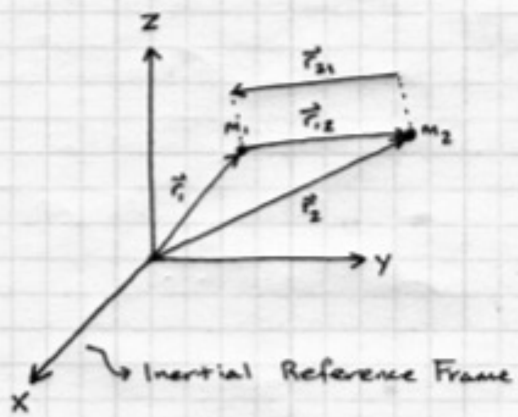
Consider the following system of two bodies with respect to an inertial reference frame:



$\vec{r}_1$: inertial position of body 1

$\vec{r}_2$: inertial position of body 2

$\vec{r}_{12}$: position vector pointing from body 1 to body 2

$\vec{r}_{21}$: position vector pointing from body 2 to body 1

m_1 : mass of body 1

m_2 : mass of body 2

An inertial reference frame is unaccelerated:

- Frame origin is at rest or moving with constant velocity.
- Frame axes are not rotating.

In an inertial frame, Newton's 2nd Law of Motion holds. It is:

$$\Sigma \vec{F} = m\ddot{\vec{r}} \quad ; \quad \ddot{\vec{r}} = \text{acceleration}$$

Note: $\underbrace{\dot{\vec{r}} \equiv \frac{d}{dt}(\vec{r})}_{\text{Velocity}}$, $\underbrace{\ddot{\vec{r}} \equiv \frac{d^2}{dt^2}(\vec{r})}_{\text{acceleration}}$

$\Sigma \vec{F}$ = sum of all forces acting upon the body with mass m

We will assume that the bodies are point masses (spherical, homogeneous gravitational fields), and so Newton's Law of Universal Gravitation describes the forces between bodies due to <u>mutual gravitation</u> as:

$$\vec{F} = \frac{G m_1 m_2}{r^2} \hat{r} = \frac{G m_1 m_2}{r^3} \vec{r}$$

$$\left(\text{since } \hat{r} = \frac{\vec{r}}{\|\vec{r}\|} = \frac{\vec{r}}{r}\right)$$

where $\hat{r}$ is the unit vector between the bodie

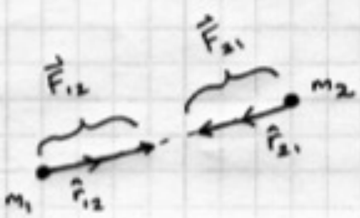
pointing ~~frome~~ one to the other.

G is the <u>Universal Gravitational Constant</u>:

$$G = 6.67428 \pm 0.0007 \times 10^{-11} \frac{m^3}{Kg\, s^2}$$

↳ 2006 $\underbrace{\text{CODATA}}_{\text{Committee on Data for Science and Technology}}$ value

For the two body system we are considering:



$\vec{F}_{12}$ = Force on body 1 du to body 2's gravit

$\vec{F}_{21}$ = Force on body 2 du to body 1's gravity

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{\|\vec{r}_{12}\|} , \quad \hat{r}_{21} = \frac{\vec{r}_{21}}{\|\vec{r}_{21}\|}$$

$$\vec{F}_{12} = \frac{G m_1 m_2}{\|\vec{r}_{12}\|^2} \hat{r}_{12} , \quad \vec{F}_{21} = \frac{G m_1 m_2}{\|\vec{r}_{21}\|^2} \hat{r}_{21}$$

Our goal is to derive the Equations of Motion (EOMs) for the bodies, so we now draw a Free Body Diagram (FBD) for each body and apply Newton's 2nd Law:

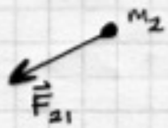
FBD for body 1:

No other forces except for the force due to body 2's gravity

$$\Rightarrow \underbrace{\sum \vec{F}_1}_{\text{Sum of the forces acting on body 1}} = \cancel{m_1}\ddot{\vec{r}}_1 = \vec{F}_{12} = \frac{G\cancel{m_1}m_2}{\|\vec{r}_{12}\|^2}\hat{r}_{12}$$

$$\boxed{\ddot{\vec{r}}_1 = \frac{Gm_2}{\|\vec{r}_{12}\|^2}\hat{r}_{12}}$$

FBD for body 2:



$$\Rightarrow \sum \vec{F}_2 = \cancel{m_2}\ddot{\vec{r}}_2 = \vec{F}_{21} = \frac{Gm_1\cancel{m_2}}{\|\vec{r}_{21}\|^2}\hat{r}_{21}$$

$$\boxed{\ddot{\vec{r}}_2 = \frac{Gm_1}{\|\vec{r}_{21}\|^2}\hat{r}_{21}}$$

Referring to the original diagram showing the bodies and their position vectors:

$$\vec{r}_1 + \vec{r}_{12} = \vec{r}_2 \qquad\qquad \vec{r}_2 + \vec{r}_{21} = \vec{r}_1$$

$$\boxed{\vec{r}_{12} = \vec{r}_2 - \vec{r}_1} \qquad\qquad \boxed{\vec{r}_{21} = \vec{r}_1 - \vec{r}_2}$$

$$\therefore\ \vec{r}_{12} = -\vec{r}_{21}$$

$$\|\vec{r}_{21}\| = \|\vec{r}_{12}\| = r$$

Recalling that $\hat{r} = \frac{\vec{r}}{\|\vec{r}\|} = \frac{\vec{r}}{r}$, we may write:

$$\boxed{\begin{aligned} \ddot{\vec{r}}_1 &= \frac{Gm_2}{r^3}(\vec{r}_2 - \vec{r}_1) \\ \ddot{\vec{r}}_2 &= -\frac{Gm_1}{r^3}(\vec{r}_2 - \vec{r}_1) = \frac{Gm_1}{r^3}(\vec{r}_1 - \vec{r}_2) \end{aligned}}$$

Which are the equations of motion for bodies 1 and 2.