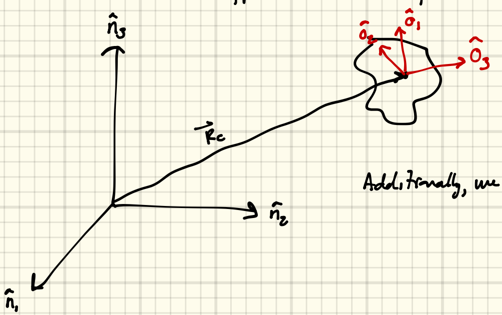
Lecture 23: Gravity Gradient Torque

Gravity-Gradient Torque:

The part of the s/c that is closer to the center of the Earth will feel a stronger gravitational acceleration than the part that is farther from Earth. This produces a torque on the s/c.

Note: Earth is approximated as a point mass.



$\{o_1, o_2, o_3\}$ = orbit frame

$\hat{o}_3 \parallel \hat{R}_c$

$\hat{o}_2 \parallel \vec{h}$

$\hat{o}_1$ completes the RH'd system

Additionally, we can also have a body-fixed frame.

Angular velocity of the O frame wrt N frame is:

$$\vec{\omega}_{ON} = n\,\hat{o}_2 \qquad n = \text{mean motion} = \sqrt{\frac{\mu}{a^3}}$$

For a body-fixed frame:

$$\vec{\omega}_{BN} = \vec{\omega}_{BO} + \vec{\omega}_{ON}$$

Get expression for gravity gradient-torque:

Gravity force acting on some mass element dm:

$$d\vec{f} = -\frac{\mu}{R^3}\vec{R}\,dm = -\frac{\mu(\vec{R}_c + \vec{r})\,dm}{|\vec{R}_c + \vec{r}|^3}$$

(reminder: $\vec{R} = \vec{R}_c + \vec{r}$)

Gravity-gradient torque about CM:

$$\vec{L} = \int_B \vec{r} \times d\vec{f} = -\mu \int \frac{\vec{r} \times \vec{R}_c}{|\vec{R}_c + \vec{r}|^3}\,dm$$

Need an expression for denominator:

$$|\vec{R}_c + \vec{r}|^{-3} = R_c^{-3}\left\{1 + \frac{2(\vec{R}_c \cdot \vec{r})}{R_c^2} + \frac{r^2}{R_c^2}\right\}^{-3/2}$$

Binomial Expansion:

$$= R_c^{-3}\left\{1 - \frac{3(\vec{R}_c \cdot \vec{r})}{R_c^2} + H.O.T.\right\}$$

Drop H.O.T.:

$$\vec{L} = \frac{\mu}{R_c^3}\vec{R}_c \times \int_B \vec{r}\left(1 - \frac{3\vec{R}_c\cdot\vec{r}}{R_c^2}\right)dm$$

$\int \vec{r}\,dm = 0$ b/c definition of CM

$$\vec{L} = \frac{3\mu}{R_c^5}\vec{R}_c \times \int -\vec{r}\,(\vec{r}\cdot\vec{R}_c)\,dm$$

Use: $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$\vec{L} = \frac{3\mu}{R_c^5}\vec{R}_c \times \int -\left(\vec{r}\times(\vec{r}\times\vec{R}_c) + (\vec{r}\cdot\vec{r})\vec{R}_c\right)dm$$

$$= \frac{3\mu}{R_c^5}\vec{R}_c \times \underbrace{\left(\int_B -[\tilde{r}][\tilde{r}]\,dm\right)}_{[I_c]}\vec{R}_c - \frac{3\mu}{R_c^5}\left(\int r^2 dm\right)\cancelto{0}{\vec{R}_c\times\vec{R}_c}$$

$$\Rightarrow \boxed{\vec{L} = \frac{3\mu}{R_c^5}\vec{R}_c \times [I_c]\vec{R}_c}$$

Gravity Gradient Torque

- Assumes 2BP gravity (point mass)
- Approximation b/c dropped the H.O.T. in the binomial expansion

Given our definition of the axes:

$${}^{O}\vec{R}_c = R_c\,\hat{o}_3$$

$${}^{O}\vec{L} = \frac{3\mu}{R_c^3}\left(-I_{23}\,\hat{o}_1 + I_{13}\,\hat{o}_2\right)$$

→ Gravity gradient never produce torque in the $\hat{o}_3$ direction

Zero torque if the orbit frame is aligned with the principal axes of the body, b/c the inertia matrix would be diagonal in this case.

Can also write the torque in the body-fixed frame, assuming principal axes:

$${}^{B}\vec{R}_c = R_{c1}\hat{b}_1 + R_{c2}\hat{b}_2 + R_{c3}\hat{b}_3$$

$${}^{B}\vec{L} = \frac{3\mu}{R_c^5}\begin{bmatrix} R_{c2}R_{c3}(I_{33}-I_{22}) \\ R_{c1}R_{c3}(I_{11}-I_{33}) \\ R_{c1}R_{c2}(I_{22}-I_{11}) \end{bmatrix}$$

- No torque if $I_{11}=I_{22}=I_{33} \Rightarrow$ sphere
- If i^{th} axis is an axis of symmetry, there is no torque about $\hat{b}_i$.
- If $\vec{R}_c$ is parallel with one of the principal axes, then ${}^{B}\vec{L} = 0$.

Attitude of S/C in response to torque:

Describe the attitude using 3-2-1 rotation matrix

ψ: yaw

θ = pitch

ϕ = roll

$c\theta = \cos\theta$

$s\psi = \sin\psi$

$$^{B}\vec{\omega}_{BN} = \begin{bmatrix} \dot{\phi} - s\theta\dot{\psi} + nc\theta s\psi \\ s\phi c\theta\dot{\psi} + c\phi\dot{\theta} + n(s\phi s\theta s\psi + c\phi c\psi) \\ c\phi c\theta\dot{\psi} - s\phi\dot{\theta} + n(c\phi s\theta s\psi - s\phi c\psi) \end{bmatrix}$$

Note: n = mean motion

Small angle assumption & linearize.

$$^{B}\vec{\omega}_{BN} = \begin{bmatrix} \dot{\phi} + n\psi \\ \dot{\theta} + n \\ \dot{\psi} - n\phi \end{bmatrix} \Rightarrow \dot{\vec{\omega}}_{BN} \simeq \begin{bmatrix} \ddot{\phi} + n\dot{\psi} \\ \ddot{\theta} \\ \ddot{\psi} - n\dot{\phi} \end{bmatrix}$$

Now write torque using ψ, ϕ, θ

$$^{B}\vec{R}_c = \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix} R_c$$

Assume circular orbit: $R_c = a$

$$^{B}\vec{L} = \frac{3}{2}n^2 \begin{bmatrix} (I_{33} - I_{22})\cos^2\theta\sin 2\phi \\ -(I_{11} - I_{33})\cos\phi\sin 2\theta \\ -(I_{22} - I_{11})\sin\phi\sin 2\theta \end{bmatrix}$$

linearize:

$$^{B}\vec{L} \simeq 3n^2 \begin{bmatrix} (I_{33} - I_{22})\phi \\ -(I_{11} - I_{33})\theta \\ 0 \end{bmatrix}$$

← does not depend on ψ (yaw)

For pitch & roll to be stabilizing, require: $I_{22} > I_{33}$ & $I_{11} > I_{33}$

$\Rightarrow$ make L_1 & $L_2 < 0$

For a cylindrical S/C, the long axis should be aligned w/ $\vec{h}$ in order to be stabilized by gravity gradient torque.