PHYS 313
HW 07: Assignment 7

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Problem 3.15:

A rectangular pipe, running parallel to the z-axis (from $-\infty$ to $+\infty$, has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specific potential $\mathrm{V}_0(y)$.

Solution

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\mathrm{V}(x,y)\right) + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\mathrm{V}(x,y)\right) = 0$$

$$\text{boundary} = \begin{cases} i: & \mathrm{V}(x,0) = 0 \\ ii: & \mathrm{V}(x,a) = 0 \\ iii: & \mathrm{V}(0,y) = 0 \\ iv: & \mathrm{V}(b,y) = \mathrm{V}_0(y) \end{cases}$$

$$\begin{aligned}
\mathrm{V}(x,y) &= \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin(ky) + D\cos(ky)\right) \\
i &\implies D = 0 \\
ii &\implies ka = n\pi, n \in \mathbb{Z} \\
iii &\implies B = -A \\
\mathrm{V}(x,y) &= AC\left(e^{n\pi\frac{x}{a}} - e^{-n\pi\frac{x}{a}}\right)\sin(n\pi\frac{y}{a}) \\
&= 2AC\sinh(n\pi\frac{x}{a})\sin(n\pi\frac{y}{a}) \\
&= \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}\frac{\sinh(n\pi\frac{x}{a})\sin(n\pi\frac{y}{a})}{n\sinh(n\pi\frac{b}{a})}
\end{aligned}$$

$$\boxed{\mathrm{V}(x,y) = \frac{4V_0}{\pi}\sum_{n=1}^{\infty}\frac{\sinh((2n-1)\,\pi\frac{x}{a})\sin((2n-1)\,\pi\frac{y}{a})}{(2n-1)\sinh((2n-1)\,\pi\frac{b}{a})}}$$

Problem 3.17:

Derive $\mathrm{P}_3(x)$ from the Rodrigues formula, and check that $\mathrm{P}_3(\cos(\theta))$ satisfiest the angular equation (3.60) for $l = 3$. Check that P_3 and P_1 are orthogonal by explicit integration.

Solution

$$\begin{aligned}
P_l(x) &= \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \\
P_3(x) &= \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 \\
(x^2-1)^3 &= x^6 - 3x^4 + 3x^2 - 1 \\
\frac{d}{dx}(x^6 - 3x^4 + 3x^2 - 1) &= 6x^5 - 12x^3 + 6x \\
\frac{d^2}{dx^2}(x^6 - 3x^4 + 3x^2 - 1) &= 30x^4 - 36x^2 + 6 \\
\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) &= 120x^3 - 72x \\
P_3(x) &= \frac{1}{2^3 3!}(120x^3 - 72x) = \frac{1}{8\cdot 6}(120x^3 - 72x) = \frac{1}{48}(120x^3 - 72x) \\
P_3(x) &= \frac{120}{48}x^3 - \frac{72}{48}x = \frac{5}{2}x^3 - \frac{3}{2}x \\
0 &= \frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d}{d\theta}P_3(\cos(\theta))\right) + l(l+1)P_3(\cos(\theta)), \\
y(\theta) &= P_3(\cos(\theta)) = \frac{5}{2}\cos(\theta)^3 - \frac{3}{2}\cos(\theta) \\
\frac{dy}{d\theta} &= \frac{5}{2}\cdot 3\cos(\theta)^2\left(-\sin(\theta)\right) - \frac{3}{2}(-\sin(\theta)) = -\frac{15}{2}\cos(\theta)^2\ \sin(\theta) + \frac{3}{2}\sin(\theta) \\
\frac{dy}{d\theta} &= \frac{3\sin(\theta)}{2}\left(1 - 5\cos(\theta)^2\right) \\
\sin(\theta)\frac{dy}{d\theta} &= \frac{3\sin()^2\theta}{2}\left(1 - 5\cos()^2\theta\right) \\
A(\theta) = \sin(\theta)\frac{dy}{d\theta} &= \frac{3}{2}\sin(\theta)^2\left(1 - 5\cos(\theta)^2\right) \\
\frac{dA}{d\theta} &= \frac{3}{2}\left[2\sin(\theta)\cos(\theta)\left(1 - 5\cos(\theta)^2\right) + \sin(\theta)^2\left(10\cos(\theta)\sin(\theta)\right)\right] \\
\frac{dA}{d\theta} &= \frac{3}{2}\left[2\sin(\theta)\cos(\theta)(1 - 5\cos(\theta)^2) + 10\sin(\theta)^3\cos(\theta)\right] \\
\frac{dA}{d\theta} &= \frac{3}{2}\cdot 2\sin(\theta)\cos(\theta)\left[(1 - 5\cos(\theta)^2) + 5\sin(\theta)^2\right] \\
(1 - 5\cos(\theta)^2) + 5(1 - \cos(\theta)^2) &= 1 - 5\cos(\theta)^2 + 5 - 5\cos(\theta)^2 = 6 - 10\cos(\theta)^2 \\
\frac{dA}{d\theta} &= 3\sin(\theta)\cos(\theta)\left(6 - 10\cos(\theta)^2\right) \\
\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dy}{d\theta}\right) &= 3\cos(\theta)\left(6 - 10\cos(\theta)^2\right) = 18\cos(\theta) - 30\cos(\theta)^3 \\
12y(\theta) &= 12\left(\frac{5}{2}\cos(\theta)^3 - \frac{3}{2}\cos(\theta)\right) = 30\cos(\theta)^3 - 18\cos(\theta) \\
0 &= \left[18\cos(\theta) - 30\cos(\theta)^3\right] + \left[30\cos(\theta)^3 - 18\cos(\theta)\right] \quad \square
\end{aligned}$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = 0 \quad \text{for } l \neq l'$$

Problem 3.17 continued on next page...

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \quad \text{and} \quad P_1(x) = x$$

$$P_3(x)P_1(x) = \left(\frac{5}{2}x^3 - \frac{3}{2}x\right)x = \frac{5}{2}x^4 - \frac{3}{2}x^2$$

$$\int_{-1}^{1} \left(\frac{5}{2}x^4 - \frac{3}{2}x^2\right) dx = \frac{5}{2}\int_{-1}^{1} x^4\,dx - \frac{3}{2}\int_{-1}^{1} x^2\,dx$$

$$\int_{-1}^{1} x^{2n}\,dx = \frac{2}{2n+1}$$

$$\int_{-1}^{1} x^4\,dx = \frac{2}{5}, \quad \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$\frac{5}{2}\cdot\frac{2}{5} - \frac{3}{2}\cdot\frac{2}{3} = 1 - 1 = 0 \quad \square$$

$$\boxed{P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x}$$

$$\boxed{\int_{-1}^{1} P_3(x)P_1(x)\,dx = 0}$$

Problem 3.18:

1. Suppose the potential is a *constant* V_0 over the surface of the sphere. Use the results of Ex:3.6 and Ex:3.7 to find the potential inside and outside the sphere.

2. Find the potential inside and outside a spherical shell that carries a uniform surface charge σ_0, using the results of Ex:3.9.

Solution

Part A

$$\begin{aligned} V(R) &= V_0, \\ \text{Inside } (r \leq R): \quad V_{\text{in}}(r) &= V_0, \\ \text{Outside } (r \geq R): \quad V_{\text{out}}(r) &= \frac{V_0\, R}{r} \end{aligned}$$

Part B

$$\begin{aligned} \sigma_0 \text{ on } r = R, Q = 4\pi R^2 \sigma_0, & \\ \text{Outside } (r \geq R): \quad V_{\text{out}}(r) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r} = \frac{R^2\sigma_0}{\epsilon_0\, r}, \\ \text{Inside } (r \leq R): \quad V_{\text{in}}(r) &= V(R) = \frac{R^2\sigma_0}{\epsilon_0\, R} = \frac{R\sigma_0}{\epsilon_0} \end{aligned}$$

Problem 3.20:

Suppose the potential $\mathrm{V_0}(\theta)$ at the surface of a sphere is specified, and there is no charge inside or outside the sphere. Show that the charge density on the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 \, \mathrm{C_l P_l}(\cos(\theta))$$

where

$$\mathrm{C_l} = \int_0^{\pi} \mathrm{V_0}(\theta) \mathrm{P_l}(\cos(\theta)) \sin(\theta) \mathrm{d}\theta$$

Solution

$$\begin{aligned}
V_{\text{in}}(r,\theta) &= \sum_{l=0}^{\infty} A_l \, r^{\,l} P_l(\cos(\theta)), \\
V_{\text{out}}(r,\theta) &= \sum_{l=0}^{\infty} B_l \, r^{-(l+1)} P_l(\cos(\theta)) \\
V_0(\theta) &= V_{\text{in}}(R,\theta) = V_{\text{out}}(R,\theta) \\
&= \sum_{l=0}^{\infty} A_l \, R^l P_l(\cos(\theta)) \\
A_l R^l &= \frac{2l+1}{2} \int_0^{\pi} V_0(\theta) P_l(\cos(\theta)) \sin(\theta) \, d\theta \equiv \frac{2l+1}{2} \, C_l, \\
\Rightarrow \quad A_l &= \frac{2l+1}{2} \, \frac{C_l}{R^l} \\
C_l &= \int_0^{\pi} V_0(\theta) P_l(\cos(\theta)) \sin(\theta) \, d\theta \\
\sigma(\theta) &= \epsilon_0 \left[- \left. \frac{\partial V_{\text{out}}}{\partial r} \right|_{r=R} + \left. \frac{\partial V_{\text{in}}}{\partial r} \right|_{r=R} \right] \\
\left. \frac{\partial V_{\text{in}}}{\partial r} \right|_{r=R} &= \sum_{l=0}^{\infty} l \, A_l \, R^{l-1} P_l(\cos(\theta)), \\
\left. \frac{\partial V_{\text{out}}}{\partial r} \right|_{r=R} &= - \sum_{l=0}^{\infty} (l+1) \, B_l \, R^{-(l+2)} P_l(\cos(\theta)) \\
A_l \, R^l &= B_l \, R^{-(l+1)} \\
B_l &= A_l \, R^{2l+1} \\
\left. \frac{\partial V_{\text{out}}}{\partial r} \right|_{r=R} &= - \sum_{l=0}^{\infty} (l+1) \, A_l \, R^{2l+1} \, R^{-(l+2)} P_l(\cos(\theta)) \\
&= - \sum_{l=0}^{\infty} (l+1) \, A_l \, R^{l-1} P_l(\cos(\theta)) \\
- \left. \frac{\partial V_{\text{out}}}{\partial r} \right|_{r=R} + \left. \frac{\partial V_{\text{in}}}{\partial r} \right|_{r=R} &= \sum_{l=0}^{\infty} \left[(l+1) + l \right] A_l \, R^{l-1} P_l(\cos(\theta)) \\
&= \sum_{l=0}^{\infty} (2l+1) A_l \, R^{l-1} P_l(\cos(\theta))
\end{aligned}$$

Problem 3.20 continued on next page. . .

$$\begin{aligned}\sigma(\theta) &= \epsilon_0 \sum_{l=0}^{\infty}(2l+1)\left(\frac{2l+1}{2}\frac{C_l}{R^l}\right) R^{l-1} P_l(\cos(\theta)) \\ &= \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2\, C_l\, P_l(\cos(\theta))\end{aligned}$$

$$\boxed{\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2\, C_l\, P_l(\cos(\theta)), \quad \text{with} \quad C_l = \int_0^{\pi} V_0(\theta) P_l(\cos(\theta))\sin(\theta)\, d\theta.}$$

Problem 3.21:

Find the potential outside a *charged* metal sphere (charge Q, radius R) placed in an otherwise uniform electric field $\mathbf{E}_0$. Explain clearly where you are setting the zero of potential.

Solution

Set the zero of the potential at infinity: $V(\infty) = 0$.

$$\begin{aligned}
V(\infty) &= 0, \\
V(r, \theta) &= \frac{Q}{4\pi\epsilon_0\, r} - E_0\, r\cos(\theta) + E_0\,\frac{R^3}{r^2}\cos(\theta), \quad r \geq R, \\
V(R, \theta) &= \frac{Q}{4\pi\epsilon_0\, R} - E_0\, R\cos(\theta) + E_0\,\frac{R^3}{R^2}\cos(\theta)
\end{aligned}$$

$$\boxed{\mathrm{V}(R, \theta) = \frac{Q}{4\pi\epsilon_0\, R}}$$

Problem 3.24:

Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find *all* solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

Solution

$$
\begin{aligned}
\nabla^2 V &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0,\\
V(r,\phi) &= R(r)\,\Phi(\phi),\\
\frac{1}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \quad &\implies \quad r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Phi''}{\Phi} = 0,\\
\frac{\Phi''}{\Phi} &= -m^2 \quad \implies \quad \Phi(\phi) = A\cos(m\phi) + B\sin(m\phi), \quad m = 0, 1, 2, \dots,\\
r^2 R'' + rR' - m^2 R &= 0,\\
\text{For } m \neq 0: \quad R(r) &= C\,r^m + D\,r^{-m},\\
\text{and for } m = 0: \quad R(r) &= C_0 + D_0\ln(r)
\end{aligned}
$$

$$
\boxed{\therefore \quad V(r,\phi) = \Big(C_0 + D_0\ln(r)\Big) + \sum_{m=1}^{\infty}\Big[\Big(C_m\,r^m + D_m\,r^{-m}\Big)\cos(m\phi) + \Big(E_m\,r^m + F_m\,r^{-m}\Big)\sin(m\phi)\Big]}
$$