Polar Plots

⇒ A different way of showing the properties of $G(j\omega)$

⇒ Bode plots $|G(j\omega)|$ and $\measuredangle G(j\omega)$ vs. ω, using logarithmic scales for $0 \le \omega < \infty$

⇒ Polar shows $G(j\omega)$ as points on complex plane as ω VARIES from $\emptyset$ to ∞ using <u>actual</u> (non-logarithmic) scales

⇒ Learn to <u>sketch</u> polar from Bode

⇒ We are aiming for something qualitatively correct, but will deliberately distort scales to make certain critical features readily apparent.

Im{G(jω)}
Re{G(jω)}
G(jω)
|G(jω)|
∡G(jω)

⇒ For each $\omega \in [0,\infty)$, $G(j\omega)$ is a <u>different</u> point on complex plane

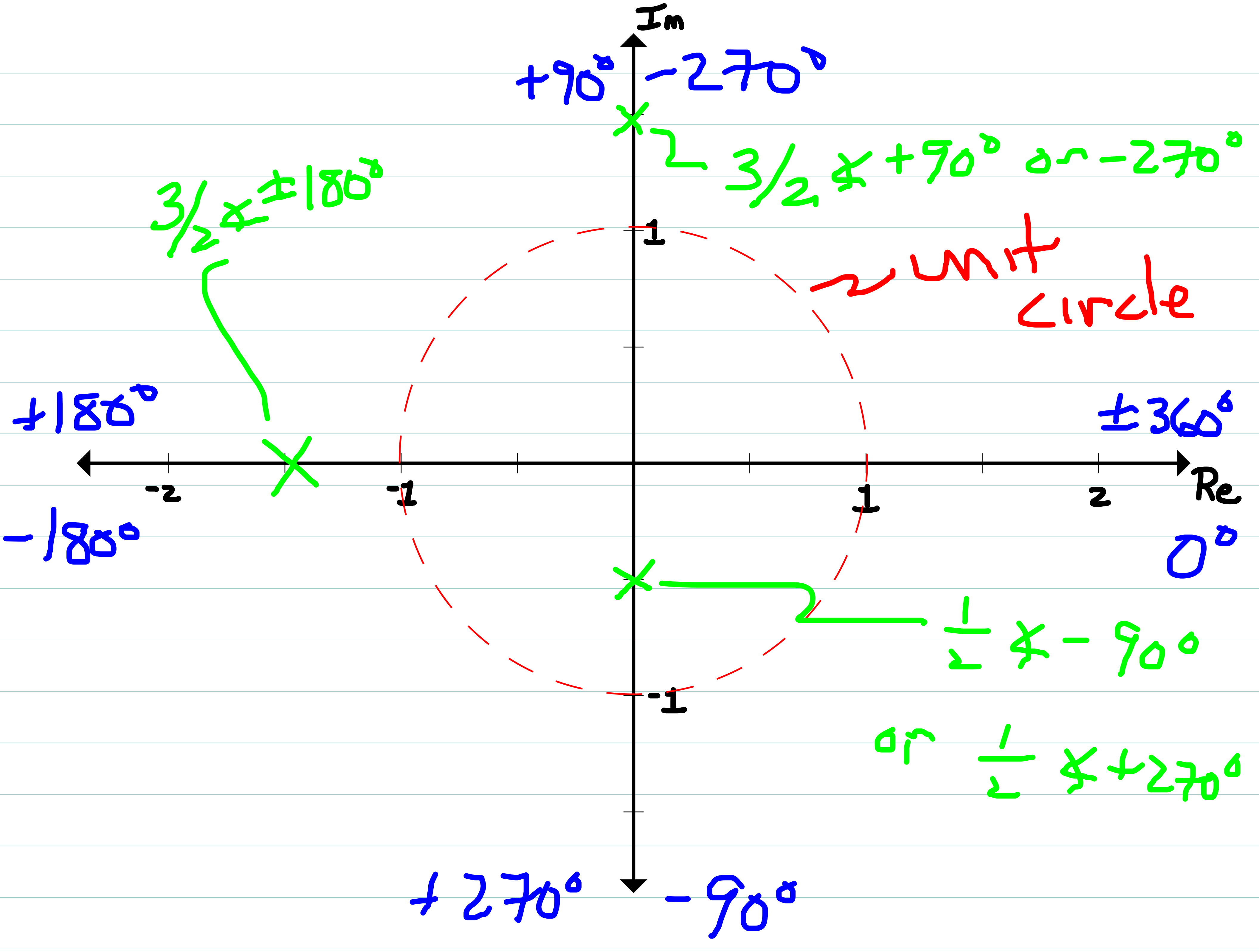
⇒ As ω varies from $\emptyset$ to ∞, these points will trace out a <u>curve</u> on complex plane.

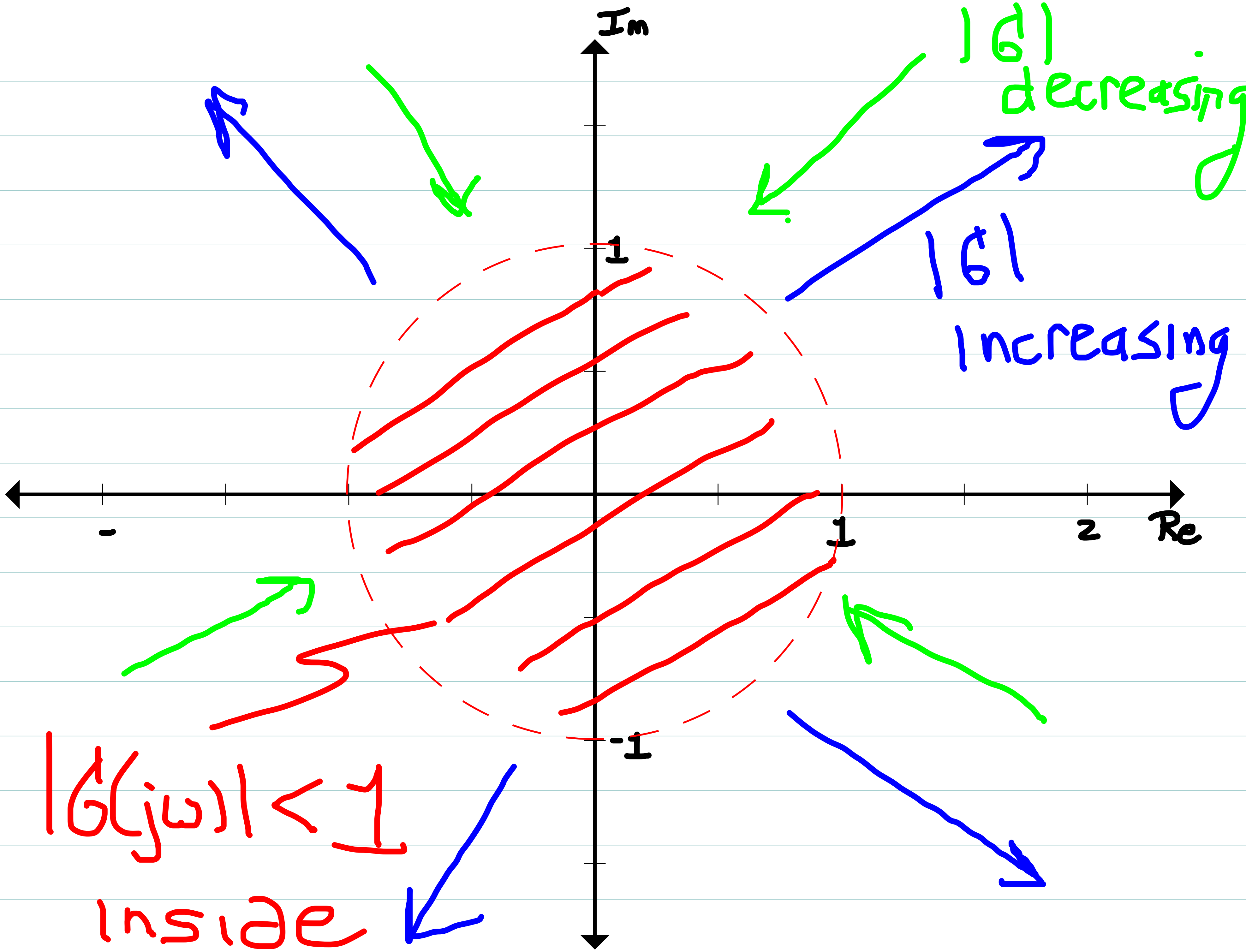
⇒ Bode diagrams show us the <u>polar coordinates</u> of the points $G(j\omega)$ for each ω

⇒ To map from Bode to polar

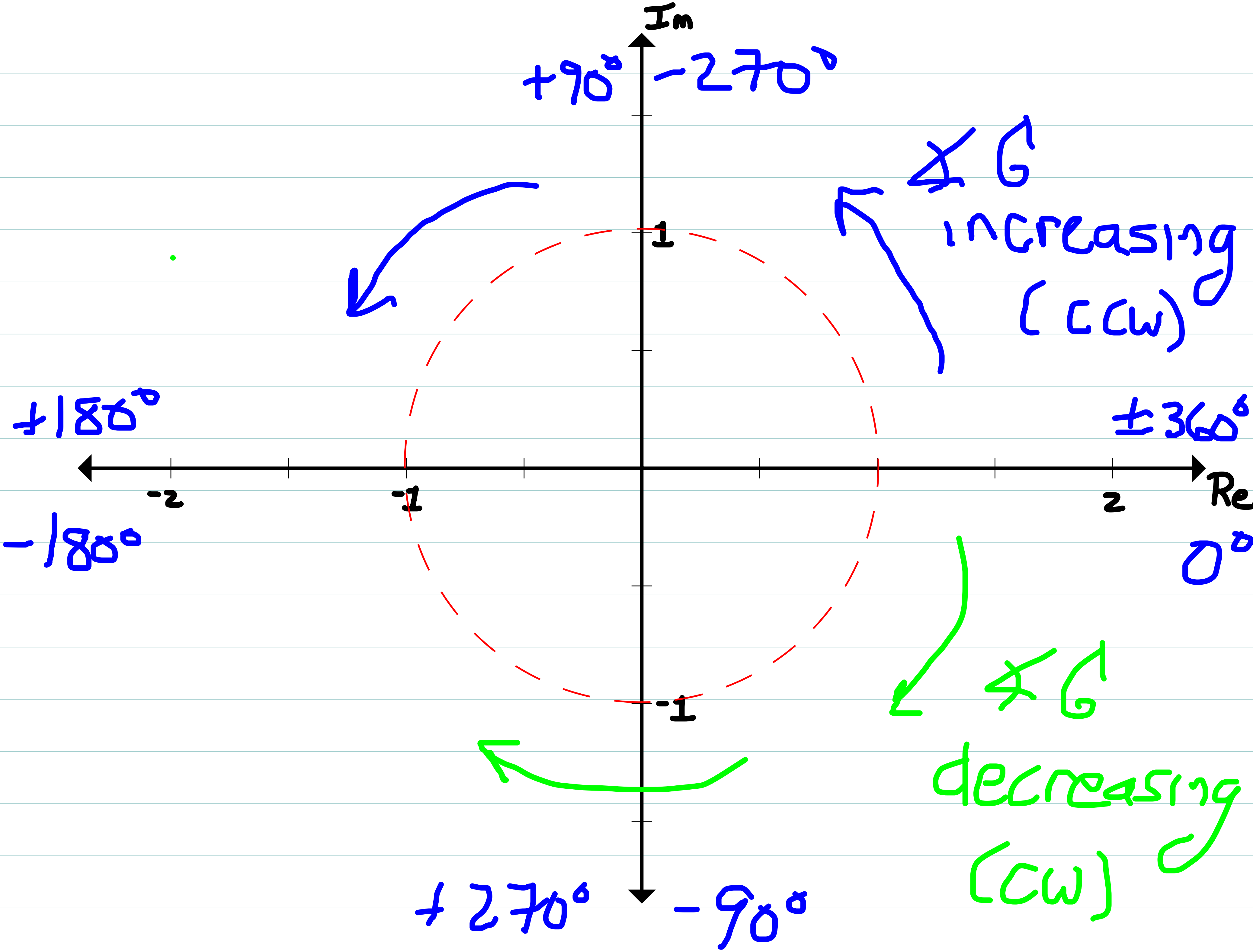
1.) Remember to convert magnitudes from dB back to actual.

2.) Remember angle convention for complex numbers.


Im
Re
+90°
−270°
$3/2 \measuredangle +90°$ or $-270°$
$3/2 \measuredangle \pm 180°$
unit circle
+180°
−180°
±360°
0°
−2
−1
1
2
1
−1
$\frac{1}{2} \measuredangle -90°$
or $\frac{1}{2} \measuredangle +270°$
+270°
−90°



Im
Re
1
-1
1
-
2
|G| decreasing
|G| increasing
|G(jω)| < 1
inside



Im
Re
+90°
−270°
+180°
−180°
±360°
0°
+270°
−90°
1
−1
−2
2
∡G increasing (CCW)
∡G decreasing (CW)

A simple Example

$$G(s) = \frac{K_B}{(1+Ts)} \qquad \begin{array}{l} T > \emptyset \text{ (min phase)} \\ K_B > 1 \end{array}$$

Alway start by thinking about low/high freq. limiting behavior:

for $\omega << \frac{1}{|T|}$: Mag slope =

Phase =

for $\omega >> \frac{1}{|T|}$: Mag slope =

Phase =

A simple Example

$$G(s) = \frac{K_B}{(1+\tau s)}$$ $\tau > \emptyset$ (min phase) $K_B > 1$

Alway start by thinking about low/high freq. limiting behavior:

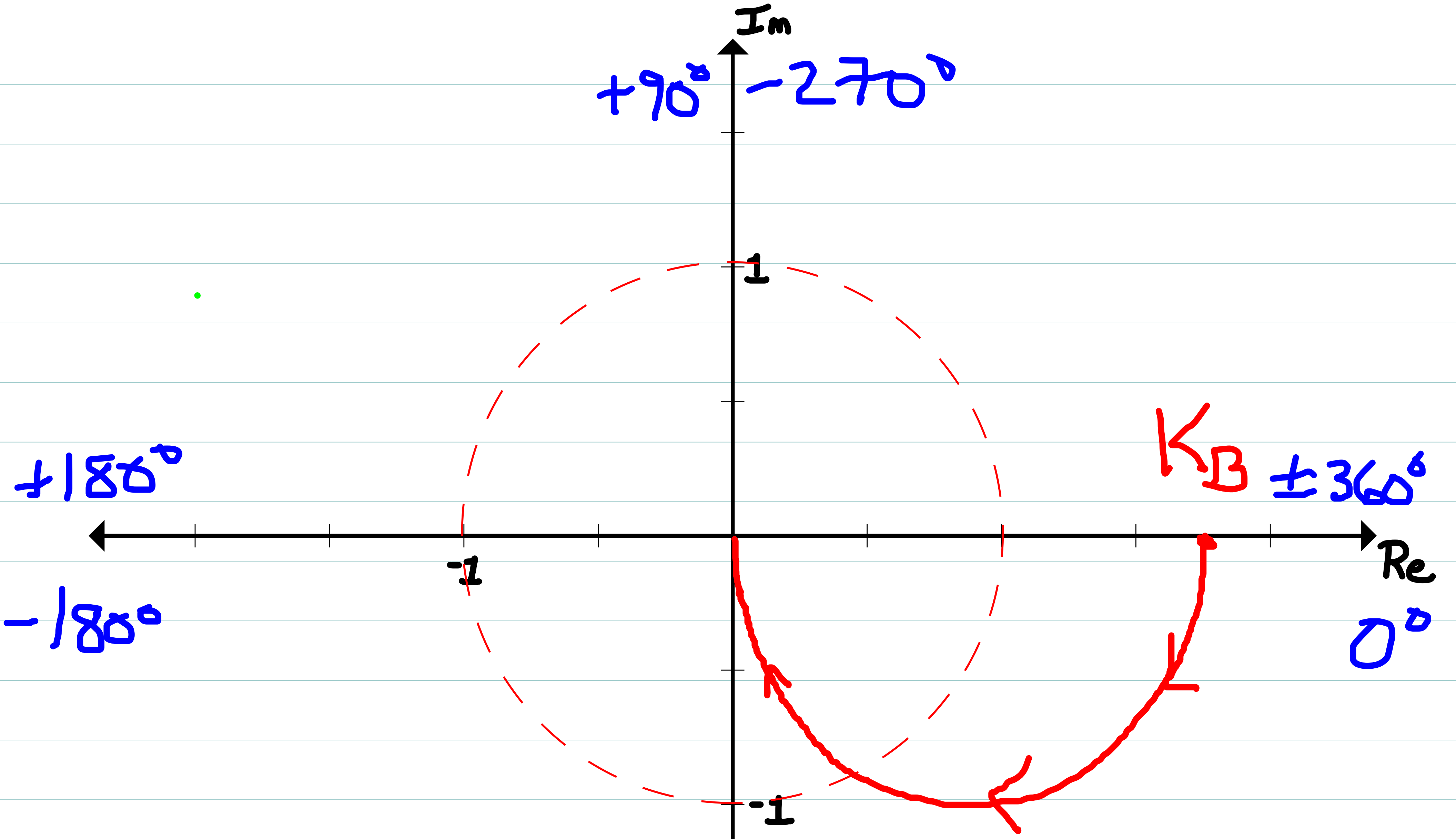
for $\omega \ll \frac{1}{|\tau|}$: Mag slope = 0 dB/dec (constant)

Phase = $0°$ (constant)

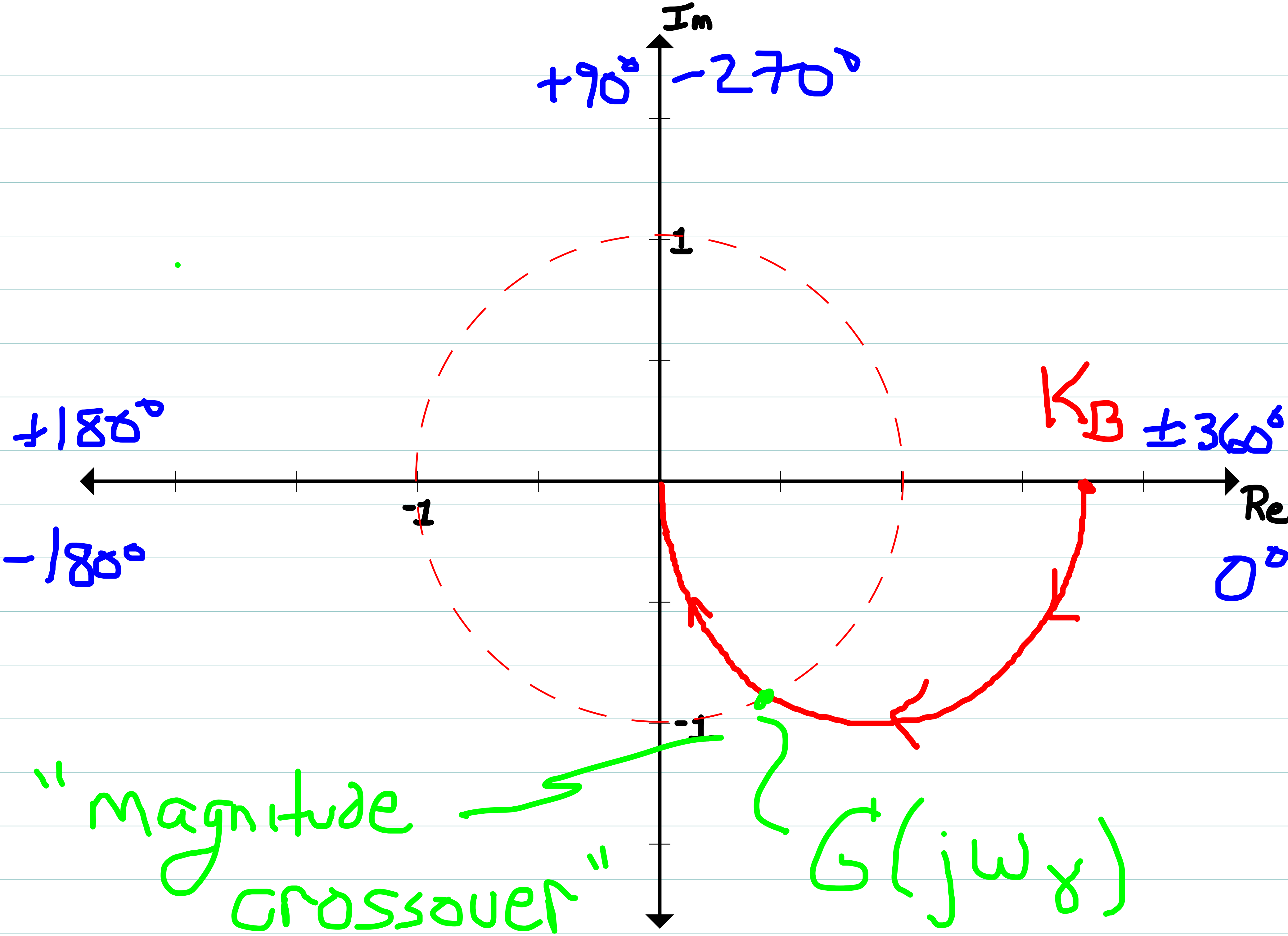
for $\omega \gg \frac{1}{|\tau|}$: Mag slope = -20 dB/dec

Phase = $-90°$

Low freq. magnitude is $|K_B| > 1$, high freq. magnitude is $\emptyset$: $\lim_{\omega \to \infty} |G(j\omega)| = \emptyset$



Arrows indicate increasing freq.



Im
+90°
-270°
1
+180°
-180°
-1
Re
K_B
±360°
0°
-1
"magnitude crossover"
$G(j\omega_\gamma)$

Magnitude Crossover

"Magnitude crossover" occurs where polar plot "punctures" the unit circle

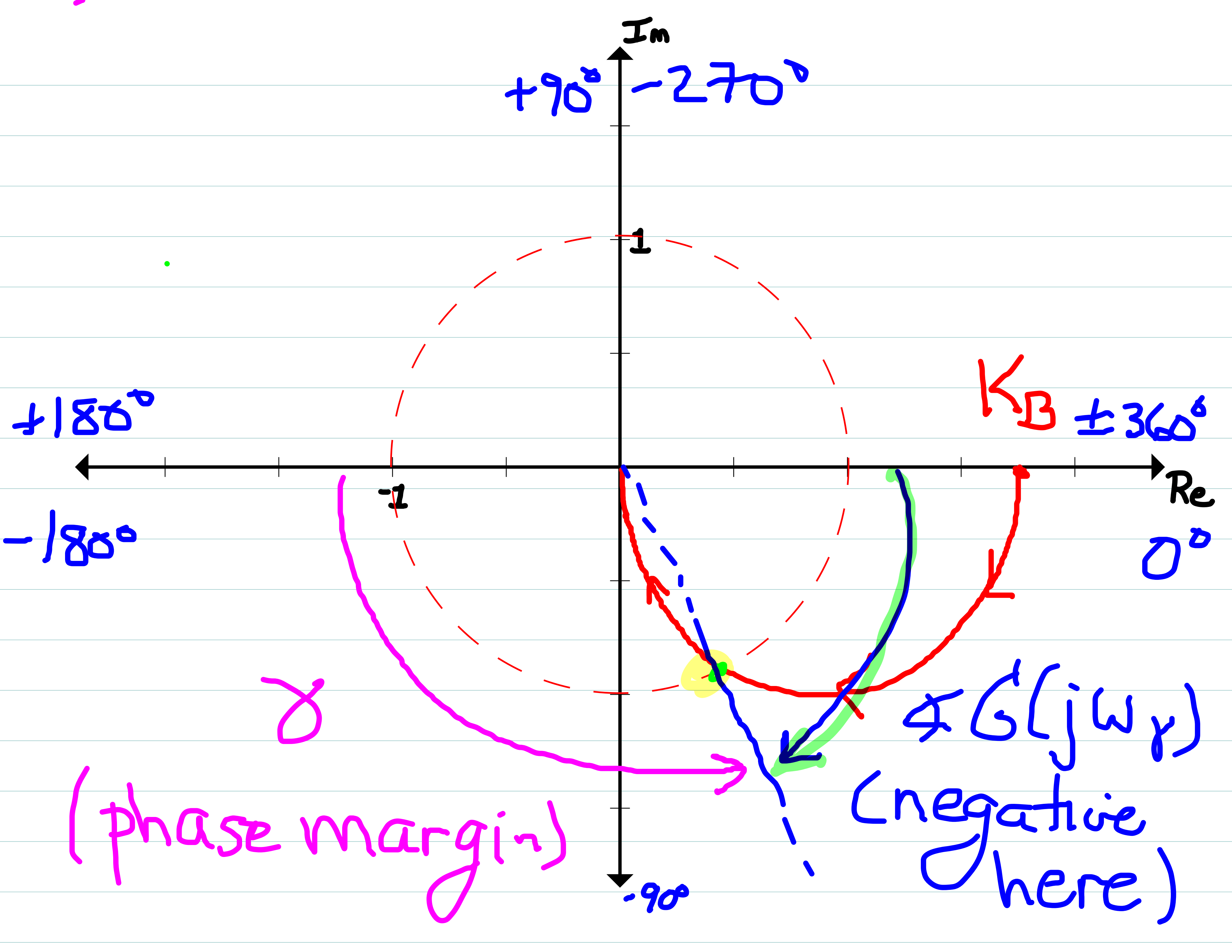
$|G(j\omega)| = 1$ at this point.

The frequency at which this occurs is the "magnitude crossover freq", termed ω_γ

Easily seen on Bode: $\boxed{\omega_\gamma \text{ is the frequency where}}$
$$|G(j\omega)| = \phi \text{ dB}$$

__Note:__ depending on the system there may be one, many, or no magnitude xover freq.

Important quantity: $\angle G(j\omega_\gamma)$: phase at magnitude xover


Im
+90°
-270°
1
+180°
-180°
-1
Re
±360°
0°
K_B
-90°
γ
(phase margin)
∡G(jω_γ)
(negative here)

Phase Margin

The **phase margin** is the angle around the unit circle from -1 to magnitude crossover point, measured positive **counter-clockwise** from -1 (or, equiv, CW from mag xover to -1)

The phase margin angle, γ, is expressed in deg (although later it will be convenient to express in rad).

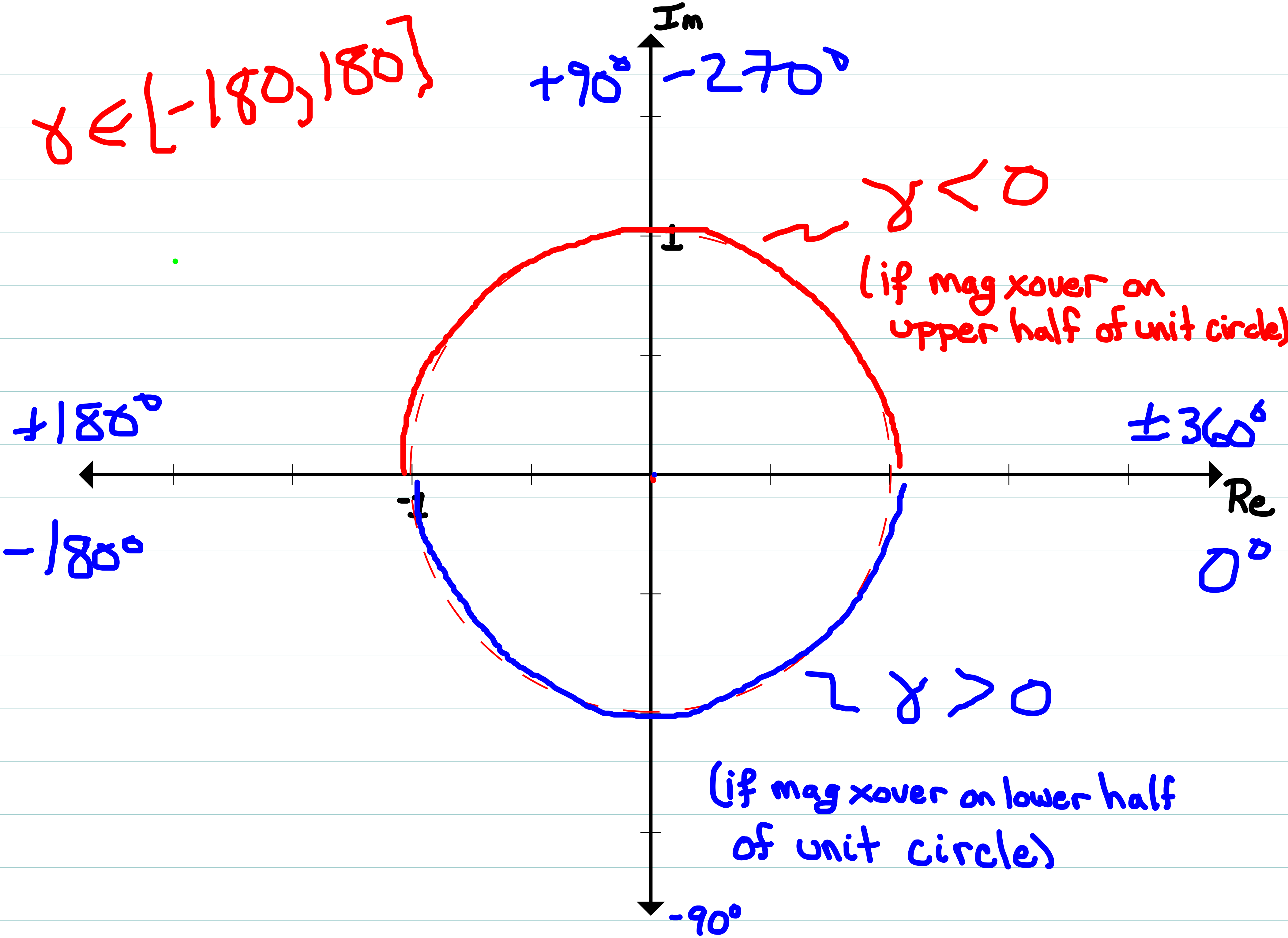
Assuming we write $\measuredangle G(j\omega_\gamma)$ in range $[0°, -360°]$ an expression for γ is:

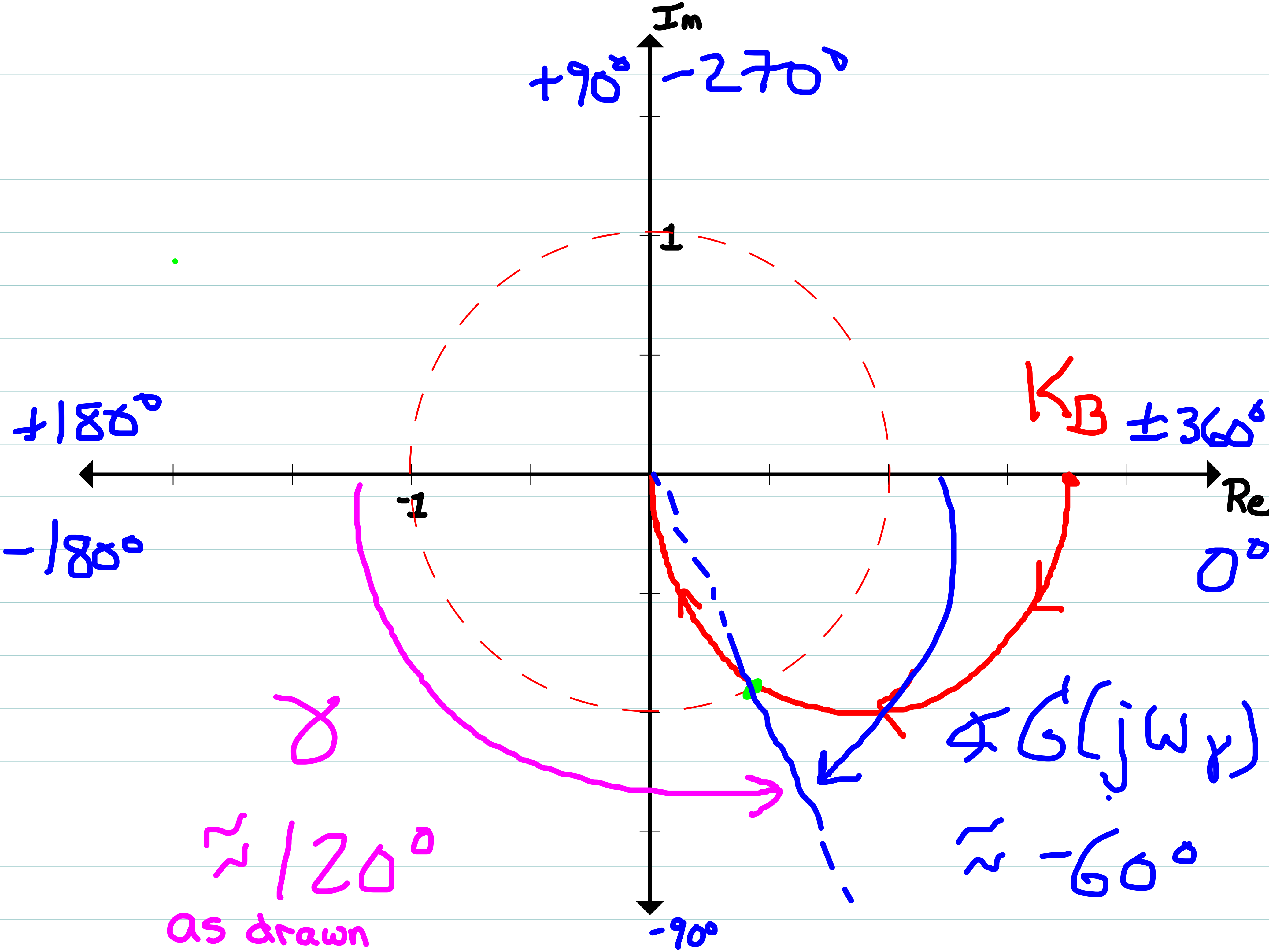
$\gamma \in [-180°, 180°]$

$$\gamma = 180° + \measuredangle G(j\omega_\gamma) \quad \left\{ \begin{array}{l} \Rightarrow \gamma > 0 \\ \text{if } \measuredangle G(j\omega_\gamma) > -180° \end{array} \right.$$

Note: Matlab will usually try to wrap phase plot $\measuredangle G(j\omega)$ so that $\measuredangle G(j\omega_\gamma)$ is in this range. Sometimes it doesn't. You can always manually add or subtract a multiple of $360°$ to get $\measuredangle G(j\omega_\gamma)$ in this range.

$\gamma \in [-180, 180]$




Im
+90°
-270°
1
+180°
-180°
-1
K_B
±360°
0°
Re
γ
≈120°
as drawn
∡G(jω_γ)
≈ -60°
-90°

Another Example

$$G(s) = \frac{K_B}{(1+\tau s)^3} \qquad K_B > 1 \quad \tau > \emptyset$$

Low freq mag: Constant at K_B

Low freq. phase: Constant at $0°$

High freq. mag slope:

High freq. phase:

Another Example

$$G(s) = \frac{K_B}{(1+\tau s)^3} \qquad K_B > 1, \quad \tau > \emptyset$$

Low freq mag: Constant at K_B

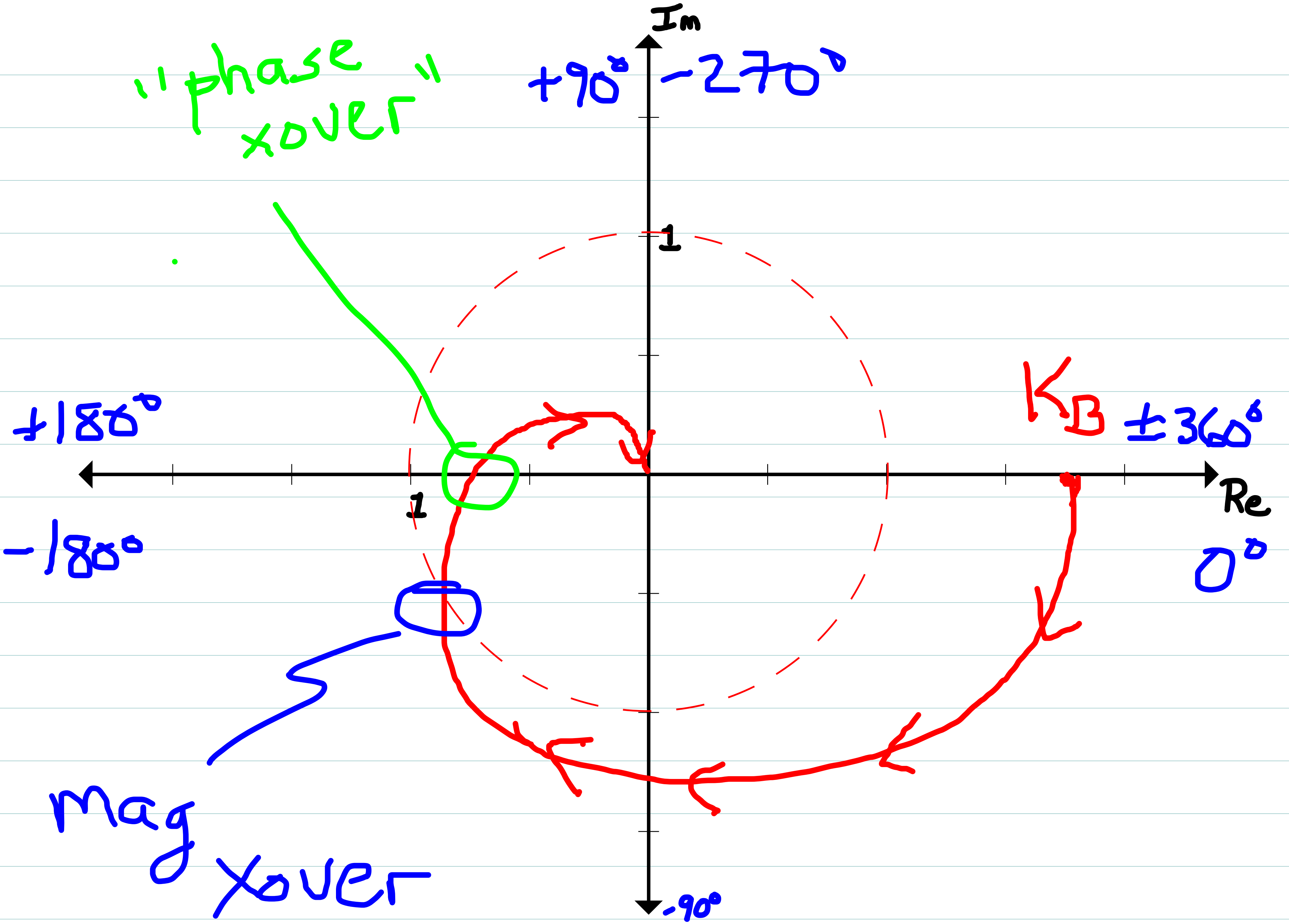
Low freq. phase: Constant at $0°$

High freq. mag slope: -60 dB/dec

High freq. phase: $-270°$

Recall: negative high freq. slope means

$$|G(j\omega)| \to \emptyset \quad \text{as} \quad \omega \to \infty$$



"phase xover"
Im
+90°
-270°
1
+180°
KB
±360°
Re
1
-180°
0°
mag xover
-90°

Phase Crossover

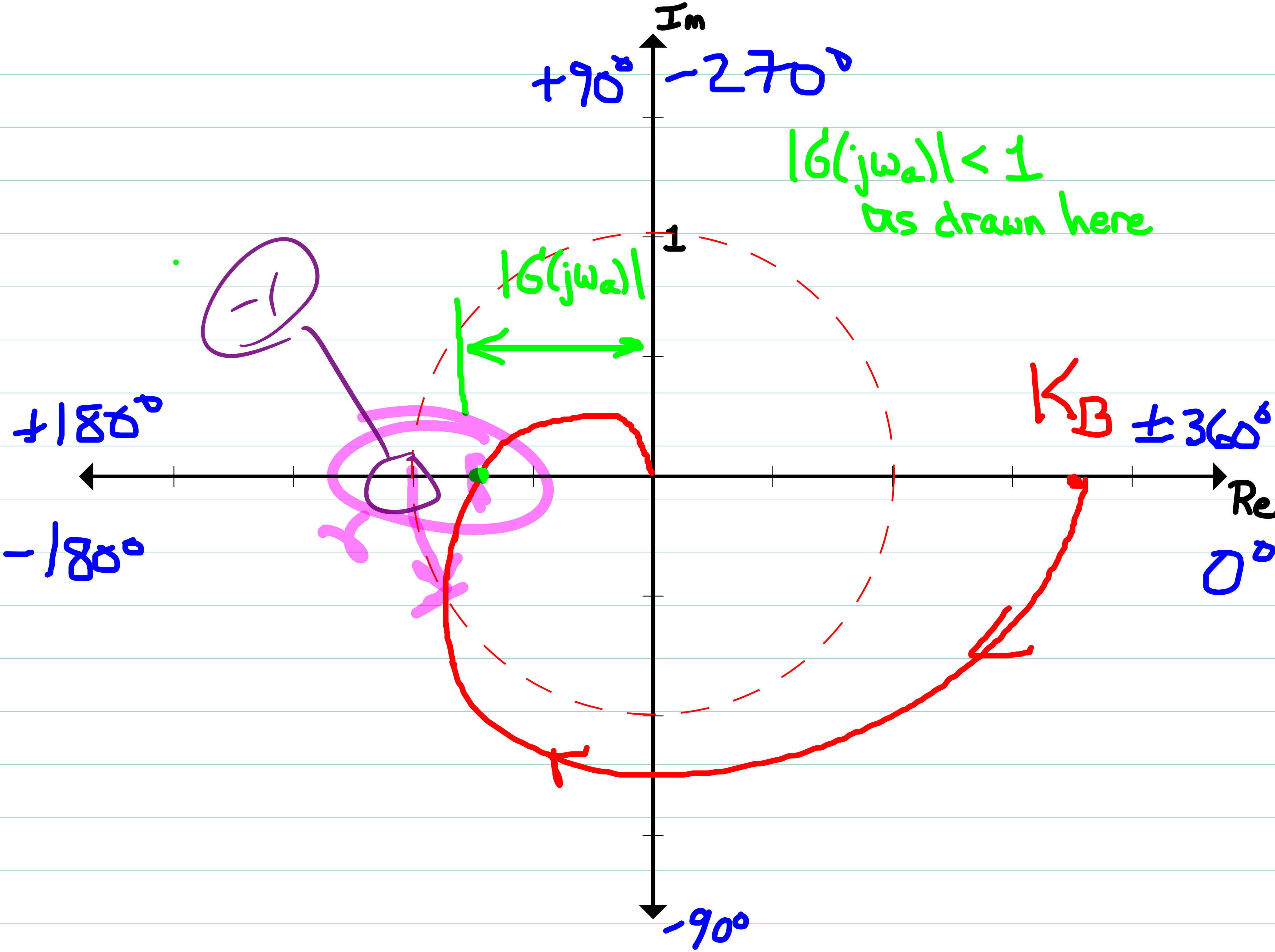
The "phase crossover" of a polar plot is the point where the plot crosses through the <u>negative real</u> axis.

This corresponds to the point where $\measuredangle G(j\omega) = -180°$

Again, easily seen from Bode phase diagrams: call ω_a "phase crossover freq." the value of ω for which $\measuredangle G(j\omega) = -180°$.

<u>Note</u>: May be one, none, or many ω_a depending on system.

Important quantity: $|G(j\omega_a)|$ magnitude at phase xover frequency



Im
+90°
-270°
$|G(j\omega_a)| < 1$
as drawn here
1
$|G(j\omega_a)|$
-1
+180°
-180°
K_B
±360°
0°
Re
-90°

Gain Margin

The gain margin, a, is defined as:

$$\boxed{a = \frac{1}{|G(j\omega_a)|}}$$

Gain margin is commonly expressed in dB:

$$a_{dB} = 20 \log a$$

$$\boxed{= -|G(j\omega_a)|_{dB}}$$

So gain margin in dB is negative of Bode magnitude at phase crossover freq.

<u>Meaning of Gain and phase margins</u>

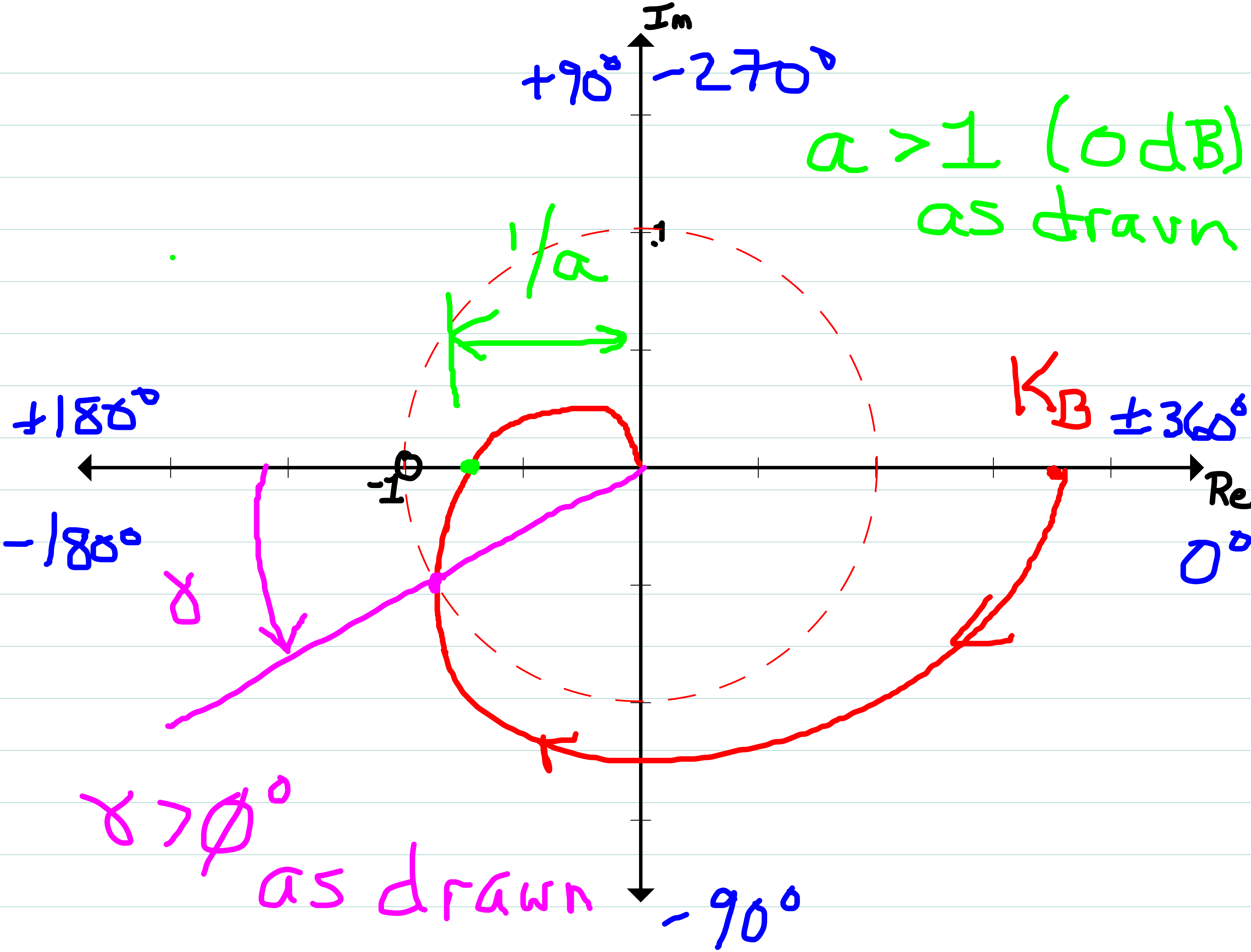
a, γ measure how close polar plot comes to point $-1+\phi j$ ("-1 point") in complex plane. Recall $-1+\phi j = 1 \measuredangle -180°$

Two "pseudo-orthogonal" directions

→ a measures distance to -1 along real Axis as a ratio $1/|G(j\omega_a)|$

⇒ γ measures distance to -1 as an angle around unit circle.

<u>Note</u>: $a>1$ ($a>\phi$ dB) means phase crossover occurs <u>inside</u> unit circle. $a<1$ ($a<\phi$ dB) means phase crossover is outside unit circle

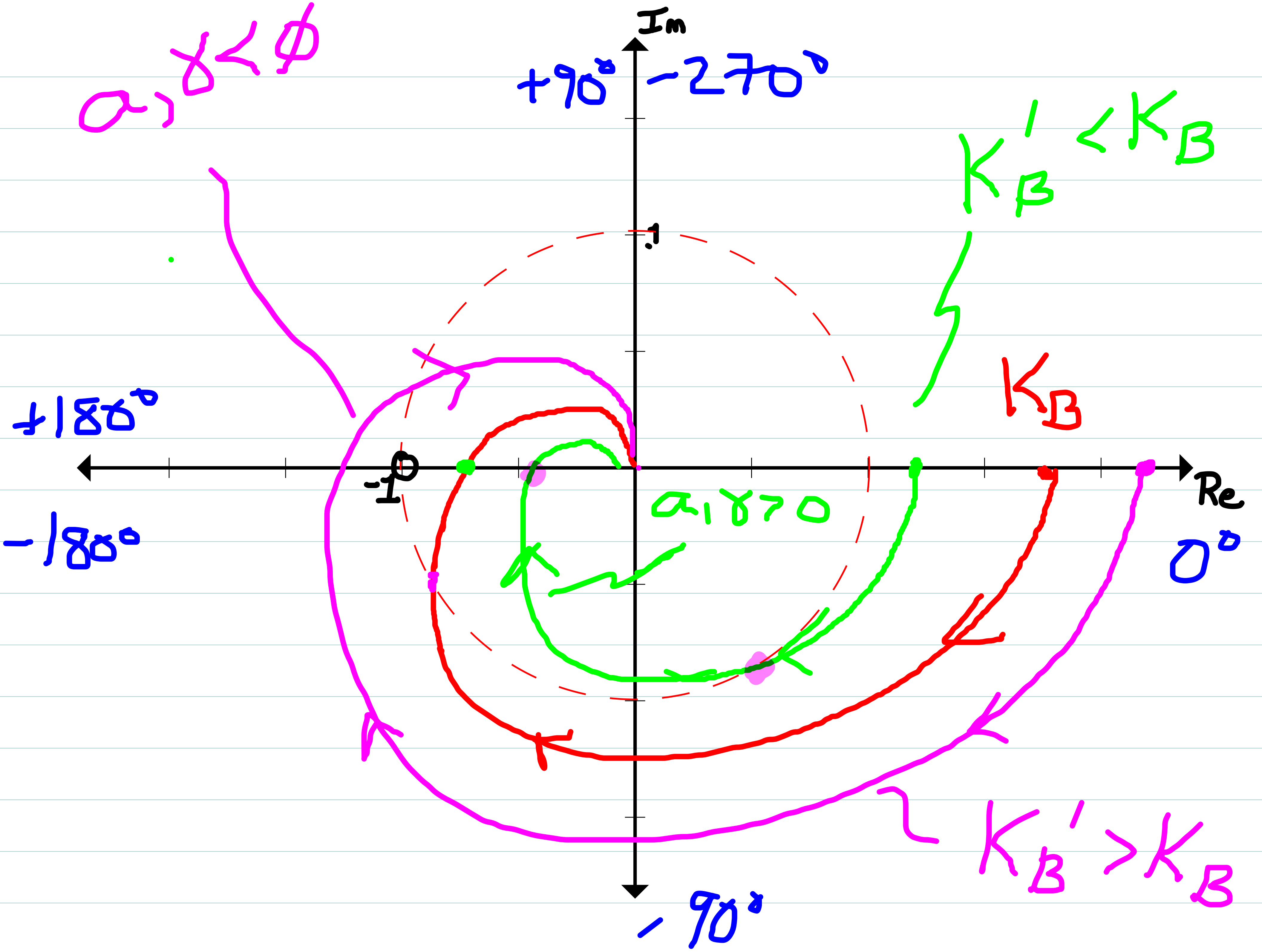


Im
Re
+90°
−270°
a > 1 (0dB)
as drawn
1/a
1
−1
+180°
−180°
±360°
0°
−90°
K_B
γ
γ > 0°
as drawn

Effect of Gain Changes

Increasing or decreasing K_B uniformly expands or contracts polar plot about the origin

$\Rightarrow$ Will generally change crossovers and margins



Im
+90°
-270°
1
$K_B' < K_B$
$a, \gamma < 0$
K_B
+180°
-180°
-1
0
Re
$a, \gamma > 0$
0°
$K_B' > K_B$
-90°

Effect of Zeros

Since they affect magnitude and phase, zeros will change shape of polar plot.

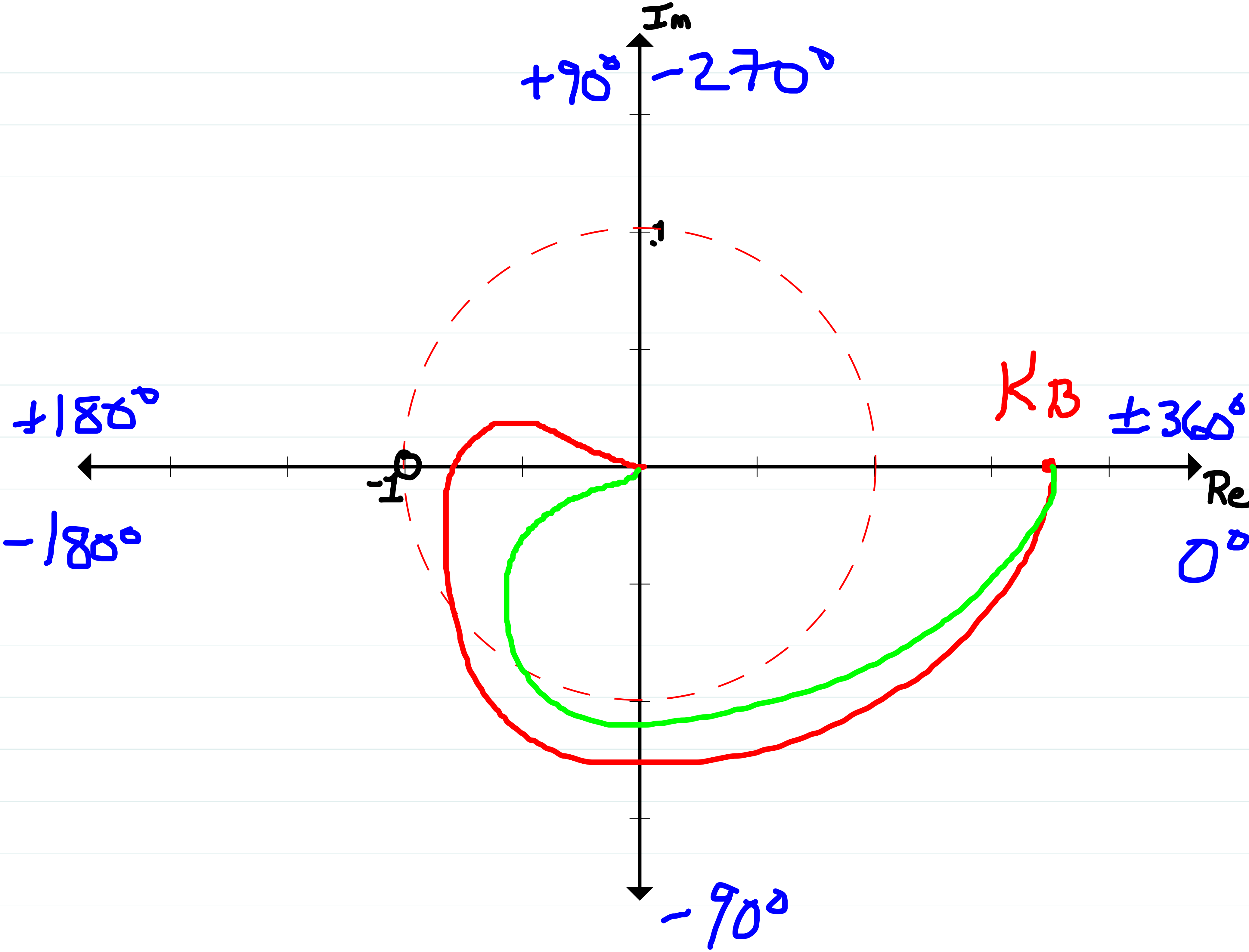
Example:

$$G(s) = K_B \frac{(T_1 s + 1)}{(T_2 s + 1)^3} \qquad K_B > 1, \quad T_1, T_2 > 0$$

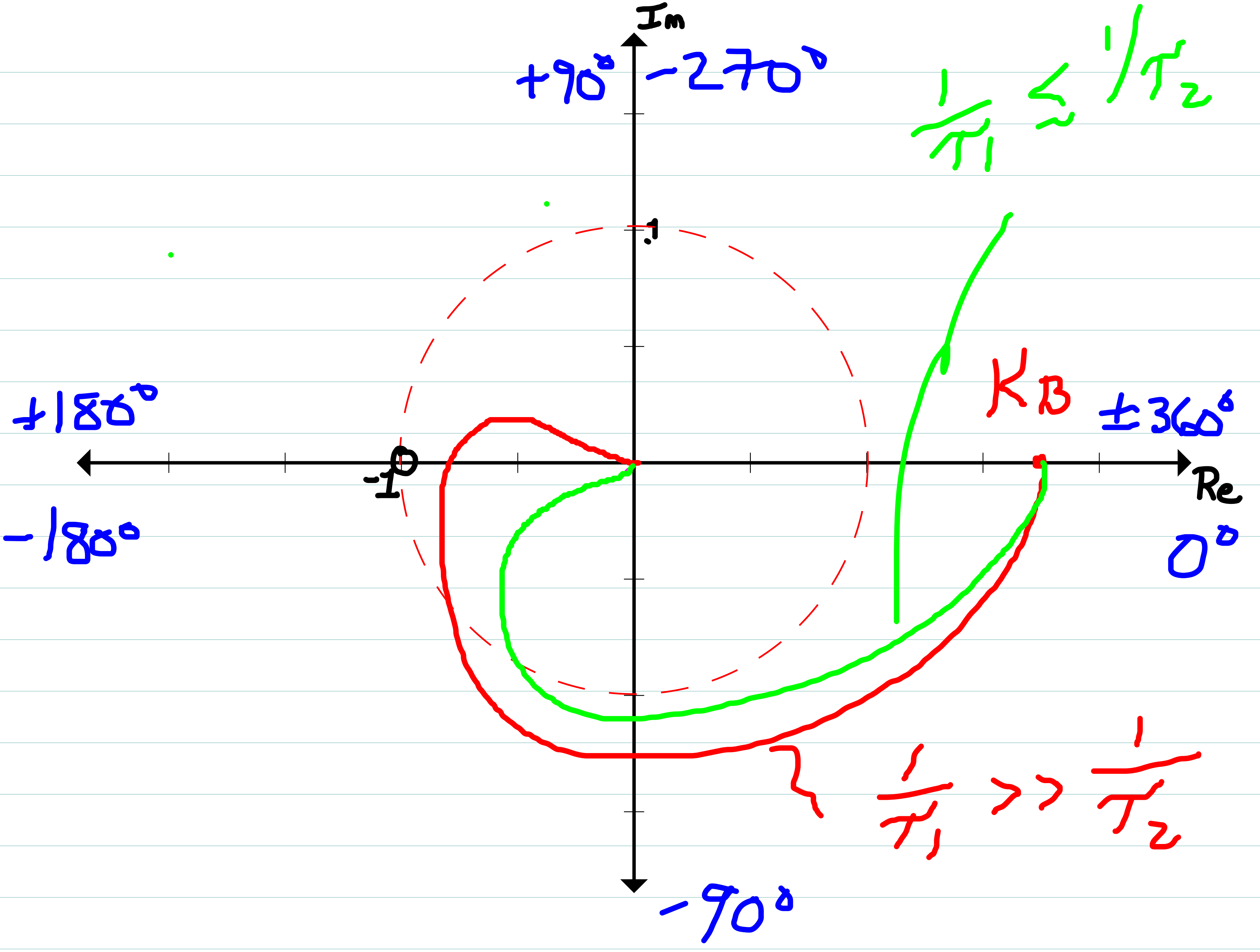
high freq phase: $-180°$ here (Why??)

But this limit may be asymptotically approached from above or below as $\omega \to \infty$

This difference can have a profound impact on shape of plot. Need to check Bode for accuracy, but can often "reason it out" for simple cases.



Im
+90°
-270°
1
+180°
-180°
-1
0
KB
±360°
0°
Re
-90°



Im
+90°
-270°
1
+180°
-180°
-1
Re
±360°
0°
-90°
KB
1/τ1 < 1/τ2
1/τ1 >> 1/τ2