Lecture 22: Aspherical Earth

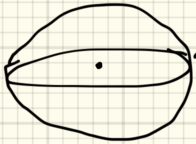
Aspherical Earth:

So far, we have assumed that the gravity fields of the bodies that we orbit are spherically symmetric, so we can approximate them as point masses.
In actuality, Earth & other planetary bodies are not perfect spheres.

This asphericity influences the orbits of S/C $\Rightarrow$ orbits don't follow predictions from 2BP.

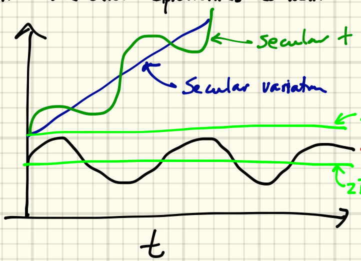
Any added perturbation (drag, aspherical gravity, 3rd body gravity) produces OE's that are not constant.

Earth is an oblate spheroid



Opposite shape: prolate spheroid

There are other asphericities as well.



In order to propagate the trajectory of the S/C about an aspherical Earth, we must modify the EOM.

2BP: $\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$

First ϕ = latitude of the S/C: $\phi = \operatorname{atan}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$

A force is the gradient of a potential field.

2BP: $U = \frac{\mu}{r}$

Spherical Harmonics Expression for Aspherical body:

$$U(r,\phi) = \frac{\mu}{r}\sum_{k=2}^{\infty} J_k\left(\frac{R}{r}\right)^k P_k(\cos\phi)$$

↑ Potential due to the Perturbing gravity force (not the 2BP mass)

R = radius of the Earth (avg) $\qquad$ P_k = Legendre Polynomials

r = S/C position

J_k = Zonal harmonics coefficient

We will consider just the "ring" of mass about Earth's equator. This is represented by the J2 coefficient. J2 is the largest spherical harmonics coefficient for Earth.

$$U(r,\phi) = \frac{\mu}{r}\left[1 - J_2\left(\frac{R}{r}\right)^2\left(\frac{3}{2}\sin^2\phi - \frac{1}{2}\right)\right]$$

$$J_2 = 0.00108248 \quad (\text{Earth})$$

$$\ddot{\vec{r}} = \vec{\nabla} U$$

When J2 is included, Ω & ω experience secular + periodic perturbations. The other OE experience periodic perturbations.

The secular variations are calculated by calculating $d\Omega/dt$ & $d\omega/dt$ and the averaging over $\nu \in 0, 360°$

$$\dot{\Omega}_{sec} = -\left[\frac{3}{2}\frac{J_2\sqrt{\mu}\,R^2}{a^{7/2}(1-e^2)^2}\right]\cos i$$

$$\dot{\omega}_{sec} = -\left[\frac{3}{2}\frac{J_2\sqrt{\mu}\,R^2}{a^{7/2}(1-e^2)^2}\right]\left(\frac{5}{2}\sin^2 i - 2\right)$$

$\cos(i) = 0$ when $i = 90° \Rightarrow \dot{\Omega}_{sec} = 0$ if $i = 90°$ ← Could still have periodic variation

Note that $\dot{\Omega} < 0$ for $i < 90° \Rightarrow$ the ascending node is moving West

"nodal regression"

$\dot{\omega} \propto \frac{5}{2}\sin^2 i - 2 \Rightarrow \dot{\omega} = 0$ if $i = 63.43°$ or $116.6°$

$\Rightarrow$ we cannot have both $\dot{\Omega}$ & $\dot{\omega} = 0$ at the same time.

Attitude Dynamics : Torque-free motion

Describe the s/c attitude using 3-2-1 Euler Angles:

Yaw: ψ

Pitch: θ

Roll: ϕ

Body-fixed coordinates that are aligned with the principal axes of the body $\Rightarrow [I]$ is diagonal

Torque free: ${}^N\vec{H}$ = constant

Specify the inertial frame s.t.: ${}^N\vec{H} = -H\hat{n}_3$

$${}^B\vec{H} = [BN]\,{}^N\vec{H}$$

↑ Direction Cosine Matrix for 3-2-1 rotation

$${}^BH_1 = H\sin\theta = I_1\omega_1$$

$${}^BH_2 = -H\sin\phi\cos\theta = I_2\omega_2$$

$${}^BH_3 = -H\cos\phi\cos\theta = I_3\omega_3$$

Note: $\vec{\omega} = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3$

$$\vec{\omega} = -\dot{\psi}\,\hat{n}_3 + \dot{\theta}\,\hat{b}_2' + \dot{\phi}\,\hat{b}_1$$

Rewrite $\vec{\omega}$ in terms of θ, ψ, ϕ and then substitute into the H_1, H_2, H_3 expressions:

Precession Rate
$$\dot{\psi} = -H\left(\frac{\sin^2\phi}{I_2} + \frac{\cos^2\phi}{I_3}\right)$$

Nutation rate
$$\dot{\theta} = \frac{H}{2}\left(\frac{1}{I_3} - \frac{1}{I_2}\right)\sin(2\phi)\cos\theta$$

$$\dot{\phi} = H\left(\frac{1}{I_1} - \frac{\sin^2\phi}{I_2} - \frac{\cos^2\phi}{I_3}\right)\sin\theta$$

If the body is axisymmetric $\Rightarrow$ Cylinder ($I_2 = I_3$)

$$\dot{\psi} = \frac{-H}{I_2}$$

$$\dot{\theta} = 0$$

$$\dot{\phi} = H\left(\frac{I_2 - I_1}{I_1 I_2}\right)\sin\theta$$