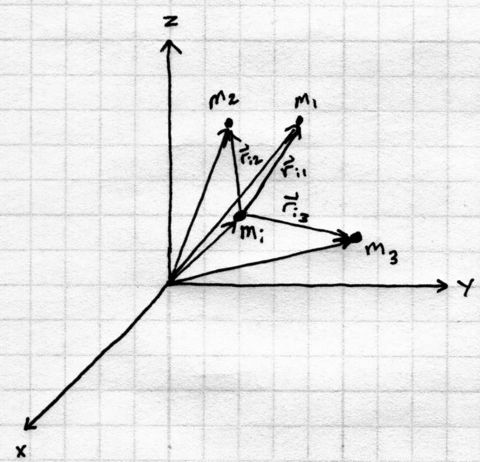
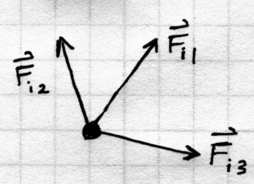
Finally, before proceeding to search for constants of the motion, we will derive the general form of the EOMs, $\ddot{\vec{r}}_i$, which is the inertial acceleration of the i^{th} body where i can equal $1, 2, \ldots, n$; n is the number of bodies in the system. In that case, the number of states is $6n$, the number of constants required for a closed form solution is $6n$, and the size of the system state vector is $6n \times 1$.

Consider the i^{th} body in a system of multiple bodies and the gravitational forces acting upon it due to 3 other bodies:



FBD



$$\vec{F}_{i1} = \frac{G m_i m_1}{r_{i1}^3}(\vec{r}_1 - \vec{r}_i)$$

$$\vec{F}_{i2} = \frac{G m_i m_2}{r_{i2}^3}(\vec{r}_2 - \vec{r}_i)$$

$$\vec{F}_{i3} = \frac{G m_i m_3}{r_{i3}^3}(\vec{r}_3 - \vec{r}_i)$$

$$\sum \vec{F}_i = m_i \ddot{\vec{r}}_i = \vec{F}_{i1} + \vec{F}_{i2} + \vec{F}_{i3}$$

$$\cancel{m_i} \ddot{\vec{r}}_i = \frac{G \cancel{m_i} m_1}{r_{i1}^3}(\vec{r}_1 - \vec{r}_i) + \frac{G \cancel{m_i} m_2}{r_{i2}^3}(\vec{r}_2 - \vec{r}_i) + \frac{G \cancel{m_i} m_3}{r_{i3}^3}(\vec{r}_3 - \vec{r}_i)$$

Factor out a $-G$ term:

$$\ddot{\vec{r}}_i = -G\left[\frac{m_1}{r_{1i}^3}(\vec{r}_i - \vec{r}_1) + \frac{m_2}{r_{2i}^3}(\vec{r}_i - \vec{r}_2) + \frac{m_3}{r_{3i}^3}(\vec{r}_i - \vec{r}_3)\right]$$

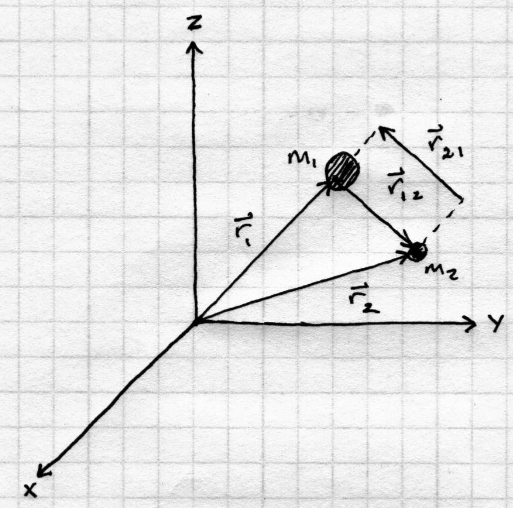
From this we can clearly see the pattern for the case of $n-1$ bodies acting upon the i^{th} body:

$$\boxed{\ddot{\vec{r}}_i = -G\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_j}{r_{ji}^3}(\vec{r}_i - \vec{r}_j) \qquad i = 1, 2, \ldots, n}$$

Note that j is not allowed to equal i in the finite sum because body i's gravity cannot act upon itself.

The Two-Body Problem

We now turn our attention to the study of a two-body system, which has many practical applications. Consider the following system:



Recall our equation for the acceleration of the i^{th} body:

$$\ddot{\vec{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^3}(\vec{r}_i - \vec{r}_j) \qquad i = 1, \ldots, n$$

We will use our equation, with $n = 2$ (two bodies) to determine $\ddot{\vec{r}}_1$ and $\ddot{\vec{r}}_2$.

For body 1: $i = 1$, $j = 1 \rightarrow$ no term, $j \neq i$

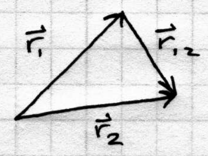
$$j = 2 \rightarrow \boxed{\ddot{\vec{r}}_1 = -G\left(\frac{m_2}{r_{21}^3}(\vec{r}_1 - \vec{r}_2)\right)}$$

↳ only one term in the series

For body 2: $i = 2$, $j = 1 \rightarrow \boxed{\ddot{\vec{r}}_2 = -G\left(\frac{m_1}{r_{12}^3}(\vec{r}_2 - \vec{r}_1)\right)}$

$j = 2 \rightarrow$ no term, $j \neq i$

Recall the vector geometry for m_1 and m_2:



$$\vec{r}_1 + \vec{r}_{12} = \vec{r}_2$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$-\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = \vec{r}_{21}$$

Now,

$$\text{let} \quad \vec{r} \equiv \vec{r}_{12}$$

$$r = \|\vec{r}\| = \|\vec{r}_{12}\| = \|-\vec{r}_{12}\|$$

Substituting $\vec{r}$ and r into our equations:

$$\ddot{\vec{r}}_1 = -G\frac{m_2}{r^3}(-\vec{r}) = G\frac{m_2}{r^3}\vec{r}$$

$$\ddot{\vec{r}}_2 = -G\frac{m_1}{r^3}\vec{r}$$

For the two-body problem we seek the motion of body 2 with respect to body 1:

$$\ddot{\vec{r}}_{12} = \ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

$$\ddot{\vec{r}} = -\frac{Gm_1}{r^3}\vec{r} - \frac{Gm_2}{r^3}\vec{r}$$

$$\ddot{\vec{r}} = -G(m_1+m_2)\frac{\vec{r}}{r^3}$$

We now define the Gravitational Parameter (μ):

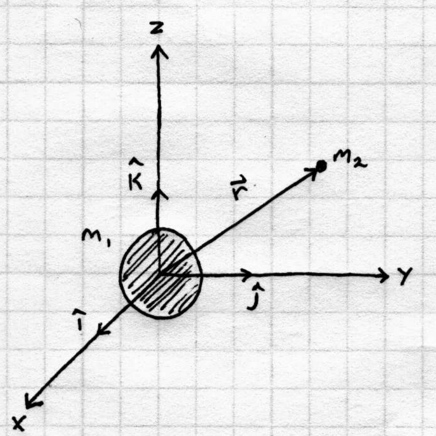
$$\mu \equiv G(m_1+m_2)$$

We will make the assumption for the two-body problem that $m_2 << m_1$, which is an excellent assumption in a variety of cases, particularly in our case of interest where m_1 is a planet, e.g., Earth, and m_2 is a spacecraft.

$$\Rightarrow \quad \mu \simeq G m_1$$

In practice, the value of μ for a planet can be determined more accurately than G or m separately.

By using $\vec{r}$, we have effectively placed the origin of our inertial frame at the center of m_1 and only concerned ourselves with the motion of m_2.



$$\boxed{\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \vec{0}}$$

Vector form of the simultaneous 2^{nd} order, non-linear, scalar differential equations in r_x, r_y, r_z

where $\vec{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$

$$\vec{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

Another approach:

$$\ddot{\vec{r}}_1 = -G\sum_{j=2}^{n} \frac{m_j}{r_{j1}^3}\vec{r}_{j1}$$

$$\ddot{\vec{r}}_2 = -G\sum_{\substack{j=1 \\ j\neq 2}}^{n} \frac{m_j}{r_{j2}^3}\vec{r}_{j2}$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\ddot{\vec{r}}_{12} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

Sub in the definitions for $\ddot{\vec{r}}_1$ & $\ddot{\vec{r}}_2$:

$$\ddot{\vec{r}}_{12} = -G\sum_{\substack{j=1 \\ j\neq 2}}^{n} \frac{m_j}{r_{j2}^3}\vec{r}_{j2} + G\sum_{j=2}^{n} \frac{m_j}{r_{j1}^3}\vec{r}_{j1}$$

$$\ddot{\vec{r}}_{12} = -\left[\frac{Gm_1}{r_{12}^3}\vec{r}_{12} + G\sum_{j=3}^{n}\frac{m_j}{r_{j2}^3}\vec{r}_{j2}\right] - \left[-\frac{Gm_2}{r_{21}^3}\vec{r}_{21} - G\sum_{j=3}^{n}\frac{m_j}{r_{j1}^3}\vec{r}_{j1}\right]$$

$$\vec{r}_{12} = -\vec{r}_{21}$$

$$\boxed{\ddot{\vec{r}}_{12} = \underbrace{\frac{-G(m_1+m_2)\vec{r}_{12}}{r_{12}^3}}_{\text{2-body term}} - G\sum_{j=3}^{n} m_j\left(\frac{\vec{r}_{j2}}{r_{j2}^3} - \frac{\vec{r}_{j1}}{r_{j1}^3}\right)}$$

Perturbing effect Caused by bodies 3→n acting on body 2.

" body 1.

Make same assumptions to get to the 2BP expression:

1. accelerations due to other masses ($j \geq 3$) are negligible
2. the larger mass has a spherical mass distribution
3. gravity is the only force acting on the masses
4. the masses of the bodies do not change
5. we describe the motion in an inertial reference frame.

Assume $m_1 \gg m_2$ (e.g., M_1 = the Earth, m_2 = Satellite)

$$\ddot{\vec{r}}_{12} = -\frac{Gm_1}{r_{12}^3}\vec{r}_{12} \quad \text{(neglecting the S/c's mass)}$$

$Gm_1 = \mu$ = Gravitational parameter

$$\boxed{\ddot{\vec{r}} = \frac{-\mu}{r^3}\vec{r}} \quad \text{EOM 2BP}$$

This is a relative position. The origin does not have to be at the center of the planet.

Energy: Energy is conserved → only gravity is acting.

Kinetic energy: $\frac{1}{2} m_2 V^2$

Potential Energy: $\Delta PE = -W = -\int_{x_i}^{x_f} F(x)\,dx$

$$= -\int_{r_{ref}}^{r} F_g\,dr = -\int_{r_{ref}}^{r} -\frac{G m_1 m_2}{r^2}\,dr$$

$$= -\frac{G m_1 m_2}{r}\Big|_{r_{ref}}^{r}$$

Define the potential energy to be zero @ $r_{ref} = \infty$

$$PE(r_{ref} = \infty) = 0$$

$$PE(r) = -\frac{\mu m_2}{r}$$

Total specific energy: $\mathcal{E} = \frac{KE + PE}{m_2} \Rightarrow \boxed{\mathcal{E} = \frac{V^2}{2} - \frac{\mu}{r}}$

Prove that $\mathcal{E}$ is conserved:

Take the dot product of $\dot{\vec{r}}$ with the EOM $\left(\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}\right)$

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} \cdot \dot{\vec{r}} = 0$$

$$\vec{V} \cdot \dot{\vec{V}} + \frac{\mu}{r^3}\vec{r} \cdot \dot{\vec{r}} = 0$$

Math-magic : $\vec{v}\cdot\dot{\vec{v}} = v\dot{v}$

Proof : $\vec{b}\cdot\dot{\vec{b}} = b\dot{b}$

Define $\vec{b} = b\hat{b}$

$$\frac{d\vec{b}}{dt} = \dot{b}\hat{b} + b\dot{\hat{b}}$$

$$\vec{b}\cdot\frac{d\vec{b}}{dt} = b\dot{b}$$

Rewriting our eqn: $v\dot{v} + \frac{\mu}{r^3} r\dot{r} = 0$

Note: $\frac{d}{dt}\left(\frac{v^2}{2}\right) = v\dot{v}$ & $\frac{d}{dt}\left(-\frac{\mu}{r}\right) = \frac{\mu}{r^2}\dot{r}$

$$v\dot{v} + \frac{\mu}{r^3} r\dot{r} = 0 = \frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right)$$

Energy is Conserved

$$\Rightarrow \frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = \frac{d}{dt}(\mathcal{E}) = 0 \quad \square$$

Angular Momentum: (Specific Angular Momentum)

→ is also Conserved. There is no torque b/c gravity acts radially.

$$\vec{r}\times\left(\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0\right)$$

$$\vec{r}\times\ddot{\vec{r}} + \vec{r}\times\frac{\mu}{r^3}\vec{r} = 0$$

$$\vec{r}\times\vec{r} = 0$$

$$\Rightarrow \vec{r}\times\ddot{\vec{r}} = 0$$

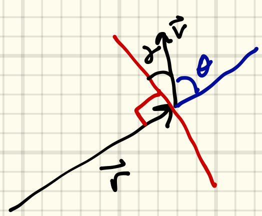
$$\frac{d}{dt}(\vec{r}\times\dot{\vec{r}}) = \cancelto{0}{\dot{\vec{r}}\times\dot{\vec{r}}} + \vec{r}\times\ddot{\vec{r}}$$

$$\Rightarrow \frac{d}{dt}(\vec{r}\times\dot{\vec{r}}) = 0$$

$$\frac{d}{dt}(\vec{r}\times\vec{v}) = 0$$

$\boxed{\vec{h} = \vec{r}\times\vec{v}}$ = Specific angular momentum

= both vector magnitude & direction are Conserved.



$h = |\vec{r}\times\vec{v}| = rv\sin\theta$

γ = Flight Path angle

$$\boxed{h = rv\cos\gamma}$$

$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$ is conserved

Spacecraft moves faster close to the central body.

We have the EOM of our satellite:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

We want to know what the orbit looks like without requiring numerical integration:

$$\ddot{\vec{r}} \times \vec{h} = \frac{\mu}{r^3}(\vec{h} \times \vec{r})$$

$$\frac{d}{dt}(\dot{\vec{r}} \times \vec{h}) = \ddot{\vec{r}} \times \vec{h} + \cancelto{0}{\dot{\vec{r}} \times \dot{\vec{h}}}$$

write the RHS as a time rate of change:

$$\frac{\mu}{r^3}(\vec{h} \times \vec{r}) = \frac{\mu}{r^3}(\vec{r} \times \vec{v}) \times \vec{r}$$

$$= \frac{\mu}{r^3}\left[\vec{v}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{r}\cdot\vec{v})\right]$$

$$= \frac{\mu}{r^3}\left(r^2\vec{v} - \vec{r}(r\dot{r})\right)$$

$$= \frac{\mu}{r}\vec{v} - \frac{\mu}{r^2}\dot{r}\vec{r}$$

Note: $\mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\mu}{r}\vec{v} - \frac{\mu\vec{r}}{r^2}\dot{r}$

Thus: $\frac{d}{dt}(\dot{\vec{r}} \times \vec{h}) = \mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$

Integrate: $\dot{\vec{r}} \times \vec{h} = \mu \frac{\vec{r}}{r} + \vec{B}$ ← integration constant

Dot with $\vec{r}$:

$$\vec{r} \cdot \dot{\vec{r}} \times \vec{h} = \frac{\mu}{r}\vec{r}\cdot\vec{r} + \vec{B}\cdot\vec{r}$$

$$\vec{a}\cdot\vec{b}\times\vec{c} = \vec{a}\times\vec{b}\cdot\vec{c}$$

$$\Rightarrow h^2 = \mu r + rB\cos\nu$$

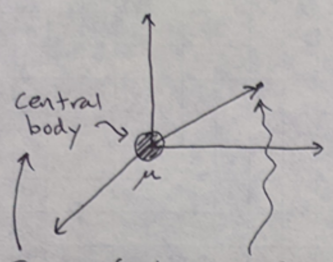
Solve for r: $r = \dfrac{h^2/\mu}{1 + B/\mu \cos\nu}$ ⟵ This is the eqn for a conic section written in polar coordinates, where $\vec{B}$ points to the point closest to the focus.

⇒ All orbits are conic sections.

$\boxed{r = \dfrac{p}{1 + e\cos\nu}}$ = the trajectory eqn.

(Restricted) <u>two-body problem EOM</u>:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$



Earth (w/spacecraft)
or, Sun (w/ Earth, asteroid, or spacecraft as in HW0)

Vector form of simultaneous 2^{nd} order, nonlinear, scalar differential equations in r_x, r_y, r_z

$$\underset{\text{position}}{\vec{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}} \qquad \underset{\text{velocity}}{\dot{\vec{r}} = \begin{bmatrix} \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \end{bmatrix}} \qquad \underset{\text{acceleration}}{\ddot{\vec{r}} = \begin{bmatrix} \ddot{r}_x \\ \ddot{r}_y \\ \ddot{r}_z \end{bmatrix}}$$

We have 6 dynamical states in this system ($r_x, r_y, r_z, \dot{r}_x, \dot{r}_y, \dot{r}_z$) so we require 6 constants of integration in order to solve the differential EOMs in closed form.

Previously, we showed that energy is conserved (provides 1 constant), angular momentum is conserved (provides 3 constants), and we have the integration constant $\vec{B}$ that points to the point closest to the focus of the conic section path of motion. However, $\vec{B}$ is not independent of the angular momentum so it provides only 1 more constant.

→ Total of 5 constants are available, but ~~[illegible]~~ we need 6

→ Cannot solve the system in closed form, need to solve numerically using a numerical integrator

To use numerical integration, we must cast our problem as a 1^{st} order system using the <u>state vector</u>, which is the position and velocity vector together.

$$\underbrace{\vec{R}}_{\text{State vector}} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \\ \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \end{bmatrix} \begin{matrix} R(1) \\ R(2) \\ R(3) \\ R(4) \\ R(5) \\ R(6) \end{matrix} \qquad \frac{d\vec{R}}{dt} = \dot{\vec{R}} = \begin{bmatrix} \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \\ \ddot{r}_x \\ \ddot{r}_y \\ \ddot{r}_z \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right\} \text{EOM} \qquad \dot{\vec{R}} = \begin{bmatrix} R(4) \\ R(5) \\ R(6) \\ R(1)\cdot -\mu/\left(\sqrt{R(1)^2 + R(2)^2 + R(3)^2}\right)^3 \\ R(2)\cdot \quad " \\ R(3)\cdot \quad " \end{bmatrix}$$

The numerical integrator solves $\int_{t_0}^{t_f} \dot{\vec{R}}\, dt$ numerically.

The $\dot{\vec{R}}$ (written in terms of $\vec{R}$, as above) is what should be written in the "propagate_2BP" function discussed in the Hint section of HW0.