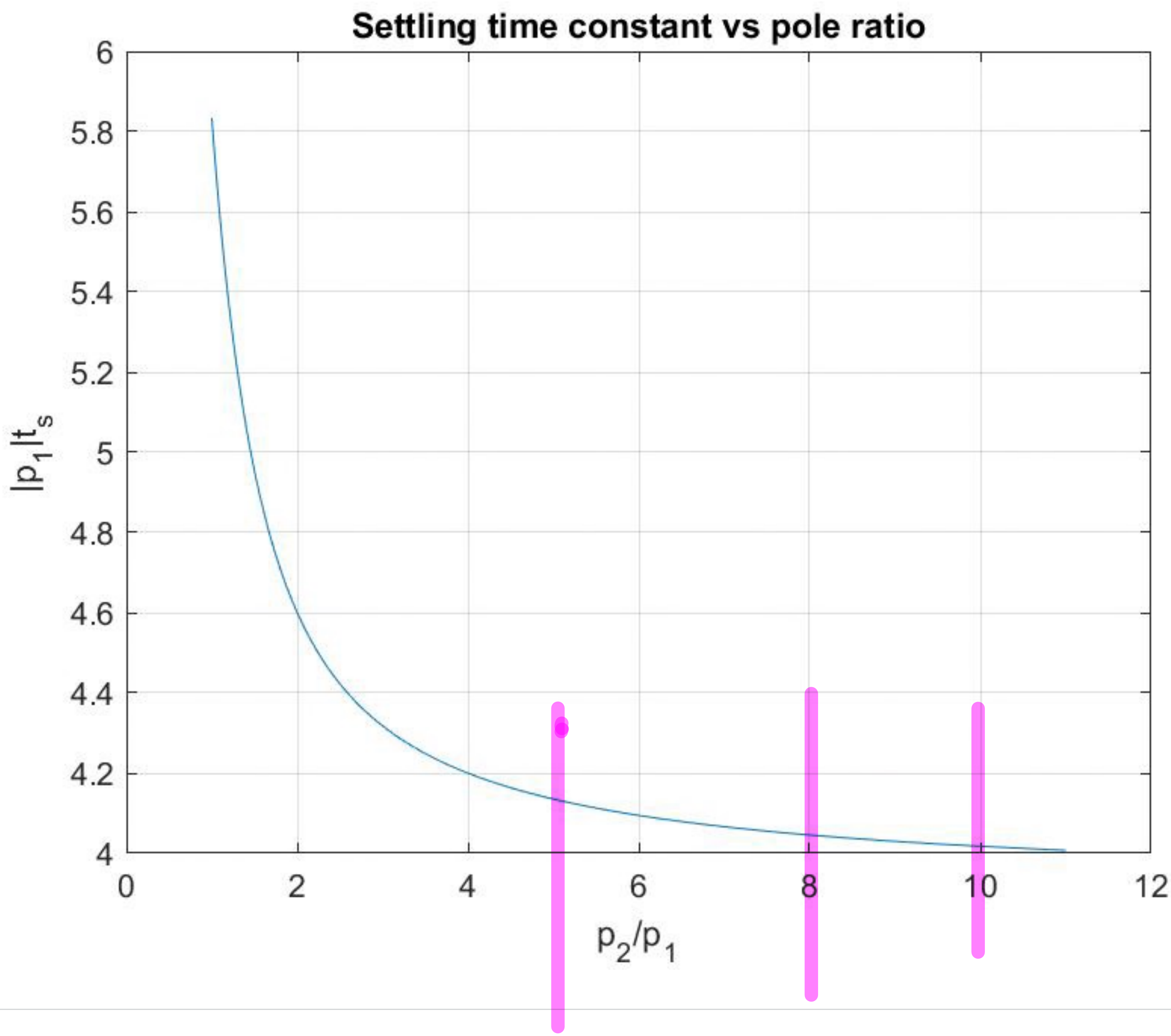
2 real poles, $C\left(\frac{p_2}{p_1}\right)$ $\quad t_s = \frac{C(p_2/p_1)}{|p_1|}$



"2nd Order" Step Responses

$$\ddot{y}(t)+\alpha_1\dot{y}(t)+\alpha_0 y(t)=\beta_0 u(t) \implies G(s)=\frac{\beta_0}{s^2+\alpha_1 s+\alpha_0}$$

2 poles, both stable if $\alpha_1>0, \alpha_0>0$.

3 possibilities for poles:

1. $\alpha_1^2<4\alpha_0 \implies p_1, p_2$ complex conjugates
2. $\alpha_1^2=4\alpha_0 \implies p_1=p_2$ repeated real
3. $\alpha_1^2>4\alpha_0 \implies p_1, p_2$ real, non-repeated

Case 1. is most interesting (and complicated) tackle this after the other two

Useful Observation (Case 1)

$p_1 = \sigma + j\omega_d \qquad \omega_d = \mathrm{Im}\{p_1\}$

Note slight change of notation! $\omega \to \omega_d$

$$s^2 + \alpha_1 s + \alpha_0 = (s - p_1)(s - \overline{p}_1)$$

$$= s^2 - (p_1 + \overline{p}_1)s + p_1\overline{p}_1$$

$$= s^2 - 2\mathrm{Re}\{p_1\}s + |p_1|^2$$

$$= s^2 - 2\sigma s + (\sigma^2 + \omega_d^2)$$

Hence:

$$\boxed{\begin{aligned} \alpha_1 &= -2\sigma = -2\mathrm{Re}\{p_1\} \\ \alpha_0 &= \sigma^2 + \omega_d^2 = |p_1|^2 \end{aligned}}$$

Rapidly identify pole location from coefs.

2^{nd} Order Response, Case 1:

$$Y(s) = \frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)} = \frac{A_1}{s} + \frac{A_2}{(s-p_1)} + \frac{\bar{A}_2}{(s-\bar{p}_1)}$$

$$A_1 = \left[sY(s)\right]_{s=0} = \frac{\beta_0}{p_1\bar{p}_1} = \frac{\beta_0}{\alpha_0} = G(0)$$

$$A_2 = \left[(s-p_1)Y(s)\right]_{s=p_1} = \frac{\beta_0}{p_1(p_1-\bar{p}_1)} = \frac{\beta_0}{(\sigma+j\omega_d)(2j\omega_d)}$$

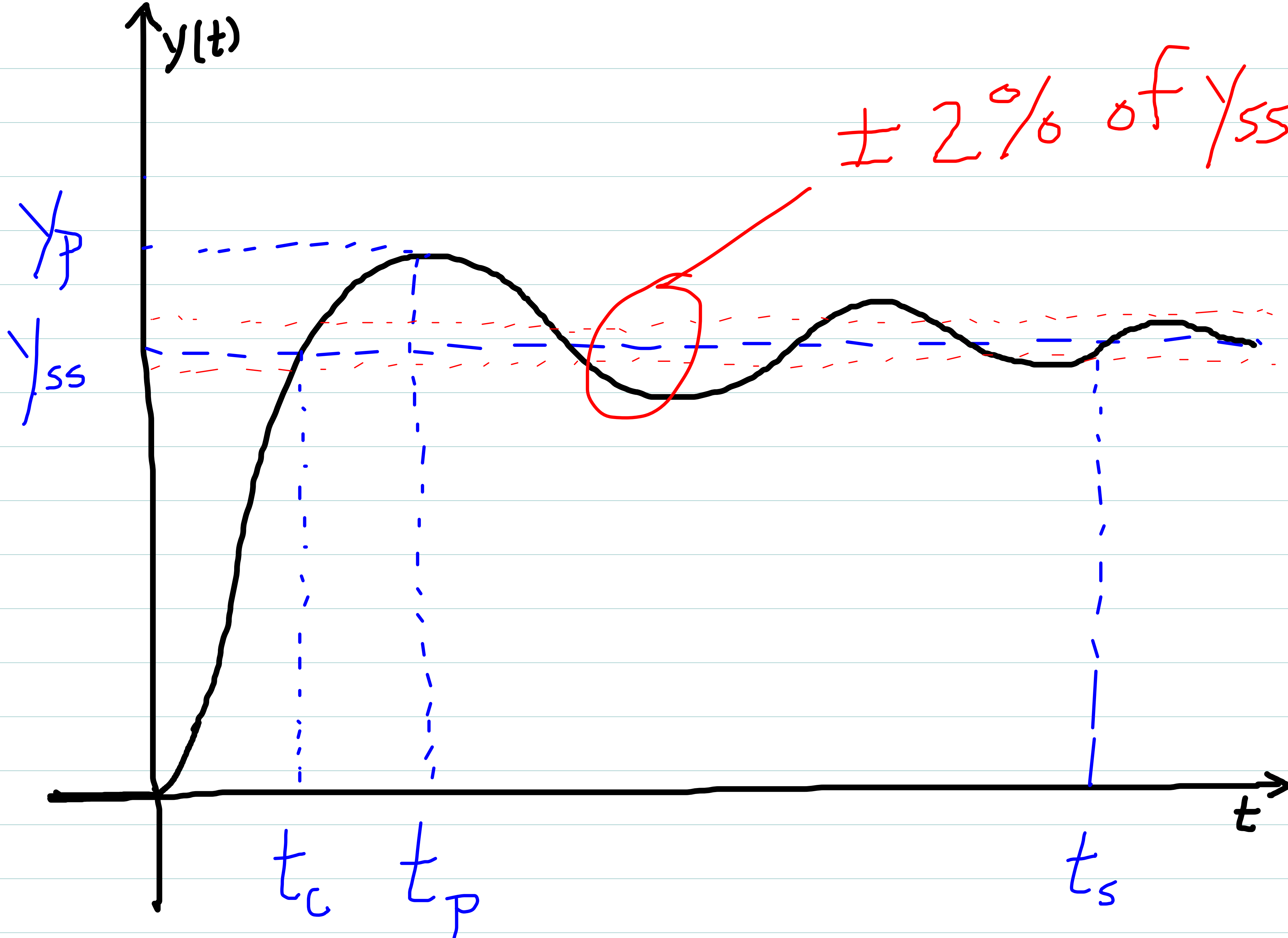
$$= \underbrace{\left(\frac{\beta_0}{2\alpha_0}\right)}_{\frac{1}{2}G(0)} \underbrace{\left(\frac{\alpha_0}{(\sigma+j\omega_d)(j\omega_d)}\right)}_{B}$$

SO:

$$y(t) = G(0) + 2|A_2|\, e^{\sigma t}\cos(\omega_d t + \measuredangle A_2)$$

OR:

$$\boxed{y(t) = G(0)\left[1 + |B|\, e^{\sigma t}\cos(\omega_d t + \measuredangle B)\right]}$$



y(t)
± 2% of y_{ss}
y_p
y_{ss}
t_c
t_p
t_s
t

General Observations

1. $y(t)$ continually oscillates about its steady-state value $y_{ss} = G(\phi)$
2. t_c = time steady-state is first crossed
3. 1^{st} oscillation is largest, and creates an initial overshoot past the steady-state.
4. This initial overshoot has peak value y_p, and occurs at time t_p
5. Settling time t_s defined where response enters ±2% tolerance band and remains within it for times thereafter

Must learn to rapidly quantify these!!

$$y(t) = G(0)\left[1 + |B| e^{\sigma t} \cos(\omega_d t + \measuredangle B)\right]$$

where:

$$B = \frac{\alpha_0}{(j\omega_d)(\sigma + j\omega_d)} = \frac{|P_1|^2}{(j\omega_d)\, P_1}$$

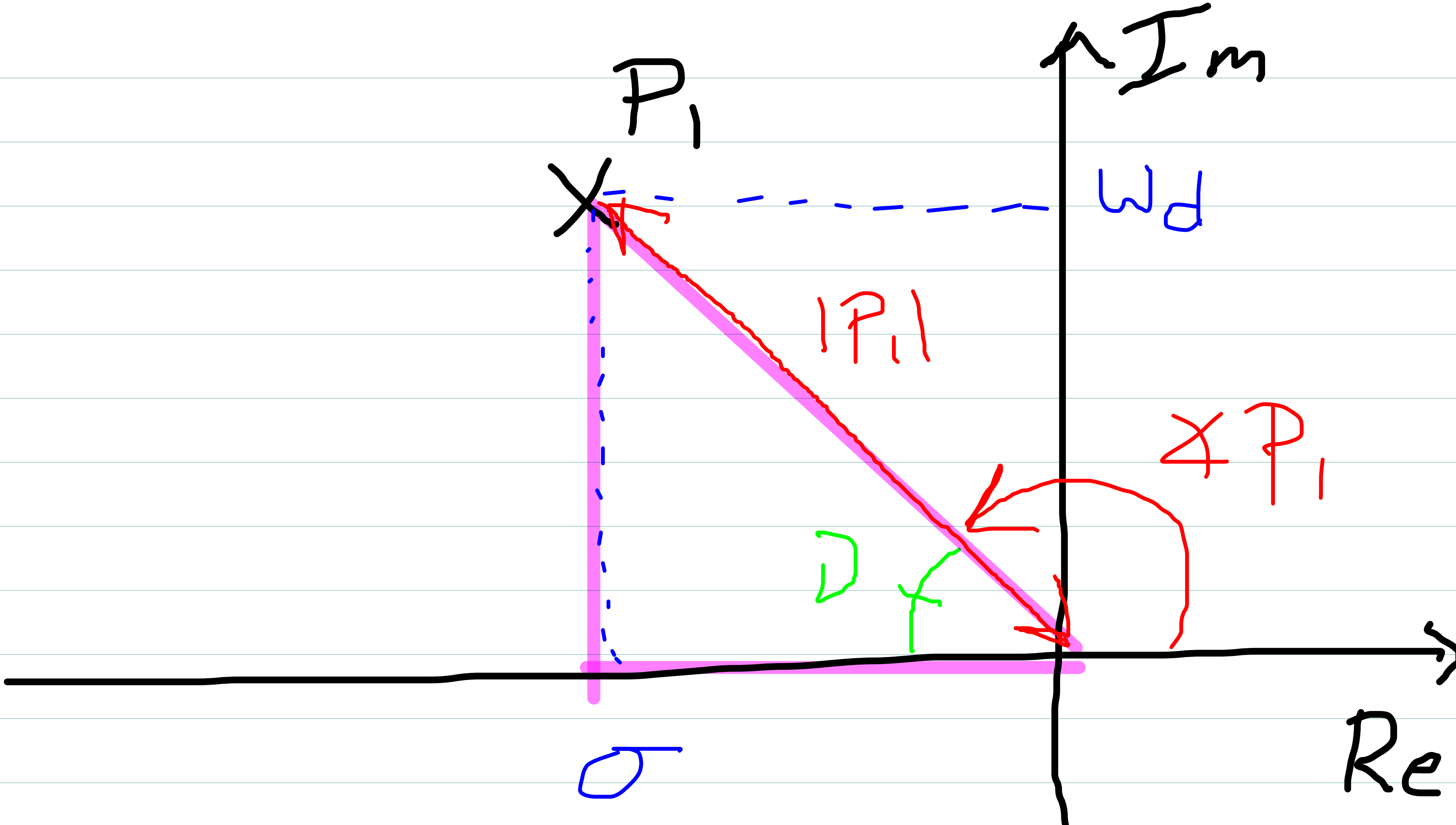
$\Rightarrow$ Transient features completely determined by location of pole $P_1 = \sigma + j\omega_d$ in complex plane

$$|B| = \frac{|P_1|^2}{|j\omega_d| \cdot |P_1|} = \frac{|P_1|}{\omega_d}$$

$$\measuredangle B = \cancelto{0}{\measuredangle |P_1|^2} - \left(\measuredangle (j\omega_d) + \measuredangle P_1\right)$$

$$= -\left(\frac{\pi}{2} + \measuredangle P_1\right)$$

must quantify this!



$\angle P_1 = \pi - \nu$

Note: $\nu \to \phi$ is supplement of $\angle P_1$

So:

$$\measuredangle B = -\left(\frac{\pi}{2} + \measuredangle P_1\right) = -\left(\frac{\pi}{2} + (\pi - \nu)\right)$$

$$= -\frac{3\pi}{2} + \nu$$

and thus:

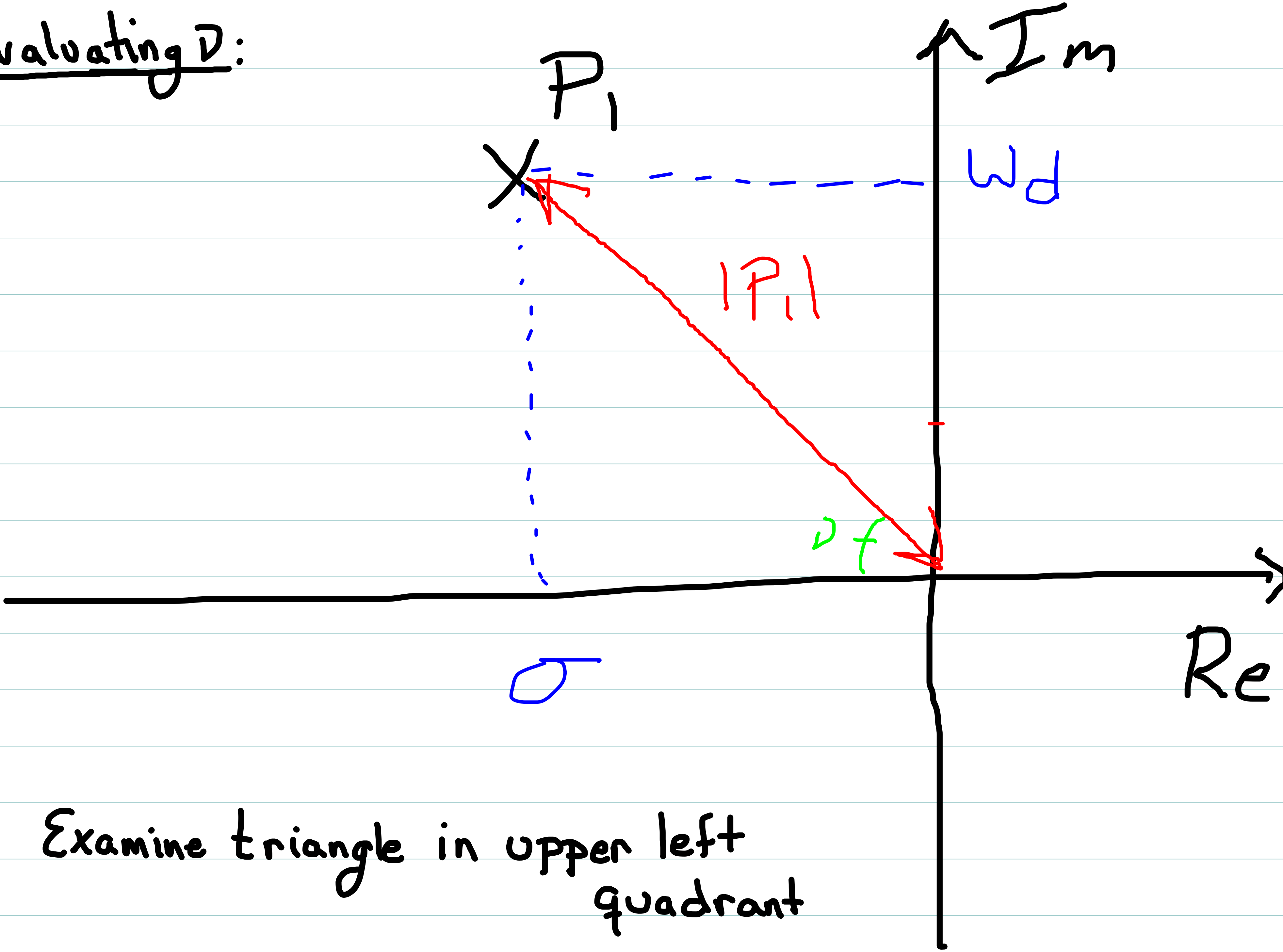
$$y(t) = G(0)\left[1 + \left(\frac{|P_1|}{\omega_d}\right) e^{\sigma t} \cos\left(\omega_d t - \frac{3\pi}{2} + \nu\right)\right]$$

so:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{|P_1|}{\omega_d}\right) e^{\sigma t} \sin(\omega_d t + \nu)\right]}$$

Need to understand how ν depends on P_1

Evaluating D:



Examine triangle in upper left quadrant

Two Useful Parameters

<u>Define:</u> $\omega_n = |p_1| = \sqrt{\sigma^2 + \omega_d^2}$ "natural" frequency

$\Rightarrow$ purely theoretical! ω_d is physical frequency of transient oscillations

<u>Define:</u> $\xi = \dfrac{|\sigma|}{\omega_n} = \dfrac{|\sigma|}{\sqrt{\sigma^2 + \omega_d^2}}$

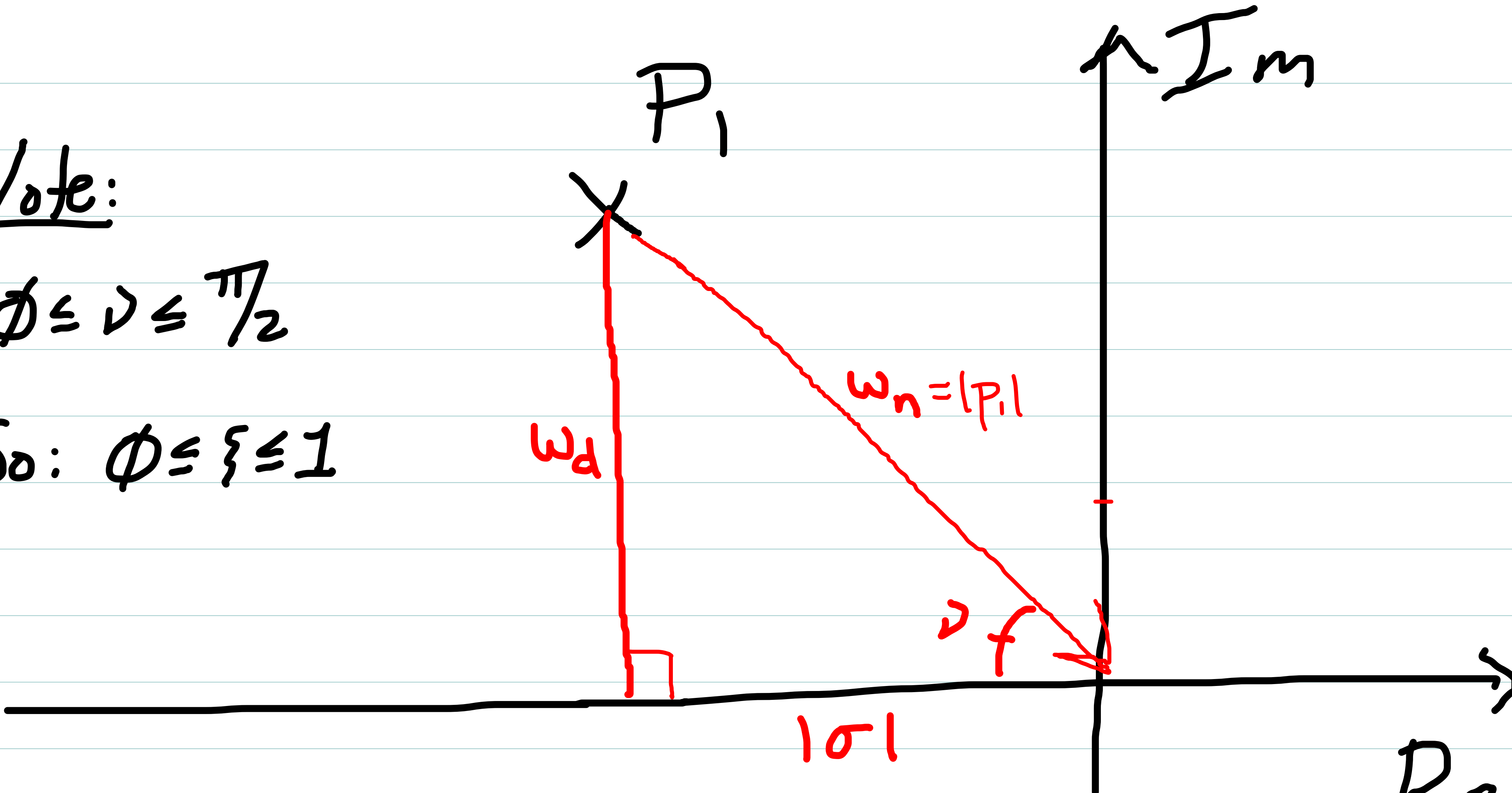
"Damping ratio"

$\Rightarrow$ A normalized measure of the number of transient oscillations observed before amplitude becomes negligible

Note:

$0 \le \nu \le \pi/2$

So: $0 \le \zeta \le 1$



$$\nu = \tan^{-1}\left(\frac{\omega_d}{|\sigma|}\right)$$

$$\nu = \sin^{-1}\left(\frac{\omega_d}{\omega_n}\right)$$

$$\nu = \cos^{-1}\left(\frac{|\sigma|}{\omega_n}\right) = \cos^{-1}\zeta$$ $\Leftarrow$ very useful!

Thus finally, the Case 1 step response is:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{\omega_n}{\omega_d}\right) e^{\sigma t} \sin(\omega_d t + \cos^{-1}\zeta)\right]}$$

We can now solve for important transient parameters

$\Rightarrow$ t_c: Solve for first $t > 0$ such that

$$y(t) = y_{ss}(t) = G(0)$$

$$\Rightarrow \sin(\omega_d t + \cos^{-1}\zeta) = 0$$

$$\Rightarrow t_c = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$$

or:

$$\boxed{t_c = \frac{\pi - \rho}{\omega_d}}$$

$\Rightarrow$ For t_p, y_p

Solve for first $t>0$ such that

$$\dot{y}(t)=0$$

$$\Rightarrow \boxed{t_p = \frac{\pi}{\omega_d}}$$

Substituting:

$$y_p = y(t_p) = G(0)\left[1 + e^{(\sigma\pi/\omega_d)}\right]$$

<u>Define:</u>

$$\boxed{\begin{aligned} M_p &= e^{(\sigma\pi/\omega_d)} \\ y_p &= G(0)\left[1+M_p\right] \end{aligned}}$$

then:

Peak Overshoot

$\Rightarrow$ M_p is the Normalized peak overshoot

$$y_p = G(0)[1+M_p] \Longrightarrow M_p = \frac{y_p - G(0)}{G(0)} = \frac{y_p - y_{ss}}{y_{ss}}$$

$\Rightarrow$ M_p is entirely determined by damping ratio ξ

$$M_p = \exp\left[\frac{\sigma\pi}{\omega_d}\right]$$

$$= \exp\left[\frac{(-\xi\omega_n)\pi}{\omega_n\sqrt{1-\xi^2}}\right]$$

OR

$$\boxed{M_p = \exp\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right]}$$

$$\%OS = 100 \times M_p$$

Summary: Case I step response; $p_1 = \sigma + j\omega_d$

"Natural" frequency: $\omega_n = \sqrt{\sigma^2 + \omega_d^2} = |p_1|$

Damping ratio: $\xi = \dfrac{|\sigma|}{\omega_n}$

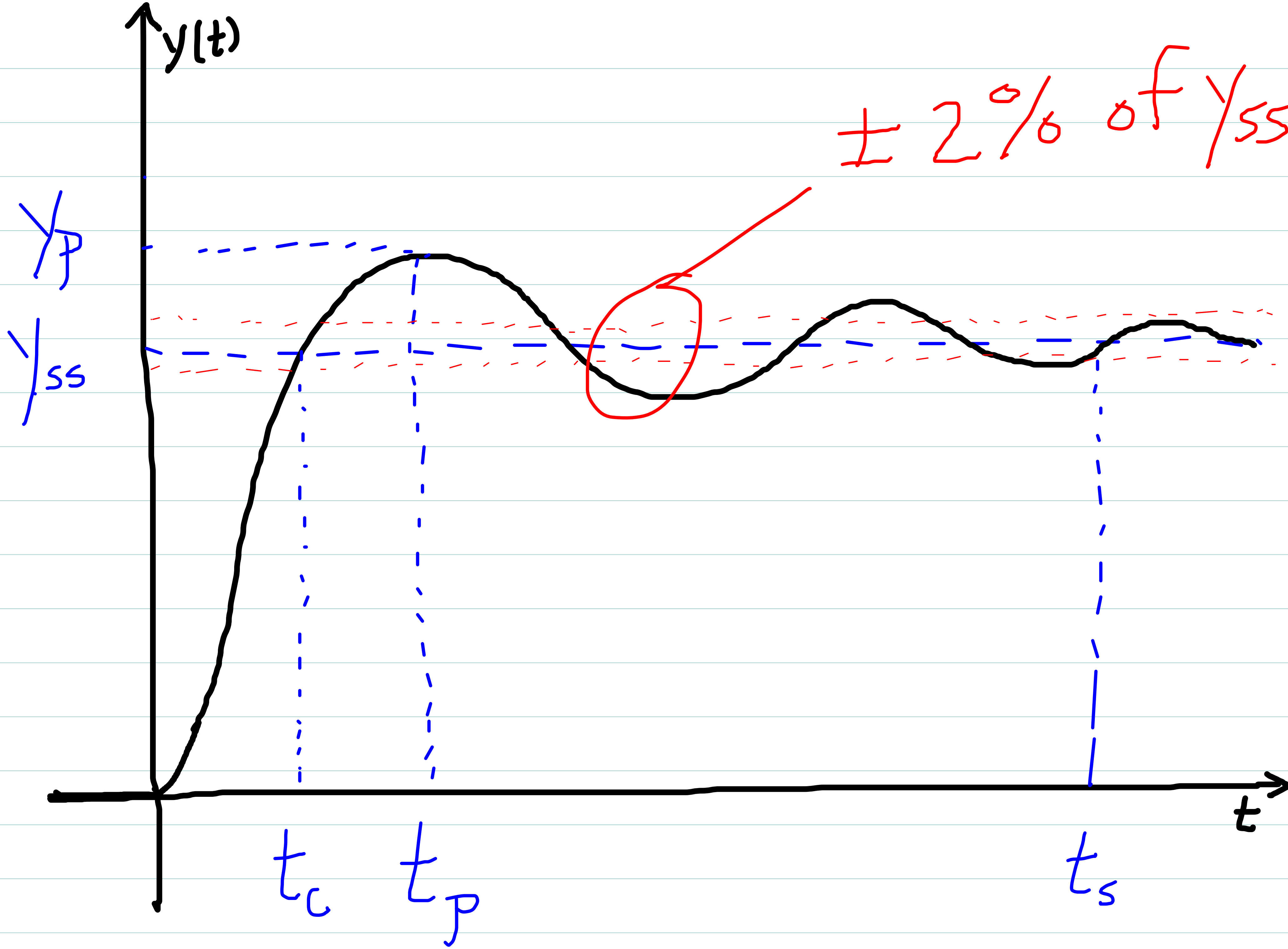
1st crossing: $t_c = \dfrac{\pi - \cos^{-1}\xi}{\omega_d} = \dfrac{\pi - \nu}{\omega_d}$, $\quad \xi = \cos\nu$

1st peak: $t_p = \dfrac{\pi}{\omega_d}$

Normalized overshoot: $M_p = \exp\left[\dfrac{\sigma\pi}{\omega_d}\right] = \exp\left[\dfrac{-\xi\pi}{\sqrt{1-\xi^2}}\right]$

$$M_p = \left[\frac{y_p - y_{ss}}{y_{ss}}\right]$$

Peak response: $y_p = y_{ss}[1 + M_p]$ $\quad y_{ss} = G(0)$ for unit step



y(t)
y_p
y_{ss}
± 2% of y_{ss}
t
t_c
t_p
t_s

Settling Time

As usual, we can use the approximation

$$\boxed{t_s \approx \frac{4}{|Re\{p_i\}|} = \frac{4}{|\sigma|}}$$

But t_s is actually a function of ξ also here:

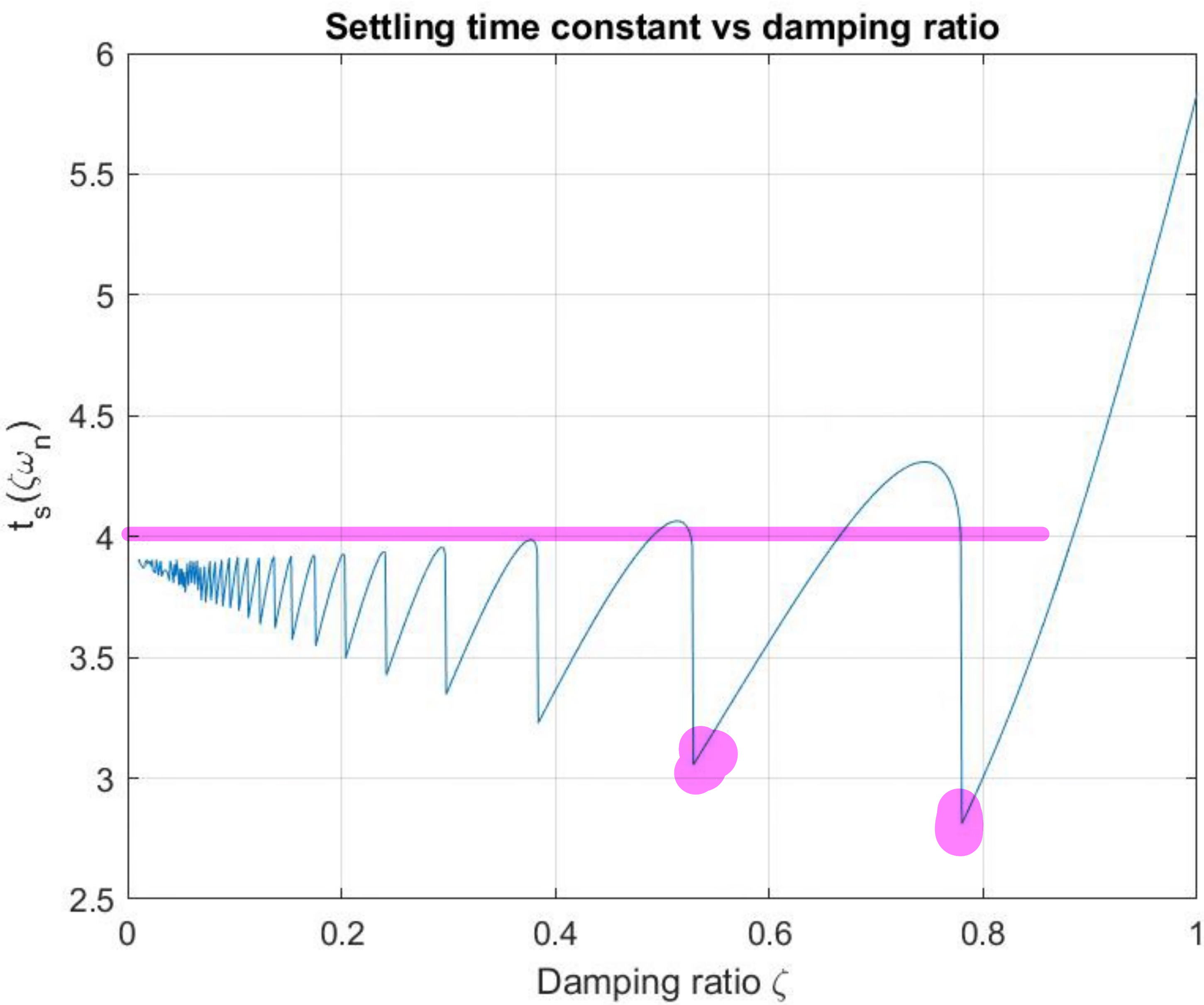
$$t_s = \frac{C(\xi)}{|\sigma|}$$

with $3 \leq C(\xi) \leq 5$ for most $0 \leq \xi < 0.9$

so 4 is an "average" value for $C(\xi)$

However for $0.95 \leq \xi \leq 1$ a better approximation is: $t_s \approx \dfrac{6}{|\sigma|}$

Complex Poles - $C(\xi)$ $t_s = \frac{C(\xi)}{\xi \omega_n}$



A few more observations

$$\zeta = \frac{|\sigma|}{\omega_n} \implies \boxed{\sigma = -\zeta\omega_n}$$

(Stable system Assumed)

$$\omega_n = \sqrt{\sigma^2 + \omega_d^2}$$

$$\implies \omega_d^2 = \omega_n^2 - \sigma^2 = \omega_n^2 - (-\zeta\omega_n)^2 = \omega_n^2(1-\zeta^2)$$

so: $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$

Then note:

$$\boxed{\begin{aligned} s^2 + \alpha_1 s + \alpha_0 &= (s-p)(s-\bar{p}) \\ &= s^2 - 2\sigma s + (\sigma^2 + \omega_d^2) \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \end{aligned}}$$

all equivalent $\Longleftarrow$

Note:

The three possible cases for 2^{nd} order step responses can be categorized by ξ:

<u>Case 1</u> (complex poles): $0 \leq \xi < 1$

$$\alpha_1^2 < 4\alpha_0 \Rightarrow (2\xi\omega_n)^2 < 4\omega_n^2 \quad ✓$$

<u>Case 2</u> (repeated real poles): $\xi = 1$

$$\alpha_1^2 = 4\alpha_0 \Rightarrow (2\overset{1}{\cancel{\xi}}\omega_n)^2 = 4\omega_n^2 \quad ✓$$

<u>Case 3</u> (distinct real poles): $\xi > 1$

$$\alpha_1^2 > 4\alpha_0 \Rightarrow 4\xi^2\omega_n^2 > 4\omega_n^2 \quad ✓$$

Note:

The three possible cases for 2^{nd} order step responses can be categorized by ξ:

Case 1 (complex poles): $0 \le \xi < 1$ "underdamped"

$$\alpha_1^2 < 4\alpha_0 \Rightarrow (2\xi\omega_n)^2 < 4\omega_n^2 \checkmark$$

Case 2 (repeated real poles): $\xi = 1$ "critically damped"

$$\alpha_1^2 = 4\alpha_0 \Rightarrow (2\cancel{\xi}^{\,1}\omega_n)^2 = 4\omega_n^2 \checkmark$$

Case 3 (distinct real poles): $\xi > 1$ "overdamped"

$$\alpha_1^2 > 4\alpha_0 \Rightarrow 4\xi^2\omega_n^2 > 4\omega_n^2 \checkmark$$

Limiting case: $\xi \to 0$

$\xi \to 0 \Rightarrow \sigma = -\xi\omega_n \to 0 \Rightarrow P_1 = j\omega_d$ (pure imaginary)

Overshoot $M_p = e^{(\sigma\pi/\omega_d)} \to 1$ (100% OS)

Peak: $y_p = G(0)[1 + M_p]$

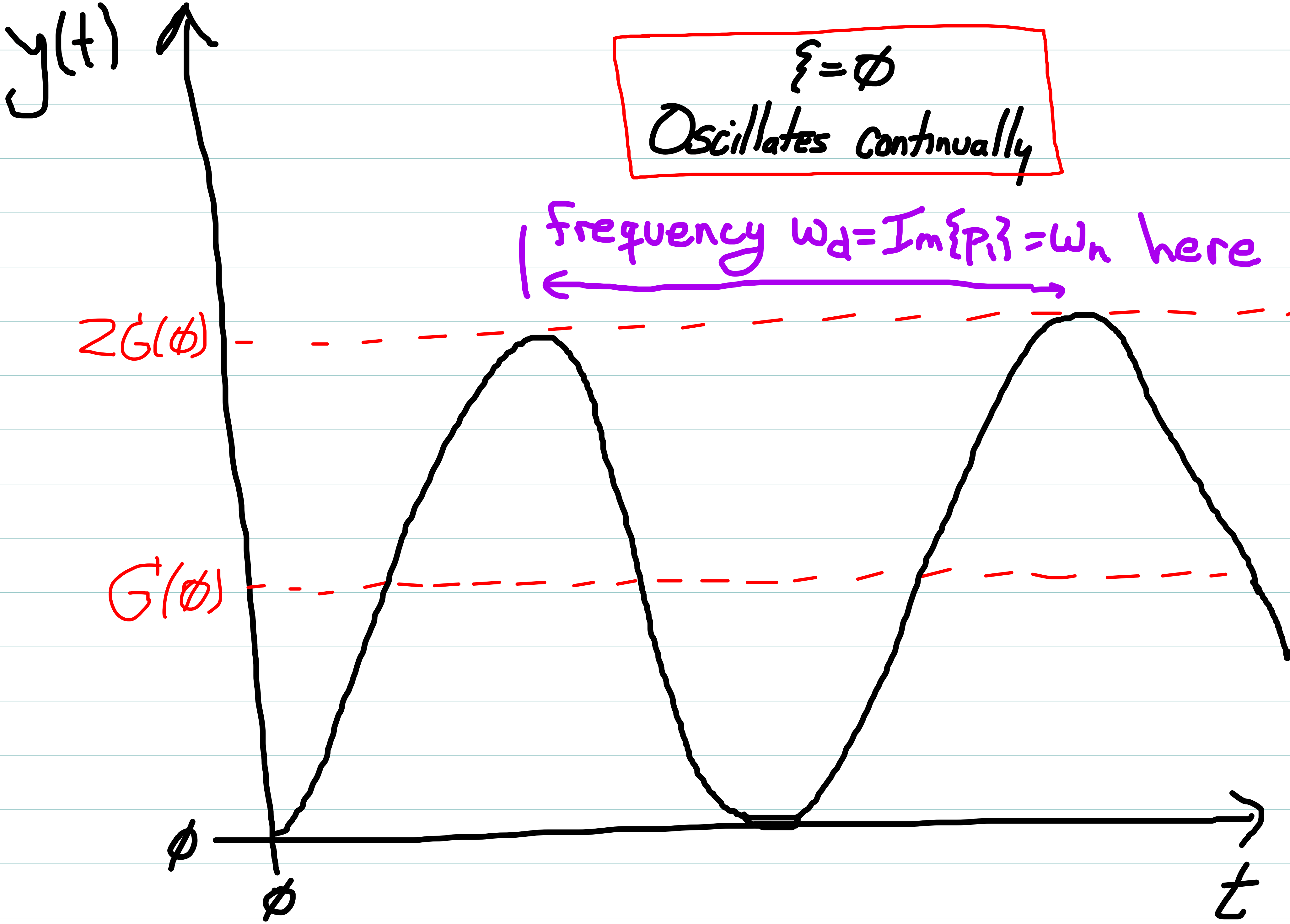
or $y_p = 2y_{ss}$

Settling time: $t_s \approx \dfrac{4}{|\sigma|} = \infty$

Never settles!

Response oscillates infinitely between 0 and $2G(0)$ with frequency $\omega_d = \omega_n\sqrt{1-\xi^2} = \omega_n$

"Undamped"


y(t)
ξ = 0
Oscillates continually
frequency ωd = Im{pi} = ωn here
2G(0)
G(0)
0
0
t

Limiting Case, $\zeta \to 1$

$\zeta \to 1 \Rightarrow \sigma = -\zeta\omega_n \to -\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \to \emptyset$$

Response does not oscillate!

Overshoot:

$$M_p = e^{(\sigma\pi/\omega_d)} = e^{-\omega_n\pi/\emptyset} = \emptyset$$

No overshoot

1st crossing:

$$t_c = \frac{\pi - \cos^{-1}\zeta}{\omega_d} = \frac{\pi/2}{\emptyset} = \infty$$

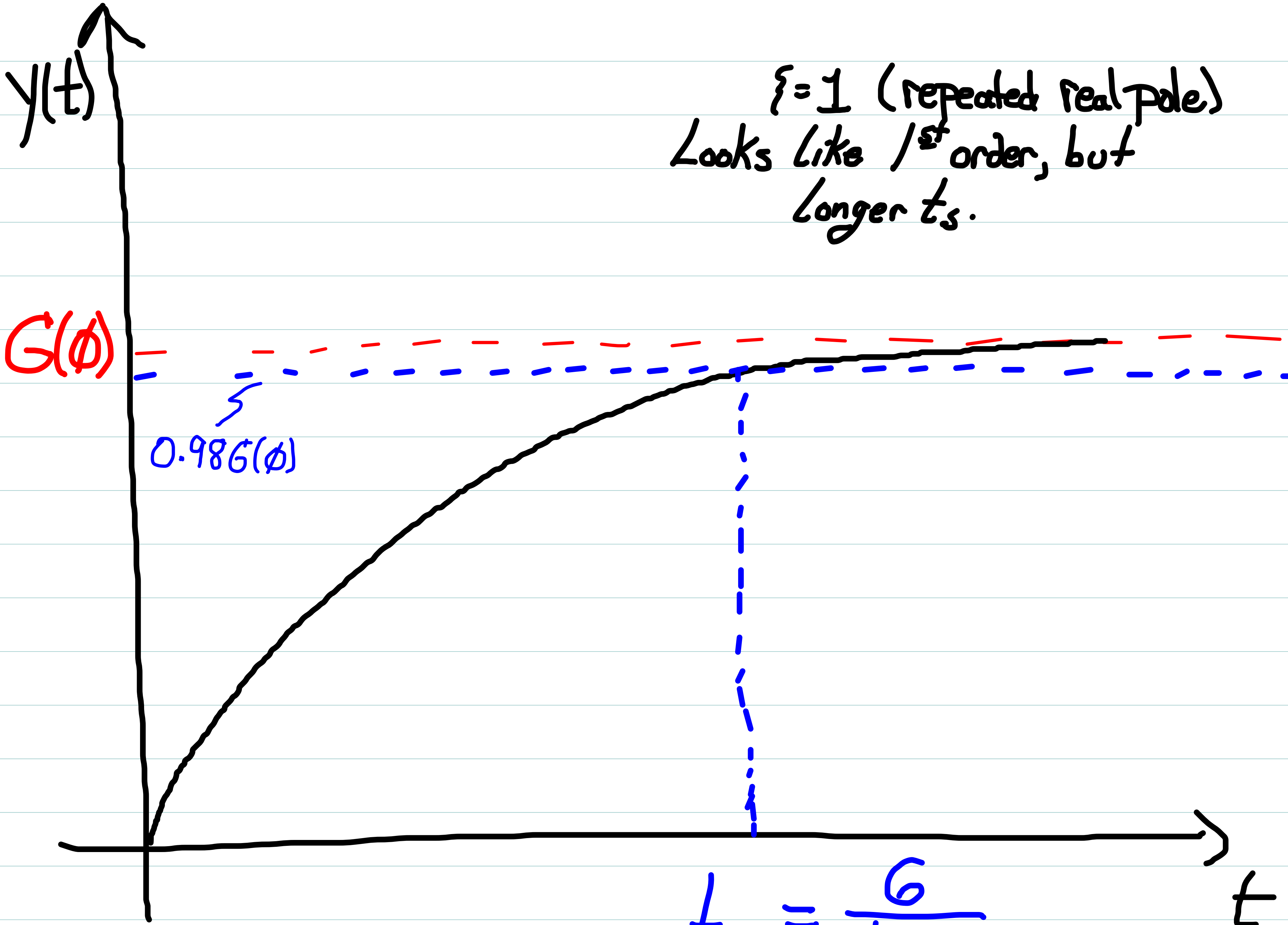
$\Rightarrow$ response asymptotes to y_{ss} from below

Settling:

$$t_s \approx \frac{6}{|\sigma|}$$

use 6 here

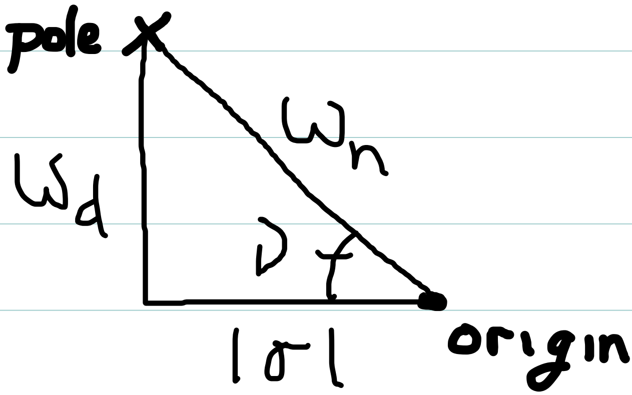
$\zeta = 1$ (repeated real pole)
Looks like 1^{st} order, but longer t_s.



$$t_s = \frac{6}{|\sigma|}$$

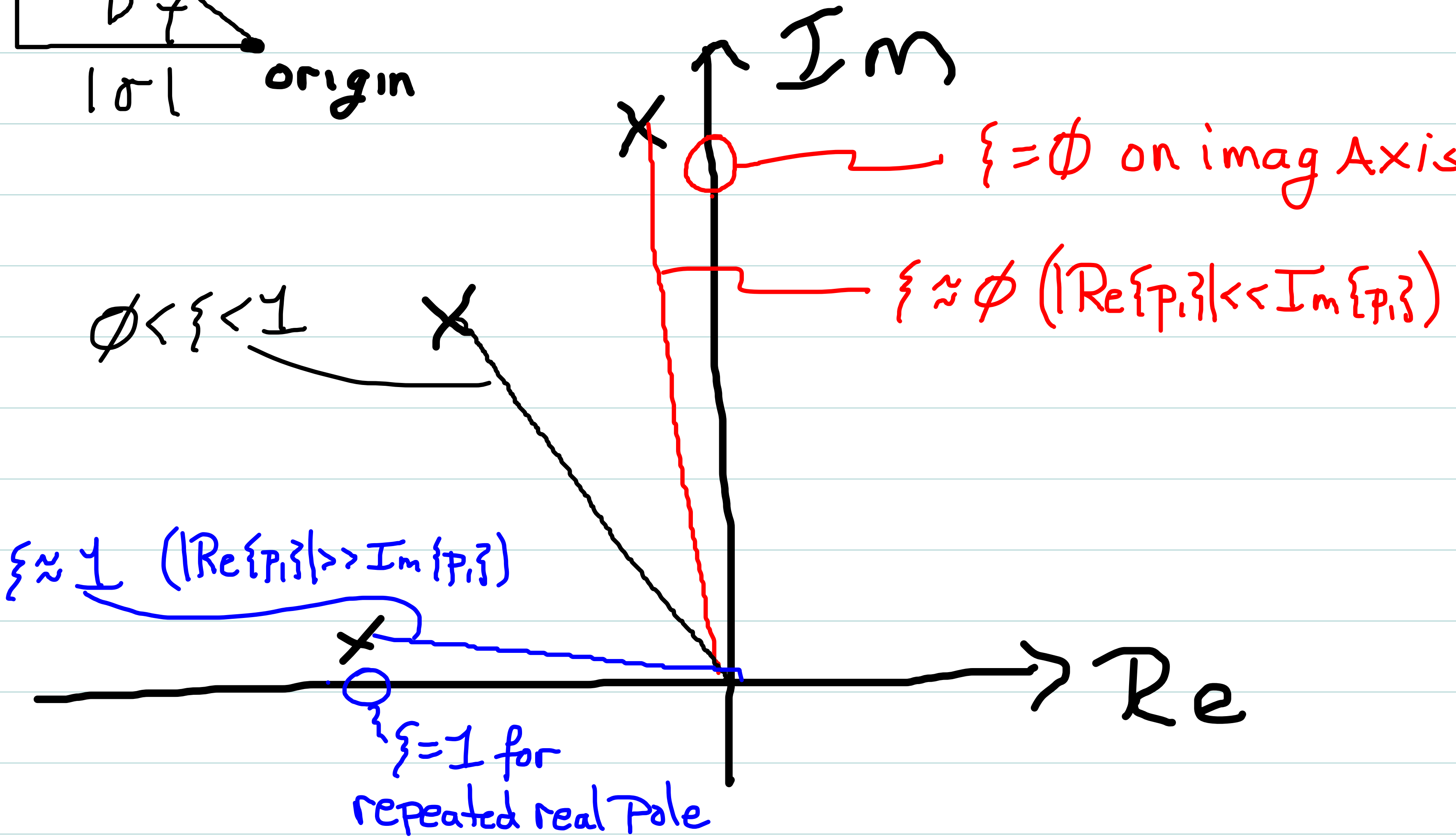
Graphical Interpretation of ξ:

$$\boxed{\xi = \cos\nu :}$$

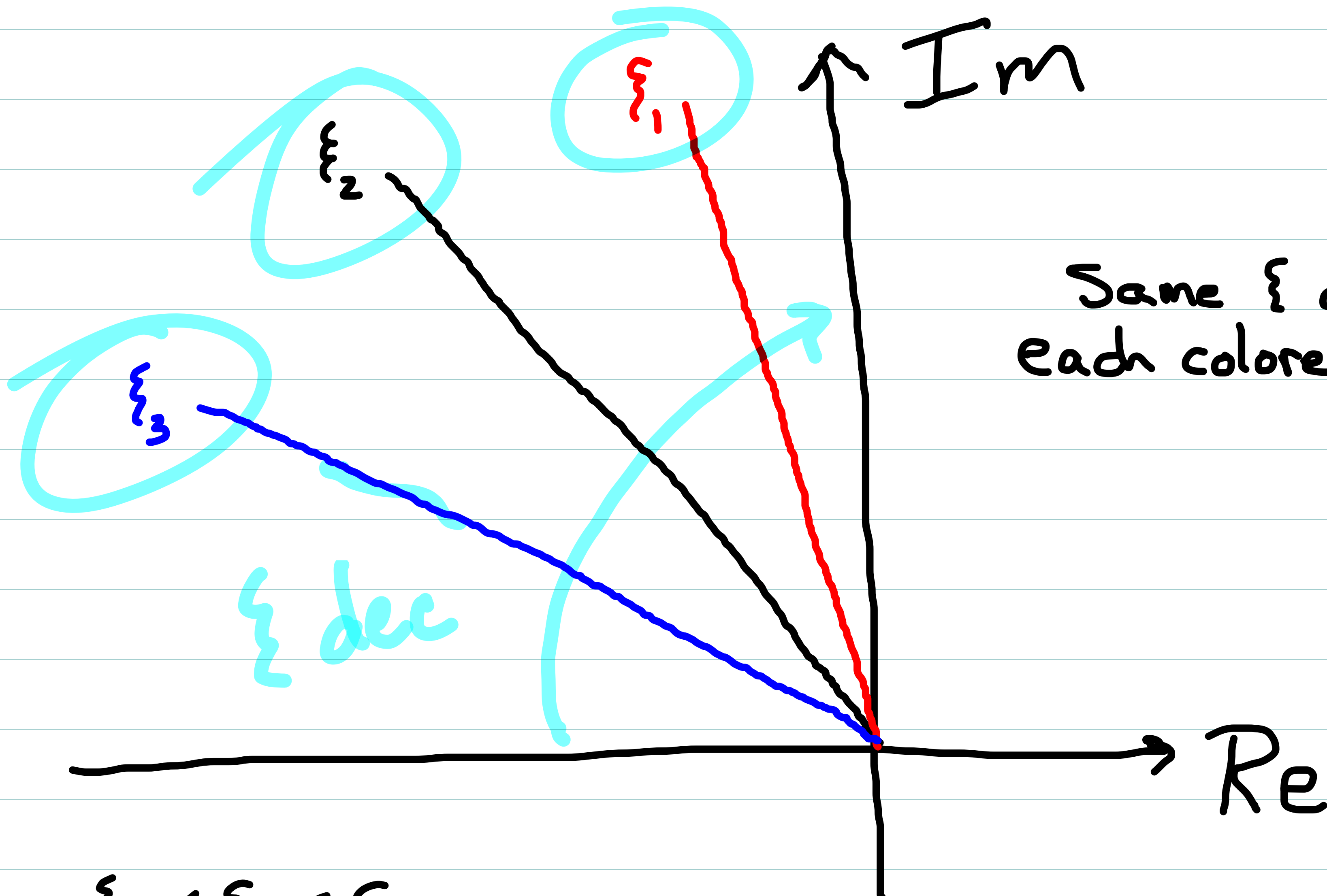


$$\xi \to 0 \Rightarrow \nu \to \pi/2$$

$$\xi \to 1 \Rightarrow \nu \to 0$$



Lines of constant ξ lie on rays in upper left quadrant of complex plane:



Same ξ on each colored line

$\xi_1 < \xi_2 < \xi_3$

⇒ 1^{st} and 2^{nd} order step responses are "building blocks" by which we can understand response of more complex systems

⇒ each real pole introduces a new decaying exponential into transient response.

⇒ each complex pole pair introduces a decaying oscillation into the transient

⇒ An arbitrary number of poles of different types will typically require numerical simulation to quantify y_p, t_r, t_p, t_s

⇒ <u>However</u> in some cases we can still accurately predict these features.

Suppose:

$$G(s) = \frac{K}{(s-p_1)(s^2+2\zeta\omega_n s+\omega_n^2)} \quad \text{with } \zeta<1$$

$$= \frac{K}{(s-p_1)(s-p_2)(s-\bar{p}_2)}$$

for a unit step input $u(t) = \mathbb{1}(t)$ we know

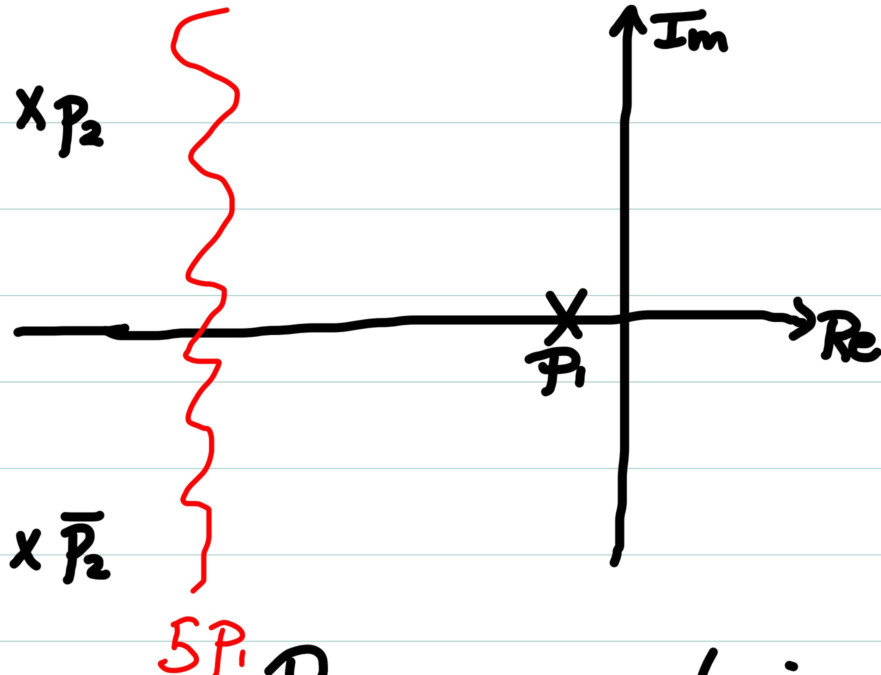
$$y_{ss} = G(0) = \frac{K}{-\omega_n^2 p_1}$$

But what can we say about y_p, t_p, t_c, t_s?

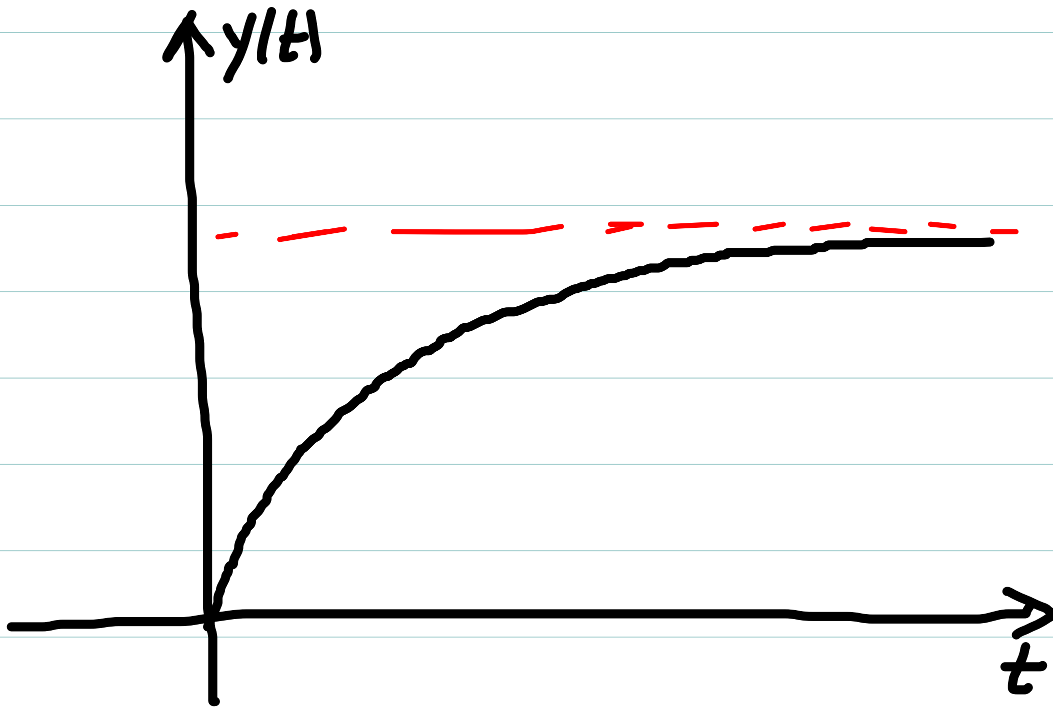
In general, not much unless either

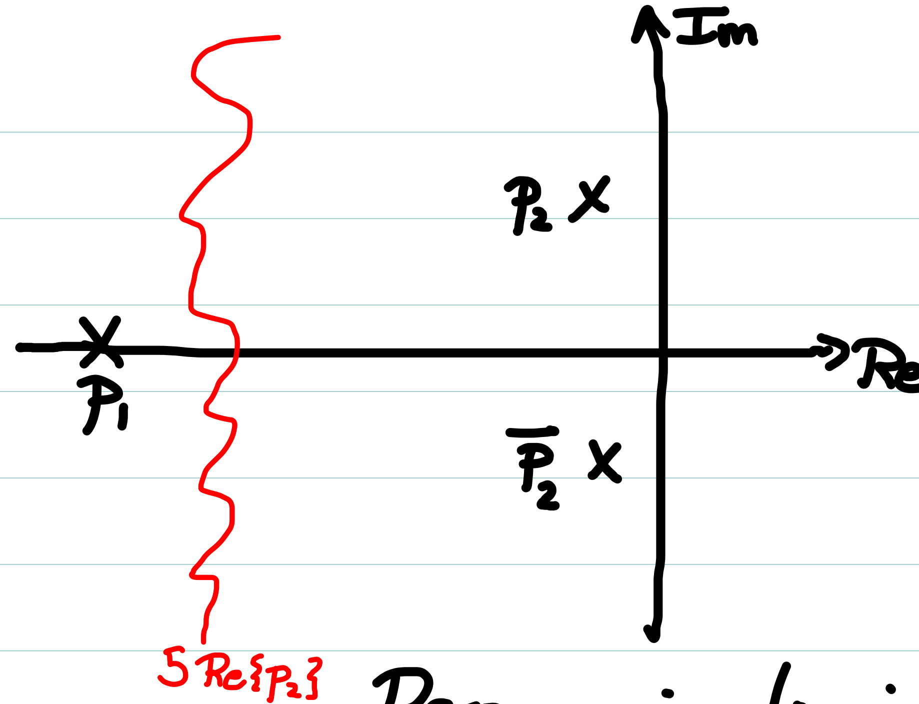
$$|p_1| > 5|Re\{p_2\}| \text{ or } |Re\{p_2\}| > 5|p_1|$$

i.e. if one of the modes is <u>dominant</u>.



Response is dominantly 1^{st} order



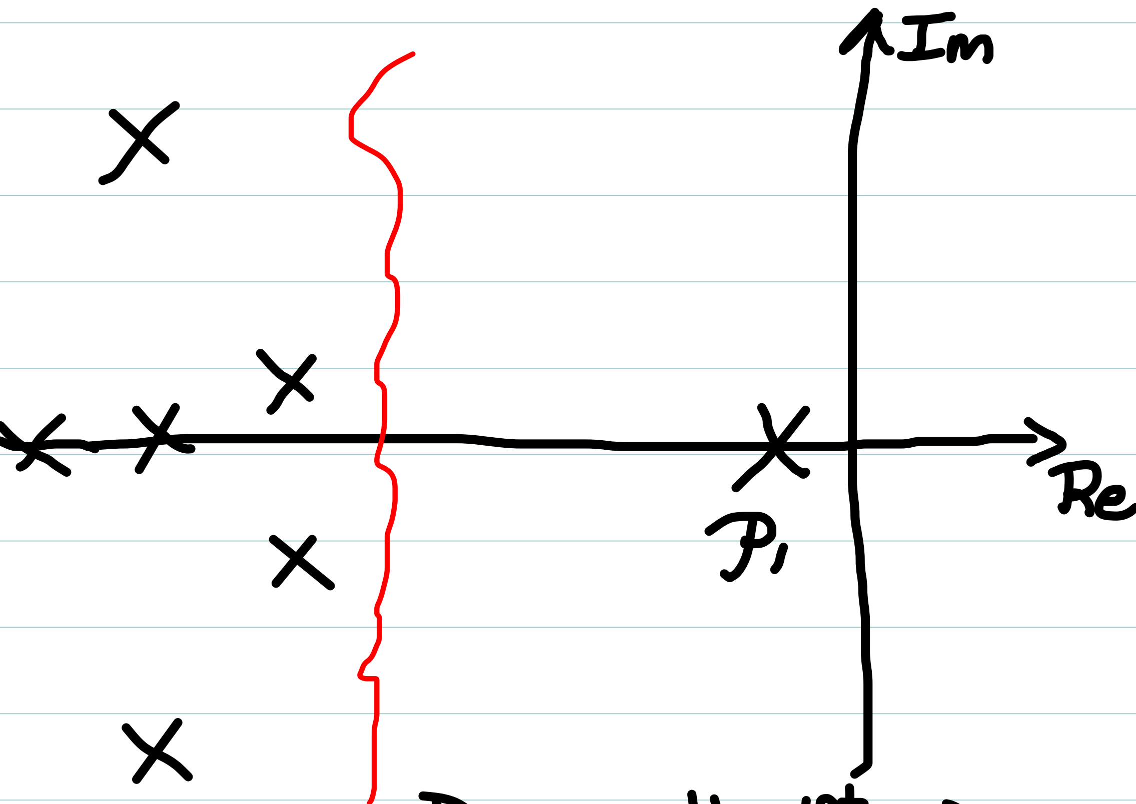


Response is dominant 2^{nd} order

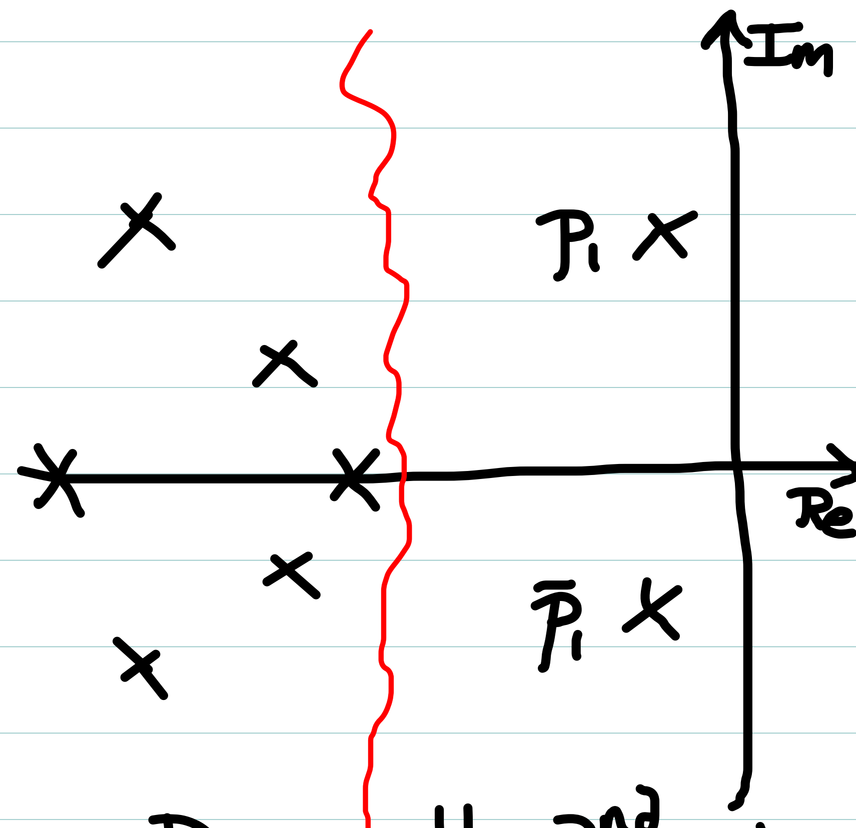
Dominant modes revisited

When a single mode is dominant, we can approximate the features of the response using just that mode

An arbitrarily complex system can be well approximated in this fashion.



Dominantly 1st order



Dominantly 2nd order

Effect of Zeros

Step response of

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Zero at $z_1 = -\beta_0/\beta_1$

3 important effects:

1. "Input absorbing" property
2. Transient suppression
3. Transient amplification

Both? Yes!

Depending on system

(I) Input absorbtion

For unit step response of stable system

$$y_{ss}(t) = G(0)$$

Suppose $z_1 = -\beta_0/\beta_1 = 0 \Rightarrow \beta_0 = 0$

zero at origin

$$G(s) = \frac{\beta_1 s}{s^2+\alpha_1 s+\alpha_0}$$

Then $y_{ss}(t) = G(0) = 0 \Leftarrow$ steady-state is zero

response contains only transient terms

In fact, $y(t)$ is the impulse response of

$$G_1(s) = \frac{\beta_1}{s^2+\alpha_1 s+\alpha_0}$$

$$U(s)=\frac{1}{s} \longrightarrow \boxed{\frac{\beta_1 s}{s^2+\alpha_1 s+\alpha_0}} \longrightarrow Y(s)$$

$$\frac{1}{s} \longrightarrow \boxed{s} \xrightarrow{U_1(s)} \overset{G_1(s)}{\boxed{\frac{\beta_1}{s^2+\alpha_1 s+\alpha_0}}} \longrightarrow Y(s)$$

$$U_1(s)=1 \longrightarrow \boxed{\frac{\beta_1}{s^2+\alpha_1 s+\alpha_0}} \longrightarrow Y(s)$$