Lecture 103: The Energy Equation in Compressible Flow

ENAE311H Aerodynamics I

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The adiabatic energy equation

Recall our conservation-of-energy equation for flow through a simple CV:

$$\dot{m}\left[h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.$$

Away from solid boundaries, there is typically no heat transfer (or shaft work), and so we can assume that the flow is adiabatic ($\dot{Q} = 0$). For air flows, we can also typically neglect the influence of gravity. With these assumptions, this equation becomes

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2.$$

If we consider condition 2 to be a stagnation point, so that $u_2 = 0$, and condition 1 to be arbitrary, we have

$$h + \frac{1}{2}u^2 = h_0 = \text{const.},$$

where h_0 is the *stagnation enthalpy*.

The stagnation temperature

Now, assuming a perfect gas, so that $h = c_p T$, this can be written as

$$c_p T + \frac{1}{2}u^2 = c_p T_0, \qquad \text{or} \qquad \frac{T_0}{T} = 1 + \frac{u^2}{2c_p T}.$$

T_0 is the *stagnation* or *total temperature*, i.e., the temperature the flow reaches if brought to rest adiabatically.

Now we use $T = a^2/(\gamma R)$ to write the above equation as

$$\frac{c_p}{\gamma R}a^2 + \frac{1}{2}u^2 = \frac{c_p}{\gamma R}a_0^2.$$

Since $c_p/(\gamma R) = 1/(\gamma - 1)$, this becomes

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma - 1}.$$

Dividing through by $a^2/(\gamma - 1)$, we then have

$$1 + \frac{\gamma - 1}{2}\frac{u^2}{a^2} = \frac{a_0^2}{a^2}, \qquad \text{or} \qquad \boxed{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2.}$$

We thus see that, for a perfect gas, T_0/T, depends only on M and γ.

The stagnation pressure and density

Let us now assume that the flow is undergoing an isentropic process. Then, from last lecture

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)}$$

From our previous equation for T_0/T, we can thus write

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)},$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}.$$

p_0 and ρ_0 are the *stagnation* (or *total*) *pressure* and *density*, i.e., the values the flow achieves if brought to rest isentropically.

Earlier we stated that a flow can be considered incompressible (constant density) at low speeds. To see at which Mach number this no longer becomes reasonable, let us assume we have an isentropic process and are willing to accept a 5% variation in density. Then the maximum Mach number allowable is given by

$$M = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{1}{0.95}\right)^{\gamma-1} - 1\right]} = 0.32.$$

Variables in adiabatic and isentropic flow

Assume we have an adiabatic flow and the Mach numbers at two points (1 and 2) are known. Then, since T_0 is constant in the flow, we can write

$$\begin{aligned}\frac{T_2}{T_1} &= \frac{T_2}{T_0}\frac{T_0}{T_1} \\ &= \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\end{aligned}$$

Thus, given T_1, we can immediately determine T_2 .

Similarly, if the flow is isentropic (p_0 constant), we can relate the pressures at points 1 and 2:

$$\begin{aligned}\frac{p_2}{p_1} &= \frac{p_2}{p_0}\frac{p_0}{p_1} \\ &= \left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{\gamma/(\gamma-1)},\end{aligned}$$

and similarly for the density.

Sonic-referenced conditions

Thus far we have used stagnation ($M = 0$) conditions to reference our flow variables, but this is by no means a unique choice. An alternative that is useful in some situations is to use the conditions the flow variables would have if accelerated/decelerated adiabatically or isentropically to sonic ($M = 1$) conditions, which we denote by *. Compared to stagnation conditions, we have:

$$\text{Adiabatic:} \quad \frac{T^*}{T_0} = \frac{2}{\gamma+1} \quad (=0.833 \text{ for } \gamma=1.4)$$

$$\text{Isentropic:} \quad \frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \quad (=0.528 \text{ for } \gamma=1.4)$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \quad (=0.634 \text{ for } \gamma=1.4)$$

The characteristic Mach number

We can also define $a^* = \sqrt{\gamma R T^*}$ and the *characteristic Mach number*, M^*, as

$$M^* = \frac{u}{a^*}.$$

Now, we can write

$$\frac{a^{*2}}{a_0^2} = \frac{T^*}{T_0} = \frac{2}{\gamma + 1}.$$

Substituting into our energy equation from earlier, we obtain

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2}.$$

Dividing through by u^2 and rearranging, we arrive at a relation between M^* and M:

$$M^{*2} = \frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2}.$$

Note that, as $M \to \infty$, $M^* \to \sqrt{(\gamma + 1)/(\gamma - 1)}$.

An airplane is flying at a standard altitude of 10,000 ft. A Pitot tube mounted at the nose measures a pressure of 2220 lb/ft^2. The airplane is flying at a high subsonic speed, faster than 300 mph. From our comments in Section 3.1, the flow should be considered *compressible*. Calculate the velocity of the airplane.

Solution

From our discussion in Section 3.4, the pressure measured by a Pitot tube immersed in an incompressible flow is the total pressure. For the same *physical reasons* discussed in Section 3.4, a Pitot tube also measures the total pressure in a high-speed subsonic compressible flow. (This is further discussed in Section 8.7.1 on the measurement of velocity in a subsonic compressible flow.) *Caution*: Because we are dealing with a compressible flow in this example, we *cannot* use Bernoulli's equation to calculate the velocity.

The flow in front of the Pitot tube is compressed isentropically to zero velocity at the mouth of the tube, hence the pressure at the mouth is the total pressure, p_0. From Equation (7.32), we can write:

$$\frac{p_0}{p_\infty} = \left(\frac{T_0}{T_\infty}\right)^{\gamma/(\gamma-1)} \tag{E.7.3}$$

where p_0 and T_0 are the total pressure and temperature, respectively, at the mouth of the Pitot tube, and p_∞ and T_∞ are the freestream static pressure and static temperature, respectively. Solving Equation (E7.3) above for T_0, we get

$$T_0 = T_\infty \left(\frac{p_0}{p_\infty}\right)^{(\gamma-1)/\gamma} \tag{E.7.4}$$

From Appendix E, the pressure and temperature at a standard altitude of 10,000 ft are 1455.6 lb/ft^2 and 483.04 °R, respectively. These are the values of p_∞ and T_∞ in Equation (E7.4). Thus, from Equation (E7.4),

$$T_0 = (483.04)\left(\frac{2220}{1455.6}\right)^{0.4/1.4} = 544.9\ ^\circ\mathrm{R}$$

From the energy equation, Equation (7.54), written in terms of temperature, we have

$$c_p\, T + \frac{V^2}{2} = c_p\, T_0 \tag{E.7.5}$$

In Equation (E7.5), both T and V are the freestream values, hence we have

$$c_p\, T_\infty + \frac{V_\infty^2}{2} = c_p\, T_0 \tag{E.7.6}$$

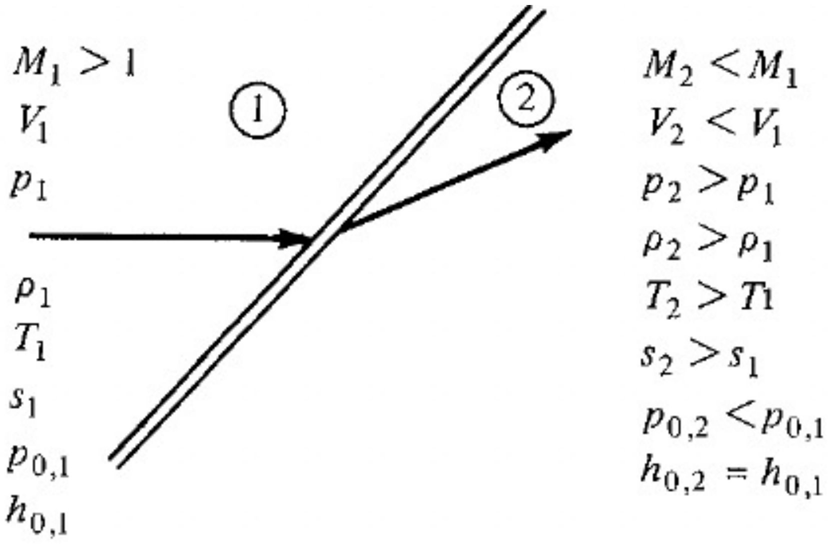
Also,

$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.4)(1716)}{0.4} = 6006\ \frac{\mathrm{ft \cdot lb}}{\mathrm{slug \cdot {}^\circ R}}$$

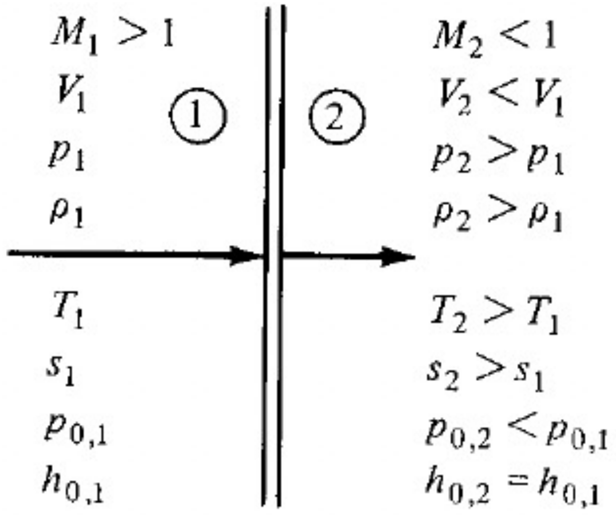
Solving Equation (E7.6) for V_∞, we have

$$\begin{aligned} V_\infty &= [2\, c_p\, (T_0 - T_\infty)]^{1/2} \\ &= [2\,(6006)(544.9 - 483.04]^{1/2} \\ &= \boxed{862\ \mathrm{ft/s}} \end{aligned}$$

Note: From this example, we see that the total pressure measured by a Pitot tube in a subsonic compressible flow is a measure of the flow velocity, but we need also the value of the flow static temperature in order to calculate the velocity. In Section 8.7, we show more fundamentally that the *ratio* of Pitot pressure to flow static pressure in a compressible flow, subsonic or supersonic, is a *direct measure* of the *Mach number*, not the velocity. But more on this later.



(a) Oblique shock wave



(b) Normal shock wave