Ch 4

Linear & Angular Momentum of a Particle

9/12/24

Recall linear momentum of particle Q is

$$^{I}\bar{p}_{Q/O} = m_Q\,^{I}\bar{v}_{Q/O}$$

$$\frac{^{I}d}{dt}\left(^{I}\bar{p}_{Q/O}\right) = \bar{F}_Q \quad \left(\text{lin. mom. form of N2L}\right)$$

If $\bar{F}_Q = 0 \Rightarrow \,^{I}\bar{v}_{Q/O}(t) = \,^{I}\bar{v}_{Q/O}(0)$ (N1L)

Separation of variables:

$$\int_{t_1}^{t_2} \cancel{dt}\,\frac{^{I}d}{\cancel{dt}}\left(^{I}\bar{p}_{Q/O}\right) = \int_{t_1}^{t_2} \bar{F}_Q\,dt$$

$$^{I}\bar{p}_{Q/O}(t_2) - \,^{I}\bar{p}_{Q/O}(t_1) = \underbrace{\int_{t_1}^{t_2} \bar{F}_Q\,dt}_{= \bar{\bar{F}}_Q(t_1,t_2)}$$

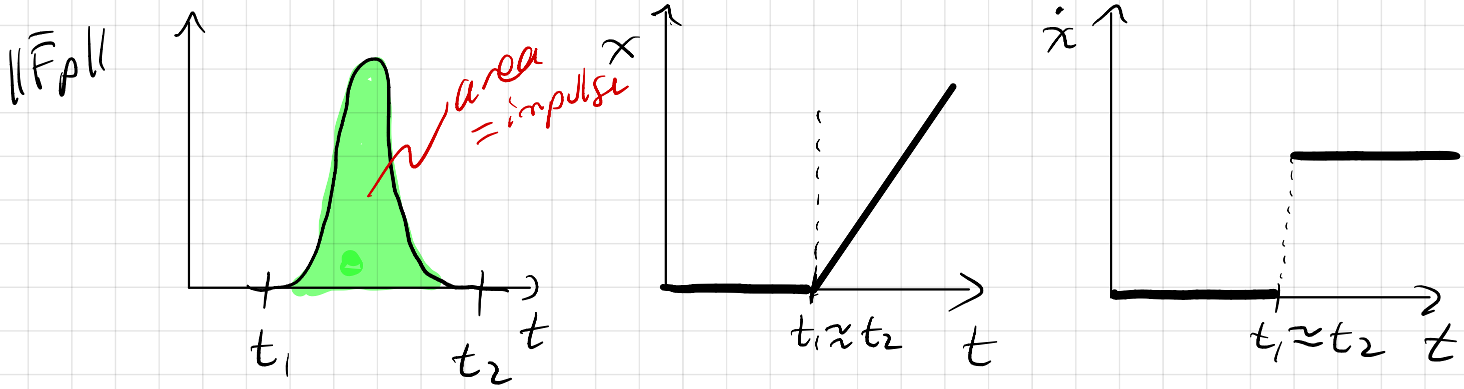
"linear impulse"

$$\bar{\bar{F}}_Q(t_1,t_2) = \int_{t_1}^{t_2} \bar{F}_Q\,dt$$

$$\boxed{^{I}\bar{p}_{Q/O}(\underset{\text{final time}}{t_2}) = \,^{I}\bar{p}_{Q/O}(\underset{\text{initial time}}{t_1}) + \bar{\bar{F}}_Q(t_1,t_2)}$$

* Conservation of Linear Momentum occurs when $\bar{F}_Q = 0 \Rightarrow \bar{\bar{F}}_Q = 0$ (N1L)

$$ {}^I\bar{v}_{Q/O}(t_2) = {}^I\bar{v}_{Q/O}(t_1) + \frac{1}{m_P}\bar{\bar{F}}_Q(t_1, t_2) $$

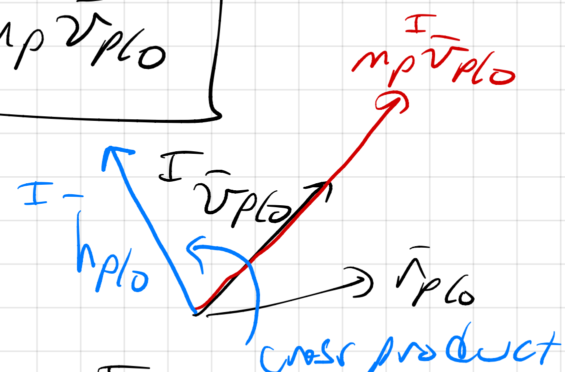


$$ \Delta t = t_2 - t_1 << 1 \Rightarrow t_2 = t_1 + \Delta t $$

"much less than"

Dfn <u>angular momentum</u> of a particle

$$ \boxed{{}^I\bar{h}_{P/O} = \bar{r}_{P/O} \times m_P\, {}^I\bar{v}_{P/O}} $$



<u>Ang Mom. Form</u> of N2L

$$ {}^I\frac{d}{dt}\left({}^I\bar{h}_{P/O}\right) = {}^I\frac{d}{dt}\left(\bar{r}_{P/O} \times m_P\, {}^I\bar{v}_{P/O}\right) $$

$$ = \underbrace{{}^I\bar{v}_{P/O} \times m_P\, {}^I\bar{v}_{P/O}}_{} ^{=0} + \bar{r}_{P/O} \times m_P\, {}^I\bar{a}_{P/O} \quad = \bar{F}_P \text{ by N2L} $$

$$ {}^I\frac{d}{dt}\left({}^I\bar{h}_{P/O}\right) = \underbrace{\bar{r}_{P/O} \times \bar{F}_P}_{= \bar{M}_{P/O}} $$

$= \bar{M}_{P/O}$ moment acting on P wrt O

$$ ** \quad \boxed{{}^I\frac{d}{dt}\left({}^I\bar{h}_{P/O}\right) = \bar{M}_{P/O}} \quad \text{Ang. Mom. form N2L} $$

Dfn the moment on P wrt O resulting $\bar{F}_P$

$$\boxed{\bar{M}_{P/O} = \bar{r}_{P/O} \times \bar{F}_P}$$

* conservation of ang. mom. occurs when $\bar{M}_{P/O} = 0$

e.g., $\bar{r}_{P/O} = 0$,

$\bar{F}_P = 0$,

or $\bar{r}_{P/O} \times \bar{F}_P = 0$ that is when $\bar{r}_{P/O}$ and $\bar{F}_P$ are parallel

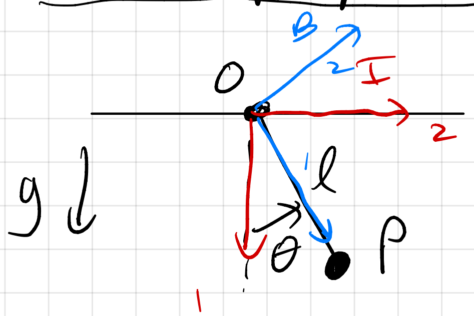
Dfn a central force $\bar{F}_P$ satisfies

$$\bar{r}_{P/O} \times \bar{F}_P = 0$$

$\Rightarrow$ If $\bar{F}_P$ is a central force then ${}^I\bar{h}_{P/O}$ is conserved

$\Rightarrow$ ${}^I\bar{h}_{P/O}(t) = {}^I\bar{h}_{P/O}(0)$

Ex 4.6 Simple pendulum using angular momentum



	$\hat{b}_1$	$\hat{b}_2$
$\hat{e}_1$	$c\theta$	$-s\theta$
$\hat{e}_2$	$s\theta$	$c\theta$

${}^I\bar{\omega}^B = \dot{\theta}\hat{b}_3$

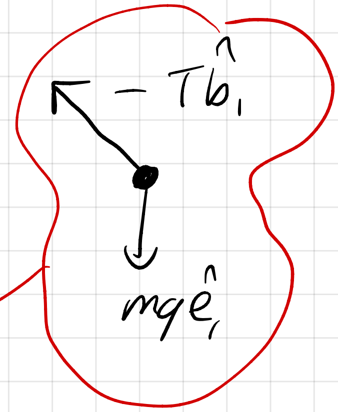
* Apply AM form of N2L $\frac{{}^Id}{dt}\left({}^I\bar{h}_{P/O}\right) = \bar{M}_{P/O}$

${}^I\bar{h}_{P/O} = \bar{r}_{P/O} \times m_P\, {}^I\bar{v}_{P/O}$ (definition A.M.)

$\bar{r}_{P/O} = l\hat{b}_1 \Rightarrow {}^I\bar{v}_{P/O} = l\dot{\theta}\hat{b}_2$

$= \frac{{}^Id}{dt}(\hat{b}_1) = \dot{\theta}\hat{b}_3 \times \hat{b}_1$

$$ {}^I\bar{h}_{P/O} = l\hat{b}_1 \times ml\dot{\theta}\hat{b}_2 $$
$$ = ml^2\dot{\theta}\,\hat{b}_3 = \hat{e}_3 $$
$$ \frac{{}^I d}{dt}\left({}^I\bar{h}_{P/O}\right) = ml^2\ddot{\theta}\,\hat{b}_3 $$



$$ \bar{M}_{P/O} = \bar{r}_{P/O} \times \bar{F}_P $$
$$ = l\hat{b}_1 \times \left(-T\hat{b}_1 + mg\hat{e}_1\right) $$
$$ = mgl\,\hat{b}_1 \times \left(C\theta\,\hat{b}_1 - S\theta\,\hat{b}_2\right) $$
$$ = -mgl\,S\theta\,\hat{b}_3 $$
$$ ml^2\ddot{\theta}\,\hat{b}_3 = -mgl\,S\theta\,\hat{b}_3 $$
$$ \hat{b}_3: \quad \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta} \quad \text{eq. mo.} $$

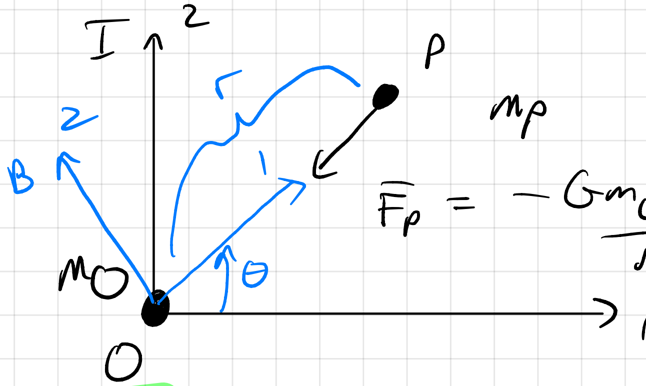
Dfn Angular Impulse

$$ \bar{\bar{M}}_{P/O}(t_1, t_2) = \int_{t_1}^{t_2} \bar{M}_{P/O}\,dt $$
$$ \frac{{}^I d}{dt}\left({}^I\bar{h}_{P/O}\right) = \bar{M}_{P/O} $$
$$ \int_{t_1}^{t_2} {}^I d\left({}^I\bar{h}_{P/O}\right) = \int_{t_1}^{t} \bar{M}_{P/O}\,dt $$
$$ {}^I\bar{h}_{P/O}(t_2) - {}^I\bar{h}_{P/O}(t_1) = \bar{\bar{M}}_{P/O}(t_1, t_2) $$
$$ \boxed{{}^I\bar{h}_{P/O}(t_2) = {}^I\bar{h}_{P/O}(t_1) + \bar{\bar{M}}_{P/O}(t_1, t_2)} $$

if $\bar{\bar{M}}_{P/O}(t_1, t_2) = 0 \Rightarrow {}^I\bar{h}_{P/O}(t_2) = {}^I\bar{h}_{P/O}(t_1)$

conservation of A.M.

Ex 4.9 Simple Satellite $\quad m_O \gg m_p$



$$\bar{F}_p = -\frac{G m_O m_p}{r^2}\hat{b}_1$$

Goal: ① find eq. mo.

② find conserved quantities

$$\bar{F}_p = m\,{}^I\bar{a}_{p/o} \qquad \bar{F}_p = \frac{{}^I d}{dt}\left({}^I\bar{p}_{p/o}\right) \qquad \bar{M}_{p/o} = \frac{{}^I d}{dt}\left({}^I\bar{h}_{p/o}\right)$$

$$\bar{r}_{p/o} \times \bar{F}_p = r\hat{b}_1 \times \frac{-G m_o m_p}{r^2}\hat{b}_1 = 0$$

$\Rightarrow {}^I\bar{h}_{p/o}$ is conserved

$$\bar{r}_{p/o} = r\hat{b}_1$$

$${}^I\bar{v}_{p/o} = \dot{r}\hat{b}_1 + r\,\dot{\theta}\hat{b}_3 \times \hat{b}_1$$

$$= \dot{r}\hat{b}_1 + r\dot{\theta}\hat{b}_2$$

$$\dot{\theta}\hat{b}_3 \times \hat{b}_2 = -\dot{\theta}\hat{b}_1$$

$${}^I\bar{a}_{p/o} = \ddot{r}\hat{b}_1 + 2\dot{r}\dot{\theta}\hat{b}_2 + r\ddot{\theta}\hat{b}_2 - r\dot{\theta}^2\hat{b}_1$$

$$-\frac{G m_o m_p}{r^2}\hat{b}_1 = m_p\left(\ddot{r} - r\dot{\theta}^2\right)\hat{b}_1 + m_p\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{b}_2$$

$$\hat{b}_1: \quad -\frac{G m_o m_p}{r^2} = m_p(\ddot{r} - r\dot{\theta}^2) \Rightarrow \boxed{\ddot{r} = r\dot{\theta}^2 - \frac{G m_O}{r^2}}$$

$$\hat{b}_2: \quad 0 = m_p(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \Rightarrow \boxed{\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}}$$

Tutorial 4.2

$${}^I\bar{h}_{p/o}(t) = {}^I\bar{h}_{p/o}(0) \qquad \text{cons. A.M.}$$

$${}^I\bar{h}_{p/o} = \bar{r}_{p/o} \times m_p\,{}^I\bar{v}_{p/o} \qquad \text{dfn.}$$

$$= r\hat{b}_1 \times m_p\left(\dot{r}\hat{b}_1 + r\dot{\theta}\hat{b}_2\right)$$

$$= m_p r^2 \dot{\theta}\hat{b}_3$$

$$\cancel{m_p}(r(t))^2\dot{\theta}(t) = \cancel{m_p}\underbrace{(r(0))^2\dot{\theta}(0)}_{h_0}$$

specific A.M.

$$\boxed{r^2\dot{\theta} = h_0}$$ cons. A.M.

$$\ddot{r} = r\frac{h_0^2}{r^4} - \frac{Gm_0}{r^2} = \frac{h_0^2}{r^3} - \frac{Gm_0}{r^2}$$

$$\ddot{\theta} = -\frac{2\dot{r}}{r}\frac{h_0}{r^2} = -\frac{2\dot{r}h_0}{r^3}$$