Lecture 16: IOD

Initial Orbit Determination:

1. New planet, asteroid, comet → what is the orbit?

   e.g. Interstellar object identified b/c $e>1 \Rightarrow$ hyperbolic orbit about our Sun.

2. Enemy (or unknown) satellite: Space Situational Awareness (SSA)

   what are the goals & capabilities of other satellites, nations?

3. Orbital debris: will the orbital debris collide with s/c?

   Strongly influenced by orbital perturbations

4. Science: thru precise orbit determination, we can get better gravity and/or tide models.

Statistical orbit determination: how to incorporate additional observations to update our estimate of the orbit.
$\Rightarrow$ grad level course

Initial orbit determination:

Method #1: Lambert's Prob: $\vec{r}_1, \vec{r}_2$, TOF $\Rightarrow \vec{v}_1, \vec{v}_2$ $\quad (\vec{r}_1, \vec{v}_1 \Rightarrow OE)$

Method #2: Gibbs Method: input: 3 position vectors

output: $\vec{v}_1$

Given: $\vec{r}_1, \vec{r}_2, \vec{r}_3$ $\quad (t_1 < t_2 < t_3)$

Find: $p, e, \hat{P}, \hat{Q}, \hat{W}$ & then $\vec{v}_i$ (Could be $\vec{v}_1, \vec{v}_2$ or $\vec{v}_3$)

The 3 positions must be coplanar.
Choose Constants s.t.:

$$C_1\vec{r}_1 + C_2\vec{r}_2 + C_3\vec{r}_3 = 0 \quad \star$$

$$\vec{r} \cdot \hat{P} = r\cos\nu$$

$$\vec{e} = e\hat{P}$$

$$\vec{r} \cdot \vec{e} = re\cos\nu$$

$$r = \frac{p}{1+e\cos\nu} \Rightarrow r + re\cos\nu = p$$

$$\vec{r} \cdot \vec{e} = p - r$$

Dot $\star$ w/ $\vec{e}$: $C_1(p-r_1) + C_2(p-r_2) + C_3(p-r_3) = 0$

Cross * eqn w/ $\vec{r}_1, \vec{r}_2, \vec{r}_3$:

$$c_2 \vec{r}_1 \times \vec{r}_2 = c_3 \vec{r}_3 \times \vec{r}_1$$

$$c_1 \vec{r}_1 \times \vec{r}_2 = c_3 \vec{r}_2 \times \vec{r}_3$$

$$c_1 \vec{r}_3 \times \vec{r}_1 = c_2 \vec{r}_2 \times \vec{r}_3$$

Multiply the scalar eqn by $\vec{r}_3 \times \vec{r}_1$ & then substitute in the cross product terms to get the eqn only in terms of c_2.

$$c_2 (p - r_1) \vec{r}_2 \times \vec{r}_3 + c_2 (p - r_2) \vec{r}_3 \times \vec{r}_1 + c_2 (p - r_3) \vec{r}_1 \times \vec{r}_2 = 0$$

Divide by c_2 & rearrange:

$$p \underbrace{\left[\vec{r}_2 \times \vec{r}_3 + \vec{r}_3 \times \vec{r}_1 + \vec{r}_1 \times \vec{r}_2\right]}_{\vec{D}} = \underbrace{r_1 \vec{r}_2 \times \vec{r}_3 + r_2 \vec{r}_3 \times \vec{r}_1 + r_3 \vec{r}_1 \times \vec{r}_2}_{\vec{N}}$$

$$p \vec{D} = \vec{N} \qquad \vec{N} \cdot \vec{D} = ND$$

$$\boxed{p = \frac{N}{D}}$$

B/c we know $\vec{r}_1, \vec{r}_2, \vec{r}_3$, we know the plane of the orbit:

$$\hat{h} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

$\Rightarrow \vec{N}$ & $\vec{D}$ are in the $\hat{h}$ direction

$\hat{w} \parallel \hat{h} \quad \Rightarrow \vec{N}, \vec{D} \parallel \hat{w}$

$$\hat{P} \times \hat{Q} = \hat{w} \Rightarrow \hat{Q} = \hat{w} \times \hat{P}$$

$$\hat{P} \parallel \hat{e}$$

$$\hat{Q} = \frac{1}{Ne} \vec{N} \times \vec{e}$$

Substitute in for $\vec{N}$:

$$Ne \hat{Q} = r_1 (\vec{r}_2 \times \vec{r}_3) \times \vec{e} + r_2 (\vec{r}_3 \times \vec{r}_1) \vec{e} + r_3 (\vec{r}_1 \times \vec{r}_2) \times \vec{e}$$

Identity: $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$

$$Ne \hat{Q} = r_1 (\vec{r}_2 \cdot \vec{e}) \vec{r}_3 - r_1 (\vec{r}_3 \cdot \vec{e}) \vec{r}_2 +$$

$$r_2 (\vec{r}_3 \cdot \vec{e}) \vec{r}_1 - r_2 (\vec{r}_1 \cdot \vec{e}) \vec{r}_3 +$$

$$r_3 (\vec{r}_1 \cdot \vec{e}) \vec{r}_2 - r_3 (\vec{r}_2 \cdot \vec{e}) \vec{r}_1$$

Using $\vec{r}\cdot\vec{e} = p - r$

$$N e\,\hat{Q} = [r_1(p-r_2) - r_2(p-r_1)]\vec{r}_3 +$$

$$[r_3(p-r_1) - r_1(p-r_3)]\vec{r}_2 +$$

$$[r_2(p-r_3) - r_3(p-r_2)]\vec{r}_1$$

$$\Rightarrow N e\,\hat{Q} = p\underbrace{\left[(r_1-r_2)\vec{r}_3 + (r_3-r_1)\vec{r}_2 + (r_2-r_3)\vec{r}_1\right]}_{\vec{S}}$$

$$N e\,\hat{Q} = p\vec{S}$$

$$\Rightarrow \boxed{e = \frac{pS}{N} \quad (\text{b/c } N = pD) \Rightarrow e = \frac{S}{D}}$$

$$\boxed{\hat{w} = \frac{\vec{N}}{N}} \quad \boxed{\hat{Q} = \frac{\vec{S}}{S}} \quad \boxed{\hat{P} = \hat{Q}\times\hat{w}}$$

We want a velocity. Start with an intermediate step in the derivation of the trajectory eqn

$$\dot{\vec{r}}\times\vec{h} = \mu\left(\frac{\vec{r}}{r} + \vec{e}\right)$$

Cross w/ $\vec{h}$: $\vec{h}\times(\dot{\vec{r}}\times\vec{h}) = \mu\left(\frac{\vec{h}\times\vec{r}}{r} + \vec{h}\times\vec{e}\right)$

Identity: $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$\vec{h}\times(\dot{\vec{r}}\times\vec{h}) = (\vec{h}\cdot\vec{h})\dot{\vec{r}} - \cancelto{0}{(\vec{h}\cdot\dot{\vec{r}})}\vec{h}$$

$$= h^2\vec{V}$$

$$h^2\vec{V} = \mu\left(\frac{\vec{h}\times\vec{r}}{r} + \vec{h}\times\vec{e}\right)$$

$$\vec{h} = h\,\hat{w}\,,\quad \vec{e} = e\hat{P}$$

$$\vec{V} = \frac{\mu}{h}\left(\frac{\hat{w}\times\vec{r}}{r} + \hat{w}\times e\hat{P}\right) \quad \text{know } \hat{P}\times\hat{Q} = \hat{w}$$

$$= \frac{\mu}{h}\left(\frac{\hat{w}\times\vec{r}}{r} + e\,\hat{Q}\right)$$

$$h = \sqrt{\mu p}\,,\quad h = \sqrt{\mu N/D}$$

$$e = \frac{S}{D}\,,\quad \hat{Q} = \frac{\vec{S}}{S}\,,\quad \hat{w} = \frac{\vec{D}}{D}$$

$$\vec{V} = \frac{\sqrt{\mu D}}{\sqrt{N}}\frac{\vec{D}\times\vec{r}}{rD} + \frac{\vec{S}}{D}\sqrt{\frac{D\mu}{N}}$$

$$\boxed{\vec{B} = \vec{D}\times\vec{r}_i}\,,\quad \boxed{L = \sqrt{\frac{\mu}{DN}}}$$

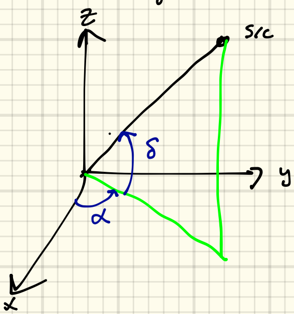
$$\boxed{\vec{V}_i = \frac{L}{r_i}\vec{B} + L\vec{S}}$$ where $\vec{r}_i$ can be $\vec{r}_1, \vec{r}_2$ or $\vec{r}_3$

Method #3: Laplace's Method: "Angles only" Proposed in 1780

(Don't need to know range → easier to just work with the angular location)

Input: $\alpha_1, \delta_1, \alpha_2, \delta_2, \alpha_3, \delta_3$

Review the angles:



α = Right ascension, measured in the xy plane E from $\hat{x}$

δ : declination measured N from the equatorial plane.