Lecture 18: Low thrust + start of s/c attitude

Low Thrust Trajectories: the maneuver can take months → years

more complicated to solve b/c the s/c $\mathcal{E}$ is constantly changing ($\vec{h}$)

↳ typically need to numerically integrate to design these trajectories.

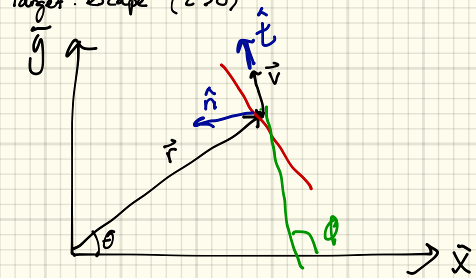
Can modify 2BP numerical integration code to include an additional thrust term.

The complicated part of low-thrust trajectory design is figuring out the control law that gets the s/c to the desired orbit in the min time with min fuel.

We will consider a special case where the s/c is thrusting tangentially to the orbit at all times.

Assume the s/c is initially in a circular orbit with radius r_0.

Target: escape ($\mathcal{E} > 0$)



Define the radius of curvature: $\rho = \frac{ds}{d\phi}$

Note: $\vec{v} = v\hat{t}$

$\dot{\vec{v}} = \dot{v}\hat{t} + v\dot{\phi}\hat{n}$

$v = \frac{ds}{dt}$ = arc-length per time

$\frac{d\phi}{dt} = \frac{d\phi}{ds}\frac{ds}{dt} = \frac{v}{\rho}$

$\dot{\vec{v}} = \dot{v}\hat{t} + \frac{v^2}{\rho}\hat{n}$

$\hat{t} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\dot{\hat{t}} = -\dot{\phi}\sin\phi\,\hat{x} + \dot{\phi}\cos\phi\,\hat{y}$

$\hat{n} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\dot{\hat{t}} = \dot{\phi}\hat{n}$

$$\dot{v}\hat{t} + \frac{v^2}{\rho}\hat{n} = \underbrace{a_T\hat{t}}_{\text{thrust}} - \underbrace{\frac{\mu}{r^2}\hat{r}}_{\text{2BP gravity}}$$

$$-\frac{\mu}{r^2}\hat{r} = -\frac{\mu}{r^2}\sin\gamma\,\hat{t} + \frac{\mu}{r^2}\cos\gamma\,\hat{n}$$

Note: $\vec{v} = \frac{ds}{dt}\hat{t}$

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\left(\frac{ds}{dt}\right)^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$v_r = \dot{r}, \quad \frac{dr}{dt} = \frac{ds}{dt}\sin\gamma \quad \Rightarrow \sin\gamma = \frac{dr}{ds}$$

$$v_\theta = r\dot{\theta} = v\cos\gamma$$

$$r\frac{d\theta}{dt} = \frac{ds}{dt}\cos\gamma \quad \Rightarrow \cos\gamma = r\frac{d\theta}{ds}$$

$$\dot{v}\hat{t} + \frac{v^2}{\rho}\hat{n} = a_T\hat{t} - \frac{\mu}{r^2}\frac{dr}{ds}\hat{t} + \frac{\mu}{r^2}r\frac{d\theta}{ds}\hat{n}$$

Separate directions:

$\hat{t}$: $\dot{v} = a_T - \frac{\mu}{r^2}\frac{dr}{ds}$ $\qquad\qquad \dot{v} = \frac{dv}{dt} = \frac{ds}{dt}\cdot\frac{dv}{ds} = v\frac{dv}{ds}$

$$\Rightarrow v\frac{dv}{ds} = a_T - \frac{\mu}{r^2}\frac{dr}{ds}$$

$\hat{n}$: $\frac{v^2}{\rho} = \frac{\mu}{r}\frac{d\theta}{ds}$

Back to $\hat{t}$:

$$\frac{d(v^2)}{ds} = 2v\frac{dv}{ds}$$

$$\boxed{a_T = \frac{1}{2}\frac{d(v^2)}{ds} + \frac{\mu}{r^2}\frac{dr}{ds}} \quad (1)$$

Given: (definition of curvature) $\frac{1}{\rho} = \frac{1}{r}\left[1 - \left(\frac{dr}{ds}\right)^2 - r\frac{d^2r}{ds^2}\right]\left[1 - \left(\frac{dr}{ds}\right)^2\right]^{-1/2}$

Substitute into the $\hat{n}$ eqn:

$$\frac{1}{r}\left[1-\left(\frac{dr}{ds}\right)^2\right]^{1/2} = \frac{d\theta}{ds} \quad \text{(expansion)}$$

$$\Rightarrow \boxed{rv^2\frac{d^2r}{ds^2} + \left(v^2 - \frac{\mu}{r}\right)\left[\left(\frac{dr}{ds}\right)^2 - 1\right] = 0} \quad ②$$

Initial Conditions: Circular initial orbit

$$r(t_0) = r_0 \qquad v^2(t_0) = v_0^2 = \frac{\mu}{r_0}$$

$$\left.\frac{dr}{ds}\right|_{t=t_0} = 0$$

Assume a_T = constant & integrate ①

$$\int a_T\, ds = \int \frac{1}{2} d(v^2) + \int \frac{\mu}{r^2} dr$$

$$a_T s = \frac{1}{2} v^2\Big|_{v_0}^{v} - \frac{\mu}{r}\Big|_{t_0}^{t}$$

$$= \frac{1}{2}(v^2 - v_0^2) - \frac{\mu}{r} + \frac{\mu}{r_0}$$

$$v^2 = 2a_T s + \mu\left(\frac{2}{r} - \frac{1}{r_0}\right)$$

Now, assume the thrust acceleration is small, s.t. $\frac{d^2r}{ds^2} \simeq 0$

then ② requires that $v^2 - \frac{\mu}{r} = 0 \Rightarrow v = \sqrt{\frac{\mu}{r}}$ = circular velocity $\Rightarrow$ implies that the orbit is always nearly circular

Plug into our eqns for r:

$$\boxed{r = r_0\left[1 - \frac{2a_T s}{v_0^2}\right]^{-1}} \quad ③$$

If you accelerate long enough, eventually the S/C will reach escape velocity:

$$v_\infty^2 = \frac{2\mu}{r}$$

Substituting v_∞ into ② $\Rightarrow$

$$2r\frac{d^2r}{ds^2} = 1 - \left(\frac{dr}{ds}\right)^2$$

Substitute ③ into above & solve for s (distance traveled before escape): $s_{esc} = \frac{v_0^2}{2a_T}\left[1 - \frac{1}{v_0}\left(20 a_T^2 r_0^2\right)^{1/4}\right]$

Substitute that distance back into ③ to get the radius when the s/c escapes:

$$\boxed{r_{esc} = \frac{r_0 V_0}{(20 a_T^2 r_0^2)^{1/4}}}$$

Time required to escape:

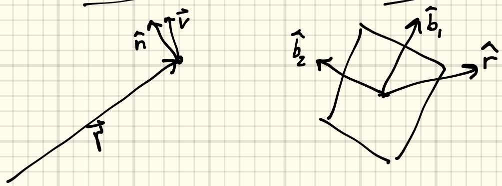
$$\boxed{t_{esc} - t_0 = \frac{V_0}{a_T}\left[1 - \left(\frac{20 a_T^2 r_0^2}{V_0^4}\right)^{1/8}\right]}$$

Spacecraft Attitude Dynamics:

Attitude: s/c orientation

Firstly, must develop a way to describe the attitude of a s/c.

Orbit: S/C:



We can use Euler Angles to describe the rotation between 2 coordinate systems.

⇒ We have already done this: we used Ω, i, ω to describe the rotation from XYZ to PQW

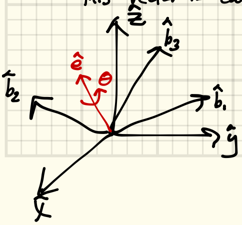
"Euler Angles" 3-1-3 rotation

Rotation Matrix: $[[R_{3\Omega}][R_{1i}][R_{3\omega}]]$ = Direction Cosine matrix

When we have 2 coordinate system, there are multiple Euler angle combinations that could be used to get the same direction cosine matrix.

Given 2 coordinate systems, there is a vector that is fixed in both systems. You can rotate between these 2 coordinate systems by executing a single rotation about this vector.

This vector is called: Euler Axis = eigenaxis = principal axis ($\hat{e}$)



Note that s/c do not typically execute maneuvers about the eigenaxis, but we use this concept to describe s/c attitude.