Lecture 8: Forces on an Airfoil and Conservation of Energy

ENAE311H Aerodynamics I

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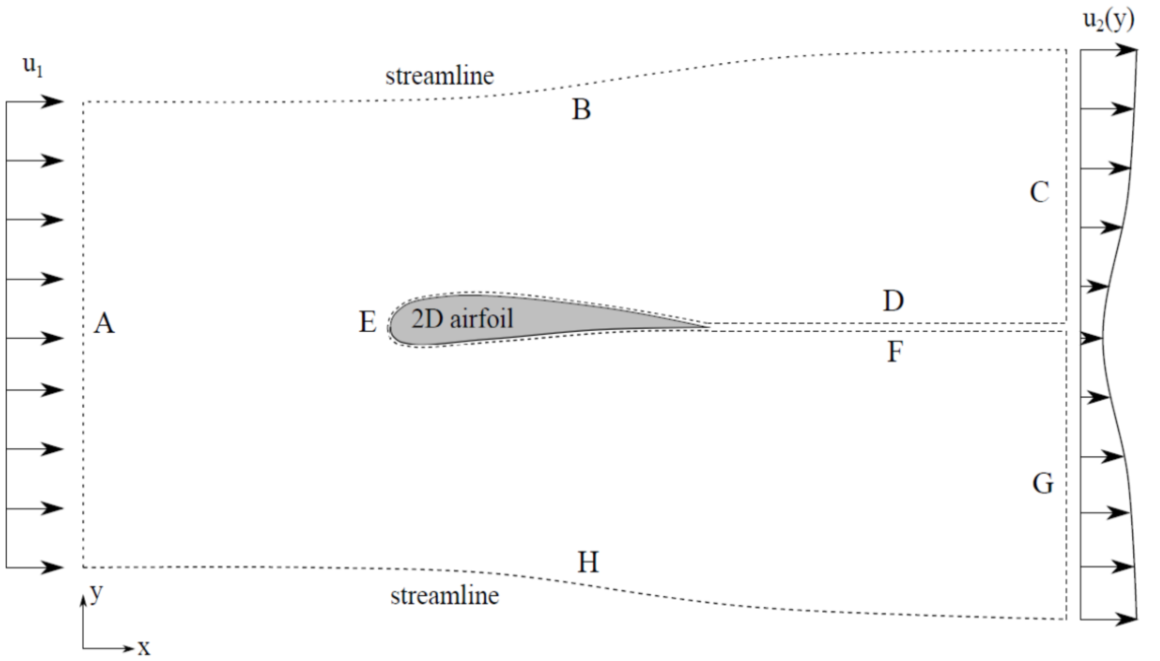
Forces on a 2D airfoil

Early efforts to determine forces acting on airfoils did not actually involve any direct force measurements, but rather relied on measurements of fluid properties downstream of the airfoil and on the wind-tunnel walls.

To see how this worked for the drag force, consider the CV shown to the right. Note the following:

1. Surfaces B, C, G, and H are sufficiently far from the airfoil that the pressure has reverted to ambient (i.e., $p = p_\infty$)
2. No matter how far downstream we go, however, a velocity deficit will remain in the wake (because of momentum transferred from the fluid to the airfoil)
3. The force exerted on the fluid along E will be equal and opposite to the drag on the airfoil, D' (Newton's 3rd law)
4. No fluid crosses B and H since they are streamlines
5. The momentum of the fluid crossing D is exactly balanced by that crossing F; also pressure forces on these two faces exactly balance.



Forces on a 2D airfoil

We can ignore the effects of friction, except along E where these combine with the pressure forces to contribute to D'. We also assume that the effects of gravity on the fluid are negligible.

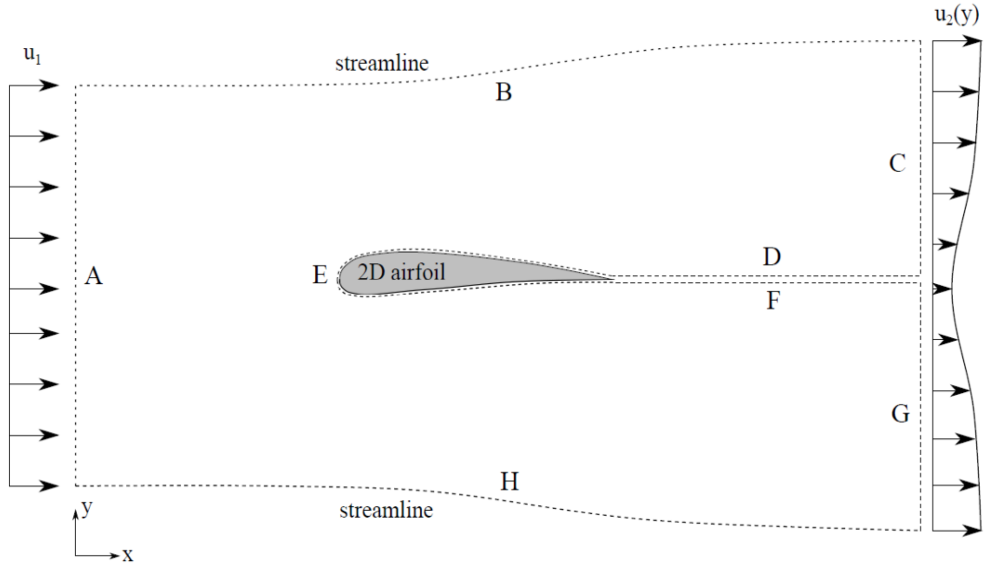
The x-component of the momentum equation then becomes

$$\cancelto{0\text{ (steady)}}{\frac{\partial}{\partial t}\iiint_{CV} v_x \rho dV} + \iint_{CS'} v_x(\rho \mathbf{v}\cdot \mathbf{dA}) = -\iint_{CS'} p dA_x - D',$$

where CS' is the control surface comprising A, B, C, G and H, and $-D'$ is the force exerted on the fluid by the airfoil along E.

Note that we have chosen B, C, G, and H so that the pressure there has equalized to ambient, so pressure is constant over CS'. Since CS' forms a closed surface, we thus have

$$\iint_{CS'} p dA_x = 0.$$



Forces on a 2D airfoil

Also, since no fluid crosses B and H, we have that $\boldsymbol{v} \cdot \boldsymbol{dA} = \mathbf{0}$ along each (so only need to consider A, C, and G for surface integral). Thus:

$$\iint_{CS'} v_x(\rho\mathbf{v} \cdot \mathbf{dA}) = -\iint_{\mathrm{A}} \rho_1 u_1^2 dA + \iint_{\mathrm{C \cup G}} \rho_2 u_2^2 dA.$$

Since the airfoil is 2D, we consider the integrals per unit depth (i.e., $dA \to dy$). Our momentum conservation equation then reduces to:

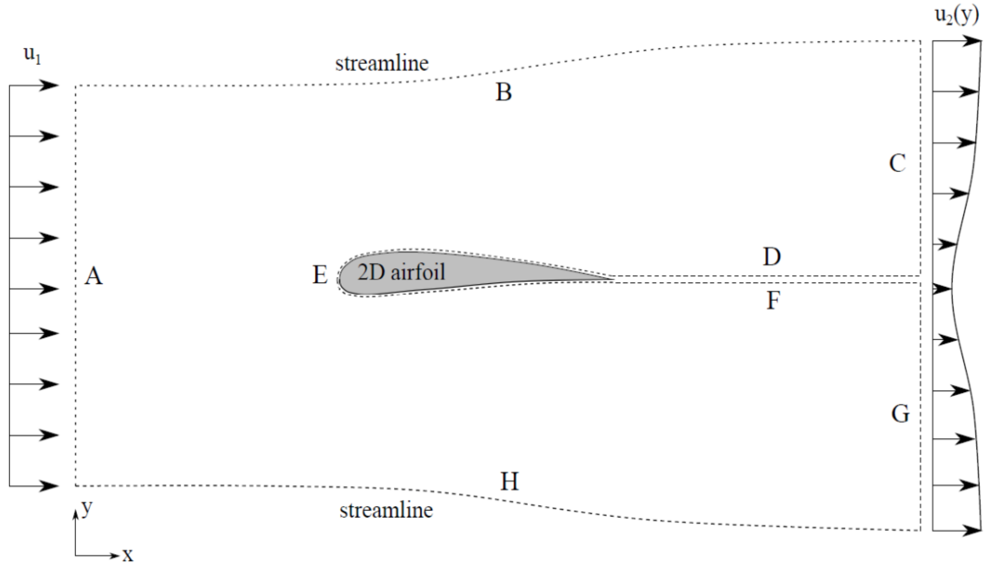
$$D' = \int_{\mathrm{A}} \rho_1 u_1^2 dy - \int_{\mathrm{C \cup G}} \rho_2 u_2^2 dy.$$

We can simplify this expression further by applying mass conservation to the same CV, which yields (per unit depth):

$$-\int_{\mathrm{A}} \rho_1 u_1 dy + \int_{\mathrm{C \cup G}} \rho_2 u_2 dy = 0.$$

Note that, since u_1 is constant, we can multiply through:

$$\int_{\mathrm{A}} \rho_1 u_1^2 dy = \int_{\mathrm{C \cup G}} \rho_2 u_1 u_2 dy.$$



Forces on a 2D airfoil

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$$\begin{aligned} D' &= \int_{\mathrm{C \cup G}} \rho_2 u_1 u_2 dy - \int_{\mathrm{C \cup G}} \rho_2 u_2^2 dy \\ &= \int_{\mathrm{C \cup G}} \underbrace{\rho_2 u_2}_{\text{mass flux per unit area}} \underbrace{(u_1 - u_2)}_{\text{velocity decrement}} dy. \end{aligned}$$

We can simplify this expression further by applying mass conservation to the same CV, which yields (per unit depth):

$$-\int_{\mathrm{A}} \rho_1 u_1 dy + \int_{\mathrm{C \cup G}} \rho_2 u_2 dy = 0.$$

Note that, since u_1 is constant, we can multiply through:

$$\int_{\mathrm{A}} \rho_1 u_1^2 dy = \int_{\mathrm{C \cup G}} \rho_2 u_1 u_2 dy.$$

or, for incompressible flow

$$D' = \rho \int_{\mathrm{C \cup G}} u_2(u_1 - u_2) dy.$$

Require only velocity measurement in wake!

Conservation of energy

We return once again to the Reynolds Transport Theorem:

$$\frac{dN_s}{dt} = \frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV + \iint_{CS} \eta\rho \mathbf{v}\cdot\mathbf{dA},$$

If the extensive system property is the total system energy, E_0, the corresponding intensive property is

$$\eta = e + \frac{V^2}{2} + gy$$

i.e., the sum of the fluid internal, kinetic, and potential (specific) energies.

The first law of thermodynamics tells us that the rate of change of energy of a system is the sum of the heat addition to and work done on the system:

$$\frac{dE_0}{dt} = \dot{Q} + \dot{W}.$$

(Note the convention that $\dot{W}$ is positive if done <u>on</u> – not by – the system.)

The RTT then becomes:

$$\frac{\partial}{\partial t}\iiint_{CV} \rho\left(e + \frac{V^2}{2} + gy\right) dV + \iint_{CS} \rho\left(e + \frac{V^2}{2} + gy\right) \mathbf{v}\cdot\mathbf{dA} = \dot{Q} + \dot{W}.$$

Work done and heat addition

In a single dimension, work done is force times distance, so rate of work done, $\dot{W}$, is force times velocity, or more generally (multiple dimensions), $\dot{W} = \boldsymbol{F} \cdot \boldsymbol{v}$.

We can thus break down $\dot{W}$ as follows:

$$\dot{W} = \underbrace{-\iint_{CS} \mathbf{v} \cdot (p\mathbf{dA})}_{\text{work done by pressure forces}} + \underbrace{\iint_{CS} \mathbf{v} \cdot (\bar{\bar{\tau}} \cdot \mathbf{dA})}_{\text{work done by shear stresses}} + \underbrace{\dot{W}_s}_{\text{shaft work}}.$$

Note that the CV can often be chosen such that the shear stress term is zero. Also, we have neglected gravitational work, since it has already been included in the potential term.

The heating term can be divided into volumetric heating, $\dot{q}$ (e.g., from radiation) and viscous heating at the CV surface:

$$\dot{Q} = \iiint_{CV} \rho \dot{q} dV + \dot{Q}_{viscous}.$$

Integral and differential forms

The integral form of our energy equation thus becomes

$$\overbrace{\frac{\partial}{\partial t}\iiint_{CV}\rho\left(e+\frac{V^2}{2}+gy\right)dV}^{\text{rate of change of energy inside CV}} + \overbrace{\iint_{CS}\rho\left(e+\frac{V^2}{2}+gy\right)\mathbf{v}\cdot\mathbf{dA}}^{\text{flux of energy through CV boundaries}} =$$

$$\underbrace{-\iint_{CS}\mathbf{v}\cdot(p\mathbf{dA})}_{\text{work done by pressure forces}} + \underbrace{\iint_{CS}\mathbf{v}\cdot(\bar{\bar{\tau}}\cdot\mathbf{dA})}_{\text{work done by shear stresses}} + \underbrace{\dot{W}_s}_{\text{shaft work}} + \underbrace{\iiint_{CV}\rho\dot{q}\,dV}_{\text{volumetric heating}} + \underbrace{\dot{Q}_{viscous}}_{\text{viscous heating}}.$$

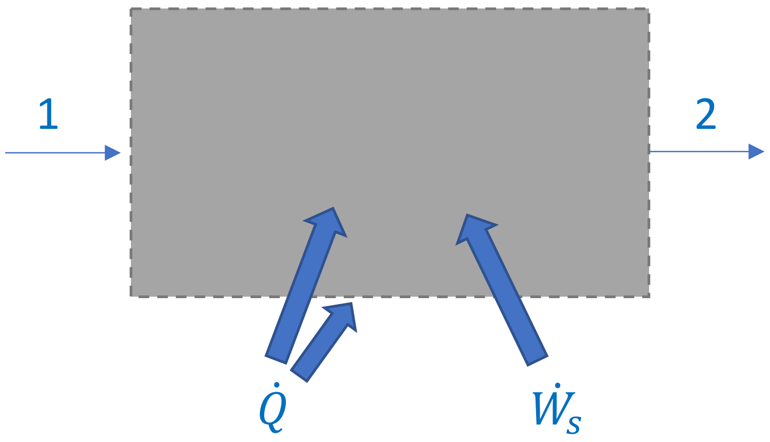
The corresponding differential form (neglecting shaft work, gravity, and viscous heating) is

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\mathbf{v}\right] = \rho\dot{q} - \nabla\cdot(\rho\mathbf{v}) + \nabla\cdot(\bar{\bar{\tau}}\cdot\mathbf{v}).$$

Simplified energy equation

In a number of useful flow configurations, the flow is steady ($\frac{\partial}{\partial t} = 0$) and we have uniform conditions across inlet and outlet. We start from the steady energy equation:

$$\iint_{CS} \rho \left(e + \frac{V^2}{2} + gy \right) \mathbf{v} \cdot \mathbf{dA} = \dot{Q} + \dot{W}$$



and use our assumption of uniform inlet/outlet conditions

$$\begin{aligned} \iint_{CS} \rho \left(e + \frac{V^2}{2} + gy \right) \mathbf{v} \cdot \mathbf{dA} &= -(e_1 + \frac{u_1^2}{2} + gy_1)\rho_1 u_1 A_1 + (e_2 + \frac{u_2^2}{2} + gy_2)\rho_2 u_2 A_2 \\ &= \dot{m} \left[e_2 - e_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1) \right], \end{aligned}$$

since (from mass conservation), $\rho_1 u_1 A_1 = \rho_2 u_2 A_2 = \dot{m}$.

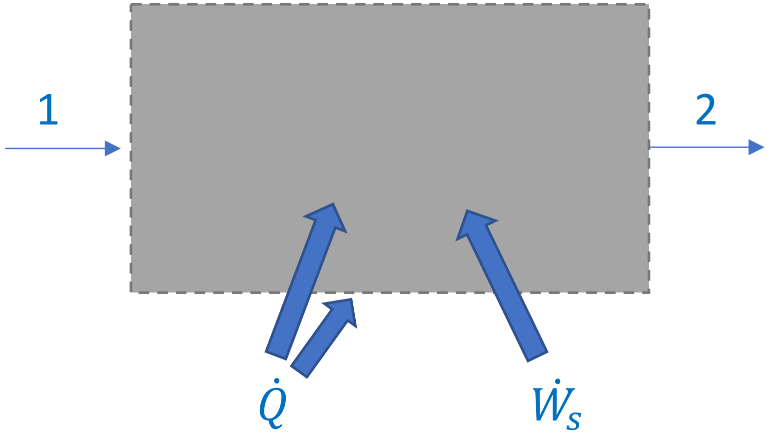
If we choose our control volume so that shear stresses don't contribute to the work, we can write

$$\begin{aligned} \dot{W} &= -\iint_{CS} \mathbf{v} \cdot (p\mathbf{dA}) + \dot{W}_s \\ &= u_1 p_1 A_1 - u_2 p_2 A_2 + \dot{W}_s \\ &= \dot{m}\frac{p_1}{\rho_1} - \dot{m}\frac{p_2}{\rho_2} + \dot{W}_s. \end{aligned}$$

Simplified energy equation

Combining these results, we have:

$$\dot{m}\left[e_2 - e_1 + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.$$



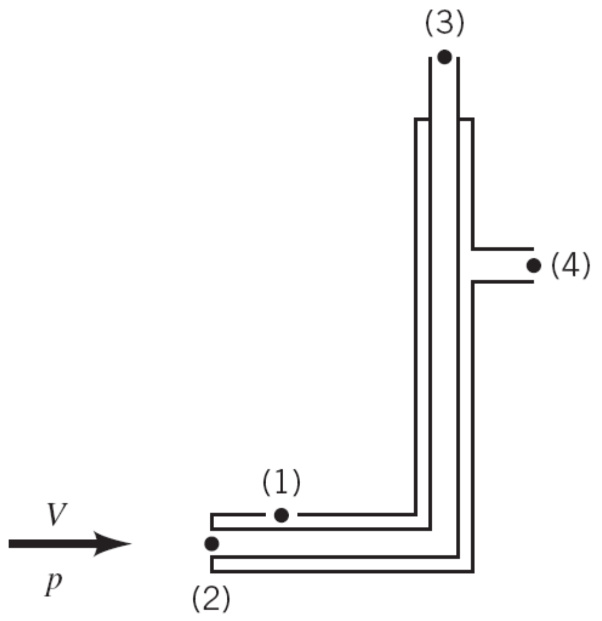
An alternative form is possible if we use the flow enthalpy, $h = e + \frac{p}{\rho}$:

$$\boxed{\dot{m}\left[h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.}$$

This equation will be useful in several important situations later in the course.

Pitot-static Tube

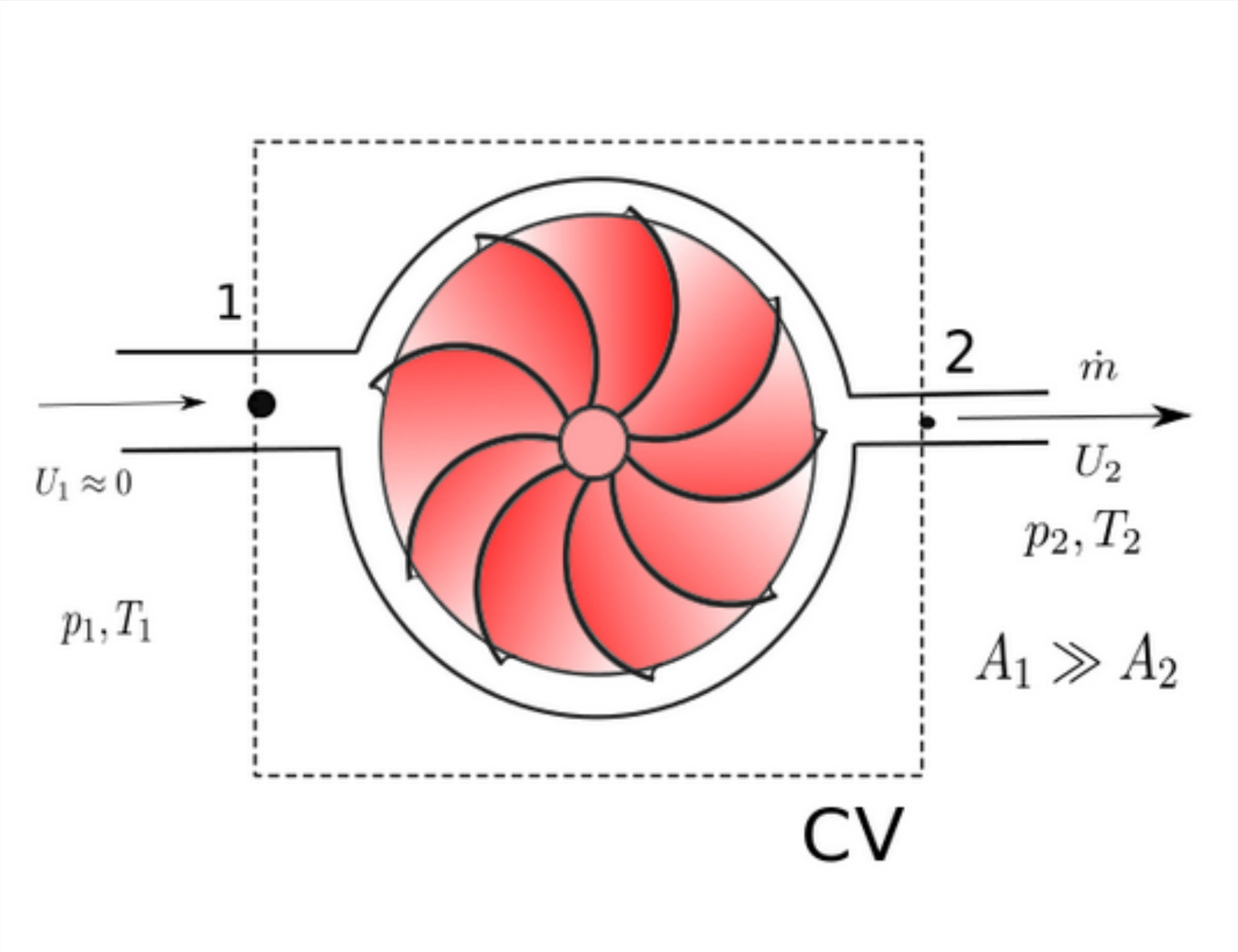
- Two concentric tubes – one with a forward facing tap and the other with a side tap



$$p_3 = p + \frac{1}{2}\rho V^2$$

$$p_4 = p$$

$$V = \sqrt{\frac{2}{\rho}\left(p_3 - p_4\right)}$$



Energy balance for a compressor (example 1)

Example 8 [edit | edit source]

Air enters compressor at inlet 1 with negligible velocity and leaves at outlet 2. The power input to the machine is P_{input} and the volume flow rate is $\dot{V}$. Find a relation for the rate of heat transfer in terms of the power, temperature, pressure, etc.

1:

$$\dot{Q} + \dot{W}_{shaft} + \underbrace{\dot{W}_{shear}}_{=0\,=\tau\cdot\vec{U}} + \underbrace{\dot{W}_{other}}_{=0} = \underbrace{\frac{\partial}{\partial t}\int e\rho dV}_{=0\ steady\ state} + \int_{cs}\left(u + \frac{p}{\rho} + \frac{U^2}{2} + gx_2\right)\rho\vec{U}\cdot\vec{n}dA$$

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{cv}\rho dV}_{=0\ steady\ state} + \int_{cs}\rho\vec{U}\cdot\vec{n}dA \rightarrow |\rho_1 U_1 A_1| = |\rho_2 U_2 A_2| = \dot{m}$$

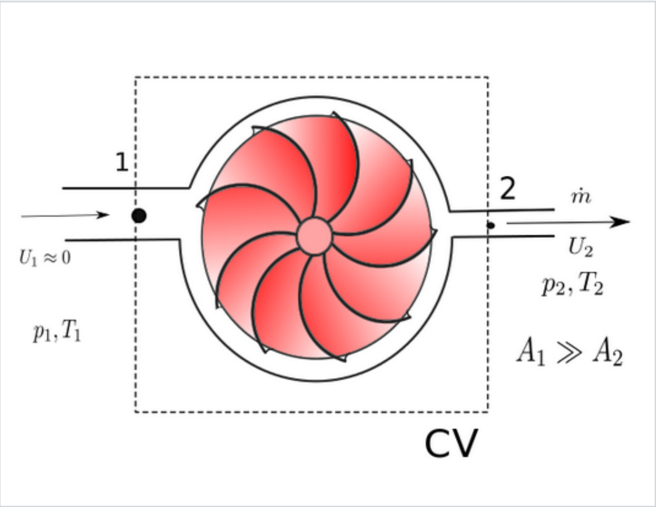
2:

$$\dot{Q} = -\dot{W}_{shaft} + \int_{cs}\left(u + \frac{p}{\rho} + \frac{U^2}{2} + gz\right)\rho\vec{U}\cdot\vec{n}dA$$

For uniform properties at 1 and 2 and inserting the inserting the relation for the enthalpy $h = u + \frac{p}{\rho}$.

$$\dot{Q} = -\dot{W}_{shaft} - \left(h_1 + \underbrace{\frac{U_1^2}{2}}_{=0} + gz_1\right)|\rho_1 U_1 A_1| + \left(h_2 + \frac{U_2^2}{2} + gz_2\right)|\rho_2 A_2 U_2|$$

$$\dot{Q} = -\dot{W}_{shaft} + \dot{m}\left[h_2 + \frac{U_2^2}{2} - h_1 + \underbrace{g(z_2 - z_1)}_{=0}\right]$$



Energy balance for a compressor (example 1)

Assuming that air behaves like an ideal gas with a constant c_p.

$$h_2 - h_1 = c_p(T_2 - T_1)$$

$$\dot{Q} = -\dot{W}_{shaft} + \dot{m}\left[c_p(T_2 - T_1) + \frac{U_2^2}{2}\right]$$

EXAMPLE 5.22 Energy—Temperature Change

GIVEN The 420-ft waterfall shown in Fig. E5.22*a* involves steady flow from one large body of water to another.

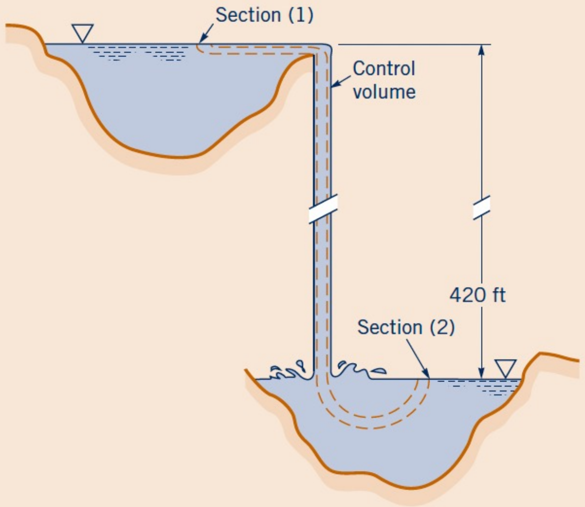
FIND Determine the temperature change associated with this flow.

SOLUTION

To solve this problem, we consider a control volume consisting of a small cross-sectional streamtube from the nearly motionless surface of the upper body of water to the nearly motionless surface of the lower body of water as is sketched in Fig. E5.22*b*. We need to determine $T_2 - T_1$. This temperature change is related to the change of internal energy of the water, $\check{u}_2 - \check{u}_1$, by the relationship

$$T_2 - T_1 = \frac{\check{u}_2 - \check{u}_1}{\check{c}} \tag{1}$$

■ **Figure E5.22*a*** [Photograph of Akaka Falls (Hawaii) courtesy of Scott and Margaret Jones.]



■ **Figure E5.22*b***

where $\check{c} = 1 \text{ Btu}/(\text{lbm} \cdot {}^\circ\text{R})$ is the specific heat of water. The application of Eq. 5.70 to the contents of this control volume leads to

$$\dot{m}\left[\check{u}_2 - \check{u}_1 + \left(\frac{p}{\rho}\right)_2 - \left(\frac{p}{\rho}\right)_1 + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1)\right]$$
$$= \dot{Q}_{\substack{\text{net} \\ \text{in}}} \tag{2}$$

We assume that the flow is adiabatic. Thus $\dot{Q}_{\text{net in}} = 0$. Also,

$$\left(\frac{p}{\rho}\right)_1 = \left(\frac{p}{\rho}\right)_2 \tag{3}$$

because the flow is incompressible and atmospheric pressure prevails at sections (1) and (2). Furthermore,

$$V_1 = V_2 = 0 \tag{4}$$

because the surface of each large body of water is considered motionless. Thus, Eqs. 1 through 4 combine to yield

$$T_2 - T_1 = \frac{g(z_1 - z_2)}{\check{c}}$$

so that with

$$\check{c} = [1 \text{ Btu}/(\text{lbm} \cdot {}^\circ\text{R})]\,(778 \text{ ft} \cdot \text{lb}/\text{Btu})$$
$$= [778 \text{ ft} \cdot \text{lb}/(\text{lbm} \cdot {}^\circ\text{R})]$$

$$T_2 - T_1 = \frac{(32.2 \text{ ft/s}^2)(420 \text{ ft})}{[778 \text{ ft} \cdot \text{lb}/(\text{lbm} \cdot {}^\circ\text{R})][32.2\,(\text{lbm} \cdot \text{ft})/(\text{lb} \cdot \text{s}^2)]}$$
$$= 0.540\ {}^\circ\text{R} \qquad \textbf{(Ans)}$$

COMMENT Note that it takes a considerable change of potential energy to produce even a small increase in temperature.

EXAMPLE 5.28 Energy—Fan Performance

GIVEN Consider the fan of Example 5.19.

FIND Show that only some of the shaft power into the air is converted into useful effects. Develop a meaningful efficiency equation and a practical means for estimating lost shaft energy.

SOLUTION

We use the same control volume used in Example 5.19. Application of Eq. 5.82 to the contents of this control volume yields

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 + w_{\substack{\text{shaft}\\ \text{net in}}} - \text{loss} \qquad \textbf{(1)}$$

As in Example 5.26, we can see with Eq. 1 that a "useful effect" in this fan can be defined as

$$\text{useful effect} = w_{\substack{\text{shaft}\\ \text{net in}}} - \text{loss}$$

$$= \left(\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1\right) \qquad \textbf{(2)} \quad \textbf{(Ans)}$$

In other words, only a portion of the shaft work delivered to the air by the fan blades is used to increase the available energy of the air; the rest is lost because of fluid friction.

A meaningful efficiency equation involves the ratio of shaft work converted into a useful effect (Eq. 2) to shaft work into the air, $w_{\text{shaft net in}}$. Thus, we can express efficiency, η, as

$$\eta = \frac{w_{\substack{\text{shaft}\\ \text{net in}}} - \text{loss}}{w_{\substack{\text{shaft}\\ \text{net in}}}} \qquad \textbf{(3)}$$

However, when Eq. 5.54, which was developed from the moment-of-momentum equation (Eq. 5.42), is applied to the contents of the control volume of Fig. E5.19, we obtain

$$w_{\substack{\text{shaft}\\ \text{net in}}} = +U_2 V_{\theta 2} \qquad \textbf{(4)}$$

Combining Eqs. 2, 3, and 4, we obtain

$$\eta = \{[(p_2/\rho) + (V_2^2/2) + gz_2] - [(p_1/\rho) + (V_1^2/2) + gz_1]\}/U_2 V_{\theta 2} \qquad \textbf{(5)} \quad \textbf{(Ans)}$$

Equation 5 provides us with a practical means to evaluate the efficiency of the fan of Example 5.19.

Combining Eqs. 2 and 4, we obtain

$$\text{loss} = U_2 V_{\theta 2} - \left[\left(\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1\right)\right] \qquad \textbf{(6)} \quad \textbf{(Ans)}$$

COMMENT Equation 6 provides us with a useful method of evaluating the loss due to fluid friction in the fan of Example 5.19 in terms of fluid mechanical variables that can be measured.