Lecture 18: Prandtl-Meyer Expansions

ENAE311H Aerodynamics I

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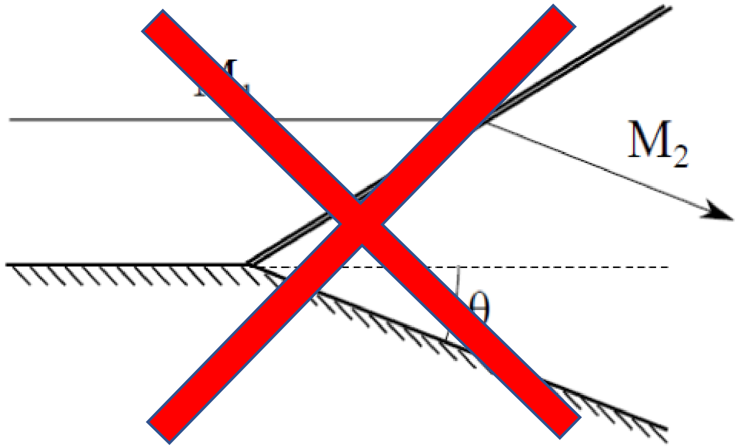
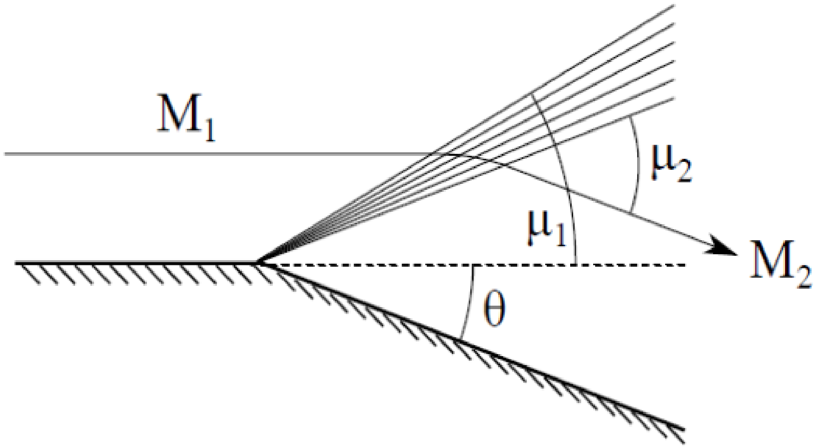
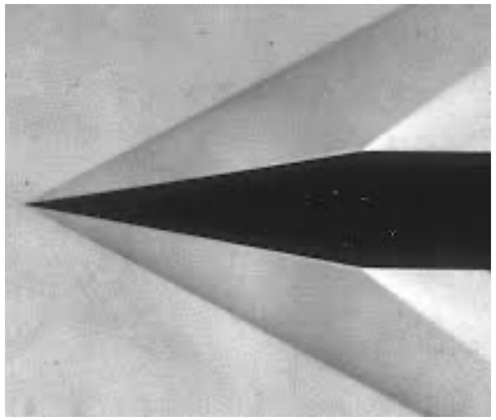
Centered expansion fans

Consider now the supersonic flow over a convex corner. Since the area is increasing, we know that this should lead to an increase in Mach number and velocity.

Unlike for a compression corner, however, this expansion cannot be achieved through a single shock, as an expansion shock would lead to a decrease in entropy.







The expansion is thus achieved via a centered expansion fan, with the flow properties changing gradually (and thus isentropically). Each ray emanating from the corner is a Mach line/wave, along which conditions are constant. The angle of the leading wave is $\mathrm{asin}(1/M_1)$, while that of the trailing wave is $\mathrm{asin}(1/M_2)$ relative to the local flow direction. Since M is increasing (and the flow is turning away), the Mach lines are diverging.

The Prandtl-Meyer function

To see how the changes in flow properties (particularly Mach number) are related to the change in flow angle, consider an infinitesimal change over one of the Mach lines.

Using the same argument as in the oblique-shock case, the tangential component of the velocity will be unchanged, while the normal component will (in this case) increase.

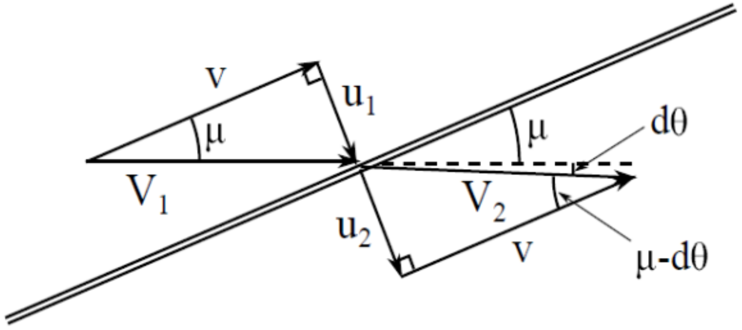
We can write:

$$\frac{V_2^2}{V_1^2} = \frac{u_2^2 + v^2}{u_1^2 + v^2} = \frac{(u_2/v)^2 + 1}{(u_1/v)^2 + 1} = \frac{\tan^2(\mu - d\theta) + 1}{\tan^2 \mu + 1}$$

$$= \frac{\cos^2 \mu}{\cos^2(\mu - d\theta)},$$

where we have used

$$\tan^2 \phi = \frac{\sin^2 \phi}{\cos^2 \phi} = \frac{1 - \cos^2 \phi}{\cos^2 \phi} = \frac{1}{\cos^2 \phi} - 1.$$



Note that $d\theta < 0$, by convention

The Prandtl-Meyer function

We have

$$\frac{V_2^2}{V_1^2} = \frac{\cos^2\mu}{\cos^2(\mu - d\theta)}$$

Now, since $d\theta$ is small, we can approximate

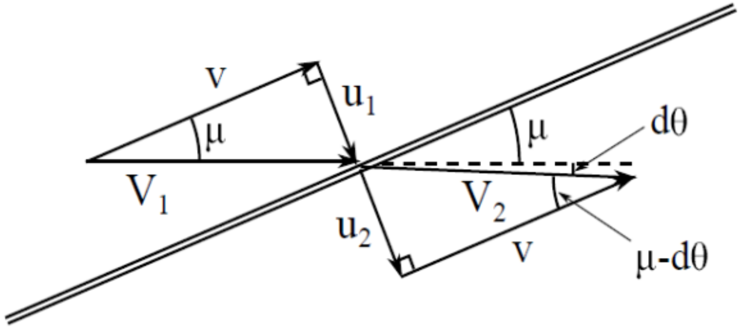
$$\begin{aligned}\cos(\mu - d\theta) &= \cos\mu\cos d\theta + \sin\mu\sin d\theta \\ &\approx \cos\mu + \sin\mu\, d\theta,\end{aligned}$$

and thus

$$\begin{aligned}\cos^2(\mu - d\theta) &\approx (\cos\mu + \sin\mu\, d\theta)^2 \\ &\approx \cos^2\mu + 2\sin\mu\cos\mu\, d\theta.\end{aligned}$$

Substituting into the above equation, we obtain

$$\begin{aligned}\frac{V_2^2}{V_1^2} &\approx \frac{\cos^2\mu}{\cos^2\mu + 2\sin\mu\cos\mu\, d\theta} \\ &= \frac{1}{1 + 2\sin\mu\, d\theta/\cos\mu}\end{aligned}$$

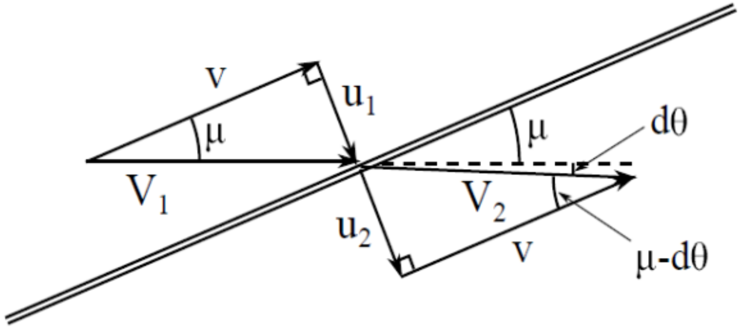


Note that $d\theta < 0$, by convention

The Prandtl-Meyer function

And thus

$$\frac{V_2}{V_1} \approx \left(1 + 2\frac{\sin\mu}{\cos\mu}d\theta\right)^{-1/2}.$$



Note that $d\theta < 0$, by convention

Since $d\theta$ is small, and using $(1+\varepsilon)^{-1/2} \approx 1 - \frac{1}{2}\varepsilon$, we can approximate this as

$$\frac{V_2}{V_1} \approx 1 - \frac{\sin\mu}{\cos\mu}d\theta.$$

Now, since $\sin\mu = 1/M$, we can write

$$\cos\mu = \sqrt{1 - \sin^2\mu} = \sqrt{1 - \frac{1}{M_1^2}} = \frac{1}{M_1}\sqrt{M_1^2 - 1}.$$

And our above expression can be written

$$\frac{V_2}{V_1} \approx 1 - \frac{d\theta}{\sqrt{M_1^2 - 1}},$$

or

$$\frac{\Delta V}{V_1} = \frac{V_2 - V_1}{V_1} \approx -\frac{d\theta}{\sqrt{M_1^2 - 1}}.$$

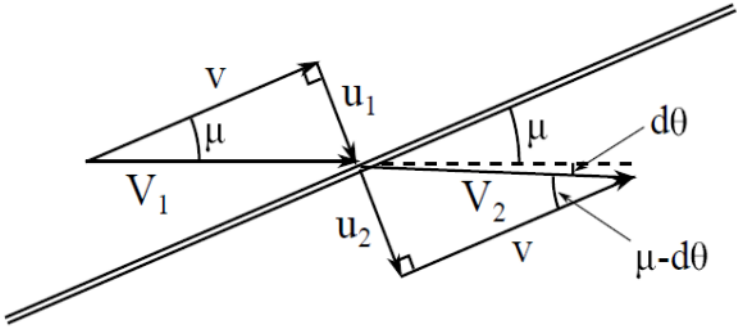
The Prandtl-Meyer function

In the limit of $d\theta \to 0$, our approximate expression

$$\frac{\Delta V}{V_1} = \frac{V_2 - V_1}{V_1} \approx -\frac{d\theta}{\sqrt{M_1^2 - 1}}.$$

becomes exact, i.e.,

$$d\theta = -\sqrt{M^2 - 1}\frac{dV}{V}.$$



Note that $d\theta < 0$, by convention

For a finite turning angle, we can thus integrate:

$$\theta = \int_0^\theta d\theta' = -\int_{V_1}^{V_2} \sqrt{M^2 - 1}\frac{dV}{V}.$$

To evaluate the integral on the RHS, we need to write dV/V in terms of the Mach number. To do so, note first that from $V = Ma$ we can write

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}.$$

Also, since the flow is adiabatic, we have

$$\left(\frac{a_0}{a}\right)^2 = \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2$$

Differentiating, we have

$$\frac{da}{a} = -\frac{(\gamma - 1)M^2}{2 + (\gamma - 1)M^2}\frac{dM}{M}$$

and so

$$\frac{dV}{V} = \frac{2}{2 + (\gamma - 1)M^2}\frac{dM}{M}.$$

The Prandtl-Meyer function

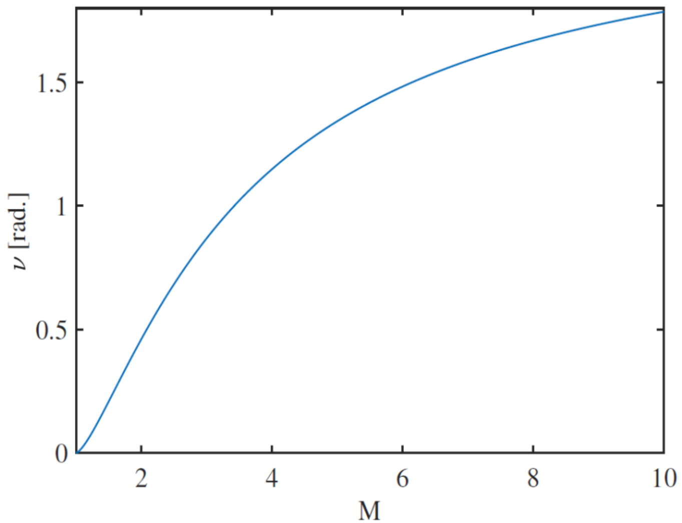
Our integral therefore becomes

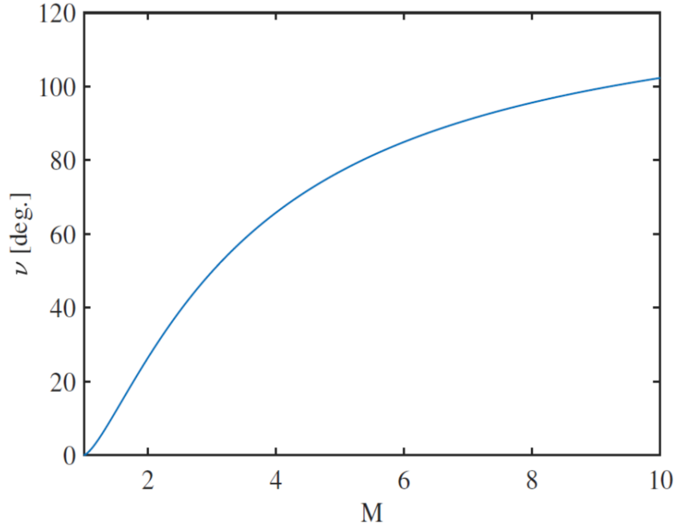
$$\theta = -\int_{M_1}^{M_2} \frac{2\sqrt{M^2-1}}{2+(\gamma-1)M^2}\frac{dM}{M}.$$

The function

$$\begin{aligned}\nu(M) &= \int \frac{2\sqrt{M^2-1}}{2+(\gamma-1)M^2}\frac{dM}{M} \\ &= \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \arctan\sqrt{M^2-1}\end{aligned}$$

is known as the *Prandtl-Meyer function*. The integration constant has been set such that $\nu(1) = 0$.





Expansions and compressions

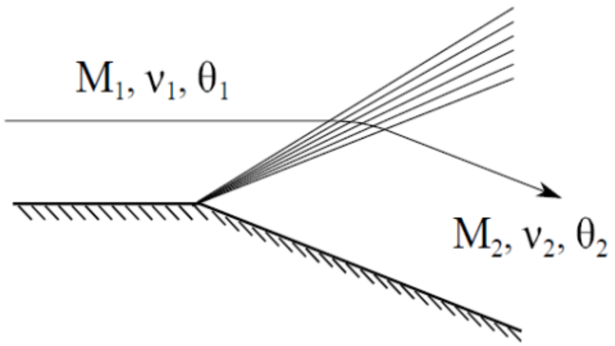
We thus have

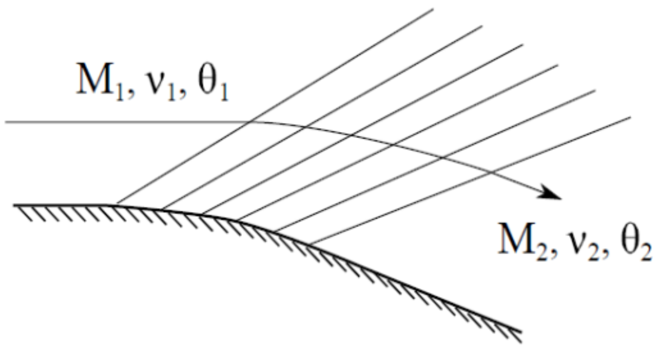
$$\theta = -[\nu(M_2) - \nu(M_1)]$$

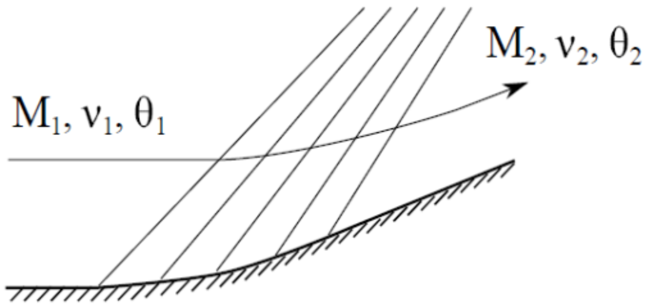
where

$$\nu(M) \quad = \quad \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan \sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \arctan \sqrt{M^2-1}$$

Since the expansion is isentropic, however, it doesn't matter whether it is achieved at a single corner or gradually over a continuous bend. Also, if a compression is gradual enough that no shocks form, it will also be isentropic and the foregoing analysis applies (with $d\theta > 0$). To avoid ambiguity in sign then, we distinguish as:







$$\text{Expansion:} \ \nu \quad = \quad \nu_1 + |\theta - \theta_1|$$

$$\text{Compression:} \ \nu \quad = \quad \nu_1 - |\theta - \theta_1|$$

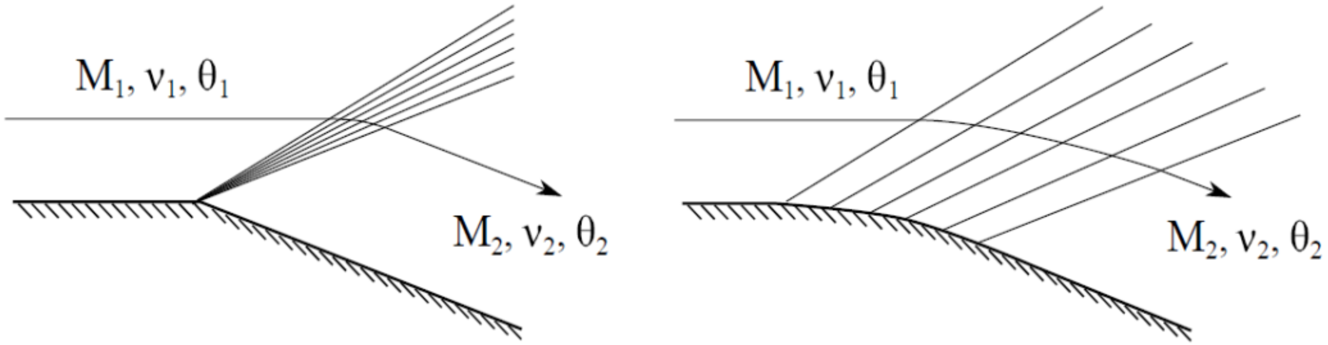
Expansions and compressions

To solve then for the flow through an expansion or isentropic compression with a given M_1 and $\Delta\theta = \theta - \theta_1$, first calculate $\nu(M_1)$ (using the Prandtl-Meyer formula directly or tables); $\nu_2 = \nu(M_2)$, can then be derived using the appropriate formula below.

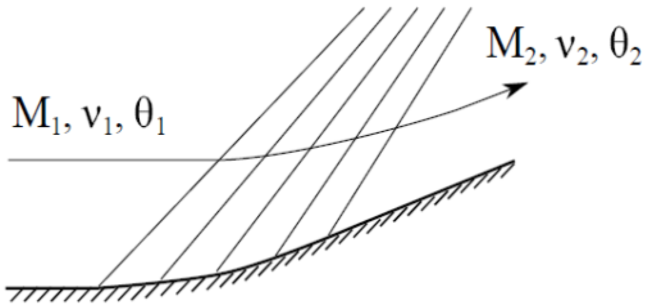
Knowing $\nu(M_2)$, M_2 can then be obtained either numerically, graphically, or through tables.

Assuming the other upstream flow properties are available, we can use the fact that T_0, p_0, and ρ_0 are all constant (flow is adiabatic and isentropic) to derive the downstream flow conditions, e.g.,

$$\frac{T_2}{T_1} = \frac{T_2}{T_0}\frac{T_0}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2} \quad \text{and} \quad \frac{p_2}{p_1} = \frac{p_2}{p_0}\frac{p_0}{p_1} = \left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{\gamma/(\gamma-1)}$$



Expansion: $\nu = \nu_1 + |\theta - \theta_1|$



Compression: $\nu = \nu_1 - |\theta - \theta_1|$

Maximum expansion angle

Finally, note that although $\nu(M)$ increases monotonically with M, it asymptotes to a maximum value given by

$$\nu_{max} = \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right).$$

For $\gamma = 1.4$, this takes the value of 130.45°.

Thus, for a flow initially at a Mach number of 1, if it encounters an expansion with a larger angle than this, it will not be able to turn all the way and will separate once it reaches ν_{max}. For flows initially at higher Mach numbers, the maximum turning angle is reduced accordingly (for Mach 5, for instance, it is 53.45°).