Input-Output

$$\ddot{y}(t) = K u(t), K = \frac{K_f K_m}{m}$$
$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$
$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) + \cdots + \alpha_n y_n(t)$$

Complex Exponents

General Form

$$\begin{aligned} z(t) &= a(t) + b(t)j \\ &= r(t)e^{j\theta(t)} \\ \sigma &: \text{amplitude envelope} \\ \omega &: \text{oscillation frequency} \\ s &: \text{complex frequency} \\ r = |A| &: \text{initial amplitude} \\ \phi = \angle A &: \text{phase shift} \\ \phi > 0 &: \text{phase lead} \\ \phi < 0 &: \text{phase lag} \end{aligned}$$

Basic Example

$$\begin{aligned} z(t) &= e^{st}, \quad s \in \mathbb{C} \\ s &= \sigma + \omega j, \quad \sigma, \theta \in \mathbb{R} \\ \mathfrak{Re}\{s\} &= \sigma \\ \mathfrak{Im}\{s\} &= \omega \\ \mathfrak{Re}\{e^{st}\} &= e^{\sigma t}\cos(\omega t) \\ \mathfrak{Im}\{e^{st}\} &= e^{\sigma t}\sin(\omega t) \end{aligned}$$

$$e^{st} = \begin{cases} e^{\sigma t} & \omega = 0 \\ e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) & \sigma = 0 \\ e^{\sigma t}[\cos(\omega t) + j\sin(\omega t)] & \text{otherwise} \end{cases}$$

Specific Example

$$\begin{aligned} z(t) &= Ae^{st}, \quad A, s \in \mathbb{C} \\ s &= \sigma + \omega j, \quad \sigma, \theta \in \mathbb{R} \\ A &= re^{j\phi} \\ Ae^{st} &= re^{\sigma t}[\cos(\omega t + \phi) + j\sin(\omega t) + \phi] \\ \mathfrak{Re}\{Ae^{st}\} &= re^{\sigma t}\cos(\omega t + \phi) \\ \mathfrak{Im}\{Ae^{st}\} &= re^{\sigma t}\sin(\omega t + \phi) \end{aligned}$$

Transfer Function

$$G(s) = \frac{q(s)}{r(s)}$$
$$q(s) = \mathscr{L}\{y(t)\} = \beta_m \prod_{i=1}^{m}(s - z_i)$$
$$r(s) = \mathscr{L}\{u(t)\} = \alpha_n \prod_{k=1}^{n}(s - p_k)$$

1. Get information on modes from homogenous response
2. Get information on forced response from evaluating $G(s)$ at specific values of s

ZPK Form

$$G(s) = K\left[\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{k=1}^{n}(s - p_k)}\right]$$

1. Zeroes: z_i satisfy $q(z_i) = 0$
2. Poles: p_k satisfy $r(p_k) = 0$
3. Gain: $K = \frac{\beta_m}{\alpha_n}$ is always real

Characteristic Polynomial

$$\begin{aligned} y &: \text{polynomial response} \\ y_h &: \text{homogenous response} \\ y_f &: \text{forced response} \\ r &: \text{characteristic polynomial} \\ p_i &: \text{roots of polynomial} \\ n &: \text{\# of roots} \\ l &: \text{\# of times roots are repeated} \end{aligned}$$

$$r(s) = (s - p_1)^l (s - p_{l+1}) \cdots (s - p_n)$$
$$y_h(t) = (C_1 + C_2 t + \cdots + C_l t^{l-1})e^{p_1 t} + \sum_{k=l+1}^{n} C_k e^{p_k t}$$
$$y(t) = y_h(t) + y_f(t)$$

Modes

1. Solutions which are possible without any input
2. Terms in solution for $y(t)$ of form e^{pt}, where $r(p) = 0$
3. First Order when $p \in \mathbb{R}$
4. Second Order when $p \in \mathbb{C}$

Stability

1. Mode is stable if: $|e^{pt}| \to 0$ as $t \to \infty \implies \sigma < 0$ (root p lies in left half of complex plane)
2. System is stable if: all modes are stable $\implies \mathfrak{Re}\{p_k\} < 0 \forall k \in \{1, \dots, n\}$
3. If the system is stable, $y_h(t) = 0$ for all initial conditions
4. Repeated modes retain the stability of their roots
5. For constant input, $y_{tr}(t) = y_h(t)$ and $y_{ss}(t) = y_f(t)$

Instability

1. Mode is unstable if: $\sigma > 0$ (root p lies in right half of complex plane)
2. System is unstable if: any mode is unstable $\implies \mathfrak{Re}\{p_k\} > 0$ for any $k \in \{1, \dots, n\}$

Marginal Stability

1. Mode is marginally stable if: $\sigma = 0$ (root p lies on complex axis)
2. Repeated marginally stable modes will increase polynomially with t

Transience

1. Transient Response: $y_{tr}(t)$
2. Terms in $y(t)$ for which: $\lim_{t\to\infty} |y_{tr}(t)| \to 0$
3. If the system is stable, $y_{tr}(t)$ contains all of $y_h(t)$ and any decaying terms of $y_f(t)$

Steady-State

1. Steady State Response: $y_{ss}(t)$
2. All other terms in $y(t)$
3. Contains all marginally stable terms of $y_h(t)$

Convergence

1. Quantifies how quickly stable modes decay to 0
2. 2 % Criterion defines the settling time:
$$t_s \text{ s.t. } |e^{pt}| \le 0.02 \forall t \ge t_s$$
3. For first order modes, $t_s = \frac{\ln(0.02)}{\sigma} \approx \frac{4}{|\sigma|}$
4. Above approximation is a good tool for second order modes, but is less accurate due to oscillations
5. Doubling time applies to unstable modes:
$$|e^{\sigma t_d}| = 2 \implies t_d \approx \frac{0.7}{\sigma}$$
6. Smaller doubling time $\iff$ "more unstable" system $\implies$ faster rate of increase in amplitude
7. Settling times decrease the further left of the imaginary axis p is
8. Doubling times decrease the further right of the imaginary axis p is

Damping Ratio

1. Only applies to second order modes
2. $\zeta = \left|\frac{\sigma}{p}\right| = \frac{|\sigma|}{\sqrt{\sigma^2 + \omega^2}}$

$$\begin{cases} 0 \le \zeta \le 1 & \text{for stable modes} \\ \zeta \approx 0 & \text{many oscillations before settled} \\ \zeta \approx 1 & \text{less that one complete oscillation} \end{cases}$$

Laplace Transform

Definition

$$f(t) = \frac{1}{2\pi j}\int F(s)e^{st}ds$$
$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Special Cases

$$\mathscr{L}\{e^{pt}\} = \frac{1}{s - p} \forall p \in \mathbb{C}$$
$$\mathscr{L}\{Ae^{at}\cos(bt + \psi)\} = \frac{C}{s - p} + \frac{\overline{C}}{s - \overline{p}}, C = \frac{A}{2}e^{j\psi}$$
$$\mathscr{L}\{c\} = \frac{c}{s} \forall c \in \mathbb{C}$$

Properties

$$\begin{aligned} \mathscr{L}\{f_1(t) + f_2(t)\} &= F_1(s) + F_2(s) \\ \mathscr{L}\{f_1(t)f_2(t)\} &\ne F_1(s)F_2(s) \\ \mathscr{L}\{f'(t)\} &= sF(s) - f(0) \\ \mathscr{L}\{f^{(k)}(t)\} &= s^k F(s) - \sum_{i=1}^{k-1} s^{k-1-i}f^{(i)}(0) \\ \mathscr{L}\{tf(t)\} &= -\frac{d}{ds}(F(s)) \\ \mathscr{L}\{te^{pt}\} &= -\frac{d}{ds}\left(\frac{1}{s - p}\right) \\ \mathscr{L}\{t^k e^{pt}\} &= \frac{k!}{(s - p)^{k+1}} \end{aligned}$$

Usage

$$\begin{aligned} r(s)Y(s) - c(s) &= q(s)U(s) - b(s) \\ r(s) &= \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0 \\ q(s) &= \beta_m s^m + \cdots + \beta_1 s + \beta_0 \\ c(s) &= n - 1 \text{ order poly from IC on } y(t) \\ b(s) &= m - 1 \text{ order poly from IC on } u(t) \\ Y(s) &= G(s)U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right] \end{aligned}$$

Inverse Laplace

Partial Fraction Expansion

$$\begin{aligned} Y(s) &= G(s)U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right] \\ &= \left[\frac{q(s)}{r(s)}\right]\left[\frac{a(s)}{h(s)}\right] + \left[\frac{c(s) - b(s)}{r(s)}\right] \\ &= \frac{q(s)a(s) + h(s)[c(s) - b(s)]}{r(s)h(s)} \\ &= \sum_{l=1}^{L}\frac{A_l}{s - d_l} \\ y(t) &= \sum_{l=1}^{L} A_l e^{d_l t} \end{aligned}$$

Residue Formula

$$A_l = [(s - d_l)Y(s)]_{s=d_l}$$
$$\overline{A_l} = \left[(s - \overline{d_l})Y(s)\right]_{s=\overline{d_l}}$$
$$A_l e^{d_l t} + \overline{A_l}e^{\overline{d_l}t} = 2|A_l|e^{\sigma t}\cos(\omega t + \angle A_l)$$

Repeated Roots

$$\begin{aligned} L &: \text{\# of roots} \\ K &: \text{\# times a root is repeated} \end{aligned}$$

$$Y(s) = \sum_{l=1}^{K}\frac{A_l}{(s - d_l)^l} + \sum_{l=K+1}^{L}\frac{A_l}{(s - d_l)}$$
$$y(t) = \sum_{l=1}^{K}\left(\frac{A_l t^{l-1}}{(l - 1)!}\right)e^{d_1 t} + \sum_{l=K+1}^{L} A_l e^{d_l t}$$

State Model

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$y(t) = C\underline{x}(t) + Du(t)$$
$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Transfer Function

$$\begin{aligned} Q(s) &= \text{Adj}(S\mathbb{I} - A) \\ r(s) &= \text{Det}(S\mathbb{I} - A) \\ G(s) &= \left[C(S\mathbb{I} - A)^{-1}B + D\right] \\ &= \frac{CQ(s)B}{r(s)} + D \\ &= \frac{CQ(s)B + Dr(s)}{r(s)} \end{aligned}$$

$$\text{Zeroes}: CQ(s)B + Dr(s) = 0$$
$$\text{Poles}: r(s) = 0 \quad (\text{Eigenvalues of } A)$$

Impulse Response

$$h(t) = Ce^{At}B + D\,\delta(t)$$

Matrix-Vector Form

$$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}\mathrm{Bu}(\tau)\mathrm{d}\tau$$
$$\mathrm{y}(t) = \mathrm{C}\underline{x}(t) + \mathrm{Du}(t)$$

Heaviside Step Function

Transfer Function

$$\mathscr{L}\{\mathrm{u}(t)\} = \frac{1}{s}$$

Impulse Response

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{u}(t)) = \delta(t)$$

Matrix-Vector Form

$$\underline{\mathrm{u}}(t) = \begin{cases} \underline{0} & t < 0 \\ \underline{1} & t \geq 0 \end{cases}$$

Dirac Delta Function

Transfer Function

$$\mathscr{L}\{\delta(t)\} = 1, \quad \mathscr{L}\{\delta'(t)\} = s$$

Impulse Response

$$\delta(t) \text{ satisfies } \int_{-\epsilon}^{\epsilon} \delta(t)\mathrm{d}t = 1, \quad \forall \epsilon > 0$$

Matrix-Vector Form

$$\underline{\delta}(t) = \begin{pmatrix} \delta(t) \\ \vdots \\ \delta(t) \end{pmatrix}$$

Step Response

First Order

$$\mathrm{y}(t) = K\left(1 - e^{-t/\tau}\right)$$
$$\tau : \text{time constant}$$
$$t_s \approx 4\tau \quad \text{(for 2\% criterion)}$$

Second Order

Poles

$$\alpha_1^2 < 4\alpha_0 \implies \text{complex conjugates}$$
$$\alpha_1^2 = 4\alpha_0 \implies \text{repeated real}$$
$$\alpha_1^2 > 4\alpha_0 \implies \text{real, non-repeated}$$

Complex Conjugates:

$$\alpha_1 = -2\sigma$$
$$\alpha_0 = \sigma^2 + \omega_d^2 = |p_1|^2$$

Repeated Real:

$$t_s = \frac{6}{|p_1|}$$

Real, Non-Repeated:

$$|p_2| \gg |p_1| \implies t_s \approx \frac{4}{|p_1|}$$
$$\text{(threshold for above is: } |p_2| > 5|p_1|\text{)}$$
$$|p_2| \approx |p_1| \implies t_s \approx \frac{6}{|p_1|}$$
$$\text{(threshold for above is: } 1 \leqslant \frac{|p_2|}{|p_1|} \leqslant 1.1\text{)}$$

Damped (Critically Damped, $\zeta = 1$)

$$\mathrm{y}(t) = 1 - (1 + \omega_n t)e^{-\omega_n t}$$

Under-Damped ($0 < \zeta < 1$)

$$\nu = \arccos(\zeta)$$
$$\mathrm{y}(t) = \mathrm{G}(0)\left[1 - \left(\frac{\omega_n}{\omega_d}e^{\sigma t}\sin(\omega_d t + \nu)\right)\right]$$

Natural (Undamped, $\zeta = 0$)

$$\mathrm{y}(t) = 1 - \cos(\omega_n t)$$

LHP Zero

1. A zero in the Left Half Plane does not induce an inverse response.
2. The step response remains monotonic though modified by the zero dynamics.

RHP Zero

1. A Right Half Plane Zero causes an initial inverse (non-minimum phase) response.
2. The response exhibits an undershoot before eventually rising to steady state.

Performance Metrics

$$M_p : \text{Maximum Overshoot } = \frac{\mathrm{y}_{\max} - \mathrm{y}_{ss}}{\mathrm{y}_{ss}} \times 100\%$$
$$t_r : \text{Rise Time (10\% to 90\% of final value)}$$
$$t_c : \text{Time steady-state is first crossed}$$
$$t_p : \text{Peak Time (time to first peak)}$$
$$t_s : \text{Settling Time (2\% criterion)}$$

Overshoot

$$M_p = e^{-\frac{\sigma}{\omega_d}\pi} \times 100\%$$
$$= e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$
$$t_c = \frac{\pi - \nu}{\omega_d}$$
$$t_p = \frac{\pi}{\omega_d}$$
$$\mathrm{y_p} = \mathrm{y_{ss}}\left[1 + M_p\right]$$

System Zeroes

Input Absorption

1. System zeroes can absorb certain input dynamics.
2. A zero at $s = z$ may cancel an input pole at $s = z$.

Transient Suppression

1. Appropriately placed zeroes can mitigate transient peaks.
2. They are used in controller design to improve system performance.

Pole Cancellation

1. Occurs when a system zero cancels a pole in the transfer function.
2. Ideal cancellation is sensitive to model uncertainties.

Frequency Response

Definition

$$\mathrm{G}(j\omega) = \mathrm{G}(s)\Big|_{s=j\omega}, \quad \omega \in \mathbb{R}$$
$$\text{Magnitude} : |\mathrm{G}(j\omega)|$$
$$\text{Phase} : \angle \mathrm{G}(j\omega)$$

Quantification

1. **Gain Margin**: Factor by which gain can be increased before instability.
2. **Phase Margin**: Additional phase lag required to reach instability.
3. These margins and the overall frequency response are visualized using Bode plots.

Bode Diagrams

Decibel Units

$$\text{Magnitude (dB)} = 20\log_{10}(|\mathrm{G}(j\omega)|)$$

Shape

Transfer Function

$$\mathrm{G}(s) = \frac{N(s)}{D(s)}$$

Zeroes

1. Each zero contributes a +20 dB/decade slope beyond its break frequency.

Poles

1. Each pole contributes a -20 dB/decade slope beyond its break frequency.

Gain

1. A constant gain K shifts the magnitude plot by $20\log_{10}(K)$ dB.

Bode Magnitude Diagrams

Shape

Transfer Function

$$|\mathrm{G}(j\omega)|$$

Zeroes

1. Zeroes add positive slopes to the magnitude plot.

Poles

1. Poles add negative slopes to the magnitude plot.

Gain

1. The overall gain sets the baseline level of the magnitude plot.