# The Geodesic Equation

What is a straight line?

(Ask the class) Two good definitions.

1. Path of shortest distance between two points
2. Path that lies along its own tangent vector
3. (Physics) Path something takes when not acted on by an external force

Let's elevate these to curved manifolds.

1. Path of extremal proper time between two space-time events
2. Path that parallel transports into itself
3. (Physics) Path something takes when acted on by nothing but "gravity"

"Geodesics": fancy term for "straight lines in curved space"

Can derive the geodesic equation using either ① or ②. Let $x^\mu(\tau)$ parameterize a path through space-time.

① is a bit messy, but basically amounts to doing the variational principle on $\tau$ (see Carroll 106).

$$\frac{d^2 x^\rho}{d\tau^2} + \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

② is easier.

$$\underbrace{\frac{dx^\mu}{d\tau}\nabla_\mu}_{\substack{\text{parallel}\\\text{transport}\\\text{along curve}}} \underbrace{\frac{dx^\nu}{d\tau}}_{\substack{\text{tangent}\\\text{vector}}} = 0$$

☆ ☆

$$\Rightarrow \boxed{\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$

This can be thought of as a differential eq. that, given an initial position and velocity, tells you the path a freely-falling object would take!

# Properties of Geodesics

Geodesic equation is generalization of $a = \frac{F}{m}$. Can even add extra (non-gravity) forces to right hand side (though you need to be careful that they are covariant).

Can also write in terms of "4-velocity":

$$U^\mu = \frac{dx^\mu}{d\tau} \Rightarrow U^\lambda \nabla_\lambda U^\mu = 0$$

Normalized such that $U^\mu U_\mu = -1$

because $d\tau^2 = -ds^2 = -g_{\mu\nu}\, dx^\mu dx^\nu$

$$\Rightarrow \quad 1 = -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

or "4-momentum":

$$p^\mu = m \frac{dx^\mu}{d\tau} \Rightarrow p^\lambda \nabla_\lambda p^\mu = 0.$$

What is this $m$? "Invariant mass", or energy of a particle at rest in a given INERTIAL frame.

Let's use geodesics to construct locally inertial coordinates

1. At any point $p$, every tangent vector $k^\mu$ defines a unique geodesic since geodesic eq. is an IVP.

   (i.e. I start here at point $p$ and throw something with $\vec{v}$. Unique path described by this, hence map of $p, \vec{v}, \tau \to x^{\hat{\mu}}$)

2. These coordinates might not cover whole manifold if manifold is geodesically incomplete, but that's okay because we only care about point $p$.

3. Linear algebra lets you diagonalize and rescale $g_{\mu\nu}$ to $\eta_{\mu\nu}$ at a single point.

4. Now we need $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} = 0$. But in coord. system we just built by throwing baseballs in various directions, all curves look like $x^{\hat{\mu}}(\tau) = k^{\tilde{\mu}} \tau$ (i.e. $\vec{x} = \vec{v}\tau$). Hence $\frac{\partial^2 x^{\hat{\mu}}}{\partial \tau^2} = 0$.

$\longrightarrow$

5. By geodesic equation, $\frac{d^2 x^{\hat{\mu}}}{d\tau^2} = -\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}} k^{\hat{\rho}} k^{\hat{\sigma}}$ for $x^{\mu} = k^{\mu}\tau$.

6. This implies $\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}} k^{\hat{\rho}} k^{\hat{\sigma}} = 0$, hence $\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}} = 0$.

7. By metric compatibility, $\nabla_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} = 0$, hence since $\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}} = 0$, $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} = 0$ <u>at point $p$</u>.

This specific construction is called "Riemann normal coordinates", but all inertial coordinates are equivalent to this up to third order.

SUPER USEFUL: Any invariant quantities you construct in inertial coordinates MUST be the same in all coordinate systems!

Ex. Invariant mass:

We define the four-velocity to be $U^\mu = \frac{dx^\mu}{d\tau}$.

We can then define a four-momentum

$$p^\mu = m \frac{dx^\mu}{d\tau}.$$

In SPECIAL relativity, we know that this "m" is just the rest energy of a particle, i.e. if someone is comoving with a particle, that is what they perceive the particle's energy to be.

$$p^\mu = (m, 0, 0, 0), \quad U^\mu_{obs} = (1, 0, 0, 0)$$

$$E = -p^\mu U_{\mu,\,particle} = m \qquad (E = mc^2!!)$$

If observer is moving, then $U^\mu{}_{obs} = (\gamma, \gamma v, 0, 0)$

$E_{kin} \sim \frac{1}{2}mv^2$ in Newton, elevate to covariant expr. $E = \underbrace{p^\mu}_{mv} \underbrace{U_\mu}_{v}$

$$\Rightarrow \quad -p^\mu U_\mu^{obs} \quad \text{in full generality}$$

$$-p^\mu p_\mu = E^2 - p^2 \equiv m^2 \quad \text{is an invariant in SR.}$$

Since this is a local invariant, must also be an invariant in GR! (All observers, on any geodesic, agree on the mass of a particle.)

# Conservation Laws

In classical mechanics, Noether's theorem gives us a sense of how a symmetry of the Lagrangian gives rise to an associated conservation law.

translation symmetry $\Rightarrow$ conservation of momentum

rotation symmetry $\Rightarrow$ conservation of angular momentum

(time) translation symmetry $\Rightarrow$ conservation of energy

We wish to find these kinds of symmetries in GR.

Isometries: Transformations under which the metric doesn't change.

(e.g. $x^1 \to x^1 + a$, clearly $(dx^1)^2$ doesn't care, gives rise to conservation of momentum in $x^1$ direction)

☆ All such transformations are related to a Killing vector. Finding Killing vectors means finding isometries. ☆

Killing vectors are defined by Killing's equation:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$$

or

$$\nabla_{(\mu} K_{\nu)} = 0 \quad \text{in fancy notation}$$

The quantity $K_\mu p^\mu$ will be conserved for all particles following geodesics.

(e.g. $X^\mu = (0,1,0,0) \Rightarrow p_x$ is conserved)

This seems random, so let's show it is both

1. true that a $K_\mu$ corresponds to leaving the metric unchanged
2. true that $K_\mu p^\mu$ is conserved along geodesics

Proof:
$$\underbrace{p^\mu}_{\text{geodesic}} \nabla_\mu (\underbrace{K_\nu p^\nu}_{\text{conserved quantity}}) = 0$$

$$\Rightarrow p^\mu (\nabla_\mu K_\nu) p^\nu + K_\nu \underbrace{p^\mu \nabla_\mu p^\nu}_{=0 \text{ by geodesic eq.}} = 0$$

$$\Rightarrow p^\mu p^\nu (\nabla_\mu K_\nu) = 0$$

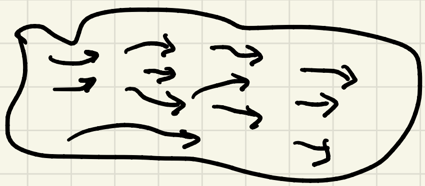
$$\Rightarrow p^\mu p^\nu \nabla_\mu K_\nu + p^\nu p^\mu \nabla_\nu K_\mu = 0$$

$$\Rightarrow p^\mu p^\nu (\nabla_{(\mu} K_{\nu)}) = 0 \Rightarrow \nabla_{(\mu} K_{\nu)} = 0$$

Now we can prove 1, but to do so, we need to introduce a new operator, the <u>Lie derivative</u>.

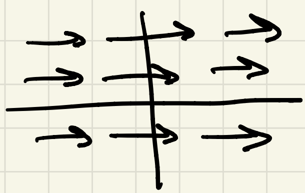
Unlike the covariant derivative, the Lie derivative is geometry-free. Basically just asks how a tensor changes in a "flow" defined by some vector field.

Flow maps manifold to itself.



Killing vector fields define flows that leave $g_{\mu\nu}$ unchanged.

$\mathbb{R}^2$



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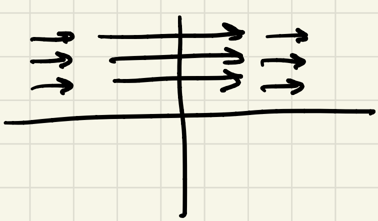
$\mathbb{R}^2$



✗ changes geometry

$$\mathcal{L}_K f = K^\mu \partial_\mu f$$

$\mathcal{L}_K g_{\mu\nu} = 2 \nabla_{(\mu} K_{\nu)}$ where $\nabla_\mu$ is covariant derivative for metric $g_{\mu\nu}$

Trivial to see that if $\mathcal{L}_K g_{\mu\nu} = 0$

$K_\mu$ satisfies $\nabla_{(\mu} K_{\nu)} = 0$.

(See Carroll 429-434 for more.)

Actually solving Killing's equation for an arbitrary metric is hard, and you often don't know when you're finished.

However, some are really easy to spot!

If the metric components don't depend on a particular coordinate, then clearly $g_{\mu\nu}$ doesn't change under shifts in that coordinate.

Ex. All symmetries of $\mathbb{R}^3$ (Carroll 138)

$$X^\mu = (1, 0, 0)$$
$$Y^\mu = (0, 1, 0)$$
$$Z^\mu = (0, 0, 1)$$
$$R = (-y, x, 0)$$
$$S = (z, 0, -x)$$
$$T = (0, -z, y)$$

Ex. Find Killing vectors of

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$

$\Rightarrow$ In an expanding universe, no simple concept of conservation of energy....