Lecture 15: Pitot Probes in Compressible Flows and Quasi-One-Dimensional Flows

ENAE311H Aerodynamics I

Christoph Brehm

Stagnation quantities through a normal shock

Our energy equation across the normal shock is

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2.$$

We thus see immediately that

$$h_{0,1} = h_{0,2},$$

and so

$$T_{0,1} = T_{0,2}.$$

This is as we would expect for an adiabatic flow.

Stagnation quantities through a normal shock

For the stagnation pressure, imagine we were to bring an element of fluid on either side of the shock to rest isentropically. Then the entropy in each case won't change, and we will have

$$s_2 - s_1 = s_{0,2} - s_{0,1}$$

Using our equation for entropy change, we then have

$$\begin{aligned} s_2 - s_1 &= c_p \ln \frac{T_{0,2}}{T_{0,1}} - R \ln \frac{p_{0,2}}{p_{0,1}} \\ &= -R \ln \frac{p_{0,2}}{p_{0,1}}, \end{aligned}$$

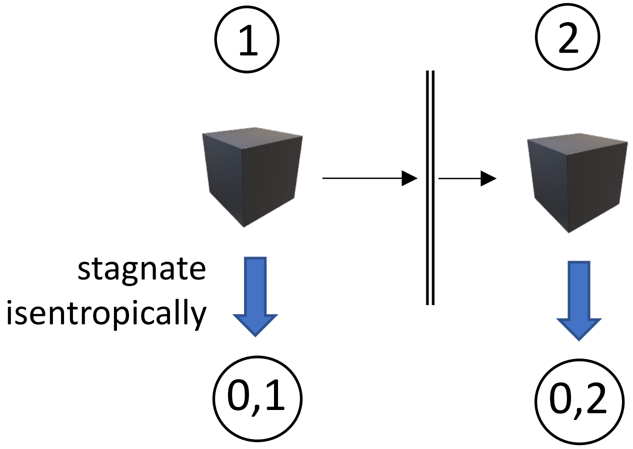
since T_0 is constant across the shock.

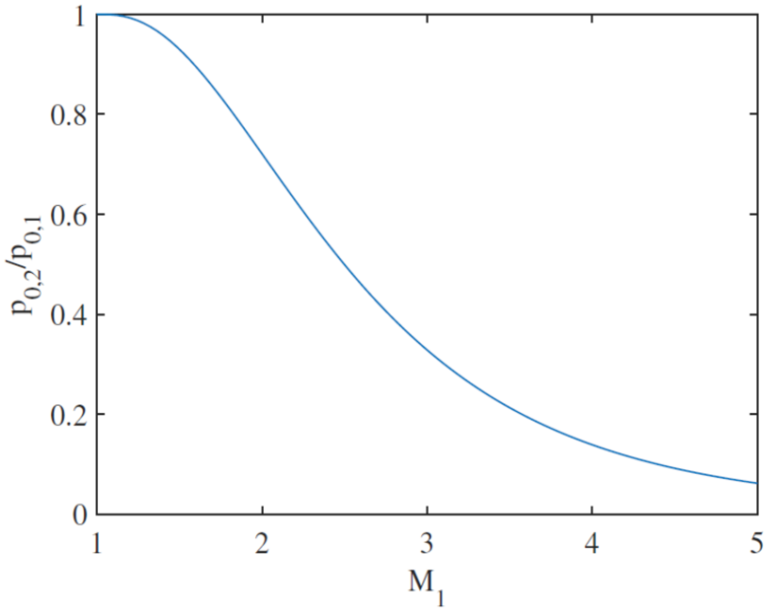
But remember:

$$\frac{s_2 - s_1}{R} = \ln \left[\left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right)^{1/(\gamma-1)} \left(\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2} \right)^{\gamma/(\gamma-1)} \right].$$

Comparing, we see

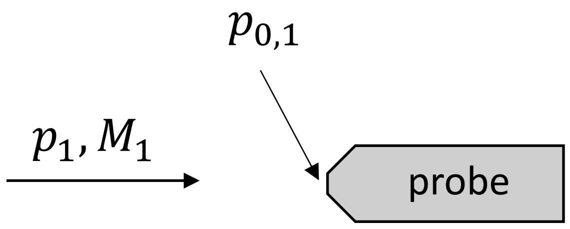
$$\frac{p_{0,2}}{p_{0,1}} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right]^{-1/(\gamma-1)} \left[\frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2} \right]^{\gamma/(\gamma-1)}$$





Pitot probes in compressible flow

Imagine we have a Pitot probe in a compressible flow. If the flow is subsonic, we can assume the deceleration from the freestream conditions is isentropic, in which case



$$\frac{p_{0,1}}{p_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\gamma/(\gamma-1)}$$

i.e., the measured Pitot pressure is the freestream total pressure.

This reaches a maximum value of 1.89 (for $\gamma = 1.4$) when $M_1 = 1$.

The Mach number can then be solved according to

$$M_1 = \left\{\frac{2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right]\right\}^{1/2}.$$

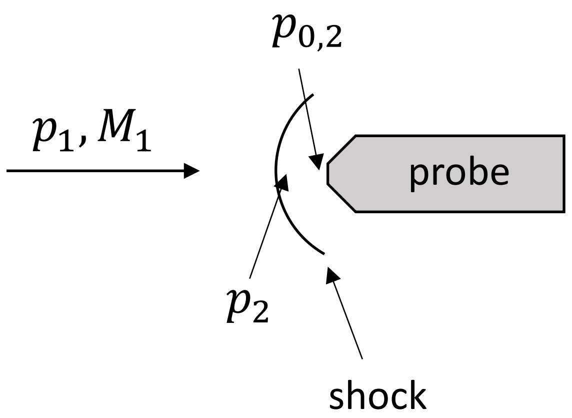
Assuming the freestream temperature (and thus sound speed) is known, the freestream velocity is given by

$$V_1 = M_1 a_1.$$

Pitot probes in compressible flow

If we have a supersonic flow, however, the situation is a little different. We assume that the stagnation streamline passes through a normal shock before reaching the probe tip. Then

$$\frac{p_{0,2}}{p_1} = \frac{p_{0,2}}{p_2}\frac{p_2}{p_1}.$$



The flow behind the shock to the probe tip can be assumed isentropic, so

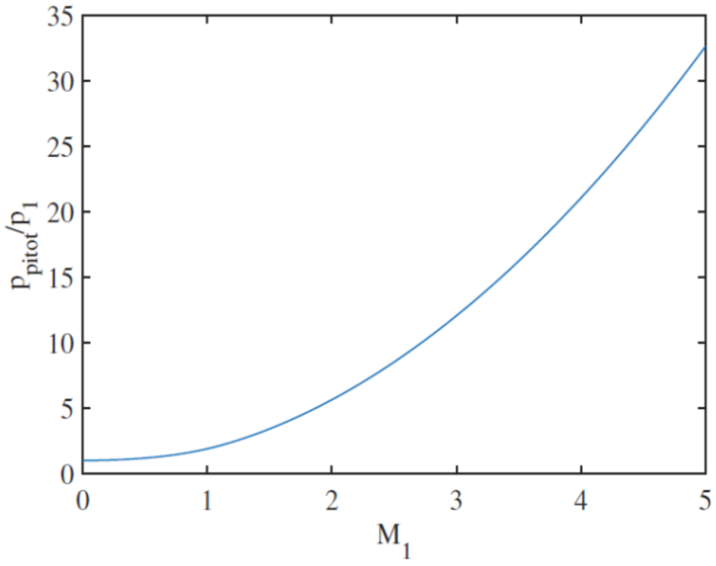
$$\frac{p_{0,2}}{p_2} = \left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{\gamma/(\gamma-1)},$$

where M_2 comes from the normal shock relations, which also give us

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1).$$

Combining, we have

$$\frac{p_{0,2}}{p_1} = \left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right]^{\gamma/(\gamma-1)} \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right].$$

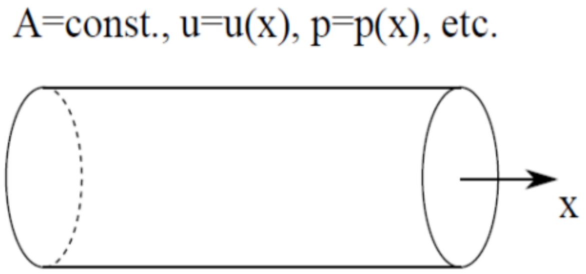


This is the Rayleigh-Pitot formula, which can be solved implicitly for M_1.

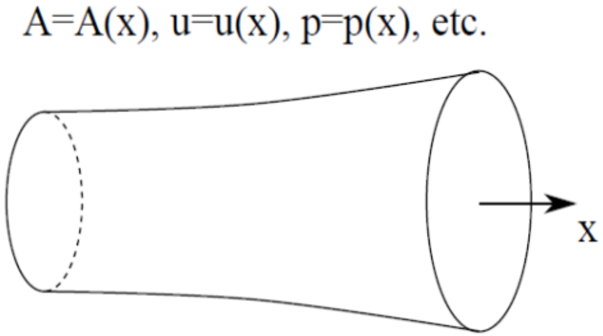
Quasi-one-dimensional flows

A normal shock wave is an example of a true one-dimensional flow: the velocity had only one component (u) and any given flow property changes only in the x direction.

To be strictly one-dimensional, a flow must be constant area; however, there is a class of flows, known as quasi-one-dimensional flows, for which area changes are gradual enough that we can approximate any changes as only taking place in single direction (and so properties are constant across a cross section normal to this direction).

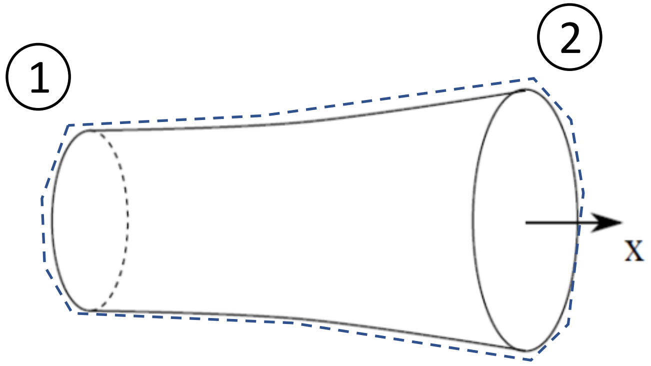


True 1-D flow



Quasi-1-D flow

Conservation equations in quasi-1-D flows



We assume the flow is steady. The continuity equation is then

$$\iint_{CS} \rho \mathbf{v} \cdot \mathbf{dA} = 0,$$

i.e.,

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2.$$

We can also write this as

$$\rho u A = \text{const.},$$

or

$$d(\rho u A) = 0.$$

Any derivatives here are to be understood as being w.r.t. x. Expanding out and dividing by $\rho u A$, we have

$$\boxed{\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0.}$$

Conservation equations in quasi-1-D flows

Instead of a formal momentum analysis, we note that Euler's equation holds for this flow (it is inviscid and adiabatic, and hence isentropic). We can thus write

$$dp + \rho u du = 0.$$

The energy equation is again

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2,$$

or

$$h + \frac{1}{2}u^2 = h_0 = \text{const.}$$

Since the flow is isentropic, we can also write

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2, \quad \Longrightarrow \quad dp = a^2 d\rho.$$

Substituting into Euler's equation:

$$a^2 \frac{d\rho}{\rho} = -u du,$$

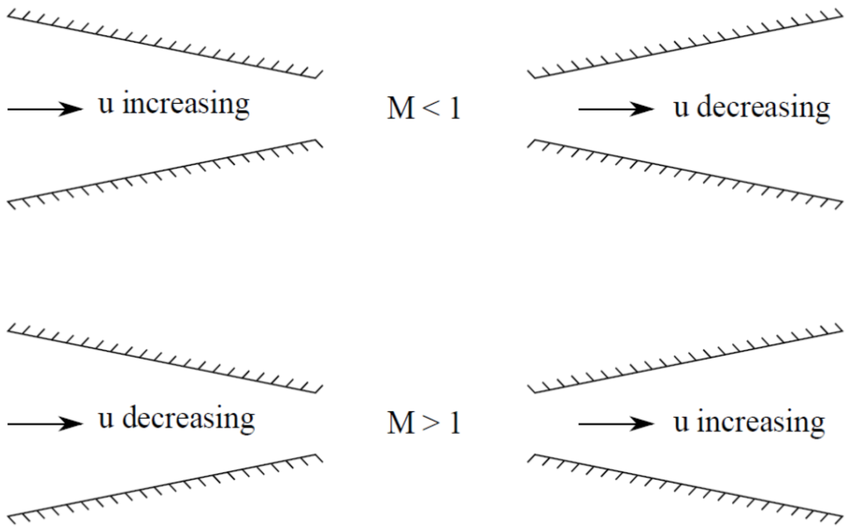
or

$$\frac{d\rho}{\rho} = -M^2 \frac{du}{u}.$$

Area-velocity relation

Combining this latest equation with our continuity equation, i.e.,

$$\frac{d\rho}{\rho} = -M^2\frac{du}{u} \quad \& \quad \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0.$$



we can write

$$(1 - M^2)\frac{du}{u} + \frac{dA}{A} = 0,$$

or

$$\boxed{\frac{du}{u} = -\frac{dA/A}{1 - M^2}.}$$

Note the following:

1. For $M = 0$, an increase in area leads to proportional decrease in velocity, and vice versa
2. For $M < 1$, increase in area leads to decrease in velocity (same qualitative behavior)
3. For $M > 1$, behavior is opposite: increase in area leads to increase in velocity
4. What about $M = 1$?

Throats in quasi-1D flows

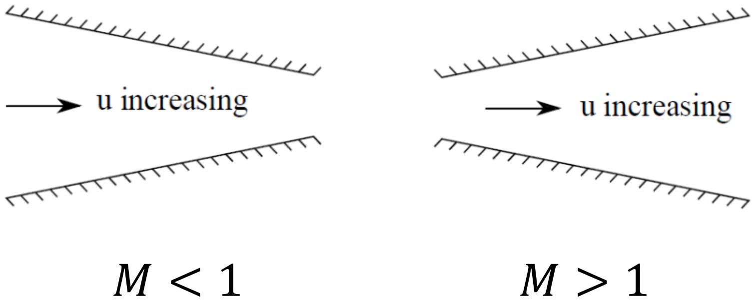
Imagine we have a geometry where the flow velocity is continuously increasing from subsonic to supersonic conditions. Then we note:

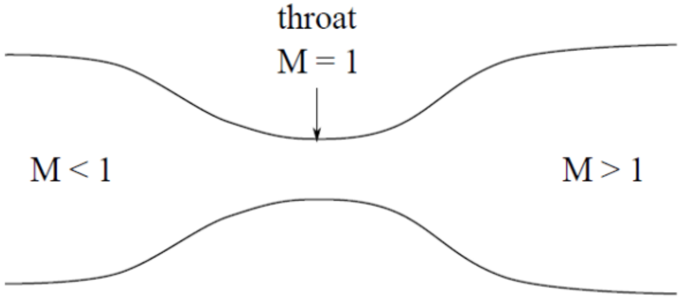
1. The area must be decreasing when $M < 1$
2. The area must be increasing when $M > 1$
3. We must have a throat ($dA/dx = 0$) when $M = 1$

It also flows from the area-velocity relation:

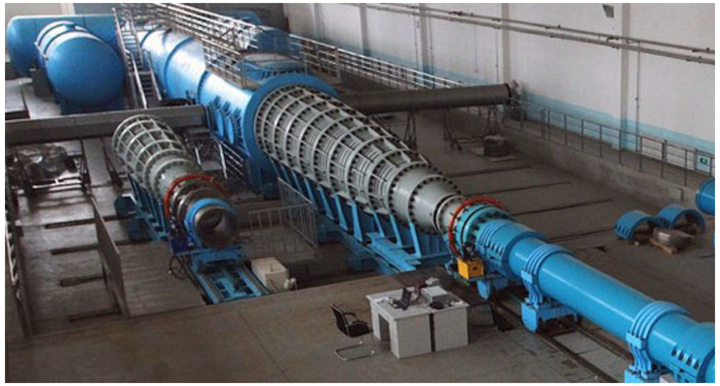
$$\frac{du}{u} = -\frac{dA/A}{1 - M^2}.$$

that dA/dx must be zero when $M = 1$ for du to remain finite.





Supersonic "de Laval" nozzle



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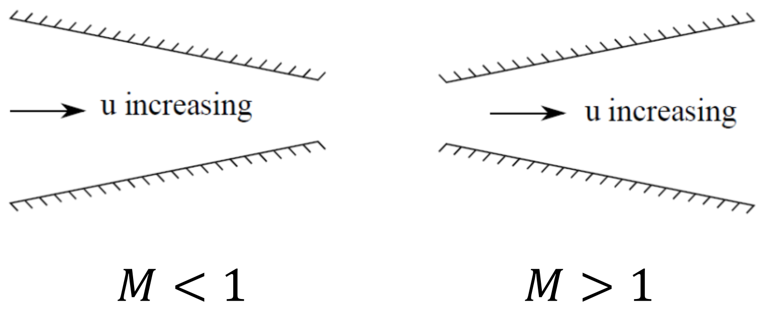
It also flows from the area-velocity relation:

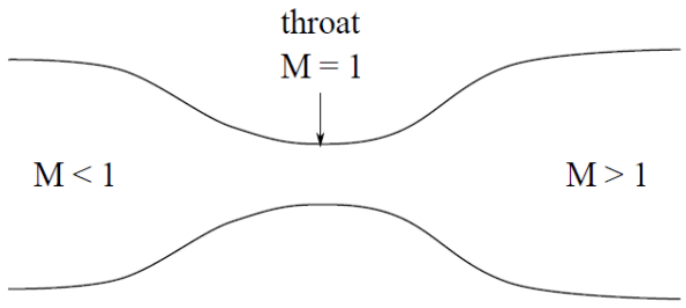
$$\frac{du}{u} = -\frac{dA/A}{1 - M^2}.$$

that dA/dx must be zero when $M = 1$ for du to remain finite.

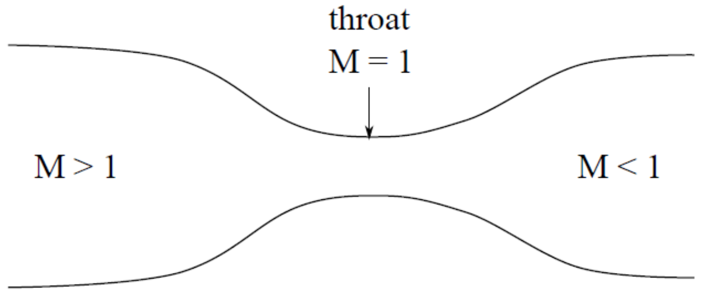
Similarly if we are transitioning isentropically from $M > 1$ to $M < 1$, can only have $M = 1$ at a throat.

Note, of course, that a throat is not a sufficient condition for sonic flow.





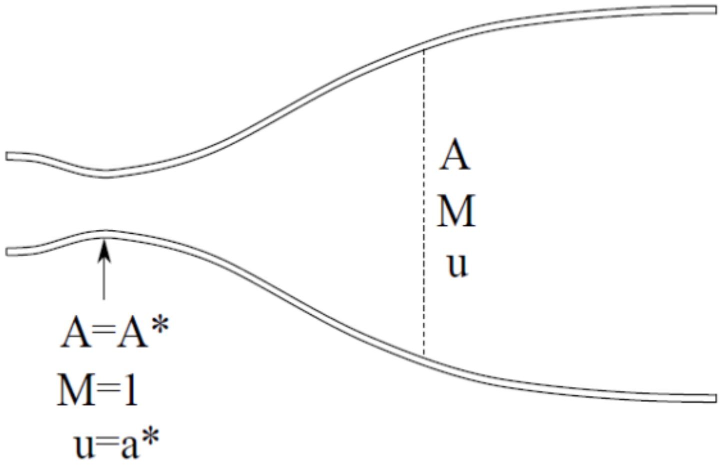
Supersonic "de Laval" nozzle



Supersonic diffuser

The area/Mach-number relationship

Imagine we have a converging-diverging supersonic nozzle with $A = A^*$ at the throat. Since at that point $M = 1$, we thus also have that $u = a = a^*$. Thus, we also have $M^* = 1$.



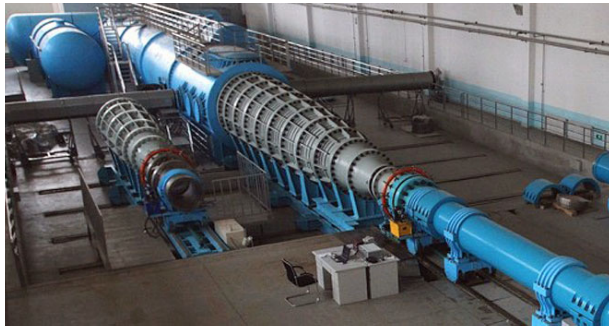
Now consider an arbitrary point downstream of the nozzle. From mass conservation we can write

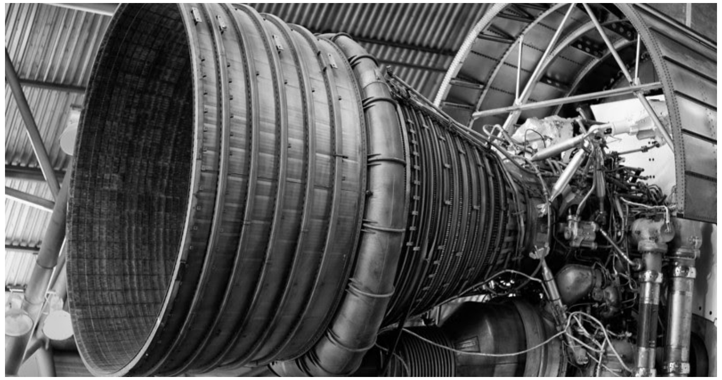
$$\rho^* a^* A^* = \rho u A$$

Rearranging, we have

$$\begin{aligned} \frac{A}{A^*} &= \frac{\rho^*}{\rho}\frac{a^*}{u} \\ &= \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{u}, \end{aligned}$$

since the total density is constant (flow is isentropic).





The area/Mach-number relationship

Starting from

$$\frac{A}{A^*} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{u}$$

We note

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)},$$

$$\frac{\rho_0}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)},$$

and

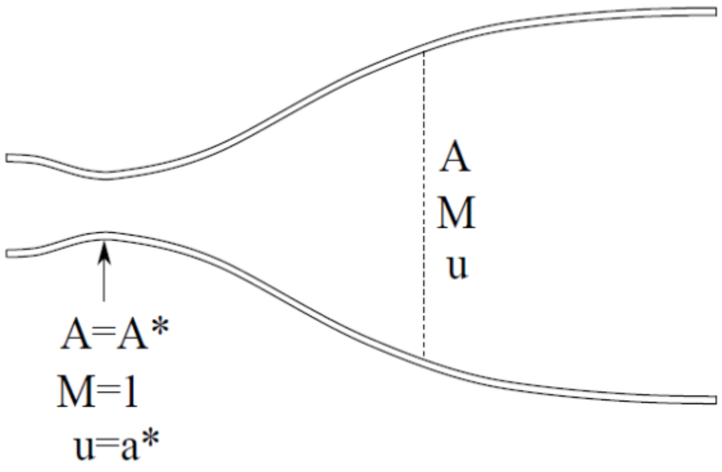
$$M^{*2} = \left(\frac{u}{a^*}\right)^2 = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}.$$

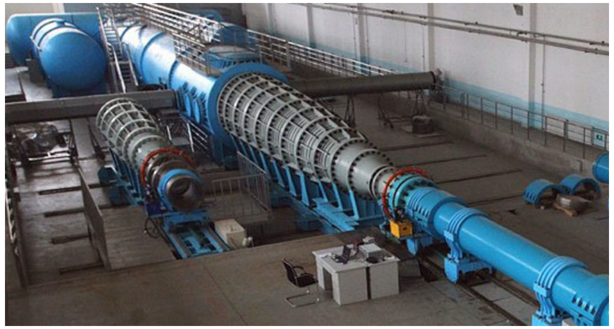
Substituting into our above expression, we have

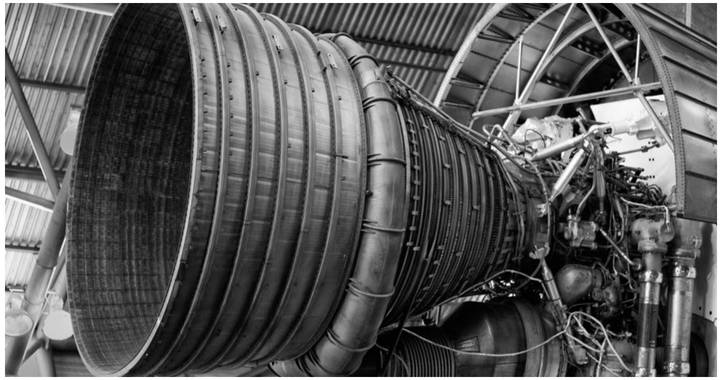
$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\left(1+\frac{\gamma-1}{2}M^2\right)^{2/(\gamma-1)}\frac{2+(\gamma-1)M^2}{(\gamma+1)M^2}.$$

Simplifying:

$$\boxed{\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}.}$$







The area/Mach-number relationship

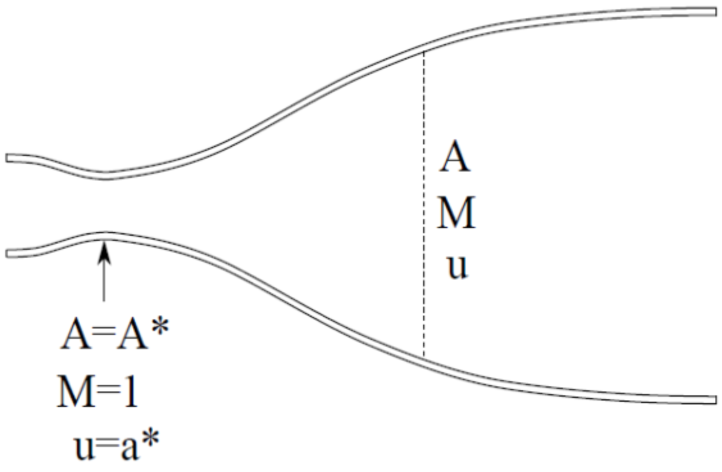
The expression

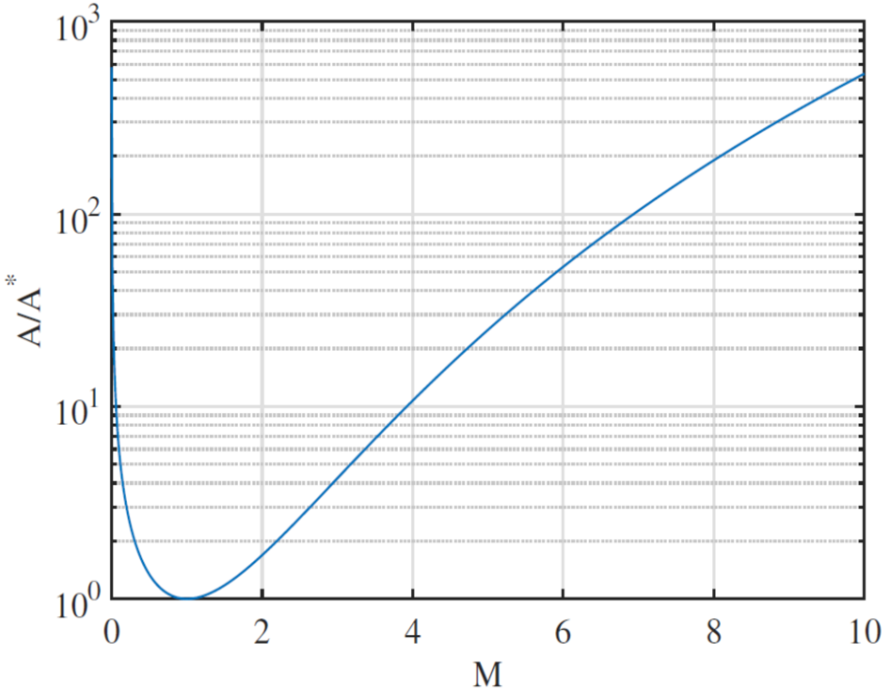
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)} .$$

is known as the area/Mach-number relation. It gives the Mach number (implicitly) as a function of the area ratio.

For $A < A^*$, there are no solutions; for $A = A^*$, there is one solution ($M = 1$); for $A > A^*$ there are two solutions (one subsonic, one supersonic). Either way, increasing A pushes M away from one.

For solving the flow inside a wind tunnel, often we will be given an area ratio and reservoir condition. Usually, the reservoir velocity will be so small that we can consider the conditions there to be the stagnation conditions. At a given point downstream, the Mach number can be obtained using the above equation. The other flow conditions can then be derived using our adiabatic/isentropic expressions for T/T_0, p/p_0, etc., since total conditions are constant throughout the nozzle.





Oblique shocks of finite strength

Note, however, that the laws of physics are invariant under a change in inertial reference frame. Thus, the conservation laws we derived for a normal shock will be valid if we use u in place of V, and the shock jump relations that we derived from them will be valid if we replace M_1 by the normal shock Mach number (remember purely thermodynamic variables aren't modified by a change in inertial reference frame), i.e.,

$$
\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2} \\
\frac{p_2}{p_1} &= 1+\frac{2\gamma}{\gamma+1}(M_{n1}^2-1) \\
\frac{T_2}{T_1} &= \left[1+\frac{2\gamma}{\gamma+1}(M_{n1}^2-1)\right]\frac{2+(\gamma-1)M_{n1}^2}{(\gamma+1)M_{n1}^2}.
\end{aligned}
$$

Note that the normal shock and Mach wave are special cases of these equations (for $\beta = \mathrm{asin}(1/M_1)$, we have simply $\rho_2/\rho_1 = p_2/p_1 = T_2/T_1 = 1$). For $\beta < \mathrm{asin}(1/M_1)$, $M_{n1} < 1$, i.e., the normal component of the velocity is subsonic, and so we can't have a shock.