Lecture 23: The Compressible Velocity Potential Equation and Its Linearized Form

ENAE311H Aerodynamics I

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The velocity potential equation

Recall our introduction of the velocity potential, ϕ, for irrotational flows, i.e., those for which

$$\boldsymbol{\xi} = \nabla \times \mathbf{v} = 0.$$

This irrotationality assumption generally holds in inviscid external flows, as long as no strongly curved shocks are present (curved shocks are sources of vorticity).

The velocity field is then given by

$$\mathbf{v} = \nabla\phi.$$

Using this relation, we can write the differential form of the steady continuity equation, i.e.,

$$\rho\nabla \cdot \mathbf{v} + \nabla\rho \cdot \mathbf{v} = 0,$$

as

$$\rho\nabla^2\phi + \nabla\rho \cdot \nabla\phi = 0.$$

In two dimensions, the expanded form of this equation is

$$\boxed{\rho}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \boxed{\frac{\partial\rho}{\partial x}}\frac{\partial\phi}{\partial x} + \boxed{\frac{\partial\rho}{\partial y}}\frac{\partial\phi}{\partial y} = 0.$$

The velocity potential equation

To eliminate ρ and its derivatives, we make use of Euler's equation, which was introduced during our derivation of the Bernoulli equation, i.e.,

$$dp = -\rho V dV = -\frac{1}{2}\rho d(V^2).$$

If the flow is irrotational (as is the case here), this equation holds along any direction in the flowfield. Using the velocity potential, we can write this as

$$dp = -\frac{1}{2}\rho d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right].$$

Now, if we additionally assume that the flow is isentropic, we can write

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2,$$

and the above equation becomes

$$d\rho = -\frac{1}{2}\frac{\rho}{a^2} d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

$$\frac{\partial\rho}{\partial x} = -\frac{1}{2}\frac{\rho}{a^2}\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] \Rightarrow \frac{\partial\rho}{\partial x} = -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x \partial y}\right)$$

$$\frac{\partial\rho}{\partial y} = -\frac{1}{2}\frac{\rho}{a^2}\frac{\partial}{\partial y}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] \Rightarrow \frac{\partial\rho}{\partial y} = -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial y \partial x} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right)$$

The velocity potential equation

Our velocity potential equation then becomes

$$a^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}-\left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2}=0.$$

This equation holds for an inviscid, irrotational, isentropic flow.

To eliminate the factor of a^2, we note that the energy equation for an adiabatic, two-dimensional flow is

$$\frac{a^2}{\gamma-1}+\frac{1}{2}(u^2+v^2)=\frac{a_0^2}{\gamma-1}.$$

This can be written as

$$a^2=a_0^2-\frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right],$$

where the stagnation sound speed, a_0, is constant and can be assumed known.

We thus have an equation purely in terms of ϕ and its derivatives. Note, however, that it is highly nonlinear, and no general solutions are available.

The linearized equation

Linearization is a technique used to transform complex, nonlinear equations into simpler linear ones.

HOWEVER, the price we pay is that any solution we obtain is no longer an exact solution to the original equation, and must not differ too much from a known, exact solution to be even approximately valid.

Take a uniform freestream flow with velocity components

$$\begin{aligned} u &= V_1, \\ v &= 0. \end{aligned}$$

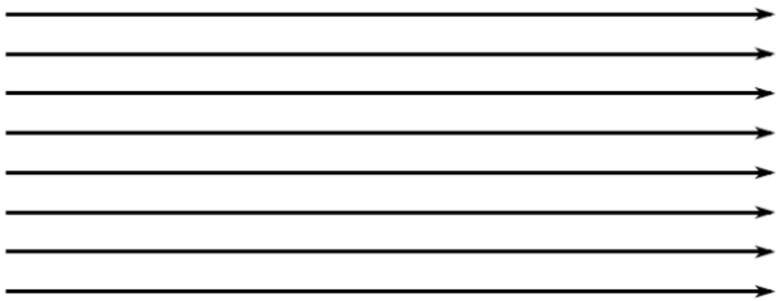
This is an exact solution to the full velocity potential equation.

Now consider the flow over a thin 2D airfoil in such a freestream.

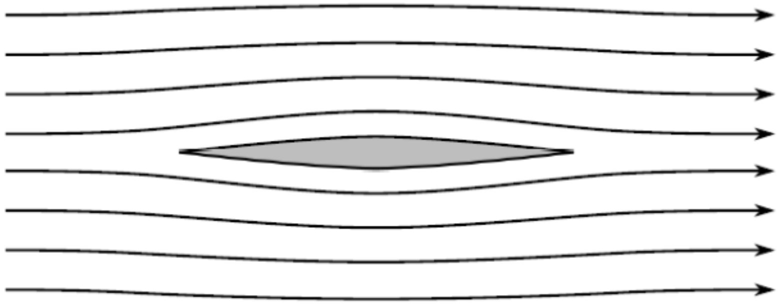
The presence of the airfoil will introduce changes in the two velocity components, i.e.,

$$\begin{aligned} u &= V_1 + \hat{u}, \\ v &= \hat{v}, \end{aligned}$$

but because the airfoil is thin, these “perturbation velocities”, $\hat{u}$ and $\hat{v}$, will be much smaller than V_1 in magnitude, i.e., $\hat{u}, \hat{v} \ll V_1$.



Uniform flow



Perturbed flow

The linearized equation

We can also define a "perturbation velocity potential", $\hat{\phi}$, which in this case takes the form

$$\phi = V_1 x + \hat{\phi}.$$

We then see that

$$\hat{u} \;=\; \frac{\partial \hat{\phi}}{\partial x}, \quad \hat{v} \;=\; \frac{\partial \hat{\phi}}{\partial y}.$$

Substituting into our full potential equation, we obtain

$$a^2 \left(\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} \right) - \left(V_1 + \frac{\partial \hat{\phi}}{\partial x} \right)^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - 2 \left(V_1 + \frac{\partial \hat{\phi}}{\partial x} \right) \frac{\partial \hat{\phi}}{\partial y} \frac{\partial^2 \hat{\phi}}{\partial x \partial y} - \left(\frac{\partial \hat{\phi}}{\partial y} \right)^2 \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0.$$

We can also write this equation in terms of perturbation velocities:

$$a^2 \left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} \right) - (V_1 + \hat{u})^2 \frac{\partial \hat{u}}{\partial x} - 2 (V_1 + \hat{u}) \, \hat{v} \frac{\partial \hat{u}}{\partial y} - \hat{v}^2 \frac{\partial \hat{v}}{\partial y} = 0.$$

The sound speed can be written as

$$a^2 = a_1^2 - \frac{\gamma - 1}{2} (2\hat{u} V_1 + \hat{u}^2 + \hat{v}^2) \qquad \text{(since } a_0^2 = a_1^2 + \frac{\gamma - 1}{2} V_1^2 \text{)}$$

The linearized equation

Substituting

$$a^2 = a_1^2 - \frac{\gamma - 1}{2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2)$$

into

$$a^2\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y}\right) - (V_1 + \hat{u})^2\frac{\partial \hat{u}}{\partial x} - 2\,(V_1 + \hat{u})\,\hat{v}\frac{\partial \hat{u}}{\partial y} - \hat{v}^2\frac{\partial \hat{v}}{\partial y} = 0$$

dividing through by a_1^2, and rearranging, we arrive at a complicated equation containing a large number of terms that involve either $\hat{u}/V_1$ or $\hat{v}/V_1$ (see Anderson, equation 11.16).

Provided the freestream Mach number is neither too large (i.e., $M_1 \lesssim 5$) nor too close to unity (i.e., in the transonic range between approximately 0.8 and 1.2), we can neglect all terms containing $\hat{u}/V_1$ or $\hat{v}/V_1$ to arrive at

$$(1 - M_1^2)\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0.$$

or

$$\boxed{(1 - M_1^2)\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0.}$$

This is the *linearized* velocity potential equation, which is much simpler than the original equation.