What about phase?

Recall:

$$\measuredangle G(j\omega) = \measuredangle K_B - N\measuredangle(j\omega) + \sum_{i=1}^{m} \measuredangle\left(1 - \frac{j\omega}{z_i}\right) - \sum_{K=N+1}^{n}\left(1 - \frac{j\omega}{p_K}\right)$$

$$\measuredangle K_B = \begin{cases} \emptyset & K_B > \emptyset \\ -180 & K_B < \emptyset \end{cases} \quad \text{for all } \omega \geq \emptyset$$

$$\measuredangle(j\omega) = 90^\circ \quad \text{for all } \omega \geq \emptyset$$

So, low frequency phase is <u>constant</u> at

$-90N$ if $K_B > \emptyset$

$-180 - 90N$ if $K_B < \emptyset$

Other poles/zeros will cause "bends" at higher freqs.

Phase response from other poles|zeros

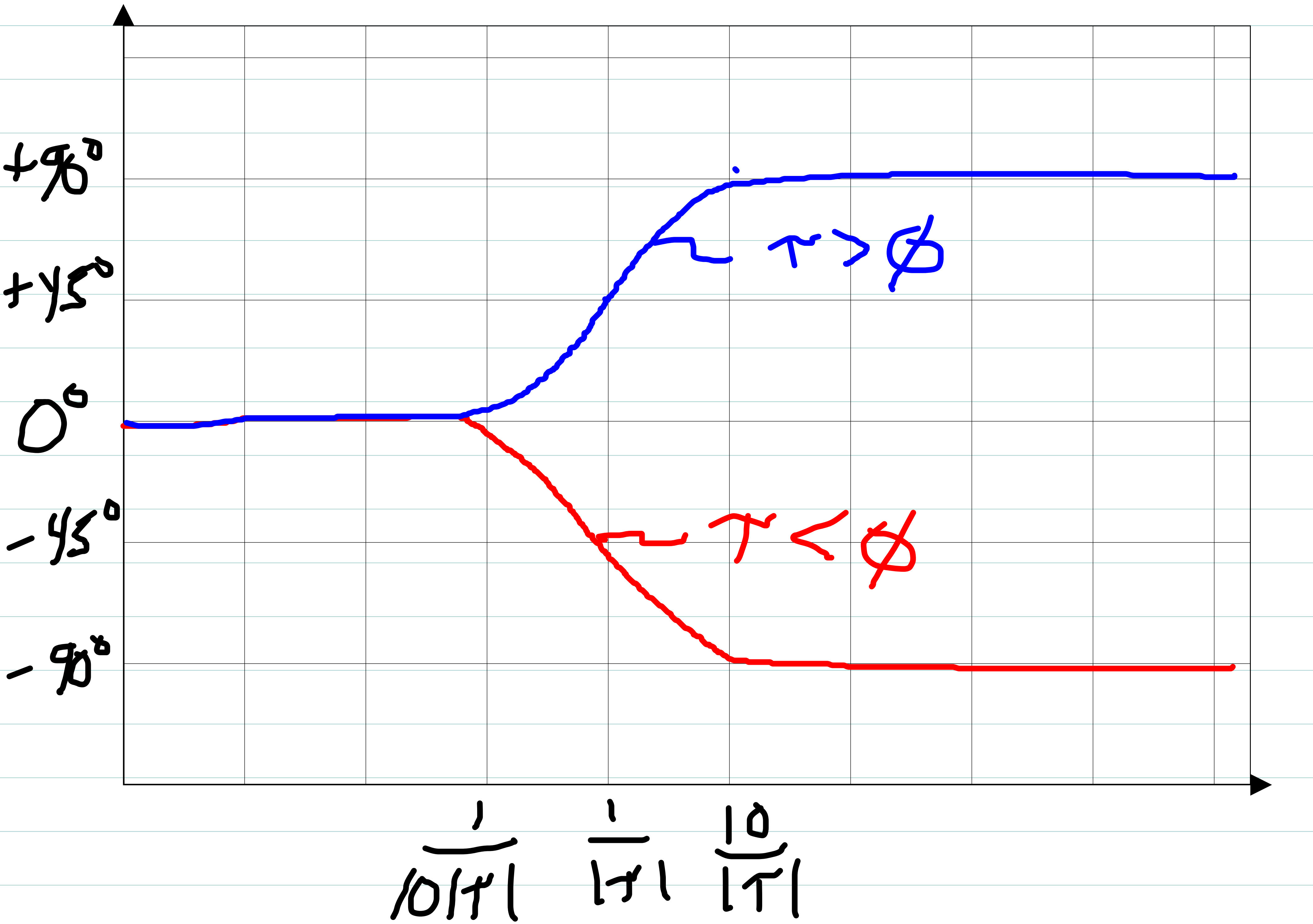
Consider again in generic form $(1+j\omega\tau)$ with $\tau = -1/z_i$ or $\tau = -1/p_k$

$$\measuredangle(1+j\omega\tau) = \tan^{-1}\omega\tau$$

$$= \begin{cases} \phi & \text{if } \omega \ll 1/|\tau| \\ +45^\circ & \text{if } \omega = 1/|\tau| \\ +9\phi^\circ & \text{if } \omega \gg 1/|\tau| \end{cases}$$

above is for $\tau > 0$. If instead $\tau < 0$

$$\measuredangle(1+j\omega\tau) = -\tan^{-1}\omega|\tau| = \begin{cases} \phi & \text{if } \omega \ll 1/|\tau| \\ -45^\circ & \text{if } \omega = 1/|\tau| \\ -90^\circ & \text{if } \omega \gg 1/|\tau| \end{cases}$$



+90°
+45°
0°
-45°
-90°
$\tau > 0$
$\tau < 0$
$\frac{1}{10|\tau|}$
$\frac{1}{|\tau|}$
$\frac{10}{|\tau|}$

Observations

⇒ Phase change due to a single factor occurs in a 2 decade band of frequencies centered at the magnitude corner frequency $1/|\tau|$

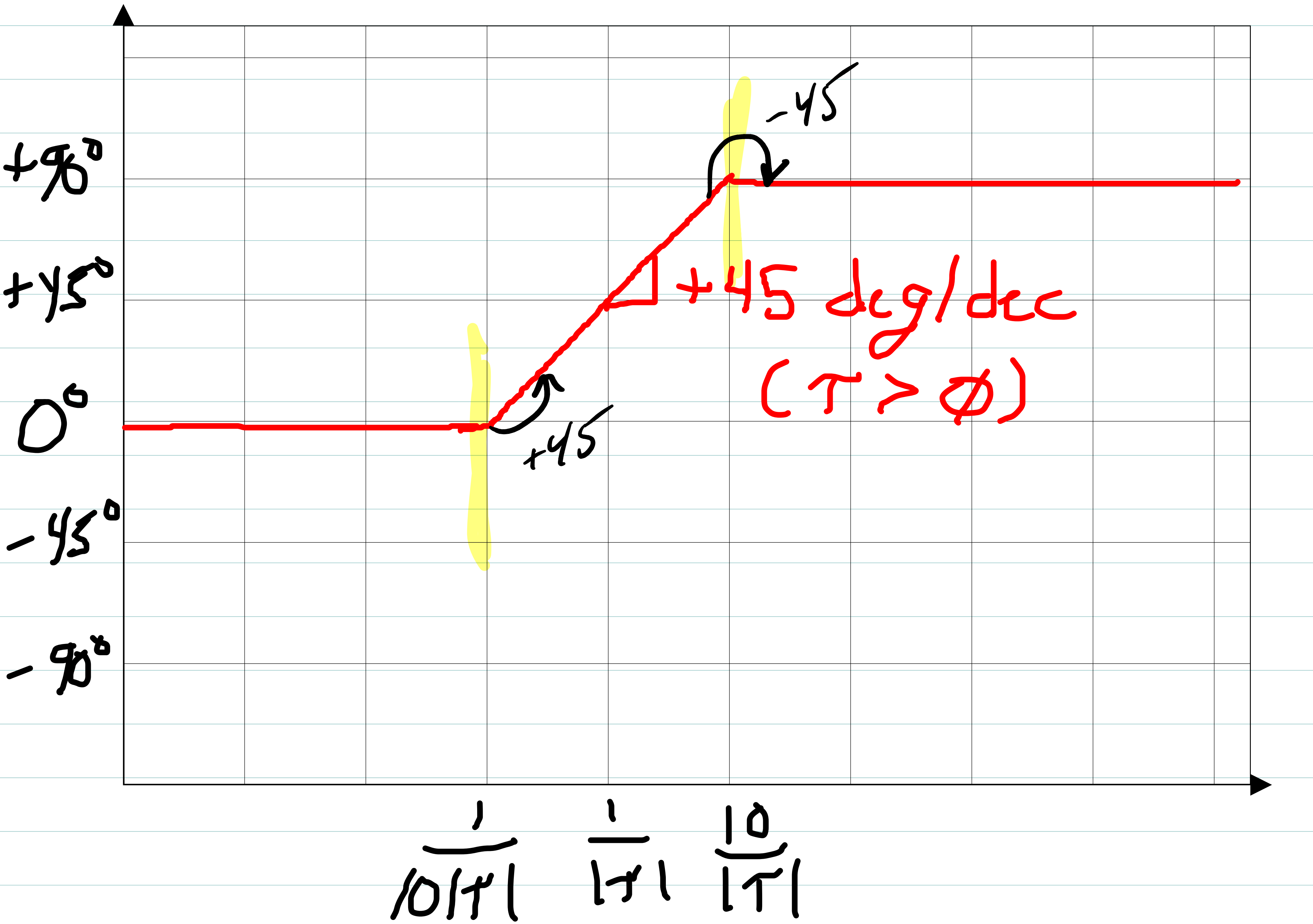
i.e. in band $\frac{1}{10|\tau|} \leq \omega \leq 10/|\tau|$

⇒ Phase is <u>constant</u> outside this band

low freq phase $\approx 0°$

h.f. phase $\approx \pm 90°$ ($+90°$ if $\tau > 0$, $-90°$ if $\tau < 0$

⇒ Phase change is approximate <u>linear</u> across band with slope $\pm 45°/\text{dec}$



+90°
+45°
0°
-45°
-90°
+45
-45
+45 deg/dec
(τ > 0)
1/(10|τ|)
1/|τ|
10/|τ|

Sign of phase change depends on:

$\Rightarrow$ whether factor is pole or zero

$\Rightarrow$ whether factor is RHP ($\tau < \emptyset$) or LHP ($\tau > \emptyset$)

Suppose all factors are LHP, $z_i < \emptyset$ $p_k < \emptyset$

then all $\tau = {}^{-1}/_{z_i}$ or ${}^{-1}/_{p_k}$ are positive.

This is called the "minimum phase" case

Then:

$\Rightarrow$ zeros cause $+90°$ phase change over band $\frac{|z_i|}{10}$ to $10|z_i|$

$\Rightarrow$ poles cause $-90°$ change over $\frac{|p_k|}{10}$ to $10|p_k|$

(Minimum Phase Systems)

Slopes of phase change are $+45°/\text{dec}$ (zeros) or $-45°/\text{dec}$ (poles) in these bands

Note phase changes in minimum phase cases mirror those for magnitude changes:

$\Rightarrow$ zeros cause positive slope changes

$\Rightarrow$ poles cause negative slope changes.

Graphical addition is again straightforward, but requires a little care:

$\Rightarrow$ slopes are nonzero only in a 2 decade band

$\Rightarrow$ bands from different factors may overlap.

Example:

$$G(s) = \frac{10s+1}{s(s+1)(s/10+1)}$$

Low freq. phase $-90°$

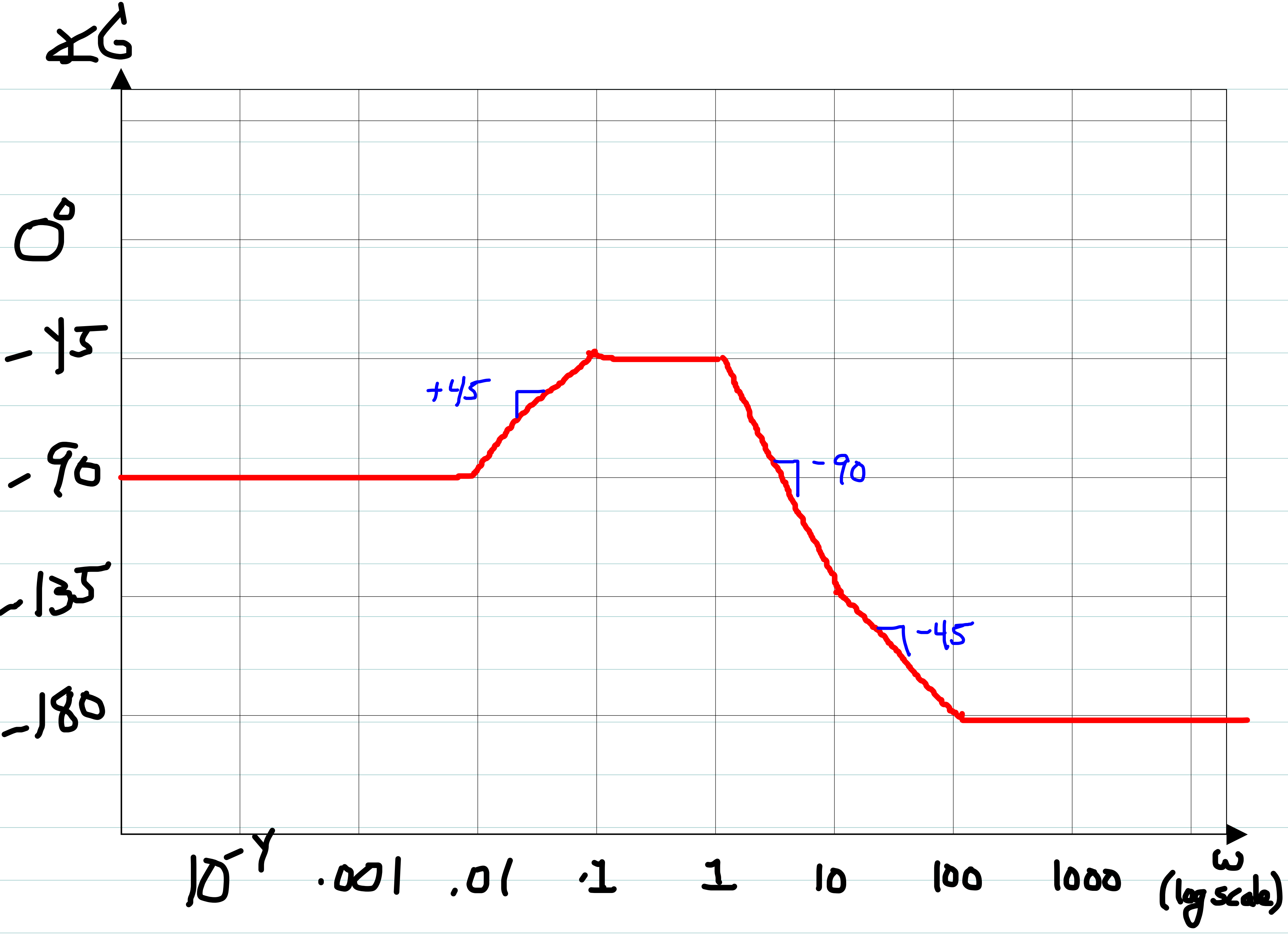
Phase changes:

$+45°$/dec in .01 to 1
$-45°$/dec in .1 to 10
$-45°$/dec in 1 to 100

Net:

$+45°$/dec in .01 to .1
$0°$/dec in .1 to 1
$-90°$/dec in 1 to 10
$-45°$/dec in 10 to 100

Constant for $\omega > 100$.



∡G
0°
-45
-90
-135
-180
+45
-90
-45
10^-4
.001
.01
.1
1
10
100
1000
ω
(log scale)

Repeated factors

Repeated factors $(1+j\omega T)^{l}$ multiply the phase changes by l, just like magnitudes.

Example:

$$G(s) = \frac{10s+1}{s(s+1)^2}$$

Changes:

$+45°/\text{dec}$ in $.01$ to 1

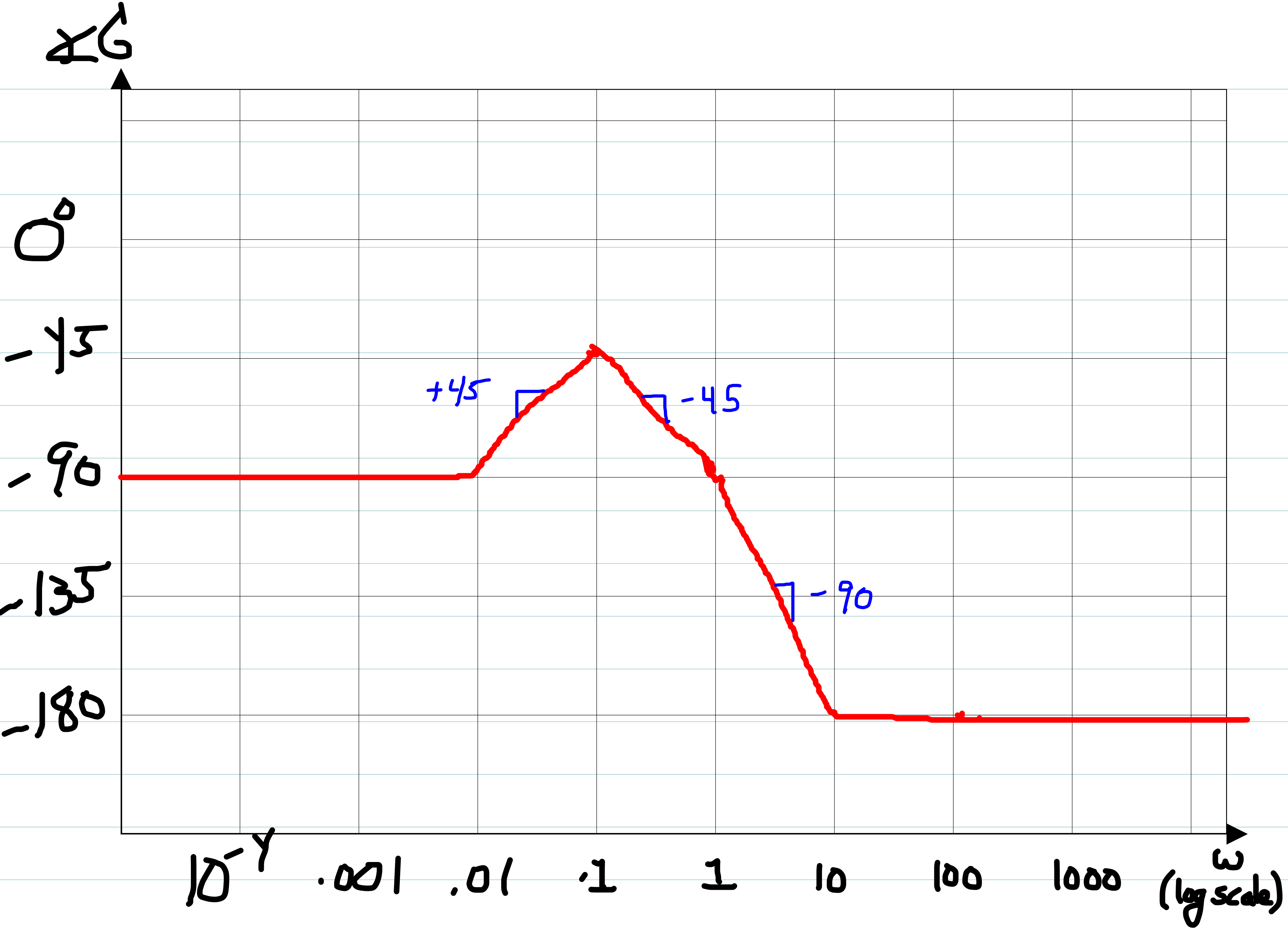
$-90°/\text{dec}$ in $.1$ to 10

Net:

$+45°/\text{dec}$ in $.01$ to $.1$

$-45°/\text{dec}$ in $.1$ to 1

$-90°/\text{dec}$ in 1 to 10



∡G
0°
-45
-90
-135
-180
+45
-45
-90
10^{-4}
.001
.01
.1
1
10
100
1000
ω
(log scale)

Summary (minimum phase)

=> Low freq. phase is $\measuredangle K_B - N\,90^\circ$

=> high freq. phase is $\measuredangle K_B - 90^\circ(n-m)$

=> Note low and high freq. phases are <u>constant</u> (slope is zero).

=> Recall typically $n > m$ for a physical system so high freq. phase is typically <u>negative</u> for a minimum phase system.

=> zeros cause $+90^\circ$ change at rate of $+45^\circ/\text{dec}$ in 2 decade band centered at $|z_i|$

=> poles cause -90° change at rate of $-45^\circ/\text{dec}$ in 2 decade band centered at $|p_k|$.

Can be tricky to accurately sketch phase

=> Overlapping change regions for multiple factors

=> No standard formula for phase change of underdamped factors

=> Helps to 1st make a table of slope changes over frequency ranges as above

=> Generally, straight-line phase sketch is less accurate than magnitude sketch.

=> Still sufficiently accurate to give us a good general idea of phase behavior.

=> We'll use Matlab when greater accuracy is required.

Non-minimum phase systems

If any poles or zeros of $G(s)$ in RHP, the system is "Non-minimum phase"

Corresponds to $\tau < \phi$ in phase analysis and $\angle(1+j\omega\tau) = -\tan^{-1}\omega/|\tau|$.

$\Rightarrow$ Phase response is negative of that seen above

$\Rightarrow$ In particular, zeros cause -90 deg phase change in 2 decade band around corner freq.

Poles cause $+90$ deg change

Opposite of minimum phase behavior, but $\boxed{\text{Corner freqs unchanged}}$ ($|z_i|$ or $|p_k|$)

Example:

$$G(s) = \frac{(10s+1)}{(1-s)}$$

Min phase zero: $+45°$/dec change in .01 to 1

Nonmin phase pole: $+45°$/dec change in .1 to 10

Net: $+45°$/dec in .01 to .1
$+90°$/dec in .1 to 1
$+45°$/dec in 1 to 10.

Note: h.f. phase is $+180°$ here. Above rule for h.f. phase in min phase systems does not apply if $G(s)$ has RHP poles or zeros

Underdamped factors

$$(s^2+2\zeta\omega_n s+\omega_n^2) \Rightarrow \left[\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1\right]$$ in Bode form.

How do we draw magnitude response when $G(s)$ contains these factors?

$\Rightarrow$ If $\frac{\sqrt{2}}{2} \leq \zeta \leq 1$, we can well approximate the response as a repeated pole at $-\omega_n$ (it isn't really, but it's a good approx to sketch this way).

$\Rightarrow$ If $0 \leq \zeta < \frac{\sqrt{2}}{2}$ a more substantial correction is needed...

$\Rightarrow$ To illustrate, suppose

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{\left[(s/\omega_n)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1\right]}$$

Let's find $|G(j\omega)|$ here

$$|G(j\omega)| = \left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2\xi\left(\frac{j\omega}{\omega_n}\right) + 1\right|^{-1}$$

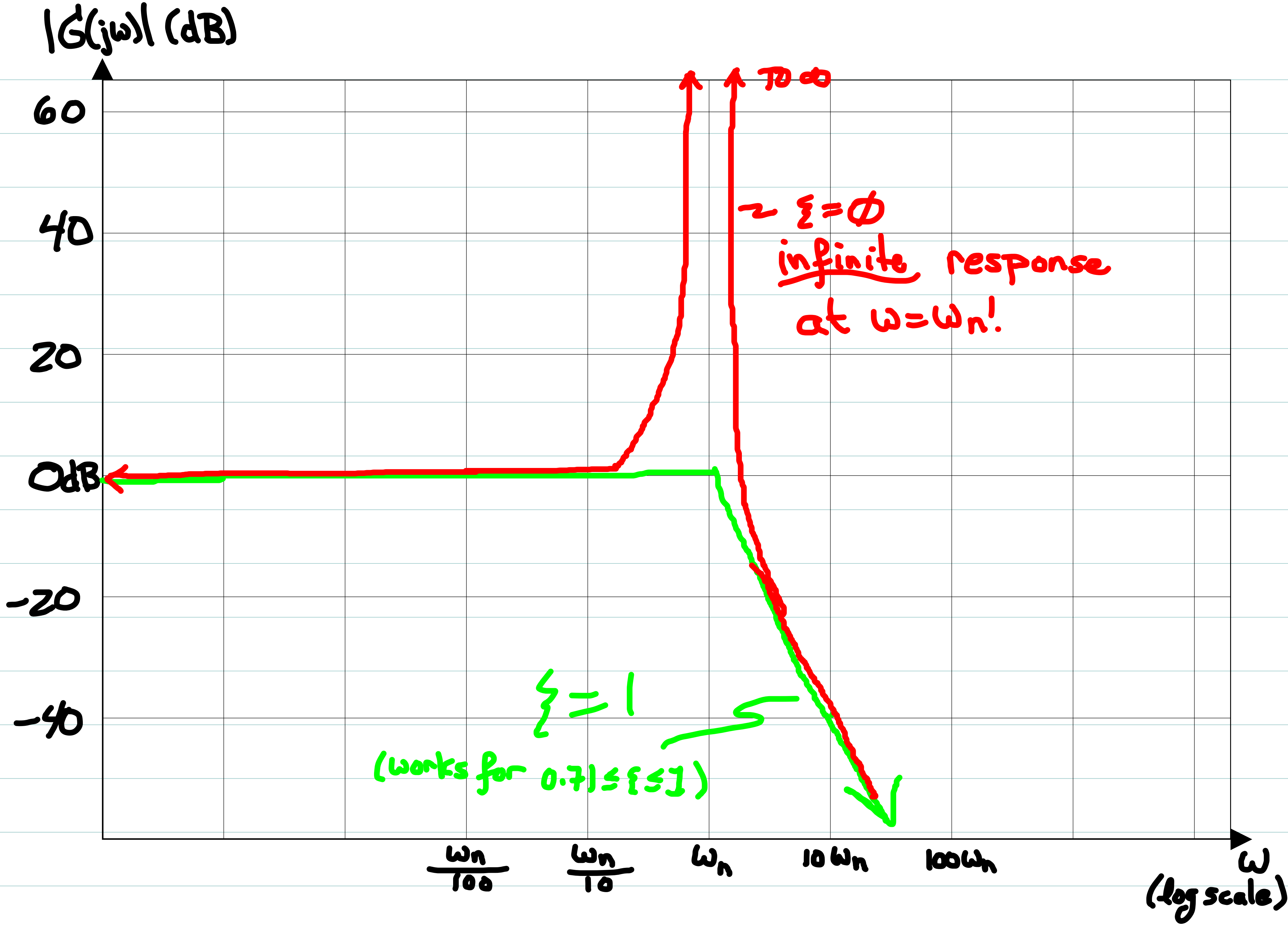
$$= \left[\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}\right]^{-1}$$

Which is ugly, so why bother?

Consider if $\xi = \emptyset$, then

$$|G(j\omega)| = \frac{1}{\left|1-\left(\frac{\omega}{\omega_n}\right)^2\right|} \approx \begin{cases} 1 & \text{if } \omega \ll \omega_n \\ \left|\frac{\omega}{\omega_n}\right|^{-2} & \omega \gg \omega_n \end{cases}$$

so that $|G(j\omega)|_{\omega=\omega_n} = \infty$!!! Definitely Something Going on!



|G(jω)| (dB)
60
40
20
0dB
-20
-40
TO ∞
ζ = 0
infinite response at ω = ωn!
ζ = 1
(works for 0.71 ≤ ζ ≤ 1)
ωn/100
ωn/10
ωn
10ωn
100ωn
ω (log scale)

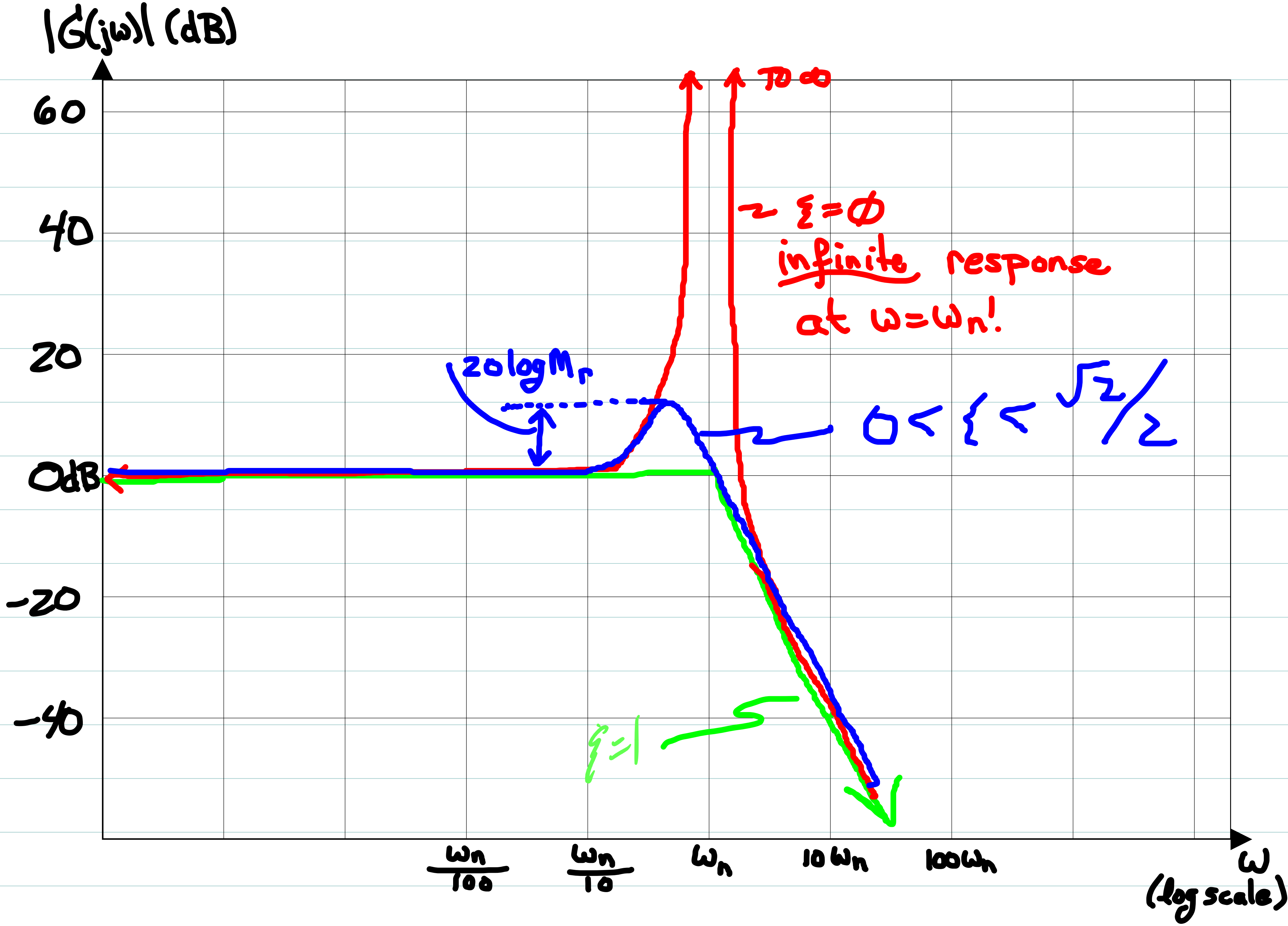
When $0 < \zeta < \sqrt{2}/2$, a similar "peaking" phenomenon occurs, but peak height is finite:

for
$$G(s) = \left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right]^{-1}$$

$\max_{\omega \geq 0} |G(j\omega)|$ occurs at:

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

and $|G(j\omega_r)| = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} \triangleq M_r$



|G(jω)| (dB)
60
40
20
0dB
-20
-40
TO ∞
ξ = 0
infinite response at ω=ωn!
20 log Mr
0 < ξ < √2/2
ξ = 1
ωn/100
ωn/10
ωn
10 ωn
100 ωn
ω (log scale)

This is the phenomenon of **resonance**

An ideal (no zeros) underdamped 2^{nd} order system with $0 \leq \xi < \sqrt{2}/2$ will exhibit output amplitudes significantly greater than the input amplitude when input frequency is close to the natural frequency ω_n.

The largest amplitude ratio will occur at the

resonant frequency $\boxed{\omega_r = \omega_n \sqrt{1-2\xi^2}} < \omega_n$

and the maximal amplitude ratio (maximal resonance) is

$$\boxed{M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}}$$

Notes:

1.) Height of peak on diagram is M_r in dB, i.e. $20\log M_r$

2.) When 2^{nd} order factor in TF with other factors, the peak is $20\log M_r$ above whatever magnitude the plot would otherwise have at ω_r. That is, M_r is a relative offset to plot, not absolute.

3.) For small ξ, say $0 < \xi < 1/10$

$$\omega_r \approx \omega_n \quad \text{and} \quad M_r \approx 1/(2\xi)$$

So $20\log M_r \approx -[6 + 20\log \xi]$ is a good approximation

i.e. at $\xi = 1/10$, $20\log M_r \approx +14$ dB

Example

$$G(s) = \left[\frac{(10s+1)}{s\left(\left(\frac{s}{10}\right)^2 + 0.2\left(\frac{s}{10}\right) + 1\right)} \right]$$

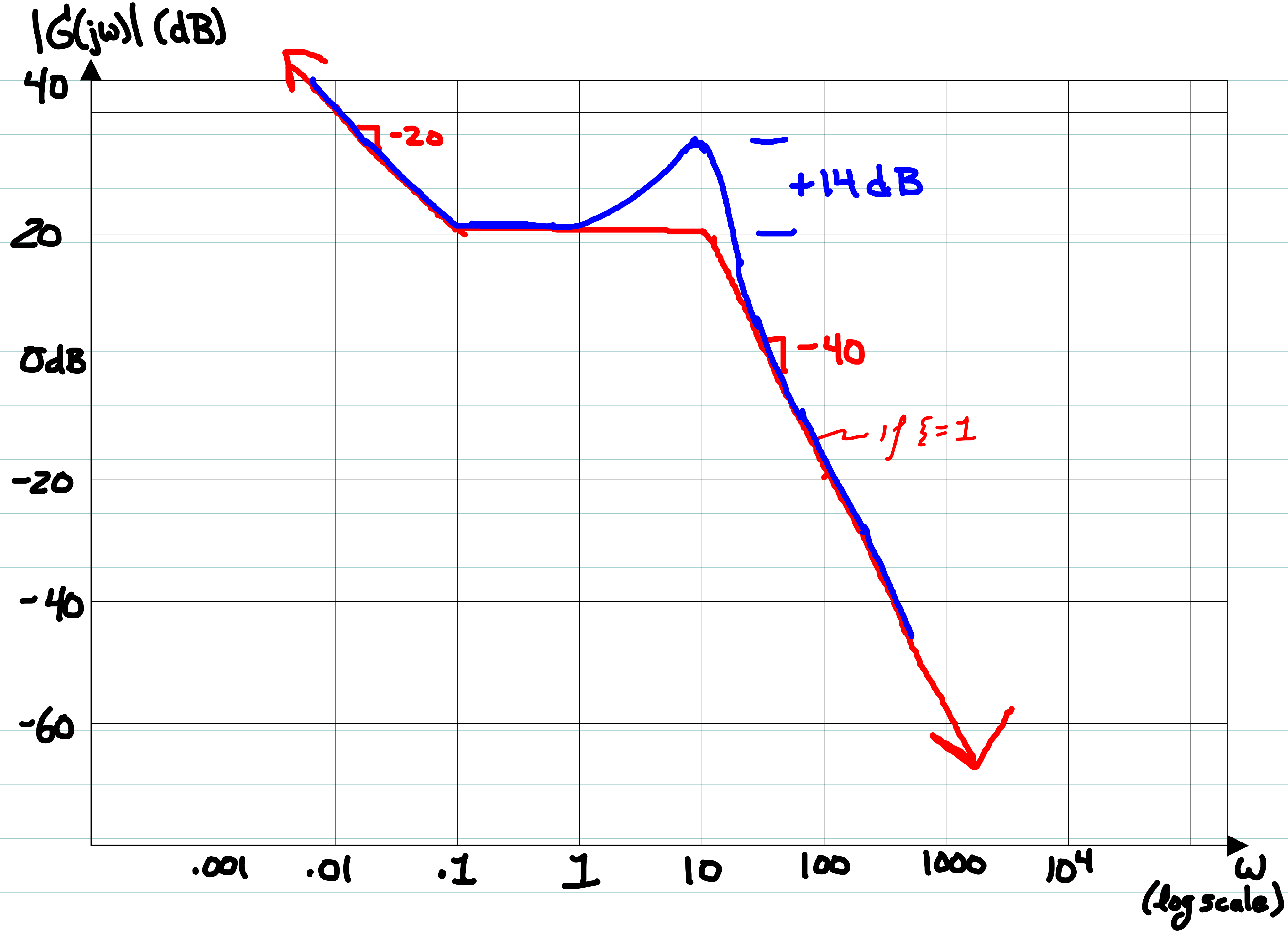
Same as example above, except:

$K_B = 1$ (instead of 10)

$\zeta = 0.1$ (instead of $\zeta = 1$)

$\omega_n = 10$

$\Rightarrow$ Expect resonant peak of height +14dB near $\omega = 10$.



|G(jω)| (dB)
40
20
0dB
-20
-40
-60
-20
+14 dB
-40
if ξ=1
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)

2^{nd} order min phase factors - phase

$$\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right)s + 1\right]^{\pm 1}$$

$\Rightarrow$ Like magnitudes, can sketch as repeated real factor ($\zeta = 1$) for $\sqrt{2}/2 \le \zeta \le 1$

$\Rightarrow$ for $0 \le \zeta < \frac{\sqrt{2}}{2}$, a more significant correction is needed

$\Rightarrow$ when $\zeta = 0$, phase changes <u>discontinuously</u> by $\pm 180°$ at $\omega = \omega_n$

<u>Example:</u>

$$G(s) = \frac{\left(s/10 + 1\right)}{\left[s^2 + 2\zeta s + 1\right]}$$

If $\zeta = 1$:

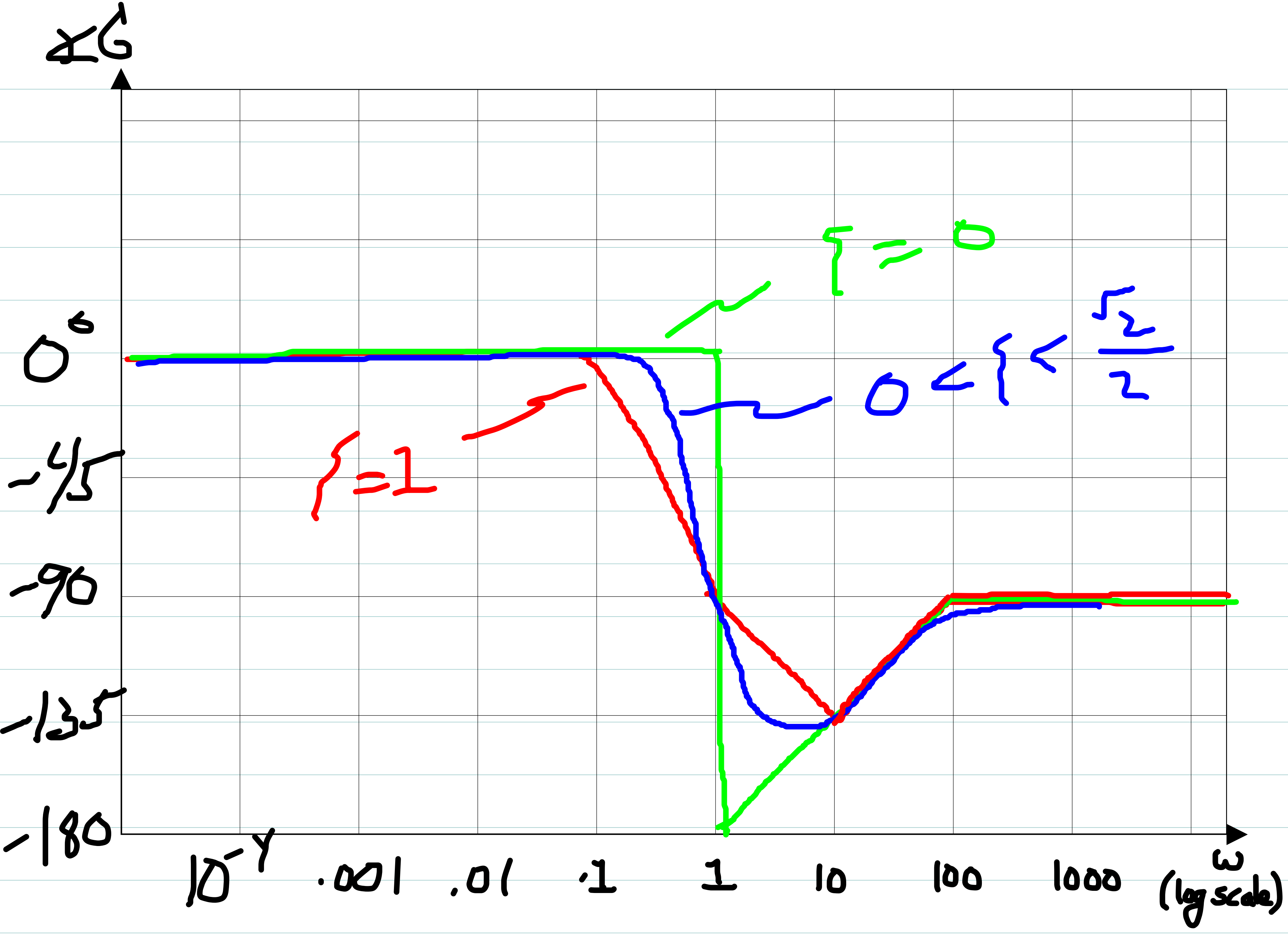
Change of $-90°/dec$ in $.1$ to 10
Change of $+45°/dec$ in 1 to 100

Net is $-90°/dec$ from $.1$ to 1
$-45°/dec$ from 1 to 10
$+45°/dec$ from 10 to 100

If $\zeta = 0$

Change of $+45°/dec$ in 1 to 100

$-180°$ drop at $\omega = \omega_n = 1$



∡G
0°
-45
-90
-135
-180
10^-4
.001
.01
.1
1
10
100
1000
ω
(log scale)
ζ = 0
ζ = 1
0 < ζ < √2/2

Notes: (2nd order phase, small ξ)

- Unlike magnitude, no useful simple formula to quantify "steepness" of phase drop for small ξ.

- Usually sketch something in between the $\xi=1$ and $\xi=0$ limits

- Necessarily qualitative - will use Matlab when precise analysis is needed.

- Note generally that we expect to see steep phase drops near frequencies where magnitude diagram shows resonant peaks!