"Pole placement" problem

Our objective in control would be easier (but not as easy as you might think!) if we could freely set all the CL poles to specified desired values.

Suppose $\lambda_1, \dots, \lambda_n$ are the desired CL poles. Then it must be the case that

$$
\begin{aligned}
1 + L(s) &= 0 \\
&= (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n) \\
&= s^n + \gamma_{n-1} s^{n-1} + \cdots + \gamma_1 s + \gamma_0 = D_T(s)
\end{aligned}
$$

i.e. Char Eq'n $1+L(s)=0$ must expand into above polynomial where γ_k coefs are determined by desired CL poles.

Root locus is "ouput pole placement"

Root locus examines this problem when $L(s) = \frac{K N(s)}{D(s)}$

where K is adjustable parameter, $N(s)$ $D(s)$ fixed, so

$$1 + L(s) = 0 \Rightarrow D(s) + K N(s) = 0$$

Let $D(s) = s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0$

Assume initially $N(s) = 1 \Rightarrow L(s) = \frac{K}{D(s)}$

Then

$$1 + L(s) = 0 \Rightarrow \boxed{s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + (\alpha_0 + K)}$$

if we want $\rightarrow$ $\boxed{s^n + \gamma_{n-1} s^{n-1} + \dots + \gamma_1 s + \gamma_0}$

Clearly we can only "match" desired polynomial in constant term, i.e. choose K so that $\gamma_0 = \alpha_0 + K$

Not enough DOF to affect other coefs $\Rightarrow$ can't solve PP.

If, more generally, $N(s) = s^{n-1} + \beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0$

then $1 + L(s) = 0$

$$= s^n + (\alpha_{n-1} + K\beta_{n-1})s^{n-1} + \cdots + (\alpha_1 s + K\beta_1)s + (\alpha_0 + K\beta_0)$$

Can affect all coefs, but not independently.

Still only 1 DOF.

Probably cannot simultaneously satisfy

$$\alpha_0 + K\beta_0 = \gamma_0$$

$$\alpha_1 + K\beta_1 = \gamma_1$$

$$\vdots$$

$$\alpha_{n-1} + K\beta_{n-1} = \gamma_{n-1}$$

except in special cases ⇒ Root locus shows these feasible sol'ns (possible CL poles) !!

Controllers with more DOF have more flexibility in "placing" CL poles.

Suppose $H(s) = K_D s + K_P$, $G(s) = \dfrac{1}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$

(PD comp)

Then

$$1 + L(s) = 0$$

$$= s^n + \alpha_{n-1} s^{n-1} + \cdots + (\alpha_1 + K_D)s + (\alpha_0 + K_P)$$

Can now independently affect 2 coefs in char poly,

i.e. match

$$\gamma_0 = \alpha_0 + K_P$$

$$\gamma_1 = \alpha_1 + K_D$$

$\Rightarrow$ Larger set of CL poles feasible.

Can we generalize this idea?

i.e. Keep introducing more DOF into $H(s)$ to allow us to achieve an arbitrary set of CL poles

$\Rightarrow$ While still keeping $H(s)$ "proper"

(i.e. $\rho(H) \geq 0 \Rightarrow$ at least as many poles as zeros)

Consider first a simple example, similar to HW problem.

Terminology:

"Proper": no more zeros than poles

"strictly proper": at least 1 more pole than zeros

Suppose $G(s) = \frac{1}{(s-1)^2}$

and we want all CL poles at -1 (i.e. $D_T(s) = (s+1)^k$)

Try $H(s) = \frac{K(s-z)}{s-p} = \frac{b_1 s + b_0}{s + a_0}$ (lead comp?)

Poles of $T(s)$ satisfy

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{b_1 s + b_0}{(s-1)^2(s+a_0)} = 0$$

or $\underbrace{(s-1)^2(s+a_0) + b_1 s + b_0}$ $= 0$ 3rd order poly

$\Rightarrow$ want this to factor as $(s+1)^3$

$$(s^2-2s+1)(s+a_0)+b_1 s+b_0$$

$$= s^3+(a_0-2)s^2+(1-2a_0+b_1)s+a_0+b_0$$

$$\overset{?}{=} s^3+\gamma_2 s^2+\gamma_1 s+\gamma_0$$ (Desired CL poly)

$$\Rightarrow \begin{cases} \gamma_0 = a_0+b_0 \\ \gamma_1 = 1-2a_0+b_1 \\ \gamma_2 = a_0-2 \end{cases}$$

Determines $b_0=\gamma_0-a_0$

Determines $b_1=\gamma_1-1+2a_0$

Determines $a_0=\gamma_2+2$

Desired CL poles

for $\lambda_1=\lambda_2=\lambda_3=-1 \Rightarrow (s+1)^3 = s^3+\underbrace{3}_{\gamma_2}s^2+\underbrace{3}_{\gamma_1}s+\underbrace{1}_{\gamma_0}$

$\gamma_2=3 \Rightarrow a_0=5$

$\gamma_1=3 \Rightarrow b_1=12$

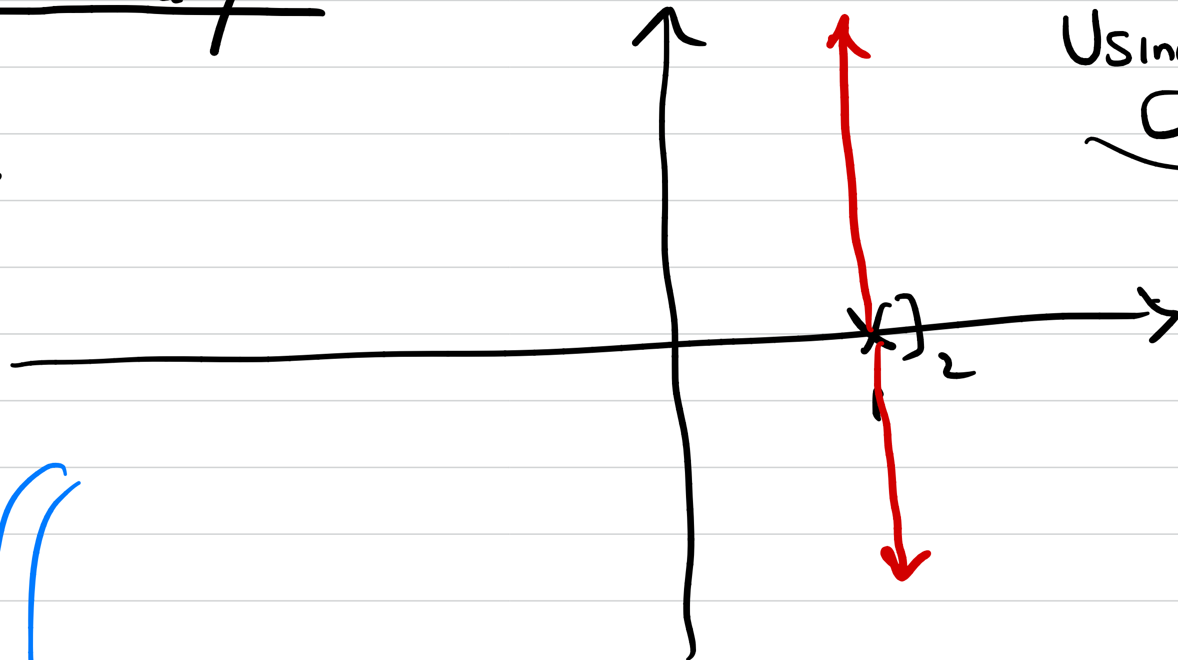
$\gamma_0=1 \Rightarrow b_0=-4$

$$H(s)=\frac{12s-4}{s+5}$$

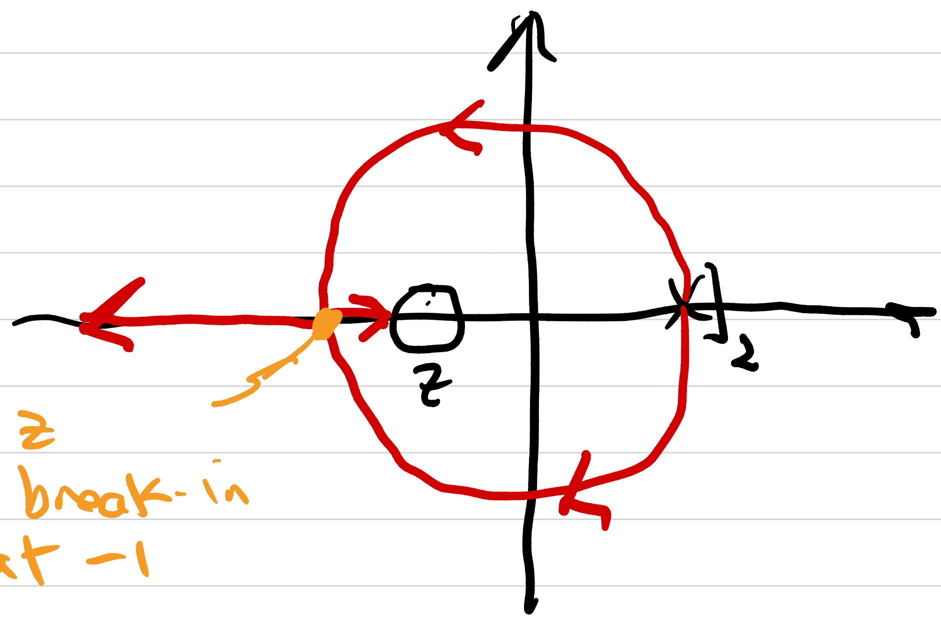
$$H(s)=\frac{K(s-1/3)}{s+5} \qquad K=12$$

Root locus analysis

$$G(s) = \frac{1}{(s-1)^2}$$



Using $H(s) = K$



Using $H(s) = K_D s + K_P$

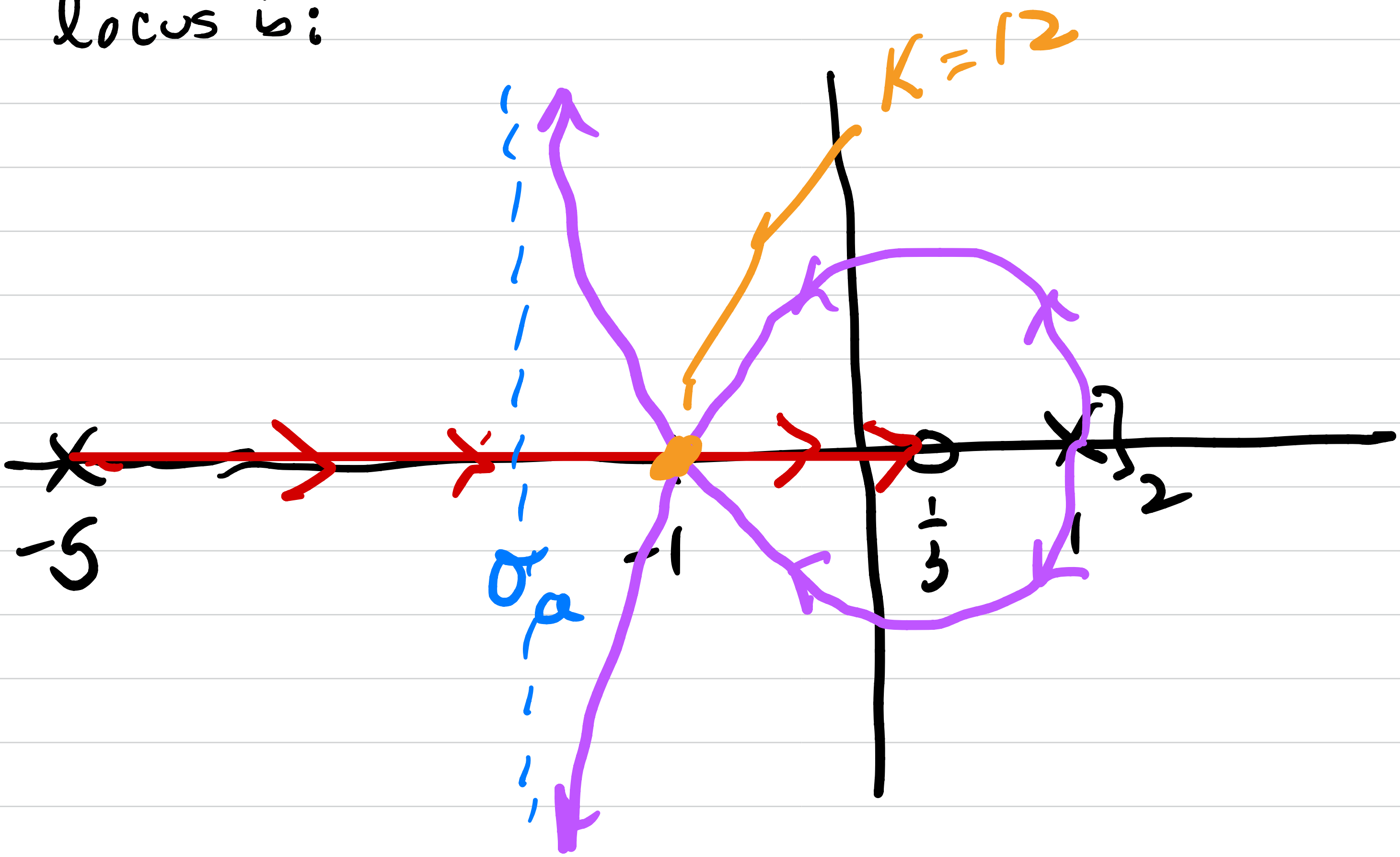
$= K(s-z)$

(PD – not proper/implementable)

choose z to get break-in at -1

But Using $H(s) = \dfrac{K(s - \frac{1}{3})}{s+5}$ | Our Design from above

locus is:



$\gamma_e = \pm 90^\circ$

$\sigma_a = \frac{-10}{6}$

Root locus interpretation:

We have added a zero and a pole in just the Right locations to "pull" the RHP branches left in a loop that meets the New branch coming from the added pole at exactly -1

$\Rightarrow$ With enough experience in root locus we might have guessed this could happen but matching DOFs in $H(s)$ polys to desired char poly for $T(s)$ made this straightforward!

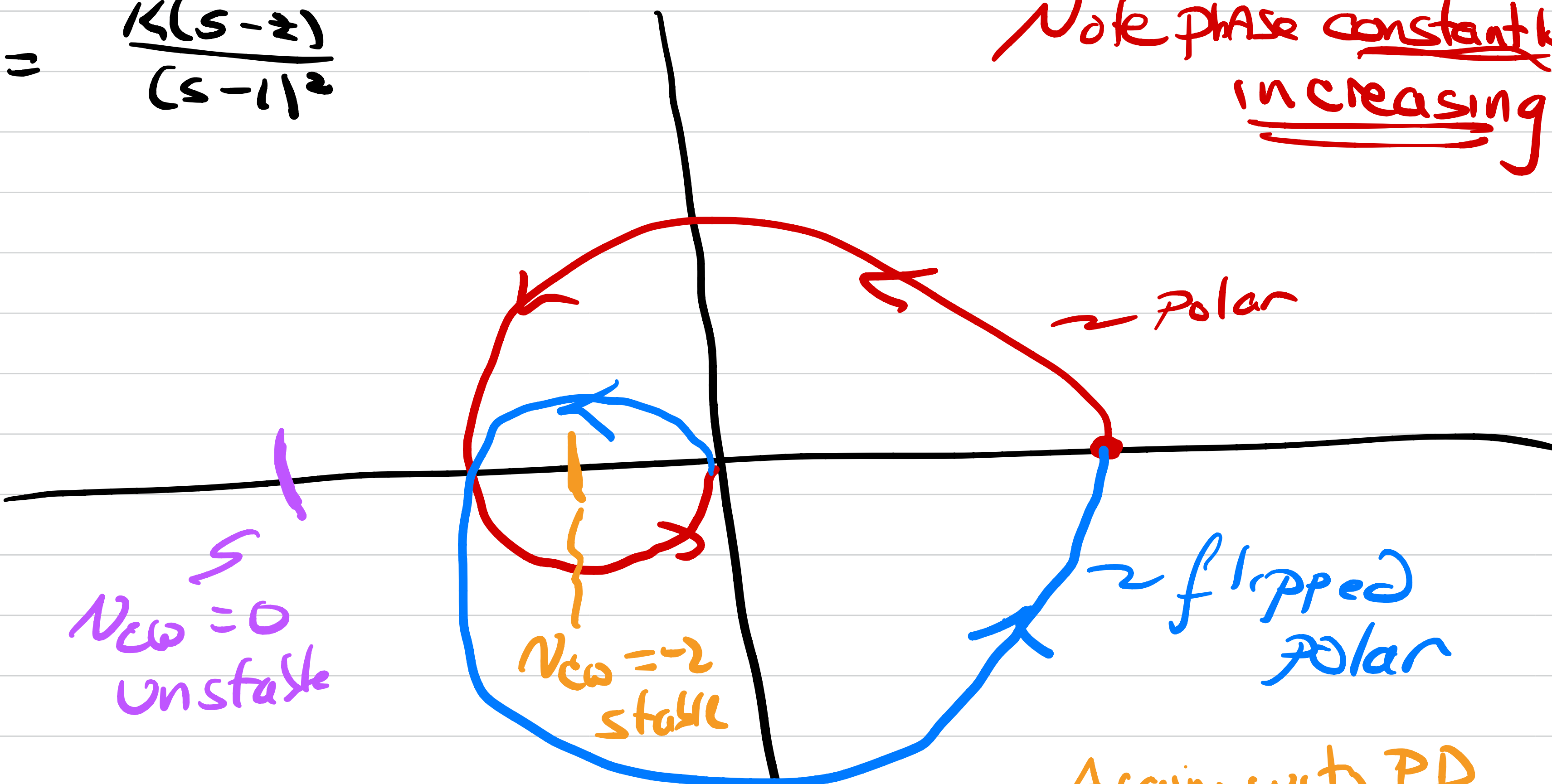
Nyquist interpretation:

$$G(s) = \frac{1}{(s-1)^2}$$

w/ $H(s) = K_D s + K_P = K(s-z) \quad (z<0)$ (PD comp)

$$L(s) = \frac{K(s-z)}{(s-1)^2}$$

Note phase constantly increasing

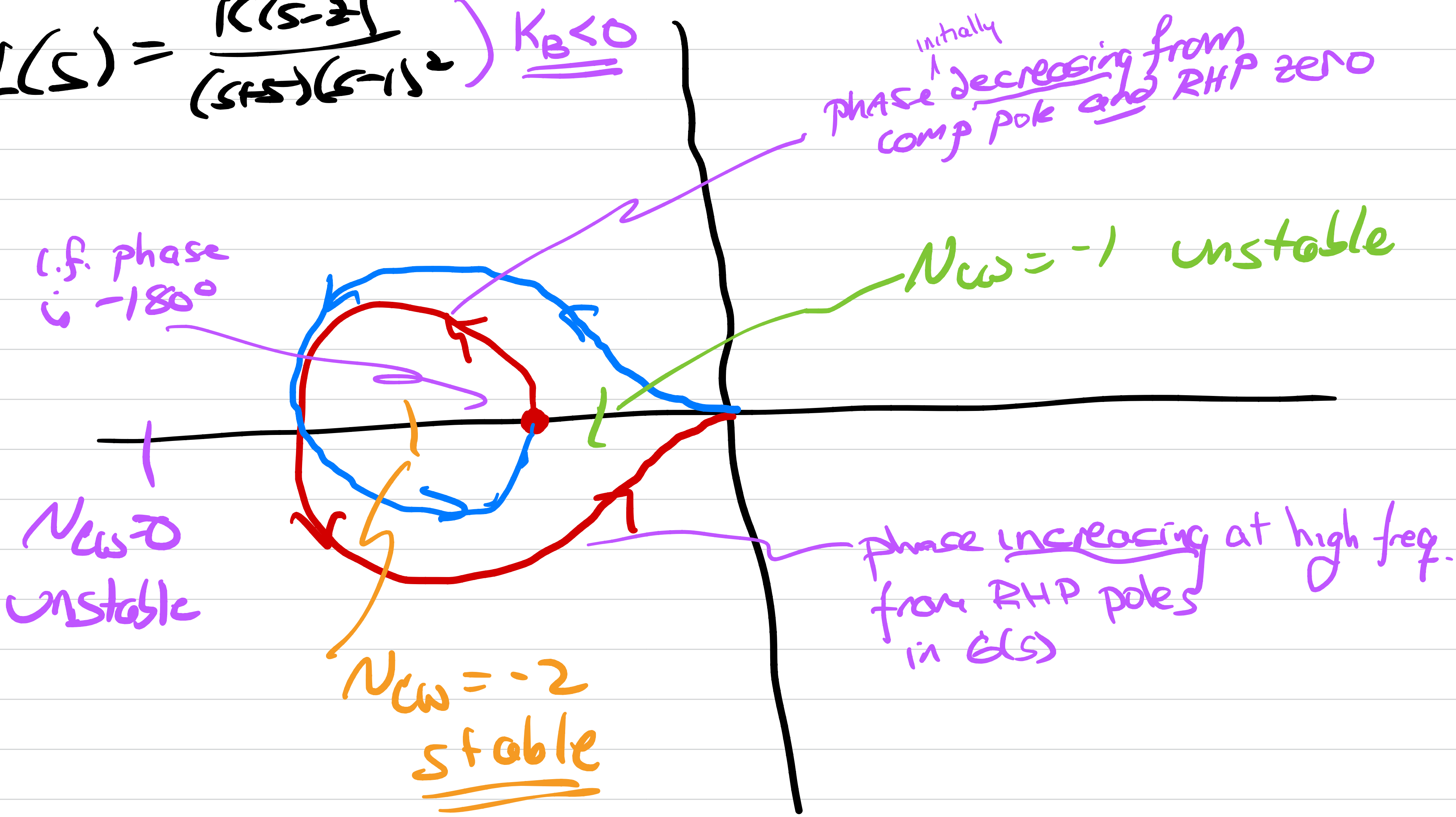


2 unstable OL poles
$\Rightarrow$ need $N_{cw}=-2$
$\Rightarrow$ 2 CCW encirclements

Again, with PD stability is straightforward
But H(s) not deriv free

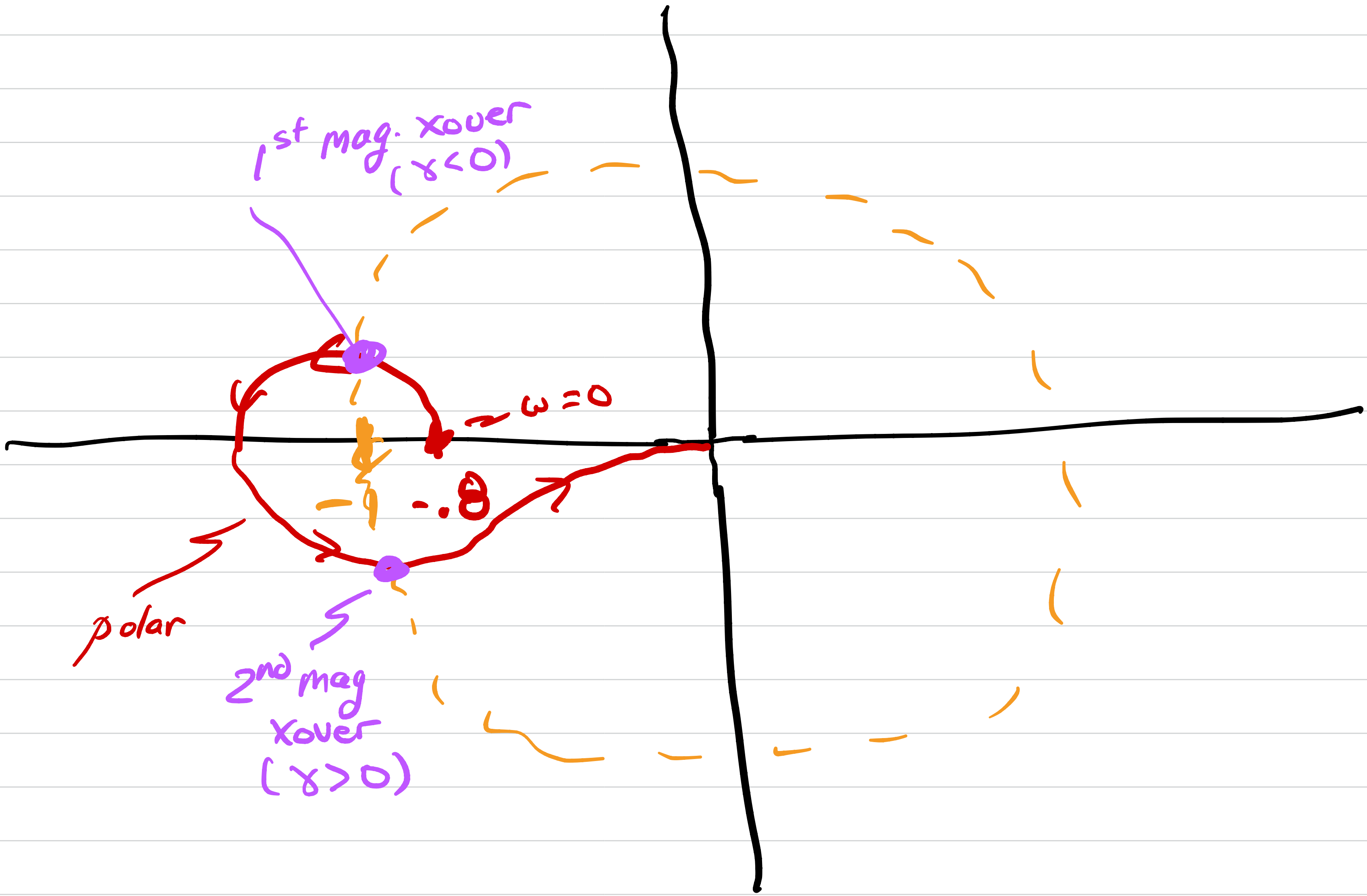
w/ $H(s) = K\frac{(s-z)}{(s+5)}$ ($\underline{z>0}$, RHP zero!)

$L(s) = \frac{K(s-z)}{(s+5)(s-1)^2}$) $\underline{\underline{K_B<0}}$



Note: H(s) (counterintuitively) contributes **negative phase** at every freq here!

Specifically w/ $H(s) = \frac{12s-4}{5s+5}$ $\left[G(s) = \frac{1}{(s-1)^2} \right]$



2 mag xovers = 2 phase margins
one positive, one negative

The above "game" – matching poly coefficients in $H(s)$ to desired coefs in char poly for $T(s)$ – yields design insights that are difficult (or even counter-intuitive) from either root locus or Bode/Nyquist perspective.

$\Rightarrow$ A useful additional tool, if it can be systematically generalized!

Generically solve

Solve

$$\underline{D_G(s)}\,\overline{D_H(s)} + \underline{N_G(s)}\,\overline{N_H(s)} = \underline{D_T(s)}$$

specified

Generically solve

$$\underline{D_G(s)}\,\overline{D_H(s)} + \underline{N_G(s)}\,\overline{N_H(s)} = \underline{D_T(s)}$$

Solve

specified

Where

$$G(s) = \frac{N_G(s)}{D_G(s)}$$ (fixed)

$$H(s) = \frac{N_H(s)}{D_H(s)}$$ free to choose

$$T(s) = \frac{N_T(s)}{D_T(s)}$$

set by desired CL poles.

"Polynomial design/matching"

When can we solve this problem generally?

Suppose $G(s)$ is n^{th} order (n poles), strictly proper and either

- $H(s)$ is proper, $(n-1)$th order

or

- $H(s)$ is strictly proper, n^{th} order

then the matching problem can be solved from

$$\underline{M}\,\underline{c} = \underline{d}$$

vector of coefs from desired char poly

vector of coefs of num + Den of $H(s)$

2n-1 or 2n total

Square Matrix of coefs of num, den in $G(s)$

(2n-1) or (2n) dimension

Example:

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$ Dim $n=2$

$$H(s) = \frac{b_1 s + b_0}{(s + a_0)}$$

$H(s)$ proper

3 DOF $= 2n-1$

$$1 + L = 0 = (s^2 + \alpha_1 s + \alpha_0)(s + a_0) + (\beta_1 s + \beta_0)(b_1 s + b_0)$$

$$= s^3 + \underbrace{(a_0 + \alpha_1 + \beta_1 b_1)}_{\gamma_2} s^2 + \underbrace{[\alpha_1 a_0 + \alpha_0 + \beta_1 b_0 + \beta_0 b_1]}_{\gamma_1} s + \underbrace{[\alpha_0 a_0 + \beta_0 b_0]}_{\gamma_0}$$

(3 COEFS)

$$\gamma_0 = \alpha_0 a_0 + \beta_0 b_0$$

$$\gamma_1 = a_1 \alpha_0 + \beta_1 b_0 + \beta_0 b_1 + \alpha_0$$

$$\gamma_2 = a_0 + \alpha_1 + \beta_1 b_1$$

$$\begin{bmatrix} \alpha_0 & \beta_0 & 0 \\ \alpha_1 & \beta_1 & \beta_0 \\ 1 & 0 & \beta_1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 - \alpha_1 \end{bmatrix}$$

M $(2n-1)\times(2n-1)$

$\underset{2n-1}{c} = \underset{2n-1}{d}$

Example #2)

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

$$H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

$\Rightarrow$ 4 DOF $= 2n$ / strictly proper

$1 + L = s^4 + \gamma_3 s^3 + \gamma_2 s^2 + \gamma_1 s + \gamma_0$

(4 coefs)

$$1 + GH = 0$$

$$= (s^2 + \alpha_1 s + \alpha_0)(s^2 + a_1 s + a_0) + (\beta_1 s + \beta_0)(b_1 s + b_0)$$

$$\beta_1 b_1 s^2 + (\beta_1 b_0 + \beta_0 b_1) s + \beta_0 b_0$$

$$s^4 + (a_1 + \alpha_1) s^3 + (a_0 + \alpha_0 + \alpha_1 a_1) s^2 + (\alpha_1 a_0 + \alpha_0 a_1) s + \alpha_0 a_0$$

$$\gamma_0 = \alpha_0 a_0 + \beta_0 b_0$$

$$\gamma_1 = (\alpha_1 a_0 + \alpha_0 a_1 + \beta_1 b_0 + \beta_0 b_1)$$

$$\gamma_2 = \beta_1 b_1 + a_0 + \alpha_0 + \alpha_1 a_1$$

$$\gamma_3 = a_1 + \alpha_1$$

$$\underset{2n \times 2n}{\underline{M}\begin{bmatrix} \alpha_0 & \beta_0 & 0 & 0 \\ \alpha_1 & \beta_1 & \alpha_0 & \beta_0 \\ 1 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}\ \underset{2n}{\underline{c}\begin{bmatrix} a_0 \\ b_0 \\ a_1 \\ b_1 \end{bmatrix}} = \underset{2n}{\underline{d}\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 - \alpha_0 \\ \gamma_3 - \alpha_1 \end{bmatrix}}$$

A different way to think about this approach

Note: Not strictly proper PDo

Recall if

$$H(s) = K_D s + K_P, \quad G(s) = \frac{1}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

(PD comp)

Then

$$1 + L(s) = 0$$

$$= s^n + \alpha_{n-1}s^{n-1} + \cdots + (\alpha_1 + K_D)s + (\alpha_0 + K_P)$$

Can now independently affect 2 coefs in char poly,

i.e. match

$$\gamma_0 = \alpha_0 + K_P$$

$$\gamma_1 = \alpha_1 + K_D$$

$\Rightarrow$ Larger set of CL poles feasible.

An (extreme + impractical) way to add sufficient DOFs is

$$H(s) = K_{n-1} s^{n-1} + \cdots + K_1 s + K_0 \quad \Big\} \; p = -(n-1)$$

all zeros, NO poles!

$$\Rightarrow \quad 1 + L(s) = 0$$

$$= s^n + (\alpha_{n-1} + K_{n-1}) s^{n-1} + \cdots + (\alpha_1 + K_1) s + (\alpha_0 + K_0)$$

We can <u>independently</u> change <u>all</u> coefs in char poly $\Rightarrow$ can "place" <u>any</u> desired set of CL poles.

$\frown$

"impractical" because above implies:

$$u(t) = K_0 e(t) + K_1 \dot{e}(t) + \underbrace{K_2 \ddot{e}(t) + \cdots + K_{n-1} e^{(n-1)}(t)}_{\text{measured???}}$$

Measurement of all these derivs is probably impractical (if not impossible)

But, equivalent measurements may be feasible.

Example: consider model of A/C pitch angle (θ) dynamics

Typical TF has 4 poles ⟹ would need to measure

$\theta, \dot{\theta}, \underbrace{\ddot{\theta}, \dddot{\theta}}_{\text{probably not measurable.}}$

However,

A state-space model which yields the same TF has states θ, α (angle of attack), $\dot{\theta}$, and v (air speed)

These 4 physical variables typically can be measured.

So, if we shift focus from controllers using output + its derivs to controllers using state meas, we may be able to increase effective DoFs in design + solve pole placement