Chapter 11
3D Rigid Bodies

11/19/24

The governing laws of motion for a particle are

Newton's Laws ② $\bar{F}_P = m_P \, {}^I\bar{a}_{P/O} = \frac{{}^I d}{dt}\left({}^I\bar{p}_{P/O}\right)$

The governing laws of motion for a rigid body

Euler's Laws ① $\bar{F}_G = m_G \, {}^I\bar{a}_{G/O} = \frac{{}^I d}{dt}\left({}^I\bar{p}_{G/O}\right)$

② $M_G = \frac{{}^I d}{dt}\left({}^I\bar{h}_G\right)$

The transport equation allows us to compute $\frac{{}^I d}{dt}$ w/o coordinates!

$$\bar{F}_G = \frac{{}^I d}{dt}\left({}^I\bar{p}_{G/O}\right) = \frac{{}^C d}{dt}\left({}^I\bar{p}_{G/O}\right) + {}^I\bar{\omega}^C \times {}^I\bar{p}_{G/O}$$

$$= \frac{{}^C d}{dt}\left(m_G \, {}^I\bar{v}_{G/O}\right) + {}^I\bar{\omega}^C \times \left(m_G \, {}^I\bar{v}_{G/O}\right)$$

In aerospace engineering it is common to use

$$\left[{}^I\bar{v}_{G/O}\right]_C = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C$$

$$\left[{}^I\bar{v}_{G/O}\right]_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_I = {}^I R^C \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C$$

$$\left[{}^I\bar{\omega}^C\right]_C = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_C$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}_C = m_G \frac{^C d}{dt} \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C + \underbrace{\begin{bmatrix} p \\ q \\ r \end{bmatrix}_C}_{\bar{\omega}} \times m_G \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C$$

$$= m_G \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}_C + m_G \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C$$

Using the transport equation on the rotational dynamics

$$\bar{M}_G = \frac{^I d}{dt}\left({}^I\bar{h}_G\right) = \frac{^C d}{dt}\left({}^I\bar{h}_G\right) + {}^I\bar{\omega}^C \times {}^I\bar{h}_G$$

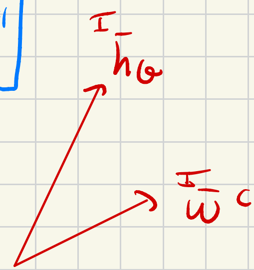
Recall in 2D, for a planar rigid body

$${}^I\bar{h}_G = \mathbb{I}_G \, {}^I\bar{\omega}^B$$

In 3D the moment of inertia MOI becomes a 3x3 matrix $[\mathbb{I}_G]_C$

$$\Rightarrow \left[{}^I\bar{h}_G\right]_C = [\mathbb{I}_G]_C \left[{}^I\bar{\omega}^C\right]_C$$

$$\begin{bmatrix} 3\times1 \end{bmatrix} = \begin{bmatrix} 3\times3 \end{bmatrix} \begin{bmatrix} 3\times1 \end{bmatrix}$$



Qualitative Rotational Dynamics

Rapidly Spinning RB

let C be the body frame

let B be the intermediate frame that shares the spin axis with C

$$ {}^I\bar{\omega}^C = \underbrace{{}^I\bar{\omega}^B}_{\text{precession}} + \underbrace{{}^B\bar{\omega}^C}_{\text{spin}} $$

E&L :

$$ \bar{M}_G = \frac{{}^I d}{dt}\left({}^I\bar{h}_G\right) \quad \text{T.E.} $$

$$ = \frac{{}^C d}{dt}\left({}^I\bar{h}_G\right) + {}^I\bar{\omega}^C \times {}^I\bar{h}_G $$

$$ = \frac{{}^B d}{dt}\left({}^I\bar{h}_G\right) + \underbrace{{}^I\bar{\omega}^B}_{?} \times {}^I\bar{h}_G $$

For a rapidly spinning rigid body, we have

$$ \| {}^B\bar{\omega}^C \| \gg \| {}^I\bar{\omega}^B \| $$

spin >> precession

Using intuition for 2D rigid body

$$ {}^I\bar{h}_G \approx \mathbb{I}_G \, {}^I\bar{\omega}^C \approx \mathbb{I}_G \, {}^B\bar{\omega}^C $$

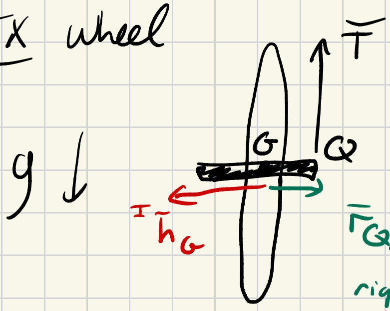
$$\bar{M}_G = \underbrace{\frac{{}^B d}{dt}\left({}^I\bar{h}_G\right)}_{\approx 0} + {}^I\bar{\omega}^B \times {}^I\bar{h}_G$$

(assuming constant spin rate)

$$\Rightarrow \boxed{\underbrace{\bar{M}_G}_{\text{out}} \approx \underbrace{{}^I\bar{\omega}^B}_{\substack{\text{unknown} \\ \text{up}}} \times \underbrace{{}^I\bar{h}_G}_{\text{left}}}$$

precesses CCW from above

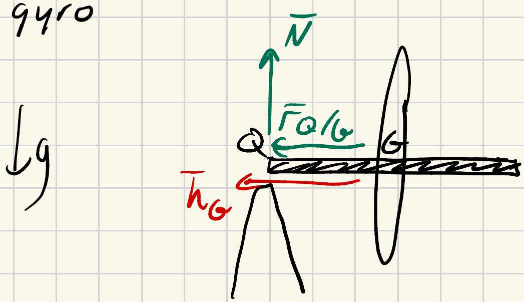
Ex wheel



$$\bar{r}_{Q/G} \times \bar{T} = \bar{M}_G$$

right × up = out

Ex gyro



$$\bar{r}_{Q/G} \times \bar{N} = \bar{M}_G$$

left × up = in

$$\bar{M}_G \approx {}^I\bar{\omega}^B \times {}^I\bar{h}_G$$

in ≈ down × left

precesses CW from above

COM of 3D RB

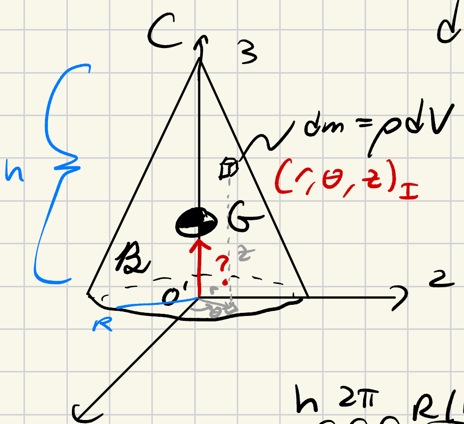
$$\lim_{N\to\infty} \bar{r}_{G/O} = \frac{1}{m_G}\sum_{k=1}^{N} m_k \bar{r}_{k/O}$$

$$= \frac{1}{m_G}\int_{B} dm\, \bar{r}_{dm/O}$$

$\rho\, dA$ (2D)
$\rho\, dV$ (3D)

Ex Cone (Cylindrical Coordinates) $(r, \theta, z)_I$

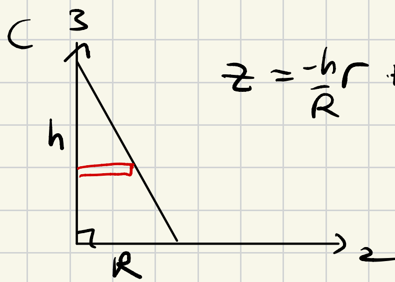
$$dV = r\,dr\,d\theta\,dz$$



$$\bar{r}_{G/O} = \frac{\rho}{m}\iiint_{B} r\,dr\,d\theta\,dz\, \bar{r}_{dm/O}$$

$$\bar{r}_{dm/O} = r c\theta\, \hat{c}_1 + r s\theta\, \hat{c}_2 + z\, \hat{c}_3$$

$$\bar{r}_{G/O} = \frac{\rho}{m}\int_0^{h}\int_0^{2\pi}\int_0^{\frac{R(h-z)}{h}} r\,dr\,d\theta\,dz \left(\cancel{r c\theta\, \hat{c}_1} + \cancel{r s\theta\, \hat{c}_2} + z\, \hat{c}_3\right)$$



$$z = -\frac{h}{R} r + h \Rightarrow r = -\frac{R}{h}(z-h)$$

$$= \frac{R(h-z)}{h}$$

$$\bar{r}_{G/O} = \left[\frac{\rho}{m}\int_0^h\int_0^{2\pi}\int_0^{\frac{R(h-z)}{h}} z\, r\,dr\,d\theta\,dz\right]\hat{c}_3 = \dots = \frac{h}{4}\,\hat{c}_3$$

$$m = \rho V = \rho\,\frac{\pi R^2 h}{3}$$

<u>Moment of Inertia of 3D RB</u>

Recall Ch. 9

$$^{I}\bar{h}_G = \sum_{k=1}^{N} m_k\left[{}^{I}\bar{\omega}^{C}\left(\bar{r}_{k/G}\cdot\bar{r}_{k/G}\right) - \bar{r}_{k/G}\left(\bar{r}_{k/G}\cdot{}^{I}\bar{\omega}^{C}\right)\right]$$

$$\left[{}^{I}\bar{h}_G\right]_c = \sum_{k=1}^{N} m_k\left(\left[{}^{I}\bar{\omega}^{C}\right]_c \|\bar{r}_{k/G}\|^2 - \left[r_{k/G}\right]_c \left(r_{k/G}\right]_c^T \left[{}^{I}\bar{\omega}^{C}\right]_c\right)$$

$$= \sum_{k=1}^{N} m_k\left(\|\bar{r}_{k/G}\|^2 \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\substack{3\times 3\\ \text{identity}}} - \underbrace{\left[\bar{r}_{k/G}\right]_c}_{3\times 1}\underbrace{\left(\bar{r}_{k/G}\right]_c^T}_{[1\times 3]}\right)\underbrace{\left[{}^{I}\bar{\omega}^{C}\right]_c}_{3\times 1}$$

$$= \left[\mathbb{I}_G\right]_c \left[{}^{I}\bar{\omega}^{C}\right)_c$$

where

$$\left[\mathbb{I}_G\right]_c = \sum_{k=1}^{N} m_k\left(\|\bar{r}_{k/G}\|^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \left[\bar{r}_{k/G}\right]_c\left[r_{k/G}\right]_c^T\right)$$

as $N \to \infty$

$$\left[\mathbb{I}_G\right]_c = \int_{\mathcal{B}} dm\left(\|\bar{r}_{dm/G}\|^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \left[\bar{r}_{dm/G}\right]_c\left[\bar{r}_{dm/G}\right]_c^T\right)$$

Ex Cartesian Coordinates $(x, y, z)_c$

$$\bar{r}_{dm/G} = x\hat{c}_1 + y\hat{c}_2 + z\hat{c}_3$$

$$\|\bar{r}_{dm/G}\|^2 = x^2 + y^2 + z^2$$

$$[\bar{r}_{dm/G}]_c [\bar{r}_{dm/G}]_c^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_c \begin{bmatrix} x & y & z \end{bmatrix}_c$$

$$= \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

$$[\mathbb{I}_G]_c = \rho \int_{\mathcal{B}} dx\,dy\,dz \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

Ex 11.6 Cube $2\ell \times 2\ell \times 2\ell = 8\ell^3$

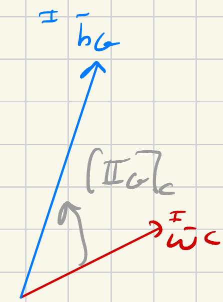
$$[\mathbb{I}_G]_c = \frac{m}{8\ell^3} \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} dx\,dy\,dz \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

$$= \cdots = \frac{2}{3} m l^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

AM of a 3D RB

$$[{}^I\bar{h}_G]_C = [\mathbb{I}_G]_C \, [{}^I\bar{\omega}^C]_C$$

Symmetric matrix



Suppose $[\mathbb{I}_G]_C = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$

$$[{}^I\bar{h}_G]_C = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_C = \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix}_C$$

* key idea: it is always (wlog) possible to choose a body frame in which the MOI matrix is diagonal.

Rotation Dynamics of 3D RB

E2L $\quad \frac{{}^I d}{dt}({}^I\bar{h}_G) = \bar{M}_G$

$$\bar{M}_G = \frac{{}^C d}{dt}({}^I\bar{h}_G) + {}^I\bar{\omega}^C \times {}^I\bar{h}_G$$

let $[\bar{M}_G]_C = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}_C$ and $[\mathbb{I}_G]_C = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}_c = {}^c\frac{d}{dt}\begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix}_c + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix}_c$$

$$\hat{c}_1: \quad \dot{\omega}_1 = \left[(I_2 - I_3)\omega_2\omega_3 + M_1\right]/I_1$$

$$\hat{c}_2: \quad \dot{\omega}_2 = \left[(I_3 - I_1)\omega_1\omega_3 + M_2\right]/I_2$$

$$\hat{c}_3: \quad \dot{\omega}_3 = \left[(I_1 - I_2)\omega_1\omega_2 + M_3\right]/I_3$$

$$\dot{\psi} = \left(\overset{\omega_2}{\cancel{q}}\, s\phi + \overset{\omega_3}{\cancel{r}}\, c\phi\right)\sec\theta$$

$$\dot{\theta} = \overset{\omega_2}{\cancel{q}}\, c\phi - \cancel{r}\, s\phi$$

$$\dot{\phi} = \left(\overset{\omega_2}{\cancel{q}}\, s\phi + \cancel{r}\, c\phi\right)\tan\theta + \overset{\omega_1}{\cancel{p}}$$

3-2-1

Free Rigid Body $\quad \bar{M}_G = 0, \quad \bar{F}_G = 0$

$$\bar{F}_G = {}^I\frac{d}{dt}\left({}^I\bar{P}_{G/0}\right) \Rightarrow {}^I\bar{P}_{G/0} \text{ conserved}$$

$$\bar{M}_G = {}^I\frac{d}{dt}\left({}^I\bar{h}_G\right) \Rightarrow {}^I\bar{h}_G \text{ conserved}$$

$$\left.\begin{aligned} \dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3/I_1 = 0 \\ \dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3/I_2 = 0 \\ \dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2/I_3 = 0 \end{aligned}\right\} \text{@ equilibrium}$$

let $I_3 > I_2 > I_1$

Stable ① $\omega_1 \neq 0,\ \omega_2 = 0,\ \omega_3 = 0$ spinning $\hat{c}_1$

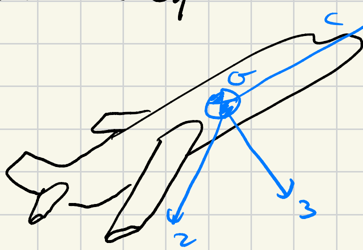
unstable ② $\omega_2 \neq 0,\ \omega_1 = 0,\ \omega_3 = 0$ spinning $\hat{c}_2$

Stable ③ $\omega_3 \neq 0,\ \omega_1 = 0,\ \omega_2 = 0$ Spinning $\hat{c}_3$

Stable ④ $\omega_1 = \omega_2 = \omega_3 = 0$ not rotating

Ⓐ key idea: spin about the intermediate axis in unstable!!

Ex 11.11 Rotational dynamics of an airplane



$$[\mathbb{I}_G]_c = \begin{bmatrix} I_1 & 0 & -I_{13} \\ 0 & I_2 & 0 \\ -I_{13} & 0 & I_3 \end{bmatrix}_c$$

Recall $\bar{M}_G = \frac{{}^I d}{dt}({}^I\bar{h}_G) = \frac{{}^c d}{dt}({}^I\bar{h}_G) + {}^I\bar{\omega}^c \times {}^I\bar{h}_G$

$$[{}^I\bar{h}_G]_c = [\mathbb{I}_G]_c\,[{}^I\bar{\omega}^c]_c$$

$$= \begin{bmatrix} I_1 & 0 & -I_{13} \\ 0 & I_2 & 0 \\ -I_{13} & 0 & I_3 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}_c$$

$$= \begin{bmatrix} I_1 p - I_{13} r \\ I_2 q \\ I_3 r - I_{13} p \end{bmatrix}_c$$

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}_C = \begin{bmatrix} I_1 \dot{p} - I_{13} \dot{r} \\ I_2 \dot{q} \\ I_3 \dot{r} - I_{13} \dot{p} \end{bmatrix} + \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} I_1 p - I_{13} r \\ I_2 q \\ I_3 r - I_{13} p \end{bmatrix}$$

$$\hat{c}_1 : \; M_1 = I_1 \dot{p} - I_{13} \dot{r} + q(I_3 r - I_{13} p) - I_2 q r$$

$$\hat{c}_2 : \; M_2 = I_2 \dot{q} + r(I_1 p - I_{13} r) - p(I_3 r - I_{13} p)$$

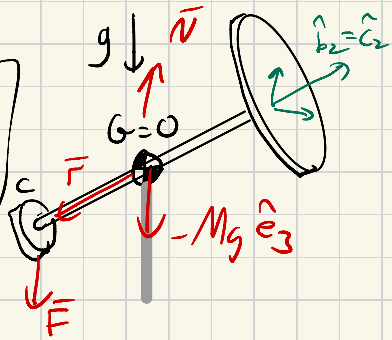
$$\hat{c}_3 : \; M_3 = I_3 \dot{r} - I_{13} \dot{p} + I_2 p q - q(I_1 p - I_{13} r)$$

$$\boxed{\begin{aligned} \dot{p} &= \dots \\ \dot{q} &= \dots \\ \dot{r} &= \dots \end{aligned}}$$

rotational dynamics of an airplane

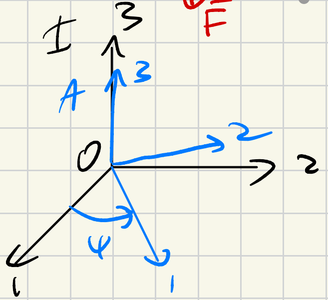
Ex 11.13

$$[\bar{M}_G]_C = \begin{bmatrix} M_1 \\ 0 \\ 0 \end{bmatrix}_C$$



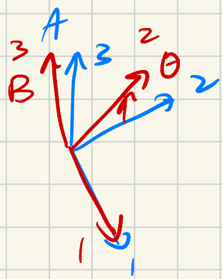
$$\bar{M}_G = \frac{{}^I d}{dt}\left({}^I \bar{h}_G\right)$$

$$[{}^I \bar{h}_G]_C = [\mathbb{I}_G]_C \left({}^I \bar{\omega}^C\right)_C = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_C$$



$${}^I \bar{\omega}^A = \dot{\psi}\, \hat{a}_3$$

$${}^I R^A = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$${}^A\bar{\omega}^B = \dot{\theta}\,\hat{b}_1$$

$${}^AR^B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

$\hat{c}_2 = \hat{b}_2$
spin axis

$C = (G, \hat{c}_1, \hat{c}_2, \hat{c}_3)$ body frame $\quad {}^B\bar{\omega}^C = \Omega\,\hat{b}_2$

$${}^BR^C = \begin{bmatrix} c\phi & 0 & +s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix}$$

$$[\mathbb{I}_G]_B = {}^BR^C\,[\mathbb{I}_G]_C\,\left({}^BR^C\right)^T$$

$$= \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_1 \end{bmatrix}_C \begin{bmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{bmatrix}$$

$$= \begin{bmatrix} c\phi I_1 & 0 & s\phi I_1 \\ 0 & I_2 & 0 \\ -s\phi I_1 & 0 & c\phi I_1 \end{bmatrix} \begin{bmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{bmatrix}$$

$$\checkmark = \begin{bmatrix} \cancelto{I_1}{c^2\phi I_1 + s^2\phi I_1} & 0 & \cancelto{0}{-s\phi c\phi I + s\phi c\phi I_1} \\ 0 & I_2 & 0 \\ \cancelto{0}{-s\phi c\phi I_1 + c\phi s\phi I_1} & 0 & \cancelto{I}{s^2\phi I_1 + c^2\phi I_1} \end{bmatrix}$$

$$\bar{M}_G = \frac{{}^{B}d}{dt}\left({}^{I}\bar{h}_G\right) + {}^{I}\bar{\omega}^{B} \times {}^{I}\bar{h}_G$$

$$\begin{bmatrix} M_1 \\ 0 \\ 0 \end{bmatrix}_B = \frac{{}^{B}d}{dt}\begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_1\omega_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_1\omega_3 \end{bmatrix}$$

$$\hat{b}_1: \quad M_1 = I_1\dot{\omega}_1 + (I_2 - I_1)\omega_2\omega_3$$

$$\hat{b}_2: \quad 0 = I_2\dot{\omega}_2$$

$$\hat{b}_3: \quad 0 = I_1\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2$$

$${}^{I}\bar{\omega}^{C} = {}^{I}\bar{\omega}^{A} + {}^{A}\bar{\omega}^{B} + {}^{B}\bar{\omega}^{C}$$

$$= \dot{\psi}\,\hat{a}_3 + \dot{\theta}\,\hat{b}_1 + \Omega\,\hat{c}_2$$

$$\left[{}^{I}\bar{\omega}^{C}\right]_C = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_C = {}^{C}R^{A}\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}_A + {}^{C}R^{B}\begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}_B + \begin{bmatrix} 0 \\ \Omega \\ 0 \end{bmatrix}_C$$

$${}^{C}R^{A} = {}^{C}R^{B}\,{}^{B}R^{A}$$

$$= \begin{bmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} = \begin{bmatrix} c\phi & s\phi s\theta & -s\phi c\theta \\ 0 & c\theta & s\theta \\ s\phi & -c\phi s\theta & c\phi c\theta \end{bmatrix}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_C = \begin{bmatrix} c\phi & s\phi s\theta & -s\phi c\theta \\ 0 & c\theta & s\theta \\ s\phi & -c\phi s\theta & c\phi c\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}_A + \begin{bmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}_B + \begin{bmatrix} 0 \\ \Omega \\ 0 \end{bmatrix}_C$$

$$\boxed{\begin{aligned} \omega_1 &= -s\phi c\theta\dot{\psi} + c\phi\dot{\theta} \\ \omega_2 &= s\theta\dot{\psi} + \Omega \\ \omega_3 &= c\phi c\theta\dot{\psi} + s\phi\dot{\theta} \end{aligned}} \rightarrow \boxed{\begin{aligned} \dot{\psi} &= \dots \\ \dot{\theta} &= \dots \\ \dot{\phi} &= \Omega \end{aligned}}$$

kinematics

$(3\text{-}1\text{-}2 \;\; \psi, \theta, \phi)^C_I$

Pinned RB

let Q be fixed in C & I, then

Recall Ch 9

$$\frac{{}^I d}{dt}\left({}^I\bar{h}_Q\right) = \bar{M}_Q$$

$$\left[{}^I\bar{h}_Q\right]_C = \left[\mathbb{I}_Q\right]_C \left[{}^I\bar{\omega}^C\right]_C$$

* parallel axis theorem (11.57)

$$\mathbb{I}_G \longrightarrow \mathbb{I}_Q$$

3D RB Work & Energy

Recall Ch 7, 9 $\quad T_O = T_{G/O} + T_G$

$$T_{G/O} = \frac{1}{2} m_G \| {}^I\bar{v}_{G/O} \|^2$$

$$T_G = \frac{1}{2} \mathbb{I}_G \| {}^I\bar{\omega}^B \|^2$$

scalar

planar RB $\quad$ 2D RB

$$T_G = \frac{1}{2}\, {}^{I}\bar{\omega}^{C} \cdot {}^{I}\bar{h}_G$$

3D RB

$$= \frac{1}{2}\left[{}^{I}\bar{\omega}^{C}\right]_C^T \left[{}^{I}\bar{h}_G\right]_C$$

$$= \frac{1}{2}\left[{}^{I}\bar{\omega}^{C}\right]_C^T \left[\mathbb{I}_G\right]_C \left[{}^{I}\bar{\omega}^{C}\right]_C$$

$$= \frac{1}{2}\begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}_C^T \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_C \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_C$$

$$= \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_2^2 \right)$$

WE #1 $\quad T_O(t_2) = T_O(t_1) + W^{(tot)}$

$+$ WE #2 $\quad U_O(t_2) = U_O(t_1) - W^{(c)}$

WE #3 $\quad E_O(t_2) = E_O(t_1) + W^{(nc)}$

☆ E_O is conserved if $W^{(nc)} = \underset{\text{translation}}{W_{G/O}^{(nc)}} + \underset{\text{rotation}}{W_G^{(nc)}} = 0$

Total Energy of Airplane

$$E_O = T_O + U_O$$

$$= T_{G/O} + T_G + U_O$$

$$T_{G/O} = \frac{1}{2} m_G \, \| {}^{I}\bar{v}_{G/O} \|^2$$

$$\left[{}^{I}\bar{v}_{G/O}\right]_C = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C$$

$$\| {}^{I}\bar{v}_{G/O} \|^2 = u^2 + v^2 + w^2$$

$$T_{G/O} = \frac{1}{2} m_G (u^2 + v^2 + w^2)$$

$$T_G = \frac{1}{2} \left[{}^{I}\bar{\omega}^{C}\right]_C^T \left[\mathbb{I}_G\right]_C \left[{}^{I}\bar{\omega}^{C}\right]_C$$

$$= \frac{1}{2} \begin{bmatrix} p & q & r \end{bmatrix}_C \begin{bmatrix} I_1 & 0 & -I_{13} \\ 0 & I_2 & 0 \\ -I_{13} & 0 & I_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} p & q & r \end{bmatrix}_C \begin{bmatrix} I_1 p - I_{13} r \\ I_2 q \\ -I_{13} p + I_3 r \end{bmatrix}$$

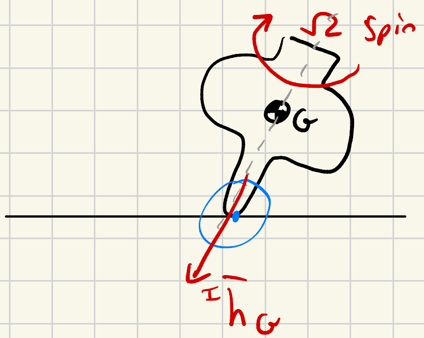
$$T_G = \frac{1}{2} \left[(I_1 p - I_{13} r) p + I_2 q^2 + (-I_{13} p + I_3 r) r \right]$$

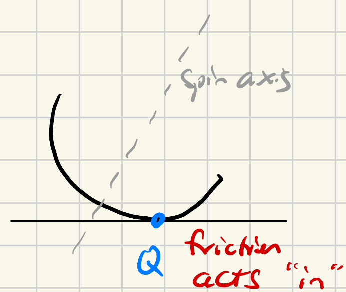
$$U_O = U_O^{(\text{weight})}$$

$$= m_G g h = m g z$$

Topnosis

Ex 11.5





$$\bar{M}_G \approx {}^{I}\bar{\omega}^{B} \times {}^{I}\bar{h}_G$$

$$\bar{r}_{Q/G} \times \bar{F} \qquad \underline{\text{out}} \times \quad \Rightarrow \text{top self-righting}$$