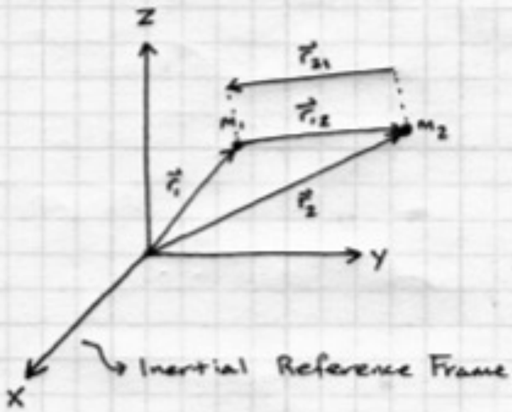
Consider the following system of two bodies with respect to an <u>inertial</u> <u>reference</u> <u>frame</u>:



$\vec{r}_1$: inertial position of body 1

$\vec{r}_2$: inertial position of body 2

$\vec{r}_{12}$: position vector pointing <u>from</u> body 1 <u>to</u> body 2

$\vec{r}_{21}$: position vector pointing <u>from</u> body 2 <u>to</u> body 1

m_1 : mass of body 1

m_2 : mass of body 2

An <u>inertial</u> <u>reference</u> <u>frame</u> is <u>unaccelerated</u>:

- Frame origin is at rest <u>or</u> moving with <u>constant</u> velocity.
- Frame axes are not rotating.

In an inertial frame, Newton's 2nd Law of Motion holds. It is:

$$\Sigma \vec{F} = m\ddot{\vec{r}} \quad ; \quad \ddot{\vec{r}} = \text{acceleration}$$

Note: $\underbrace{\dot{\vec{r}} \equiv \frac{d}{dt}(\vec{r})}_{\text{Velocity}}$, $\underbrace{\ddot{\vec{r}} \equiv \frac{d^2}{dt^2}(\vec{r})}_{\text{acceleration}}$

$\Sigma \vec{F}$ = sum of all forces acting upon the body with mass m

We will assume that the bodies are point masses (spherical, homogeneous gravitational fields), and so Newton's Law of Universal Gravitation describes the forces between bodies due to <u>mutual</u> <u>gravitation</u> as:

$$\vec{F} = \frac{G m_1 m_2}{r^2}\hat{r} = \frac{G m_1 m_2}{r^3}\vec{r}$$

$$\left(\text{since } \hat{r} = \frac{\vec{r}}{\|\vec{r}\|} = \frac{\vec{r}}{r}\right)$$

where $\hat{r}$ is the unit vector between the bodie

pointing ~~from~~ one to the other.

G is the <u>Universal Gravitational Constant</u>:

$$G = 6.67428 \pm 0.0007 \times 10^{-11} \frac{m^3}{kg\, s^2}$$

↳ 2006 $\underbrace{\text{CODATA}}_{}$ value

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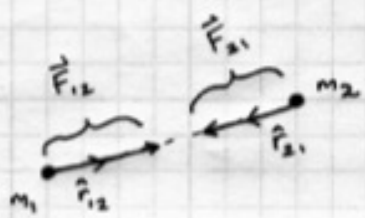
For the two body system we are considering:



$\vec{F}_{12}$ = Force on body 1 due to body 2's gravit

$\vec{F}_{21}$ = Force on body 2 du to body 1's gravity

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{\|\vec{r}_{12}\|}, \quad \hat{r}_{21} = \frac{\vec{r}_{21}}{\|\vec{r}_{21}\|}$$

$$\vec{F}_{12} = \frac{G m_1 m_2}{\|\vec{r}_{12}\|^2}\hat{r}_{12}, \quad \vec{F}_{21} = \frac{G m_1 m_2}{\|\vec{r}_{21}\|^2}\hat{r}_{21}$$

Our goal is to derive the Equations of Motion (EOMs) for the bodies, so we now draw a Free Body Diagram (FBD) for each body and apply Newton's 2nd Law:

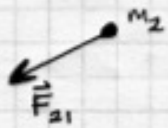
FBD for body 1:

No other forces except for the force due to body 2's gravity

$$\Rightarrow \underbrace{\sum \vec{F}_1}_{\text{Sum of the forces acting on body 1}} = \cancel{m_1}\ddot{\vec{r}}_1 = \vec{F}_{12} = \frac{G\cancel{m_1}m_2}{\|\vec{r}_{12}\|^2}\hat{r}_{12}$$

$$\boxed{\ddot{\vec{r}}_1 = \frac{Gm_2}{\|\vec{r}_{12}\|^2}\hat{r}_{12}}$$

FBD for body 2:



$$\Rightarrow \sum \vec{F}_2 = \cancel{m_2}\ddot{\vec{r}}_2 = \vec{F}_{21} = \frac{Gm_1\cancel{m_2}}{\|\vec{r}_{21}\|^2}\hat{r}_{21}$$

$$\boxed{\ddot{\vec{r}}_2 = \frac{Gm_1}{\|\vec{r}_{21}\|^2}\hat{r}_{21}}$$

Referring to the original diagram showing the bodies and their position vectors:

$$\vec{r}_1 + \vec{r}_{12} = \vec{r}_2 \qquad\qquad \vec{r}_2 + \vec{r}_{21} = \vec{r}_1$$

$$\boxed{\vec{r}_{12} = \vec{r}_2 - \vec{r}_1} \qquad\qquad \boxed{\vec{r}_{21} = \vec{r}_1 - \vec{r}_2}$$

$$\therefore \vec{r}_{12} = -\vec{r}_{21}$$

$$\|\vec{r}_{21}\| = \|\vec{r}_{12}\| = r$$

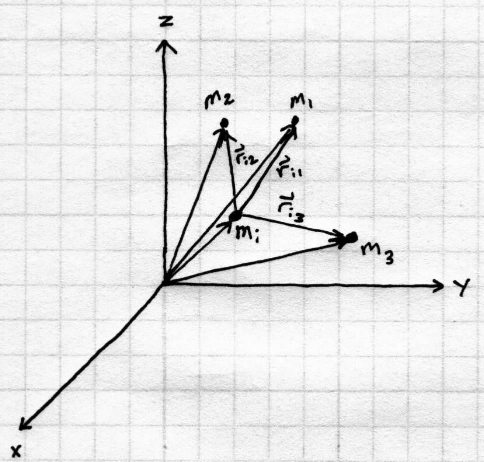
Recalling that $\hat{r} = \frac{\vec{r}}{\|\vec{r}\|} = \frac{\vec{r}}{r}$, we may write:

$$\boxed{\begin{aligned} \ddot{\vec{r}}_1 &= \frac{Gm_2}{r^3}(\vec{r}_2 - \vec{r}_1) \\ \ddot{\vec{r}}_2 &= -\frac{Gm_1}{r^3}(\vec{r}_2 - \vec{r}_1) = \frac{Gm_1}{r^3}(\vec{r}_1 - \vec{r}_2) \end{aligned}}$$

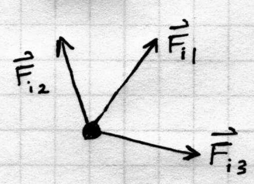
Which are the equations of motion for bodies 1 and 2.

Finally, before proceeding to search for constants of the motion, we will derive the general form of the EOMs, $\ddot{\vec{r}}_i$, which is the inertial acceleration of the i^{th} body where i can equal $1, 2, \ldots, n$; n is the number of bodies in the system. In that case, the number of states is $6n$, the number of constants required for a closed form solution is $6n$, and the size of the system state vector is $6n \times 1$.

Consider the i^{th} body in a system of multiple bodies and the gravitational forces acting upon it due to 3 other bodies:



FBD



$$\vec{F}_{i1} = \frac{G m_i m_1}{r_{i1}^3}(\vec{r}_1 - \vec{r}_i)$$

$$\vec{F}_{i2} = \frac{G m_i m_2}{r_{i2}^3}(\vec{r}_2 - \vec{r}_i)$$

$$\vec{F}_{i3} = \frac{G m_i m_3}{r_{i3}^3}(\vec{r}_3 - \vec{r}_i)$$

$$\sum \vec{F}_i = m_i \ddot{\vec{r}}_i = \vec{F}_{i1} + \vec{F}_{i2} + \vec{F}_{i3}$$

$$\cancel{m_i} \ddot{\vec{r}}_i = \frac{G \cancel{m_i} m_1}{r_{i1}^3}(\vec{r}_1 - \vec{r}_i) + \frac{G \cancel{m_i} m_2}{r_{i2}^3}(\vec{r}_2 - \vec{r}_i) + \frac{G \cancel{m_i} m_3}{r_{i3}^3}(\vec{r}_3 - \vec{r}_i)$$

Factor out a $-G$ term:

$$\ddot{\vec{r}}_i = -G\left[\frac{m_1}{r_{1i}^3}(\vec{r}_i-\vec{r}_1)+\frac{m_2}{r_{2i}^3}(\vec{r}_i-\vec{r}_2)+\frac{m_3}{r_{3i}^3}(\vec{r}_i-\vec{r}_3)\right]$$

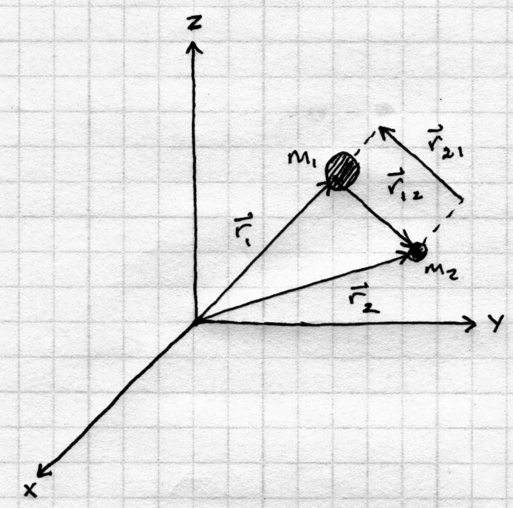
From this we can clearly see the pattern for the case of $n-1$ bodies acting upon the i^{th} body:

$$\boxed{\ddot{\vec{r}}_i = -G\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{m_j}{r_{ji}^3}(\vec{r}_i-\vec{r}_j) \qquad i=1,2,\ldots,n}$$

Note that j is not allowed to equal i in the finite sum because body i's gravity cannot act upon itself.

The Two-Body Problem

We now turn our attention to the study of a two-body system, which has many practical applications. Consider the following system:



Recall our equation for the acceleration of the i^{th} body:

$$\ddot{\vec{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^3}(\vec{r}_i - \vec{r}_j) \qquad i = 1, \ldots, n$$

We will use our equation, with $n = 2$ (two bodies) to determine $\ddot{\vec{r}}_1$ and $\ddot{\vec{r}}_2$.

For body 1: $i = 1$, $j = 1 \rightarrow$ no term, $j \neq i$

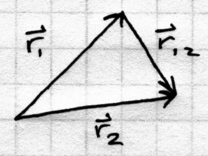
$j = 2 \rightarrow$ $\boxed{\ddot{\vec{r}}_1 = -G\left(\frac{m_2}{r_{21}^3}(\vec{r}_1 - \vec{r}_2)\right)}$

↳ only one term in the series

For body 2: $i = 2$, $j = 1 \rightarrow$ $\boxed{\ddot{\vec{r}}_2 = -G\left(\frac{m_1}{r_{12}^3}(\vec{r}_2 - \vec{r}_1)\right)}$

$j = 2 \rightarrow$ no term, $j \neq i$

Recall the vector geometry for m_1 and m_2:



$$\vec{r}_1 + \vec{r}_{12} = \vec{r}_2$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$-\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = \vec{r}_{21}$$

Now,

let $\vec{r} \equiv \vec{r}_{12}$

$$r = \|\vec{r}\| = \|\vec{r}_{12}\| = \|-\vec{r}_{12}\|$$

Substituting $\vec{r}$ and r into our equations:

$$\ddot{\vec{r}}_1 = -G\frac{m_2}{r^3}(-\vec{r}) = G\frac{m_2}{r^3}\vec{r}$$

$$\ddot{\vec{r}}_2 = -G\frac{m_1}{r^3}\vec{r}$$

For the two-body problem we seek the motion of body 2 with respect to body 1:

$$\ddot{\vec{r}}_{12} = \ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

$$\ddot{\vec{r}} = -\frac{Gm_1}{r^3}\vec{r} - \frac{Gm_2}{r^3}\vec{r}$$

$$\ddot{\vec{r}} = -G(m_1 + m_2)\frac{\vec{r}}{r^3}$$

We now define the Gravitational Parameter (μ):

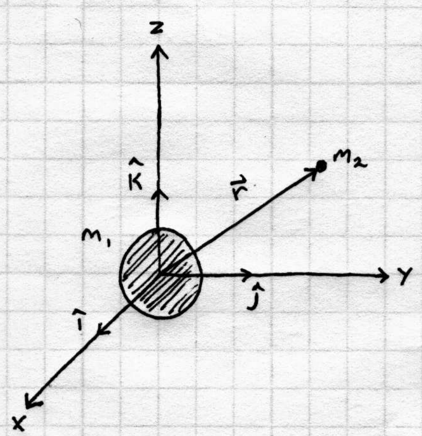
$$\mu \equiv G(m_1 + m_2)$$

We will make the assumption for the two-body problem that $m_2 << m_1$, which is an excellent assumption in a variety of cases, particularly in our case of interest where m_1 is a planet, e.g., Earth, and m_2 is a spacecraft.

$$\Rightarrow \quad \mu \simeq G m_1$$

In practice, the value of μ for a planet can be determined more accurately than G or m separately.

By using $\vec{r}$, we have effectively placed the origin of our inertial frame at the center of m_1 and only concerned ourselves with the motion of m_2.



$$\boxed{\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \vec{0}}$$

Vector form of the simultaneous 2nd order, non-linear, scalar differential equations in r_x, r_Y, r_z

where $\vec{r} = r_x \hat{i} + r_Y \hat{j} + r_z \hat{k}$

$$\vec{r} = \begin{bmatrix} r_x \\ r_Y \\ r_z \end{bmatrix}$$

Another approach:

$$\ddot{\vec{r}}_1 = -G\sum_{j=2}^{n} \frac{m_j}{r_{j1}^3}\vec{r}_{j1}$$

$$\ddot{\vec{r}}_2 = -G\sum_{\substack{j=1\\ j\neq 2}}^{n} \frac{m_j}{r_{j2}^3}\vec{r}_{j2}$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\ddot{\vec{r}}_{12} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

Sub in the definitions for $\ddot{\vec{r}}_1$ & $\ddot{\vec{r}}_2$:

$$\ddot{\vec{r}}_{12} = -G\sum_{\substack{j=1\\ j\neq 2}}^{n} \frac{m_j}{r_{j2}^3}\vec{r}_{j2} + G\sum_{j=2}^{n} \frac{m_j}{r_{j1}^3}\vec{r}_{j1}$$

$$\ddot{\vec{r}}_{12} = -\left[\frac{Gm_1}{r_{12}^3}\vec{r}_{12} + G\sum_{j=3}^{n}\frac{m_j}{r_{j2}^3}\vec{r}_{j2}\right] - \left[-\frac{Gm_2}{r_{21}^3}\vec{r}_{21} - G\sum_{j=3}^{n}\frac{m_j}{r_{j1}^3}\vec{r}_{j1}\right]$$

$$\vec{r}_{12} = -\vec{r}_{21}$$

$$\boxed{\ddot{\vec{r}}_{12} = \underbrace{\frac{-G(m_1+m_2)\vec{r}_{12}}{r_{12}^3}}_{\text{2-body term}} - G\sum_{j=3}^{n} m_j\left(\frac{\vec{r}_{j2}}{r_{j2}^3} - \frac{\vec{r}_{j1}}{r_{j1}^3}\right)}$$

2-body term

Perturbing effect caused by bodies 3→n acting on body 2.

" body 1.

Make same assumptions to get to the 2BP expression:

1. accelerations due to other masses ($j \geq 3$) are negligible
2. the larger mass has a spherical mass distribution
3. gravity is the only force acting on the masses
4. the masses of the bodies do not change
5. we describe the motion in an inertial reference frame.

Assume $m_1 \gg m_2$ (e.g., M_1 = the Earth, m_2 = Satellite)

$$\ddot{\vec{r}}_{12} = -\frac{Gm_1}{r_{12}^3}\vec{r}_{12} \quad \text{(neglecting the S/C's mass)}$$

$Gm_1 = \mu$ = Gravitational parameter

$$\boxed{\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}} \quad \text{EOM 2BP}$$

This is a relative position. The origin does not have to be at the center of the planet.

Energy: Energy is conserved → only gravity is acting.

Kinetic energy: $\frac{1}{2} m_2 V^2$

Potential Energy: $\Delta PE = -W = -\int_{x_i}^{x_f} F(x)\,dx$

$$= -\int_{r_{ref}}^{r} F_g \, dr = -\int_{r_{ref}}^{r} -\frac{G m_1 m_2}{r^2} \, dr$$

$$= -\frac{G m_1 m_2}{r} \Big|_{r_{ref}}^{r}$$

Define the potential energy to be zero @ $r_{ref} = \infty$

$$PE(r_{ref} = \infty) = 0$$

$$PE(r) = -\frac{\mu m_2}{r}$$

Total specific energy: $\varepsilon = \frac{KE + PE}{m_2} \Rightarrow \boxed{\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}}$

Prove that ε is conserved:

Take the dot product of $\dot{\vec{r}}$ with the EOM $\left(\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}\right)$

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} + \frac{\mu}{r^3} \vec{r} \cdot \dot{\vec{r}} = 0$$

$$\vec{V} \cdot \dot{\vec{V}} + \frac{\mu}{r^3} \vec{r} \cdot \dot{\vec{r}} = 0$$

Math-magic : $\vec{v}\cdot\dot{\vec{v}} = v\dot{v}$

Proof: $\vec{b}\cdot\dot{\vec{b}} = b\dot{b}$

Define $\vec{b} = b\hat{b}$

$$\frac{d\vec{b}}{dt} = \dot{b}\hat{b} + b\dot{\hat{b}}$$

$$\vec{b}\cdot\frac{d\vec{b}}{dt} = b\dot{b}$$

Rewriting our eqn: $v\dot{v} + \frac{\mu}{r^3}r\dot{r} = 0$

Note: $\frac{d}{dt}\left(\frac{v^2}{2}\right) = v\dot{v}$ & $\frac{d}{dt}\left(-\frac{\mu}{r}\right) = \frac{\mu}{r^2}\dot{r}$

$$v\dot{v} + \frac{\mu}{r^3}r\dot{r} = 0 = \frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right)$$ Energy is Conserved

$$\Rightarrow \frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = \frac{d}{dt}(\mathcal{E}) = 0 \quad \square$$

Angular Momentum: (Specific Angular Momentum)

$\rightarrow$ is also Conserved. There is no torque b/c gravity acts radially.

$$\vec{r}\times\left(\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0\right)$$

$$\vec{r}\times\ddot{\vec{r}} + \vec{r}\times\frac{\mu}{r^3}\vec{r} = 0$$

$\vec{r}\times\vec{r} = 0$

$\Rightarrow \vec{r}\times\ddot{\vec{r}} = 0$

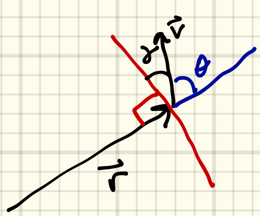
$$\frac{d}{dt}(\vec{r}\times\dot{\vec{r}}) = \cancelto{0}{\dot{\vec{r}}\times\dot{\vec{r}}} + \vec{r}\times\ddot{\vec{r}}$$

$$\Rightarrow \frac{d}{dt}(\vec{r}\times\dot{\vec{r}}) = 0$$

$$\frac{d}{dt}(\vec{r}\times\vec{v}) = 0$$

$\boxed{\vec{h} = \vec{r}\times\vec{v}}$ = Specific angular momentum

= both vector magnitude & direction are Conserved.



$h = |\vec{r}\times\vec{v}| = rv\sin\theta$

γ = Flight Path angle

$\boxed{h = rv\cos\gamma}$

$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$ is conserved

Spacecraft moves faster close to the central body.

We have the EOM of our satellite:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

We want to know what the orbit looks like without requiring numerical integration:

$$\ddot{\vec{r}} \times \vec{h} = \frac{\mu}{r^3}(\vec{h} \times \vec{r})$$

$$\frac{d}{dt}(\dot{\vec{r}} \times \vec{h}) = \ddot{\vec{r}} \times \vec{h} + \cancel{\dot{\vec{r}} \times \dot{\vec{h}}}^{\,0}$$

write the RHS as a time rate of change:

$$\frac{\mu}{r^3}(\vec{h} \times \vec{r}) = \frac{\mu}{r^3}(\vec{r} \times \vec{v}) \times \vec{r}$$

$$= \frac{\mu}{r^3}\left[\vec{v}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{r}\cdot\vec{v})\right]$$

$$= \frac{\mu}{r^3}\left(r^2\vec{v} - \vec{r}(r\dot{r})\right)$$

$$= \frac{\mu}{r}\vec{v} - \frac{\mu}{r^2}\dot{r}\vec{r}$$

Note: $\mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\mu}{r}\vec{v} - \frac{\mu\vec{r}}{r^2}\dot{r}$

Thus: $\frac{d}{dt}(\dot{\vec{r}} \times \vec{h}) = \mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$

Integrate: $\dot{\vec{r}} \times \vec{h} = \mu\frac{\vec{r}}{r} + \vec{B}$ ← integration constant

Dot with $\vec{r}$:

$$\vec{r}\cdot\dot{\vec{r}} \times \vec{h} = \frac{\mu}{r}\vec{r}\cdot\vec{r} + \vec{B}\cdot\vec{r}$$

$$\vec{a}\cdot\vec{b} \times \vec{c} = \vec{a} \times \vec{b}\cdot\vec{c}$$

$$\Rightarrow h^2 = \mu r + rB\cos\nu$$

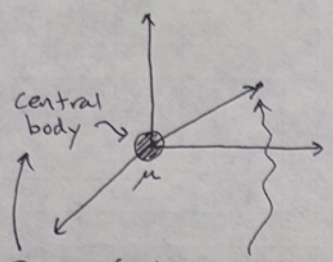
Solve for r: $r = \dfrac{h^2/\mu}{1 + B/\mu \cos\nu}$ ⟵ This is the eqn for a conic section written in polar coordinates, where $\vec{B}$ points to the point closest to the focus.

⇒ <u>All orbits are conic sections.</u>

$\boxed{r = \dfrac{p}{1 + e\cos\nu}}$ = the trajectory eqn.

(Restricted) <u>two-body problem EOM</u>:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$



Earth (w/spacecraft)
or, Sun (w/ Earth, asteroid, or spacecraft as in HW0)

Vector form of simultaneous 2nd order, nonlinear, scalar differential equations in r_x, r_y, r_z

$$\vec{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \quad \dot{\vec{r}} = \begin{bmatrix} \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \end{bmatrix} \quad \ddot{\vec{r}} = \begin{bmatrix} \ddot{r}_x \\ \ddot{r}_y \\ \ddot{r}_z \end{bmatrix}$$

position — velocity — acceleration

we have 6 dynamical states in this system ($r_x, r_y, r_z, \dot{r}_x, \dot{r}_y, \dot{r}_z$) so we require 6 constants of integration in order to solve the differential EOMs in closed form.

Previously, we showed that energy is conserved (provides 1 constant), angular momentum is conserved (provides 3 constants), and we have the integration constant $\vec{B}$ that points to the point closest to the focus of the conic section path of motion. However, $\vec{B}$ is not independent of the angular momentum so it provides only 1 more constant.

→ Total of 5 constants are available, but we need 6

→ Cannot solve the system in closed form, need to solve numerically using a numerical integrator

To use numerical integration, we must cast our problem as a 1st order system using the <u>state vector</u>, which is the position and velocity vector together.

$$\vec{R} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \\ \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \end{bmatrix} \begin{matrix} R(1) \\ R(2) \\ R(3) \\ R(4) \\ R(5) \\ R(6) \end{matrix} \qquad \frac{d\vec{R}}{dt} = \dot{\vec{R}} = \begin{bmatrix} \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \\ \ddot{r}_x \\ \ddot{r}_y \\ \ddot{r}_z \end{bmatrix} \begin{matrix} \\ \\ \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \text{EOM} \end{matrix} \qquad \dot{\vec{R}} = \begin{bmatrix} R(4) \\ R(5) \\ R(6) \\ R(1) \cdot \dfrac{-\mu}{\left(\sqrt{R(1)^2 + R(2)^2 + R(3)^2}\right)^3} \\ R(2) \cdot \quad " \\ R(3) \cdot \quad " \end{bmatrix}$$

<u>State vector</u>

The numerical integrator solves $\int_{t_0}^{t_f} \dot{\vec{R}}\, dt$ numerically.

The $\dot{\vec{R}}$ (written in terms of $\vec{R}$, as above) is what should be written in the "propagate_2BP" function discussed in the Hint section of HW0.

Lecture 3

Trajectory Eqn: $r = \dfrac{p}{1+e\cos\nu}$

p = semi-latus rectum = h^2/μ

e = eccentricity = "how un-circular is the conic section"

$e=0$ = circular orbit

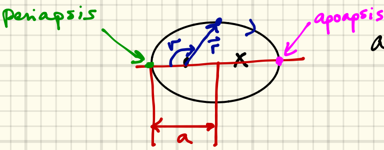
$0<e<1$ = ellipse

$e=1$: parabola

$e>1$: hyperbola

ν = true anomaly = the angle from periapsis to the s/c's location:

periapsis = the point on the orbit that is closest to the central body.



a = Semi-major axis

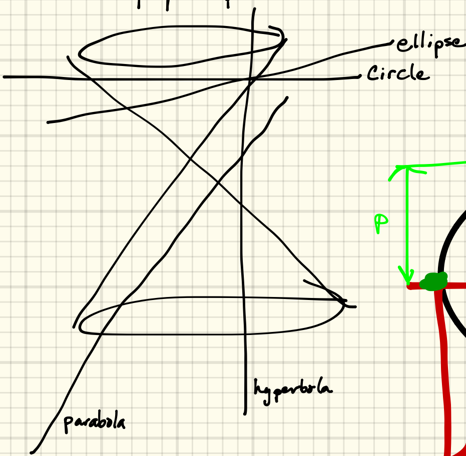
perigee = point closest to Earth

perijove = " " Jupiter

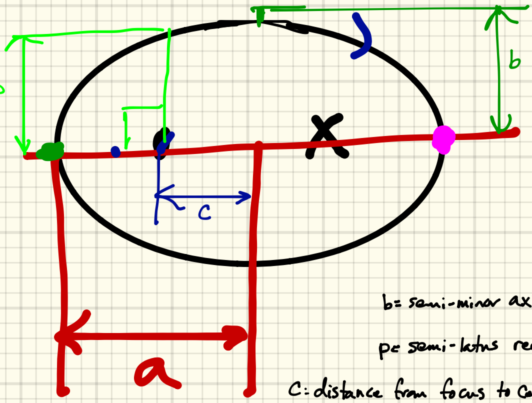
perihelion = " Sun

periwinkle = close to purple

apoapsis = point on the orbit that is farthest from the focus



Degenerate Conics: line, point



b = semi-minor axis

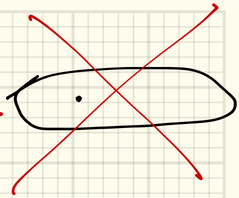
p = semi-latus rectum

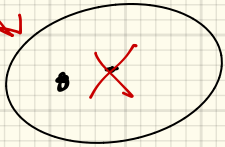
c: distance from focus to center

$e = \dfrac{c}{a}$

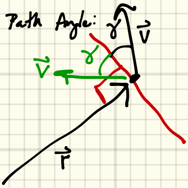
Characteristics of a good ellipse:

1. periapsis should be the point closest to the central body
2. The planet should be at a focus, not the center
3. The flight path angle at periapsis & apoapsis is zero.





Flight Path Angle: γ



— : $\gamma < 0$, $\dot{r} < 0$

— : $\gamma > 0$, $\dot{r} > 0$

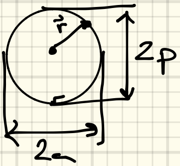
$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$, we now have an eqn for r.

For each location on a given orbit (a, e, etc), there is only 1 possible velocity magnitude.

(v)

- $\mathcal{E} < 0$ for circles & ellipses
- $\mathcal{E} = 0$ for parabola
- $\mathcal{E} > 0$ for a hyperbolic orbit

Circle:



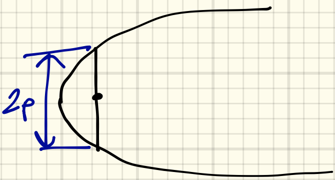
$a = r$

$e = 0$

periapsis & apoapsis are undefined.

velocity is constant

Parabola:



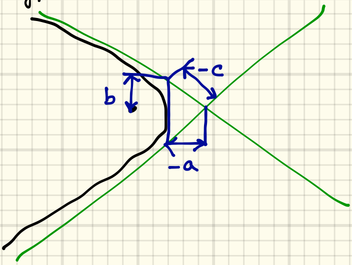
$a = \infty$

$c = \infty$

$e = 1$

Parabola has a velocity of 0 @ $r = \infty$

Hyperbola:



$e > 1$
$a < 0$
$V @ r=\infty > 0$

For all conics (except parabola):

$$p = a(1-e^2)$$

Radius of periapsis:

$$r_p = \frac{p}{1+e\cos\nu} = \frac{p}{1+e\cos(0)} = \frac{p}{1+e} = \frac{a(1-e^2)}{1+e}$$

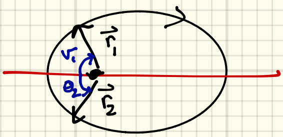
$$\Rightarrow \boxed{r_p = a(1-e)}$$

Radius of Apoapsis (same procedure):

$$\nu_a = 180°$$

$$r_a = \frac{p}{1-e} \Rightarrow \boxed{r_a = a(1+e)}$$

If $0 < \nu < 180°$, $\gamma > 0$
elseif $180 < \nu < 360°$, $\gamma < 0$



$\theta_2 = \nu_1$

$\gamma_1 = -\gamma_2$

Another expression for energy:

$$\vec{h} = \vec{r} \times \vec{v}$$

What is the expression for $|\vec{h}|$ at periapsis: $\gamma @ \nu=0$ is 0.

$$h = rv\cos\gamma = r_p v_p$$

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v_p^2}{2} - \frac{\mu}{r_p}$$

$$r_p = a(1-e)$$

$$h^2 = r_p^2 v_p^2 \Rightarrow v_p^2 = h^2/r_p^2$$

$$\varepsilon = \frac{h^2}{2r_p^2} - \frac{\mu}{a(1-e)}$$

$$\boxed{h = \sqrt{\mu p}}$$

$$\varepsilon = \frac{\mu a(1-e^2)}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)} = \frac{\mu}{2a}\left[\frac{-e^2+2e-1}{e^2-2e+1}\right]$$

$$\Rightarrow \boxed{\varepsilon = -\frac{\mu}{2a}}$$

$$\boxed{\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}}$$

Lecture 4

Example: A S/C is orbiting Earth on an orbit with an eccentricity of 0.2. The radius of periapsis is 1000 km altitude. What is the speed at periapsis, the radius of apoapsis & the speed @ apoapsis?

$r_p = 6378 + 1000 = 7378 \text{ km}$

Speed at r_p: $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$

$r_p = a(1-e) \Rightarrow a = r_p/(1-e) = 9222.5 \text{ km}$

$\frac{V_p^2}{2} - \frac{\mu}{r_p} = -\frac{\mu}{2a} \Rightarrow V_p = \sqrt{\frac{2\mu}{r_p} - \frac{\mu}{a}}$

Earth $\Rightarrow \mu = 3.986 \times 10^5 \text{ km}^3/\text{s}^2 \Rightarrow \boxed{V_p = 8.052 \text{ km/s}}$

Radius of Apoapsis: $r_a = a(1+e) \Rightarrow \boxed{r_a = 11067 \text{ km}}$

Speed at apoapsis: from energy eqn: $V_a = \sqrt{\frac{2\mu}{r_a} - \frac{\mu}{a}} \Rightarrow \boxed{V_a = 5.368 \text{ km/s}}$

Example: A S/C is orbiting Earth. At periapsis, it has an altitude of 1500 km and a velocity of 8.5 km/s. What is the eccentricity of the orbit? What is the flight path angle & speed of the S/C when its altitude is 3000 km and $\nu < 180°$?

Eccentricity: $r_p = a(1-e)$

$r_p = 1500 + 6378 = 7878 \text{ km}$

$V_p = 8.5 \text{ km/s}$

$\mathcal{E} = \frac{V_p^2}{2} - \frac{\mu}{r_p} = -\frac{\mu}{2a}$

$\Rightarrow a = -\frac{\mu}{2}\left[\frac{V_p^2}{2} - \frac{\mu}{r_p}\right]^{-1} \Rightarrow a = 27{,}543.6 \text{ km}$

$1 - e = r_p/a \Rightarrow e = 1 - r_p/a \Rightarrow \boxed{e = 0.714}$

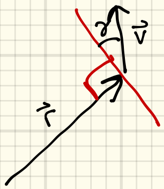
Velocity @ $r = 3000 + 6378 = 9378 \text{ km}$

$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$

$V = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \Rightarrow \boxed{V = 8.399 \text{ km/s}}$

Flight Path Angle: γ

$h = rv\cos\gamma$



From the previous part, we know r, v at this location

$r = 9378 \text{ km}$

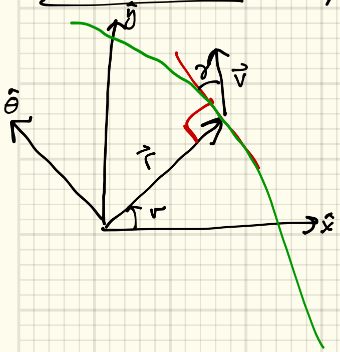
$v = 8.399 \text{ km/s}$

$h = rv\cos\gamma = r_p v_p \cos\gamma_p \,, \quad \gamma_p = 0$

$= r_p v_p$

$\cos\gamma = \dfrac{r_p v_p}{rv} \Rightarrow \boxed{\gamma = 31.77^\circ}$ b/c $\nu < 180^\circ$, $\gamma > 0$

Period of an Ellipse: time required to traverse the orbit



From Kinematics:

$\vec{r} = r\hat{r}$

$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$

$\hat{r} = \cos\nu\,\hat{x} + \sin\nu\,\hat{y}$

$\hat{\theta} = -\sin\nu\,\hat{x} + \cos\nu\,\hat{y}$

$\dot{\hat{r}} = -\dot{\nu}\sin\nu\,\hat{x} + \dot{\nu}\cos\nu\,\hat{y}$

$\dot{\hat{r}} = \dot{\nu}\hat{\theta}$

$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\nu}\hat{\theta}$

$h = rv\cos\gamma$

$v\cos\gamma = r\dot{\nu}$ (Component of the velocity in the $\hat{\theta}$ direction)

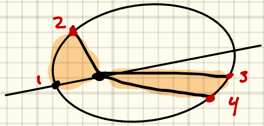
$\Rightarrow h = r^2\dot{\nu} = r^2\dfrac{d\nu}{dt} \quad \Rightarrow dt = \dfrac{r^2}{h}d\nu$ $\quad$ $r = f(\nu)$ thru the trajectory eqn $r = \dfrac{p}{1+e\cos\nu}$

Area of a circle $= \pi r^2$

Portion of a circle $= \dfrac{\nu}{2\pi}\pi r^2$ (ν in rad here)

$dA = \dfrac{1}{2}r^2 d\nu$

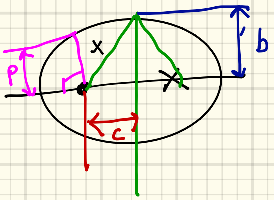
$dt = \dfrac{2}{h}dA$ ← Proves Kepler's 2nd law: equal areas in equal times



Suppose the 2 shaded areas are the same size, then it will take the spacecraft the same amount of time to go from $r_1 \rightarrow r_2$ and from $r_3 \rightarrow r_4$

The area of an ellipse is: $A = \pi a b$ (b = semi-minor axis)

Integrating $dt = \frac{2}{h} dA \Rightarrow TP = \frac{2\pi a b}{h}$



$e = \frac{c}{a}$

Length of string $= 2r_a = 2a(1+e)$

$= 2x + 2c = 2x + 2ea$

$\cancel{2}a(1+e) = \cancel{2}x + \cancel{2}ea$

$a + ea = x + ea \Rightarrow x = a$

$$a^2 = b^2 + c^2 \Rightarrow b = \sqrt{a^2 - c^2} = \sqrt{a^2 - e^2a^2} = \sqrt{a^2(1-e^2)} = \sqrt{ap}$$

$h = \sqrt{\mu p}$

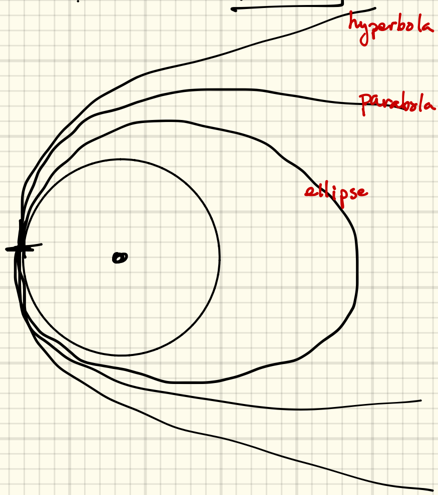
$$TP = \frac{2\pi a\sqrt{ap}}{\sqrt{\mu p}} \Rightarrow \boxed{TP = 2\pi\sqrt{\frac{a^3}{\mu}}}$$

Velocity of a circular orbit:

r = constant $\Rightarrow$ v = constant

$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$ $\qquad a = r$ (for circular orbit)

$$\frac{v_c^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2r} \Rightarrow \boxed{v_c = \sqrt{\frac{\mu}{r}}}$$



Parabola: $\mathcal{E} = 0$

The escape speed is the speed of a parabolic orbit at a given radius (r).

$$\mathcal{E} = \frac{V_{esc}^2}{2} - \frac{\mu}{r} = \cancel{\frac{V_\infty^2}{2}}^{\,0} - \cancel{\frac{\mu}{r_\infty}}^{\,0} = 0$$

$$\boxed{V_{esc} = \sqrt{\frac{2\mu}{r}}}$$

Hyperbola: The hyperbola has some non-zero velocity at infinity. The hyperbolic excess speed is this velocity.

hyperbolic excess speed = Speed at $r = \infty$

$$\mathcal{E} = \frac{V_\infty^2}{2} - \cancel{\frac{\mu}{r_\infty}}^{\,0,\ b/c\ r_\infty = \infty} = -\frac{\mu}{2a} \quad \Rightarrow \boxed{V_\infty = \sqrt{-\frac{\mu}{a}}}$$

Note $a < 0$ for hyperbola

Lecture 5 - Orbital Elements

Orbital Elements:

$\vec{r}, \vec{v}$

Orbital elements are an alternate set of coordinates that are used instead of $\vec{r}, \vec{v}$ (Cartesian coordinates)
B/c 6 Cartesian coordinates $\Rightarrow$ 6 orbital elements

We have already been using 3 of the orbital elements: a, e, ν

We have so far just thought of orbits in 2D $\rightarrow$ ie in the orbital plane

The other 3 orbital elements describe the orientation of the orbital plane in 3D space & the orientation of the orbit in the orbital plane.

i = inclination: angle between $\vec{h}$ & $\hat{z}$ $\quad (\vec{h} = \vec{r} \times \vec{v})$

Ω = longitude of the ascending node: angle from $\hat{x}$ to the ascending node

ascending node is the point on the orbit where the orbital plane crosses the equatorial plane & the s/c is ascending (aka "moving towards $\hat{z}$)

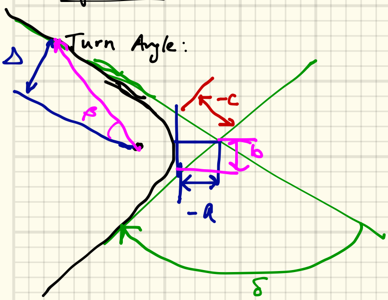
ω = argument of periapsis: angle from the ascending node to periapsis (in the orbital plane) in the direction of spacecraft motion.

$a, e, i, \Omega, \omega, \nu$

All the orbital elements are constant except ν, unless we execute a maneuver.

Hyperbolas (continued)

Turn Angle:



Turn angle: δ

$\sin\left(\frac{\delta}{2}\right) = \frac{a}{c} \qquad e = \frac{c}{a}$

$$\boxed{\sin\left(\frac{\delta}{2}\right) = \frac{1}{e}}$$

Δ = miss distance

$h = rv\cos\gamma$

At ∞, V is parallel to the asymptotes

$h = rv\cos\gamma = V_\infty r\sin\beta = V_\infty \Delta \qquad \boxed{h = V_\infty \Delta}$

Canonical Units: useful when doing calculations by hand

method of nondimensionalization

1 AU = Astronomical Unit = distance from the Sun to the Earth

= 149,597,871 km

→ interplanetary trajectories

DU = distance units =

often, radius of the Earth

$\mu = 1 \ DU^3/TU^2$

We pick a DU (some # of km) and we know that $\mu = 1 \ DU^3/TU^2$. So we can solve for TU.

1 DU = 6378 km

$\mu = 1 \ DU^3/TU^2 = 3.986 \times 10^5 \ km^3/s^2$

$$1 \ \frac{DU^3}{TU^2} \cdot \left(\frac{6378 \ km}{1 \ DU}\right)^3 \cdot \left(\frac{1 \ TU}{X \ sec}\right)^2 = 3.986 \times 10^5 \ \frac{km^3}{s^2}$$

$\Rightarrow$ solve for X. $\Rightarrow$ 1 TU = X sec

Then, express $\vec{r}, \vec{v}$, etc (a) in terms of DU & DU/TU.

Lecture 6 - Sketching Orbits

Inclination:

if $i = 0, 180°$ = equatorial $\qquad$ $i = 90°$ = polar orbit

$0 < i < 90°$: Prograde

$90 < i < 180°$ = Retrograde

Eccentricity:

Can also be defined as a vector that points from the focus towards periapsis

Starting with an intermediate step from our derivation of the trajectory eqn:

$$\vec{r} \times \vec{h} = \mu \frac{\vec{r}}{r} + \vec{B} \qquad \vec{e} = \vec{B}/\mu \qquad r = \frac{p}{1 + e\cos\nu}$$

Solve for $\vec{B}$: $\vec{B} = \vec{v} \times \vec{h} - \mu \frac{\vec{r}}{r}$

$$\vec{h} = \vec{r} \times \vec{v}$$

$$\mu \vec{e} = \vec{v} \times (\vec{r} \times \vec{v}) - \mu \frac{\vec{r}}{r}$$

$$= (\vec{v} \cdot \vec{v})\vec{r} - (\vec{r} \cdot \vec{v})\vec{v} - \mu \frac{\vec{r}}{r}$$

$$\boxed{\vec{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{r} - (\vec{r} \cdot \vec{v})\vec{v}\right]}$$

Another expression for e:

$$p = a(1 - e^2) \Rightarrow e = \sqrt{1 - p/a}$$

$$p = h^2/\mu \ , \ \mathcal{E} = -\frac{\mu}{2a} \Rightarrow a = -\frac{\mu}{2\mathcal{E}}$$

$$e = \sqrt{1 + \frac{h^2/\mu}{\mu/2\mathcal{E}}} \Rightarrow \boxed{e = \sqrt{1 + \frac{2\mathcal{E}h^2}{\mu^2}}}$$

Other orbital elements: We talked about "Keplerian" orbital elements

If our orbit is in the equatorial plane, then we don't have an ascending node.

Π = longitude of periapsis: angle from $\hat{x}$ to $\hat{e}$

$$\Pi = \Omega + \omega$$

For a circular orbit, there is no periapsis.

u: argument of latitude: angle (in the plane of the orbit) between the ascending node & the position.

$$u = \omega + \nu$$

ℓ : true longitude : angle from $\hat{x}$ to $\vec{r}$

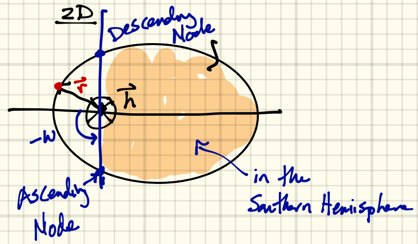
$$\ell = \Omega + \omega + \nu$$

Sketching Orbits:

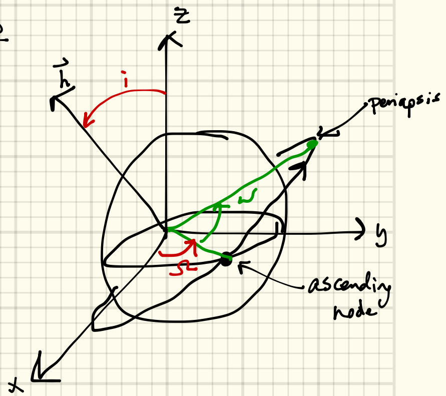
Earth orbit: $i = 20°$, $a = 20{,}000$ km, $\omega = 90°$, $e = 0.5$, $\Omega = 45°$, $\nu = 10°$

$r_p = a(1-e) = 10{,}000$ km > 6378 km $\rightarrow$ will not hit Earth

2D



3D

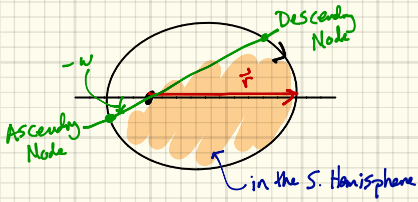


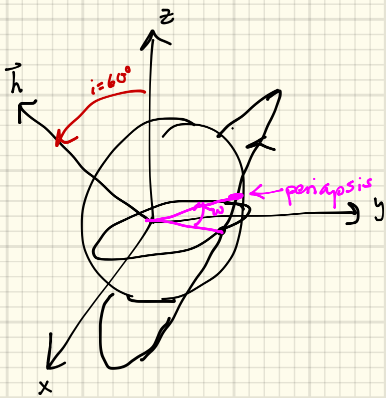
S/C is in the S. Hemisphere if:

$360° < \omega + \nu < 180$: S/C is in the N. Hemisphere

Earth orbiting:

$a = 20{,}000$ km, $e = 0.5$, $i = 60°$, $\Omega = 60°$, $\omega = 10°$, $\nu = 180°$

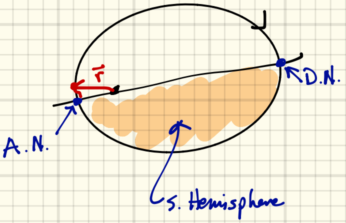


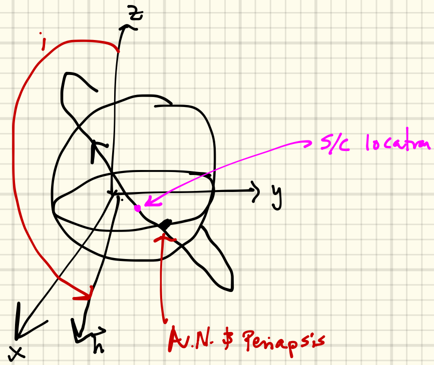


Earth orbiting:

$a = 20{,}000\,km$
$e = 0.5$
$i = 160°$

$\Omega = 45°$
$\omega = 0°$
$\nu = 10°$





Lecture 7: Cartesian to/from Orbital Elements

Longitude of the Ascending Node: Ω : measured from the Greenwich meridian

Right Ascension of the Ascending Node (RAAN): measured from the vernal equinox

Conversion from Cartesian to OE's:

Cartesian State: $[\vec{r}, \vec{v}] = [x, y, z, v_x, v_y, v_z]$

Also know μ.

a|

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

e| $\vec{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{r} - (\vec{r}\cdot\vec{v})\vec{v}\right]$

take the magnitude to get e.

i| Angle between $\hat{z}$ & $\vec{h}$

$$\frac{\vec{h}\cdot\hat{z}}{h} = \cos(i) \qquad \vec{h} = \vec{r}\times\vec{v}$$

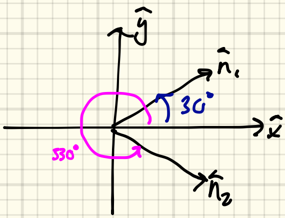
Ω| Angle from $\hat{x}$ to ascending node

$\vec{n} = \hat{z} \times \vec{h}$ (points at the ascending node)

$\cos(\Omega) = \hat{n}\cdot\hat{x}$ $\qquad 0 \le \Omega \le 360°$

$\cos(30) = \cos(330)$

Need to check calculator output



If $\hat{n}\cdot\hat{y} > 0$, $0 < \Omega < 180°$

If $\hat{n}\cdot\hat{y} < 0$, $180 < \Omega < 360°$

$\underline{\omega}$ | angle from ascending node to periapsis
$\vec{n}$ $\vec{e}$

$$\cos(\omega) = \frac{\vec{n}\cdot\vec{e}}{ne}$$

Do the quadrant check
If $\vec{e}\cdot\hat{z} > 0$, $0 < \omega < 180°$.
else $180 < \omega < 360°$

$\underline{\nu}$ | From periapsis to s/c location:

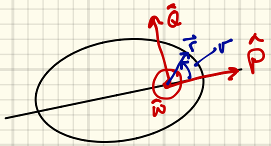
$$\cos(\nu) = \frac{\vec{e}\cdot\vec{r}}{er}$$

Do the quadrant check:
if $\vec{r}\cdot\vec{v} > 0$, then $0 < \nu < 180°$ (b/c $|\vec{r}|$ is growing here)

else, $180 < \nu < 360°$.

OE to Cartesian:

We know: $a, e, i, \Omega, \omega, \nu$ & μ



Use the perifocal frame: $\hat{P}, \hat{Q}, \hat{W}$
$\hat{P}$: points towards periapsis
$\hat{W}$: points along $\vec{h}$
$\hat{Q}$: Completes the RH'd system

1. Express $\vec{r}, \vec{v}$ in the PQW frame

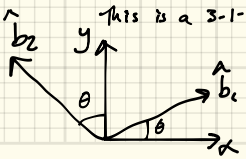
2. Rotate into the XYZ frame.

1. $\vec{r} = r\cos\nu\,\hat{P} + r\sin\nu\,\hat{Q}$ $\qquad r = \dfrac{p}{1+e\cos\nu}$, $p = a(1-e^2)$

$$\vec{v} = \sqrt{\frac{\mu}{p}}\left[-\sin\nu\,\hat{P} + (e+\cos\nu)\hat{Q}\right]$$

2. Rotate from PQW to XYZ.

We will write the rotation matrix from XYZ → PQW & then take the inverse.
This is a 3-1-3 rotation



$\hat{b}_1 = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} + 0\,\hat{z}$
$\hat{b}_2 = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} + 0\,\hat{z}$
$\hat{b}_3 = \hat{z}$

$$[R_3] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

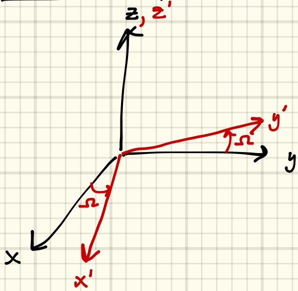
$$[R_2] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$${}^{C}\vec{a} = [BC][AB]\,{}^{A}\vec{a}$$

Another property of rotation matrices: $[R]^{-1} = [R]^{T}$ (b/c orthonormal matrix)

From $XYZ \rightarrow PQW$: 3-1-3 (z-x-z)

Rotation #1: Rotate about $\hat{z}$ by Ω.

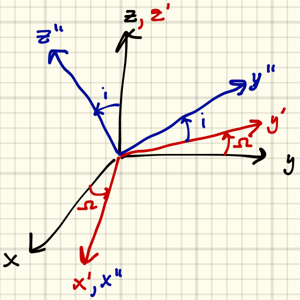


x' points at the ascending node

$$[R_\Omega] = \begin{bmatrix} c\Omega & s\Omega & 0 \\ -s\Omega & c\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

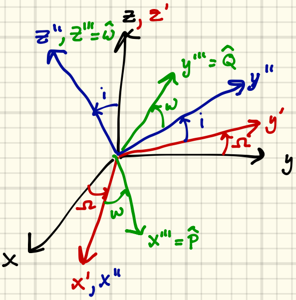
Rotation #2: Rotation about x' by i

After this rotation, $\hat{z}'' \parallel \vec{h} \parallel \hat{w}$



$$[R_i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & ci & si \\ 0 & -si & ci \end{bmatrix}$$

Rotation #3: Rotate about $\hat{z}''$ by ω



$$[R_\omega] = \begin{bmatrix} c\omega & s\omega & 0 \\ -s\omega & c\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⇒ To Rotate from XYZ → PQW

$$^{P}\vec{b} = [R_\omega][R_i][R_\Omega]\,^{I}\vec{b}$$

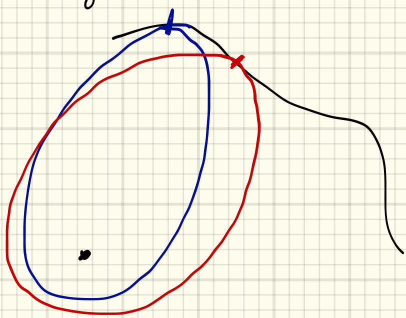
We want to rotate $\vec{r}, \vec{v}$ from PQW → XYZ

$$^{I}\vec{r} = \left[[R_\omega][R_i][R_\Omega]\right]^{T}\,^{P}\vec{r}$$

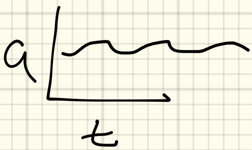
Keys: 1. Transpose
2. Rotation matrices in the correct order

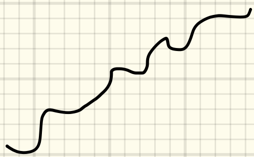
$$^{I}\vec{v} = \left[[R_\omega][R_i][R_\Omega]\right]^{T}\,^{P}\vec{v}$$

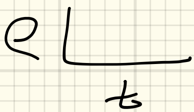
Osculating Orbital elements: orbital elements calculated at each time step.



Plot the OE's as a $f(t)$ → Constant for 2BP, but varying if other forces are included.







Lecture 8: TOF

<u>Time of Flight</u>: Time for the s/c to travel from point A to point B on a given orbit

The period for an ellipse: $\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}}$

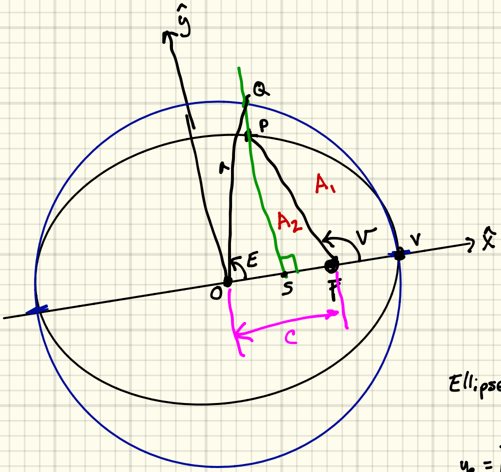
Kepler's 2nd law states that the line joining the planet to the sun sweeps out equal areas in equal times.

$$\frac{t-T}{A_1} = \frac{\mathbb{P}}{\underbrace{\pi ab}_{\text{Area of an ellipse}}}$$

t = time when the s/c is at Point B

T = time of the last periapsis passage

A_1 = area of the portion of the orbit from periapsis to point B.



E = eccentric anomaly

Need an expression for A_1.

$A_1 = \text{Area}(PSV) - A_2$

Base of $A_2 = c - a\cos E$

$= ea - a\cos E$

Note: $e = \frac{c}{a} \Rightarrow$

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y_e = \sqrt{\frac{a^2b^2 - b^2x^2}{a^2}}$

$= \frac{b}{a}\sqrt{a^2 - x^2}$

Circle: $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$

$y_c = \sqrt{a^2 - x^2}$

$\Rightarrow \frac{y_e}{y_c} = \frac{b}{a}$

Height of $A_2 = \frac{b}{a}(a\sin E) = b\sin E$

Area A_2: $\frac{ba}{2}(e\sin E - \sin E\cos E)$

$\text{Area}(PSV) = \frac{b}{a}(\text{Area}(QSV))$

$\text{Area}(QSV) = \text{Area}(QOV) - \text{Area}(QOS)$

$\text{Area}(QOV) = \pi a^2 \frac{E}{2\pi} = \frac{1}{2}Ea^2$ (E in RADIANS!!)

$\text{Area}(QOS) = \frac{1}{2}(a\cos E)(a\sin E)$

Plug into eqns for A_1: $A_1 = \frac{ba}{2}(E - e\sin E)$

Plug A_1 into Kepler's law eqn:

$$\frac{t-T}{\frac{ba}{2}(E-e\sin E)} = \frac{2\pi}{\pi ab}\sqrt{\frac{a^3}{\mu}}$$

$$\boxed{t-T=\sqrt{\frac{a^3}{\mu}}(E-e\sin E)}$$ Time of flight from periapsis to E.

Must use E in RADIANS!

Would like an expression for E as $f(\nu)$:

$$\boxed{\cos(E)=\frac{e+\cos\nu}{1+e\cos\nu}}$$ Half-plane check: if $\nu > 180°$, then $E > 180° = \pi$

$M = E - e\sin E$ = Mean Anomaly $\qquad M = n(t-T)$

$n=\sqrt{\frac{\mu}{a^3}}$ = mean motion

avg angular rate as the s/c goes around the orbit

TOF between 2 arbitrary points on the orbit:

s/c @ ν_0, E_0 @ t_0

s/c @ ν, E @ t $\qquad t > t_0$

$$t-t_0=\sqrt{\frac{a^3}{\mu}}\left[2\pi k+(E-e\sin E)-(E_0-e\sin E_0)\right]$$

k = # of times that the s/c passes thru periapsis.

Parabola: $t-t_0=\frac{1}{2\sqrt{\mu}}\left[pD+\frac{1}{3}D^3)-(pD_0+\frac{1}{3}D_0^3)\right]$

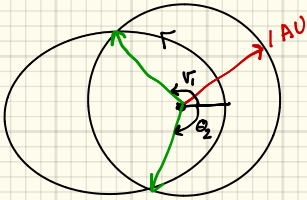
$D=\sqrt{p}\tan(\nu/2)$ = parabolic eccentric anomaly RAD

Hyperbola: $t-t_0=\sqrt{\frac{(-a)^3}{\mu}}\left[(e\sinh F-F)-(e\sinh F_0-F_0)\right]$

$\cosh F=\frac{e+\cos\nu}{1+e\cos\nu}$ $\qquad$ if $\nu<\pi$, $F>0$

else $F<0$

Example: A probe in an elliptical orbit about the Sun. Perihelion is 0.5 AU, aphelion is 2.5 AU. How many days in each orbit is the s/c 1 AU or closer to the sun?



B/c the orbit is symmetric about the semi-major axis,

$$v_1 = \theta_2$$

To get the time when the s/c is 1 AU or closer to the Sun, Just calculate TOF from $v=0$ to $v=v_1$ & multiply by 2.

$$t-T=\sqrt{\frac{a^3}{\mu}}\left[E-e\sin E\right] \qquad \cos E=\frac{e+\cos v}{1+e\cos v}$$

$$r(@v_1)=1\text{ AU}=\frac{p}{1+e\cos v} \qquad p=a(1-e^2)$$

$$2a=r_p+r_a=3\text{ AU} \Rightarrow a=1.5\text{ AU}=\frac{3}{2}\text{ AU}$$

$$r_p=a(1-e) \Rightarrow e=1-\frac{r_p}{a}=1-\frac{1}{2}\cdot\frac{2}{3}=1-\frac{1}{3}=\frac{2}{3}$$

$$r=\frac{a(1-e^2)}{1+e\cos v} \Rightarrow 1+e\cos v=\frac{a(1-e^2)}{r} \Rightarrow \frac{a(1-e^2)}{re}-\frac{1}{e}=\cos v$$

$$\cos v=\frac{a(1-e^2)}{re}-\frac{1}{e}$$
$$=\frac{3}{2}\frac{(1-4/9)3}{2}-\frac{3}{2}=-\frac{1}{4}$$

Plug $\cos v$ into expression for $\cos E \Rightarrow \cos E=\frac{1}{2} \Rightarrow E=60°=\frac{60\pi}{180}$ rad

B/c we are using canonical units: $\mu=1\frac{AU^3}{TU^2}$

Plug into $t-T$ eqn $\Rightarrow t-T=0.863$ TU

TOF when s/c is ≤ 1 AU from the Sun is $\underline{2(t-T)=1.726\text{ TU}}$

Need to convert from TU to days.

Knowing the dimensional value of μ_{sun} (km^3/s^2), we can solve for the conversion from TU to sec.

$1\text{ TU}=5.024\times10^6$ sec

$\underline{\text{TOF}\simeq 100\text{ days}}$

Lecture 9: Coordinate Frames & Hohmann transfers

Coordinate Systems:

Earth-Centered Inertial: (ECI)

x: vernal equinox

y: completes the RH'd system

z: perpendicular to equatorial plane

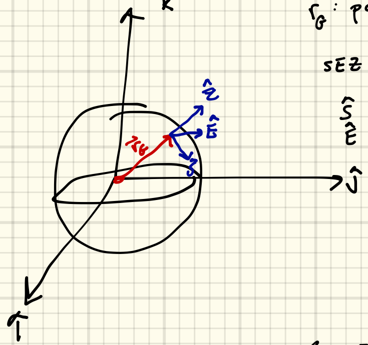
Perifocal Frame: PQW

Earth-Centered Earth-Fixed: (ECEF)

Same as ECI, but the frame rotates with the Earth.

in this case, $\hat{x}$ always points towards the greenwich meridian.

Topocentric Horizon: SEZ



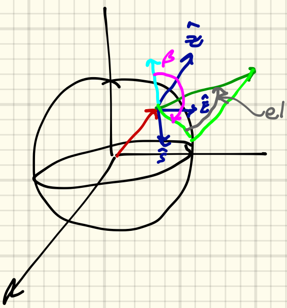
$\vec{r}_G$: position of the ground station

SEZ origin is at $\vec{r}_G$

$\hat{S}$ points South

$\hat{E}$ points East

$\hat{Z} \parallel \vec{r}_G$



β = azimuth = measured clockwise from North to the projection in the SE plane.

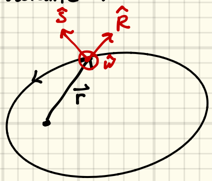
el: elevation: measured from the SE projection to the vector

Heliocentric Ecliptic: origin = sun

$\hat{X}$: vernal equinox

x, y plane is the ecliptic

Satellite: RSW



$\hat{R}$ is along the position vector

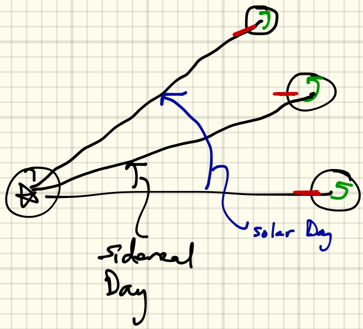
$\hat{W}$ is along $\vec{h}$

$\hat{S}$ completes the RH'd system

Time:

Solar Day: Amount of time between 2 sunrises
slightly more time than is required for Earth to rotate 360°.

Sidereal Day: time required for Earth to rotate 360° about its axis.



Julian Date: Interval of time (measured in Days) from Jan 1 4713 B.C. at Noon
- often used when calculating the position of planetary bodies
  - day starts at noon
- Need a lot of sigfigs, b/c values are in the millions

Maneuvers: ΔV is the magnitude of the velocity change from orbit A to orbit B.

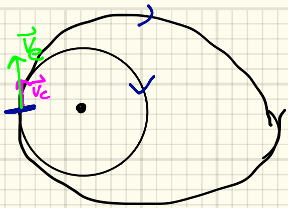
We solve for both ΔV magnitude & also the direction of the burn.

Assume instantaneous maneuvers.

Larger $\Delta V \Rightarrow$ more fuel.

$\boxed{\text{To transfer between 2 orbits using a single, instantaneous maneuver, the 2 orbits must intersect.}}$

At the intersection point, change the velocity <u>vector</u> from its initial value to the value on orbit B at that point.



Velocity of this ellipse at periapsis is greater than the velocity of the circle (From $\mathcal{E}$ eqn $a_{ellipse} > a_{circle}$)

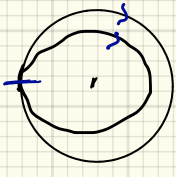
Velocity of the ellipse at periapsis and the circle are in the same direction b/c γ is the same ($\gamma = 0$) for both orbits at that point.

Initially on the circular orbit.

Execute a maneuver to get on the elliptical orbit, need to increase velocity

$$\Delta V = V_e - V_c$$

This type of maneuver where the velocity vector direction does not change is called a "tangential" maneuver.



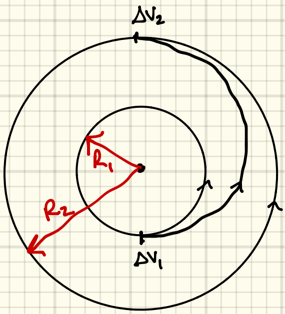
Transfer from circle to ellipse:

S/C needs to slow down

Velocity vectors are in the same direction = tangential maneuver

$$\Delta V = V_c - V_e$$

Hohmann Transfer: Cheapest maneuver between 2 circular orbits:



Half an ellipse connects R_1 & R_2

ΔV_1 & ΔV_2 are both tangential maneuvers

$$\Delta V_1 = V_{tp} - V_{c1}$$

V_{tp} = velocity of the transfer orbit at periapsis

$$V_{c1} = \sqrt{\frac{\mu}{R_1}}$$

$$\mathcal{E} = \frac{V_{tp}^2}{2} - \frac{\mu}{R_1} = -\frac{\mu}{R_1 + R_2} \Rightarrow V_{tp} = \sqrt{\frac{2\mu}{R_1} - \frac{2\mu}{R_1 + R_2}}$$

$$\boxed{\Delta V_1 = \sqrt{\frac{2\mu}{R_1} - \frac{2\mu}{R_1 + R_2}} - \sqrt{\frac{\mu}{R_1}}}$$ → increase S/C velocity by this amount

$$\Delta V_2 = V_{c2} - V_{ta}$$

$$V_{c2} = \sqrt{\frac{\mu}{R_2}}$$

V_{ta} = velocity of transfer orbit at apoapsis

$$\mathcal{E} = \frac{V_{ta}^2}{2} - \frac{\mu}{R_2} = -\frac{\mu}{R_1 + R_2} \Rightarrow V_{ta} = \sqrt{\frac{2\mu}{R_2} - \frac{2\mu}{R_1 + R_2}}$$

$$\boxed{\Delta V_2 = \sqrt{\frac{\mu}{R_2}} - \sqrt{\frac{2\mu}{R_2} - \frac{2\mu}{R_1 + R_2}}}$$

$$\boxed{\Delta V_{tot} = \Delta V_1 + \Delta V_2}$$

Lecture 11: More maneuvers

Tangential Maneuvers:

EX | Initial orbit: $r_p = 8000$ km, $r_a = 10{,}000$ km

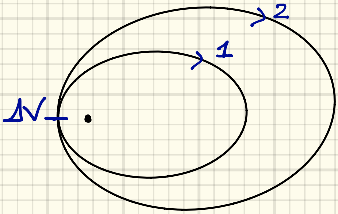
Final orbit: $r_p = 8000$ km, $r_a = 12{,}000$ km

Assume orbiting Earth

Both orbits have the same orbital plane

$\Rightarrow (\Omega, i)$

$\omega_1 = \omega_2$



B/c orbits intersect at r_p, that is where the burn will occur.

$V_{p2} > V_{p1}$

$\Delta V = V_{p2} - V_{p1}$

$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow V_p = \sqrt{\frac{2\mu}{r_p} - \frac{2\mu}{2a}}$$

Orbit 1: $2a = 18{,}000$ km $= r_{a_1} + r_{p_1}$

orbit 2: $2a = 20{,}000$ km

$$V_{p_1} = \sqrt{\frac{2\mu}{r_{p1}} - \frac{2\mu}{18000}}$$

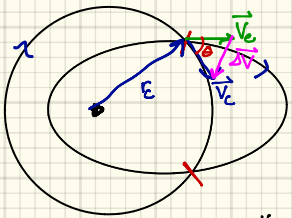
$$V_{p2} = \sqrt{\frac{2\mu}{r_{p2}} - \frac{2\mu}{20{,}000}}$$

Nontangential, in-plane:

EX | Initial orbit: $r_p = 8000$ km, $r_a = 10{,}000$ km

Final orbit: Circle: $r_c = 9000$ km

Same orbital plane



$\vec{V_e} + \vec{\Delta V} = \vec{V_c}$

Non-tangential maneuver: change velocity magnitude & direction

Law of Cosines:

$$\Delta V^2 = V_1^2 + V_2^2 - 2V_1V_2\cos\theta$$

$V_1 = V_e$ $\qquad$ $V_2 = V_c = \sqrt{\frac{\mu}{r_c}}$

$$\boxed{\Delta V^2 = V_e^2 + V_c^2 - 2V_eV_c\cos(-\gamma_1)}$$

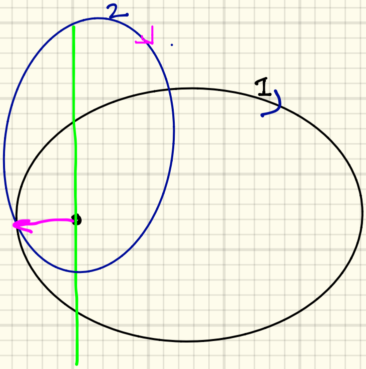
$$\frac{V_e^2}{2} - \frac{\mu}{r_c} = -\frac{\mu}{(r_a + r_p)} \Rightarrow V_e = \sqrt{\frac{2\mu}{r_c} - \frac{2\mu}{r_a + r_p}}$$

Find θ using FPA: $\theta = \gamma_2 - \gamma_1$ $\qquad$ $\gamma_2 = \gamma_c = 0$

$$h = \sqrt{\mu p} = rV\cos\gamma \Rightarrow \sqrt{\mu p_1} = r_c V_e \cos\gamma_1$$

Ex | Initial orbit: $r_p = 8000$ km, $r_a = 10{,}000$ km $\omega_1 = 90°$ $\Omega_1 = \Omega_2$, $i_1 = i_2$

Final orbit: $p = 8000$ km, $e = 0.3$ $\omega_2 = 0°$



$\omega_1 + \nu_1 = \omega_2 + \nu_2$

$90 + 0 = 0 + 90$

The two orbits intersect at periapsis of orbit 1 $\Rightarrow r_{p_1} = 8000$ km

Velocities are not in the same direction

$\Rightarrow$ law of cosines

$\Delta v^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta$

$v_1^2 = v_{p_1}^2$ = velocity at periapsis of orbit 1.

$\frac{v_{p_1}^2}{2} - \frac{\mu}{r_{p_1}} = \frac{-\mu}{r_{p_1} + r_{a_1}}$ $\Rightarrow$ Solve for v_{p_1}.

v_2^2 = velocity of orbit 2 at the semi-latus rectum of orbit 2 = r_{p_1}

$\frac{v_2^2}{2} - \frac{\mu}{r_{p_1}} = \frac{-\mu}{2a_2}$

$\uparrow$ solve for v_2

Need a_2: $p = a(1-e^2)$

$p_2 = a_2(1-e_2^2)$

$\frac{8000}{(1-0.3^2)} = \underline{a_2}$

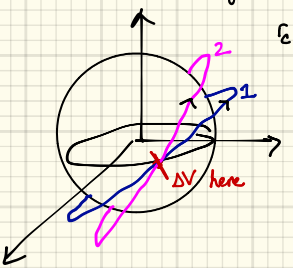
$\theta = \gamma_2 - \gamma_1$, $\gamma_1 = 0$ (b/c at periapsis)

$h = \sqrt{\mu p_2} = r_{p_1} v_2 \underline{\cos\gamma_2}$

Non-tangential, out-of-plane: want to change Ω, i

Difficult to visualize intersection points when $\Omega_1 \neq \Omega_2$. Focus primarily on changing inclination

2 circular orbits $\rightarrow$ only difference between orbits is $i_1 \neq i_2$ $(\Omega_1 = \Omega_2)$



$r_{c_1} = r_{c_2}$

Orbits intersect at the ascending & descending nodes

$\Rightarrow$ burn occur at ascending node

v_1 & v_2 are not in the same direction $\Rightarrow$ law of cosines

$\Delta v^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta$

$v_1 = v_2$ b/c $r_{c_1} = r_{c_2}$ $\rightarrow$ $v = \sqrt{\frac{\mu}{r_c}}$

v, r_c

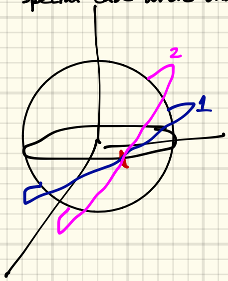
$\Delta v^2 = 2v^2(1-\cos\theta)$ $\theta = \Delta i$

Ex] Change i, a, e

Initial orbit: $r_c = 8000\,km$, $i = 10°$ $\quad \Omega_1 = \Omega_2$

Final orbit: $r_p = 8000\ km$, $r_a = 12000\ km$, $\omega = 0$, $i = 15°$

Special case where orbits intersect at the asc. node, and $\gamma_1 = \gamma_2 = 0$ at that point



These orbits do not intersect at the descending node

$$\Delta v^2 = v_1^2 + v_2^2 - 2 v_1 v_2 \cos\theta$$

$$v_1 = \sqrt{\frac{\mu}{r_c}}$$

$$v_2 \neq v_1 = \sqrt{\frac{2\mu}{r_p} - \frac{2\mu}{r_a + r_p}}$$

$\gamma_1 = \gamma_2 = 0$ at intersection

$\Rightarrow \theta = \Delta i$

Lecture 11 - finishing out-of-plane + Patched conics

Plane Change ellipse, not at periapsis

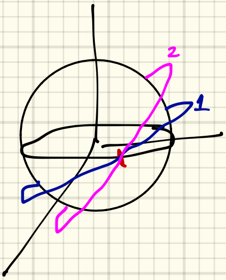
Initial orbit: $r_p = 8000$ km, $r_a = 12{,}000$ km, $\Omega_1 = 30°$, $i = 20°$, $\omega_1 = 20°$

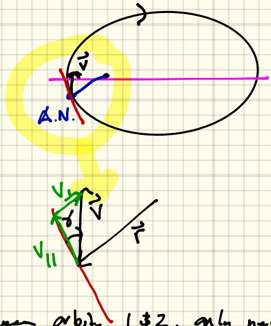
Final orbit: $r_p = 8000$ km, $r_a = 12{,}000$ km, $\Omega_2 = \Omega_1$, $i = 30°$, $\omega_2 = \omega_1$

↑ Same a, e

↑ periapsis is not at the ascending node

These 2 orbits will intersect at the ascending & descending nodes





$V_\perp$ stays the same between orbits 1 & 2, only need to rotate the $V_{\parallel}$ component of velocity

$V_{\parallel} = V\cos\gamma$

$\Delta V^2 = v_1^2 + v_2^2 - 2v_1v_2\cos\theta$

$v_1 = v_2$ but we only need to rotate the $V_{\parallel}$ component of velocity

$\Delta V^2 = 2v_{\parallel}^2(1-\cos\theta)$

$= 2v^2\cos^2\gamma(1-\cos\theta)$

Angle between $(\vec{V}_{\parallel})_1$ & $(\vec{V}_{\parallel})_2 = \Delta i$

$\Delta V = v\cos\gamma\sqrt{2(1-\cos\Delta i)}$

Half-angle identity: $\sin\left(\frac{\Delta i}{2}\right) = \sqrt{\frac{1-\cos\Delta i}{2}}$

$$\boxed{\Delta V = 2v\cos\gamma\sin\left(\frac{\Delta i}{2}\right)}$$

Single-impulse, instantaneous

Maneuvers: Law of Cosines

- Where does the maneuver occur?
- What are v_1, v_2, θ?

Tangential:

- in-plane only
- $\theta = 0$
- Change the velocity magnitude
- a, e
- if not at r_p or r_a, ω, ν

Non-tangential:

in-plane:

- Change velocity direction and (maybe) magnitude
- $\theta = \gamma_2 - \gamma_1$
- Change: a, e, ω, ν

out-of-plane:

- Change the velocity direction and (maybe) magnitude
- Change any OE
- θ = complicated

Ex. w/ simple θ:

1. inclination change for circular orbit
2. a, e, i change where $\omega = 0$ or $180°$
3. inclination only change for $e \neq 0$, $\omega \neq 0, 180°$, $a_1 = a_2$, $e_1 = e_2$

$$\Delta v = 2 v \cos\gamma \sin(\Delta i / 2)$$

Patched Conics: interplanetary maneuvers

Execute a Hohmann transfer from Earth to Mars

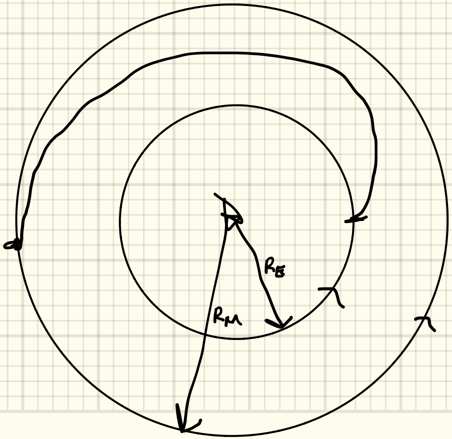
At Earth, S/C is initially on a parking orbit $r_{EP} = 7{,}000$ km

At Mars, parking orbit $r_{MP} = 7{,}000$ km (circular)

"Patching" together different Conic sections

Split the trajectory into 3 parts:

1. hyperbola to escape from Earth
2. ellipse about the Sun ← start here
3. hyperbola to get captured at Mars



R_E = radius of Earth's orbit about the Sun (circular)

R_M = radius of Mars' orbit (circular)

Calculate the velocity at periapsis & apoapsis of ellipse

$$\frac{V_{tp}^2}{2} - \frac{\mu_s}{R_E} = -\frac{\mu_s}{R_E + R_M} \qquad \mu = \mu_{sun}$$

$$\Rightarrow V_{tp}$$

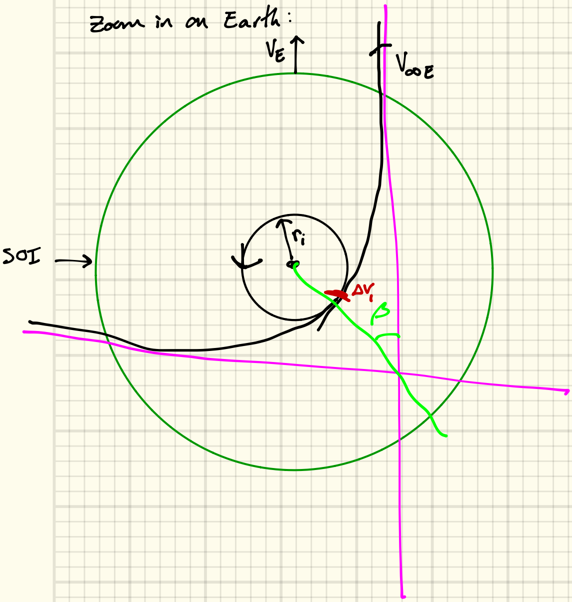
$$\frac{V_{ta}^2}{2} - \frac{\mu_s}{R_M} = -\frac{\mu_s}{R_E + R_M} \qquad \Rightarrow V_{ta}$$

Also, velocities of planets:

$$V_E = \sqrt{\frac{\mu_S}{R_E}}$$

$$V_M = \sqrt{\frac{\mu_S}{R_M}}$$

Zoom in on Earth:



r_i = radius of the initial parking orbit

SOI = Sphere of Influence

Only using Earth's gravity here

From the Earth's perspective, $r_{SOI} = \infty$

Assume periapsis of hyperbola is at r_i.

Execute a tangential maneuver to go from the circular parking orbit on to the hyperbola.

$$V_i = \sqrt{\frac{\mu_E}{r_i}}$$

$$\frac{V_{hp}^2}{2} - \frac{\mu_E}{r_i} = \frac{V_{\infty E}^2}{2} - \cancelto{0}{\frac{\mu_E}{r_{SOI}}}$$

V_{hp} = velocity of hyperbola at periapsis

$V_{\infty E} = V_{tp} - V_E$ (we know both of these from heliocentric phase on prior page)

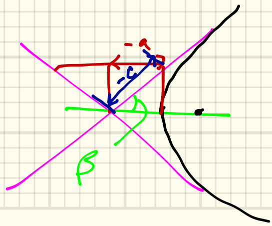
$$\Delta V_1 = V_{hp} - V_i$$

Given $V_{\infty E}$ & r_p (of the hyperbola), we can solve for the eccentricity.

$$r_p = \frac{h^2/\mu_E}{1 + e\cos\nu}, \quad \nu = 0 \text{ b/c periapsis.}$$

$$h = \frac{\mu_E\sqrt{e^2-1}}{V_{\infty E}}$$

$$\Rightarrow \boxed{e = 1 + \frac{r_p V_\infty^2}{\mu}}$$



$$\cos\beta = \frac{-a}{-c}$$

$$e = \frac{c}{a}$$

$$\boxed{\cos\beta = \frac{1}{e}}$$

Definition of sphere of influence:

$$\boxed{r_{SOI} = \left(\frac{m_s}{m_p}\right)^{2/5} a}$$

m_s = mass of the secondary (smaller)

m_p = mass of the primary (sun, larger body)

a = semi-major axis of the secondary about the primary.

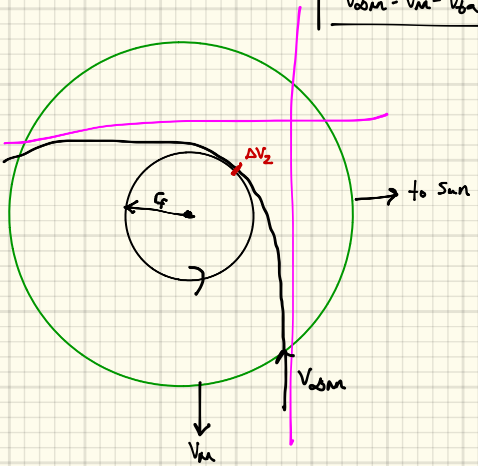
Arrival: Zoom in on Mars.

From the heliocentric phase, we know that s/c is traveling slower than Mars when it gets to Mars.

From the heliocentric phase, we know $V_{ta} = \sqrt{\frac{-2\mu_s}{R_E + R_M} + \frac{2\mu_s}{R_M}}$

$$V_M = \sqrt{\frac{\mu_s}{R_M}}$$

$$\boxed{V_{\infty M} = V_M - V_{ta}}$$



$V_{\infty M}$ goes in the opposite direction as V_M b/c the s/c is traveling slower than the planet. The planet "catches up" to the s/c.

$$V_f = \sqrt{\frac{\mu_M}{r_f}}$$

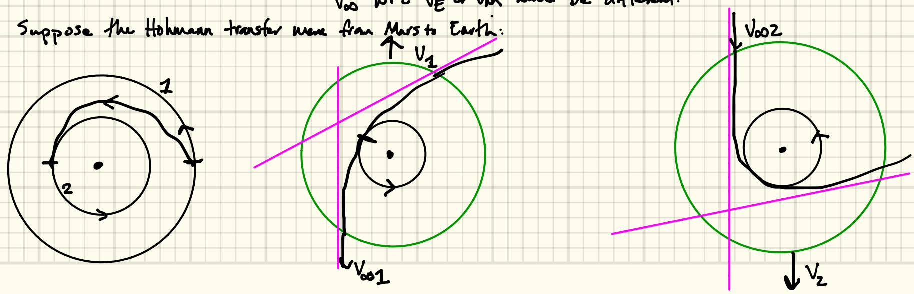
$$\frac{V_{ph}^2}{2} - \frac{\mu_M}{r_f} = \frac{V_{\infty M}^2}{2} - \cancelto{0}{\frac{\mu_M}{r_{SOI}}}$$

$\Delta V_2 = V_{ph} - V_f$.

Note: we are assuming that the hyperbola's periapsis is r_f.

Second simplifying assumption: Hohmann transfer for the heliocentric phase
would follow the same procedure for a different transfer, but the orientation of V_∞ wrt V_E or V_M would be different.

Suppose the Hohmann transfer were from Mars to Earth:



Kepler's Prod

Kepler's Prediction Problem:

3 major concepts related to TOF:

1. How long does it take to get between 2 locations on an orbit?

$\Rightarrow$ TOF eqn

Known: OE, $\vec{r}_0$, $\vec{r}_1$ Unknown: TOF

2. Given an orbit & an initial state, where will the s/c be at some later time?

$\Rightarrow$ Kepler's Pred Prob

Known: OE, $\vec{r}_0$, TOF Unknown: $\vec{r}_1$

3. Given 2 positions & a TOF, what is the orbit?

$\Rightarrow$ Lambert's Prob (also relevant to maneuver design)

Known: $\vec{r}_0$, $\vec{r}_1$, TOF Unknown: OE's

Kepler's Prediction Problem:

Know: OE, $\vec{r}_0$, TOF $\rightarrow$ Find $\vec{r}_1$

Unfortunately, there is no analytical solution to this problem. Must iterate.

Given $\nu_0 \rightarrow E_0$ (Eccentric Anomaly)

M = mean anomaly (rad) = angular location of the s/c if it's angular velocity were constant

$= n(t-T)$

n = mean motion $= \sqrt{\frac{\mu}{a^3}}$

$t-T$ = time since last periapsis passage.

$M = E - e\sin E$

Know: OE's $\Rightarrow n$

TOF is known

Thus: ΔM is known: $M_1 = M_0 + n(TOF)$

If M_1 is known, solve for E_1 (E_1 = location of the s/c $\vec{r}_1$)

<u>But</u>: $M = E - e\sin E$ $\leftarrow$ Cannot be solved analytically for E. Must solve iteratively

Will not go thru the full derivation.

<u>Algorithm for solving Kepler's Prediction Prob:</u>

1. Guess χ_n (a function of s/c position)

2. Calculate $t_n = f(\chi_n)$

3. If $(t_1 - t_n) >$ tolerance

   Calculate χ_{n+1}

4. Repeat steps 2 & 3 until $(t_1 - t_n) \leq$ tolerance

5. Calculate $\vec{r}_1$ & $\vec{v}_1$ using Lagrange Coefficients (which are a $f(\chi_n)$)

Define $\dot{\chi} = \sqrt{\mu}/r$

Pseudo-Code:

Input: $\vec{r}_0, \vec{v}_0, \Delta t$

$\mathcal{E} = \frac{v_0^2}{2} - \frac{\mu}{r_0} \rightarrow$ tells us the type of Conic

Solve for a: $\mathcal{E} = -\frac{\mu}{2a}$

If circle or Ellipse:

$$\chi_n = \frac{\sqrt{\mu}}{a} \Delta t$$

If parabola:

$$\vec{h} = \vec{r}_0 \times \vec{v}_0$$

$$p = h^2/\mu$$

$$\cot(2s) = 3\sqrt{\frac{\mu}{p^3}}\Delta t$$

s & w here are just intermediate variables (not OE's)

$$\tan^3(w) = \tan(s)$$

$$\chi_n = \sqrt{p}\, 2\cot(2w)$$

If hyperbola:

$$\chi_n = \mathrm{sign}(\Delta t)\sqrt{-a}\,\ln\left[\frac{-2\mu\Delta t/a}{(\vec{r}_0 \cdot \vec{v}_0) + \mathrm{sign}(\Delta t)\sqrt{-\mu a}\,(1 - r_0/a)}\right]$$

$$\psi = \chi^2/a$$

Calculate C & S:

if $\psi > 1E{-6}$

$$C = \frac{1-\cos(\sqrt{\psi})}{\psi}$$

$$S = \frac{\sqrt{\psi} - \sin(\sqrt{\psi})}{\sqrt{\psi^3}}$$

elseif $\psi < -1E{-6}$

$$C = \frac{1-\cosh(\sqrt{-\psi})}{\psi}$$

$$S = \frac{\sinh(\sqrt{-\psi}) - \sqrt{-\psi}}{\sqrt{(-\psi)^3}}$$

else $C = 1/2$

$S = 1/6$

while $|t-\Delta t| > 10$ sec (You can play with this tolerance)

(Need to update χ)

$$r = \chi^2 C + \frac{\vec{r}_0\cdot\vec{v}_0}{\sqrt{\mu}}\chi(1-\psi S) + r_0(1-\psi C)$$

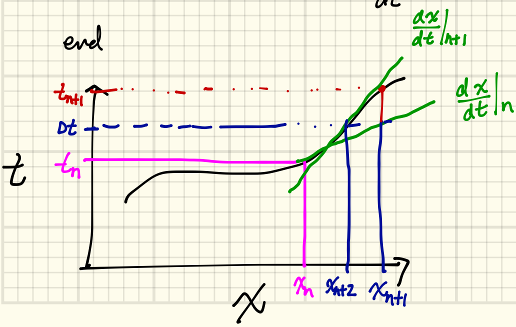
$$\frac{d\chi}{dt} = \sqrt{\mu}/r$$

$$t = \frac{1}{\sqrt{\mu}}\left[\chi^3 S + \frac{\vec{r}_0\cdot\vec{v}_0}{\sqrt{\mu}}\chi^2 C + r_0\chi(1-\psi S)\right]$$

$$\chi_{n+1} = \chi_n + (\Delta t - t)\frac{d\chi}{dt}$$ (updates χ)

end

$$\chi_{n+1} = \chi_n + \left.\frac{d\chi}{dt}\right|_n (\Delta t - t_n)$$



Newton-Raphson Method to update χ_n.

Lagrange Coefficients:

$$f = 1 - \frac{\chi^2}{r_0} C$$

$$g = t - \frac{\chi^3}{\sqrt{\mu}} S$$

$$\dot{f} = \frac{\sqrt{\mu}}{r_0 r} \chi (\psi S - 1)$$

$$\dot{g} = 1 - \frac{\chi^2}{r} C$$

Output:

$$\vec{r} = f \vec{r}_0 + g \vec{v}_0$$

$$\vec{v} = \dot{f} \vec{r}_0 + \dot{g} \vec{v}_0$$

Lecture 15 - Lambert's Prob

TOF-related Problems:

1. Calculate the TOF between 2 points on a known orbit

2. Kepler's Pred Problem: Given $\vec{r}_1$ & $\vec{v}_1$ find $\vec{r}_2, \vec{v}_2$ @ some later time

3. Given 2 known positions $(\vec{r}_1, \vec{r}_2)$ and a time of flight, find the orbit that links these positions. Lambert's Problem = Gauss' Problem

How should you check to see if your Kepler's Pred Problem code is working?

Use your 2BP integrator: IC's $\vec{r}_1, \vec{v}_1$ & integrate for the desired TOF.

Kepler's Pred Problem can be less computationally expensive than the numerical integrator

Lambert's Problem: Given $\vec{r}_1$ & $\vec{r}_2$ & TOF $\rightarrow$ what is the orbit that fits?

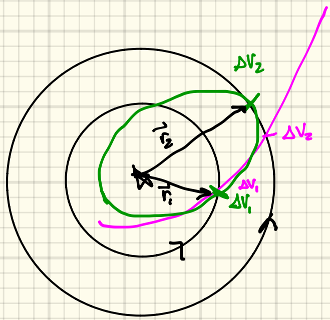
If we know $\vec{r}_1, \vec{v}_1 \Rightarrow$ OE

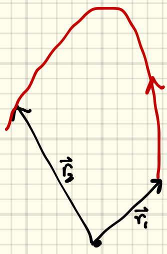
Motivated by desire to predict the orbit of Ceres

Modern Uses:

1. Orbit Determination: Given observations of a satellite, figure out what orbit it's on.

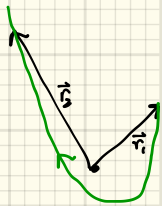
2. Trajectory Design: Given departure & arrival positions & a TOF, calculate the departure and arrival velocities.





Short-way transfer

$\Delta\nu < \pi$



Long-way transfer

$\Delta\nu > \pi$

Just given $\vec{r}_1$ & $\vec{r}_2$, there are an infinite number of possible transfers.

Given $\vec{r}_1, \vec{r}_2$ & TOF, there are 2 possible transfers (short-way & long-way).

Lambert's Problem: Bi-section Method
Pseudo-Code Algorithm:

Input: $\vec{r}_0, \vec{r}_f$, TOF, DM $\rightarrow$ DM = direction of motion

DM = 1 for short-way
DM = -1 for long way

Output: $\vec{v}_0, \vec{v}_f$

$$\cos \Delta\nu = \frac{\vec{r}_0 \cdot \vec{r}_f}{r_0 r_f}$$

$$A = DM \sqrt{r_0 r_f (1 + \cos\Delta\nu)}$$

If $\Delta\nu = 0$, $A = 0 \Rightarrow$ Error

Else:

$\psi = 0$

$C_2 = \frac{1}{2}$, $C_3 = \frac{1}{6}$ $\quad \begin{bmatrix} C_2 = C \\ C_3 = S \end{bmatrix}$

$\psi_{up} = 4\pi^2$

$\psi_{low} = -4\pi$

while $|TOF - \Delta t| > 1 \times 10^{-6}$

$$y = r_0 + r_f + \frac{A(\psi C_3 - 1)}{\sqrt{C_2}}$$

If $A > 0$ & $y < 0$

increase ψ_{low}

end

$$\chi = \sqrt{\frac{y}{C_2}}$$

$$\Delta t = \frac{\chi^3 C_3 + A\sqrt{y}}{\sqrt{\mu}}$$

If $\Delta t < TOF$

$\psi_{low} = \psi$

else $\psi_{up} = \psi$

end

$$\psi_{new} = \frac{\psi_{up} + \psi_{low}}{2}$$

If $\psi > 1 \times 10^{-6}$

$$C_2 = \frac{1 - \cos\sqrt{\psi}}{\psi}$$

$$C_3 = \frac{\sqrt{\psi} - \sin\sqrt{\psi}}{\sqrt{\psi^3}}$$

else if $\psi < -1 \times 10^{-6}$

$$C_2 = \frac{1 - \cosh\sqrt{-\psi}}{\psi}$$

$$C_3 = \frac{\sinh\sqrt{-\psi} - \sqrt{-\psi}}{\sqrt{(-\psi)^3}}$$

else $C_2 = 1/2$, $C_3 = 1/6$

end
end (while)

New Δt is within some tolerance of TOF

Calculate: $f = 1 - \frac{y}{r_0}$

$$g = A\sqrt{\frac{y}{\mu}}$$

$$\dot{g} = 1 - \frac{y}{r_f}$$

Output: $\vec{v}_0 = \frac{\vec{r}_f - f\vec{r}_0}{g}$

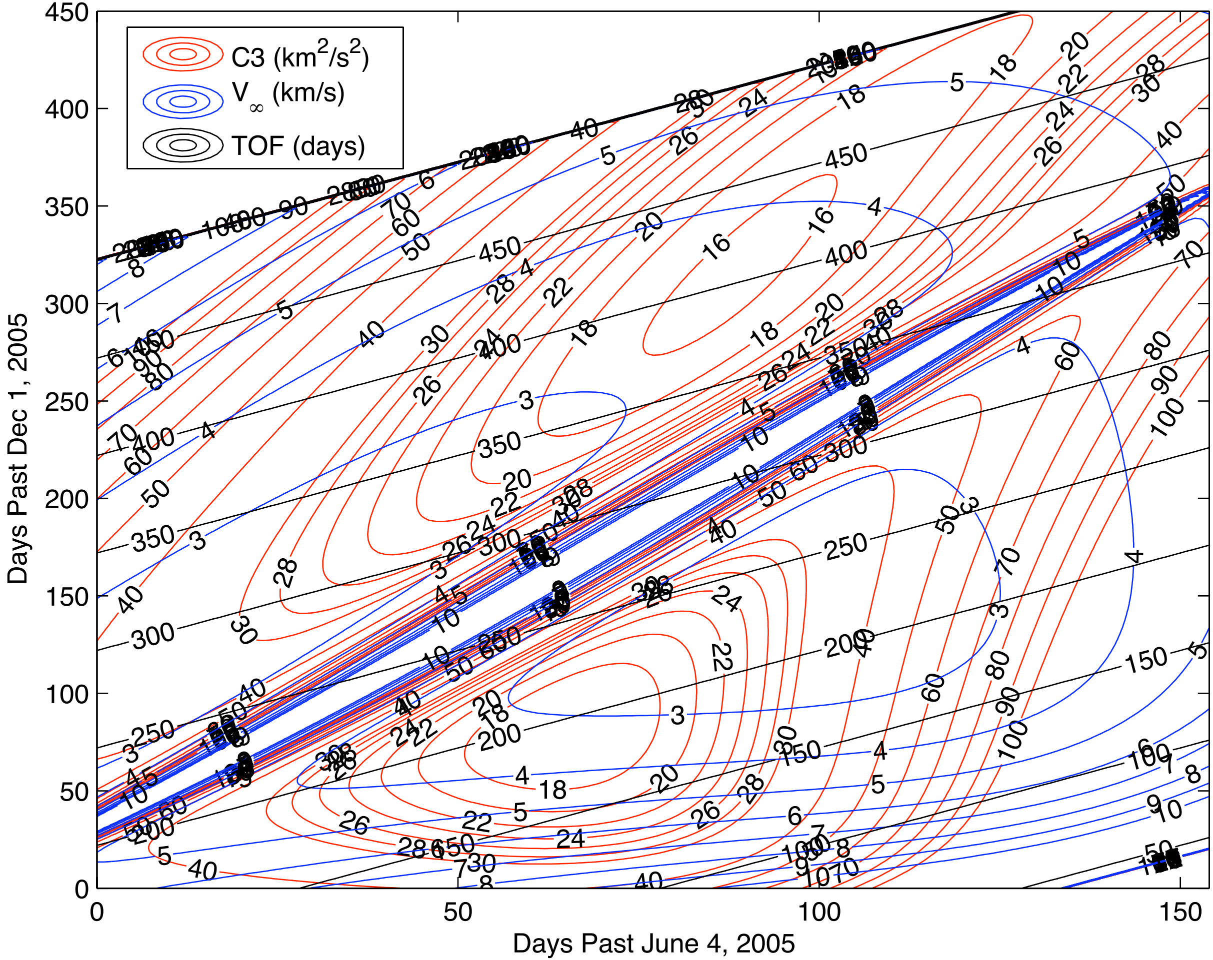
$$\vec{v}_f = \frac{\dot{g}\vec{r}_f - \vec{r}_0}{g}$$

How to check if Lambert's Prob code is working:

Given $\vec{r}_0$ & $\vec{r}_f$, TOF

From Lambert: $\vec{v}_0, \vec{v}_f$

Plug $\vec{r}_0, \vec{v}_0$ into your 2BP integrator → should get $\vec{r}_f, \vec{v}_f$



C3 (km^2/s^2)
V_∞ (km/s)
TOF (days)
Days Past Dec 1, 2005
Days Past June 4, 2005

For Porkchop plot:

a series of launch dates $\Rightarrow \vec{r}_0$ (position of Earth at the launch date)

a series of arrival dates $\Rightarrow \vec{r}_f$ (position of Mars at the arrival date)

TOF = arrival date - launch (departure) date

$\Rightarrow$ Calculate $\vec{v}_0$ & $\vec{v}_f$ for each transfer using the Lambert solver

We also know the velocities of Earth & Mars

$$\vec{v}_{\infty E} = \vec{v}_0 - \vec{v}_E \qquad C3 = |\vec{v}_{\infty E}|^2$$

$$\vec{v}_{\infty M} = \vec{v}_f - \vec{v}_M \qquad v_\infty = |\vec{v}_{\infty M}|$$

Lecture 16: IOD

Initial Orbit Determination:

1. New planet, asteroid, comet → what is the orbit?

   e.g. Interstellar object identified b/c $e > 1$ ⇒ hyperbolic orbit about our Sun.

2. Enemy (or unknown) satellite: Space Situational Awareness (SSA)

   what are the goals & capabilities of other satellites, nations?

3. Orbital debris: will the orbital debris collide with s/c?

   Strongly influenced by orbital perturbations

4. Science: thru precise orbit determination, we can get better gravity and/or tide models.

Statistical orbit determination: how to incorporate additional observations to update our estimate of the orbit.
⇒ grad level course

Initial orbit determination:

Method #1: Lambert's Prob: $\vec{r}_1, \vec{r}_2$, TOF ⇒ $\vec{v}_1, \vec{v}_2$ $\quad (\vec{r}_1, \vec{v}_1 \Rightarrow OE)$

Method #2: Gibbs Method: input: 3 position vectors

output: $\vec{v}_1$

Given: $\vec{r}_1, \vec{r}_2, \vec{r}_3$ $\quad (t_1 < t_2 < t_3)$

Find: $p, e, \hat{P}, \hat{Q}, \hat{W}$ & then $\vec{v}_i$ (Could be $\vec{v}_1, \vec{v}_2$ or $\vec{v}_3$)

The 3 positions must be coplanar.
Choose constants s.t.:

$$C_1\vec{r}_1 + C_2\vec{r}_2 + C_3\vec{r}_3 = 0 \quad \star$$

$$\vec{r} \cdot \hat{P} = r\cos\nu$$

$$\vec{e} = e\hat{P}$$

$$\vec{r} \cdot \vec{e} = re\cos\nu$$

$$r = \frac{p}{1+e\cos\nu} \Rightarrow r + re\cos\nu = p$$

$$\vec{r} \cdot \vec{e} = p - r$$

Dot $\star$ w/ $\vec{e}$: $C_1(p - r_1) + C_2(p - r_2) + C_3(p - r_3) = 0$

Cross ~~*~~ eqn w/ $\vec{r}_1, \vec{r}_2, \vec{r}_3$:

$$C_2 \vec{r}_1 \times \vec{r}_2 = C_3 \vec{r}_3 \times \vec{r}_1$$

$$C_1 \vec{r}_1 \times \vec{r}_2 = C_3 \vec{r}_2 \times \vec{r}_3$$

$$C_1 \vec{r}_3 \times \vec{r}_1 = C_2 \vec{r}_2 \times \vec{r}_3$$

Multiply the scalar eqn by $\vec{r}_3 \times \vec{r}_1$ & then substitute in the cross product terms to get the eqn only in terms of C_2.

$$C_2(p-r_1)\vec{r}_2 \times \vec{r}_3 + C_2(p-r_2)\vec{r}_3 \times \vec{r}_1 + C_2(p-r_3)\vec{r}_1 \times \vec{r}_2 = 0$$

Divide by C_2 & rearrange:

$$p\underbrace{\left[\vec{r}_2 \times \vec{r}_3 + \vec{r}_3 \times \vec{r}_1 + \vec{r}_1 \times \vec{r}_2\right]}_{\vec{D}} = \underbrace{r_1 \vec{r}_2 \times \vec{r}_3 + r_2 \vec{r}_3 \times \vec{r}_1 + r_3 \vec{r}_1 \times \vec{r}_2}_{\vec{N}}$$

$$p\vec{D} = \vec{N} \qquad \vec{N} \cdot \vec{D} = ND$$

$$\boxed{p = \frac{N}{D}}$$

B/c we know $\vec{r}_1, \vec{r}_2, \vec{r}_3$, we know the plane of the orbit:

$$\hat{h} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

$\Rightarrow \vec{N}$ & $\vec{D}$ are in the $\hat{h}$ direction

$\hat{w} \parallel \hat{h} \quad \Rightarrow \vec{N}, \vec{D} \parallel \hat{w}$

$$\hat{P} \times \hat{Q} = \hat{w} \Rightarrow \hat{Q} = \hat{w} \times \hat{P}$$

$$\hat{P} \parallel \hat{e}$$

$$\hat{Q} = \frac{1}{Ne} \vec{N} \times \vec{e}$$

Substitute in for $\vec{N}$:

$$Ne\hat{Q} = r_1(\vec{r}_2 \times \vec{r}_3) \times \vec{e} + r_2(\vec{r}_3 \times \vec{r}_1)\vec{e} + r_3(\vec{r}_1 \times \vec{r}_2) \times \vec{e}$$

Identity: $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

$$Ne\hat{Q} = r_1(\vec{r}_2 \cdot \vec{e})\vec{r}_3 - r_1(\vec{r}_3 \cdot \vec{e})\vec{r}_2 +$$

$$r_2(\vec{r}_3 \cdot \vec{e})\vec{r}_1 - r_2(\vec{r}_1 \cdot \vec{e})\vec{r}_3 +$$

$$r_3(\vec{r}_1 \cdot \vec{e})\vec{r}_2 - r_3(\vec{r}_2 \cdot \vec{e})\vec{r}_1$$

Using $\vec{r}\cdot\vec{e} = p - r$

$$Ne\hat{Q} = [r_1(p-r_2) - r_2(p-r_1)]\vec{r}_3 +$$

$$[r_3(p-r_1) - r_1(p-r_3)]\vec{r}_2 +$$

$$[r_2(p-r_3) - r_3(p-r_2)]\vec{r}_1$$

$$\Rightarrow Ne\hat{Q} = p\underbrace{[(r_1-r_2)\vec{r}_3 + (r_3-r_1)\vec{r}_2 + (r_2-r_3)\vec{r}_1]}_{\vec{S}}$$

$$Ne\hat{Q} = p\vec{S}$$

$$\Rightarrow \boxed{e = \frac{pS}{N} \qquad (b/c\ N = pD) \Rightarrow e = \frac{S}{D}}$$

$$\boxed{\hat{w} = \frac{\vec{N}}{N}} \qquad \boxed{\hat{Q} = \frac{\vec{S}}{S}} \qquad \boxed{\hat{P} = \hat{Q}\times\hat{w}}$$

We want a velocity. Start with an intermediate step in the derivation of the trajectory eqn

$$\dot{\vec{r}}\times\vec{h} = \mu\left(\frac{\vec{r}}{r} + \vec{e}\right)$$

Cross w/ $\vec{h}$: $\vec{h}\times(\dot{\vec{r}}\times\vec{h}) = \mu\left(\frac{\vec{h}\times\vec{r}}{r} + \vec{h}\times\vec{e}\right)$

Identity: $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$\vec{h}\times(\dot{\vec{r}}\times\vec{h}) = (\vec{h}\cdot\vec{h})\dot{\vec{r}} - \cancelto{0}{(\vec{h}\cdot\dot{\vec{r}})}\vec{h}$$

$$= h^2\vec{v}$$

$$h^2\vec{v} = \mu\left(\frac{\vec{h}\times\vec{r}}{r} + \vec{h}\times\vec{e}\right)$$

$$\vec{h} = h\hat{w}, \quad \vec{e} = e\hat{P}$$

$$\vec{v} = \frac{\mu}{h}\left(\frac{\hat{w}\times\vec{r}}{r} + \hat{w}\times e\hat{P}\right) \qquad \text{know } \hat{P}\times\hat{Q} = \hat{w}$$

$$= \frac{\mu}{h}\left(\frac{\hat{w}\times\vec{r}}{r} + e\hat{Q}\right)$$

$$h = \sqrt{\mu p}, \quad h = \sqrt{\mu N/D}$$

$$e = \frac{S}{D}, \quad \hat{Q} = \frac{\vec{S}}{S}, \quad \hat{w} = \frac{\vec{D}}{D}$$

$$\vec{v} = \frac{\sqrt{\mu D}}{\sqrt{N}}\frac{\vec{D}\times\vec{r}}{rD} + \frac{\vec{S}}{D}\sqrt{\frac{D\mu}{N}}$$

$$\boxed{\vec{B} = \vec{D}\times\vec{r}_i}, \quad \boxed{L = \sqrt{\frac{\mu}{DN}}}$$

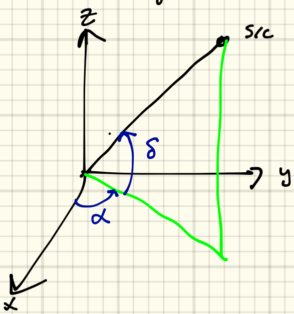
$$\boxed{\vec{V}_i = \frac{L}{r_i}\vec{B} + L\vec{S}}$$ where $\vec{r}_i$ can be $\vec{r}_1, \vec{r}_2$ or $\vec{r}_3$

Method #3: Laplace's Method: "Angles only" Proposed in 1780

(Don't need to know range → easier to just work with the angular location)

Input: $\alpha_1, \delta_1, \alpha_2, \delta_2, \alpha_3, \delta_3$

Review the angles:

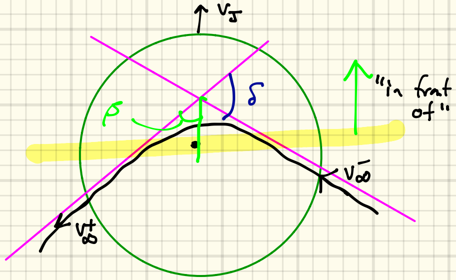


α = Right ascension, measured in the xy plane E from $\hat{x}$

δ : declination, measured N from the equatorial plane.

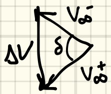
13/14: Flybys

Flybys: Consider a flyby of Jupiter.



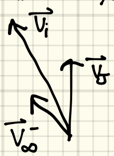
$V_\infty^- = V_\infty^+$ (from Jupiter's perspective)

δ = turn angle



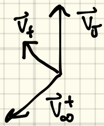
Heliocentric/inertial frame:

Initial s/c velocity:



$$\vec{V}_i = \vec{V}_\infty^- + \vec{V}_J$$

Final s/c velocity:



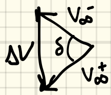
$$\vec{V}_f = \vec{V}_J + \vec{V}_\infty^+$$

$\Rightarrow$ This flyby decreased the s/c velocity magnitude ($V_f < V_i$) & changed its direction

"Leading edge" flyby: where periapsis of the hyperbola is "in front of" the planet

: decrease heliocentric velocity

Can use law of cosines to calculate the ΔV of the flyby

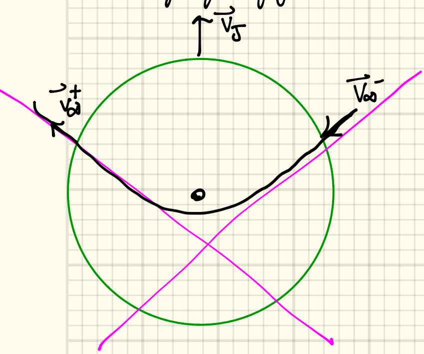


$$V_\infty^- = V_\infty^+ = V_\infty$$

$$\Delta V^2 = v_1^2 + v_2^2 - 2 v_1 v_2 \cos\theta$$

$$\Delta V^2 = 2 v_\infty^2 (1 - \cos\delta)$$

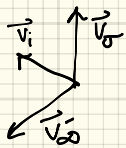
Trailing Edge Flyby:



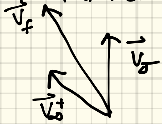
Trailing Edge: periapsis of hyperbola is behind planet

$|\vec{V}_\infty^-| = |\vec{V}_\infty^+|$

Initial Heliocentric:



Final Heliocentric:



Trailing edge flyby increases the heliocentric velocity.

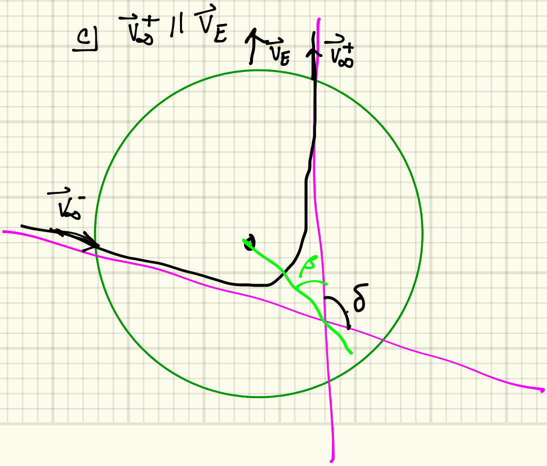
Example: Earth Flyby: $e = 1.3$
$r_p = 40{,}000$ km

a. Calculate v_∞
b. Calculate β.
c. Sketch the flyby that will maximize the s/c's final, inertial velocity.
d. Calculate the ΔV.

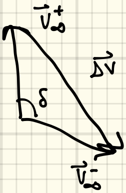
a) $e = 1 + \frac{r_p v_\infty^2}{\mu} \Rightarrow V_\infty = 1.729$ km/s

b) $\cos\beta = \frac{1}{e} \Rightarrow \beta = 39.7°$

c) $\vec{V}_\infty^+ \parallel \vec{V}_E$



d.



law of Cosines, $|\vec{V}_\infty^-| = |\vec{V}_\infty^+|$

$\Delta V^2 = 2 V_\infty^2 (1 - \cos\delta)$

$\delta = 180 - 2\beta = 100.57°$

$\Delta V = 2.66$ km/s

Need a force to change velocity:

$$\vec{F} = m\vec{a} = m_{s/c} \frac{d\vec{v}}{dt}$$

$$\Delta V_{sc} = \left| \int d\vec{v} \right| = \frac{1}{m_{sc}} \left| \int \vec{F}_{s/c}\, dt \right|$$

Note: $\vec{F}_{sc} = -\vec{F}_p$ (force on planet)

What is the ΔV of the planet?

$$\frac{1}{m_p} \left| -\int \vec{F}_{sc}\, dt \right| = \Delta V_p$$

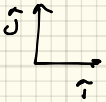
From above: $\left| \int \vec{F}_{sc}\, dt \right| = m_{sc} \Delta V_{sc}$

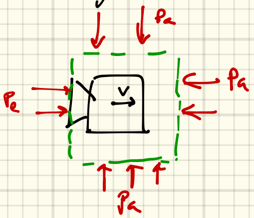
$$\Delta V_p = \frac{m_{sc} \Delta V_{sc}}{m_p}$$

$m_p \gg m_{sc} \Rightarrow \Delta V_p \simeq 0 \quad \square$

Lecture 17 - Rocket Eqn

Rocket eqn: This is why we calculate ΔV.





rocket w/ velocity V & mass m.

it expells exhaust ΔM at speed V_e.

All forces & velocities are in the $\hat{i}$ direction.

Momentum of the system @ $t+\Delta t$ − initial momentum = external impulse

$$\left[(m-\Delta m)(V+\Delta V)\hat{i} + \Delta m(-V_e)\hat{i}\right] - mV\hat{i} = (p_e - p_a)A_e \Delta t\,\hat{i}$$

Pressure = F/A

A_e = area of the exit nozzle

Define $\dot{m}_e$ = exhaust mass flow rate > 0

Assuming $\dot{m}$ = constant

Note $\dot{m}$ is the time rate of change of the s/c mass $= -\dot{m}_e$

$$\left[(m-\dot{m}_e\Delta t)(V+\Delta V)\hat{i} + \dot{m}_e\Delta t(-V_e)\hat{i}\right] - mV\hat{i} = (p_e - p_a)A_e\Delta t\,\hat{i}$$

$$m\Delta V - \dot{m}_e\Delta t(V+V_e) - \underbrace{\dot{m}_e\Delta t\Delta V}_{=0 \text{ b/c it's the product of 2 small terms}} = (p_e - p_a)A_e\Delta t$$

Divide by Δt:

$$\underbrace{m\frac{dv}{dt}}_{=T} - \dot{m}_e(v+v_e) = (p_e - p_a)A_e$$

$$T = m\frac{dv}{dt} = \dot{m}_e(v+v_e) + (p_e - p_a)A_e$$

Define an effective exhaust velocity:

$$T = \dot{m}_e V_{eff}$$

$$V_{eff} = V + V_e + \frac{(p_e - p_a)A_e}{\dot{m}_e}$$

Note: $g_0 = 9.81\ m/s^2$

Specific Impulse:

$$\boxed{I_{sp} = \frac{T}{\dot{m}_e g_0}} = \frac{\text{Thrust}}{\text{rate of propellant weight consumption}}$$

Units: seconds

Large I_{sp} if we have a large thrust or if $\dot{m}_e$ is small

Typical chemical propulsion:

Solid: $I_{sp} \sim 200-300\,s$

liquid: $I_{sp} \sim 250-450\,s$

Electric Propulsion: $I_{sp} \sim 1000-5000$ sec, but low thrust

$$T = I_{sp}\dot{m}_e g_0 = -I_{sp} g_0 \frac{dm}{dt}$$

$$\Rightarrow \frac{dm}{dt} = \frac{-T}{I_{sp} g_0}$$

$$\frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g\sin\gamma$$

gravity on rocket launching from Earth's surface

$$\frac{dv}{dt} = \frac{-I_{sp} g_0}{m}\frac{dm}{dt} - \frac{D}{m} - g\sin\gamma$$

Integrate wrt time:

$$\Delta V = I_{sp} g_0 \ln\left(\frac{m_i}{m_f}\right) - \Delta V_D - \Delta V_G$$

ΔV due to gravity

ΔV due to drag

If we are in free space, we can neglect drag & gravity.

$$\boxed{\Delta V = I_{sp} g_0 \ln\left(\frac{m_i}{m_f}\right)}$$ ← Approximation!

B/c dropped H.O.T.

& T = const.

We can rearrange to solve for the mass ratio ($n = m_i/m_f$)

$$\frac{m_i}{m_f} = \exp\left(\frac{\Delta V}{I_{sp} g_0}\right)$$

Solve for Δm required to produce ΔV: $\Delta m = m_i - m_f$

$$\boxed{\frac{\Delta m}{m_i} = 1 - \exp\left(\frac{-\Delta V}{I_{sp} g_0}\right)}$$

As $\Delta V \uparrow$, exponential term $\downarrow$ & $\frac{\Delta m}{m_i} \rightarrow 1 \Rightarrow$ which means that the whole s/c mass is fuel mass

Example of benefits of Electric Propulsion: Dawn Mission: W/ Chemical Propulsion: 26,000 kg

W/ electric propulsion $\Rightarrow$ 1,240 kg

$$m_i = m_E + m_P + m_{PL}$$

m_E: empty mass (structures)

m_P: propellant mass

m_{PL}: Payload mass

Define payload ratio: $\lambda = \dfrac{m_{PL}}{m_i - m_{PL}}$

Structural ratio: $\epsilon = \dfrac{m_E}{m_i - m_{PL}}$

Mass fraction: $n = \dfrac{m_i}{m_f} = \dfrac{m_E + m_P + m_{PL}}{m_E + m_{PL}}$

$$\Rightarrow n = \frac{1+\lambda}{\epsilon+\lambda}$$

For a chemical rocket with $I_{sp} = 300\,s$, $\epsilon = 0.1$, $\lambda = 0.05$ $\Rightarrow$ What's the max ΔV provided?

$\Delta V = 5.7$ km/s

Low Thrust Trajectories: the maneuver can take months → years

more complicated to solve b/c the s/c $\mathcal{E}$ ($\vec{h}$) is constantly changing

↳ typically need to numerically integrate to design these trajectories.

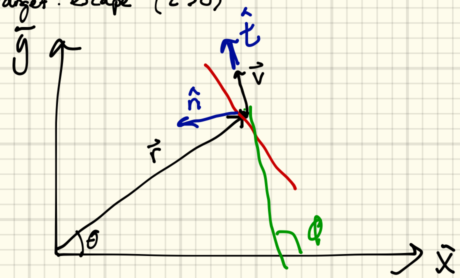
Can modify 2BP numerical integration code to include an additional thrust term.

The complicated part of low-thrust trajectory design is figuring out the control law that gets the s/c to the desired orbit in the min time with min fuel.

We will consider a special case where the s/c is thrusting tangentially to the orbit at all times.

Assume the s/c is initially in a circular orbit with radius r_0.

Target: escape ($\mathcal{E} > 0$)



Define the radius of curvature: $\rho = \dfrac{ds}{d\phi}$

Note: $\vec{v} = v\hat{t}$

$\dot{\vec{v}} = \dot{v}\hat{t} + v\dot{\phi}\hat{n}$

$v = \dfrac{ds}{dt}$ = arc-length per time

$$\frac{d\phi}{dt} = \frac{d\phi}{ds}\frac{ds}{dt} = \frac{v}{\rho}$$

$$\dot{\vec{v}} = \dot{v}\hat{t} + \frac{v^2}{\rho}\hat{n}$$

$\hat{t} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\dot{\hat{t}} = -\dot{\phi}\sin\phi\,\hat{x} + \dot{\phi}\cos\phi\,\hat{y}$

$\hat{n} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\dot{\hat{t}} = \dot{\phi}\hat{n}$

Lecture 18: Low thrust + start of s/c attitude

Low Thrust Trajectories: the maneuver can take months → years

more complicated to solve b/c the s/c $\mathcal{E}$ ($\vec{h}$) is constantly changing

↳ typically need to numerically integrate to design these trajectories.

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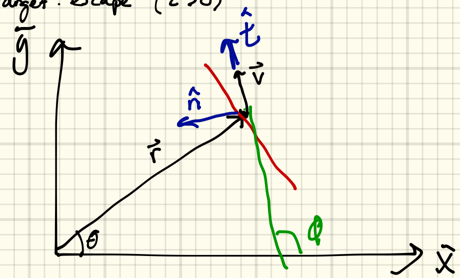


Define the radius of curvature: $\rho = \frac{ds}{d\phi}$

Note: $\vec{v} = v\hat{t}$

$\dot{\vec{v}} = \dot{v}\hat{t} + v\dot{\phi}\hat{n}$

$v = \frac{ds}{dt}$ = arc-length per time

$\frac{d\phi}{dt} = \frac{d\phi}{ds}\frac{ds}{dt} = \frac{v}{\rho}$

$\dot{\vec{v}} = \dot{v}\hat{t} + \frac{v^2}{\rho}\hat{n}$

$\hat{t} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\dot{\hat{t}} = -\dot{\phi}\sin\phi\,\hat{x} + \dot{\phi}\cos\phi\,\hat{y}$

$\hat{n} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\dot{\hat{t}} = \dot{\phi}\hat{n}$

$$\dot{v}\hat{t} + \frac{v^2}{\rho}\hat{n} = \underbrace{a_T\hat{t}}_{\substack{\vec{a}_T \\ \text{thrust}}} - \underbrace{\frac{\mu}{r^2}\hat{r}}_{\substack{\text{2BP} \\ \text{gravity}}}$$

$$-\frac{\mu}{r^2}\hat{r} = -\frac{\mu}{r^2}\sin\gamma\,\hat{t} + \frac{\mu}{r^2}\cos\gamma\,\hat{n}$$

Note: $\vec{v} = \frac{ds}{dt}\hat{t}$

$$\vec{r} = r\hat{r}$$
$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$
$$\left(\frac{ds}{dt}\right)^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$v_r = \dot{r}$, $\frac{dr}{dt} = \frac{ds}{dt}\sin\gamma \Rightarrow \sin\gamma = \frac{dr}{ds}$

$v_\theta = r\dot{\theta} = v\cos\gamma$

$r\frac{d\theta}{dt} = \frac{ds}{dt}\cos\gamma \Rightarrow \cos\gamma = r\frac{d\theta}{ds}$

$$\dot{v}\hat{t} + \frac{v^2}{\rho}\hat{n} = a_T\hat{t} - \frac{\mu}{r^2}\frac{dr}{ds}\hat{t} + \frac{\mu}{r^2}r\frac{d\theta}{ds}\hat{n}$$

Separate directions:

$\hat{t}$: $\dot{v} = a_T - \frac{\mu}{r^2}\frac{dr}{ds}$ $\qquad\qquad$ $\dot{v} = \frac{dv}{dt} = \frac{ds}{dt}\cdot\frac{dv}{ds} = v\frac{dv}{ds}$

$$\Rightarrow v\frac{dv}{ds} = a_T - \frac{\mu}{r^2}\frac{dr}{ds}$$

$\hat{n}$: $\frac{v^2}{\rho} = \frac{\mu}{r}\frac{d\theta}{ds}$

Back to $\hat{t}$:

$$\frac{d(v^2)}{ds} = 2v\frac{dv}{ds}$$

$$\boxed{a_T = \frac{1}{2}\frac{d(v^2)}{ds} + \frac{\mu}{r^2}\frac{dr}{ds}} \quad 1$$

Given:
(definition of curvature) $\frac{1}{\rho} = \frac{1}{r}\left[1 - \left(\frac{dr}{ds}\right)^2 - r\frac{d^2r}{ds^2}\right]\left[1 - \left(\frac{dr}{ds}\right)^2\right]^{-1/2}$

Substitute into the $\hat{n}$ eqn:

$$\frac{1}{r}\left[1-\left(\frac{dr}{ds}\right)^2\right]^{1/2} = \frac{d\theta}{ds} \quad \text{(expansion)}$$

$$\Rightarrow \boxed{rv^2\frac{d^2r}{ds^2} + \left(v^2-\frac{\mu}{r}\right)\left[\left(\frac{dr}{ds}\right)^2-1\right]=0} \quad ②$$

Initial Conditions: Circular initial orbit

$$r(t_0)=r_0 \qquad v^2(t_0)=v_0^2=\frac{\mu}{r_0}$$

$$\left.\frac{dr}{ds}\right|_{t=t_0}=0$$

Assume a_T = constant & integrate ①

$$\int a_T\, ds = \int \frac{1}{2} d(v^2) + \int \frac{\mu}{r^2} dr$$

$$a_T s = \frac{1}{2} v^2\Big|_{v_0}^{v} - \frac{\mu}{r}\Big|_{t_0}^{t}$$

$$= \frac{1}{2}(v^2-v_0^2) - \frac{\mu}{r} + \frac{\mu}{r_0}$$

$$v^2 = 2a_T s + \mu\left(\frac{2}{r}-\frac{1}{r_0}\right)$$

Now, assume the thrust acceleration is small, s.t. $\frac{d^2r}{ds^2} \simeq 0$

then ② requires that $v^2-\frac{\mu}{r}=0 \Rightarrow v=\sqrt{\frac{\mu}{r}}$ = circular velocity $\Rightarrow$ implies that the orbit is always nearly circular

Plug into our eqns for r:

$$\boxed{r=r_0\left[1-\frac{2a_T s}{v_0^2}\right]^{-1}} \quad ③$$

If you accelerate long enough, eventually the s/c will reach escape velocity:

$$V_\infty^2 = \frac{2\mu}{r}$$

Substituting V_∞ into ② $\Rightarrow$

$$2r\frac{d^2r}{ds^2} = 1-\left(\frac{dr}{ds}\right)^2$$

Substitute ③ into above & solve for s (distance traveled before escape): $S_{esc} = \frac{V_0^2}{2a_T}\left[1-\frac{1}{V_0}\left(20a_T^2 r_0^2\right)^{1/4}\right]$

Substitute that distance back into (3) to get the radius when the s/c escapes:

$$\boxed{r_{esc} = \frac{r_0 V_0}{\left(20 a_T^2 r_0^2\right)^{1/4}}}$$

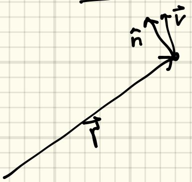
Time required to escape:

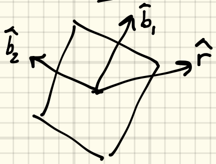
$$\boxed{t_{esc} - t_0 = \frac{V_0}{a_T}\left[1 - \left(\frac{20 a_T^2 r_0^2}{V_0^4}\right)^{1/8}\right]}$$

Spacecraft Attitude Dynamics:

Attitude: s/c orientation

Firstly, must develop a way to describe the attitude of a s/c.

Orbit: 

S/C: 

We can use Euler Angles to describe the rotation between 2 coordinate systems.

$\Rightarrow$ We have already done this: we used Ω, i, ω to describe the rotation from XYZ to PQW

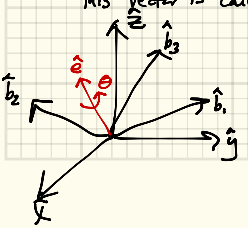
"Euler Angles" 3-1-3 rotation

Rotation Matrix: $\left[[R_{3\Omega}][R_{1i}][R_{3\omega}]\right]$ = Direction Cosine matrix

When we have 2 coordinate system, there are multiple Euler angle combinations that could be used to get the same direction cosine matrix.

Given 2 coordinate systems, there is a vector that is fixed in both systems. You can rotate between these 2 coordinate systems by executing a single rotation about this vector.

This vector is called: Euler Axis = eigenaxis $\hat{e}$ = principal axis



Note that s/c do not typically execute maneuvers about the eigenaxis, but we use this concept to describe s/c attitude.

Lecture 19: Quarternions

Note: $\hat{e} = e_{b1}\hat{b}_1 + e_{b2}\hat{b}_2 + e_{b3}\hat{b}_3$

$\hat{e} = e_{n1}\hat{x} + e_{n2}\hat{y} + e_{n3}\hat{z}$

b/c the vector is fixed in both frames: $e_{bi} = e_{ni} = e_i$

$^B\vec{e} = [C]^I\vec{e}$

$\vec{e}$ is the eigenvector of $[C]$ corresponding to the eigenvalue of $+1$

We would prefer not to find the eigenvectors of a 3x3 matrix → time consuming

Instead, we can derive this principal axis directly from the rotation matrix.

Full derivation is in Schaub & Junkins (Sec 3.4)

Note: the eigenvector is normalized ⇒ length $= 1 = e_1^2 + e_2^2 + e_3^2$

$\cos\theta = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1)$, C_{ij} is the i^{th} row, j^{th} column of the rotation matrix

θ is the required rotation angle about the principal axis.

$$\vec{e} = \frac{1}{2\sin\theta}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix}$$

$\vec{e}$ = principal axis, unit length

$\pm\theta, \pm\vec{e}$, but we choose θ such that the rotation vector $= \theta\hat{e}$

A single s/c orientation can be described with 2 possible combinations of θ & $\hat{e}$:

$\theta, \hat{e}$ or $-\theta, -\hat{e}$

Quarternions: Euler Parameters

Key idea: Quarternions describe s/c attitude by describing the rotation matrix between the body-fixed & inertial frames.

$$\boxed{\begin{aligned} q_1 &= e_1\sin(\theta/2) \\ q_2 &= e_2\sin(\theta/2) \\ q_3 &= e_3\sin(\theta/2) \\ q_4 &= \cos(\theta/2) \end{aligned}}$$

Note: $\boxed{q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1}$

Benefit of quarternions is that they are non-singular: they do not have a divide by zero.

- also work well for large & small rotations.

Can also map directly between rotation matrix & quaternions

$$[C] = \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_2q_1-q_3q_4) & 1-2(q_1^2+q_3^2) & 2(q_2q_3+q_1q_4) \\ 2(q_3q_1+q_2q_4) & 2(q_3q_2-q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

$$\beta_0 = q_4 = \pm \frac{1}{2}(1+C_{11}+C_{22}+C_{33})^{1/2}$$

$$\beta_1 = q_1 = \frac{1}{4q_4}(C_{23}-C_{32})$$

$$\beta_2 = q_2 = \frac{1}{4q_4}(C_{31}-C_{13})$$

$$\beta_3 = q_3 = \frac{1}{4q_4}(C_{12}-C_{21})$$

$\pm$ term gives 2 sets of quaternions that describe a single orientation (ie a single rotation)

Note: quaternions are not singular, but this mapping has a singularity (when $q_4 = 0$)

If $\theta = 180° \Rightarrow q_4 = 0. \Rightarrow$ body-fixed coordinate system is opposite the inertial system $\Rightarrow$ S/C is upside down

If $q_4 < 0$, $\theta > 180°$ $\rightarrow$ S/C has traveled thru "upside down" to get to the current orientation

"long-way" rotation

$\boxed{\Rightarrow \text{quaternions have some path dependence}}$

If $\beta_0 = q_4 = \cos(360/2) = -1 \Rightarrow$ S/C has executed a complete rotation

$\boxed{\vec{\beta} \text{ \& } -\vec{\beta} \text{ describe the same orientation}}$

$\beta_0 = \cos(\theta/2) \quad \beta_i = e_i \sin(\theta/2)$

$\beta_0' = \cos(-\theta/2) = \cos(\theta/2) = \beta_0$

$\beta_i' = -e_i \sin(-\theta/2) = -e_i(-\sin(\theta/2)) = e_i \sin(\theta/2) = \beta_i$

For a long-way rotation: $\hat{e}$, $\theta' = \theta - 2\pi$

$$\beta_0' = \cos(\theta'/2) = \cos(\theta/2 - \pi) = -\cos(\theta/2) = -\beta_0$$

$$\beta_i' = e_i \sin(\theta'/2) = e_i \sin(\theta/2 - \pi) = -e_i \sin(\theta/2) = -\beta_i$$

If β_0 is close to 1, you are describing a small rotation.

Another benefit of quarternions is that there is a linear mapping to angular momentum:

$$\begin{bmatrix}\dot\beta_0\\ \dot\beta_1\\ \dot\beta_2\\ \dot\beta_3\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\beta_0 & -\beta_1 & -\beta_2 & -\beta_3\\ \beta_1 & \beta_0 & -\beta_3 & \beta_2\\ \beta_2 & \beta_3 & \beta_0 & -\beta_1\\ \beta_3 & -\beta_2 & \beta_1 & \beta_0\end{bmatrix}\begin{bmatrix}0\\ w_1\\ w_2\\ w_3\end{bmatrix}$$

this is the angular velocity of the s/c's rotation

The first column is multiplied by zero, but we use it because it makes the matrix orthogonal.

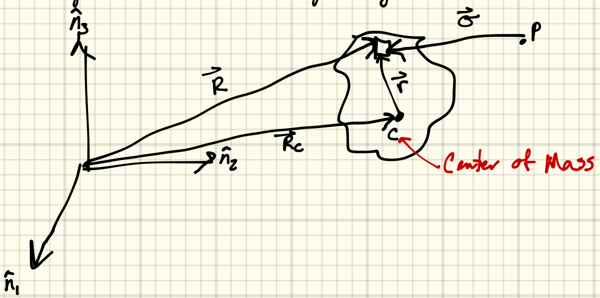
$\vec\beta$ is a unit vector $\Rightarrow$ matrix is orthonormal.

Why do we like orthonormal matrices?

B/c the inverse is the transpose. Use this to get $\vec w$ as $f(\dot{\vec\beta})$

Rigid body Dynamics:

Angular Momentum of the Rigid body:



Find the angular momentum about some arbitrary point P.

Note: $\vec\sigma = \vec R - \vec R_p$

Angular momentum: $\vec H_p = \int_B \vec\sigma \times \dot{\vec\sigma}\, dm$

Take the derivative of $\vec H_p$:

$$\dot{\vec H}_p = \int_B \cancelto{0}{\dot{\vec\sigma}\times\dot{\vec\sigma}}\, dm + \int_B \vec\sigma \times \ddot{\vec\sigma}\, dm$$

$$= \int_B \vec\sigma \times \ddot{\vec R}\, dm - \int_B \vec\sigma\, dm \times \ddot{\vec R}_p$$

Note: $\int_B \vec R\, dm = M\vec R_c$

$\int_B \vec\sigma\, dm = \int_B \vec R\, dm - \int_B dm\, \vec R_p \Rightarrow \int_B \vec\sigma\, dm = M(\vec R_c - \vec R_p)$

External torque applied to the system:

$$\vec{L}_P = \int_B \vec{\sigma} \times \ddot{\vec{R}}\, dm$$

$$\Rightarrow \boxed{\dot{\vec{H}}_P = \vec{L}_P - M(\vec{R}_c - \vec{R}_P) \times \ddot{\vec{R}}_P}$$

If $\vec{R}_c = \vec{R}_P$ -or- $\ddot{\vec{R}}_P = 0$, then $\dot{\vec{H}}_P = \vec{L}_P$.

Lecture 20: More Rigid Body Dynamics

Take the angular momentum about the origin of the coordinate system O:

$$\vec{H}_0 = \int_B \vec{R} \times \dot{\vec{R}}\, dm$$

$$\vec{R} = \vec{R}_c \times \vec{r}$$

$$\vec{H}_0 = \int_B (\vec{R}_c + \vec{r}) \times (\dot{\vec{R}}_c + \dot{\vec{r}})\, dm$$

$$= \int \vec{R}_c \times \dot{\vec{R}}_c\, dm + \int \vec{r}\, dm \times \dot{\vec{R}}_c + \vec{R}_c \times \int \dot{\vec{r}}\, dm + \int \vec{r} \times \dot{\vec{r}}\, dm$$

Remember that $\vec{r}$ points from the CM to some point on the body.

$$\Rightarrow \int \vec{r}\, dm = 0$$

Also, the body is rigid: $\int \dot{\vec{r}}\, dm = 0$

$$\boxed{\vec{H}_0 = \underbrace{M \vec{R}_c \times \dot{\vec{R}}_c}_{\text{Ang. Mom. of the CM about the origin.}} + \underbrace{\int_B \vec{r} \times \dot{\vec{r}}\, dm}_{\text{Ang. Mom. about the CM. (ie the rotation or spin of the body)}}}$$

Angular momentum with respect to the CM:

$$\vec{H}_c = \int_B \vec{r} \times \dot{\vec{r}}\, dm$$

Using the transport theorem:

$$\dot{\vec{r}} = {}^I\frac{d\vec{r}}{dt} = {}^B\frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r}$$

$\vec{\omega}$ is the rotation of the body frame wrt the inertial frame

B/c rigid body: ${}^B\frac{d\vec{r}}{dt} = 0$

$$\dot{\vec{r}} = {}^I\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\vec{H}_c = \int_B \vec{r} \times (\vec{\omega} \times \vec{r})\, dm$$

Use the tilde matrix = skew-symmetric matrix notation

$\vec{a}$ is a vector: $[\tilde{\vec{a}}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

$$\vec{a} \times \vec{b} = [\tilde{\vec{a}}]\vec{b}$$

$$\vec{H}_C = \underbrace{\left(\int_B -[\tilde{r}][\tilde{r}]\,dm\right)}\vec{\omega}$$

The integral term is the inertia matrix:

$$\boxed{\vec{H}_C = [I_C]\vec{\omega}}$$

$$[I_C] = \int_B \begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} dm$$

We would like to diagonalize $[I_C]$.

Diagonalize $[I_C]$ by rotating the body fixed coordinate system to a principal axis system.

1. Calculate the eigenvalues & eigenvectors for the inertia matrix.
2. Make sure the vectors are unit vectors & form a RH'd system.
3. The rotation matrix is:

$$[C] = \begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \\ \vec{v}_3^{\,T} \end{bmatrix} \qquad (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \text{eigenvectors}$$

4. The inertia matrix is:

$$[I] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \lambda_1, \lambda_2, \lambda_3 = \text{eigenvalues}$$

Principal inertia matrix.

Euler's Rotational EOM:

$$\dot{\vec{H}}_C = \frac{{}^I d\vec{H}_C}{dt} = \frac{{}^B d\vec{H}_C}{dt} + \vec{\omega}\times\vec{H}_C = \vec{L}_C$$

$$\vec{H}_C = [I_C]\vec{\omega}$$

$$\frac{{}^B d\vec{H}_C}{dt} = \underbrace{\cancel{\frac{{}^B d}{dt}[I]}}_{0}\vec{\omega} + [I]\frac{{}^B d\vec{\omega}}{dt} = [I]\dot{\vec{\omega}}$$

$$\frac{{}^I d\vec{\omega}}{dt} = \frac{{}^B d\vec{\omega}}{dt} + \underbrace{\cancel{\vec{\omega}\times\vec{\omega}}}_{0}$$

$$[I]\dot{\vec{\omega}} + \vec{\omega}\times[I]\vec{\omega} = \vec{L}_C$$

$$\boxed{[I]\dot{\vec{\omega}} = -[\tilde{\omega}][I]\vec{\omega} + \vec{L}_C}$$ Valid if calculated about CM -or- an arbitrary inertial point.

If $[I]$ is diagonal (principal body-fixed coordinates):

$$\boxed{\begin{aligned} I_{11}\dot{\omega}_1 &= -(I_{33}-I_{22})\,\omega_2\omega_3 + L_1 \\ I_{22}\dot{\omega}_2 &= -(I_{11}-I_{33})\,\omega_3\omega_1 + L_2 \\ I_{33}\dot{\omega}_3 &= -(I_{22}-I_{11})\,\omega_1\omega_2 + L_3 \end{aligned}}$$

← Tell us how rotation rate change given a torque.

– Even if $\vec{L}=0$, ${}^B\vec{\omega}$ is changing if the body is rotating about more than 1 axis.

In other words, $\dot{\vec{\omega}}=0$ if $\vec{L}=0$ & rotating about just one axis (eg. $\omega_2=\omega_3=0$)

★ The rotation rate (or spin) in the body fixed frame is constant only if torque is zero AND the body is rotating about only 1 of the principal axes.

Kinetic Energy:

$$T=\frac{1}{2}\int_B \dot{\vec{R}}\cdot\dot{\vec{R}}\,dm \qquad \vec{R}=\vec{R}_c+\vec{r}$$

$$T=\frac{1}{2}\int_B dm\,\dot{\vec{R}}_c\cdot\dot{\vec{R}}_c + \dot{\vec{R}}_c\cdot\cancelto{0}{\int_B \dot{\vec{r}}\,dm} + \frac{1}{2}\int_B \dot{\vec{r}}\cdot\dot{\vec{r}}\,dm$$

$$T=\underbrace{\frac{1}{2}M\dot{\vec{R}}_c\cdot\dot{\vec{R}}_c}_{\text{Translational}} + \underbrace{\frac{1}{2}\int_B \dot{\vec{r}}\cdot\dot{\vec{r}}\,dm}_{\text{Rotational}}$$

$$T_{rot}=\frac{1}{2}\int_B \dot{\vec{r}}\cdot\dot{\vec{r}}\,dm \qquad {}^I\frac{d\vec{r}}{dt}=\cancelto{0,\ \text{Rigid body}}{{}^B\frac{d\vec{r}}{dt}}+\vec{\omega}\times\vec{r}$$

$$=\frac{1}{2}\int_B(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})\,dm$$

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})$$

$$T_{rot}=\frac{1}{2}\vec{\omega}\cdot\underbrace{\int_B \vec{r}\times(\vec{\omega}\times\vec{r})\,dm}_{=\vec{H}_c}$$

$$=\frac{1}{2}\vec{\omega}\cdot\vec{H}_c=\frac{1}{2}\vec{\omega}^T[I]\vec{\omega}$$

$$\boxed{T=\frac{1}{2}M\dot{\vec{R}}_c\cdot\dot{\vec{R}}_c+\frac{1}{2}\vec{\omega}^T[I]\vec{\omega}}$$

Torque-free Motion:

$$\vec{L}_c=0 \Rightarrow \dot{\vec{H}}_c=0={}^I\frac{d\vec{H}_c}{dt}.$$

★ However, in the body-fixed frame, $\vec{H}_c$ may be rotating.
The magnitude of $\vec{H}_c$ is constant in all frames.

Assume principal axes for the body-fixed frame $\Rightarrow$ $[I]$ is diagonal

$${}^B\vec{H}_c = I_{11}\omega_1\hat{b}_1 + I_{22}\omega_2\hat{b}_2 + I_{33}\omega_3\hat{b}_3$$

B/c $|\vec{H}_c| = \text{constant}$

$$H^2 = I_{11}^2\omega_1^2 + I_{22}^2\omega_2^2 + I_{33}^2\omega_3^2$$

↳ Eqn for the surface of an ellipsoid.

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

kinetic energy is also constant:

$$T_{rot} = T = \tfrac{1}{2}I_{11}\omega_1^2 + \tfrac{1}{2}I_{22}\omega_2^2 + \tfrac{1}{2}I_{33}\omega_3^2$$

${}^B\vec{\omega}(t)$ must satisfy both $H = \text{constant}$ & $T = \text{constant}$

We will graphically investigate how the body-fixed angular velocity (due to the rotation of the spacecraft) will change ($\vec{L}_c = 0$)

Lecture 21: Polhode Plots

From last lecture, we knew that even though $T_c = 0$, the $\dot{\vec{w}} \neq 0$ (if $\vec{w}$ has nonzero components in more than 1 direction).

Ex. if $\vec{w} = [w_1, 0, 0] \Rightarrow \dot{\vec{w}} = 0$

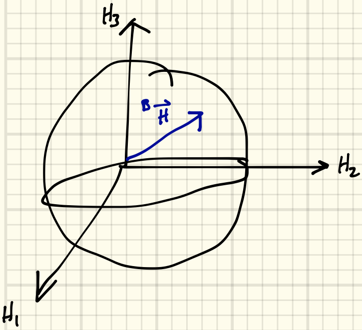
if $\vec{w} = [w_1, w_2, 0] \Rightarrow \dot{\vec{w}} \neq 0$

But, $\vec{w}$ can't be any random value, it must satisfy constraints on conservation of angular momentum magnitude & kinetic energy.

Our goal is to graphically investigate how $\vec{w}$ is changing.

Known: $H^2 = H_1^2 + H_2^2 + H_3^2$

$\Rightarrow 1 = \frac{H_1^2}{H^2} + \frac{H_2^2}{H^2} + \frac{H_3^2}{H^2}$ ← eqn for a sphere



B/c H is conserved, ${}^B\vec{H}$ must point to a point on the surface of this sphere.

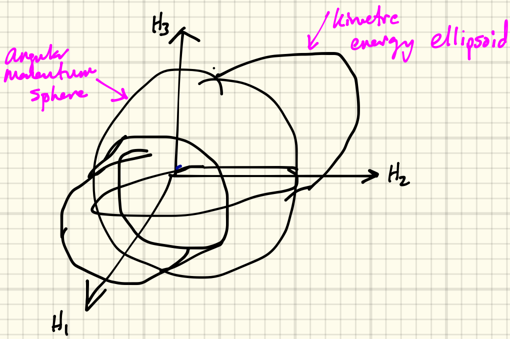
$\vec{H}$ contains information about $[I]$ & $\vec{w}$.

We know $[I]$ is constant in the body-fixed frame.

We can also rewrite the kinetic energy expression in terms of H:

$$1 = \frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T}$$ → Eqn for ellipsoid: $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

$\Rightarrow {}^B\vec{H}$ must also point to a point on the surface of the kinetic energy ellipsoid.



${}^B\vec{H}$ must point to an intersection between the KE ellipsoid & H sphere.

Polhode Plots

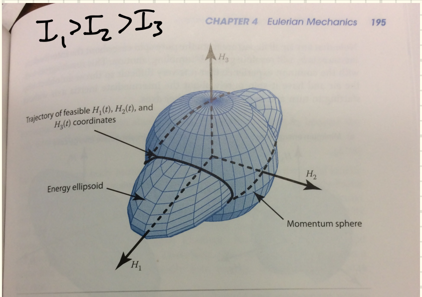


Fig. 4.10 General intersection of the momentum sphere and the energy ellipsoid.

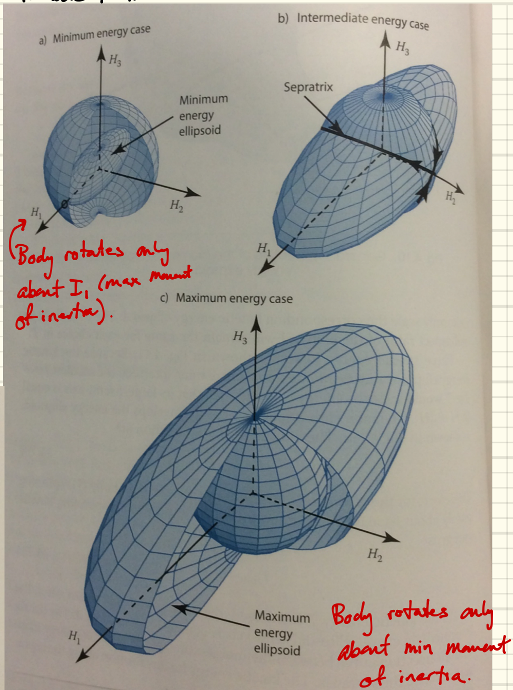


Fig. 4.11 Special cases of kinetic energy ellipsoid and momentum sphere intersections.

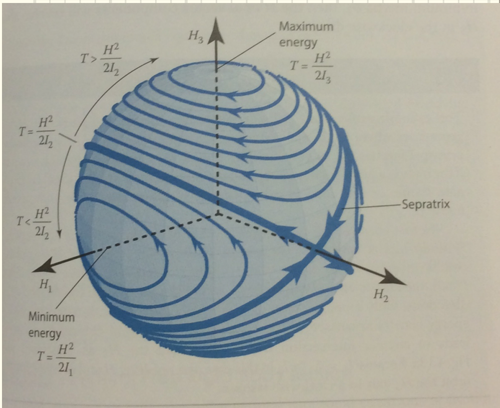


Fig. 4.12 Family of energy ellipsoid and momentum sphere intersections.

Review: ${}^I\vec{H}$ = constant

${}^B H$ = constant (sphere), but ${}^B\vec{H} \neq$ constant

${}^B[I]$ = constant b/c rigid body

${}^B\dot{\vec{w}} \neq 0$ even though $\vec{L} = 0$ if $\vec{w}$ has nonzero components in more than 1 direction.

In the body-fixed frame, w is transferred between components

Since ${}^I\vec{H}$ = constant & ${}^I\dot{\vec{w}} \neq$ constant $\Rightarrow$ ${}^I[I] \neq$ constant

$\Rightarrow$ the body is rotating

Polhode plots assume that the body-fixed frame is a principal axes

${}^B\vec{H}$ must point to an intersection between the angular momentum sphere and KE ellipsoid.

Lecture 22: Aspherical Earth

Asphercal Earth:

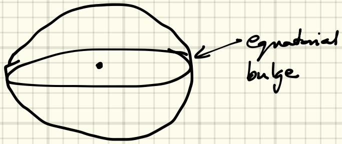
So far, we have assumed that the gravity fields of the bodies that we orbit are spherically symmetric, so we can approximate them as point masses.

In actuality, Earth & other planetary bodies are not perfect spheres.

This asphericity influences the orbits of s/c $\Rightarrow$ orbits don't follow predictions from 2BP.

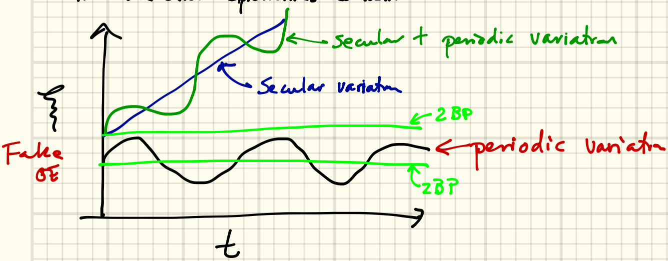
Any added perturbation (drag, aspherical gravity, 3rd body gravity) produces OE's that are not constant.

Earth is an oblate spheroid



Opposite shape: prolate spheroid

There are other asphericities as well.



In order to propagate the trajectory of the s/c about an aspherical Earth, we must modify the EOM.

2BP: $\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$

First ϕ = latitude of the s/c: $\phi = \text{atan}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$

A force is the gradient of a potential field.

2BP: $U = \frac{\mu}{r}$

Spherical Harmonics Expression for Aspherical body:

$$U(r,\phi) = \frac{\mu}{r}\sum_{k=2}^{\infty} J_k\left(\frac{R}{r}\right)^k P_k(\cos\phi)$$

↑ Potential due to the perturbing gravity force (not the 2BP mass)

R = radius of the Earth (avg)

P_k = Legendre Polynomials

r = s/c position

J_k = Zonal harmonics coefficient

We will consider just the "ring" of mass about Earth's equator. This is represented by the J2 coefficient. J2 is the largest spherical harmonics coefficient for Earth.

$$U(r,\phi) = \frac{\mu}{r}\left[1 - J_2\left(\frac{R}{r}\right)^2\left(\frac{3}{2}\sin^2\phi - \frac{1}{2}\right)\right]$$

$$J_2 = 0.00108248 \quad \text{(Earth)}$$

$$\ddot{\vec{r}} = \vec{\nabla} U$$

When J2 is included, Ω & ω experience secular + periodic perturbations. The other OE experience periodic perturbations.

The secular variations are calculated by calculating $d\Omega/dt$ & $d\omega/dt$ and then averaging over $\nu \in 0, 360°$

$$\dot{\Omega}_{sec} = -\left[\frac{3}{2}\frac{J_2\sqrt{\mu}R^2}{a^{7/2}(1-e^2)^2}\right]\cos i$$

$$\dot{\omega}_{sec} = -\left[\frac{3}{2}\frac{J_2\sqrt{\mu}R^2}{a^{7/2}(1-e^2)^2}\right]\left(\frac{5}{2}\sin^2 i - 2\right)$$

$\cos(i) = 0$ when $i = 90° \Rightarrow \dot{\Omega}_{sec} = 0$ if $i = 90°$ ← Could still have periodic variation

Note that $\dot{\Omega} < 0$ for $i < 90° \Rightarrow$ the ascending node is moving west

"nodal regression"

$\dot{\omega} \propto \frac{5}{2}\sin^2 i - 2 \Rightarrow \dot{\omega} = 0$ if $i = 63.43°$ or $116.6°$

$\Rightarrow$ we cannot have both $\dot{\Omega}$ & $\dot{\omega} = 0$ at the same time.

Attitude Dynamics : Torque-free motion

Describe the s/c attitude using 3-2-1 Euler Angles:

Yaw: ψ

Pitch: θ

Roll: ϕ

Body-fixed coordinates that are aligned with the principal axes of the body $\Rightarrow [I]$ is diagonal

Torque free: ${}^N\vec{H}$ = constant

Specify the inertial frame s.t.: ${}^N\vec{H} = -H\hat{n}_3$

$${}^B\vec{H} = [BN]\,{}^N\vec{H}$$

↑ Direction Cosine Matrix for 3-2-1 rotation

$${}^B H_1 = H\sin\theta = I_1\omega_1$$
$${}^B H_2 = -H\sin\phi\cos\theta = I_2\omega_2$$
$${}^B H_3 = -H\cos\phi\cos\theta = I_3\omega_3$$

Note: $\vec{\omega} = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3$

$\vec{\omega} = -\dot{\psi}\,\hat{n}_3 + \dot{\theta}\,\hat{b}_2' + \dot{\phi}\,\hat{b}_1$

Rewrite $\vec{\omega}$ in terms of θ, ψ, ϕ and then substitute into the H_1, H_2, H_3 expressions:

Precession Rate $\dot{\psi} = -H\left(\frac{\sin^2\phi}{I_2} + \frac{\cos^2\phi}{I_3}\right)$

Nutation rate $\dot{\theta} = \frac{H}{2}\left(\frac{1}{I_3} - \frac{1}{I_2}\right)\sin(2\phi)\cos\theta$

$\dot{\phi} = H\left(\frac{1}{I_1} - \frac{\sin^2\phi}{I_2} - \frac{\cos^2\phi}{I_3}\right)\sin\theta$

If the body is axisymmetric $\Rightarrow$ Cylinder $(I_2 = I_3)$

$$\dot{\psi} = \frac{-H}{I_2}$$

$$\dot{\theta} = 0$$

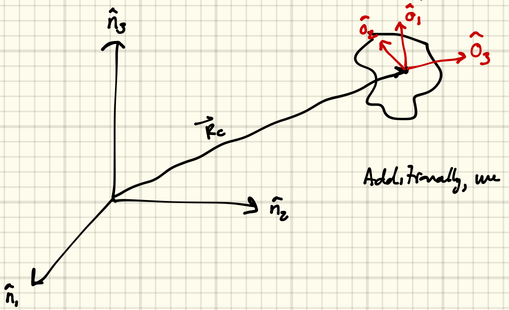
$$\dot{\phi} = H\left(\frac{I_2 - I_1}{I_1 I_2}\right)\sin\theta$$

Lecture 23: Gravity Gradient Torque

Gravity-Gradient Torque:

The part of the S/C that is closer to the center of the Earth will feel a stronger gravitational acceleration than the part that is farther from Earth. This produces a torque on the S/C.

Note: Earth is approximated as a point mass.



$\{o_1, o_2, o_3\}$ = orbit frame

$\hat{o}_3 \parallel \hat{R}_c$

$\hat{o}_2 \parallel \vec{h}$

$\hat{o}_1$ completes the RH'd system

Additionally, we can also have a body-fixed frame.

Angular velocity of the O frame wrt N frame is:

$$\vec{\omega}_{ON} = n\,\hat{o}_2 \qquad n = \text{mean motion} = \sqrt{\frac{\mu}{a^3}}$$

For a body-fixed frame:

$$\vec{\omega}_{BN} = \vec{\omega}_{BO} + \vec{\omega}_{ON}$$

Get expression for gravity gradient-torque:

Gravity force acting on some mass element dm:

$$d\vec{f} = -\frac{\mu}{R^3}\vec{R}\,dm = -\frac{\mu(\vec{R}_c+\vec{r})\,dm}{|\vec{R}_c+\vec{r}|^3}$$

(reminder: $\vec{R} = \vec{R}_c + \vec{r}$)

Gravity-gradient torque about CM:

$$\vec{L} = \int_B \vec{r}\times d\vec{f} = -\mu\int \frac{\vec{r}\times\vec{R}_c}{|\vec{R}_c+\vec{r}|^3}\,dm$$

Need an expression for denominator:

$$|\vec{R}_c+\vec{r}|^{-3} = R_c^{-3}\left\{1+\frac{2(\vec{R}_c\cdot\vec{r})}{R_c^2}+\frac{r^2}{R_c^2}\right\}^{-3/2}$$

Binomial Expansion:

$$= R_c^{-3}\left\{1-\frac{3(\vec{R}_c\cdot\vec{r})}{R_c^2}+H.O.T.\right\}$$

Drop H.O.T.:

$$\vec{L} = \frac{\mu}{R_c^3}\vec{R}_c \times \int_B \vec{r}\left(1 - \frac{3\vec{R}_c\cdot\vec{r}}{R_c^2}\right)dm$$

$\int \vec{r}\,dm = 0$ b/c definition of CM

$$\vec{L} = \frac{3\mu}{R_c^5}\vec{R}_c \times \int -\vec{r}\,(\vec{r}\cdot\vec{R}_c)\,dm$$

Use: $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$\vec{L} = \frac{3\mu}{R_c^5}\vec{R}_c \times \int -\left(\vec{r}\times(\vec{r}\times\vec{R}_c) + (\vec{r}\cdot\vec{r})\vec{R}_c\right)dm$$

$$= \frac{3\mu}{R_c^5}\vec{R}_c \times \underbrace{\left(\int_B -[\tilde{r}][\tilde{r}]\,dm\right)}_{[I_c]}\vec{R}_c - \frac{3\mu}{R_c^5}\left(\int r^2 dm\right)\cancelto{0}{\vec{R}_c \times \vec{R}_c}$$

$$\Rightarrow \boxed{\vec{L} = \frac{3\mu}{R_c^5}\vec{R}_c \times [I_c]\vec{R}_c}$$

Gravity Gradient Torque

- Assumes 2BP gravity (point mass)
- Approximation b/c dropped the H.O.T. in the binomial expansion

Given our definition of the axes:

$${}^{O}\vec{R}_c = R_c\,\hat{o}_3$$

$${}^{O}\vec{L} = \frac{3\mu}{R_c^3}\left(-I_{23}\,\hat{o}_1 + I_{13}\,\hat{o}_2\right)$$

→ Gravity gradient never produce torque in the $\hat{o}_3$ direction

Zero torque if the orbit frame is aligned with the principal axes of the body, b/c the inertia matrix would be diagonal in this case.

Can also write the torque in the body-fixed frame, assuming principal axes:

$${}^{B}\vec{R}_c = R_{c1}\hat{b}_1 + R_{c2}\hat{b}_2 + R_{c3}\hat{b}_3$$

$${}^{B}\vec{L} = \frac{3\mu}{R_c^5}\begin{bmatrix} R_{c2}R_{c3}(I_{33}-I_{22}) \\ R_{c1}R_{c3}(I_{11}-I_{33}) \\ R_{c1}R_{c2}(I_{22}-I_{11}) \end{bmatrix}$$

- No torque if $I_{11}=I_{22}=I_{33} \Rightarrow$ sphere
- If i^{th} axis is an axis of symmetry, there is no torque about $\hat{b}_i$.
- If $\vec{R}_c$ is parallel with one of the principal axes, then ${}^{B}\vec{L} = 0$.

Attitude of S/C in response to torque:

Describe the attitude using 3-2-1 rotation matrix

ψ: yaw

θ = pitch

ϕ = roll

$c\theta = \cos\theta$

$s\psi = \sin\psi$

$$^{B}\vec{\omega}_{BN} = \begin{bmatrix} \dot{\phi} - s\theta\dot{\psi} + nc\theta s\psi \\ s\phi c\theta\dot{\psi} + c\phi\dot{\theta} + n(s\phi s\theta s\psi + c\phi c\psi) \\ c\phi c\theta\dot{\psi} - s\phi\dot{\theta} + n(c\phi s\theta s\psi - s\phi c\psi) \end{bmatrix}$$

Note: n = mean motion

Small angle assumption & linearize.

$$^{B}\vec{\omega}_{BN} = \begin{bmatrix} \dot{\phi} + n\psi \\ \dot{\theta} + n \\ \dot{\psi} - n\phi \end{bmatrix} \Rightarrow \dot{\vec{\omega}}_{BN} \simeq \begin{bmatrix} \ddot{\phi} + n\dot{\psi} \\ \ddot{\theta} \\ \ddot{\psi} - n\dot{\phi} \end{bmatrix}$$

Now write torque using ψ, ϕ, θ

$$^{B}\vec{R}_c = \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix} R_c$$

Assume circular orbit: $R_c = a$

$$^{B}\vec{L} = \frac{3}{2}n^2 \begin{bmatrix} (I_{33} - I_{22})\cos^2\theta\sin 2\phi \\ -(I_{11} - I_{33})\cos\phi\sin 2\theta \\ -(I_{22} - I_{11})\sin\phi\sin 2\theta \end{bmatrix}$$

linearize:

$$^{B}\vec{L} \simeq 3n^2 \begin{bmatrix} (I_{33} - I_{22})\phi \\ -(I_{11} - I_{33})\theta \\ 0 \end{bmatrix}$$

← does not depend on ψ (yaw)

For pitch & roll to be stabilizing, require: $I_{22} > I_{33}$ & $I_{11} > I_{33}$

$\Rightarrow$ make L_1 & $L_2 < 0$

For a cylindrical S/C, the long axis should be aligned w/ $\vec{h}$ in order to be stabilized by gravity gradient torque.