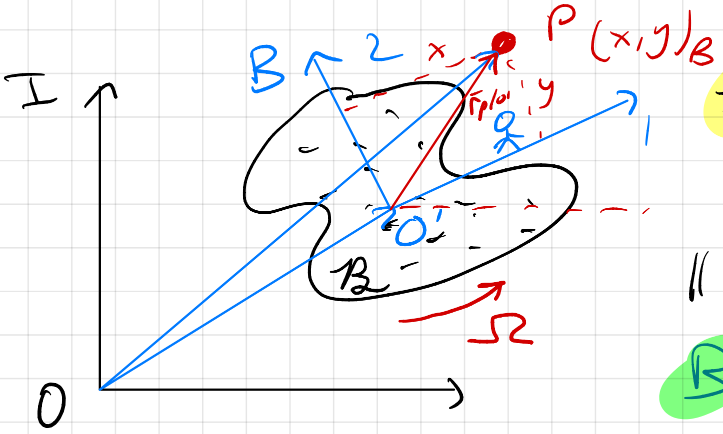
Relative Motion in a Rotating Frame

10/22/24



$\mathcal{B}$ rigid body

$k = 1, \dots, N$

$\|\bar{r}_{k/j}\|$ constant $\forall\, k, j$

B body frame $(O'; \hat{b}_1, \hat{b}_2, \hat{b}_3)$

✱ key idea: B is fixed to $\mathcal{B}$, which free to translate & rotate (in the plane)

$$\bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}$$

$$^{I}\bar{v}_{P/O} = {}^{I}\bar{v}_{O'/O} + {}^{I}\bar{v}_{P/O'}$$

$$^{I}\bar{v}_{P/O'} = \frac{^{I}d}{dt}\left(\bar{r}_{P/O'}\right)$$

$$= \frac{^{I}d}{dt}\left(x\hat{b}_1 + y\hat{b}_2\right)$$

$$= \dot{x}\hat{b}_1 + x\Omega\hat{b}_2 + \dot{y}\hat{b}_2 - y\Omega\hat{b}_1$$

$$= \underbrace{\dot{x}\hat{b}_1 + \dot{y}\hat{b}_2}_{= \frac{^{B}d}{dt}\left(\bar{r}_{P/O'}\right)} + \underbrace{\Omega\left(x\hat{b}_2 - y\hat{b}_1\right)}_{= {}^{I}\bar{\omega}^{B} \times \bar{r}_{P/O'}}$$

$$\frac{^{I}d}{dt}\hat{b}_1 = {}^{I}\bar{\omega}^{B} \times \hat{b}_1$$

$$= \Omega\hat{b}_3 \times \hat{b}_1$$

$$= \Omega\hat{b}_2$$

$$\Omega\hat{b}_3 \times \left(x\hat{b}_1 + y\hat{b}_2\right) \overset{\checkmark}{=} \Omega\left(x\hat{b}_2 - y\hat{b}_1\right)$$

$$\frac{^{I}d}{dt}\left(\bar{r}_{P/O'}\right) = \frac{^{B}d}{dt}\left(\bar{r}_{P/O'}\right) + {}^{I}\bar{\omega}^{B} \times \left(\bar{r}_{P/O'}\right)$$

$$\boxed{\frac{^{I}d}{dt}(\cdot) = \frac{^{B}d}{dt}(\cdot) + {}^{I}\bar{\omega}^{B} \times (\cdot)}$$

transport equation

✱ memorize

✱ works with any vector ✱

$$^{I}\bar{v}_{P/O} = {}^{I}\bar{v}_{O'/O} + \frac{^{I}d}{dt}\left(\bar{r}_{P/O'}\right)$$

$$= {}^{I}\bar{v}_{o'/o} + \underbrace{\frac{{}^{B}d}{dt}\left(\bar{r}_{p/o'}\right)}_{= {}^{B}\bar{v}_{p/o'}} + {}^{I}\bar{\omega}^{B} \times \bar{r}_{p/o'}$$

$${}^{I}\bar{a}_{p/o} = {}^{I}\bar{a}_{o'/o} + {}^{I}\bar{a}_{p/o'}$$

$$= {}^{I}\bar{a}_{o'/o} + \frac{{}^{I}d}{dt}\left({}^{I}\bar{v}_{p/o'}\right)$$

$$= {}^{I}\bar{a}_{o'/o} + \frac{{}^{I}d}{dt}\left({}^{B}\bar{v}_{p/o'} + {}^{I}\bar{\omega}^{B} \times \bar{r}_{p/o'}\right)$$

$$= {}^{I}\bar{a}_{o'/o} + \frac{{}^{B}d}{dt}\left({}^{B}\bar{v}_{p/o'} + {}^{I}\bar{\omega}^{B} \times \bar{r}_{p/o'}\right) + {}^{I}\bar{\omega}^{B} \times \left({}^{B}\bar{v}_{p/o'} + {}^{I}\bar{\omega}^{B} \times \bar{r}_{p/o'}\right)$$

$$\frac{{}^{I}d}{dt}\left({}^{I}\bar{\omega}^{B}\right) = \frac{{}^{B}d}{dt}\left({}^{I}\bar{\omega}^{B}\right) + \underbrace{{}^{I}\bar{\omega}^{B} \times \left({}^{I}\bar{\omega}^{B}\right)}_{=0}$$

$$\frac{{}^{I}d}{dt}\left({}^{I}\bar{\omega}^{B}\right) = \frac{{}^{B}d}{dt}\left({}^{I}\bar{\omega}^{B}\right) = {}^{I}\bar{\alpha}^{B}$$ angular acceleration

$$\boxed{{}^{I}\bar{a}_{p/o} = {}^{I}\bar{a}_{o'/o} + {}^{B}\bar{a}_{p/o'} + {}^{I}\bar{\alpha}^{B} \times \bar{r}_{p/o'} + 2\,{}^{I}\bar{\omega}^{B} \times {}^{B}\bar{v}_{p/o'} + {}^{I}\bar{\omega}^{B} \times \left({}^{I}\bar{\omega}^{B} \times \bar{r}_{p/o'}\right)}$$

See App. B p. 640

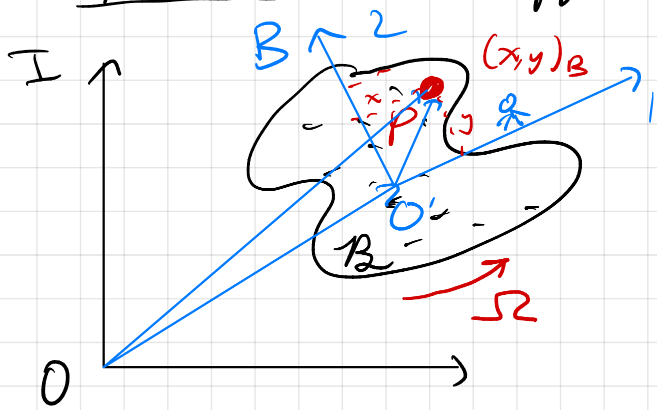
$2\,{}^{I}\bar{\omega}^{B} \times {}^{B}\bar{v}_{p/o'}$ Coriolis Acceleration

${}^{I}\bar{\omega}^{B} \times {}^{I}\bar{\omega}^{B} \times \bar{r}_{p/o'}$ Centripetal Acceleration

$$F_p = m_p\,{}^{I}\bar{a}_{p/o}$$

$$F_p - m_p\, 2\,{}^{I}\bar{\omega}^{B} \times {}^{B}\bar{v}_{p/o'} - m_p\,{}^{I}\bar{\omega}^{B} \times \left({}^{I}\bar{\omega}^{B} \times \bar{r}_{p/o'}\right)\Big)$$
$$- m_p\,{}^{I}\bar{a}_{o'/o} - m_p\,{}^{I}\bar{\alpha}^{B} \times \bar{r}_{p/o'} = m_p\,{}^{B}\bar{a}_{p/o'}$$

Special Case: Suppose P is fixed to $\mathcal{B}$



$\bar r_{P/O'} = x\hat b_1 + y\hat b_2$

$^B\bar v_{P/O'} = \frac{^Bd}{dt}(x\hat b_1 + y\hat b_2)$

$= 0$ because $\dot x = 0$

$\dot y = 0$

$\frac{^Bd}{dt}(\hat b_1) = 0$

$\frac{^Bd}{dt}(\hat b_2) = 0$

$\bar r_{P/O} = \bar r_{O'/O} + \bar r_{P/O'}$

$^I\bar v_{P/O'} = \frac{^Id}{dt}(\bar r_{P/O'})$

$\underbrace{\frac{^Bd}{dt}(\bar r_{P/O'})}_{=0} + {}^I\bar\omega^B \times (\bar r_{P/O'})$

$= \frac{^Id}{dt}(x\hat b_1 + y\hat b_2)$

$= x\Omega\hat b_2 - y\Omega\hat b_1$

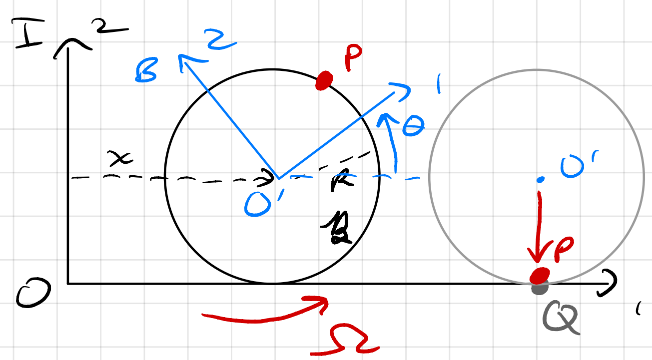
$= \Omega(x\hat b_2 - y\hat b_1)$

$= {}^I\bar\omega^B \times \bar r_{P/O'}$

$$\boxed{{}^I\bar v_{P/O} = {}^I\bar v_{O'/O} + {}^I\bar\omega^B \times \bar r_{P/O'}}$$

Special case only: P is fixed to $\mathcal{B}$

Ex 8.2 Rolling without Slipping



Let P be fixed to $\mathcal{B}$

Let Q be fixed in I

*key idea: at the instant P touches Q

$^I\bar v_{P/O} = {}^I\bar v_{Q/O} = 0$

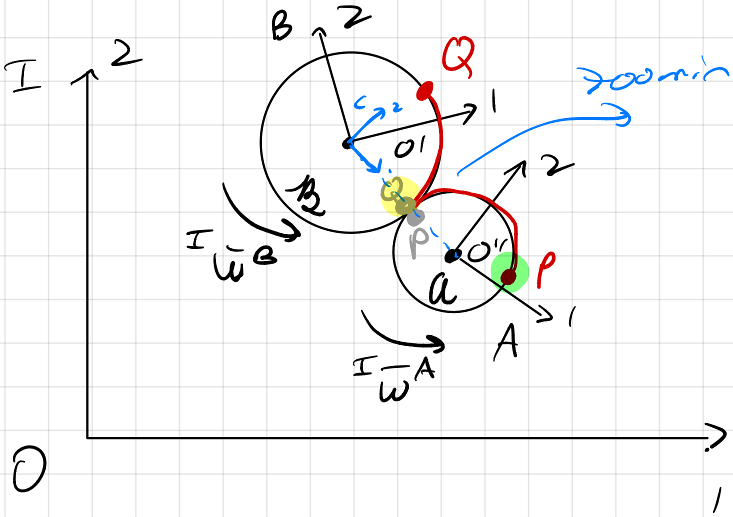
$\Rightarrow 0 = {}^I\bar v_{O'/O} + {}^I\bar\omega^B \times \bar r_{P/O'}$

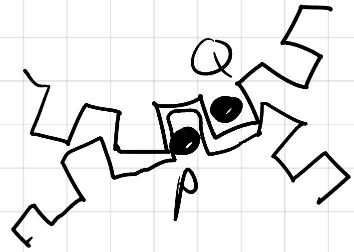
$0 = \dot x\hat e_1 + \Omega\underbrace{\hat b_3}_{=\hat e_3} \times (-R\hat e_2)$

$0 = \dot x\hat e_1 + \Omega R\hat e_1 \Rightarrow \boxed{\dot x = -\Omega R = -\dot\theta R}$

rolling without slipping constraint

Ex 8.3 Gears





* key idea: at the instant P & Q touch, ${}^I\bar{v}_{P/O} = {}^I\bar{v}_{Q/O}$

$${}^I\bar{v}_{P/O} = {}^I\bar{v}_{O''/O} + {}^I\bar{\omega}^A \times \bar{r}_{P/O''}$$

$${}^I\bar{v}_{Q/O} = {}^I\bar{v}_{O'/O} + {}^I\bar{\omega}^B \times \bar{r}_{Q/O'}$$

$$\cancel{{}^I\bar{v}_{O''/O}} + {}^I\bar{\omega}^A \times \bar{r}_{P/O''} = \cancel{{}^I\bar{v}_{O'/O}} + {}^I\bar{\omega}^B \times \bar{r}_{Q/O'}$$

$$\Omega_A \hat{a}_3 \times (-R_A \hat{c}_1) = \Omega_B \hat{b}_3 \times (R_B \hat{c}_1)$$

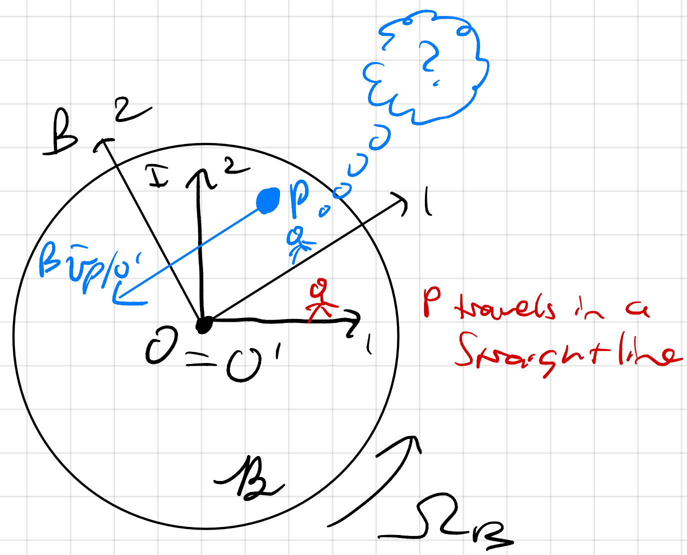
$\hat{c}_2$: $\boxed{-\Omega_A R_A = \Omega_B R_B}$ gear equation

Coriolis Acceleration

$$-2\,{}^I\bar{\omega}^B \times {}^B\bar{v}_{P/O'}$$

$$\bar{F}_P = 0$$

→ turns to the right



Centripetal Acceleration

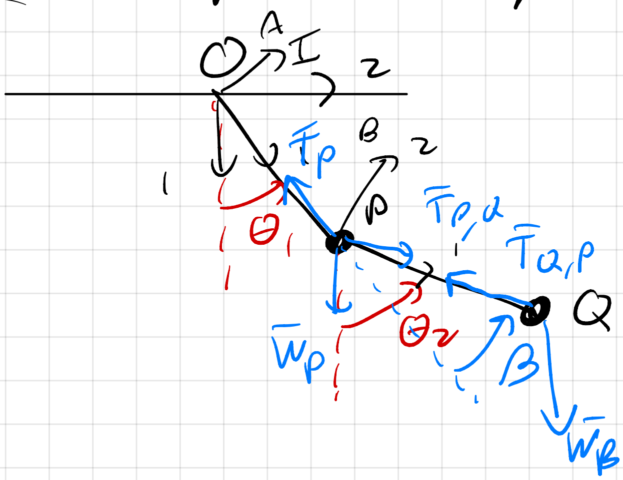
$$-{}^I\bar{\omega}^B \times \left({}^I\bar{\omega}^B \times \bar{r}_{P/O'}\right)$$

→ moves away from the origin

$O' = O$ $\quad {}^{I}\bar{v}_{P/O} = \frac{{}^{I}d}{dt}\left(\bar{r}_{P/O}\right)$

$$= \underbrace{\frac{{}^{B}d}{dt}\left(r_{P/O}\right)}_{\text{initial velocity}} + {}^{I}\bar{\omega}^{B} \times \bar{r}_{P/O}$$

Ex 8.9 Simple double pendulum



$\bar{F}_P = m_P \, {}^{I}\bar{a}_{P/O}$

$\bar{F}_Q = m_Q \, {}^{I}\bar{a}_{Q/O}$

$\ddot{\theta}_1, \ddot{\theta}_2, T_{Q,P}, T_P$

$${}^{I}\bar{\omega}^{A} = \dot{\theta}_1 \hat{a}_3$$

$${}^{I}\bar{\omega}^{B} = \dot{\theta}_2 \hat{b}_3$$

$${}^{A}\bar{\omega}^{B} = \underbrace{\left(\dot{\theta}_2 - \dot{\theta}_1\right)}_{\beta} \hat{b}_3$$

vector addition property

$${}^{I}\bar{\omega}^{B} = {}^{I}\bar{\omega}^{A} + {}^{A}\bar{\omega}^{B}$$

$$\dot{\theta}_2 \overset{\checkmark}{=} \dot{\theta}_1 + \dot{\theta}_2 - \dot{\theta}_1$$