PHYS 313

HW 04: Assignment 4

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Problem 2.22:

Find the potential a distance s from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.

Solution

$$
\begin{aligned}
V(s) - V(s_0) &= -\int_{s_0}^{s} \frac{2k\lambda}{s'}\, ds' \\
&= -2k\lambda \ln \frac{s}{s_0} \\
V(s) &= 2k\lambda \ln \frac{s_0}{s} \quad \text{(choosing } V(s_0) = 0\text{)} \\
\frac{dV}{ds} &= -\frac{2k\lambda}{s} \quad \Longrightarrow \quad \nabla V = -\frac{2k\lambda}{s}\, \hat{s} \\
-\nabla V &= \frac{2k\lambda}{s}\, \hat{s} = \mathbf{E}(s)
\end{aligned}
$$

$$
\boxed{V(\mathscr{r}(s)) = 2k\lambda \ln \frac{s_0}{\mathscr{r}(s)} \quad , \quad \mathbf{E}(\mathscr{r}(s)) = \frac{2k\lambda}{\mathscr{r}(s)}\, \boldsymbol{\mathscr{r}}(s)}
$$

Problem 2.26:

A conical surface (an empty ice-cream cone) carries a uniform surface charge σ. The height of the cone is h, as is the radius of the top. Find the potential difference between points $\boldsymbol{a}$ (the vertex) and $\boldsymbol{b}$ (the center of the top).

Solution

$$\text{For point } \boldsymbol{a} \text{ (vertex):} \quad V(\boldsymbol{a}) = k\sigma \int_0^{2\pi} d\theta \int_0^L \frac{dA}{r_a} \quad , \quad r_a = s,\ dA = s \sin\alpha \, ds \, d\theta$$

$$= k\sigma \int_0^{2\pi} d\theta \int_0^L \frac{s \sin\alpha \, ds}{s} = 2\pi k\sigma \sin\alpha \, L$$

For a cone with height h and top radius h:

$$\tan\alpha = \frac{h}{h} = 1 \quad \Longrightarrow \quad \alpha = 45^\circ, \quad \sin\alpha = \frac{1}{\sqrt{2}}, \quad L = \sqrt{h^2 + h^2} = h\sqrt{2}$$

$$\therefore \quad V(\boldsymbol{a}) = 2\pi k\sigma \frac{1}{\sqrt{2}} (h\sqrt{2}) = 2\pi k\sigma h$$

$$\text{For point } \boldsymbol{b} \text{ (center of top):} \quad V(\boldsymbol{b}) = k\sigma \int_0^{2\pi} d\theta \int_0^L \frac{dA}{r_b(s)}$$

$$\text{where } r_b(s) = \sqrt{s^2 - \sqrt{2}\, h\, s + h^2} = \sqrt{\left(s - \frac{h}{\sqrt{2}}\right)^2 + \frac{h^2}{2}}$$

$$\text{Let } u = s - \frac{h}{\sqrt{2}}, \quad ds = du, \quad s = u + \frac{h}{\sqrt{2}},$$

$$\text{with } u : -\frac{h}{\sqrt{2}} \to \frac{h}{\sqrt{2}}, \quad dA = \frac{s}{\sqrt{2}} \, du \, d\theta$$

$$\Longrightarrow \quad V(\boldsymbol{b}) = \frac{2\pi k\sigma}{\sqrt{2}} \int_{-h/\sqrt{2}}^{h/\sqrt{2}} \frac{u + \frac{h}{\sqrt{2}}}{\sqrt{u^2 + \frac{h^2}{2}}} \, du$$

$$= \frac{2\pi k\sigma}{\sqrt{2}} \left[\underbrace{\int_{-h/\sqrt{2}}^{h/\sqrt{2}} \frac{u \, du}{\sqrt{u^2 + \frac{h^2}{2}}}}_{=0} + \frac{h}{\sqrt{2}} \int_{-h/\sqrt{2}}^{h/\sqrt{2}} \frac{du}{\sqrt{u^2 + \frac{h^2}{2}}} \right]$$

$$= \frac{2\pi k\sigma h}{2} \cdot 2 \int_0^{h/\sqrt{2}} \frac{du}{\sqrt{u^2 + \frac{h^2}{2}}}$$

$$= \pi k\sigma h \cdot 2 \ln \left(\frac{h/\sqrt{2} + \sqrt{(h/\sqrt{2})^2 + \frac{h^2}{2}}}{h/\sqrt{2}} \right)$$

$$= 2\pi k\sigma h \ln \left(\frac{h/\sqrt{2} + h}{h/\sqrt{2}} \right) = 2\pi k\sigma h \ln \left(\sqrt{2} + 1 \right)$$

$$\boxed{V(\boldsymbol{\mathscr{r}} b) - V(\boldsymbol{\mathscr{r}} a) = 2\pi k\sigma h \Big[\ln \left(\sqrt{2} + 1 \right) - 1 \Big]}$$

Problem 2.28:

Use the following equation to calculate the potential inside a uniformly charged solid sphere of radius R and total charge q:

$$\mathrm{V}\left(r\right) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r'}\right)}{\mathscr{r}}\mathrm{d}\tau'$$

Solution

$$\begin{aligned}
\rho &= \frac{q}{\frac{4}{3}\pi R^3} = \frac{3q}{4\pi R^3} \\
V(r) &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int_0^r \rho\left(4\pi r'^2\,dr'\right) + \int_r^R \frac{\rho\left(4\pi r'^2\,dr'\right)}{r'}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{4\pi\rho}{r}\left(\frac{r^3}{3}\right) + 4\pi\rho\int_r^R r'\,dr'\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{4\pi\rho\, r^2}{3} + 4\pi\rho\left(\frac{R^2 - r^2}{2}\right)\right] \\
&= \frac{\rho}{\epsilon_0}\left(\frac{r^2}{3} + \frac{R^2 - r^2}{2}\right) \\
&= \frac{3q}{4\pi R^3\epsilon_0}\left(\frac{r^2}{3} + \frac{R^2 - r^2}{2}\right) \\
&= \frac{q}{4\pi\epsilon_0 R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)
\end{aligned}$$

$$\boxed{V(\mathscr{r}(r)) = \frac{q}{4\pi\epsilon_0 R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)}$$

Problem 2.33:

Consider an infinite chain of point charges, $\pm q$ (with alternating signs), strung out along the x axis, each a distance a from its nearest neighbors. Find the work per particle required to assemble this system.

Solution

$$\begin{aligned}
\text{For a charge at } x = 0: \quad V(0) &= \frac{1}{4\pi\epsilon_0} \sum_{n \neq 0} \frac{q_n}{|na|} \\
&= \frac{1}{4\pi\epsilon_0} \left[\frac{-q}{a} + \frac{-q}{a} + \frac{q}{2a} + \frac{q}{2a} + \frac{-q}{3a} + \frac{-q}{3a} + \cdots \right] \\
&= \frac{q}{4\pi\epsilon_0 a} \, 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \frac{q}{4\pi\epsilon_0 a} \, 2\big(-\ln 2\big) \\
&= -\frac{q \ln 2}{2\pi\epsilon_0 a}
\end{aligned}$$

$$\text{Work per particle:} \quad U = \frac{1}{2} \, q \, V(0) = -\frac{q^2 \ln 2}{4\pi\epsilon_0 a}$$

$$\boxed{U = -\frac{q^2 \ln 2}{4\pi\epsilon_0 a}}$$

Problem 2.35:

Here is a fourth way of computing the energy of a uniformly charged solid sphereL Asseble it like a snowball, layer by layer, each time bringing in an infinitesimal charge dq from far away and smearing it uniformly over the surface, thereby increasing the radius. How much work dW does it take to build up the radius by an amount dr? Integrate this to find the work necessary to create the entire sphere of radius R and total charge q.

Solution

$$\begin{aligned}
q(r) &= \frac{q}{R^3} r^3 \\
\mathrm{d}q &= \frac{d}{dr}\left(\frac{q}{R^3} r^3\right) dr = 3q\,\frac{r^2}{R^3}\,dr \\
V(r) &= \frac{1}{4\pi\epsilon_0}\frac{q(r)}{r} = \frac{1}{4\pi\epsilon_0}\frac{q\,r^2}{R^3} \\
\mathrm{d}W &= V(r)\,\mathrm{d}q = \frac{1}{4\pi\epsilon_0}\frac{q\,r^2}{R^3}\cdot 3q\,\frac{r^2}{R^3}\,dr = \frac{3q^2}{4\pi\epsilon_0 R^6} r^4\,dr \\
W &= \int_0^R \mathrm{d}W = \frac{3q^2}{4\pi\epsilon_0 R^6}\int_0^R r^4\,dr = \frac{3q^2}{4\pi\epsilon_0 R^6}\cdot\frac{R^5}{5} \\
&= \frac{3q^2}{20\pi\epsilon_0 R}
\end{aligned}$$

$$\boxed{U = \frac{3q^2}{20\pi\epsilon_0 R}}$$

Problem 2.38:

A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b). The shell carries no net charge.

1. Find the surface charge density σ at R, at a, and at b.
2. Find the potential at the center, using infinity as a reference point.
3. Now the outer surface is touched to a grounding wire, which drains off charge and lowers its potential to zero (same as at infinity). How do your answers to the previous two parts change?

Solution

Part A

$$\begin{aligned}\sigma_R &= \frac{q}{4\pi R^2},\\ \sigma_a &= -\frac{q}{4\pi a^2},\\ \sigma_b &= \frac{q}{4\pi b^2} \quad \text{(since the shell is neutral:} \quad -q+q=0).\end{aligned}$$

$$\boxed{\sigma_R = \frac{q}{4\pi R^2}, \quad \sigma_a = -\frac{q}{4\pi a^2}, \quad \sigma_b = \frac{q}{4\pi b^2}}$$

Part B

$$\begin{aligned}V(0) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{R} + \frac{(-q)}{a} + \frac{q}{b}\right]\\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{a} + \frac{1}{b}\right).\end{aligned}$$

$$\boxed{V(0) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{a} + \frac{1}{b}\right)}$$

Part C

Grounding the outer surface forces $V(b) = 0 \implies$ its net charge is $Q_b = 0$.

Then, by Gauss' law, the inner surface still has $-q$ and the sphere $+q$.

Thus, the revised surface charge densities are:

$$\begin{aligned}\sigma_R &= \frac{q}{4\pi R^2},\\ \sigma_a &= -\frac{q}{4\pi a^2},\\ \sigma_b &= 0.\end{aligned}$$

Problem 2.38 continued on next page. . .

And the potential at the center becomes:

$$V(0) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R} - \frac{q}{a}\right).$$

$$\boxed{\sigma_R = \frac{q}{4\pi R^2}, \quad \sigma_a = -\frac{q}{4\pi a^2}, \quad \sigma_b = 0, \quad V(0) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{a}\right)}$$