Reliability Analysis

Module 3: Elements of Component Reliability

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Objectives for Module 3

- Introduce methods for assessing reliability of components
- Define key terms: Reliability, MTTF, MRL, hazard rate, etc.
- Discuss & use several probability distributions commonly used in reliability engineering

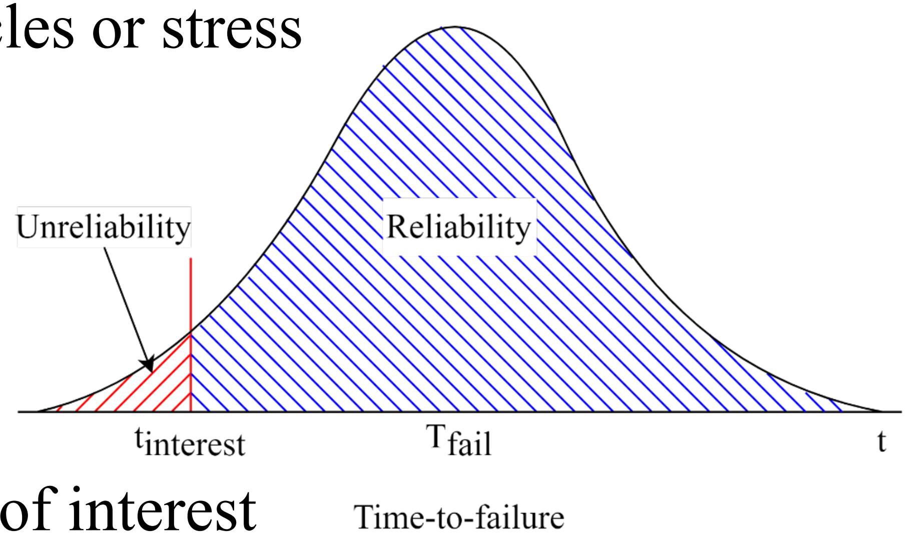
Design for reliability (Robustness)

- Extend item life by controlling or eliminating potential failure modes
  - Designing stronger, more durable items
  - Reducing harmful environmental conditions, uses, etc.
  - Minimizing / controlling loads and stresses
  - Condition monitoring and maintenance programs
- This increases reliability, i.e., the probability of successful achievement of item's function (mission)

Recall: probabilistic definition of reliability

- **Reliability:** The ability of an item to perform its expected mission under designated operating conditions for a designated period of time, number of cycles or stress

$$\boxed{R(t) = \Pr(T_{fail} > t_{interest})}$$



Where:

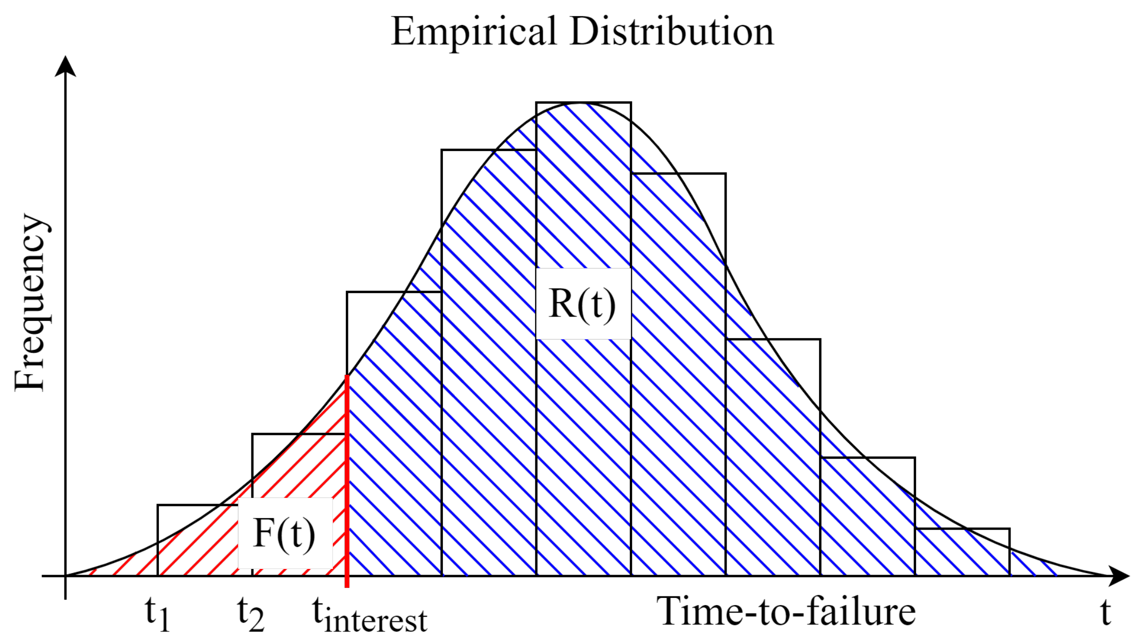
- $t_{interest}$ = mission time or time of interest
- T_{fail} = time-to-failure, cycle-to-failure, stress-to-failure, etc.

Note:

For many component reliability problems: *Time* is an *aggregate* "agent" of the failure; implies conditions are not necessary to model

Intuition: Defining reliability for non-repairable items

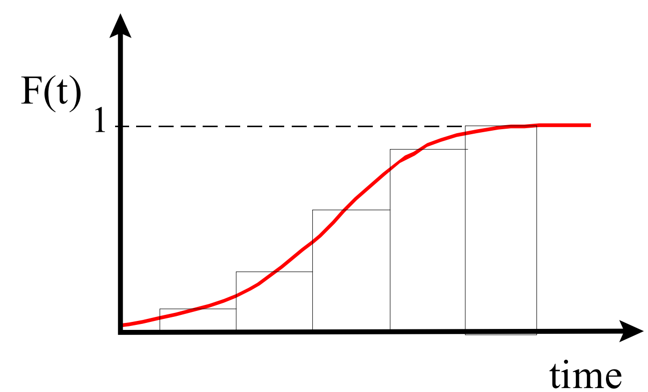
- Let there be many identical items that are subjected to life tests. As time goes by, each item will function for some time and then fail. We plot the frequency of time to failure (of these non-repairable items), we get a continuous time to failure (TTF) distribution.



f(t) = probability density function (pdf) for the *failure times*

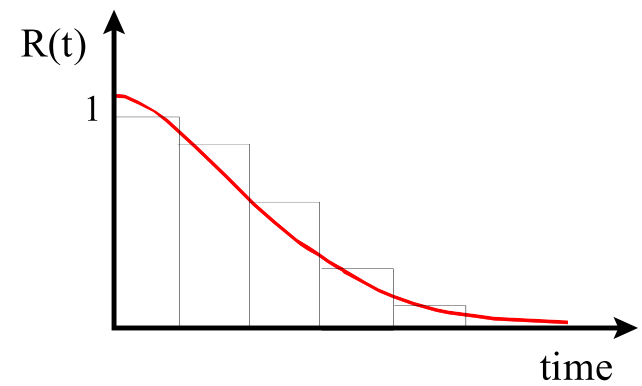
Definition of unreliability

- **Unreliability** of an item: The item fails before the mission is complete. That is, the **failure time** (T_{fail}) of the item is less than the **mission time** (t) of the system.
  - In mathematical notation: the probability that the item fails at or before *t*. (or "sometime up to mission time *t*"):
  - $\Pr\left(T_{fail} \leq t\right) = F(t) = \int_0^t f(x)dx$ ← CDF for continuous variable *T*
  - $\Pr\left(N_{fail} \leq n\right) = F(n) = \sum_{i=1}^{n} f(i)$ ← CDF for discrete variable *N* (e.g., number of cycles).

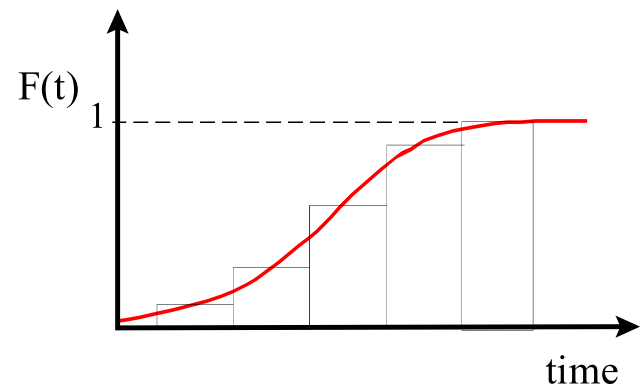


Definition of reliability

- The **Reliability** function (or **survival function**) item is therefore:
  - $R(t) = \Pr(T_{\text{fail}} > t) = 1 - F(t)$
    $$= \int_t^{\infty} f(x)dx$$



  - Since these are probabilities:
  - $0 \leq F(t) \leq 1$
  - $0 \leq R(t) \leq 1$
  - $f(t) \geq 0 \quad -\infty < t < \infty$
  - $\int_{-\infty}^{\infty} f(x)dx = 1$



Sidebar on notation

- For continuous distributions: $\Pr(a > X)$ is the same as $\Pr(a \geq X)$
- For discrete distributions:
  - You must know & define the edge of your set (i.e., "greater than" vs. "greater than or equal to.")

- Be very careful working with a distribution where the difference between "$< a$" and "$\leq a$" matters! (E.g., if the item fails exactly at time $T = t$)
  - Some reliability publications use $R(\mathrm{n}) = \Pr(\mathrm{N} \geq n)$ which results in $R(\mathrm{n}) \neq 1 - \mathrm{F}(\mathrm{n})$

*In the field, notation may not be consistent or readily apparent.
Always check that: $\mathrm{R(t)} + \mathrm{F(t)} = 1$
and adjust accordingly.

Reliability definitions: Conditional reliability

- **Conditional Reliability Function**

If an item has survived to time *t*, what is the probability that the item will survive for additional time *x*?

$$R(t + x|t) = \frac{R(t + x)}{R(t)}$$

Reliability definitions: MTTF

- **Mean-Time-To-Failure (MTTF)**
- In a population of items, the items will not all fail at the same time. As such, we get a distribution of time-to-failure.
- The **expected (mean)** time-to-failure is called MTTF, and sometimes the expected life.

We can estimate MTTF from a distribution:

$$\mathrm{MTTF} = \mathrm{E(t)} = \int_0^{\infty} t f(t) dt \quad \text{or} \quad E(n) = \sum_{i=1}^{\infty} n_i Pr(n_i)$$

We can also prove that $\mathbf{MTTF} = \int_0^{\infty} \mathbf{R}(\boldsymbol{t}) \boldsymbol{dt}$ for cases where $\lim_{t \to \infty} R(t) = 0$

Reliability definitions (cont.)

- **Mean-Time-Between-Failure (MTBF)** – Applies only to repairable components.

When the items are repairable, we have two random variables.

i. Time-between-failures of an item,

ii. Time-to-repair.

This topic will be discussed later when we discuss repairable items (Chapter 7).

Reliability definitions: MRL

- **Mean Residual Life**

If an item has survived up to a certain time, *t*, then the expected remaining time to failure (i.e., expected remaining life) is called the residual mean time to failure, or simply the **mean residual life** (MRL):

$$\mathrm{MRL(t)} = \int_0^\infty R(x|t)dx = \frac{1}{\mathrm{R}(t)}\int_t^\infty \mathrm{R(t')dt'}$$

$$Note: t' = x + t$$

$$\text{If } t = 0, \text{then } MRL = MTTF = \int_0^\infty R(t)dt$$

Hazard rate (failure rate) defined

- ***Hazard rate (or failure rate)*** is the instantaneous rate of failure for an item of age t over a period of time (Δt) as Δt tends to zero.
- Think of it as: the conditional probability that an item which has survived until time t will fail during the following small time interval (Δt)

$$h(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{F(t + \Delta t) - F(t)}{R(t)} = \frac{f(t)}{R(t)}$$

(Since $\Pr(t_1 < \mathrm{T} < \mathrm{t}_2 | \mathrm{T} > \mathrm{t}_1) = \frac{F(\mathrm{t}_2) - F(\mathrm{t}_1)}{\mathrm{R}(\mathrm{t}_1)} = \frac{\Delta F(t)}{R(t)}$)

- **Interpretation**: the greater the hazard rate between times t_1 and t_2, the greater the chance of failure in this time interval

Reliability definitions (cont.)

- It can be shown that *h(t)* can be expressed in terms of the reliability function (using definition of derivative of the ln of a function).

$$h(t) = \frac{f(t)}{R(t)} = -\frac{d}{dt}\left[\ln\left(R(t)\right)\right]$$

- As with the pdf and cdf, a **cumulative hazard function *H(t)*** can be obtained from the integral of the hazard function:

$$H(t) = \int_0^t h(x)dx$$

- We can use H(t) to obtain another expression for reliability:

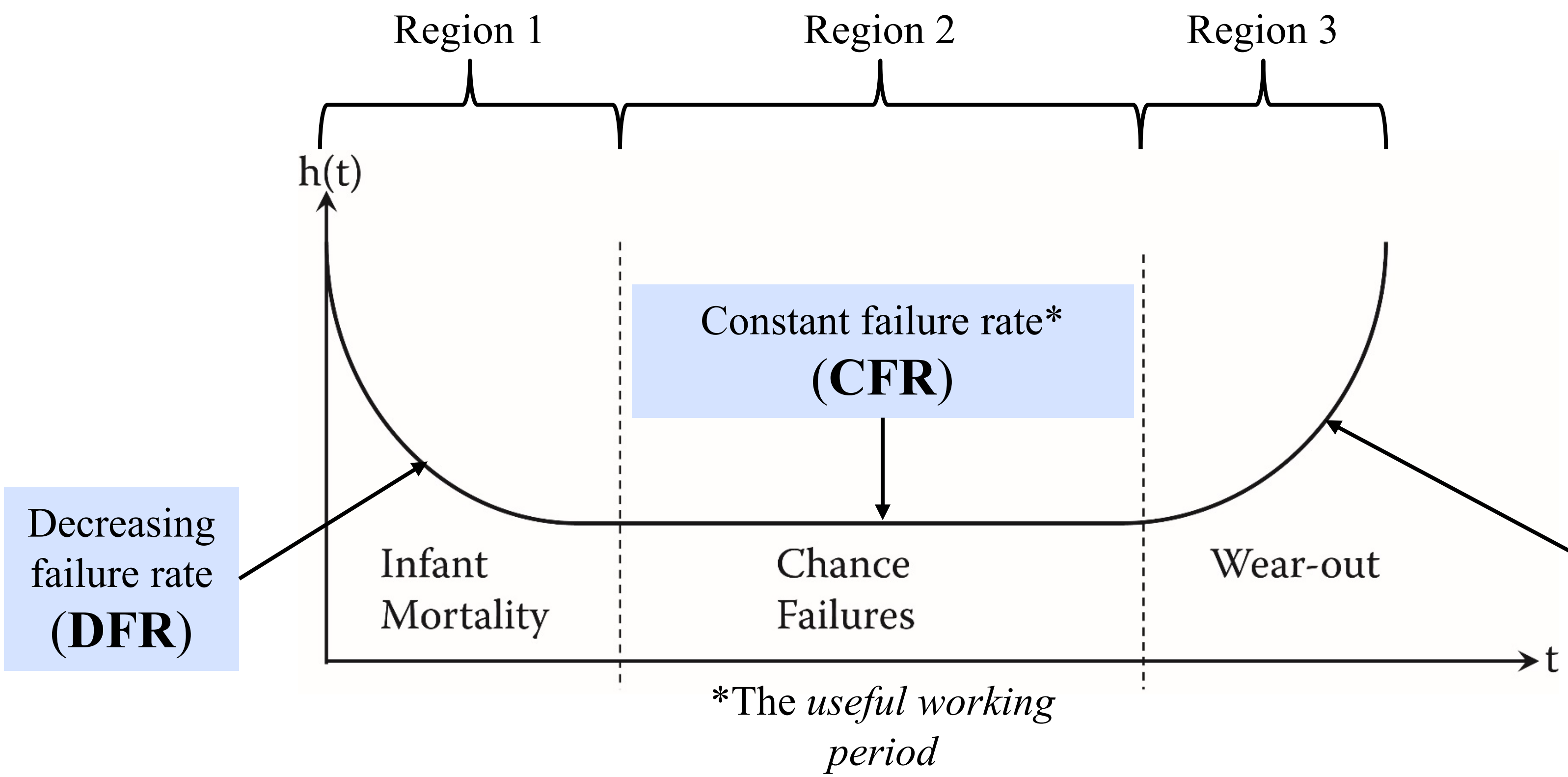
$$R(t) = e^{-\int_0^t h(x)dx} = e^{-H(t)}$$

Reminder: Key relationships

In terms of…	f(t)	F(t)	R(t)	h(t)	H(t)
f(t) =	-	$\frac{dF(t)}{dt}$	$-\frac{dR(t)}{dt}$	$h(t)e^{\left[-\int_0^t h(x)dx\right]}$	$\frac{dH(t)}{dt}e^{[-H(t)]}$
F(t) =	$\int_{-\infty}^{t} f(x)dx$	-	$1 - R(t)$	$1 - e^{\left[-\int_0^t h(x)dx\right]}$	$1 - e^{[-H(t)]}$
R(t) =	$\int_{t}^{\infty} f(x)dx$	$1 - F(t)$	-	$e^{\left[-\int_0^t h(x)dx\right]}$	$e^{[-H(t)]}$
h(t) =	$\frac{f(t)}{\int_t^\infty f(x)dx}$	$\frac{\frac{dF(t)}{dt}}{1-F(t)}$	$-\frac{\frac{dR(t)}{dt}}{R(t)}$	-	$\frac{dH(t)}{dt}$
H(t) =	$-ln\left[\int_t^\infty f(x)dx\right]$	$-ln[1 - F(t)]$	$-ln[R(t)]$	$\int_0^t h(x)dx$	-

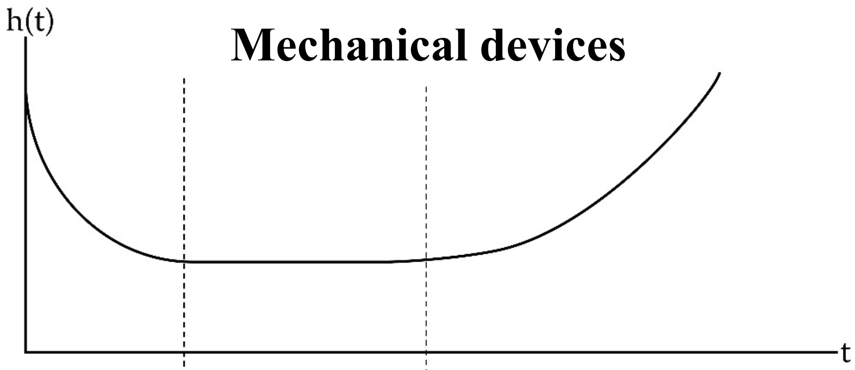
Hazard rate trends: Bathtub curve

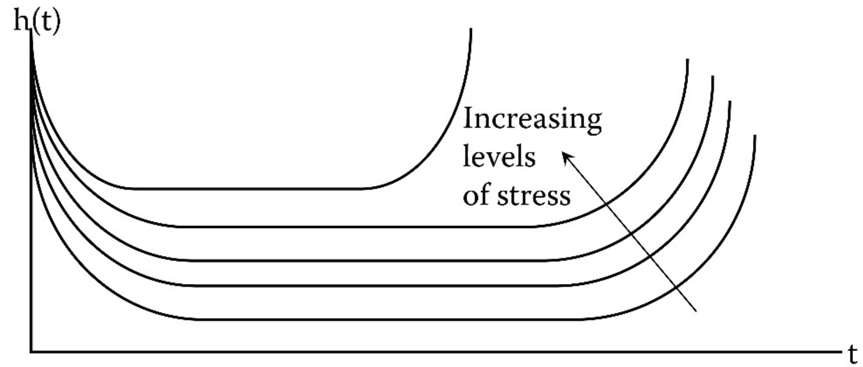
- The hazard rate *h(t)* shows changes in the probability of failure over the lifetime of a component
- In practice, *h(t)* often forms the **bathtub curve**

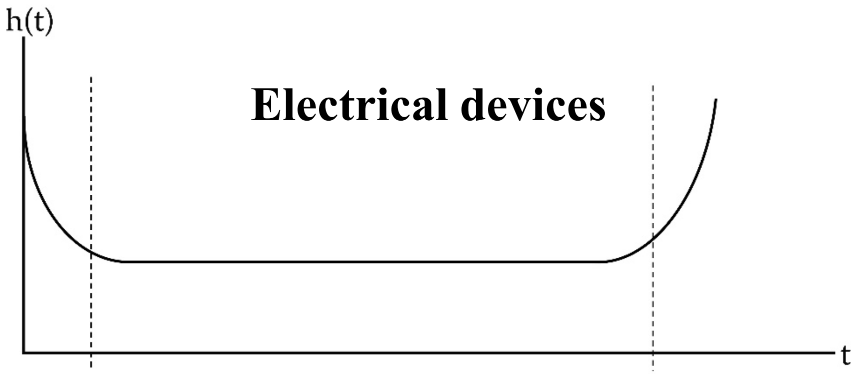


Typical bathtub curves for devices

- The hazard rate *h(t)* can also show differences between component types (e.g., mechanical vs. electrical) and the effects of stress:







Note: electrical devices tend to have longer useful working period (CFR) and shorter burn-in (DFR) and wear-out (IFR) periods.

Increasing the stress level shortens the useful life period.

Bathtub curve (cont.)

- The curve is characterized by three distinct regions:
  - **Region 1 (DFR)** The failure occurs early in the life where the item has a high, but decreasing failure rate. Failures are caused due to initial defects like defective design, poor material, etc.
  - **Region 2 (CFR)** The failures are random, i.e., difficult to predict and failure rate is approximately constant over the time. This is the "useful working period" of the component.
  - **Region 3 (IFR)** As the item reaches the end of life, it starts to deteriorate and wear out. Failures keep increasing and results in a fast-increasing hazard rate.

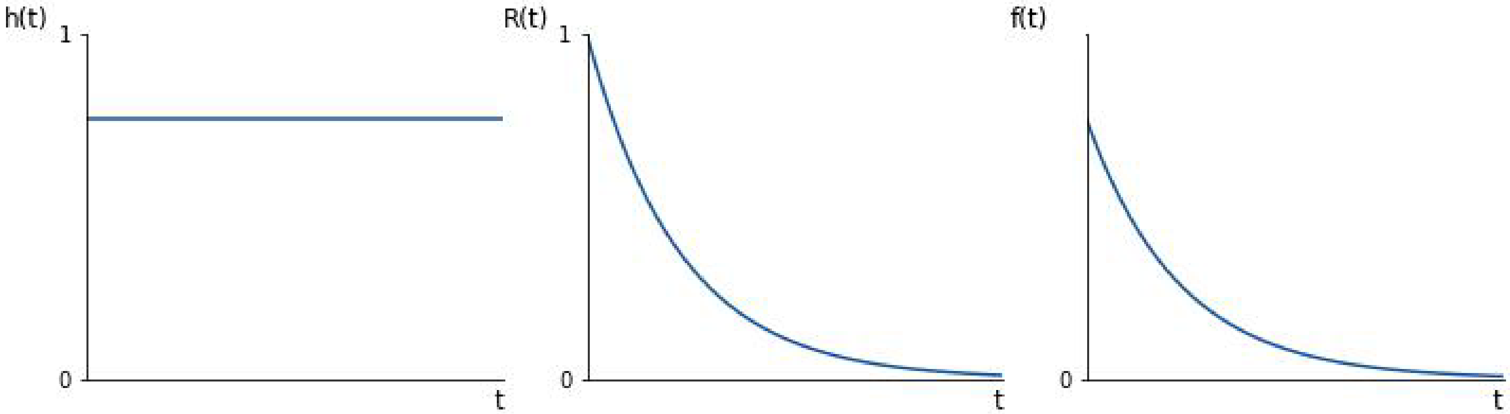
Three distinct hazard rates

- **Case I – Constant failure rate (λ)**
  - That is, $h(t) = \lambda$, and:

$$\mathrm{R(t) = e^{-\int_0^t h(x)dx} = e^{-\int_0^t \lambda dx} = e^{-\lambda \int_0^t dx} = e^{-\lambda t}}$$

$$\mathrm{f(t) = h(t) \cdot R(t) = \lambda e^{-\lambda t}}$$

This is the **exponential distribution**



So, the random failure region can be modeled by an **exponential distribution model**.

Three distinct hazard rates (cont.)

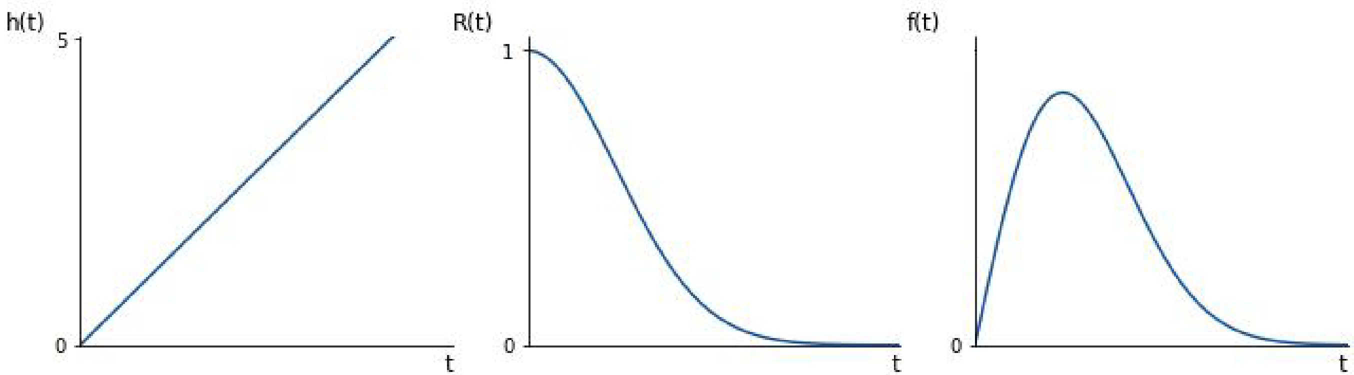
- **Case II – Increasing failure rate**
  - For example, a linearly increasing hazard rate:

$$h(t) = \beta t \text{ (where } \beta \text{ is a constant), and:}$$

$$R(t) = e^{-\int_0^t h(x)dx} = e^{-\int_0^t \beta x dx} = e^{-\beta \int_0^t x dx} = e^{-\frac{1}{2}\beta t^2}$$

$$f(t) = R(t) \cdot h(t) = e^{-\frac{1}{2}\beta t^2} \cdot \beta t = \beta t e^{-\frac{1}{2}\beta t^2}$$

  - This is the Rayleigh distribution. The wear out region in Bathtub curve can be modeled by this distribution. Other distributions may also be used, depending on failure data.



Three distinct hazard rates (cont.)

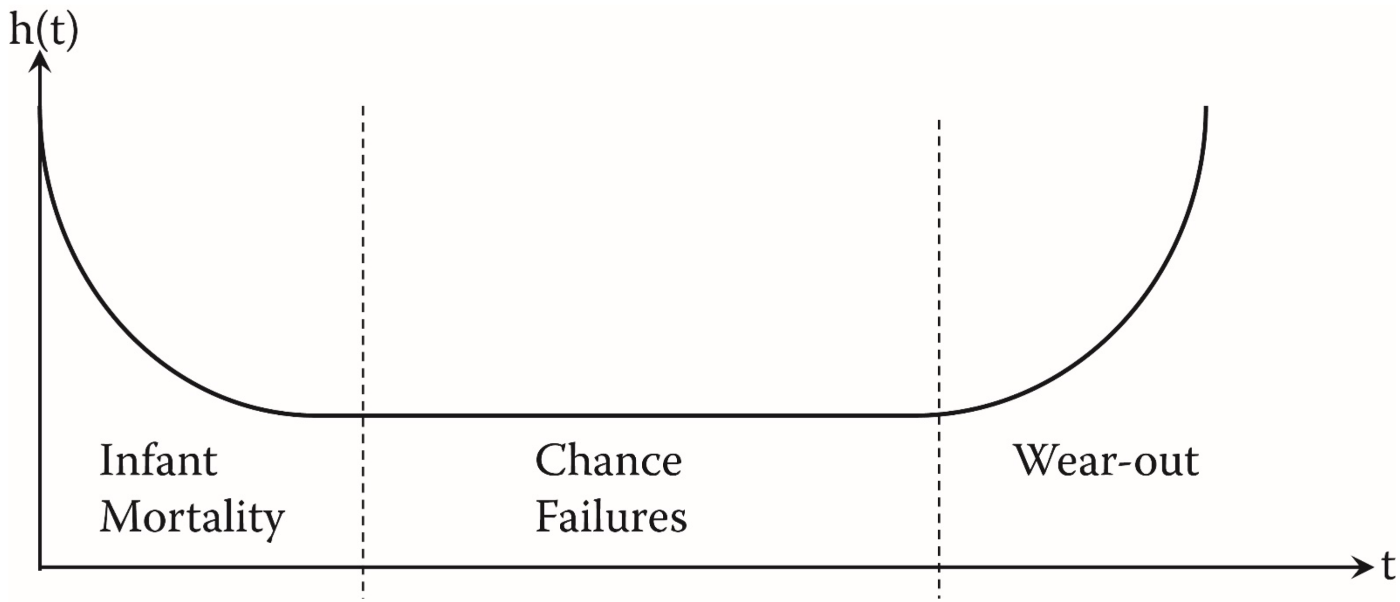
- **Case III – Decreasing failure rate**

This case is similar to Case II, only the hazard rate is a decreasing function of time.

$$h(t) = -\beta t$$

Recapitulation

- $R(t) = Pr(T > t) = e^{-\int_0^t h(x)dx}$ where *T* is a random variable, represented by a time-to-failure distribution f(t).
- MTTF is the mean of pdf f(t) representing time-to-failure
- $h(t) = \frac{f(t)}{R(t)}$



Common distributions in component reliability

- An item's reliability may be represented by many probability distributions; popular distributions in reliability include:
  - Exponential
  - Weibull
  - Gamma
  - Normal
  - Lognormal
- The distributions have their own form of reliability, hazard rate, and MTTF functions.
- Each has merits for different situations.

There is no one-size-fits-all distribution for *any* item

Exponential distribution in reliability modeling

- $T \sim \exp(\lambda)$

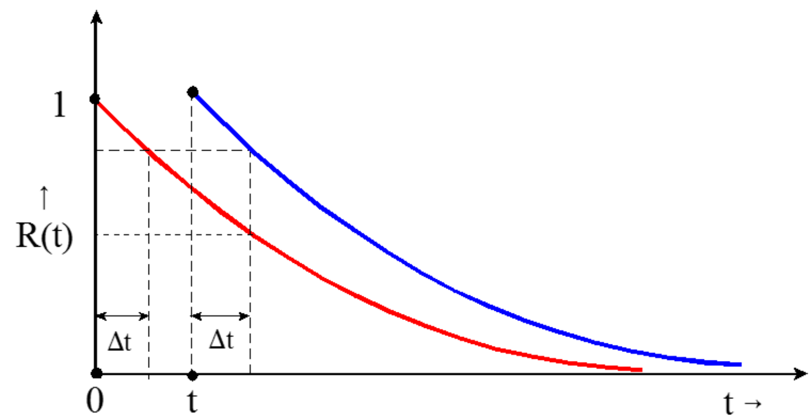
$$f(t) = \lambda e^{-\lambda t}$$

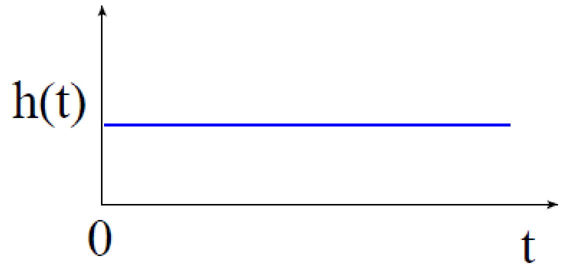
$$F(t) = 1 - e^{-\lambda t}$$

$$R(t) = e^{-\lambda t}$$

$$h(t) = \lambda$$

$$\text{MTTF} = \frac{1}{\lambda}$$





- Exponential distribution has no memory. It is often used for mathematical simplicity.

Uses of the exponential distribution

- Early efforts at collecting reliability data assumed constant failure rates and thus many reliability handbooks provide only CFRs components.
- **Electronic Components**: Some electronic components like capacitors or integrated circuits have been found to follow an exponential distribution.
- **Random Shocks**: Time to the occurrence of random shocks. An example is the failure of a vehicle tire due to puncture from a nail (random shock). The probability of failure in each mile is independent of how many miles the tire has travelled (memoryless). The failure rate when the tire is new is the same as when the tire is old.
- Modeling complex systems (due to need for mathematical simplicity)

In general, many items do not have a constant failure rate, for example due to wear or early failures. Thus, the exponential distribution is often inappropriate to model most life distributions, particularly mechanical components.

Example: Exponential TTF

- **Example:** Consider a component with exponentially distributed time-to-failure, $T \sim \exp(\lambda)$ where $\lambda = 1 \times 10^{-3} \text{hr}^{-1}$. Calculate reliability at 700 hours and calculate MTTF.

Example: Exponential TTF

- **Solution:** Consider a component with exponentially distributed time-to-failure, $T \sim \exp(\lambda)$ where $\lambda = 1 \times 10^{-3} \text{hr}^{-1}$. Calculate reliability at 700 hours and MTTF.
- Calculate R(t = 700hrs) and MTTF

$$\mathrm{R}(\mathrm{t} = 700) = \mathrm{e}^{-\lambda \mathrm{t}} = \mathrm{e}^{-10^{-3} \cdot 700} = \mathrm{e}^{-.7} = \mathbf{0.4966}$$

$$MTTF = \frac{1}{10^{-3}} = \mathbf{1000\ hrs}$$

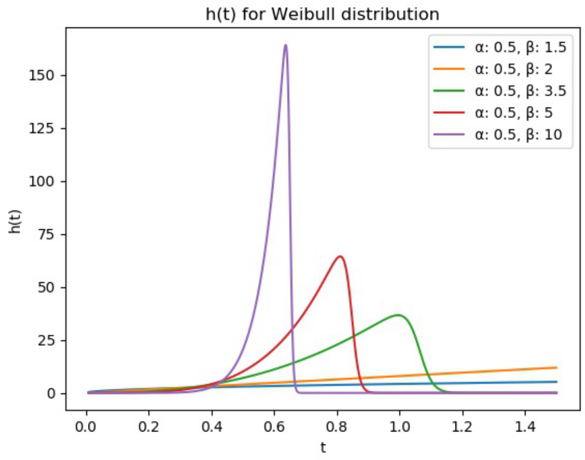
Weibull distribution in reliability modeling

$$f(t) = \frac{\beta t^{\beta-1}}{\alpha^{\beta}} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}$$

$$R(t) = e^{-\left(\frac{t}{\alpha}\right)^{\beta}}$$

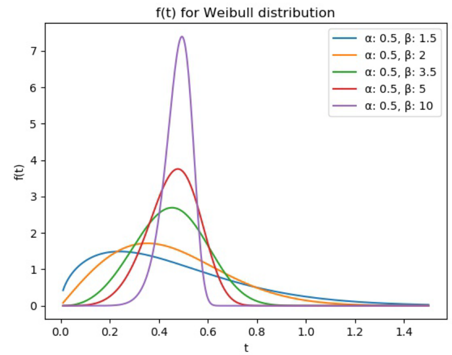
$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{\beta t^{\beta-1}}{\alpha^{\beta}} e^{-\left(\frac{t}{\alpha}\right)\beta}}{e^{-\left(\frac{t}{\alpha}\right)\beta}} = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}$$

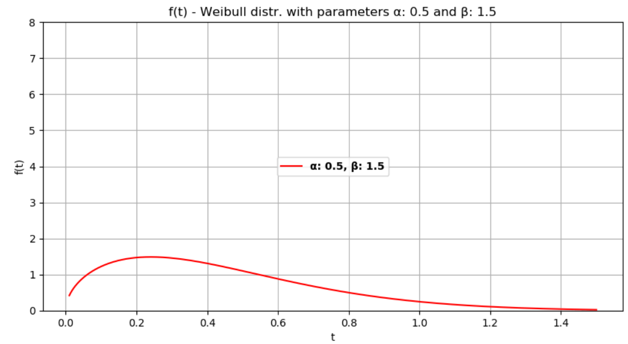
$$MTTF = \alpha\Gamma\left(\frac{\beta+1}{\beta}\right)$$



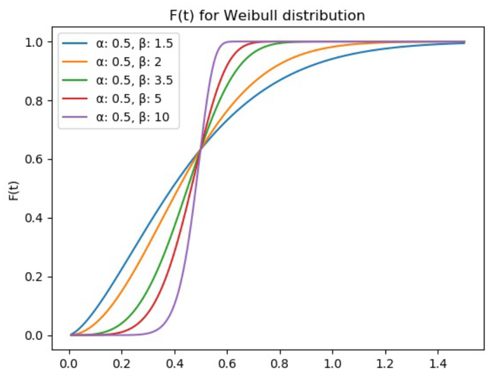
- Where,
  - α = scale parameter
  - β = shape parameter

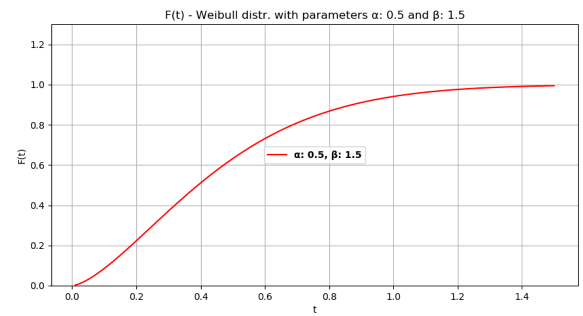
Weibull distribution (cont.)



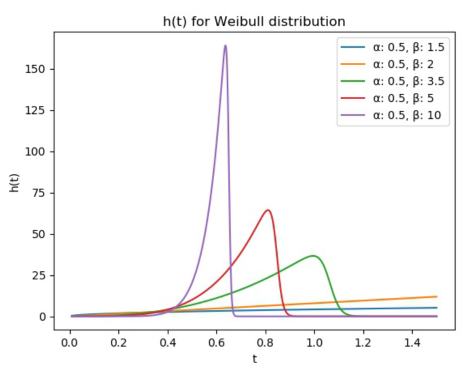


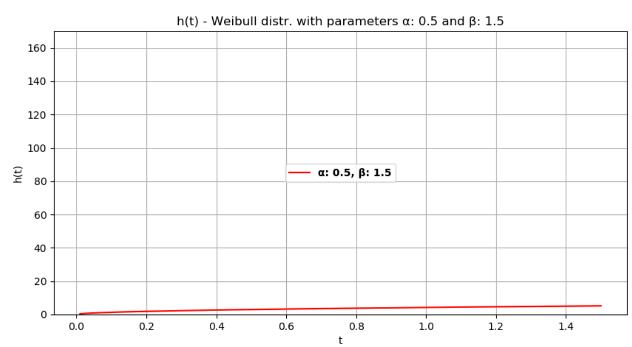
f(t)





F(t)





h(t)

Uses of the Weibull distribution

- Prevalent in modeling time to failure of many types of basic component (capacitors, ball bearings, motors, transistors, etc.)
- Corrosion caused failure
- Weakest link models

- Also see 3-parameter Weibull distribution used commonly.

Example: Weibull TTF

- **Example:** Lab testing shows that a time-to-failure model of T$\sim weibull(\alpha = 230\ hr, \beta = 2.1)$ is a good fit for a set of newly manufactured roller bearings.

a) Calculate the reliability of a bearing at 240 hours

b) Calculate the MTTF of the bearings.

c) If time: Plot the hazard rate.

Example: Weibull TTF

- **Example Solution:** Lab testing shows that a time-to-failure model of $T \sim weibull(\alpha = 230\ hr, \beta = 2.1)$ is a good fit for a set of newly manufactured roller bearings.

a) Calculate the reliability of a bearing at 240 hours

$$R(t = 240) = e^{-\left(\frac{x}{\alpha}\right)^{\beta}} = e^{-\left(\frac{240}{230}\right)^{2.1}} = \mathbf{0.335}$$

b) Calculate the MTTF of the bearings

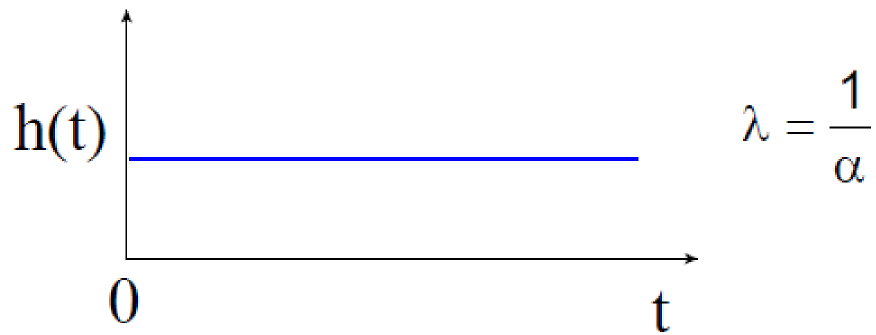
$$MTTF = \alpha\Gamma\left(1 + \frac{1}{\beta}\right) = 230 \cdot \Gamma(1.476) = 230 \cdot 0.886 = \mathbf{203.7\ hrs}$$

Use of Weibull distribution

- **Bathtub Case I** – $\beta = 1$

  That is, $h(t) = \frac{1}{\alpha}\left(\frac{t}{\alpha}\right)^{1-1} = \frac{1}{\alpha}$

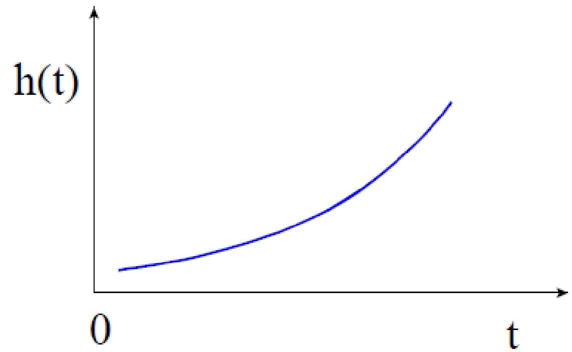
- So it is the same as an exponential distribution with $\lambda = \frac{1}{\alpha}$ and $\text{MTTF} = \alpha$.



Use of Weibull distribution (cont.)

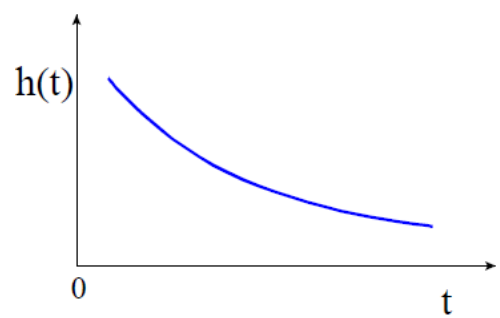
- **Bathtub Case II** – $\beta > 1$

  That is, h(t) is monotonically increasing



- **Bathtub Case III** – $\beta < 1$

  That is, h(t) is monotonically decreasing



Weibull distribution hazard rates

- **1 < β < 2**: The hazard rate increases less as time increases
- **β = 2**: The hazard rate increases with a linear relationship to time
- **β > 2**: The hazard rate increases more as time increases
- **β < 3.447798**: The distribution is positively skewed (Tail to right)
- **β ≈ 3.447798**: The distribution is approximately symmetrical
- **3 < β < 4**: The distribution approximates a normal distribution
- **β > 10**: The distribution approximates a smallest extreme value distribution

Gamma distribution in reliability modeling

$$f(t) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} t^{\alpha-1} e^{-t/\beta} \qquad \text{(For integer } \alpha \text{ recall that: } \Gamma(\alpha) = (\alpha-1)!\text{)}$$

$$R(t) = \begin{cases} 1 - \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} \displaystyle\int_0^t x^{\alpha-1}\, e^{\left(-\frac{x}{\beta}\right)} dx & t, \alpha, \beta \geq 0 \text{ and } \alpha \text{ is continuous} \\ e^{-\frac{t}{\beta}} \displaystyle\sum_{n=0}^{\alpha-1} \dfrac{t^n}{\beta^n n!} & t, \alpha, \beta \geq 0 \text{ and } \alpha \textit{ is} \text{ an integer} \end{cases}$$

$$h(t) = \frac{f(t)}{R(t)} = \begin{cases} \dfrac{t^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha, \frac{t}{\beta})} e^{\left(-\frac{t}{\beta}\right)} & t, \alpha, \beta > 0 \text{ and } \alpha \text{ is continuous} \\ \dfrac{t^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha) \sum_{n=0}^{\alpha-1} \frac{t^n}{\beta^n n!}} & t, \alpha, \beta > 0 \text{ and } \alpha \text{ is an integer} \end{cases}$$

Where:

- α = shape parameter given by number of events required before failure, maintenance, etc. (α=n)
- β = scale parameter given by mean time to occurrence of one event (sometimes written as $\lambda = 1/\beta$)(and α, β, t ≥ 0)

Uses of gamma distribution

- Time to failure after being subjected to α random Poisson events
- Sum of α independent exponential variables
- Time between maintenance for an item maintained every α uses
- TTF of a system with standby components and independent failures
- Distribution of time between maintenance of items, instruments, and systems after they have been used α times.

Example: Gamma distribution

- **Example**: A generator manufacturer suggests conducting an oil change once every 140 days. (Assume that the time between oil change, T_o, follows an exponential distribution). If there is also a rule in place that says that after 5 oil changes, the oil filter must be replaced,

a) What is the distribution representing the time to replacement of the oil filter (T_R)?

b) What is the mean time to replacement of the oil filter?

c) What is the probability that the generator does not need a new oil filter at its 200-day inspection (that is, what is the reliability at this time)?

Example: Gamma distribution (cont.)

- **Solution**:

a) What is the distribution representing the time to replacement of the oil filter (T_R)?

- For T_R we have a mean time to adjustment of 140 days. Let this be the β value or mean time to occurrence.
- Likewise, α represents the number of required before the occurrence or replacement and this has been identified as 5.
- Therefore, the distribution of T_R is gamma with:

$$\alpha = 5 \text{ event}, \beta = 140 \text{ hr}$$
$$\boldsymbol{T_R \sim gamma(5, 140)}$$

b) What is the mean time to replacement of the oil filter?

- Simply compute the MTTR as,

$$\text{MTTR} = E(T_R) = \alpha\beta$$
$$= 5 \cdot 140$$
$$= \mathbf{700 \text{ days}}$$

Example: Gamma distribution (cont.)

- **Solution:**

c) What is the probability that the engine does not need a new oil filter at its 200 day inspection (that is, what is the reliability at 200 days)?

- Calculate the reliability where t = 200 days as,

$$R(t) = \exp\left(-\frac{t}{\beta}\right)\sum_{n=0}^{\alpha-1}\frac{t^n}{\beta^n n!}$$

$$= \exp\left(-\frac{200}{140}\right)\sum_{n=0}^{5-1}\frac{200^n}{140^n n!}$$

$$= e^{\left(-\frac{200}{140}\right)}\left[\frac{(200/140)^0}{0!}+\frac{(200/140)^1}{1!}+\frac{(200/140)^2}{2!}+\frac{(200/140)^3}{3!}+\frac{(200/140)^4}{4!}\right]$$

$$= 0.24 \cdot \ (1 + 1.43 + 1.02 + 0.486 + 0.174)$$

$$= \mathbf{0.985}$$

Normal distribution in reliability modeling

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right]}$$

$$R(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right]}}{1 - \Phi\left(\frac{t-\mu}{\sigma}\right)}$$

Where:

- μ is the mean and σ is the standard deviation

Uses of Normal distribution

- Stress-strength modeling
- Simple repair/inspection tasks with "typical" duration and variance symmetrical about the mean
- Modeling processes that a sum of a large number of independent random variables (see: central limit theorem)
  - Approximation to other distributions

Example: Normal distribution

- **Example:** A component's stress to failure is normally distributed with MSTF of 20 kg/cm^2 and standard deviation of 3 kg/cm^2.

a) Find the reliability at a stress level of 25 kg/cm^2.

b) Find the probability of failure between stress levels 25 and 28 kg/cm^2.

Example: Normal distribution (cont.)

- **Solution:** A component's stress to failure is normally distributed with MSTF of 20 kg/cm^2 and standard deviation of 3 kg/cm^2.

a) Find the reliability at a stress level of 25 kg/cm^2.

$$R\left(25\frac{kg}{cm^2}\right) = 1 - \Phi\left(\frac{25-20}{3}\right) = 1 - \Phi(1.667) = 0.048$$

b) Find the probability of failure between stress levels 25 and 28 kg/cm^2.

$$F(25 < S < 28) = \Phi(2.667) - \Phi(1.667) = 0.044$$

Example: Normal distribution (cont.)

c) Find the conditional probability of failure before 28kg/cm^2, given the component has survived up to a 25 kg/cm^2 stress.

Example: Normal distribution (cont.)

c) **Solution:** Find the conditional probability of failure before 28kg, given the component has survived under a 25 kg/cm^2 stress.

$$\Pr(S < 28|S > 25) = \frac{\Pr(S < 28 \cap S > 25)}{\Pr(S > 25)} = \frac{\Pr(25 < S < 28)}{\Pr(S > 25)}$$

$$\Pr(S < 28|S > 25) = \frac{0.044}{0.048} = \mathbf{0.92}$$

The conditional survival function is:

$$\Pr(S > x + s|S > s) = \frac{\Pr(S > 28)}{\Pr(S > 25)} = \frac{R(28)}{R(25)} = \frac{0.0038}{0.048} = \mathbf{0.08}$$

Lognormal distribution in reliability modeling

$$f(t) = \frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\ln t-\mu)^2}$$

$$R(t) = 1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right)$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\sigma t\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2\right]}}{1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right)} = \frac{\phi(\frac{\ln t - \mu}{\sigma})}{t\sigma(1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right))}$$

Where the two parameters are:

$$\mu = E(\ln(t)) \text{ and } \sigma^2 = var(\ln(t))$$

- Remember: if $T \sim lognorm(\mu, \sigma^2)$ then $\ln(T) \sim norm(\mu, \sigma^2)$

- Note that you can calculate the mean and variance of the lognormal distribution from the parameters of the lognormal distribution:

$$E(T)\text{=}MTTF = e^{\mu+\frac{\sigma^2}{2}}$$

$$Stdev(T) = \left(e^{\sigma^2} - 1\right)^{\frac{1}{2}} \times MTTF$$

Uses of the lognormal distribution

- The lognormal distribution is good for occurrence of events that may vary by several orders of magnitude, such as the time to finish a repair task.
- It is also good for modeling failure modes or processes that are a result of multiplicative errors (e.g., fatigue cracks)
- Human reliability analysis modeling – time to failure and repair
- Particle sizes in breakage
- Electronic components

Example: Lognormal distribution

- **Example:** The time that it takes an operator to shutdown a system is lognormally distributed with $\mu = 2.0273, \sigma^2 = 0.4608$. Determine probability of shutdown within $t = 20$ seconds (that is, that the operator successfully shuts down the system within 20 seconds).

a) Define the failure events and the success event in mathematical notation.

b) Solve.

Example: Lognormal distribution (cont.)

- **Solution:** The shutdown time is lognormally distributed with $\mu = 2.0273, \sigma^2 = 0.4608$, meaning:
  - $T_{shutdown} \sim \text{lognorm}(2.0273, \sqrt{(0.4608)})$
- Determine probability of shutdown by $t = 20$ seconds.

a) The failure event is $(T_{shutdown} > t)$ and the success event is $(T_{shutdown} \leq t)$.

b) $z_1 = \frac{\ln t - \mu}{\sigma} = \frac{\ln(20) - 2.0273}{\sqrt{0.4608}} = 1.427$

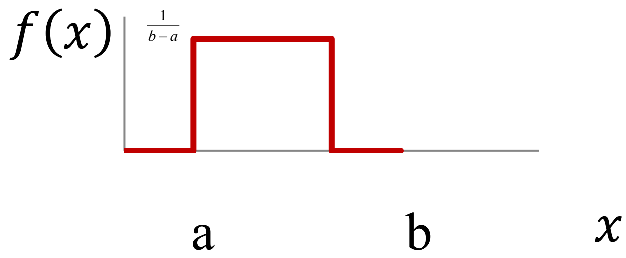
In Excel: $\text{LOGNORM.DIST}(20, 2.0273, \sqrt{0.4608},\ \text{TRUE})$

$$\boxed{\begin{gathered}\Pr(T > 20) = 1 - F(20) = 0.077 \Rightarrow Unreliability \\ \Pr(T \leq 20) = F(20) = 0.923 \Rightarrow Reliability\end{gathered}}$$

Continuous uniform distribution in reliability modeling

$$f(t) = \begin{cases} \dfrac{1}{b-a} & a \le t \le b \\ 0 & t < a \text{ or } t > b \end{cases}$$

$$R(t) = \begin{cases} 1 & t < a \\ \dfrac{b-t}{b-a} & a \le t \le b \\ 0 & t > b \end{cases}$$



$$\text{MTTF} = \frac{a+b}{2}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{b-a}}{\frac{b-t}{b-a}} = \frac{1}{b-t}, \qquad a \le t \le b$$

where,

- a = minimum value
- b = maximum value

Uses of uniform distribution in reliability modeling

- When a prior is unknown a uniform distribution can be used as an uninformative prior.
- Random number generation.

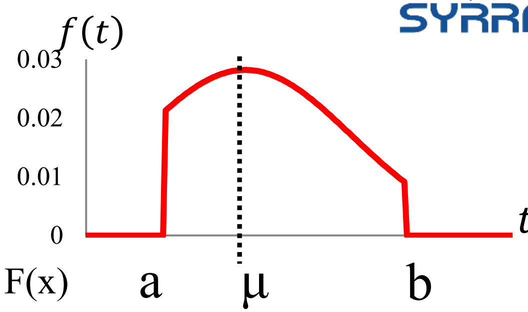
Truncated normal distribution in reliability modeling

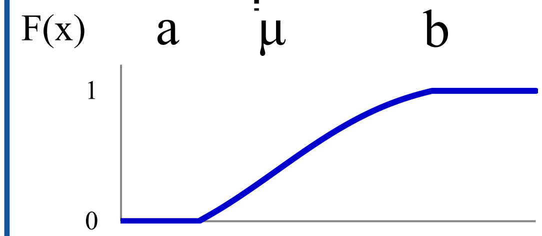
$$f(t) = \frac{f_{Norm}(t)}{F_{norm}(t=b) - F_{norm}(t=a)} = \frac{\frac{1}{\sigma}\phi(z_t)}{\Phi(z_b) - \Phi(z_a)}$$

$$R(t) = \begin{cases} 1 & t < a \\ \dfrac{\Phi(z_b) - \Phi(z_t)}{\Phi(z_b) - \Phi(z_a)} & a \le t \le b \\ 0 & t > b \end{cases}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\sigma}\phi(z_t)[\Phi(z_b) - \Phi(z_t)]}{[\Phi(z_b) - \Phi(z_a)]^2}, \qquad a \le t \le b$$

$$\text{MTTF} = \frac{\frac{1}{\sigma}[\phi(z_a) - \phi(z_b)]}{\Phi(z_b) - \Phi(z_a)}$$





Where:

- $-\infty < t < \infty, -\infty < \mu_t < \infty, \sigma_t^2 > 0$
- a = minimum value
- b = maximum value

$$z_t = \frac{t-\mu_t}{\sigma_t}$$

$$z_a = \frac{a-\mu_t}{\sigma_t}$$

$$z_b = \frac{b-\mu_t}{\sigma_t}$$

Uses of truncated normal distribution

- Life or time distributions with a known minimum and maximum life or time (e.g. product life spans, repair times)
- Measurements under an inspection threshold (e.g. flaw sizes, pit diameters)

Beta distribution in reliability modeling

$$f(t) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1} & 0 \le t \le 1 \\ 0 & t < 0 \text{ or } t > 1 \end{cases}$$

$$R(t) = \begin{cases} 1 & t < 0 \\ 1 - \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t x^{\alpha-1}(1-x)^{\beta-1} dx & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}}{1 - \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t x^{\alpha-1}(1-x)^{\beta-1} dx}$$
$$= \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta) - B_t(t|\alpha,\beta)}$$

Where:

α = shape parameter, ($\alpha > 0$)
β = shape parameter, ($\beta > 0$)

Where:

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$B_t(t|\alpha,\beta) = \int_0^t x^{\alpha-1}(1-x)^{\beta-1} dx$$

Uses of the beta distribution

- The beta distribution is a good choice when given a set of percentages of times-to-failure instead of actual times.
- Likelihood ratios and proportion modeling
- Used frequently as a Bayesian conjugate prior for probabilities.

Example: Beta distribution

- **Example:** An energy lab designed a cutting-edge wind turbine that is completely different from any previous design. In the preliminary lab tests of 115 turbines, 92 turbines experienced no failure.

a) Find the expected failure probability, given that they can be modeled with a beta distribution.

b) Given the collected data and using the beta distribution model, what is the probability that the failure rate is below 0.15?

Example: Beta distribution (cont.)

- **Solution:** An energy lab designed a cutting-edge wind turbine that is completely different from any previous design. In the preliminary lab tests of 115 turbines, 92 turbines experienced no failure.

a) Find the expected failure probability, given that they can be modeled with a Beta distribution.

$\alpha = 115 - 92 = 23, \beta = 92.$

Expected probability of failure, X, is $E(X) = \alpha/(\alpha + \beta) = 0.2$

b) Given the collected data and using the eta distribution model, what is the probability that the failure rate is below 0.15?

$\Pr(x < 0.15) = 0.082$

In Excel: BETA.DIST(0.15, 23, 92, TRUE)=0.082

Relationship between distributions