

## Input-Output

$$\ddot{y}(t) = K u(t), K = \frac{K_f K_m}{m}$$

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) + \dots + \alpha_n y_n(t)$$

## Complex Exponents

### General Form

$$z(t) = a(t) + b(t)j$$

$$= r(t)e^{j\theta(t)}$$

$\sigma$  : amplitude envelope

$\omega$  : oscillation frequency

$s$  : complex frequency

$r = |A|$  : initial amplitude

$\phi = \angle A$  : phase shift

$\phi > 0$  : phase lead

$\phi < 0$  : phase lag

### Basic Example

$$z(t) = e^{st}, \quad s \in \mathbb{C}$$

$$s = \sigma + \omega j, \quad \sigma, \theta \in \mathbb{R}$$

$$\Re\{s\} = \sigma$$

$$\Im\{s\} = \omega$$

$$\Re\{e^{st}\} = e^{\sigma t} \cos(\omega t)$$

$$\Im\{e^{st}\} = e^{\sigma t} \sin(\omega t)$$

$$e^{st} = \begin{cases} e^{\sigma t} & \omega = 0 \\ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) & \sigma = 0 \\ e^{\sigma t} [\cos(\omega t) + j \sin(\omega t)] & \text{otherwise} \end{cases}$$

### Specific Example

$$z(t) = Ae^{st}, \quad A, s \in \mathbb{C}$$

$$s = \sigma + \omega j, \quad \sigma, \theta \in \mathbb{R}$$

$$A = re^{j\phi}$$

$$Ae^{st} = re^{\sigma t} [\cos(\omega t + \phi) + j \sin(\omega t + \phi)]$$

$$\Re\{Ae^{st}\} = re^{\sigma t} \cos(\omega t + \phi)$$

$$\Im\{Ae^{st}\} = re^{\sigma t} \sin(\omega t + \phi)$$

## Transfer Function

$$G(s) = \frac{q(s)}{r(s)}$$

$$q(s) = \mathcal{L}\{y(t)\} = \beta_m \prod_{i=1}^m (s - z_i)$$

$$r(s) = \mathcal{L}\{u(t)\} = \alpha_n \prod_{k=1}^n (s - p_k)$$

1. Get information on modes from homogenous response
2. Get information on forced response from evaluating  $G(s)$  at specific values of  $s$

### ZPK Form

$$G(s) = K \left[ \frac{\prod_{i=1}^m (s - z_i)}{\prod_{k=1}^n (s - p_k)} \right]$$

1. Zeroes:  $z_i$  satisfy  $q(z_i) = 0$
2. Poles:  $p_k$  satisfy  $r(p_k) = 0$
3. Gain:  $K = \frac{\beta_m}{\alpha_n}$  is always real

## Characteristic Polynomial

$y$  : polynomial response

$y_h$  : homogenous response

$y_f$  : forced response

$r$  : characteristic polynomial

$p_i$  : roots of polynomial

$n$  : # of roots

$l$  : # of times roots are repeated

$$r(s) = (s - p_1)^{l_1} (s - p_{l+1}) \dots (s - p_n)$$

$$y_h(t) = (C_1 + C_2 t + \dots + C_l t^{l-1}) e^{p_1 t} + \sum_{k=l+1}^n C_k e^{p_k t}$$

$$y(t) = y_h(t) + y_f(t)$$

## Modes

1. Solutions which are possible without any input
2. Terms in solution for  $y(t)$  of form  $e^{pt}$ , where  $r(p) = 0$
3. First Order when  $p \in \mathbb{R}$
4. Second Order when  $p \in \mathbb{C}$

### Stability

1. Mode is stable if:  $|e^{pt}| \rightarrow 0$  as  $t \rightarrow \infty \implies \sigma < 0$  (root  $p$  lies in left half of complex plane)
2. System is stable if: all modes are stable  $\implies \Re\{p_k\} < 0 \forall k \in \{1, \dots, n\}$
3. If the system is stable,  $y_h(t) = 0$  for all initial conditions
4. Repeated modes retain the stability of their roots
5. For constant input,  $y_{tr}(t) = y_h(t)$  and  $y_{ss}(t) = y_f(t)$

### Instability

1. Mode is unstable if:  $\sigma > 0$  (root  $p$  lies in right half of complex plane)
2. System is unstable if: any mode is unstable  $\implies \Re\{p_k\} > 0$  for any  $k \in \{1, \dots, n\}$

### Marginal Stability

1. Mode is marginally stable if:  $\sigma = 0$  (root  $p$  lies on complex axis)
2. Repeated marginally stable modes will increase polynomially with  $t$

### Transience

1. Transient Response:  $y_{tr}(t)$
2. Terms in  $y(t)$  for which:  $\lim_{t \rightarrow \infty} |y_{tr}(t)| \rightarrow 0$
3. If the system is stable,  $y_{tr}(t)$  contains all of  $y_h(t)$  and any decaying terms of  $y_f(t)$

### Steady-State

1. Steady State Response:  $y_{ss}(t)$
2. All other terms in  $y(t)$
3. Contains all marginally stable terms of  $y_h(t)$

### Convergence

1. Quantifies how quickly stable modes decay to 0
2. 2% Criterion defines the settling time:  
 $t_s$  s.t.  $|e^{pt}| \leq 0.02 \forall t \geq t_s$
3. For first order modes,  $t_s = \frac{\ln(0.02)}{\sigma} \approx \frac{4}{|\sigma|}$
4. Above approximation is a good tool for second order modes, but is less accurate due to oscillations
5. Doubling time applies to unstable modes:  
 $|e^{\sigma t_d}| = 2 \implies t_d \approx \frac{0.7}{\sigma}$
6. Smaller doubling time  $\iff$  "more unstable" system  $\implies$  faster rate of increase in amplitude
7. Settling times decrease the further left of the imaginary axis  $p$  is
8. Doubling times decrease the further right of the imaginary axis  $p$  is

### Damping Ratio

$$\omega_n = \sqrt{\sigma^2 + \omega_d^2} = |p_1|$$

$$\zeta = \frac{|\sigma|}{\omega_n}$$

$$\zeta = \begin{cases} 0 \leq \zeta \leq 1 & \text{for stable modes} \\ \zeta \approx 0 & \text{many oscillations before settled} \\ \zeta \approx 1 & \text{less than one complete oscillation} \end{cases}$$

## Laplace Transform

### Definition

$$f(t) = \frac{1}{2\pi j} \int F(s) e^{st} ds$$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

### Special Cases

$$\mathcal{L}\{e^{pt}\} = \frac{1}{s-p} \forall p \in \mathbb{C}$$

$$\mathcal{L}\{Ae^{at} \cos(bt + \psi)\} = \frac{C}{s-p} + \frac{\bar{C}}{s-\bar{p}}, C = \frac{A}{2} e^{j\psi}$$

$$\mathcal{L}\{c\} = \frac{c}{s} \forall c \in \mathbb{C}$$

## Properties

$$\mathcal{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$$

$$\mathcal{L}\{f_1(t)f_2(t)\} \neq F_1(s)F_2(s)$$

$$\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f^{(k)}(t)\} = s^k F(s) - \sum_{i=1}^k s^{i-1} f^{(k-i)}(0)$$

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds} (F(s))$$

$$\mathcal{L}\{te^{pt}\} = -\frac{d}{ds} \left( \frac{1}{s-p} \right)$$

$$\mathcal{L}\{t^k e^{pt}\} = \frac{k!}{(s-p)^{k+1}}$$

### Usage

$$r(s)Y(s) - c(s) = q(s)U(s) - b(s)$$

$$r(s) = \alpha_n s^n + \dots + \alpha_1 s + \alpha_0$$

$$q(s) = \beta_m s^m + \dots + \beta_1 s + \beta_0$$

$$c(s) = n - 1 \text{ order poly from IC on } y(t)$$

$$b(s) = m - 1 \text{ order poly from IC on } u(t)$$

$$Y(s) = G(s)U(s) + \left[ \frac{c(s) - b(s)}{r(s)} \right]$$

## Inverse Laplace

### Partial Fraction Expansion

$$\begin{aligned} Y(s) &= G(s)U(s) + \left[ \frac{c(s) - b(s)}{r(s)} \right] \\ &= \left[ \frac{q(s)}{r(s)} \right] \left[ \frac{a(s)}{h(s)} \right] + \left[ \frac{c(s) - b(s)}{r(s)} \right] \\ &= \frac{q(s)a(s) + h(s)[c(s) - b(s)]}{r(s)h(s)} \end{aligned}$$

$$= \sum_{l=1}^L \frac{A_l}{s - d_l}$$

$$y(t) = \sum_{l=1}^L A_l e^{d_l t}$$

### Residue Formula

$$A_l = [(s - d_l)Y(s)]_{s=d_l}$$

$$\overline{A}_l = [(s - \overline{d}_l)Y(s)]_{s=\overline{d}_l}$$

$$A_l e^{d_l t} + \overline{A}_l e^{\overline{d}_l t} = 2|A_l| e^{\sigma t} \cos(\omega t + \angle A_l)$$

### Repeated Roots

$L$  : # of roots

$K$  : # times a root is repeated

$$Y(s) = \sum_{l=1}^K \frac{A_l}{(s - d_l)^l} + \sum_{l=K+1}^L \frac{A_l}{(s - d_l)}$$

$$y(t) = \sum_{l=1}^K \left( \frac{A_l t^{l-1}}{(l-1)!} \right) e^{d_l t} + \sum_{l=K+1}^L A_l e^{d_l t}$$

## State Model

$$\dot{\underline{x}}(t) = A \underline{x}(t) + B u(t)$$

$$y(t) = C \underline{x}(t) + D u(t)$$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

### Transfer Function

$$Q(s) = \text{Adj}(sI - A)$$

$$r(s) = \text{Det}(sI - A)$$

$$G(s) = [C(sI - A)^{-1}B + D]$$

$$= \frac{CQ(s)B}{r(s)} + D$$

$$= \frac{CQ(s)B + Dr(s)}{r(s)}$$

$$\text{Zeroes : } CQ(s)B + Dr(s) = 0$$

$$\text{Poles : } r(s) = 0 \quad (\text{Eigenvalues of } A)$$

## Impulse Response

$$h(t) = Ce^{At}B + D\delta(t)$$

## Matrix-Vector Form

$$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = C\underline{x}(t) + Du(t)$$

## Heaviside Step Function

### Transfer Function

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

## Impulse Response

$$\frac{d}{dt}(u(t)) = \delta(t)$$

## Matrix-Vector Form

$$\underline{u}[t] = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

## Dirac Delta Function

### Transfer Function

$$\mathcal{L}\{\delta(t)\} = 1$$

## Impulse Response

$$\delta(t) \text{ satisfies } \int_{-\epsilon}^{\epsilon} \delta(t)dt = 1, \quad \forall \epsilon > 0$$

## Matrix-Vector Form

$$\underline{\delta}[t] = \begin{pmatrix} \delta(t) \\ \vdots \\ \delta(t) \end{pmatrix}$$

## Step Response

### First Order

$$y(t) = K(1 - e^{-t/\tau})$$

$\tau$  : time constant

$$t_s \approx 4\tau \quad (\text{for 2\% criterion})$$

### Second Order

#### Poles

$$\alpha_1^2 < 4\alpha_0 \implies \text{complex conjugates}$$

$$\alpha_1^2 = 4\alpha_0 \implies \text{repeated real}$$

$$\alpha_1^2 > 4\alpha_0 \implies \text{real, non-repeated}$$

Complex Conjugates:

$$\alpha_1 = -2\sigma$$

$$\alpha_0 = \sigma^2 + \omega_d^2 = |p_1|^2$$

Repeated Real:

$$t_s = \frac{6}{|p_1|}$$

Real, Non-Repeated:

$$|p_2| \gg |p_1| \implies t_s \approx \frac{4}{|p_1|}$$

(threshold for above is:  $|p_2| > 5|p_1|$ )

$$|p_2| \approx |p_1| \implies t_s \approx \frac{6}{|p_1|}$$

(threshold for above is:  $1 \leq \frac{|p_2|}{|p_1|} \leq 1.1$ )

### Damped (Critically Damped, $\zeta = 1$ )

$$y(t) = 1 - (1 + \omega_n t)e^{-\omega_n t}$$

### Under-Damped ( $0 < \zeta < 1$ )

$$\nu = \arccos(\zeta)$$

$$y(t) = G(0) \left[ 1 - \left( \frac{\omega_n}{\omega_d} e^{\sigma t} \sin(\omega_d t + \nu) \right) \right]$$

### Natural (Undamped, $\zeta = 0$ )

$$y(t) = 1 - \cos(\omega_n t)$$

### LHP Zero

1. A zero in the Left Half Plane does not induce an inverse response.
2. The step response remains monotonic though modified by the zero dynamics.

### RHP Zero

1. A Right Half Plane Zero causes an initial inverse (non-minimum phase) response.
2. The response exhibits an undershoot before eventually rising to steady state.

## Performance Metrics

$$M_p : \text{Maximum Overshoot} = \frac{y_{\max} - y_{ss}}{y_{ss}} \times 100\%$$

$t_r$  : Rise Time (10 % to 90 % of final value)

$t_c$  : Time steady-state is first crossed

$t_p$  : Peak Time (time to first peak)

$t_s$  : Settling Time (2 % criterion)

## Overshoot

$$M_p = e^{-\frac{\sigma}{\omega_d} \pi} \times 100\%$$

$$= e^{\frac{-\zeta \pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$t_c = \frac{\pi - \nu}{\omega_d}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$y_p = y_{ss} [1 + M_p]$$

## System Zeroes

### Input Absorption

1. System zeroes can absorb certain input dynamics.
2. A zero at  $s = z$  may cancel an input pole at  $s = z$ .

### Transient Suppression

1. Appropriately placed zeroes can mitigate transient peaks.
2. They are used in controller design to improve system performance.

### Pole Cancellation

1. Occurs when a system zero cancels a pole in the transfer function.
2. Ideal cancellation is sensitive to model uncertainties.

## Frequency Response

### Definition

$$G(j\omega) = G(s) \Big|_{s=j\omega}, \quad \omega \in \mathbb{R}$$

Magnitude :  $|G(j\omega)|$

Phase :  $\angle G(j\omega)$

### Quantification

1. **Gain Margin**: Factor by which gain can be increased before instability.
2. **Phase Margin**: Additional phase lag required to reach instability.
3. These margins and the overall frequency response are visualized using Bode plots.

## Bode Diagrams

### Decibel Units

$$\text{Magnitude (dB)} = 20 \log_{10} (|G(j\omega)|)$$

### Shape

#### Transfer Function

$$G(s) = \frac{N(s)}{D(s)}$$

### Zeroes

1. Each zero contributes a +20 dB/decade slope beyond its break frequency.

### Poles

1. Each pole contributes a -20 dB/decade slope beyond its break frequency.

### Gain

1. A constant gain  $K$  shifts the magnitude plot by  $20 \log_{10}(K)$  dB.

## Bode Magnitude Diagrams

### Shape

#### Transfer Function

$$|G(j\omega)|$$

### Zeroes

1. Zeroes add positive slopes to the magnitude plot.

### Poles

1. Poles add negative slopes to the magnitude plot.

### Gain

1. The overall gain sets the baseline level of the magnitude plot.