Metric (a very special tensor)

To start building towards a good derivative, we need a notion of the "distance" between two nearby points.

this is differential geometry

This is provided by the metric.

Symmetric rank $(0,2)$ - tensor $g_{\mu\nu}$.

Since it is $(0,2)$, spanned by dual vector product $dx^\mu \otimes dx^\nu$.

$$ds^2 = g_{\mu\nu}(dx^\mu \otimes dx^\nu)$$

line element

components

basis

We've seen this before! SR: $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

where $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, the Minkowski metric.

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ldots$$

Now ANY $g_{\mu\nu}$ is allowed!!!

Ex. Flat $\mathbb{R}^2$

$$ds^2 = dx^2 + dy^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

Convert to radial coordinates two ways

1. $x = r\cos\theta$ $\qquad x^0 = x \quad x^{0'} = r$

$y = r\sin\theta$ $\qquad x^1 = y \quad x^{1'} = \theta$

$$\frac{dx}{dr} = \cos\theta \qquad \frac{dy}{dr} = \sin\theta$$

$$\frac{dx}{d\theta} = -r\sin\theta \qquad \frac{dy}{d\theta} = r\cos\theta$$

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = \begin{pmatrix} \frac{\partial x^\mu}{\partial r} \frac{\partial x^\nu}{\partial r} g_{\mu\nu} & \cdot \\ \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \\ & \ddots \end{pmatrix} = \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$$

2. $dx = \cos\theta \, dr + (-r\sin\theta)\, d\theta$

$dy = \sin\theta \, dr + (r\cos\theta)\, d\theta$

$$ds^2 = (\cos^2\theta + \sin^2\theta)\, dr^2 + r^2(\sin^2\theta + \cos^2\theta)\, d\theta^2$$

$$ds^2 = dr^2 + r^2 d\theta^2 \Rightarrow g_{\mu'\nu'} = \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$$

Length of a vector is just given by

$$|V| = g_{\mu\nu} V^\mu V^\nu$$

(Ex) Prove this is a scalar, i.e. invariant under coordinate transformations.

Does it work for polar coord.s? $\quad V^\mu = \begin{pmatrix} 1 \\ \frac{\pi}{4} \end{pmatrix}$

$g_{\mu\nu} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$ at origin $\Rightarrow |V| = 1$ $\quad V^{\mu'} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

$g_{\mu'\nu'} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ cartesian $\Rightarrow |V| = 1$

The length of a path $\gamma = x^\mu(\lambda)$ is

$$\int \sqrt{g_{\mu\nu}\, x^\mu(\lambda)\, x^\nu(\lambda)}\; d\lambda$$

just like in SR.

Connecting back to physics for a second, this is actually just the proper time, even in GR.

(How much time elapses on the clock of an observer moving along $x^\mu(\lambda)$)

Derivatives?

Natural to ask how a vector field changes along some direction in manifold.

Need a notion of differentiation.

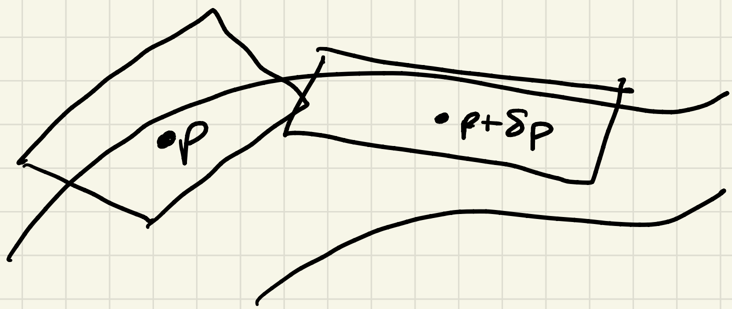
We run into an immediate problem when trying to define the derivative.

Intuitively, we want

$$\lim_{\delta p \to 0} \frac{V(p+\delta p) - V(p)}{\delta p} \quad \ldots \quad \text{but}$$

T_p and $T_{p+\delta p}$ are TWO DIFFERENT SPACES.

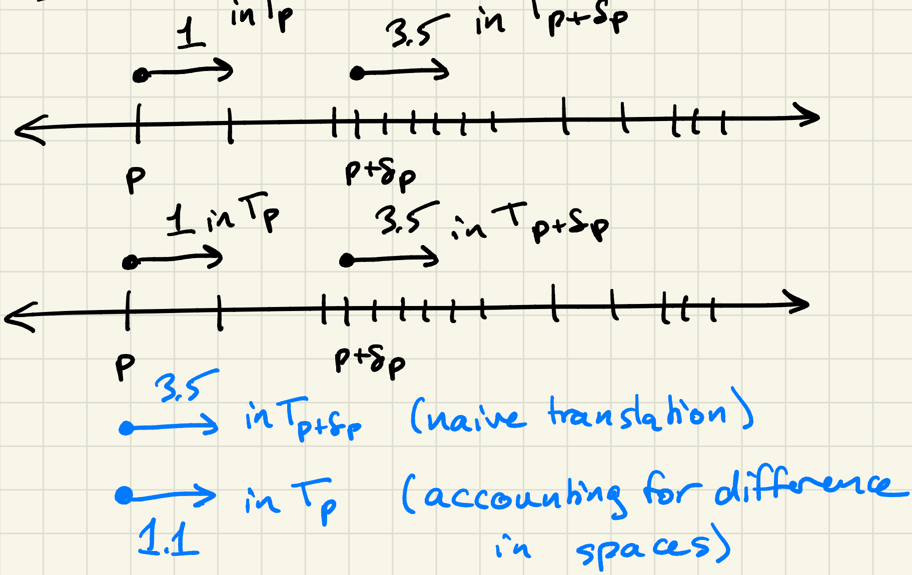
It makes no sense to try to subtract two things in different spaces.



Furthermore, as we will see, since any coordinate transformation is allowed, p has no idea what's happening with the transform at $p+\delta p$. (e.g. 1-manifold)

We need to find a way to put both vectors in the same space in a way that knows about how the coordinates change in the δ_p interval.

Ex. 1 - manifold



How will we know if we're successfully working at a single point? Derivative of vector should transform covariantly.

ALL PHYSICS SHOULD BE COORDINATE-IND. $\Rightarrow$ covariant

$(\text{derivative})_\mu V^\nu$ should live in $T_p^* \otimes T_p$, hence be a tensor. $\text{deriv}_\mu V^\nu \, dx^\mu \otimes \partial x^\nu$

SHOULD NOT CHANGE

However, the naive guess of $\partial_\mu V^\nu$ doesn't work.

Let's see this in math. Taking a naive derivative of a vector field gives:

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right)$$

This acts on both!!!

$$= \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}}_{\text{bad!}} + \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial}{\partial x^\mu} V^\nu}_{\text{good!}}$$

(this is the part arising from not mapping btw T_p and $T_{p+\delta p}$)

Let's do the dumbest possible thing ...

just define a new derivative that subtracts off the bad part.

We define some non-tensor quantity $\Gamma^{\nu}_{\mu\lambda}$ that transforms like

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\lambda}}$$

and define the "covariant derivative"

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda}$$

Ex. Verify this transforms covariantly.

Properties: - $\nabla_\mu \phi = \partial_\mu \phi$ on scalar

- commutes w/ contractions
- can be extended to tensors

$$\nabla_\mu T^\nu{}_\lambda = \partial_\mu T^\nu{}_\lambda + \Gamma^{\nu}_{\mu\sigma} T^\sigma{}_\lambda - \Gamma^{\rho}_{\mu\lambda} T_\rho{}^\nu$$

We wish to find a connection that

① exists

② gives good properties

What do I mean by "good properties"?

A natural choice would be to find a connection that in some sense treats the manifold in a "flat-ish" way.

↳ Good property: $\nabla_\mu g_{\sigma\lambda} = 0$

"Metric compatability"

- On small scales, manifold looks flat ... Equivalence Principle

↳ Good property: $\Gamma^{\mu}_{\sigma\lambda} = \Gamma^{\mu}_{\lambda\sigma}$

"Torsion free"

- Defines a <u>unique</u> connection

Ex. How many eq.s in $\nabla_\mu g_{\sigma\lambda} = 0$? How many ind. components in $\Gamma^{\mu}_{\lambda\sigma}$? (40 and 40 → unique?)

"The Metric Connection"

See Carroll pg. 99 for proof but can show that unique metric connection is

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)}$$

Provides unique, good, covariant derivative

$$\nabla_\mu g_{\sigma\rho} = 0 \qquad (\text{Ex. Prove it!})$$

$$\nabla_\mu \Phi = \partial_\mu \Phi$$

scalar

Reduces to ∂_μ in flat space.

Locally Inertial Coordinates

At any point p, can find coordinates $\hat{x}^\mu$ such that $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ and $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} = 0$ (but you can't kill second derivatives).

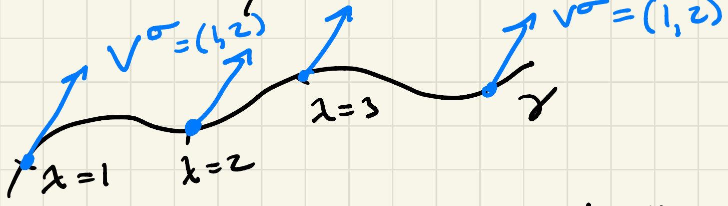
We'll derive how to construct these in a little while.

Also defines notion of "parallel transport":

How to slide a vector around in $\mathbb{R}^2$?

If I give you a path $\gamma = x^\mu(\lambda)$,

how would you slide a vector around?



What is this in math? $\dfrac{dx^\mu}{d\lambda} \partial_\mu V^\sigma = 0$

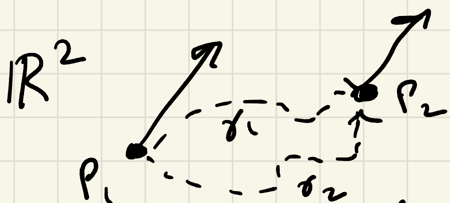
The Rule for Making Anything Covariant:

If you see a ∂_μ, make it a ∇_μ.

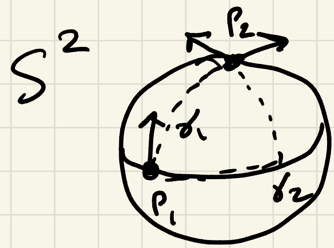
"Parallel transport": Defined by $\dfrac{dx^\mu}{d\lambda} \nabla_\mu V^\sigma = 0$

Ex. Prove that for a metric-compatible connection, the inner product $|V| = g_{\mu\nu} V^\mu V^\nu$ is conserved.

★ ONLY IN FLAT SPACE IS PARALLEL TRANSPORT PATH-INDEPENDENT. ★



path-independent
$\Rightarrow$ flat, as we will see

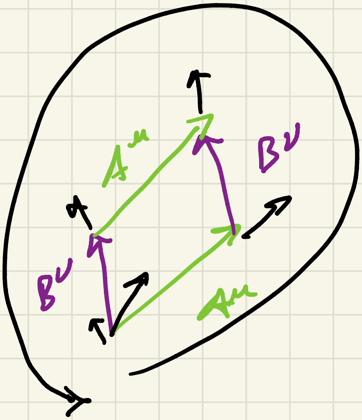


In fact, difference in parallel transported vectors over two different paths <u>defines</u> curvature!

Curvature

How much does a vector differ when parallel transported along two different paths?

$\Longleftrightarrow$ How much does it change going around a loop?



δV depends on V, A, B, so relation must have some four-index tensor...

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

At an infinitesimal scale, this can be rephrased as how covariant derivatives commute.

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$$

Ex. Compute expression for $R^\rho{}_{\sigma\mu\nu}$ (assuming torsion-free connection)

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}}$$

RIEMANN CURVATURE TENSOR

If and only if $R^\rho{}_{\sigma\mu\nu} = 0$, space-time is flat. $\Rightarrow$ "Flat" = there exists a coordinate system in which $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere

Properties of $R_{\rho\sigma\mu\nu}$ $\left(= g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu}\right)$

1. $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$

2. $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$

3. $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

4. $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$

5. $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$ (Bianchi identity)

Can perform contractions to get

$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ (Ricci tensor)

$R = R^\lambda{}_\lambda$ (Ricci scalar)

$G^{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ (Einstein tensor)

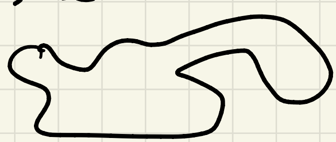
Intrinsic versus extrinsic curvature?

Intrinsic: curvature as seen by observer in manifold

Extrinsic: curvature as seen by observer in embedding space

Riemann tensor measures INTRINSIC curvature.

Ex. Is this a curved manifold? (1 dimensional)



No, because all indices of $R^\rho{}_{\sigma\mu\nu}$ are the same, but we know $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$, so the only consistent solution is $R^\rho{}_{\sigma\mu\nu} = 0$.

This is true for all 1D spaces.

$\Rightarrow$ 1D manifolds are all flat.