ENRE 447 - 0101
Homework 02:

Due on February 24th, 2025 at 03:30 PM

Dr. Groth, 03:30 PM

Vai Srivastava

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Problem 1:

Find the constant c so that

$$\mathrm{f}(x,y) = \begin{cases} cxy & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint pdf of X and Y. Find the following:

1. Are X and Y independent?
2. $\mathrm{E}(X)$
3. $\mathrm{E}(Y)$
4. $\mathrm{E}(XY)$
5. $\mathrm{Var}(X)$
6. $\mathrm{Var}(Y)$
7. $\mathrm{Cov}(X,Y)$

Solution

$$\int_0^1 \int_0^1 cxy\,dx\,dy = c\Big(\int_0^1 x\,dx\Big)\Big(\int_0^1 y\,dy\Big) = c\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{c}{4} = 1,$$
$$\implies \quad c = 4. \qquad \square$$

Part A

$$f_X(x) = \int_0^1 4xy\,dy = 4x\left(\frac{1}{2}\right) = 2x, \quad 0 \le x \le 1,$$
$$f_Y(y) = \int_0^1 4xy\,dx = 2y, \quad 0 \le y \le 1. f(x,y) = 4xy = (2x)(2y) = f_X(x)f_Y(y)$$

$\therefore X$ and Y are independent. $\square$

Part B

$$\mathrm{E}(X) = \int_0^1 x\,f_X(x)\,dx = \int_0^1 x\,(2x)\,dx = 2\int_0^1 x^2\,dx = 2\left(\frac{1}{3}\right) = \frac{2}{3}. \qquad \square$$

Part C

$$\mathrm{E}(Y) = \frac{2}{3}. \qquad \square$$

Part D

$$\mathrm{E}(XY) = \int_0^1 \int_0^1 xy\,(4xy)\,dx\,dy = 4\left(\int_0^1 x^2\,dx\right)\left(\int_0^1 y^2\,dy\right)$$
$$= 4\left(\frac{1}{3}\right)^2 = \frac{4}{9}. \qquad \square$$

Problem 1 continued on next page...

Part E

$$\begin{aligned}
\operatorname{Var}(X) &= \operatorname{E}\left(X^2\right) - \left(\operatorname{E}(X)\right)^2, \\
\operatorname{E}\left(X^2\right) &= \int_0^1 x^2(2x)\,dx = 2\int_0^1 x^3\,dx = 2\left(\frac{1}{4}\right) = \frac{1}{2}, \\
\operatorname{Var}(X) &= \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}. \qquad \square
\end{aligned}$$

Part F

$$\operatorname{Var}(Y) = \frac{1}{18}. \qquad \square$$

Part G

$$\operatorname{Cov}(X, Y) = \operatorname{E}(XY) - \operatorname{E}(X)\operatorname{E}(Y) = \frac{4}{9} - \left(\frac{2}{3}\right)^2 = 0. \qquad \square$$

Problem 2:

Given that Pr = 0.006

1. No engine failure in 1000 flights.
2. At least one failure in 1000 flights.
3. At least two failures in 1000 flights.

Solution

We model the number of failures by a Binomial distribution $X \sim \text{Bin}(1000, 0.006)$.

Part A

$$P(X = 0) = (1 - p)^n = (0.994)^{1000} = 0.002434. \qquad \square$$

Part B

$$P(\text{at least one failure}) = 1 - P(X = 0) = 1 - (0.994)^{1000} = 0.9976. \qquad \square$$

Part C

$$\begin{aligned} P(\text{at least 2 failures}) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - (0.994)^{1000} - \binom{1000}{1}(0.006)(0.994)^{999} = 0.9829. \qquad \square \end{aligned}$$

Problem 3:

The manufacturer of a type pump states that, on average, this type of pump experiences 3.0 failures per $100,000$ operational hours. At a factory with many of these pumps, they will accumulate $200,000$ operational hours this year. Find the probability that there will be each of the following amounts of pump failures at the factory this year.

1. 0
2. 2
3. 6
4. 8
5. Between 4 and 8
6. Fewer than 3

Solution

Part A

$$P(X=0) = e^{-6}\frac{6^0}{0!} = e^{-6} = 0.002479. \qquad \square$$

Part B

$$P(X=2) = e^{-6}\frac{6^2}{2!} = 18e^{-6} = 0.04462. \qquad \square$$

Part C

$$P(X=6) = e^{-6}\frac{6^6}{6!} = 0.1606. \qquad \square$$

Part D

$$P(X=8) = e^{-6}\frac{6^8}{8!} = 0.1033. \qquad \square$$

Part E

$$P(4 \leq X \leq 8) = \sum_{k=4}^{8} e^{-6}\frac{6^k}{k!} = 0.4589. \qquad \square$$

Part F

$$P(X<3) = \sum_{k=0}^{2} e^{-6}\frac{6^k}{k!} = e^{-6}\left(1+6+\frac{6^2}{2}\right) = 25e^{-6} = 0.06197. \qquad \square$$

Problem 4:

If the diameter of a given kind of ball bearings are normally distributed with the mean 0.6140 in and standard deviation 0.0025 in, determine the percentage of ball bearings with diameters:

1. Between 0.610 and 0.618 in, inclusive
2. Greater than 0.617 in
3. Less than 0.608 in
4. Equal to 0.615 in

Solution

Part A

$$z_1 = \frac{0.610 - 0.6140}{0.0025} = -1.6, \quad z_2 = \frac{0.618 - 0.6140}{0.0025} = 1.6,$$
$$P(0.610 \leq X \leq 0.618) = \Phi(1.6) - \Phi(-1.6) \approx 0.9452 - 0.0548 = 0.8904. \qquad \square$$

Part B

$$z = \frac{0.617 - 0.6140}{0.0025} = 1.2,$$
$$P(X > 0.617) = 1 - \Phi(1.2) \approx 1 - 0.8849 = 0.1151. \qquad \square$$

Part C

$$z = \frac{0.608 - 0.6140}{0.0025} = -2.4,$$
$$P(X < 0.608) = \Phi(-2.4) \approx 0.0082. \qquad \square$$

Part D

$$P(X = 0.615) = 0 \quad \text{(since the distribution is continuous).} \qquad \square$$

Problem 5:

Assume that T, the random variable that denotes life in hours of specified component, has a cumulative density function (cdf) of

$$\mathrm{F}(t) = \begin{cases} 1 - \frac{100}{t} & t \geq 100 \\ 0 & t < 100 \end{cases}$$

Determine the following:

1. PDF $\mathrm{f}(t)$
2. Reliability function $\mathrm{R}(t)$
3. MTTF (Using a practical upper limit of 1 million hrs to avoid trivial solution)

Solution

Part A

$$f(t) = \frac{d}{dt}F(t) = \frac{d}{dt}\left(1 - \frac{100}{t}\right) = \frac{100}{t^2}, \quad t \geq 100. \qquad \square$$

Part B

$$R(t) = 1 - F(t) = \frac{100}{t}, \quad t \geq 100. \qquad \square$$

Part C

$$\begin{aligned} \text{MTTF} &= \int_{100}^{10^6} R(t)\,dt = 100\int_{100}^{10^6} \frac{1}{t}\,dt \\ &= 100\Big[\ln t\Big]_{100}^{10^6} = 100\ln\left(\frac{10^6}{100}\right) = 100\ln(10^4) = 400\ln(10) \\ &= 921.034 \text{ hours} \qquad \square \end{aligned}$$

Problem 6:

A manufacturer uses the exponential distribution to model the number of cycles to failure for a product. The product has $\lambda = 0.003$ failures/cycle.

1. What is the mean cycle to failure for this product?
2. If the product survives for 300 cycles, what is the probability that it will fail sometimes after 500 cycles? If operational data show that 1000 components have survived 300 cycles, how many of these would be expected to fail after 500 cycles?

Solution

Part A

$$\mathrm{E}\left(T\right) = \frac{1}{\lambda} = \frac{1}{0.003} \approx 333.33 \text{ cycles.} \qquad \square$$

Part B

$$P(T > 500 \mid T > 300) = e^{-\lambda(500-300)} = e^{-0.003 \cdot 200} = e^{-0.6}.$$

Thus, for 1000 components:

$$\text{Expected amount} = 1000\, e^{-0.6} = 548.8 \implies 549 \text{ components.} \qquad \square$$

Problem 7:

Time to failure of a relay follows a Weibull distribution with $\alpha = 10$ years, $\beta = 0.5$. Find the following:

1. Pr (failure after 1 year)
2. Pr (failure after 10 years)
3. MTTF

Solution

$$P(t) = e^{-(t/10)^{0.5}}.$$

Part A

$$P(\text{failure by 1 year}) = P(1) = e^{-(1/10)^{0.5}} = e^{-1/\sqrt{10}} = 0.7289. \qquad \square$$

Part B

$$P(\text{failure by 10 years}) = P(10) = e^{-(10/10)^{0.5}} = e^{-1} = 0.3679. \qquad \square$$

Part C

$$\text{MTTF} = \alpha\,\Gamma\left(1 + \frac{1}{\beta}\right) = 10\,\Gamma\left(1 + 2\right) = 10\,\Gamma(3) = 10 \cdot 2 = 20 \text{ years}. \qquad \square$$

Problem 8:

An electronic device has a time to failure modeled by the lognormal distribution with parameters $\mu = 5.8$ and $\sigma = 1.2$.

1. Find the MTTF
2. If this device is used in an application which requires it to be replaced when its reliability falls below 0.9, when should the device be replaced?
3. Find the hazard function for the device at the time calculated in the previous part.

Solution

Part A

$$\text{MTTF} = \mathrm{E}(T) = e^{\left(\mu+\frac{\sigma^2}{2}\right)} = e^{\left(5.8+\frac{1.44}{2}\right)} = e^{(5.8+0.72)} = e^{(6.52)} = 678.6. \qquad \square$$

Part B

$$\begin{aligned}
R(t) =& 1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right) \\
1 - \Phi&\left(\frac{\ln t - 5.8}{1.2}\right) = 0.9 \quad \Longrightarrow \quad \Phi\left(\frac{\ln t - 5.8}{1.2}\right) = 0.1 \\
z_{0.1} &\approx -1.2816 \\
\frac{\ln t - 5.8}{1.2} &= -1.2816 \\
\ln t &= 5.8 - 1.2816(1.2) \approx 5.8 - 1.5379 = 4.2621, \\
t &\approx \exp(4.2621) \approx 71.0 \text{ (hours).} \qquad \square
\end{aligned}$$

Part C

$$\begin{aligned}
h(t) &= \frac{f(t)}{R(t)}, \\
f(t) &= \frac{1}{t\,\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln t - \mu)^2}{2\sigma^2}\right) \\
t &\approx 71.0, R(71.0) \approx 0.9 \\
h(71.0) &\approx \frac{f(71.0)}{0.9} \approx 0.00229. \qquad \square
\end{aligned}$$