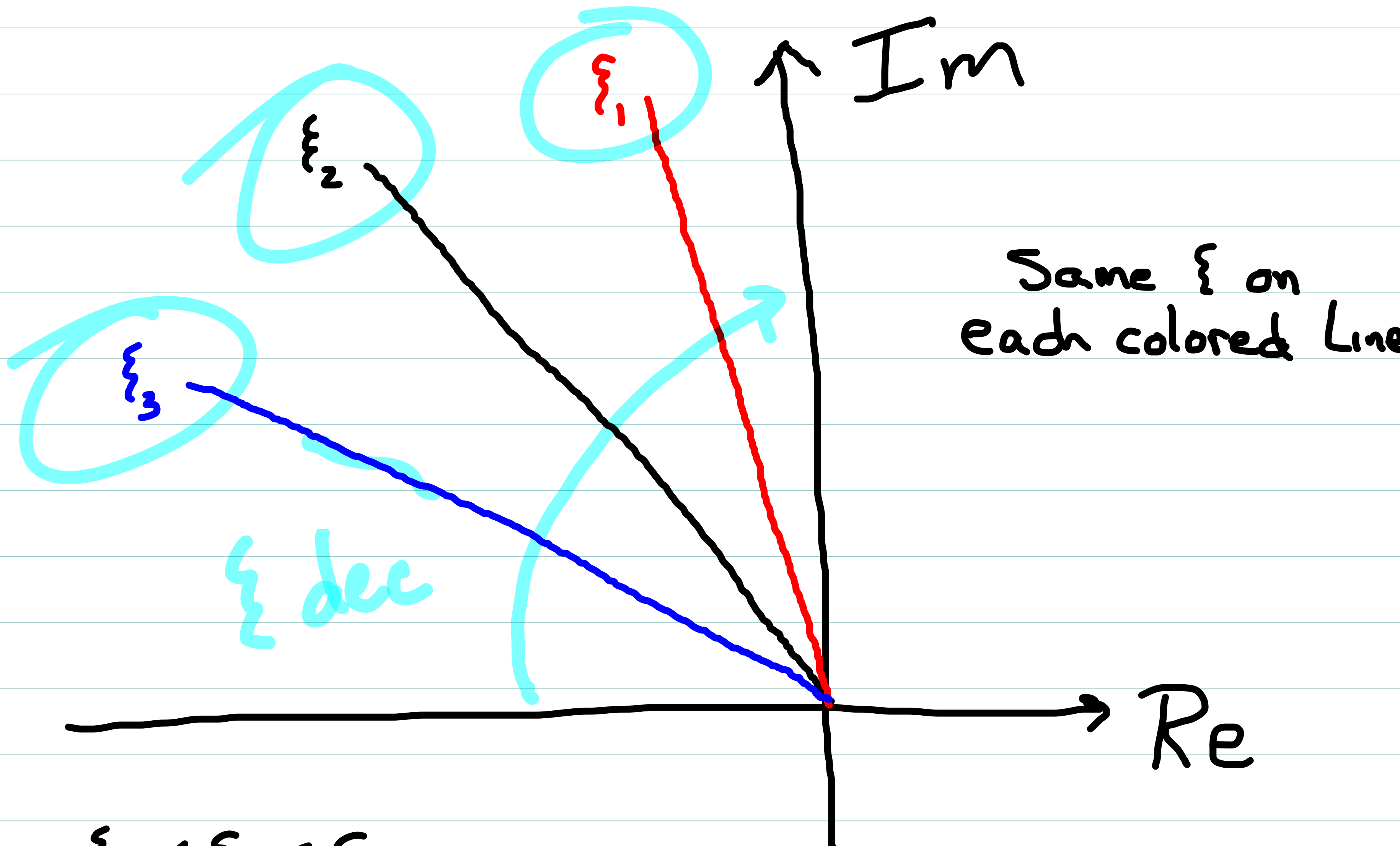
$$\zeta \to 0 \implies \nu \to \pi/2$$

$$\zeta \to 1 \implies \nu \to 0$$





Lines of constant ξ lie on rays in upper left quadrant of complex plane:



$\xi_1 < \xi_2 < \xi_3$

=> 1^{st} and 2^{nd} order step responses are "building blocks" by which we can understand response of more complex systems

=> each real pole introduces a new decaying exponential into transient response.

=> each complex pole pair introduces a decaying oscillation into the transient

=> An arbitrary number of poles of different types will typically require numerical simulation to quantify y_p, t_r, t_p, t_s

=> <u>However</u> in some cases we can still accurately predict these features.

Suppose:

$$G(s) = \frac{K}{(s-p_1)(s^2+2\zeta\omega_n s+\omega_n^2)} \quad \text{with } \zeta<1$$

$$= \frac{K}{(s-p_1)(s-p_2)(s-\bar{p}_2)}$$

for a unit step input $u(t) = \mathbb{1}(t)$ we know

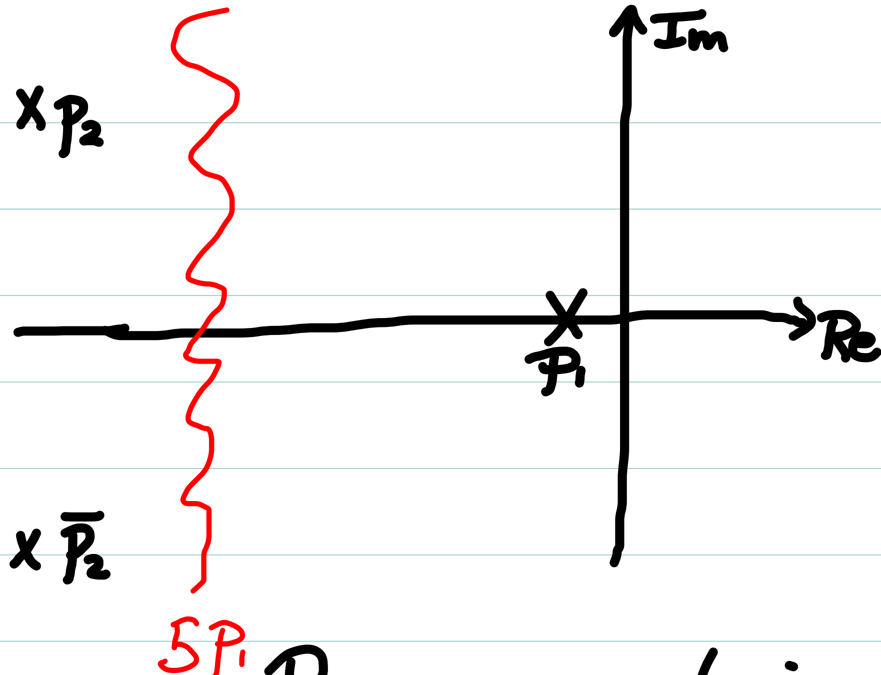
$$y_{ss} = G(0) = \frac{K}{-\omega_n^2 p_1}$$

But what can we say about y_p, t_p, t_c, t_s?

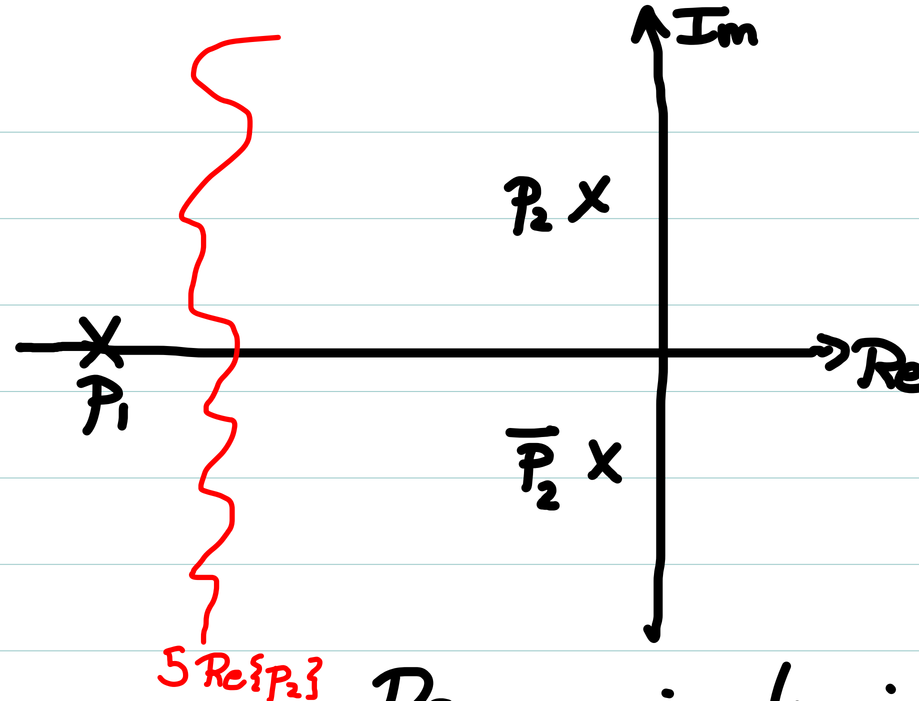
In general, not much unless either

$$|p_1| > 5|Re\{p_2\}| \text{ or } |Re\{p_2\}| > 5|p_1|$$

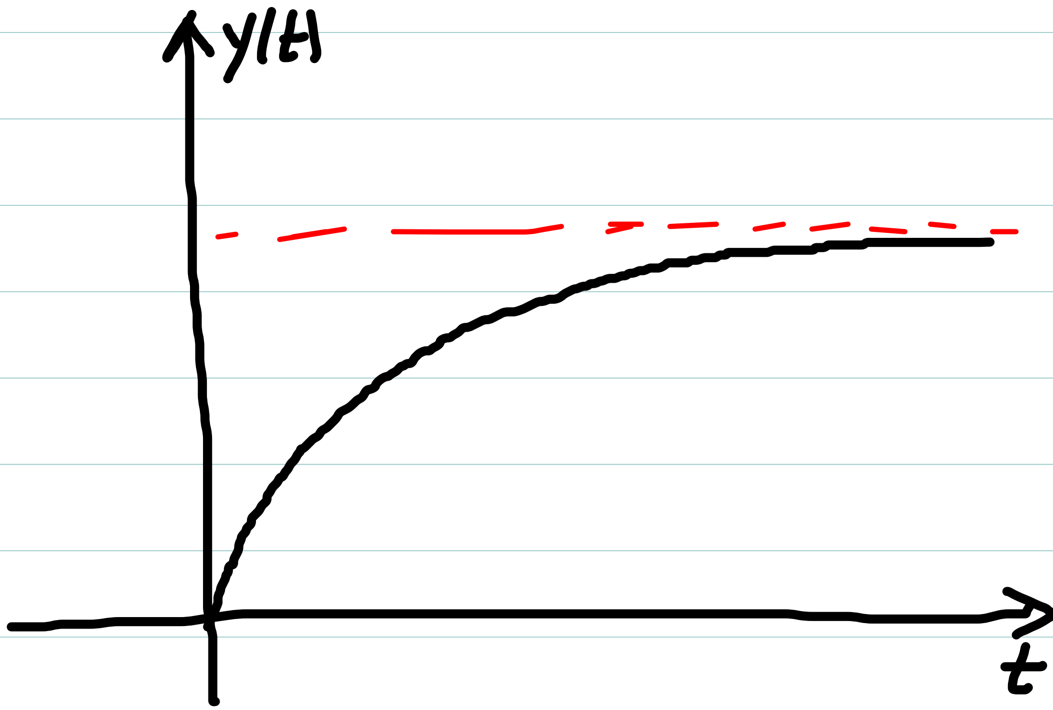
i.e. if one of the modes is <u>dominant</u>.



Response is dominantly 1^{st} order



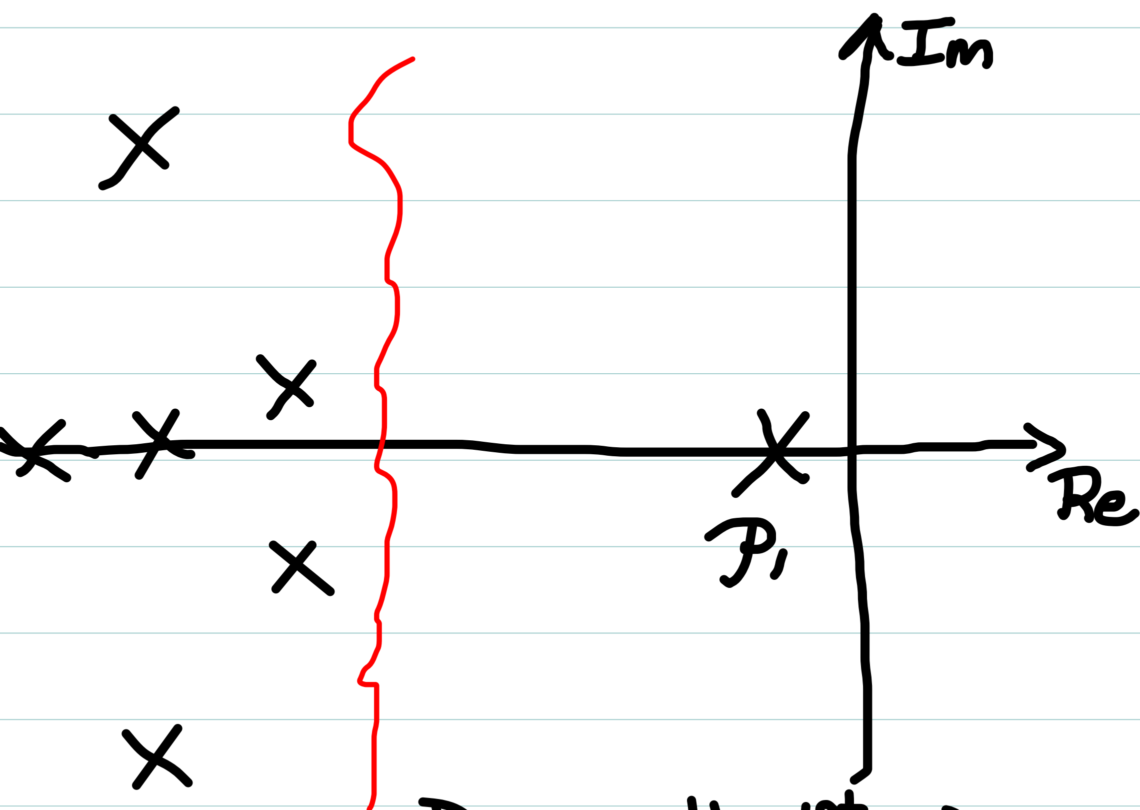
Response is dominant 2^{nd} order



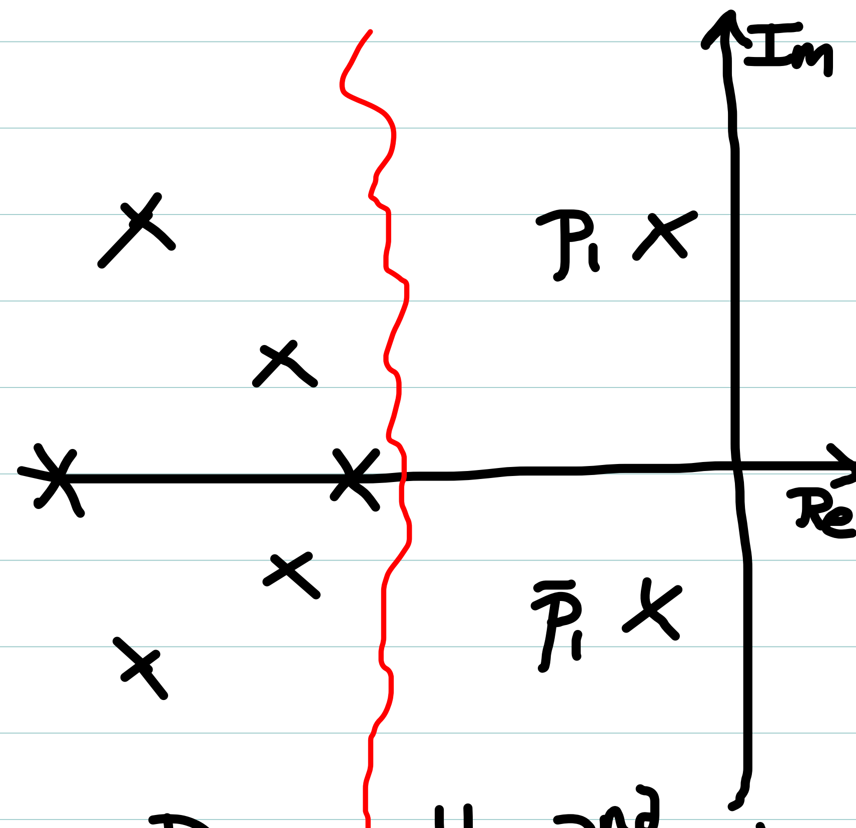
Dominant modes revisited

When a single mode is dominant, we can approximate the features of the response using just that mode

An arbitrarily complex system can be well approximated in this fashion.



Dominantly 1st order



Dominantly 2nd order

Effect of Zeros

Step response of

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Zero at
$z_1 = -\beta_0/\beta_1$

3 important effects:

1. "Input absorbing" property
2. Transient suppression
3. Transient amplification

Both?
Yes!

Depending on system

(I.) Input absorbtion

For unit step response of stable system

$$y_{ss}(t) = G(\emptyset)$$

Suppose $z_1 = -\beta_0/\beta_1 = \emptyset \implies \beta_0 = \emptyset$

zero at origin

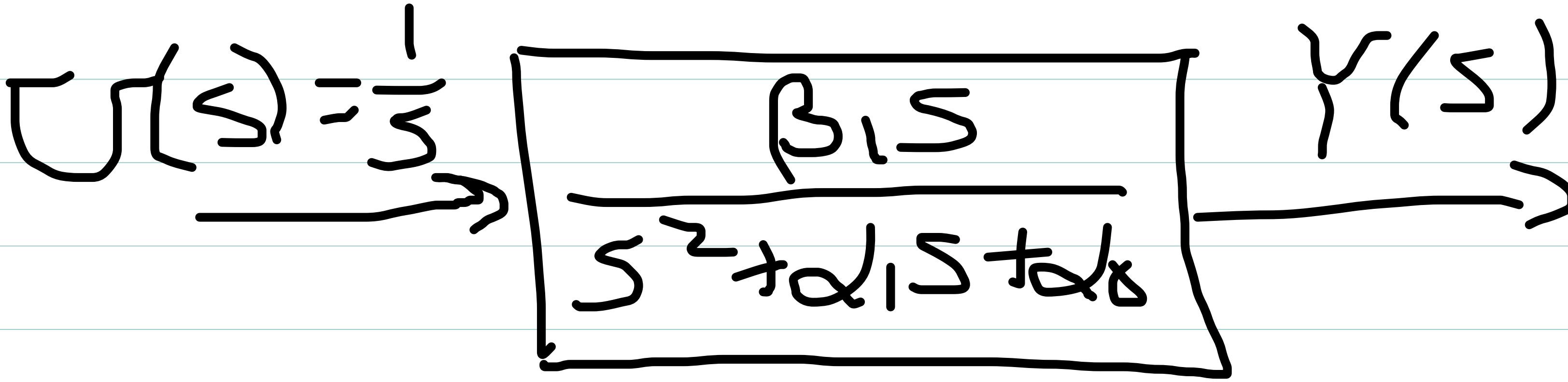
$$G(s) = \frac{\beta_1 s}{s^2 + \alpha_1 s + \alpha_0}$$

Then $y_{ss}(t) = G(\emptyset) = \emptyset \Leftarrow$ steady-state is zero

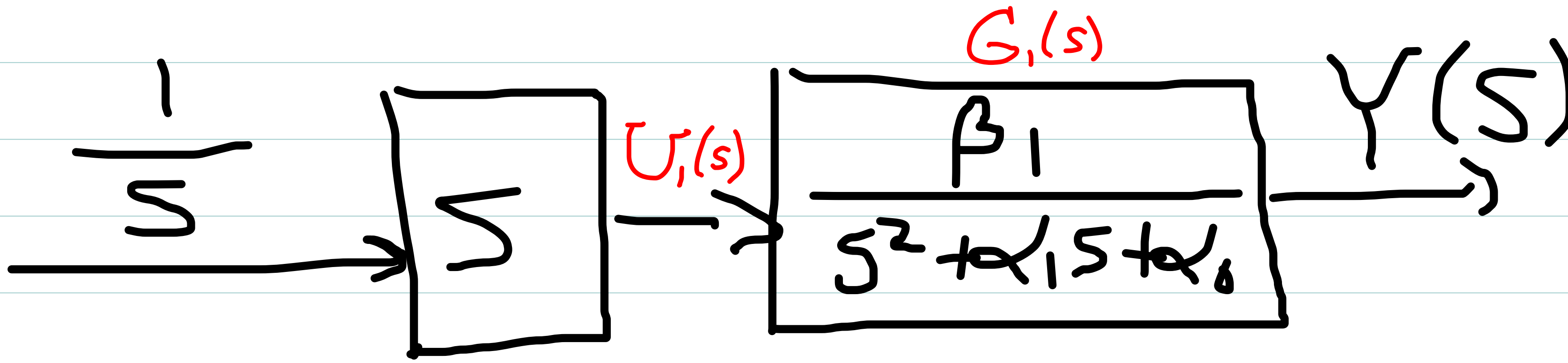
response contains only transient terms

In fact, $y(t)$ is the impulse response of

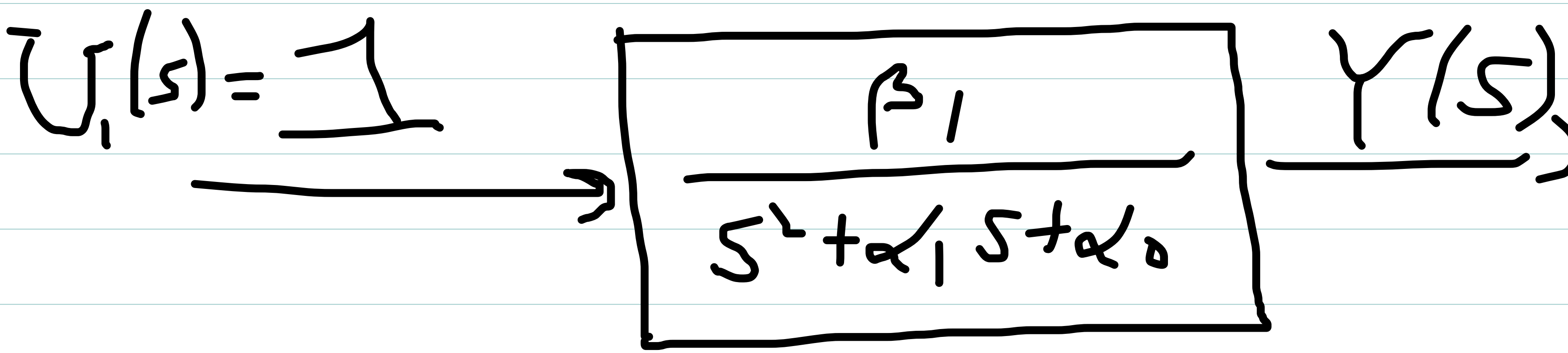
$$G_1(s) = \frac{\beta_1}{s^2 + \alpha_1 s + \alpha_0}$$



$U(s) = \frac{1}{s}$
$\frac{\beta_1 s}{s^2 + \alpha_1 s + \alpha_0}$
$Y(s)$



$\frac{1}{s}$
s
$U_1(s)$
$G_1(s)$
$\frac{\beta_1}{s^2 + \alpha_1 s + \alpha_0}$
$Y(s)$



$U_1(s) = 1$
$\frac{\beta_1}{s^2 + \alpha_1 s + \alpha_0}$
$Y(s)$

=> 1^{st} and 2^{nd} order step responses are "building blocks" by which we can understand response of more complex systems

=> each real pole introduces a new decaying exponential into transient response.

=> each complex pole pair introduces a decaying oscillation into the transient

=> An arbitrary number of poles of different types will typically require numerical simulation to quantify y_p, t_r, t_p, t_s

=> <u>However</u> in some cases we can still accurately predict these features.

Suppose:

$$G(s) = \frac{K}{(s-p_1)(s^2+2\zeta\omega_n s+\omega_n^2)} \quad \text{with } \zeta<1$$

$$= \frac{K}{(s-p_1)(s-p_2)(s-\bar{p}_2)}$$

for a unit step input $u(t) = \mathbb{1}(t)$ we know

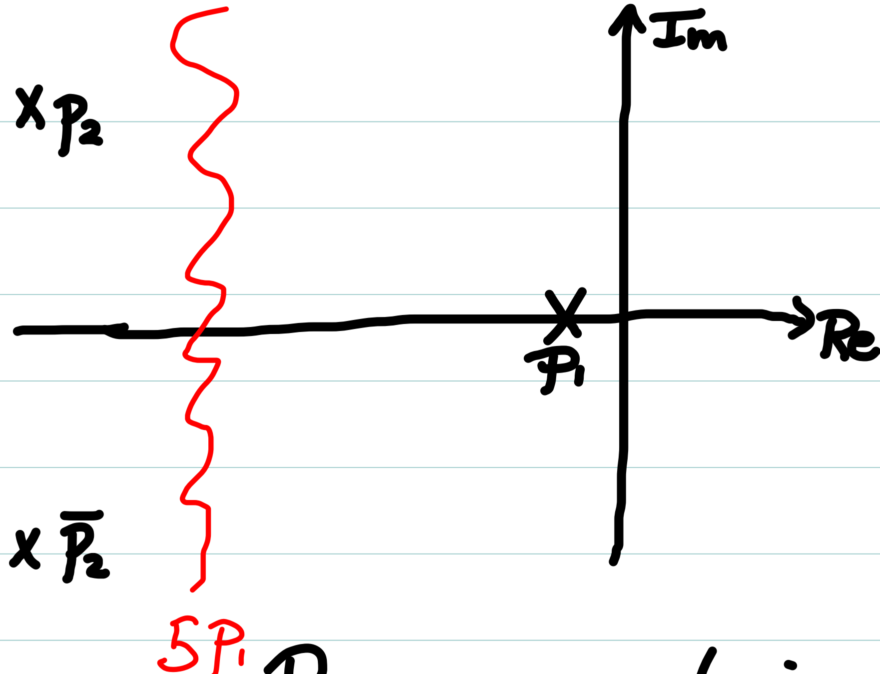
$$y_{ss} = G(0) = \frac{K}{-\omega_n^2 p_1}$$

But what can we say about y_p, t_p, t_c, t_s?

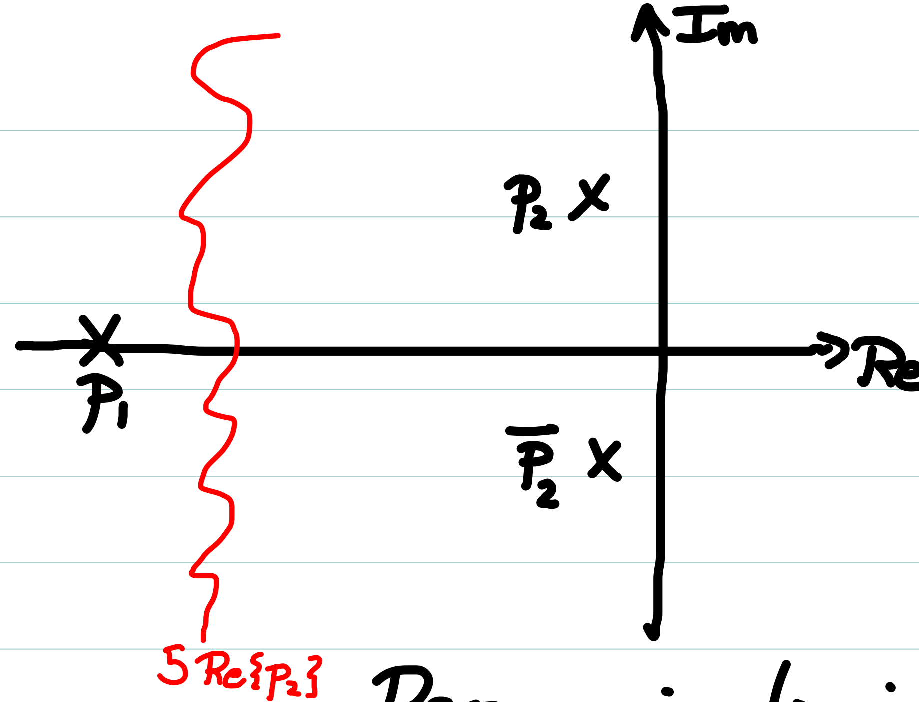
In general, not much unless either

$$|p_1| > 5|Re\{p_2\}| \text{ or } |Re\{p_2\}| > 5|p_1|$$

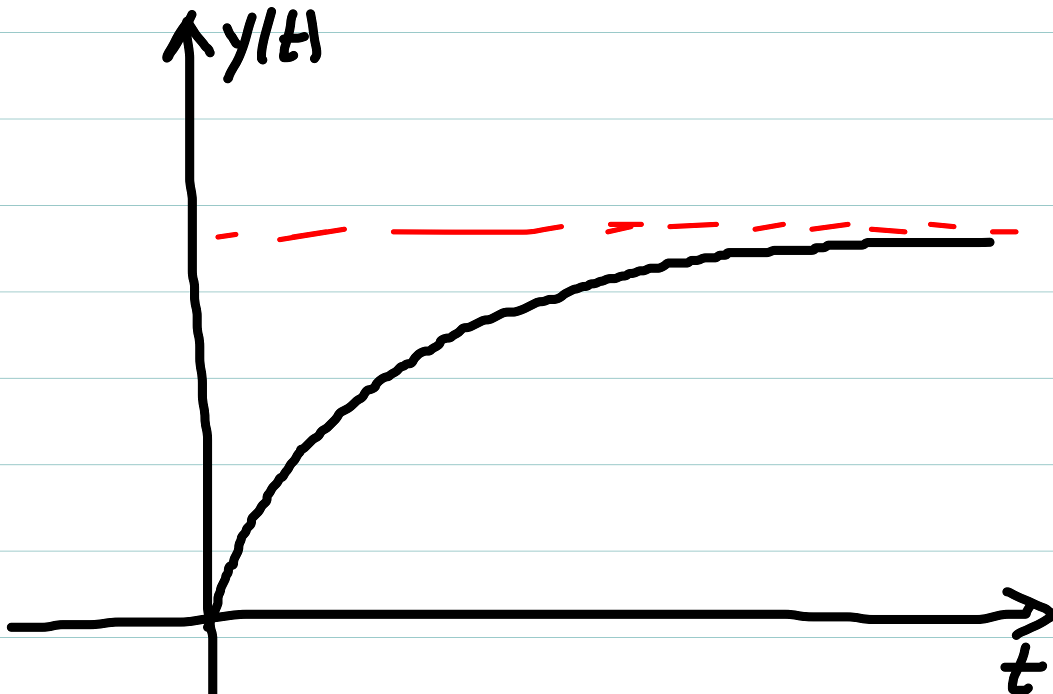
i.e. if one of the modes is <u>dominant</u>.



Response is dominantly 1^{st} order



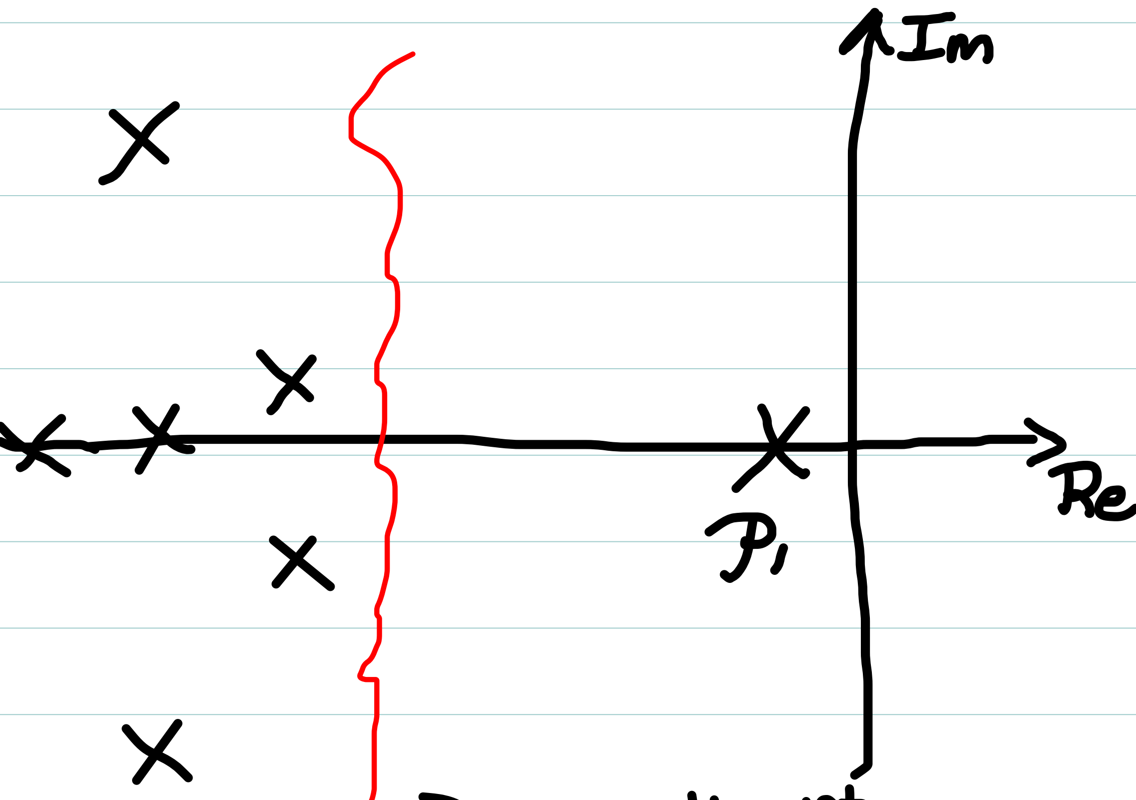
Response is dominant 2^{nd} order



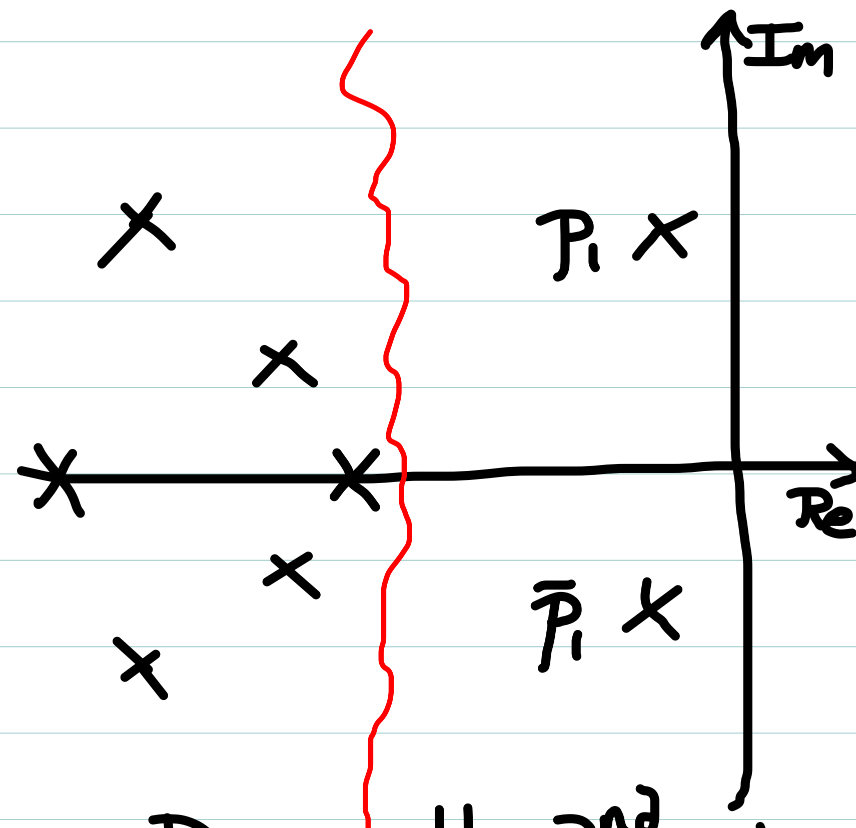
Dominant modes revisited

When a single mode is dominant, we can approximate the features of the response using just that mode

An arbitrarily complex system can be well approximated in this fashion.



Dominantly 1st order



Dominantly 2nd order

Effect of Zeros

Step response of

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Zero at $z_1 = -\beta_0/\beta_1$

3 important effects:

1. "Input absorbing" property
2. Transient suppression
3. Transient amplification

Both? Yes!

Depending on system

(I) Input absorbtion

For <u>unit</u> step response of stable system

$$y_{ss}(t) = G(\phi)$$

Suppose $z_1 = {}^{-\beta_0}/_{\beta_1} = \phi \implies \beta_0 = \phi$

$$G(s) = \frac{\beta_1 s}{s^2 + \alpha_1 s + \alpha_0}$$

zero at origin

Then $y_{ss}(t) = G(\phi) = \phi \Leftarrow$ steady-state is zero response contains only transient terms

In fact, $y(t)$ is the <u>impulse</u> response of

$$G_1(s) = \frac{\beta_1}{s^2 + \alpha_1 s + \alpha_0}$$

Effect of Zeros

Step response of

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Zero at $z_1 = -\beta_0 / \beta_1$

3 important effects:

1. "Input absorbing" property
2. Transient suppression
3. Transient amplification

Both? Yes!

Depending on system

② Transient Suppression

Suppose $s^2 + \alpha_1 s + \alpha_0 = (s-p_1)(s-p_2)$ p_1, p_2 real

So

$$G(s) = \frac{\beta_1 (s - z_1)}{(s-p_1)(s-p_2)}$$

Suppose $z_1 \approx p_1$, i.e. $|z_1 - p_1| = \varepsilon \ll 1$

We know $y(t) = G(\phi) + A_1 e^{p_1 t} + A_2 e^{p_2 t}$

where $A_1 = \left[(s-p_1)Y(s)\right]_{s=p_1} = \dfrac{\beta_1 (p_1 - z_1)}{p_1 (p_1 - p_2)}$ is small

$\sim \ll 1$

so, for sufficiently small ε, the $e^{p_1 t}$ term in transient is negligible, and response is equivalent to a <u>1^{st} order</u> system with single pole p_2

Pole-zero Cancellation

Algebraically, if $z_1 \approx p_1$

$$G(s) = \frac{\beta_1 \cancel{(s-z_1)}}{\cancel{(s-p_1)}(s-p_2)} \approx \frac{\beta_1}{(s-p_2)}$$

Usually, if

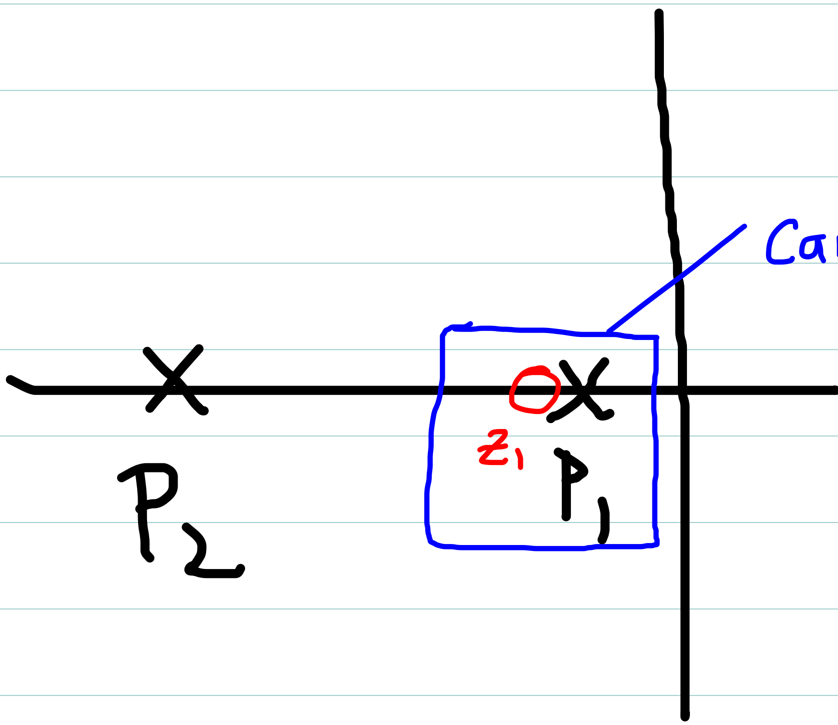
$$0.9 \leq \left|\frac{z_1}{p_1}\right| \leq 1.1$$

i.e. zero location within 10% of pole location

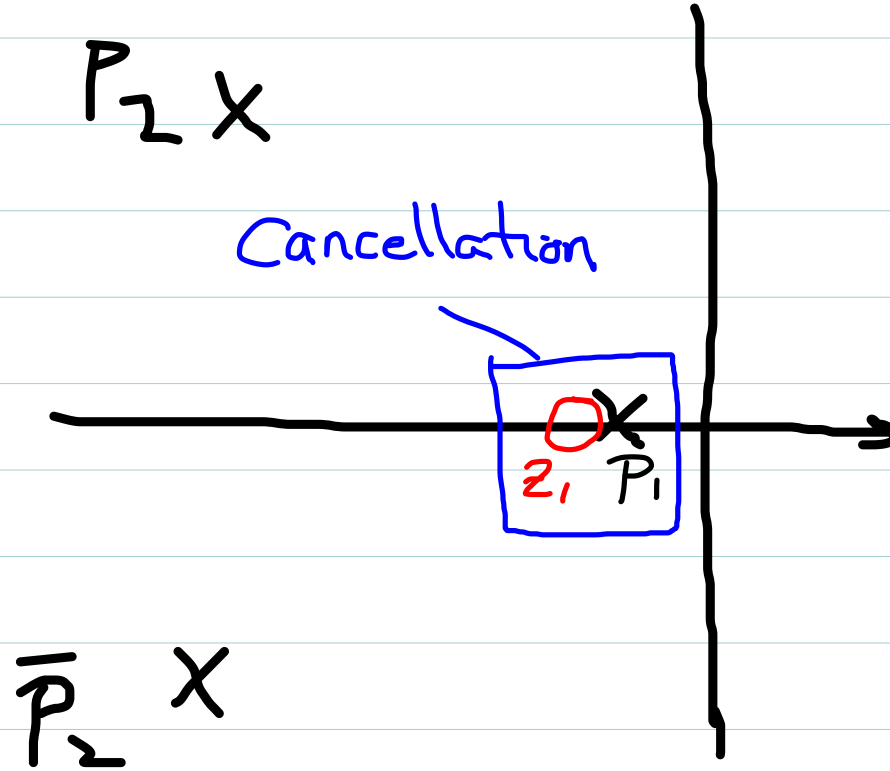
this is a good approximation

Cancellation and Dominance

Pole-zero cancellations can change dominance calculation



"fast" pole becomes dominant



2^{nd} order poles become dominant

Cancellation is never exact!

$\Rightarrow$ z_1, p_1 come from <u>different</u> coefs. in diff'l eq'n.

$\Rightarrow$ These coefs come from physical properties of system whose values are not known precisely.

$\Rightarrow$ Cancellation should always be considered <u>approx</u>.

$\Rightarrow$ If p_1 is stable, it is a good approximation to cancel it

$$A_1 e^{p_1 t} \propto \varepsilon e^{p_1 t}$$

This term starts small, and gets smaller as t increases

But

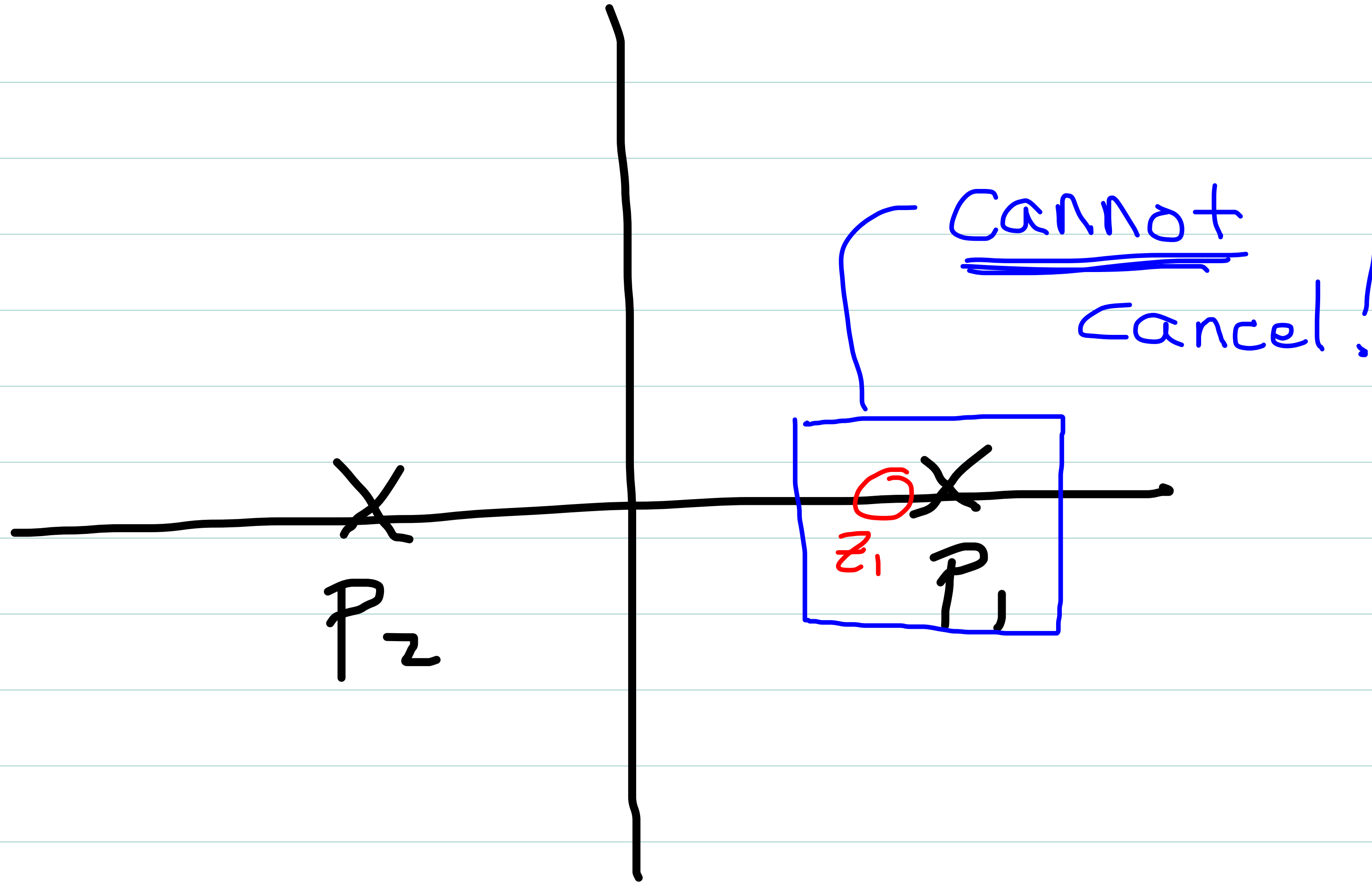
Suppose p_1 Not stable: $p_1 > 0$

then $A_1 e^{p_1 t} \propto \varepsilon e^{p_1 t}$

may start small, but increases w/o bound as t increases

Term will diverge to ∞, regardless how small ε is!

$$\boxed{\text{Pole-zero cancellation can } \underline{\text{Never}} \text{ be performed in RHP}}$$


Cannot
cancel!
z_1
P_1
P_2

Moreover...

Generally, if ICs on y(t) are not all zero

$$Y(s) = G(s)U(s) + \frac{C(s)}{r(s)}$$

Nonzero

Will contribute terms to y(t) which contain unstable mode <u>even if</u> this mode "cancels" in G(s)

<u>Moral</u>: Can <u>never</u> "cancel" an unstable mode

! ! !

Effects of zeros on step response

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}, \quad \text{zero at } z_1 = \frac{-\beta_0}{\beta_1}$$

① Input absorbtion (if $\beta_0 = \emptyset \Rightarrow z_1 = \emptyset$)

② Transient suppression via pole-zero cancellation

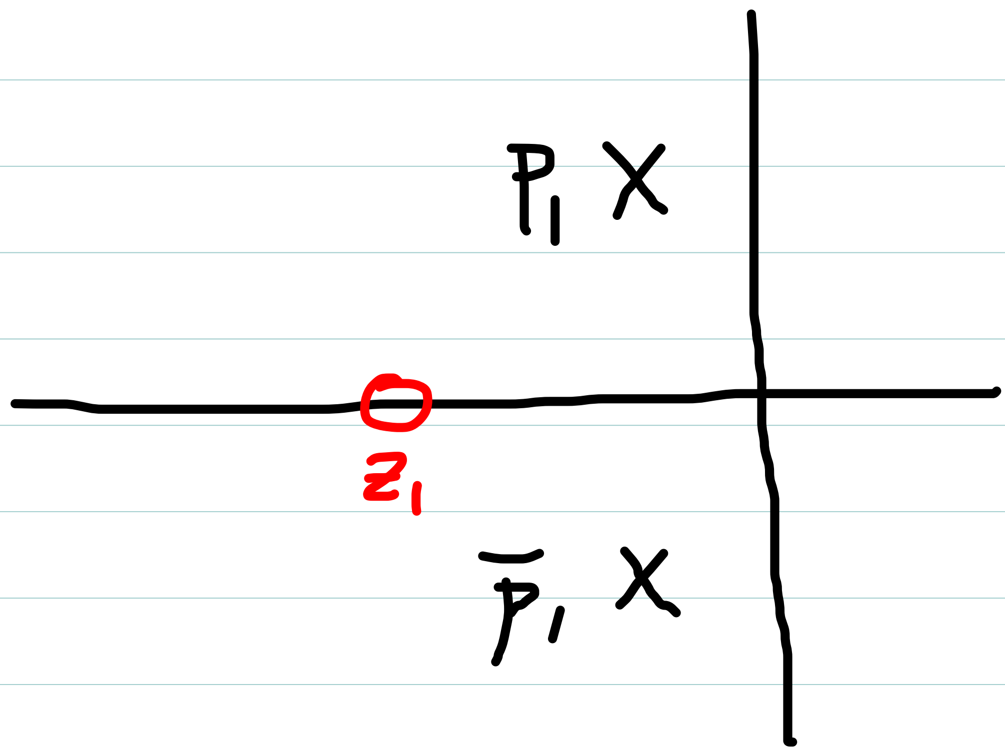
$\Rightarrow$ if $s^2 + \alpha_1 s + \alpha_0 = (s - p_1)(s - p_2)$; p_1, p_2 real
and $z_1 \approx p_1$ (or p_2)

③ Transient amplification $\Rightarrow$ examine this now.

③ Transient Amplification

Now suppose $s^2 + \alpha_1 s + \alpha_0 = (s - p_1)(s - \bar{p}_1)$

$p_1 = \sigma + j\omega_d, \quad \omega_d \neq 0$



Pole-zero cancellation cannot occur here

What <u>is</u> the effect of the zero?

$$Y(s) = \frac{\beta_1 s + \beta_0}{s(s-p_1)(s-\bar{p}_1)} = \frac{\beta_1 s}{s(s-p_1)(s-\bar{p}_1)} + \frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)}$$

$$= \left[\left(\frac{\beta_1}{\beta_0}\right) s\right]\left[\frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)}\right] + \left[\frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)}\right]$$

Let

$$Y_1(s) = \left[\frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)}\right]$$

So

$$Y(s) = \left(\frac{\beta_1}{\beta_0}\right)\left[s\,Y_1(s)\right] + Y_1(s)$$

$$\Longrightarrow \boxed{y(t) = \left(\frac{\beta_1}{\beta_0}\right)\dot{y}_1(t) + y_1(t)}, \quad y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\}$$

<u>Note:</u> $y_1(t)$ is ideal 2^{nd} order step response

$$y(t) = \left(\frac{\beta_1}{\beta_0}\right)\dot{y}_1(t) + y_1(t)$$

or equivalently:

$$\boxed{y(t) = y_1(t) - \left(\frac{1}{z_1}\right)\dot{y}_1(t)} \quad \left(z_1 = -\beta_0/\beta_1\right)$$

Where $y_1(t)$ is the "ideal" (no zero) step response

The total response $y(t)$ is the sum of the ideal response, and a fraction of the <u>derivative</u> of this response.

Suppose 1^{st} $z_1 < 0$ (LHP zero)

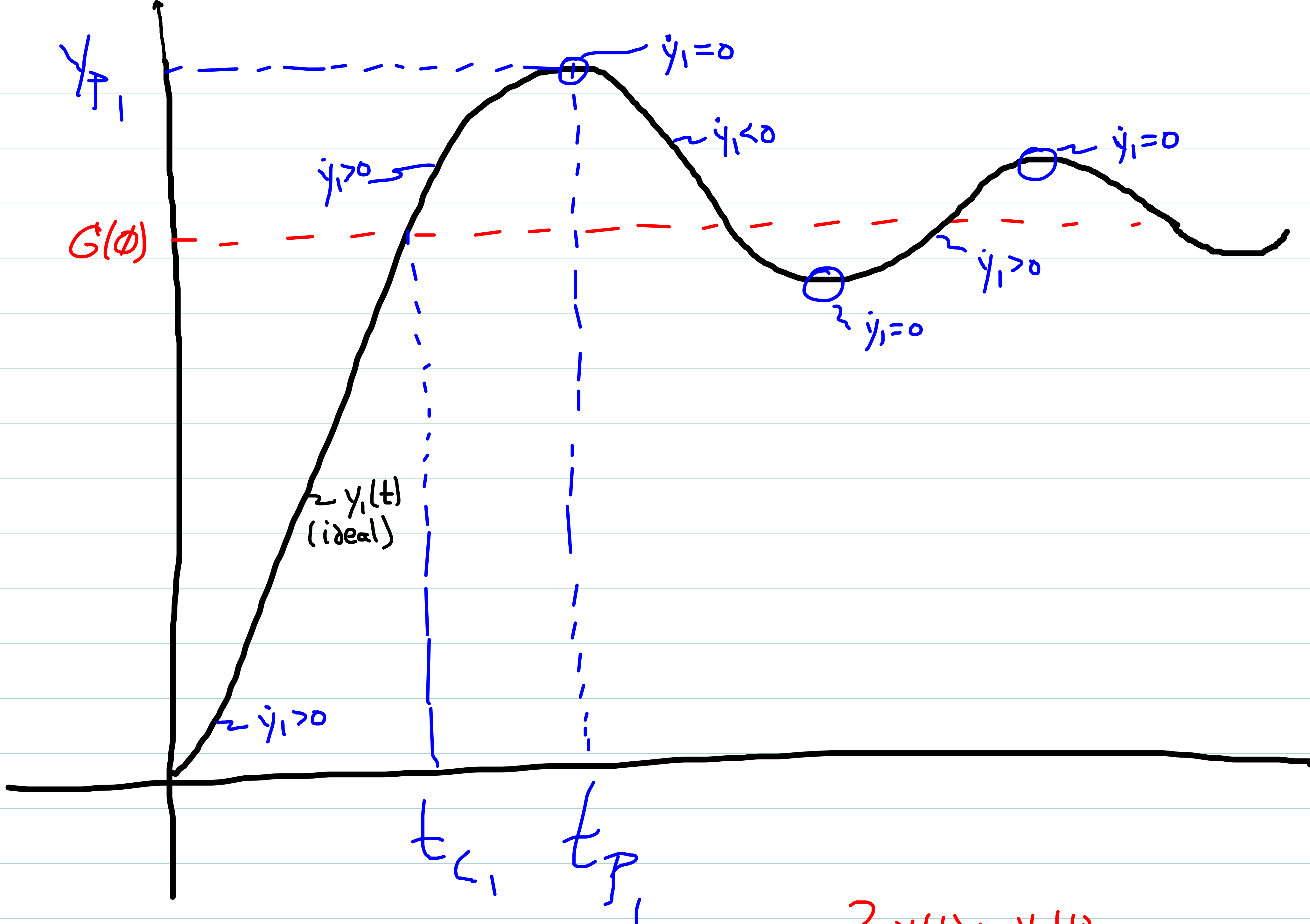
then $z_1 < 0$ and $\left(-\frac{1}{z_1}\right) > 0$ so we can write

$$y(t) = y_1(t) + \left(\frac{1}{|z_1|}\right) \dot{y}_1(t)$$

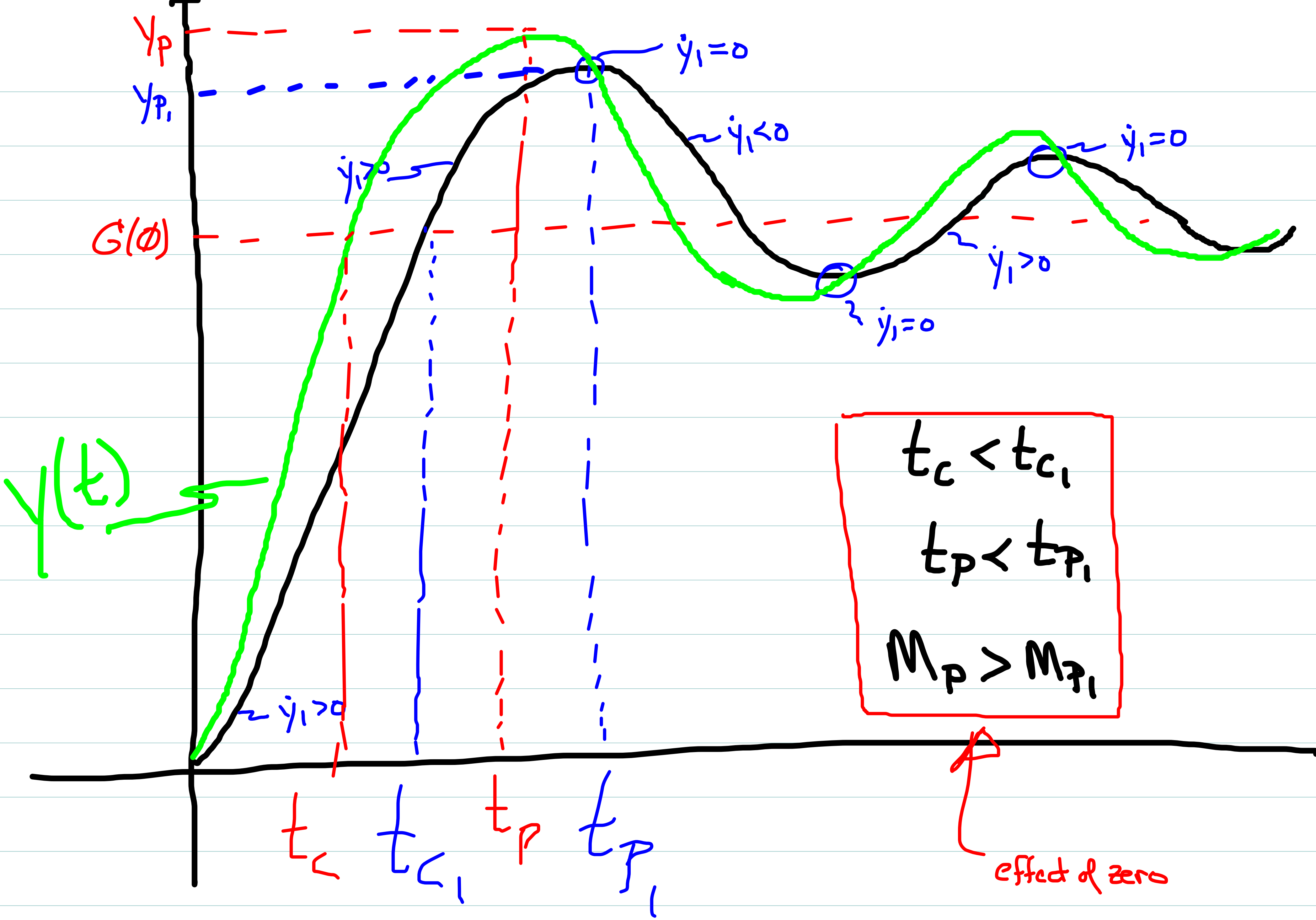
Derivative <u>adds</u> to total response. To understand effect of this, must examine behavior of $\dot{y}_1(t)$

Note that $\dot{y}_1(t) \to \emptyset$ As $t \to \infty$, so the steady-state of the new response will be the <u>Same</u> as the ideal response

$$y_{ss} = G(\emptyset)$$



<u>Note:</u> $\dot{y}_1(t) > \phi$ for all $\phi \le t < t_{P_1}$ } $y(t) > y_1(t)$ in this region



y_p
y_{p_1}
$G(\emptyset)$
$y(t)$
$\dot{y}_1 = 0$
$\dot{y}_1 < 0$
$\dot{y}_1 > 0$
$\dot{y}_1 = 0$
$\dot{y}_1 > 0$
$\dot{y}_1 = 0$
$\dot{y}_1 > 0$
t_c
t_{c_1}
t_p
t_{p_1}
$t_c < t_{c_1}$
$t_p < t_{p_1}$
$M_p > M_{p_1}$
effect of zero

$$y(t) = \left(\frac{\beta_1}{\beta_0}\right) \dot{y}_1(t) + y_1(t)$$

or equivalently:

$$\boxed{y(t) = y_1(t) - \left(\frac{1}{z_1}\right) \dot{y}_1(t)} \quad (z_1 = -\beta_0/\beta_1)$$

Where $y_1(t)$ is the "ideal" (no zero) step response

The total response $y(t)$ is the sum of the ideal response, and a fraction of the <u>derivative</u> of this response.

Suppose 1^{st} $z_1 < 0$ (LHP zero)

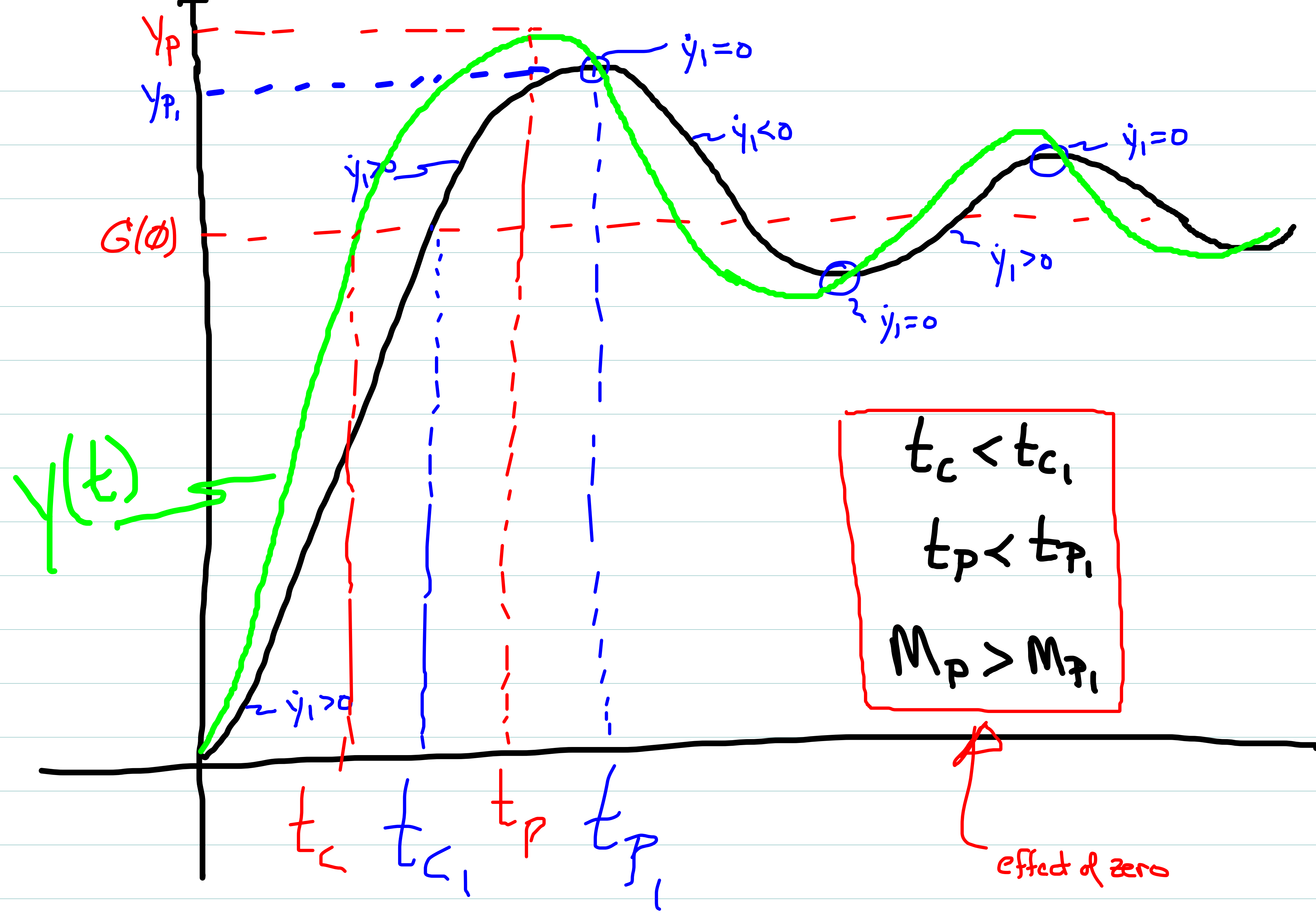
then $z_1 < 0$ and $\left(-\frac{1}{z_1}\right) > 0$ so we can write

$$y(t) = y_1(t) + \left(\frac{1}{|z_1|}\right) \dot{y}_1(t)$$

Derivative <u>adds</u> to total response. To understand effect of this, must examine behavior of $\dot{y}_1(t)$

Note that $\dot{y}_1(t) \to 0$ As $t \to \infty$, so the steady-state of the new response will be the <u>Same</u> as the ideal response

$$y_{ss} = G(0)$$



y_p
y_{p_1}
$G(0)$
$y(t)$
$\dot{y}_1 = 0$
$\dot{y}_1 < 0$
$\dot{y}_1 > 0$
$\dot{y}_1 = 0$
$\dot{y}_1 > 0$
$\dot{y}_1 = 0$
$\dot{y}_1 > 0$
t_c
t_{c_1}
t_p
t_{p_1}
$t_c < t_{c_1}$
$t_p < t_{p_1}$
$M_p > M_{p_1}$
effect of zero

Summary of observations

A LHP zero changes a 2^{nd} order step response by:

$\Rightarrow$ Increasing overshoot y_p and M_p

$\Rightarrow$ decreasing t_c and t_p

In a sense, system "responds" faster (crosses y_{ss} more quickly), but price is greater overshoot!

$\Rightarrow$ Note: tricky to quantify exact changes to t_c, t_p, y_p based on z_1

$\Rightarrow$ However, note change from "ideal" response is proportional to $\frac{1}{|z_1|}$

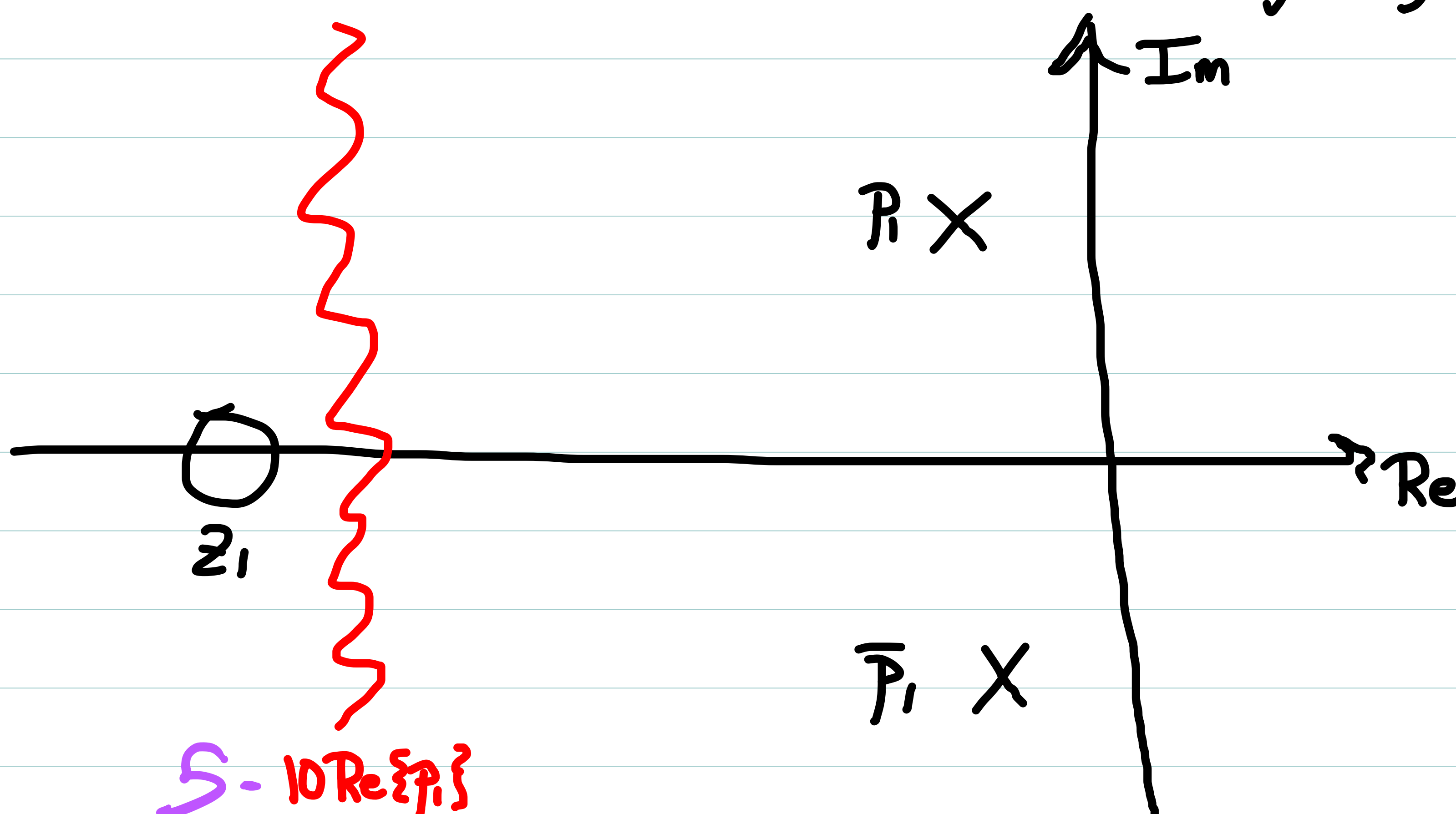
$\Rightarrow$ The further z_1 is from imag axis, the smaller the effect

Rule of Thumb

Effect of zero in this case is negligible if

$$|z_1| > \cancel{10}\,5\,|\mathrm{Re}\{p_1\}|$$

i.e. zero is 10 times further into LHP than complex poles.



Question

$\Rightarrow$ A zero increases (amplifies) the overshoot of a 2^{nd} order system with $\zeta < 1$ (complex poles).

$\Rightarrow$ Can it actually <u>create</u> overshoot in a system with 2 real poles $(\zeta \geq 1)$?

$\Rightarrow$

Question

⇒ A zero increases (amplifies) the overshoot of a 2nd order system with $\zeta < 1$ (complex poles).

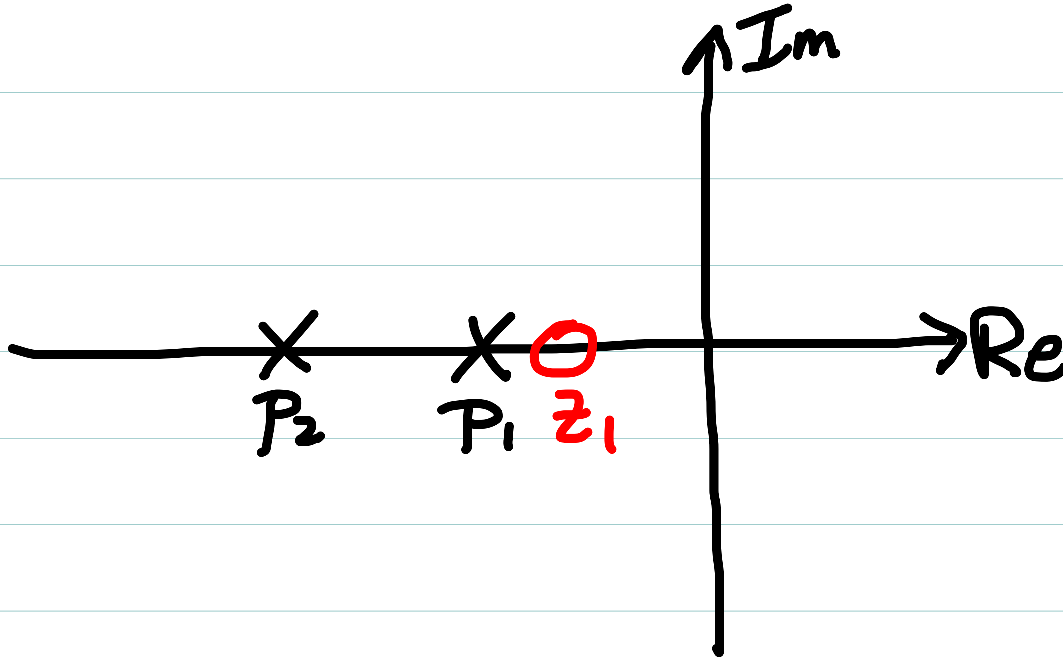
⇒ Can it actually create overshoot in a system with 2 real poles ($\zeta \geq 1$)?

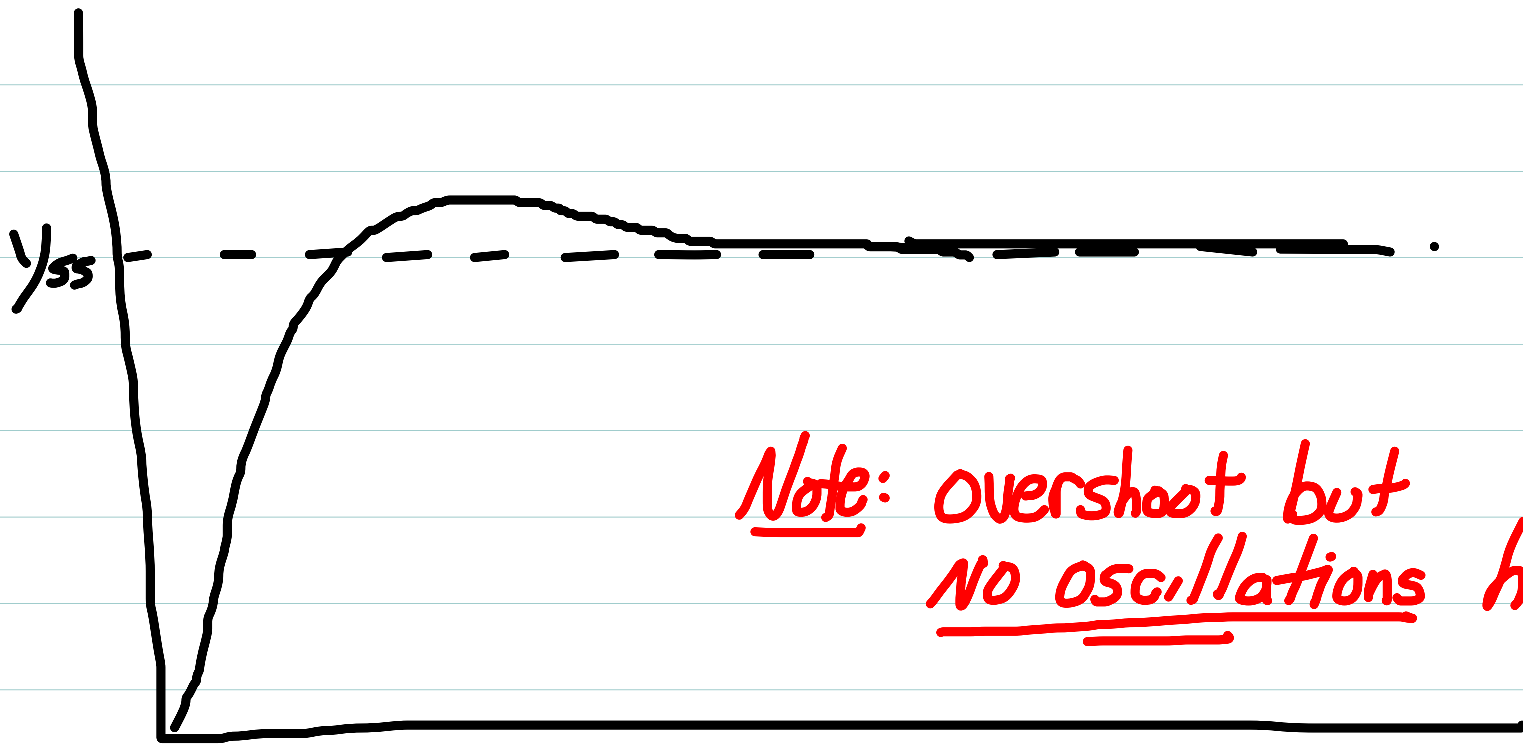
⇒ Yes!

⇒ With 2 real poles p_1 and p_2, $y_p > y_{ss}$ if

$$|z_1| < \min(|p_1|, |p_2|)$$

i.e. if zero is closer to imag axis than both ~~either~~ of the two poles.





Note: overshoot but NO oscillations here

Back to 2^{nd} order ($\zeta < 1$ case)

Suppose $z_1 > \phi$, i.e. z_1 in RHP, then

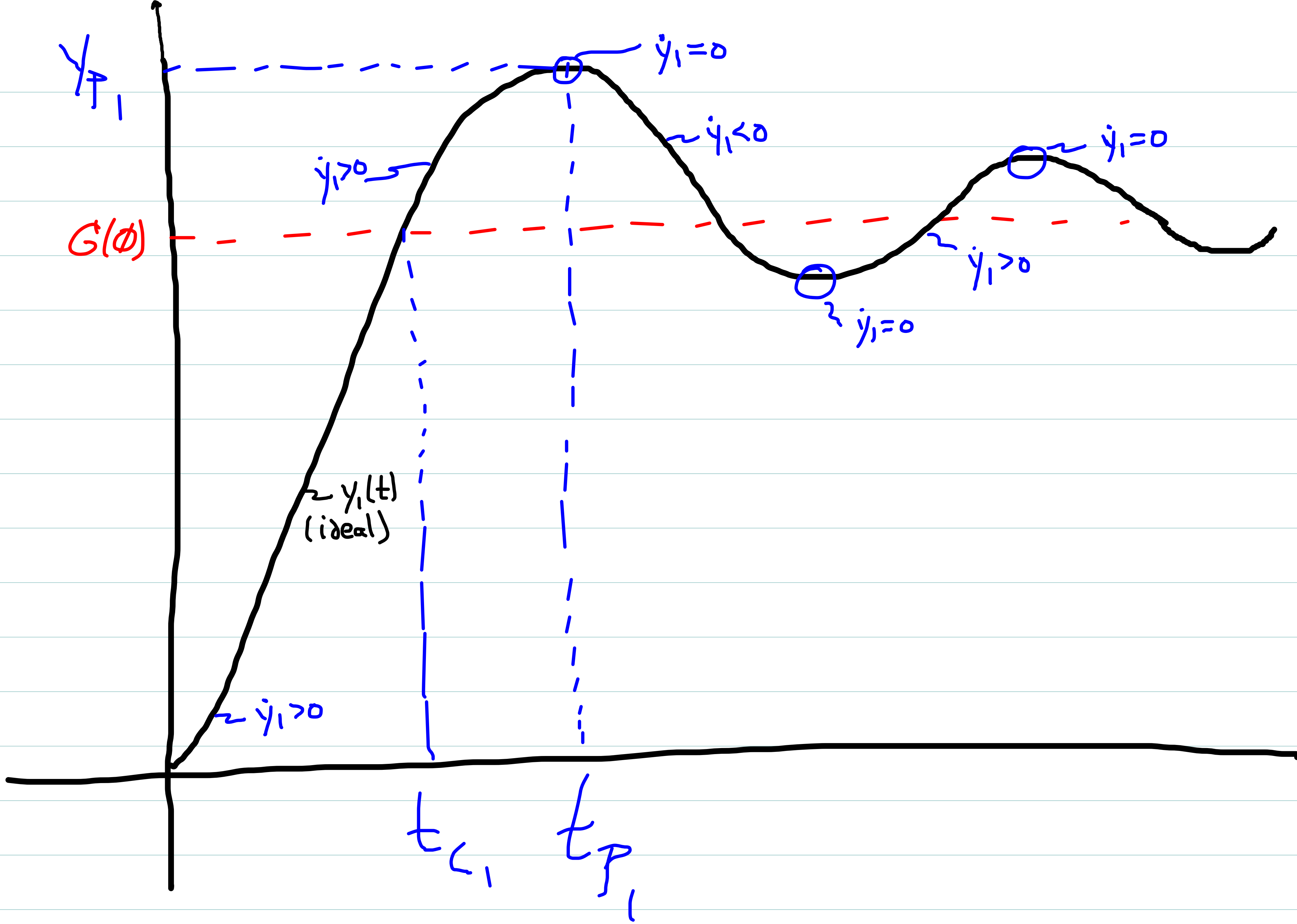
$$y(t) = y_1(t) - \left(\frac{1}{z_1}\right) \dot{y}_1(t)$$

So we are subtracting the derivative from the ideal response.

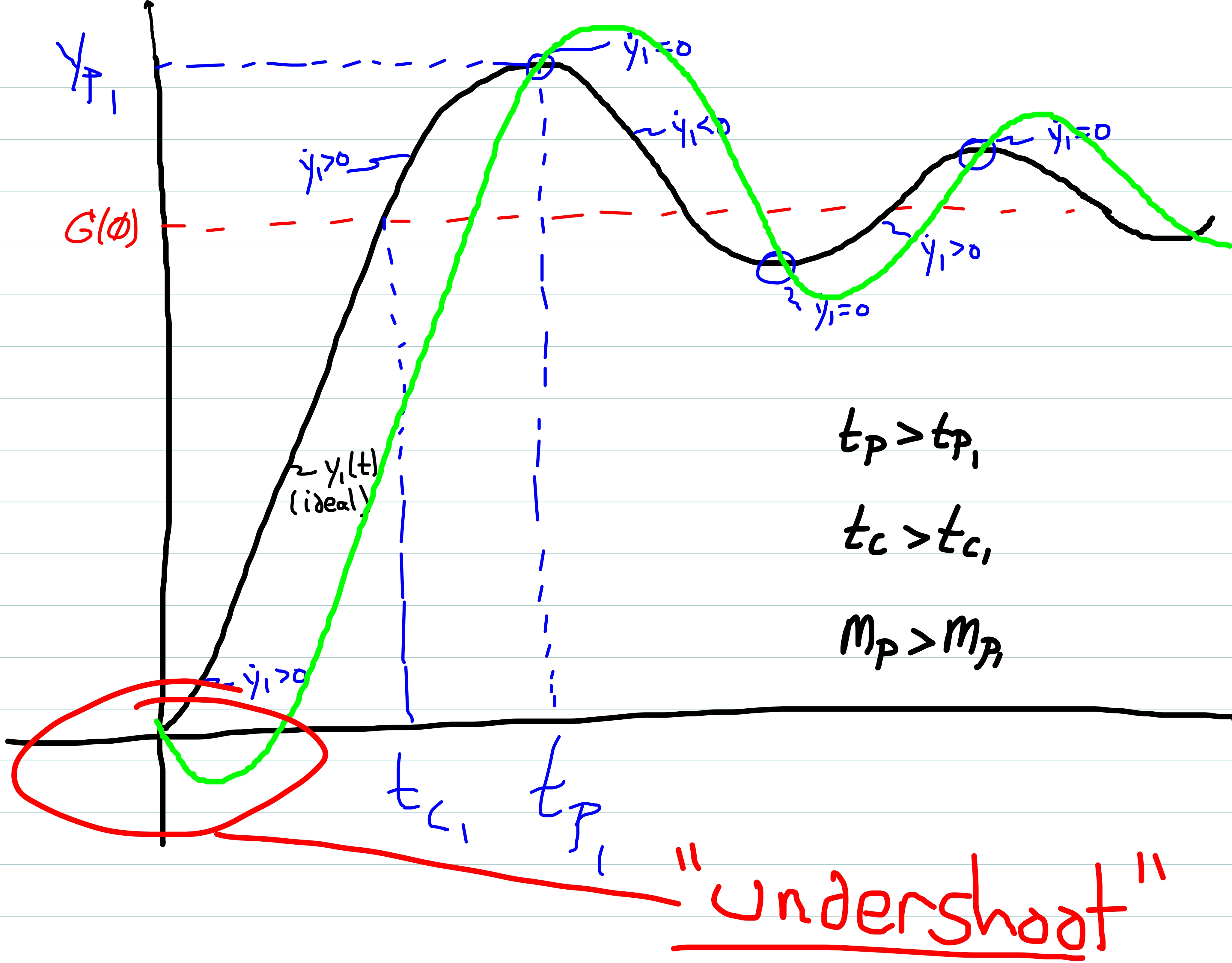
But recall $\dot{y}_1(t) \geq \phi$ for $\phi < t < t_p$,

And $y_1(t) \approx \phi$ for t close to zero

Seems to suggest that $y(t)$ may become negative for times near $t = \phi$...?



y_{P_1}
$\dot{y}_1 = 0$
$\dot{y}_1 < 0$
$\dot{y}_1 = 0$
$\dot{y}_1 > 0$
$G(0)$
$\dot{y}_1 > 0$
$\dot{y}_1 = 0$
$y_1(t)$ (ideal)
$\dot{y}_1 > 0$
t_{c_1}
t_{P_1}



$t_p > t_{p_1}$

$t_c > t_{c_1}$

$M_p > M_{p_1}$

Observations (RHP zero)

=> Again, the peak response is greater

=> However, t_c and t_p have increased

=> Appearence of a new feature: "undershoot"

=> Response initially heads "in wrong direction" before ultimately returning to the same steady-state

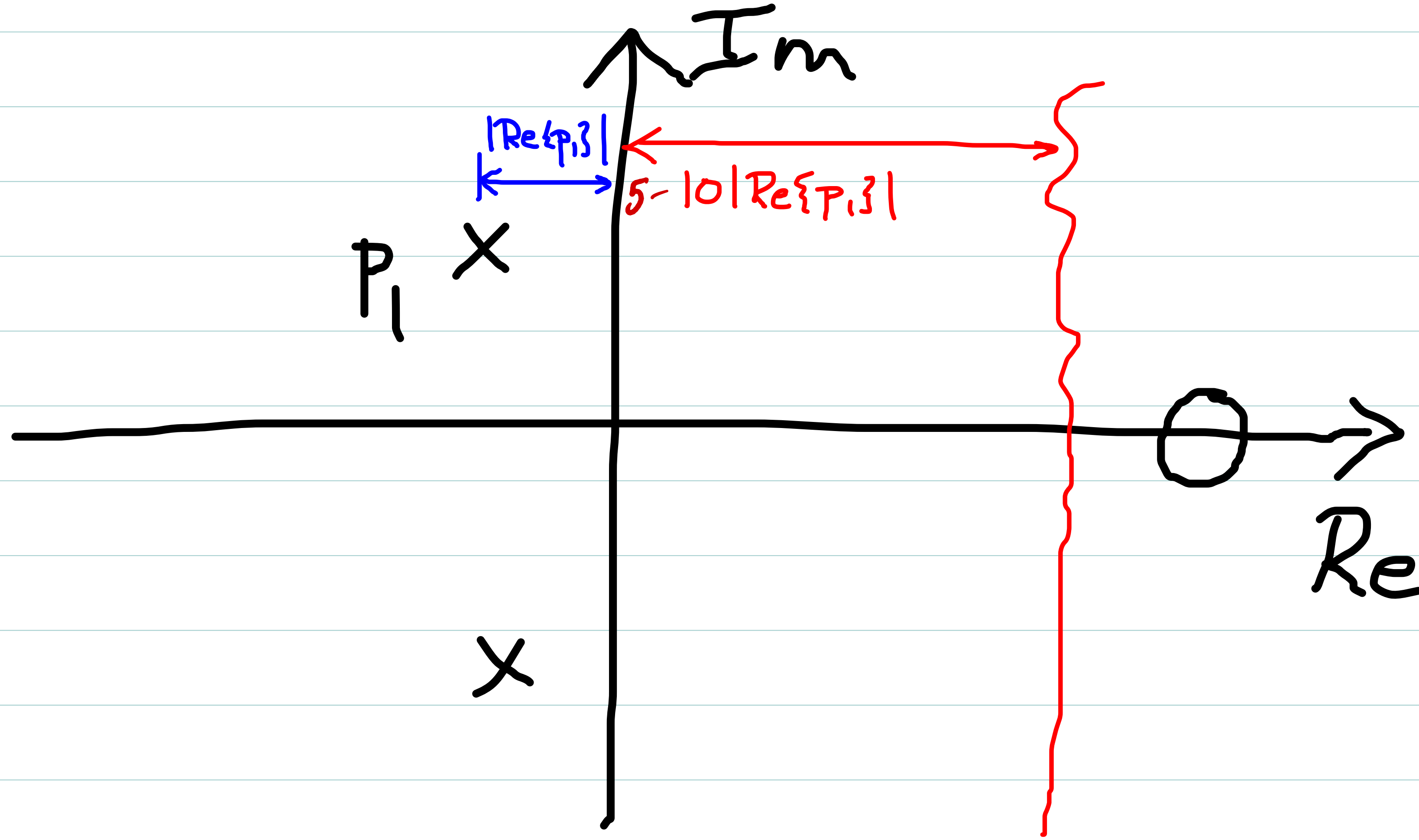
=> Such behavior is NOT UNstable

=> It is, however, very tricky to design controllers for such systems.

Effect is still proportional to $\frac{1}{|z_1|}$

hence diminishes as z_1 moves further from Im axis

Again negligible if $|z_1| > 10|\mathrm{Re}\{p_1\}|$



Effect on settling time

How a zero, either LHP or RHP, affects t_s is difficult to predict.

$\Rightarrow$ Often, but not always, t_s is longer with zero due to increased amplitude of transient oscillations

$\Rightarrow$ No hard and fast rule here

$\Rightarrow$ Primary effect is increased overshoot and:

- reduction of t_c, t_p (LHP)
- undershoot, with increase of t_c, t_p (RHP)

Performance Specifications

⇒ Step inputs representative for many desired behaviors

- move to new pointing angle (spacraft)
- Move to new altitude or heading (aircraft)

⇒ Required performance often specified as <u>upper limits</u> on acceptable t_s, M_p

- System must settle quickly enough, and not overshoot too much.

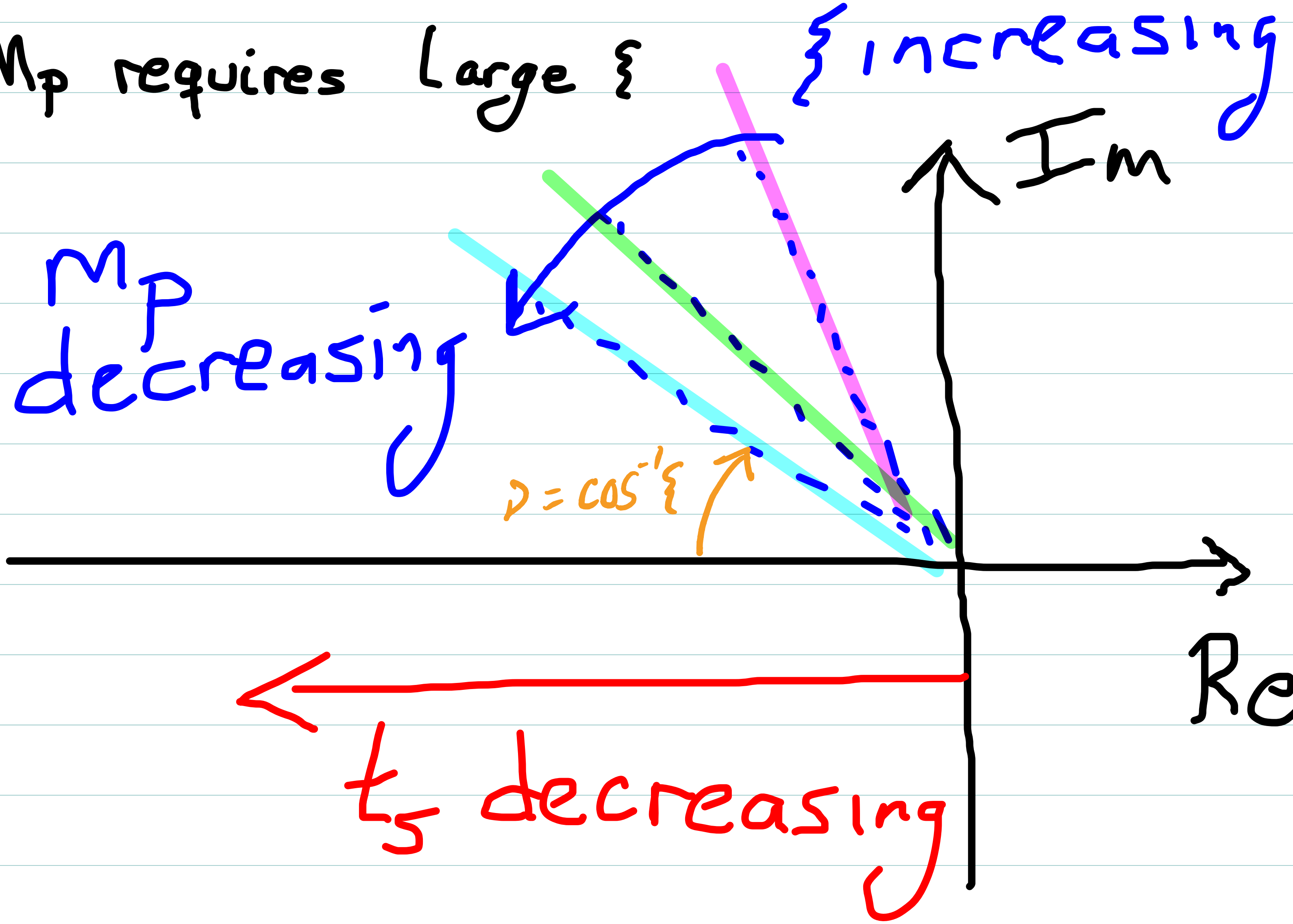
⇒ <u>Recall</u>:

- t_s inversely proportional to $|Re\{p_i\}|$
- M_p a decreasing function of ξ

$$t_s \approx \frac{4}{|Re\{p_1\}|} \quad , \quad \boxed{M_p = \exp\left[\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right]}$$

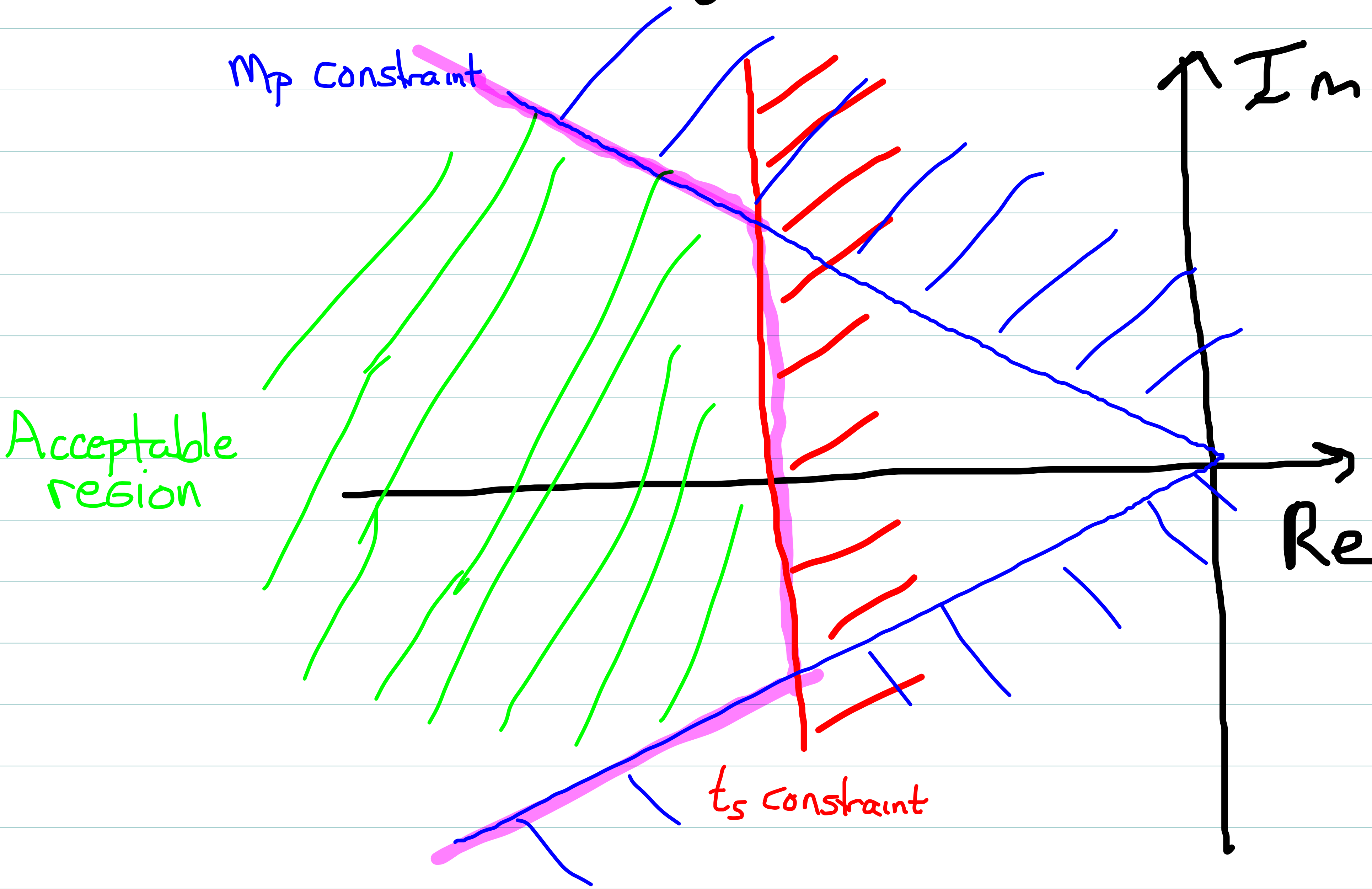
$\Rightarrow$ small t_s requires large $|Re\{p_1\}|$

$\Rightarrow$ small M_p requires large ζ



$\Rightarrow$ Upper bound on t_s gives lower bound on $|\mathrm{Re}\{p_i\}|$

$\Rightarrow$ Upper bound on M_p gives lower bound on ζ



Desireable Pole Locations

⇒ "Good" poles satisfy all transient performance constraints (upper bounds on t_s, M_p)

⇒ "Bad" poles are unstable

⇒ "Ugly" poles are stable, but have too much overshoot or take too long to settle.

⇒ Most aerospace system have natural dynamics which are "bad" or "ugly"

⇒ Goal of control is to make these systems "good"

Feedback "moves" poles

⇒ Already seen this on previous homeworks.

⇒ But it can be tricky!

Suppose $u(t) = K(y_d(t) - y(t))$

If system is modeled with $Y(s) = G(s)U(s)$

where $$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Then poles are moved to roots of

$$r_{CL}(s) = s^2 + (\alpha_1 + K\beta_1)s + (\alpha_0 + K\beta_0)$$

⇒ Tricky to predict movement of poles for all possible values of $K, \alpha_0, \alpha_1, \beta_0, \beta_1$

⇒ Even more complicated for $G(s)$ with additional poles and/or zeros

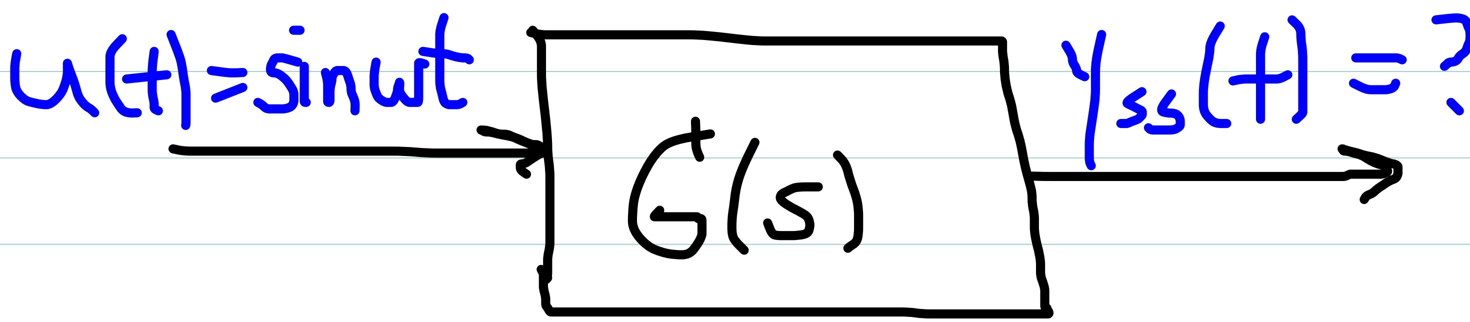
⇒ Need a more systematic tool to predict effectiveness of a control strategy.

⇒ One approach is BASED on a more careful analysis of the behavior of $G(j\omega)$.

Sinusoidal Response

Here we wish to understand the properties of the steady-state response of a stable system when $u(t) = \sin \omega t$.

Note: our focus is shifting (temporarily) away from the transient response



$$\Longrightarrow y_{ss}(t) = \mathrm{Im}\{G(j\omega)e^{j\omega t}\}$$

Sinusoidal Response

Here we wish to understand the properties of the steady-state response of a $\boxed{\text{stable}}$ system when $\boxed{u(t)=\sin\omega t.}$

Note: our focus is shifting (temporarily) away from the transient response

$$u(t)=\sin\omega t \longrightarrow \boxed{G(s)} \longrightarrow y_{ss}(t)=\,?$$

Of course, we've already solved this problem:

$$u(t) = \sin\omega t = \operatorname{Im}\{e^{j\omega t}\}$$

$$\Rightarrow y_f(t) = \operatorname{Im}\{G(j\omega)e^{j\omega t}\} = |G(j\omega)|\sin(\omega t + \measuredangle G(j\omega))$$

Then $y(t) = y_f(t) + y_h(t)$

But if system is stable, $y_h(t) \to \phi$ as $t \to \infty$ for any set of initial cond'ns.

Hence $y_{tr}(t) = y_h(t)$ leaving us with

$$y_{ss}(t) = |G(j\omega)|\sin(\omega t + \measuredangle G(j\omega))$$

So:

$$u(t) = \sin\omega t \implies y_{ss}(t) = |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

<u>Note:</u>

$y_{ss}(t)$ is sinusoidal at same frequency as $u(t)$

<u>But:</u>

Amplitude and phase of $y_{ss}(t)$ different.

Now, more generally suppose:

$$u(t) = B\sin(\omega t + \psi) = \mathrm{Im}\{U e^{j\omega t}\}, \quad U = Be^{j\psi}$$

then

$$y_{ss}(t) = \mathrm{Im}\{G(j\omega) U e^{j\omega t}\}$$

$$= |G(j\omega)| \cdot |U| \sin(\omega t + \angle G(j\omega) + \angle U)$$

or

$$y_{ss}(t) = |G(j\omega)| B \sin(\omega t + \angle G(j\omega) + \psi)$$

Thus generally:

$$u(t) = B\sin(\omega t + \psi) \implies y_{ss}(t) = A\sin(\omega t + \varphi)$$

where: $A = |G(j\omega)| B$

$$\varphi = \measuredangle G(j\omega) + \psi$$

Define:

Amplitude ratio: A/B (ratio of output ampl. to input ampl.

Phase shift: $\varphi - \psi$ (Diff. between output and input phase)

Then note:

$$\boxed{\begin{aligned} A/B &= |G(j\omega)| \\ \varphi - \psi &= \measuredangle G(j\omega) \end{aligned}}$$

So generally

$|G(j\omega)|$ quantifies the ratio between output and input amplitude

$\measuredangle G(j\omega)$ quantifies the change in phase of output compared to input

Note: these are frequency dependent

i.e. the amplitude ratio and phase shift depend on frequency of input.

Very useful to quantify this dependence!

Example

$$G(s) = \frac{3}{s+2}$$

$$|G(j\omega)| = \frac{3}{\sqrt{\omega^2+4}} \qquad \measuredangle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

$\omega = 1/2 \Rightarrow |G(j/2)| = 3/\sqrt{4.25} \approx 1.46$

$\measuredangle G(j/2) = -\tan^{-1}(1/4) = -.245 \text{ rad or } -14.04^\circ$

$\omega = 2 \Rightarrow |G(2j)| = 3/\sqrt{8} \approx 1.06$

$\measuredangle G(2j) = -\tan^{-1}(1) = -\pi/4 = -45^\circ$

$\omega = 20 \Rightarrow |G(20j)| = 3/\sqrt{404} = 0.15$

$\measuredangle G(20j) = -\tan^{-1}(10) = -1.47 \approx -84.3^\circ$

⇒ Want to learn to predict these changes based on ZPK structure of $G(s)$

⇒ Useful also to visualize graphically

⇒ Three methods

1. Plot $|G(j\omega)|$ and $\measuredangle G(j\omega)$ vs. $\omega \geq 0$ (2 plots)

2. Plot $G(j\omega)$ as ω varies from 0 to ∞ as points in complex plane.

3. Plot $|G(j\omega)|$ vs. $\measuredangle G(j\omega)$ for $0 \leq \omega < \infty$

⇒ Want to learn to <u>predict</u> these changes based on ZPK structure of $G(s)$

⇒ Useful also to visualize <u>graphically</u>

⇒ Three methods

1. Plot $|G(j\omega)|$ and $\measuredangle G(j\omega)$ vs. $\omega \geq 0$ (2 plots) "Bode diagrams"

2. Plot $G(j\omega)$ as ω varies from 0 to ∞ as points in complex plane. "Polar diagram"

3. Plot $|G(j\omega)|$ vs. $\measuredangle G(j\omega)$ for $0 \leq \omega < \infty$ "Nichols chart"

Bode is most fundamental, start there

⇒ Want to see behavior for large range of $\omega \geq 0$

⇒ $|G(j\omega)|$ will vary enormously in size

⇒ Use logarithmic scales for plots.

⇒ Horizontal Axis on Bode diagram is freq on a log scale

⇒ equally spaced divisions on this scale are factors of 10 apart.

⇒ We call one of these divisions a "decade"

$\left.\begin{matrix} 1/10 \to 1 \\ 2 \to 20 \end{matrix}\right\}$ one decade

$\left.\begin{matrix} 1/10 \to 10 \\ 2 \to 200 \end{matrix}\right\}$ two decades

Decibels

$|G(j\omega)|$ is shown on Bode diagrams in special units called decibels.

Def'n: For any real number $X \geq 0$

$$X_{db} = 20 \log X$$

Conversely

$$X = 10^{(X_{db}/20)}$$

Example (from above): $X = 1.46 \Rightarrow X_{db} = 3.25$

$X = 1.06 \Rightarrow X_{dB} = 0.51$

$X = 0.15 \Rightarrow X_{dB} = -16.5$

Common Shorthand

$$X = 0.15 = -16.5\,dB$$

Note common conversions

X	X (dB)
.01	-40
.1	-20
1	0
10	20
100	40

Important $\Rightarrow$

Zero on dB axis means magnitude of 1!!

Bode diagrams show

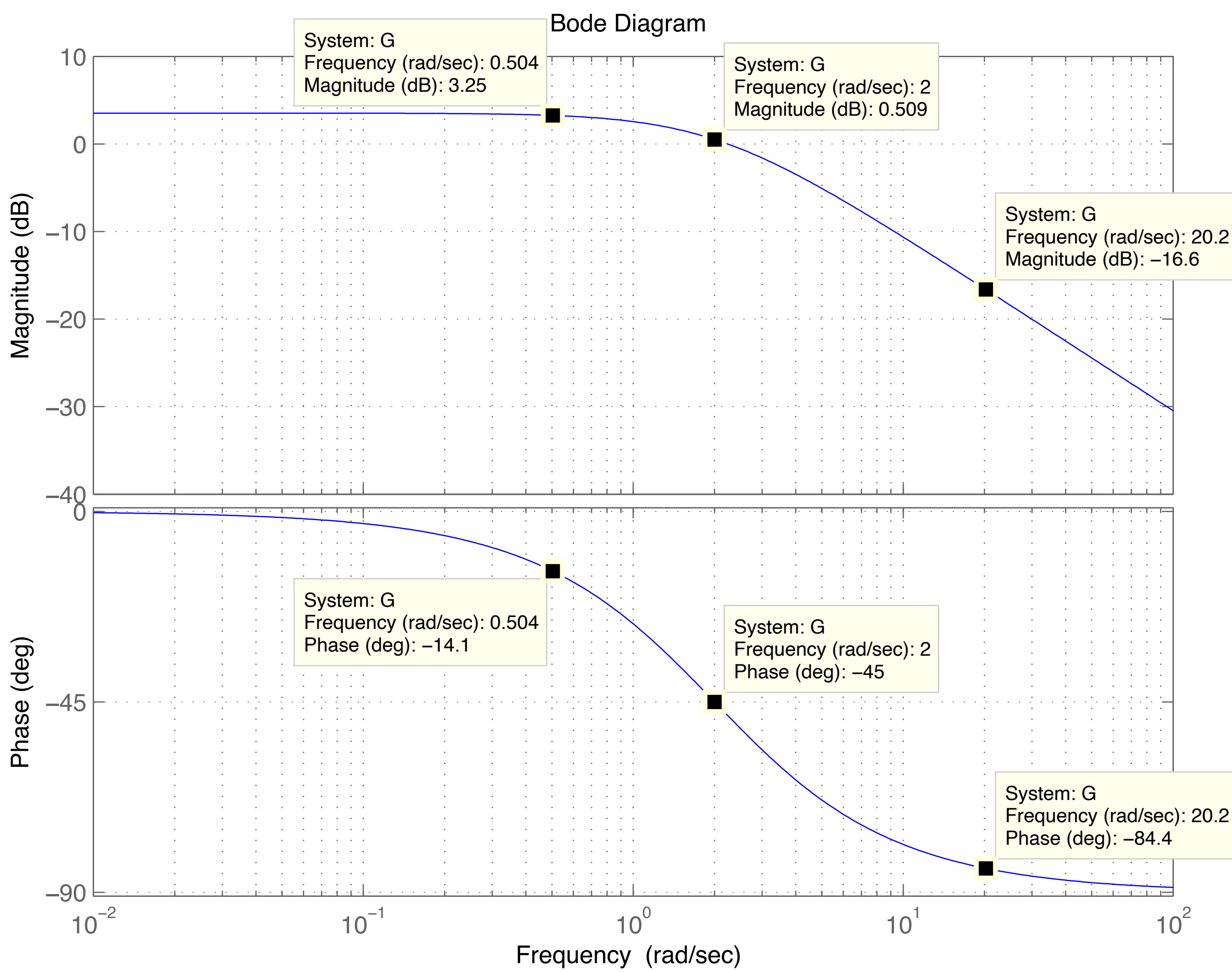
1. $|G(j\omega)|$ in dB vs ω on a log scale

2. $\measuredangle G(j\omega)$ in deg " " "

See example

Note: there are no negative frequencies on a Bode diagram!

The left limit of the horizontal scale corresponds to $\omega \to \emptyset$!


Bode Diagram
System: G
Frequency (rad/sec): 0.504
Magnitude (dB): 3.25
System: G
Frequency (rad/sec): 2
Magnitude (dB): 0.509
System: G
Frequency (rad/sec): 20.2
Magnitude (dB): −16.6
Magnitude (dB)
10
0
−10
−20
−30
−40
System: G
Frequency (rad/sec): 0.504
Phase (deg): −14.1
System: G
Frequency (rad/sec): 2
Phase (deg): −45
System: G
Frequency (rad/sec): 20.2
Phase (deg): −84.4
Phase (deg)
0
−45
−90
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Recap: Frequency Response Analysis

$$u(t) = B\sin(\omega t + \psi) \implies y_{ss}(t) = A\sin(\omega t + \varphi)$$

$$A = B\,|G(j\omega)|, \quad \varphi = \measuredangle G(j\omega) + \psi$$

Bode diagrams: show

$|G(j\omega)|$ (dB) vs. ω (log scale) "Magnitude diagram"

$\measuredangle G(j\omega)$ (deg) vs. ω (log scale) "Phase diagram"

Want to learn to rapidly predict the shapes of these diagrams from the ZPK structure of transfer function $G(s)$

How?

Will Show:

(1) Effect of each pole p_K and zero z_i is concentrated in a narrow band of frequencies near $\omega = |p_K|$ (or $|z_i|$, as appropriate)

$\Rightarrow$ remember: $\omega \geq 0$ on Bode diagrams. There are no negative frequencies shown!

(2) Effect of individual poles/zeros on *total* Bode diagrams are <u>additive</u>

"Bode form" of transfer function

ZPK form:

$$G(s) = K\left[\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{k=1}^{n}(s-p_k)}\right]$$

Bode form:

$$G(s) = K_B \frac{\prod_{i=1}^{m}(1-s/z_i)}{s^N \prod_{k=N+1}^{n}(1-s/p_k)}$$

N = # of poles at origin "Type" of system

K_B = "Bode gain"; note $N=\emptyset \Rightarrow K_B = G(\emptyset)$

Bode and ZPK forms are two different ways of writing the <u>same</u> transfer function

<u>Example</u>:

$$G(s) = \frac{5(s+2)}{s(s+3)(s+4)} \qquad \text{(ZPK)}$$

(Bode) $$= \left(\frac{5}{6}\right)\left[\frac{(1+s/2)}{s(1+s/3)(1+s/4)}\right]$$

Here $N=1$ and $K_B = 5/6$

<u>Algebraically equivalent</u> to ZPK form.

i.e. both are the <u>same</u> TF

So:

$$G(j\omega) = K_B \left[\frac{\prod_{i=1}^{m} \left(1 - j\omega/z_i\right)}{(j\omega)^N \prod_{k=N+1} \left(1 - j\omega/p_k\right)} \right]$$

for any real $\omega \geq 0$, $G(j\omega)$ is complex and so are each individual factor (except K_B, which is real)

recall for any $s_1, s_2 \in \mathbb{C}$

$$\boxed{\begin{aligned} &\measuredangle(s_1 s_2) = \measuredangle s_1 + \measuredangle s_2 \\ &\measuredangle\left(\frac{s_1}{s_2}\right) = \measuredangle s_1 - \measuredangle s_2 \\ &\measuredangle s_1^N = N \measuredangle s_1 \end{aligned}}$$

Thus:

$$\measuredangle G(j\omega) = \measuredangle K_B + \sum_{i=1}^{m} \measuredangle(1 - j\omega/z_i) - N\measuredangle(j\omega) - \sum_{k=N+1}^{n} \measuredangle(1 - j\omega/p_k)$$

Note: (1.) Each factor contributes additively

(2.) Zeros add to angle, poles subtract

(3) $\measuredangle K_B$ same for any ω:

$$\measuredangle K_B = \emptyset \ (K_B > \emptyset), \quad \measuredangle K_B = \pm 180^\circ \ (K_B < \emptyset)$$

(3.) $\measuredangle(j\omega)$ is same for any $\omega \geq \emptyset$

$$\measuredangle(j\omega) = 90^\circ$$

(4.) Changes to $\measuredangle G(j\omega)$ as ω varies depends on specific z_i and nonzero p_k.

What about Magnitudes?

Recall: for $s_1, s_2 \in \mathbb{C}$

$$|s_1 s_2| = |s_1|\,|s_2|$$

$$\left|\frac{s_1}{s_2}\right| = \frac{|s_1|}{|s_2|}$$

$$|s_1^N| = |s_1|^N$$

So:

$$|G(j\omega)| = |K_B| \frac{\prod_{i=1}^{m} |1 - j\omega/z_i|}{|j\omega|^N \prod_{k=N+1}^{n} |1 - j\omega/p_k|}$$

UGLY . . .

But Bode shows $|G(j\omega)|$ in dB

i.e. $20\log|G(j\omega)|$

Now recall:

$$\log(xy) = \log x + \log y$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\log(x^N) = N\log x$$

Hence in dB:

$$|G(j\omega)|_{dB} = |K_B|_{dB} + \boxed{\sum_{i=1}^{m} |1 - j\omega/z_i|_{dB}} - N|j\omega|_{dB} - \boxed{\sum_{K=N+1}^{n} \left|1 - \frac{j\omega}{P_K}\right|_{dB}}$$

Notes:

1. Magnitudes in dB are <u>additive</u> for each factor
2. zeros <u>add</u> to magnitude, poles <u>subtract</u>
3. $|K_B|$ is constant for all ω, like with phase
4. $|j\omega|$ is <u>Not</u> constant, unlike phase.

So, we see effect of individual parts of $G(s)$ contribute additively to

$$\angle G(j\omega) \quad \text{and} \quad |G(j\omega)|_{dB}$$

Look at effect of individual factors

Look at how each $(1 - j\omega/z_i)$ or $(1 - j\omega/p_k)$ changes with ω.

To simplify notation, we'll look at $(1 + j\omega\tau)$, where $\tau = -1/z_i$ or $\tau = -1/p_k$ as appropriate

Then:

$$|1 + j\omega\tau| = \sqrt{1 + \omega^2\tau^2}$$

and

$$\measuredangle(1 + j\omega\tau) = \tan^{-1}\omega\tau$$

Study how these vary with ω

Consider first magnitude

$$|1+j\omega\tau| = \sqrt{1+(\omega\tau)^2} = \begin{cases} 1 & \text{if } \omega \ll 1/|\tau| \\ \sqrt{2} & \text{if } \omega = 1/|\tau| \\ \omega|\tau| & \text{if } \omega \gg 1/|\tau| \end{cases}$$

and thus:

$$|1+j\omega\tau|_{dB} = \begin{cases} \emptyset & \omega \ll 1/|\tau| \quad \text{"Low freq. Limit"} \\ 3 & \omega = 1/|\tau| \\ 20\log\omega|\tau| & \omega \gg 1/|\tau| \quad \text{"high freq Limit"} \end{cases}$$

Look at 3rd case:

$$20\log\omega|\tau| = 20\left[\log\omega + \log|\tau|\right]$$

Note when $\omega = 1/|\tau|$, $\log\omega = -\log|\tau|$ & 3rd case evaluates to $\emptyset$.

Also:

in high freq limit $\omega >> 1/|\tau|$

$$|1+j\omega\tau|_{dB} = 20\left[\log\omega + \log|\tau|\right]$$

Suppose we have two freqs, ω_1, ω_2 both $>> 1/|\tau|$

with $\omega_2 = 10\omega_1$, then:

$$|1+j\omega_2\tau|_{dB} = |1+j(10\omega_1)\tau|_{dB}$$

$$= 20\left[\log(10\omega_1) + \log|\tau|\right]$$

$$= 20\left[\log\omega_1 + \log 10 + \log|\tau|\right]$$

$$= 20\left[\log\omega_1 + \log|\tau|\right] + 20$$

So

$$|1+j\omega_2\tau|_{dB} = |1+j\omega_1\tau|_{dB} + 20 \Leftarrow \text{+20 dB increase}$$

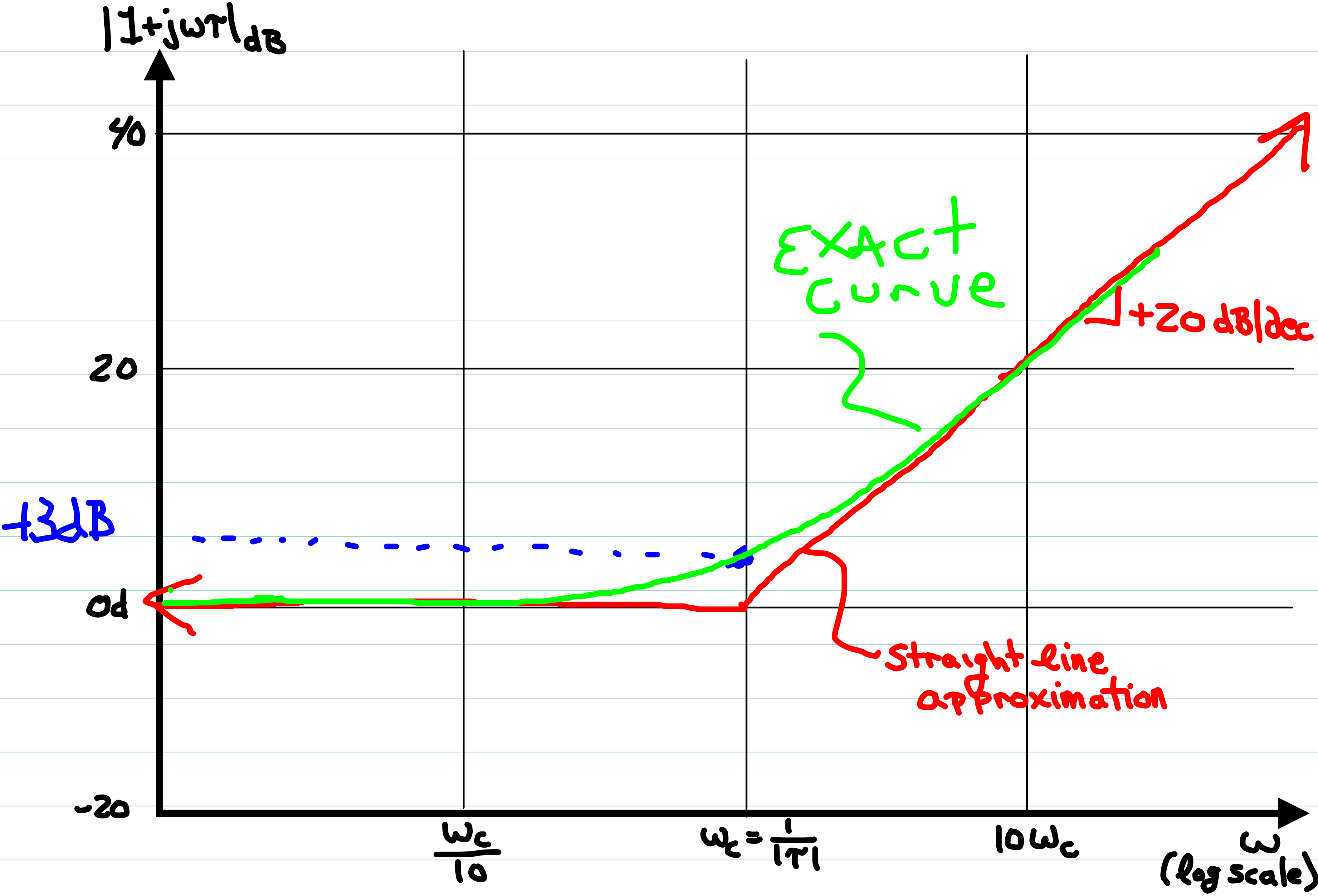
Hence:

in high frequency region $|1+j\omega T|_{dB}$ increases by 20dB for every factor of 10 increase in frequency (decade)

$\Rightarrow$ graph has a slope of 20dB/decade in high freq. region

$\Rightarrow$ Recall graph is constant at $\emptyset$ dB in low freq. region

$\Rightarrow$ The two limiting cases come together at the "corner frequency", $\omega_c = \dfrac{1}{|T|}$.



$|1+j\omega\tau|_{dB}$
40
20
+3dB
0d
-20
$\frac{\omega_c}{10}$
$\omega_c = \frac{1}{|\tau|}$
$10\omega_c$
ω
(log scale)
EXACT curve
+20 dB/dec
Straight line approximation

Things to note:

- Graph changes slope by +20 dB/dec
- Think in terms of this slope change, not the total shape
- Recall $(1+j\omega\tau)$ is a generic representation of a factor of $G(s)$, either

$$\left(1-\frac{j\omega}{z_i}\right) \quad \text{or} \quad \left(1-\frac{j\omega}{p_K}\right)$$

i.e. $\tau = -1/z_i$ or $\tau = -1/p_K$

Thus the corner freq. $\omega_c = 1/|\tau| = |z_i|$ or $|p_K|$

$\boxed{\text{Corner freq is the absolute VALUE of a pole or zero of } G(s)}$

$\Rightarrow$ Because $|G(j\omega)|_{dB}$ is the sum of the effects of the individual terms $|1 - j\omega/z_i|_{dB}$ $|1 - j\omega/p_k|_{dB}$

each pole or zero will create a "corner" on the complete graph

$\Rightarrow$ The total graph will have corners at every freq. corresponding to $|z_i|$ and $|p_k|$.

$\Rightarrow$ zeros add to overall $|G(j\omega)|_{dB}$ $\Rightarrow$ slope changes of $+20$ dB/dec at $\omega = |z_i|$, $i = 1 \ldots m$

$\Rightarrow$ Poles subtract from overall $|G(j\omega)|_{dB}$ $\Rightarrow$ Slope changes of -20 dB/dec at $\omega = |p_k|$.

Example #1

$$G(s) = (10s+1)(s/10+1)$$

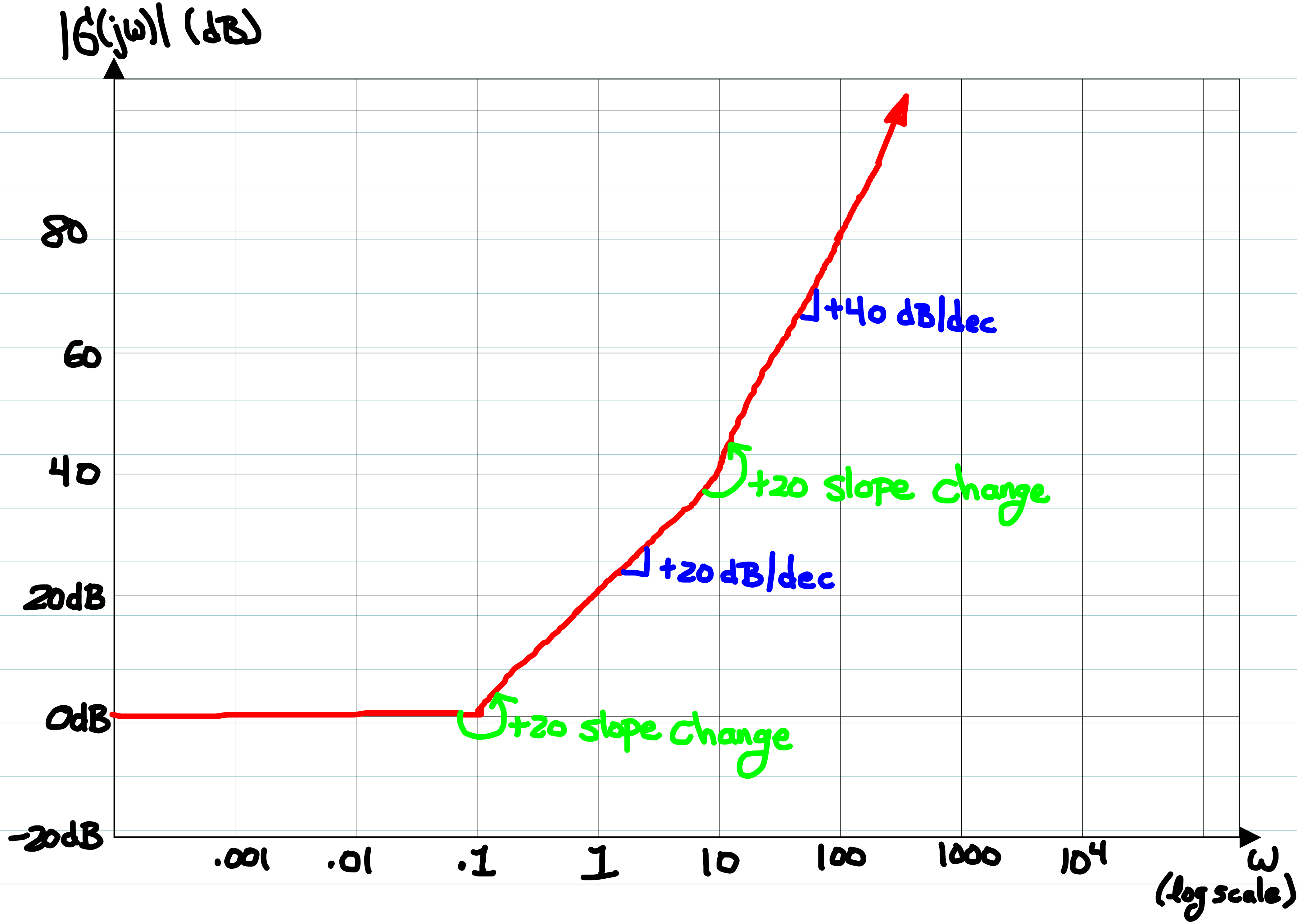
No poles; zeros at $z_1 = -10$, $z_2 = 1/10$

$|G(j\omega)|_{dB}$ will show +20 dB/dec changes at

$\omega = 1/10$ and $\omega = 10$

Below $\omega = 1/10$ the graph will be constant at 0 dB.

Graph bends up by +20 dB/dec at $\omega = 1/10$, and again at $\omega = 10$.


|G(jω)| (dB)
80
60
40
20dB
0dB
-20dB
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)
+20 slope change
+20 dB/dec
+20 slope change
+40 dB/dec

Example #2

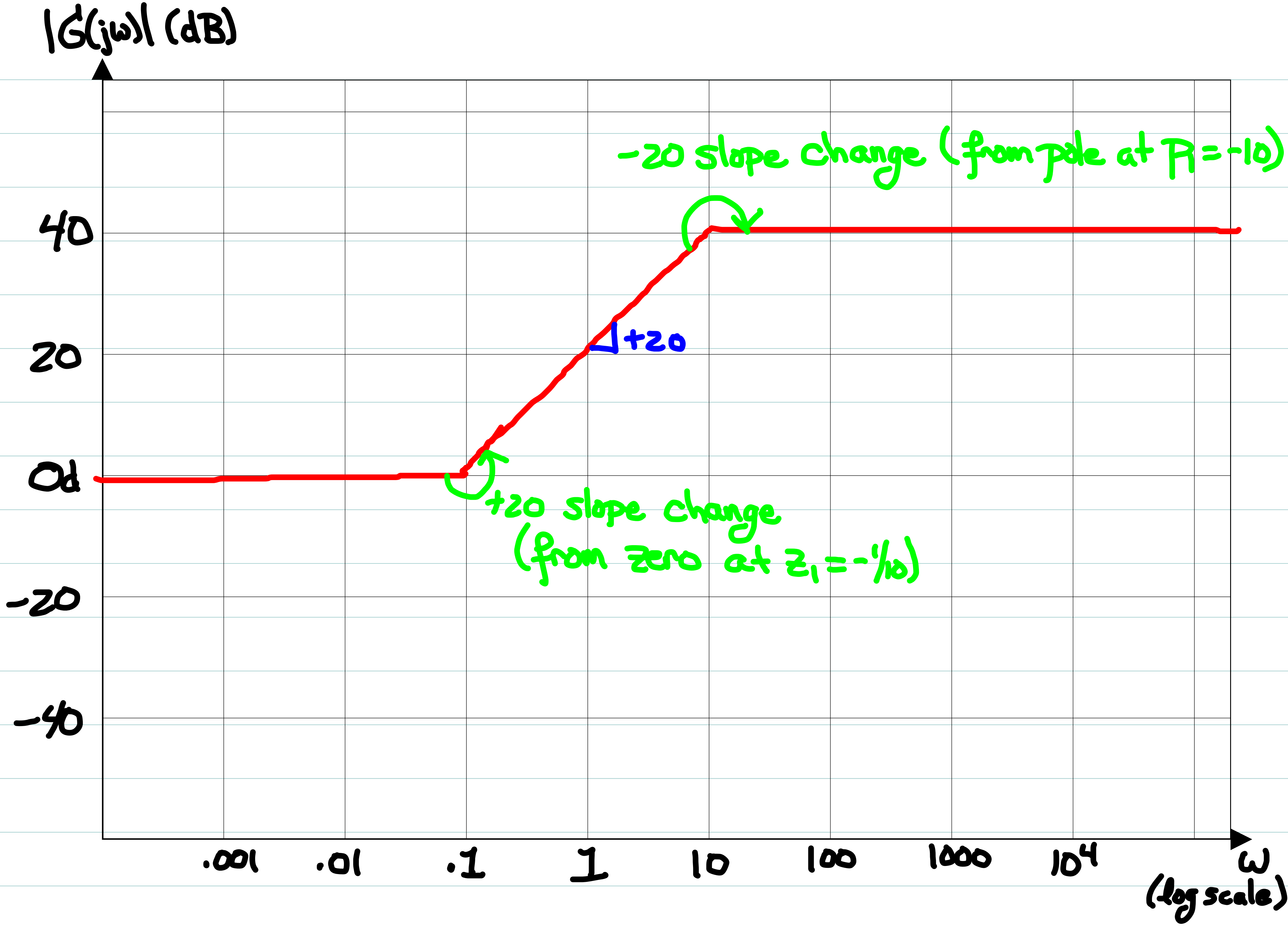
$$G(s) = \frac{(10s+1)}{(s/10+1)}$$

Zero at $z_1 = -1/10$, pole at $p_1 = -10$

Corners at $\omega = 1/10$ and $\omega = 10$ again

But now: at $\omega = 1/10$ slope <u>increases</u> by $+20$ dB/dec

at $\omega = 10$ slope <u>decreases</u> by -20 dB/dec



|G(jω)| (dB)
40
20
0d
-20
-40
-20 slope change (from pole at $P_1 = -10$)
+20
+20 slope change
(from zero at $z_1 = -1/10$)
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)

Gain effect is additive also, and <u>constant</u> for all ω:

$$|K_B(1+j\omega\tau)|_{dB} = |K_B|_{dB} + |1+j\omega\tau|_{dB}$$

$\Rightarrow$ entire graph shifts <u>up or down</u> by $|K_B|_{dB} = 20\log|K_B|$

shifts <u>up</u> if $|K_B|_{dB} > \emptyset$

shifts <u>down</u> if $|K_B|_{dB} < \emptyset$

Gain effect is additive also, and <u>constant</u> for all ω:

$$|K_B(1+j\omega\tau)|_{dB} = |K_B|_{dB} + |1+j\omega\tau|_{dB}$$

$\Rightarrow$ entire graph shifts <u>up or down</u> by $|K_B|_{dB} = 20\log|K_B|$

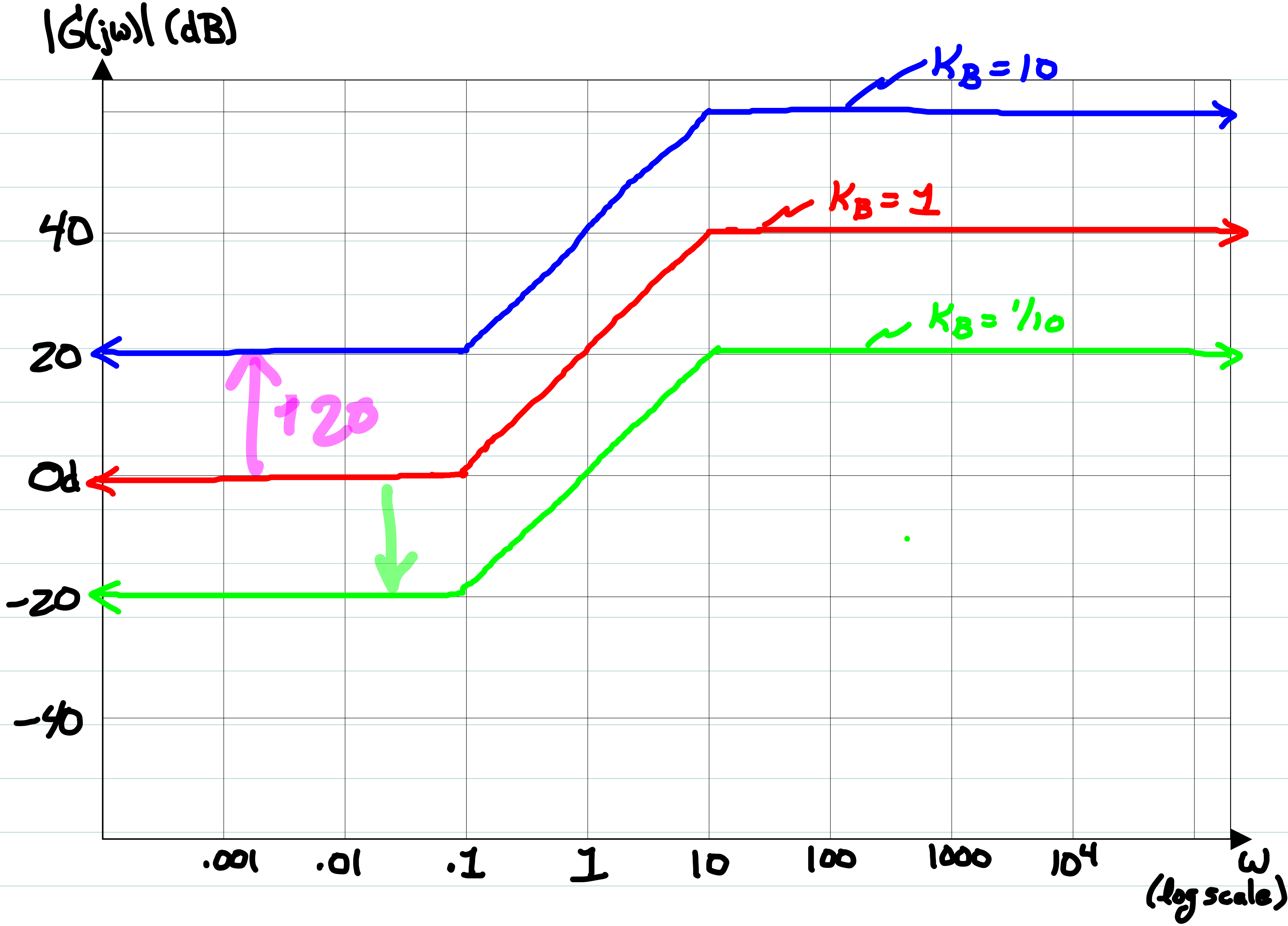
shifts <u>up</u> if $|K_B|_{dB} > \emptyset \Rightarrow |K_B| > 1$

Shifts <u>down</u> if $|K_B|_{dB} < \emptyset \Rightarrow |K_B| < 1$

Remember the <u>sign</u> of K_B has <u>No effect</u> on the magnitude diagram!

<u>Example #3:</u>

$$G(s) = K_B\left[\frac{(10s+1)}{(s/10+1)}\right]$$



|G(jω)| (dB)
$K_B = 10$
$K_B = 1$
$K_B = 1/10$
40
20
0d
-20
-40
+20
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)

Repeated factors

$(1+j\omega\tau)^{\ell}$, ℓ integer ≥ 1

$$|(1+j\omega\tau)^{\ell}|_{dB} = 20 \log |1+j\omega\tau|^{\ell}$$
$$= (20\ell) \log |1+j\omega\tau|$$

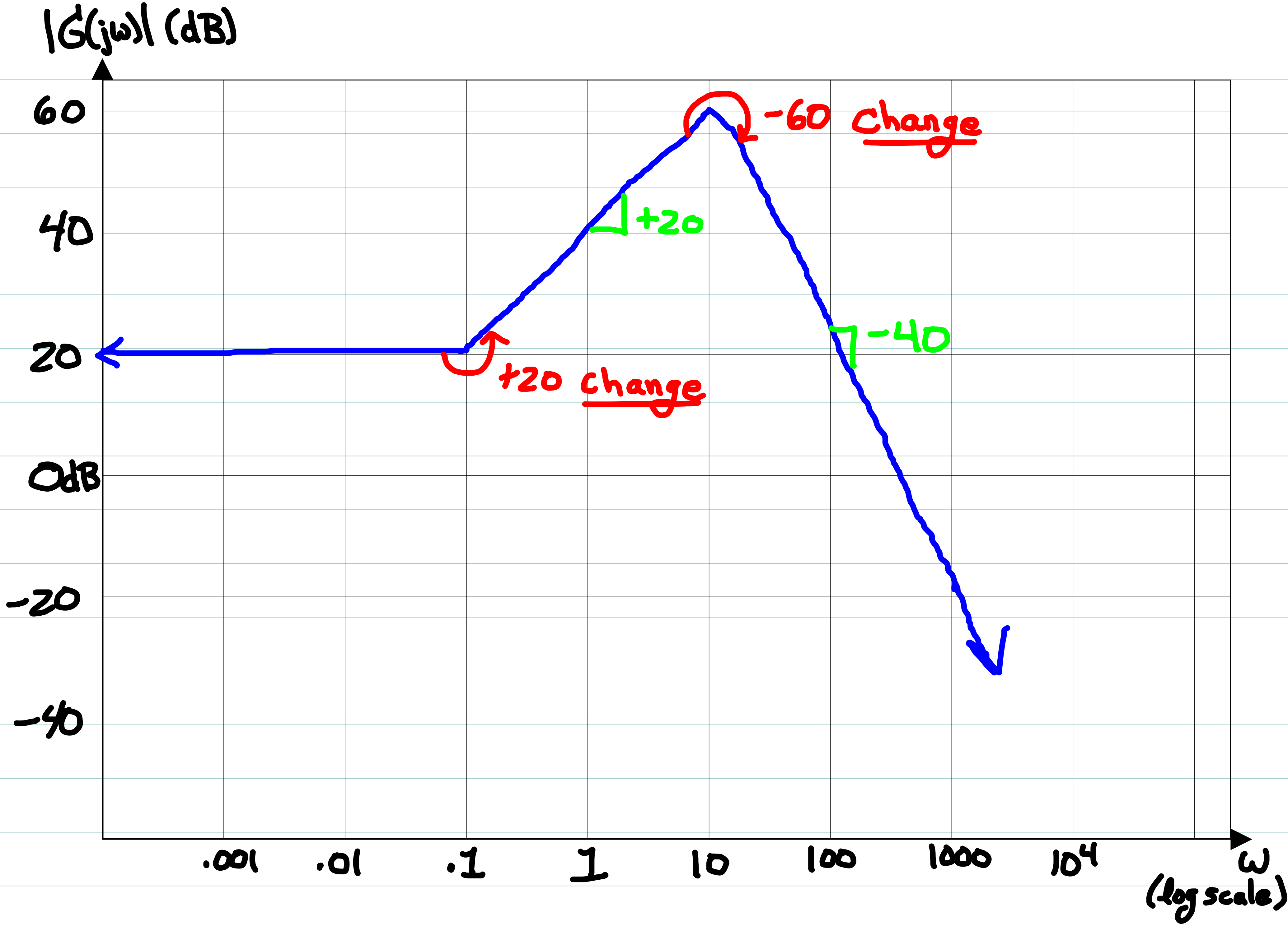
$\Rightarrow$ slope change is $\pm 20\ell$ at $\omega = |1/\tau|$

(positive for zero, negative for pole)

Example #4:

$$G(s) = 10\left[\frac{(10s+1)}{(s/10+1)^3}\right]$$

+20 slope change at $\omega = 1/10$, $\underline{-60}$ change at $\omega = 10$.



|G(jω)| (dB)
60
40
20
0dB
-20
-40
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)
+20 change
+20
-60 change
-40

Summary (so far)

⇒ Poles p_K and zeros z_i cause changes in $|G(j\omega)|_{dB}$ graph at corner frequencies $|p_K|$ and $|z_i|$

⇒ Slope of graph changes at these corners

⇒ Zero corners "bend up", i.e. change slope by +20 dB/dec

⇒ Pole corners "bend down", i.e. change slope by −20dB/dec

⇒ If $|K_B| \neq 1$, entire graph is raised or lowered by $|K_B|_{dB}$

Poles/zeros at origin

Poles at origin (type $N > \emptyset$) or zeros at origin ($N < \emptyset$)

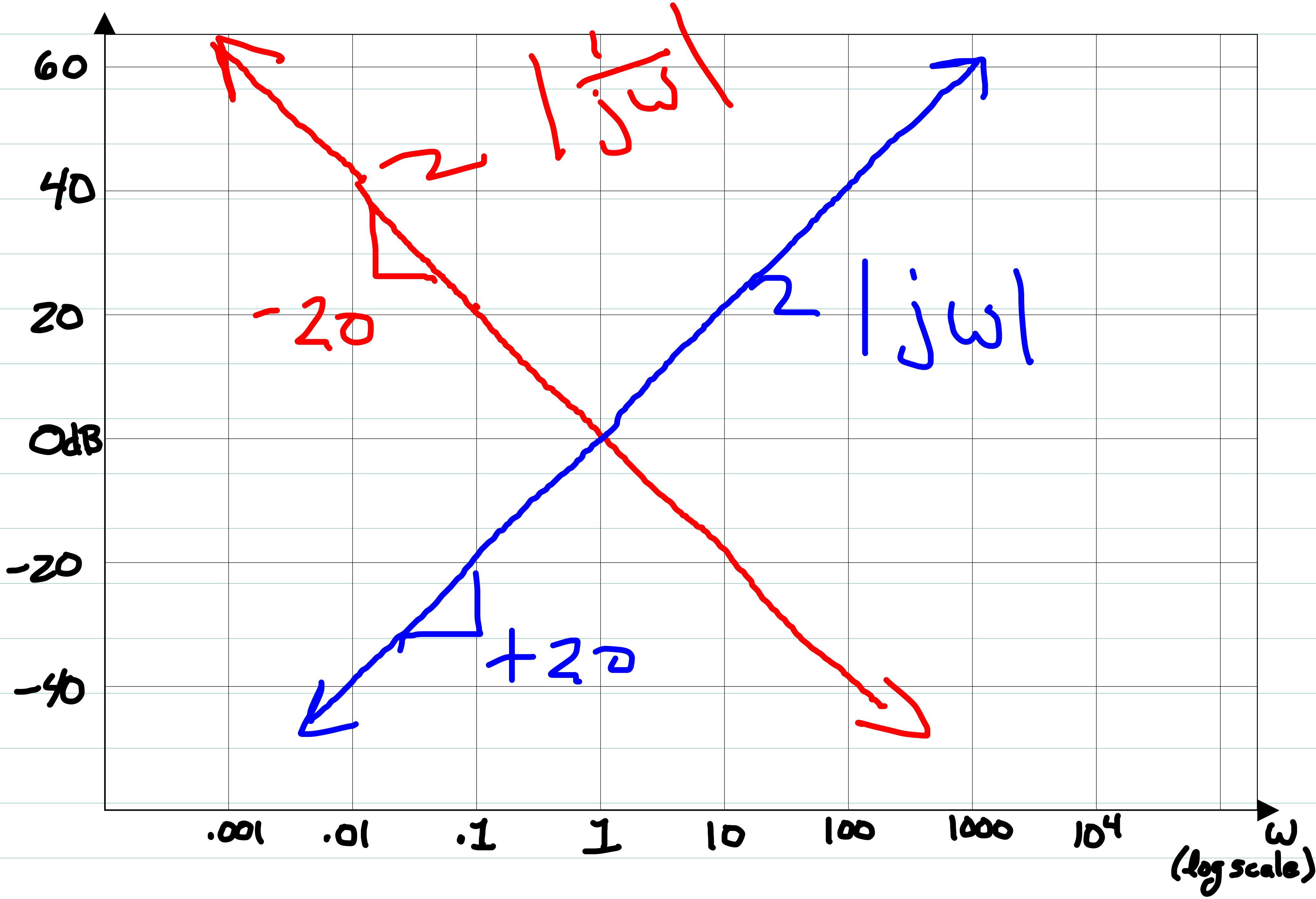
have corner frequencies at $\omega = 0$

$\Rightarrow$ infinitely far to left on horizontal frequency Axis.

These factors do not produce "visible" corners, instead contribute a constant slope of $-20N$ dB/dec for all freqs.

Note also: $|(j\omega)^N| = 1$ at $\omega = 1$ for any N

so graph of $|(j\omega)^N|_{dB}$ will pass through 0 dB at $\omega = 1$



60
40
20
0dB
-20
-40
.001
.01
.1
1
10
100
1000
10^4
ω (log scale)
∠ |jω|
-20
∠ |jω|
+20

For $G(s)$ with poles/zeros at origin:

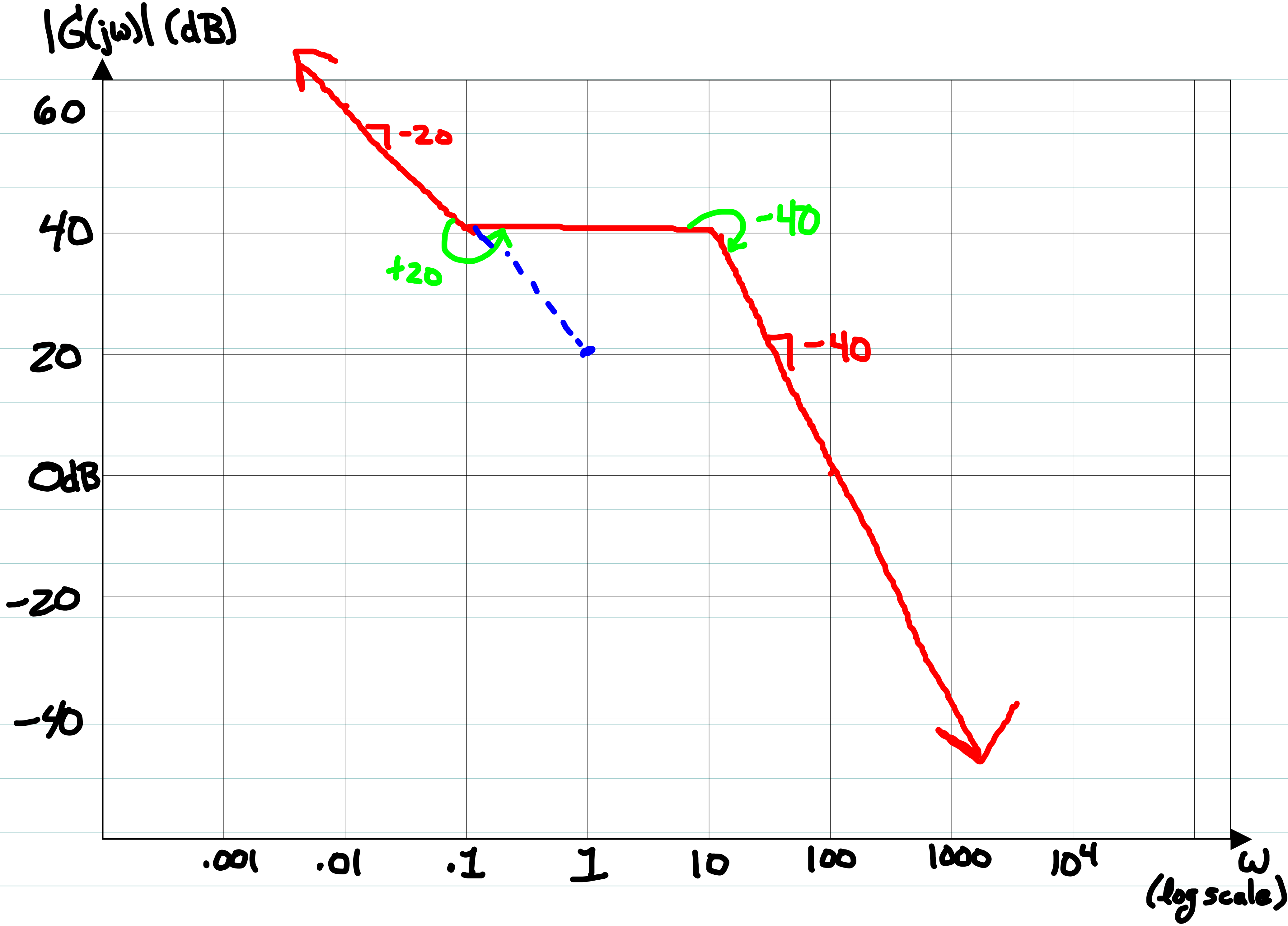
⇒ Start diagram by sketching effect of these poles at low frequencies

⇒ Note if $|K_B| \neq 1$, then this low frequency asymptote will pass through $|K_B|_{dB}$ at $\omega = 1$

⇒ Then add bends due to nonzero z_i and p_K as usual.

Example:

$$G(s) = 10\left[\frac{(10s+1)}{s(s/10+1)^2}\right]$$



|G(jω)| (dB)
60
40
20
0dB
-20
-40
-20
+20
-40
-40
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)

What about phase?

Recall:

$$\measuredangle G(j\omega) = \measuredangle K_B - N\measuredangle(j\omega) + \sum_{i=1}^{m} \measuredangle\left(1 - \frac{j\omega}{z_i}\right) - \sum_{k=N+1}^{n} \left(1 - \frac{j\omega}{p_k}\right)$$

$$\measuredangle K_B = \begin{cases} \emptyset & K_B > \emptyset \\ -180 & K_B < \emptyset \end{cases} \quad \text{for all } \omega \geq \emptyset$$

$$\measuredangle (j\omega) = 90^\circ \quad \text{for all } \omega \geq \emptyset$$

So, low frequency phase is <u>constant</u> at

$$-90N \quad \text{if } K_B > \emptyset$$

$$-180 - 90N \quad \text{if } K_B < \emptyset$$

Other poles/zeros will cause "bends" at higher freqs.

Phase response from other poles/zeros

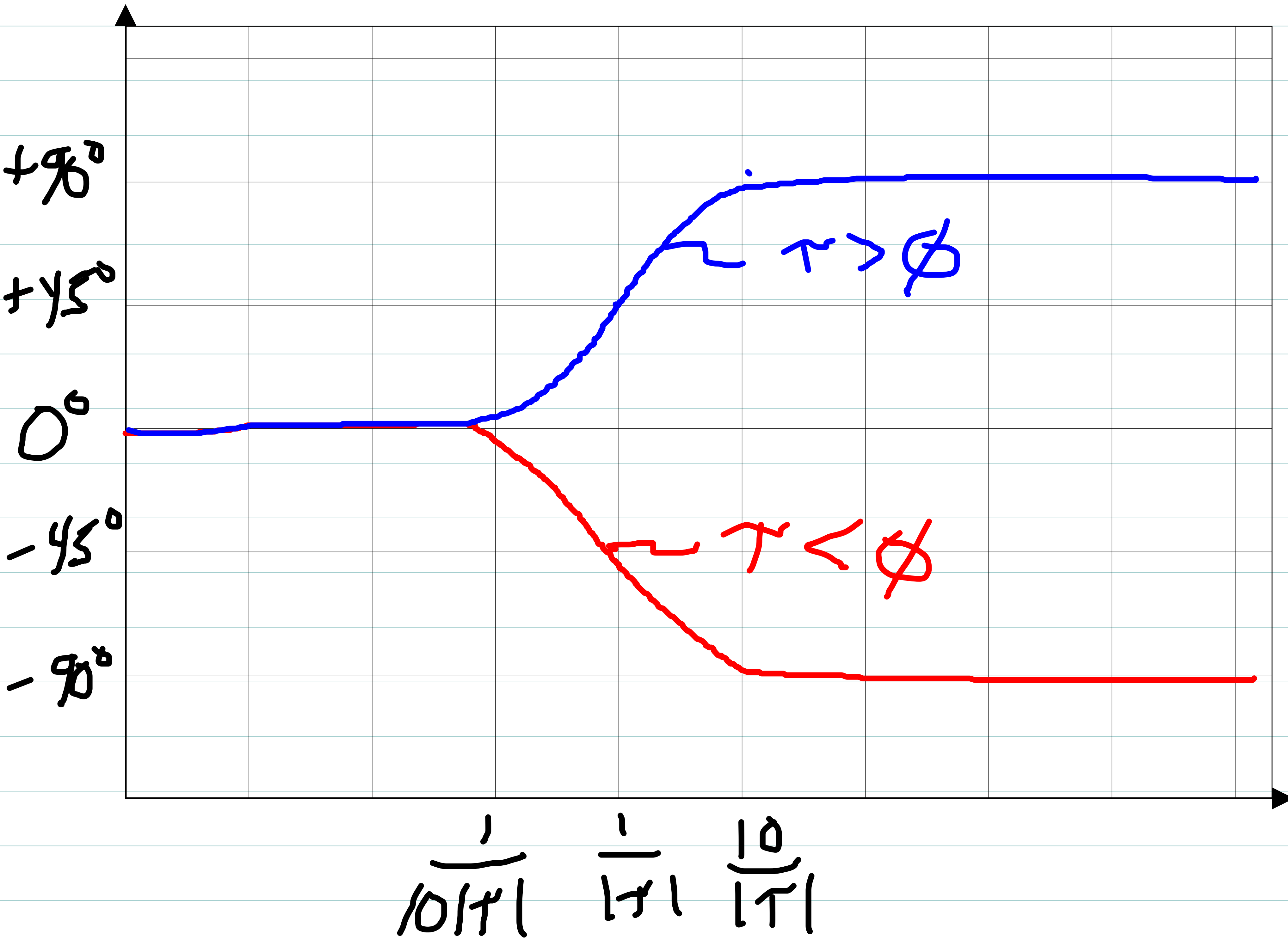
Consider again in generic form $(1+j\omega\tau)$ with $\tau = -1/z_i$ or $\tau = -1/p_k$

$$\measuredangle(1+j\omega\tau) = \tan^{-1}\omega\tau$$

$$= \begin{cases} 0 & \text{if } \omega \ll 1/|\tau| \\ +45^\circ & \text{if } \omega = 1/|\tau| \\ +90^\circ & \text{if } \omega \gg 1/|\tau| \end{cases}$$

above is for $\tau > 0$. If instead $\tau < 0$

$$\measuredangle(1+j\omega\tau) = -\tan^{-1}\omega|\tau| = \begin{cases} 0 & \text{if } \omega \ll 1/|\tau| \\ -45^\circ & \text{if } \omega = 1/|\tau| \\ -90^\circ & \text{if } \omega \gg 1/|\tau| \end{cases}$$



+90°
+45°
0°
-45°
-90°
τ > ∅
τ < ∅
1/10|τ|
1/|τ|
10/|τ|

Observations

⇒ Phase change due to a single factor occurs in a 2 decade band of frequencies centered at the magnitude corner frequency $1/|\tau|$

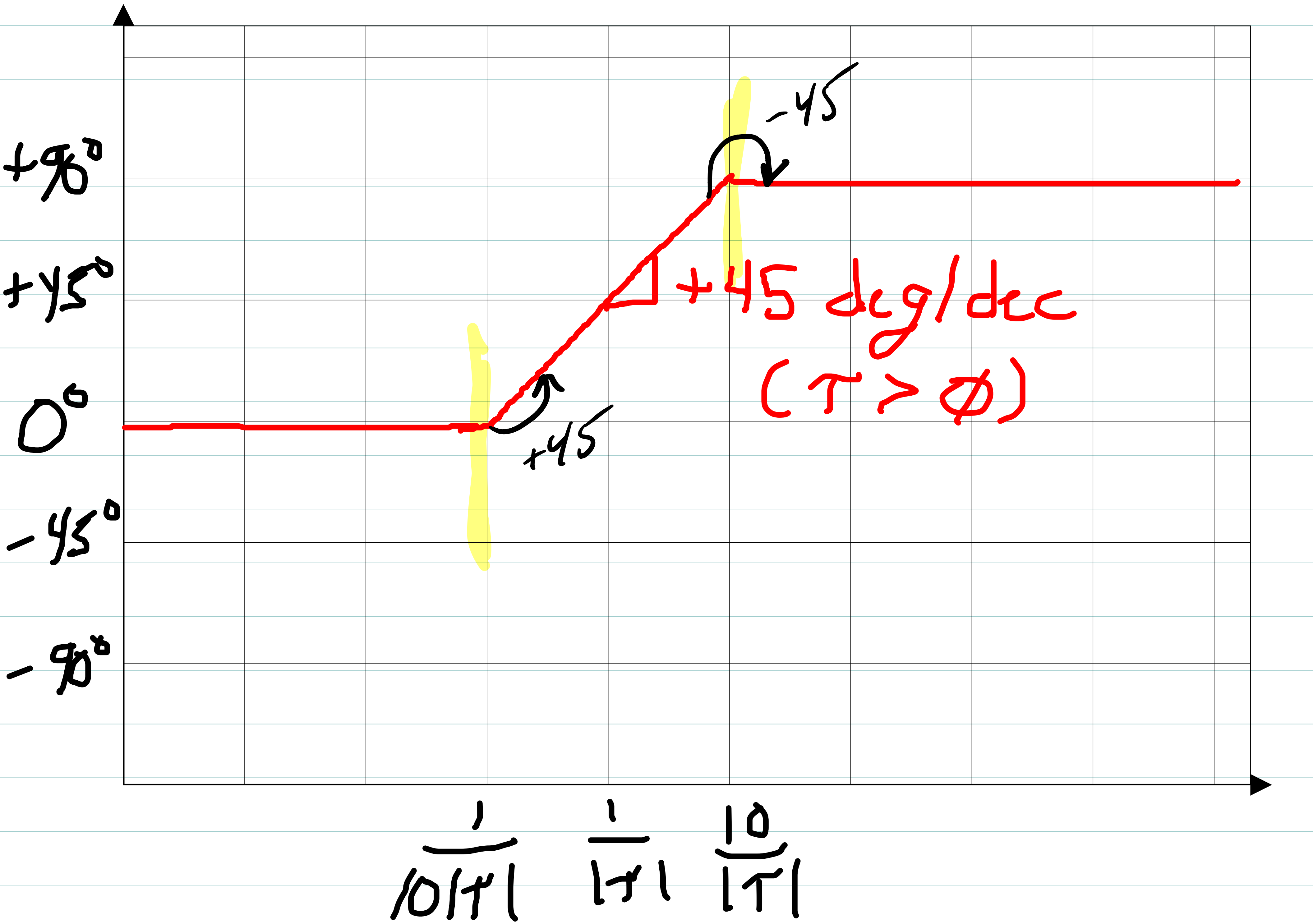
i.e. in band $\frac{1}{10|\tau|} \leq \omega \leq 10/|\tau|$

⇒ Phase is constant outside this band

low freq phase $\approx 0°$

h.f. phase $\approx \pm 90°$ ($+90°$ if $\tau > \phi$, $-90°$ if $\tau < \phi$

⇒ Phase change is approximate linear across band with slope $\pm 45°/\text{dec}$



+90°
+45°
0°
-45°
-90°
-45
+45
+45 deg/dec
(τ > 0)
1/(10|τ|)
1/|τ|
10/|τ|

Sign of phase change depends on:

=> whether factor is pole or zero

=> whether factor is RHP ($\tau < \phi$) or LHP ($\tau > \phi$)

Suppose all factors are LHP, $z_i < \phi$ $p_k < \phi$

then all $\tau = {}^{-1}/_{z_i}$ or ${}^{-1}/_{p_k}$ are positive.

This is called the "minimum phase" case

Then:

=> zeros cause $+90°$ phase change over band $\frac{|z_i|}{10}$ to $10|z_i|$

=> poles cause $-90°$ change over $\frac{|p_k|}{10}$ to $10|p_k|$

(Minimum Phase Systems)

<u>Slopes</u> of phase change are $+45°$/dec (zeros) or $-45°$/dec (poles) in these bands

Note phase changes in minimum phase cases mirror those for magnitude changes:

⇒ zeros cause <u>positive</u> slope changes
⇒ poles cause <u>negative</u> slope changes.

Graphical addition is again straightforward, but requires a little care:

⇒ slopes are nonzero only in a 2 decade band

⇒ bands from different factors may overlap.

Example:

$$G(s) = \frac{10s+1}{s(s+1)(s/10+1)}$$

Low freq. phase $-90°$

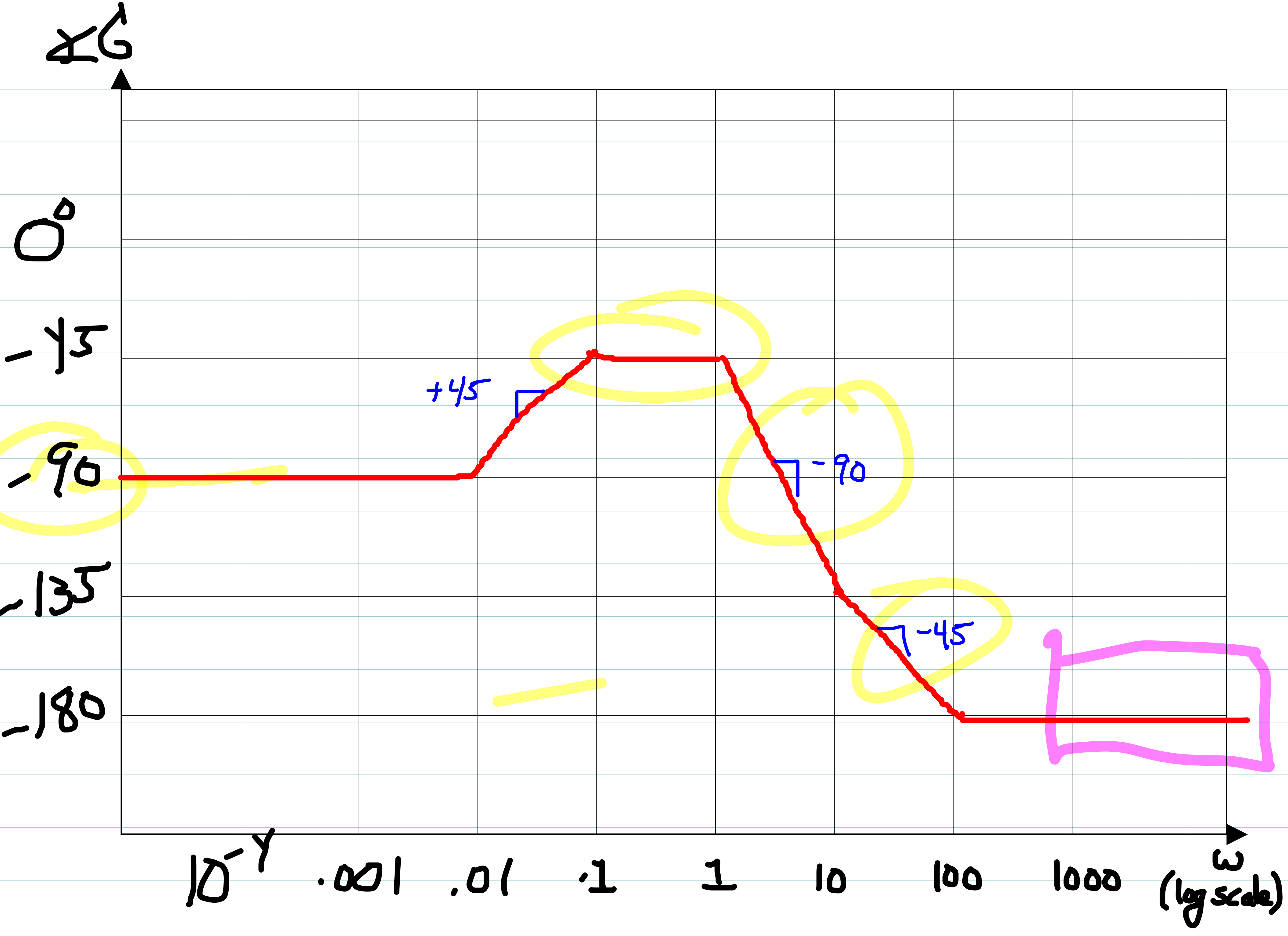
Phase changes:

$+45°$/dec in .01 to 1
$-45°$/dec in .1 to 10
$-45°$/dec in 1 to 100

Net:

$+45°$/dec in .01 to .1
$0°$/dec in .1 to 1
$-90°$/dec in 1 to 10
$-45°$/dec in 10 to 100

constant for $\omega > 100$.


∡G
0°
-45
-90
-135
-180
+45
-90
-45
10^-4
.001
.01
.1
1
10
100
1000
ω (log scale)

Repeated factors

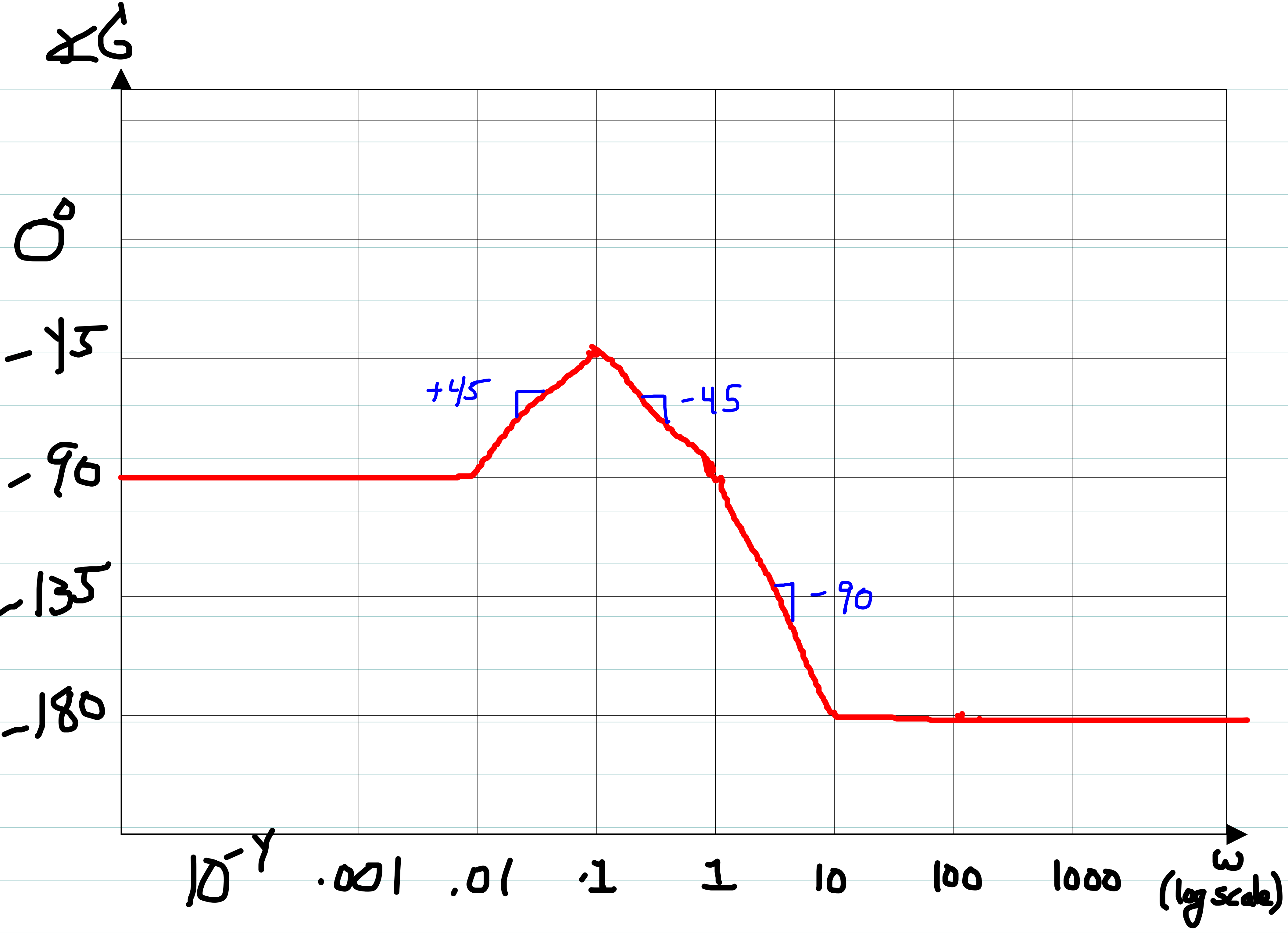
Repeated factors $(1+j\omega T)^l$ multiply the phase changes by l, just like magnitudes.

Example:

$$G(s) = \frac{10s+1}{s(s+1)^2}$$

Changes: +45°/dec in .01 to 1
−90°/dec in .1 to 10

Net: +45°/dec in .01 to .1
−45°/dec in .1 to 1
−90°/dec in 1 to 10


∡G
0°
-45
-90
-135
-180
+45
-45
-90
10^{-4}
.001
.01
.1
1
10
100
1000
ω
(log scale)

Summary (minimum phase)

=> Low freq. phase is $\angle K_B - N\,90°$

"relative degree"

$\#$poles $-$ $\#$zeros

=> high freq. phase is $\angle K_B - 90°(n-m)$

=> Note low and high freq. phases are constant (slope is zero).

=> Recall typically $n > m$ for a physical system so high freq. phase is typically negative for a minimum phase system.

=> zeros cause $+90°$ change at rate of $+45°$/dec in 2 decade band centered at $|z_i|$

=> poles cause $-90°$ change at rate of $-45°$/dec in 2 decade band centered at $|P_k|$.

Can be tricky to accurately sketch phase

=> Overlapping change regions for multiple factors

=> No standard formula for phase change of underdamped factors

=> Helps to 1st make a table of slope changes over frequency ranges as above

=> Generally, straight-line phase sketch is less accurate than magnitude sketch.

=> Still sufficiently accurate to give us a good general idea of phase behavior.

=> We'll use Matlab when greater accuracy is required.