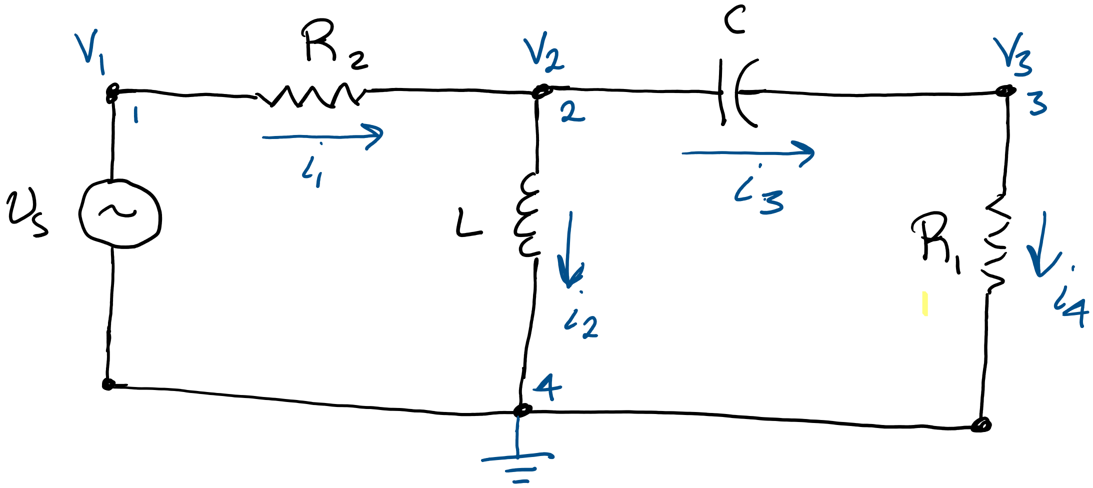
AC Circuit Analysis

Consider the 2 mesh



Start with node 2 and apply KCL,

$$I_1 - I_2 - I_3 = 0$$

$$\Rightarrow -\frac{V_2 - V_1}{R_2} - I_2 - C\frac{d}{dt}\left[V_2 - V_3\right] = 0$$

Differentiate,

$$-\frac{1}{R_2}\frac{dV_2}{dt} + \frac{1}{R_2}\frac{dV_1}{dt} - \frac{dI_2}{dt} + C\frac{d^2V_3}{dt^2} - C\frac{d^2V_2}{dt^2} = 0$$

$$\frac{dI_2}{dt} = \frac{1}{L}(V_2 - 0) = \frac{V_2}{L}$$

$\therefore$

$$-\frac{1}{R_2}\frac{dV_2}{dt} + \frac{1}{R_2}\frac{dV_1}{dt} - \frac{1}{L}V_2 + C\frac{d^2V_3}{dt^2} - C\frac{d^2V_2}{dt^2} = 0$$

Now,

Now,

$$V_1 = V_{01} e^{j\omega t}$$

$$V_2 = V_{02} e^{j(\omega t + \phi_2)} = V_{02} e^{j\phi_2} e^{j\omega t}$$

$$V_3 = V_{03} e^{j(\omega t + \phi_3)} = V_{03} e^{j\phi_3} e^{j\omega t}$$

Substitute & factor out $e^{j\omega t}$

$$\Big[-\frac{j\omega}{R_2} V_{02} e^{j\phi_2} + \frac{j\omega}{R_2} V_{01} e^{j\cancelto{0}{\phi_1}} - \frac{1}{L} V_{02} e^{j\phi_2} + (-\omega^2) C V_{03} e^{j\phi_3}$$

$$- (-\omega^2) C V_{02} e^{j\phi_2} \Big] e^{j\omega t} = 0$$

Observation 1: When we substitute $V = V_0 e^{j\omega t} e^{j\phi}$ the $e^{j\omega t}$ will always be a common factor & divide out. The only part that matters,

$$V_0 e^{j\phi}$$

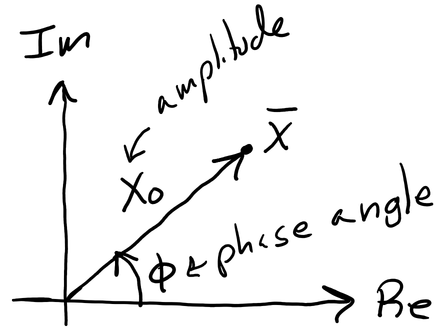
Likewise, $I = I_0 e^{j\omega t} e^{j\phi}$, we only need $I_0 e^{j\phi}$.

Given a harmonic function (harmonic signal)

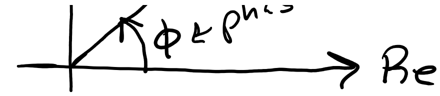
$$X(t) = X_0 e^{j\phi} e^{j\omega t}$$

define the "phasor" of the function as

$$\underbrace{\bar{X}}_{\text{phasor}} = \underbrace{X_0}_{\text{amplitude}} e^{\overbrace{j\phi}^{\text{phase angle}}}$$



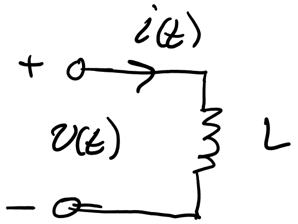
A phasor is just a complex number that encodes phasor amplitude



A phasor is just a complex number that encodes amplitude & phase information.

Phasor relationships for common circuit elements

Inductor



$$v(t) = L\frac{di}{dt}$$

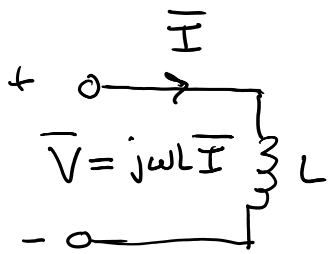
$$\Rightarrow V(t) = L\frac{dI}{dt}$$

$$\Rightarrow V_0 e^{j\theta} e^{j\omega t} = L\frac{d}{dt}\left[I_0 e^{j\phi} e^{j\omega t}\right]$$

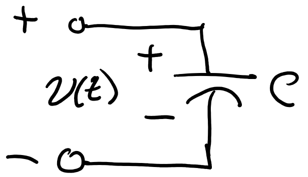
$$\Rightarrow V_0 e^{j\theta} \cancel{e^{j\omega t}} = j\omega L\, I_0 e^{j\phi} \cancel{e^{j\omega t}}$$

$$\Rightarrow V_0 e^{j\theta} = j\omega L\, I_0 e^{j\phi}$$

$$\Rightarrow \bar{V} = j\omega L \bar{I}$$



Capacitor:



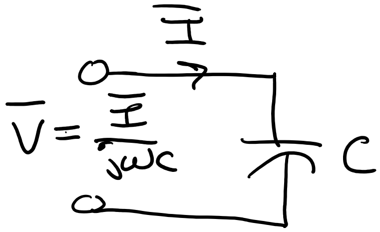
$$i(t) = C\frac{dv}{dt}$$

$$\Rightarrow I_0 e^{j\theta} e^{j\omega t} = C\frac{d}{dt}\left[V_0 e^{j\phi} e^{j\omega t}\right]$$

$$\Rightarrow I_0 e^{j\theta} = j\omega C\, V_0 e^{j\phi}$$

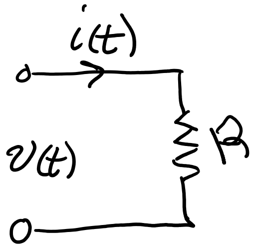
$$\Rightarrow \bar{I} = j\omega C \bar{V}$$

$$\Rightarrow \bar{V} = \frac{1}{j\omega C}\bar{I}$$



Resistor:

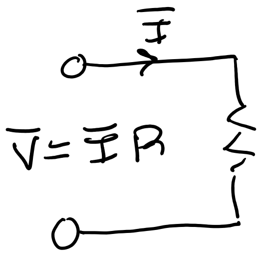
Resistor:



$$v(t) = i(t)R$$

$$\Rightarrow V_0 e^{j\theta} \cancel{e^{j\omega t}} = I_0 e^{j\phi} \cancel{e^{j\omega t}} R$$

$$\Rightarrow \bar{V} = \bar{I} R$$



Define "impedance" of a circuit element as

$$\bar{Z} = \frac{\bar{V}}{\bar{I}}$$

voltage phasor

current phasor

impedance

Impedance is a restance-like quantity and it has units of Ohms.

Impedance of comman circuit elements

Capacitor:

$$\bar{V} = \frac{1}{j\omega C}\bar{I} \Rightarrow \bar{Z}_C = \frac{\bar{V}}{\bar{I}} = \frac{1}{j\omega C}$$

Inductor

$$\bar{V} = j\omega L \bar{I} \Rightarrow \bar{Z}_L = \frac{\bar{V}}{\bar{I}} = j\omega L$$

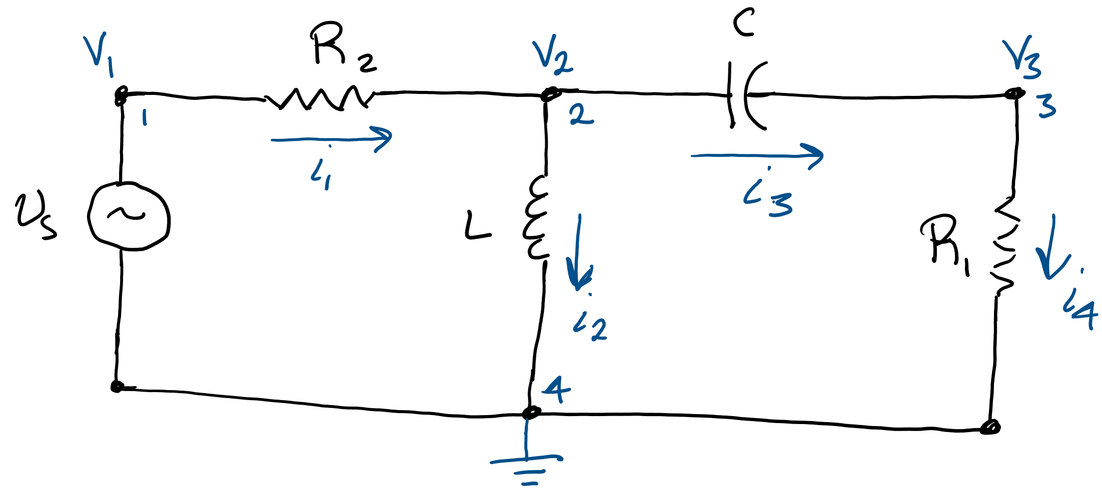
Resistor

$$\bar{Z}_R = \frac{\bar{V}}{\bar{I}} = R$$

Resistor

$$\bar{V} = \bar{I} R \quad \Rightarrow \quad \bar{Z}_R = \frac{\bar{V}}{\bar{I}} = R$$

Return to the original circuit



Node 2:

$$\bar{I}_1 - \bar{I}_2 - \bar{I}_3 = 0$$

$$\Rightarrow \frac{\bar{V}_1 - \bar{V}_2}{\bar{Z}_{R_2}} - \left[\frac{\bar{V}_2 - 0}{\bar{Z}_L}\right] - \left[\frac{\bar{V}_2 - \bar{V}_3}{\bar{Z}_C}\right] = 0$$

$$\Rightarrow \frac{\bar{V}_1 - \bar{V}_2}{R_2} - \frac{\bar{V}_2}{j\omega L} - \frac{\bar{V}_2 - \bar{V}_3}{\frac{1}{j\omega C}} = 0$$

$$\Rightarrow \left(\frac{1}{j\omega L} + j\omega C + \frac{1}{R_2}\right)\bar{V}_2 - j\omega C\,\bar{V}_3 = \frac{1}{R_2}\bar{V}_1$$

Node 3:

$$\bar{I}_3 - \bar{I}_4 = 0$$

$$\Rightarrow \quad \bar{V}_2 - \bar{V}_3 \qquad \bar{V}_3 \qquad = 0$$

$$\Rightarrow \quad \frac{\bar{V}_2 - \bar{V}_3}{\frac{1}{j\omega C}} - \frac{\bar{V}_3}{R_1} = 0$$

$$\Rightarrow \quad j\omega C\, \bar{V}_2 - (j\omega C + R_1)\, \bar{V}_3 = 0$$

2 equations, 2 unknowns

solve for $\bar{V}_2$ and $\bar{V}_3$