Recap (DE Review)

Any constant coef linear diff'l eq'n has sol'n:

$$y(t) = y_h(t) + y_f(t)$$

where

$$y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t}$$] homogeneous response

$$r(p_k) = 0, \quad k = 1, \ldots, n$$] p_k roots of char. poly $r(s)$

and $y_f(t)$ depends on specific forcing function (input)
("forced response")

For the specific case that $u(t) = Ue^{st}$, $U, s \in \mathbb{C}$

then $$y_f(t) = G(s)u(t),$$ $G(s) = \frac{q(s)}{r(s)}$ "transfer function"

$$= Ye^{st}$$

with $$Y = G(s)U$$ (ordinary complex number multiplication!)

Complex math yields real sol'ns

Note that both $y_h(t)$ and $y_f(t)$ are complex-valued functions as we have written them

But physical systems will have only real-valued inputs and outputs.

For $y_f(t)$, note that we can express a real input as the real or imag part of a complex input:

$$u(t) = Ae^{\sigma t}\sin(\omega t+\varphi) = Im\{\bar{U}e^{st}\}$$

with $\bar{U} = Ae^{j\varphi}$ and $s = \sigma + j\omega$

The corresponding real $y_f(t) = Im\{G(s)\bar{U}e^{st}\}$

and similarly if input is COSINOIDAl we use the real part of the complex number calculation

Complex => real, cont

for $y_h(t)$: Sol'n contains terms e^{pt}, $r(p)=0$

This will be complex if root p is complex, i.e.

$p=\sigma+j\omega$, $\omega\neq 0$.

However, if this is true then $\bar{p}=\sigma-j\omega$ will <u>also</u> be a root of $r(s)$, i.e. $r(\bar{p})=r(p)=0$

Complex roots of polynomials always occur in "conjugate pairs"

So sol'n for $y_h(t)$ will really look like

$$y_h(t)=C_1e^{pt}+C_2e^{\bar{p}t}+(\cdots\text{other terms})$$

Real-valued homogeneous response

So

$$y_h(t) = \underbrace{C_1 e^{pt} + C_2 e^{\bar{p}t}} + (\cdots \text{ other terms})$$

Fact: It is always the case that $C_1 = \overline{C_2}$,

i.e. the coefs. of conjugate terms are themselves conjugates

(This is b/c boundary cond'ns on DE are also real)

$\Rightarrow$ so, $$y_h(t) = ce^{pt} + \bar{c}e^{\bar{p}t} + (\cdots)$$

$$= \underbrace{ce^{pt} + \overline{ce^{pt}}}_{\text{sum of conjugate terms}} + (\cdots)$$

$$= \underbrace{2\,\mathrm{Re}\{ce^{pt}\}} + (\cdots)$$

$\Rightarrow$ The two complex terms from conjugate roots $p, \bar{p}$ combine to form a real function of time!

Conclusion

$\Rightarrow$ When $p = \sigma + j\omega$, $\omega \neq 0$ is a root of char poly $r(s)$, the homog. sol'n will contain the <u>real</u>-valued function

$$2\,\mathrm{Re}\{c e^{pt}\} = A e^{\sigma t}\cos(\omega t + \varphi)$$

where $A = 2|c|$, $\varphi = \angle c$ are determined by ICs.

$\Rightarrow$ $\left\{ \begin{array}{l} \text{Complex roots of } r(s) \text{ correspond to } \underline{\text{real}} \\ \underline{\text{oscillations}} \text{ in homog. response.} \end{array} \right\}$

(Note, there may be several pairs of conjugate roots in $r(s)$, resulting in <u>multiple</u> oscillations w/ different frequencies + damping)

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$$= Y e^{st}$$

with $Y = G(s)U$ (ordinary complex number multiplication!)

Boundary/initial conditions

Undetermined coefs c_K in $y_h(t)$ determined by boundary conditions $y(0), \dot{y}(0), \ldots, y^{(n-1)}(0)$

For simple problems, can often solve for c_K by substituting general form $y(t) = y_h(t) + y_f(t)$, differentiating, and matching stated B/Cs.

$\Rightarrow$ Results in a system of n equations in the n coefs c_K which can be solved (lin. algebra)

Warning: There are situations where this approach to compute c_K will not work.

Will cover this situation shortly.

Particularly if $u(t)$ is discontinuous at $t=0$ and one or more derivs of $u(t)$ appear in DE

Example

$$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = 2\dot{u}(t) + u(t)$$

where $y(0) = \dot{y}(0) = 0$, $\quad u(t) = 3\cos(2t - \pi/2)$

By inspection:

$$y(t) = \overbrace{c_1 e^{-t} + c_2 e^{-4t}}^{y_h(t)} + \overbrace{A\cos(2t + \varphi)}^{y_f(t)}$$

Only remaing problem is to calculate c_1, c_2, A, φ

Note: A, φ in $y_f(t)$ determined by $u(t)$, and are independent of c_1, c_2

With a little more calculation:

$$y(t) = \frac{1}{10}\left[7e^{-4t} - 4e^{-t}\right] + \underbrace{\left(\frac{3\sqrt{17}}{10}\right)}_{A}\cos\left(2t \underbrace{- \pi/2 - \tan^{-1} 1/4}_{\varphi}\right)$$

Forced response

(zero)

Here $u(t) = 3\cos(2t - \pi/2) = 3e^{0t}\cos(2t - \pi/2)$

$= \mathrm{Re}\{U e^{st}\}$ with $U = 3e^{-\pi/2 j}$

$s = 0 + 2j$

and $y_f(t) = \mathrm{Re}\{G(s) U e^{st}\}$

$= \mathrm{Re}\{Y e^{st}\}$, $Y = G(s)U$

$= |Y| e^{0t} \cos(2t + \angle Y)$ ($s = 0 + 2j$ from input)

All we need to do is the complex number multiplication $Y = G(s)U$ and convert to polar form $|Y|, \angle Y$

$\Rightarrow$ We have U, still need transfer f'n $G(s)$

Diff'l eq'n is

$$\dddot{y} + 5\dot{y} + 4y = 2\dot{u} + u$$

$$G(s) = \frac{q(s)}{r(s)}$$

$q(s) =$

$r(s) =$

so finally $G(s) =$

Note that $r(s) = (s+1)(s+4)$

which also gives us the general form for $y_h(t)$

$$c_1 e^{-t} + c_2 e^{-4t}$$

Evaluate $G(s)$ at complex freq of input, $s = 2j$ here

$$G(2j) = \left. \frac{2s+1}{s^2+5s+4} \right|_{s=2j}$$

$$= \frac{1+4j}{10j}$$

$$= \frac{1}{10}(1-4j)$$

How...?

Here $u(t) = 3\cos(2t - \pi/2) = \mathrm{Re}\{U e^{st}\}$

with $s = 2j$ and $U = 3e^{-\pi/2 j}$

So $y_f(t) = \mathrm{Re}\{G(2j)(3e^{-\pi/2 j})(e^{2jt})\}$

with here: $G(s) = \dfrac{2s+1}{s^2+5s+4}$

$Y = G(2j)U$

$\Rightarrow G(2j) = \dfrac{1+4j}{(2j)^2+10j+4} = \frac{1}{10}(4-j) = \left(\frac{\sqrt{17}}{10} \measuredangle -\tan^{-1}\left(\frac{1}{4}\right)\right)$

Hence:

$$y_f(t) = \overbrace{\frac{3\sqrt{17}}{10}}^{|Y|} \cos(2t \overbrace{- \pi/2 - \tan^{-1}(1/4)}^{\measuredangle Y})$$

So we know $y_f(t)$ exactly at this point

Homogeneous Sol'n

We have $r(s) = s^2 + 5s + 4$ (denom poly of $G(s)$)

Or: $r(s) = (s+1)(s+4)$

So $p_1 = -1$, $p_2 = -4$ and $y_h(t) = c_1 e^{-t} + c_2 e^{-4t}$

Then $y(t) = y_f(t) + y_h(t)$

$$= \frac{3\sqrt{17}}{10} \cos\left(2t - \pi/2 - \tan^{-1}(1/4)\right) + c_1 e^{-t} + c_2 e^{-4t}$$

So $y(0) = c_1 + c_2 - \frac{3}{10} = 0$ (specified)

and $\dot{y}(0) = -c_1 - 4c_2 + 12/5 = 0$ (specified)

Impose Boundary Cond'ns

Equivalently

$$\begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/10 \\ -12/5 \end{bmatrix}$$

$\Rightarrow$ (linear algebra):

$$c_1 = -4/10, \quad c_2 = 7/10$$

So that:

$$y(t) = \frac{1}{10}\left[7e^{-4t} - 4e^{-t}\right] + \frac{3\sqrt{17}}{10}\cos\left(2t - \frac{\pi}{2} - \tan^{-1}\frac{1}{4}\right)$$

as claimed

final result

Recap

General sol'n of LTI DE is:

$$y(t) = y_h(t) + y_f(t)$$

Forced response $y_f(t)$ depends on $u(t)$

Homogeneous response is <u>independent</u> of $u(t)$:

$$\left. y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t} \quad \text{where } r(p_k) = \emptyset \right\} \text{for } \underline{\text{any}}\ u(t)$$

Specific coefs c_k depend on initial conditions and $u(t)$.

Repeated roots of $r(s)$

Above formula for $r(s)$ assumes the roots p_k are <u>non-repeated</u>

Suppose instead that there are repeated roots, for example:

$$r(s) = (s-p_1)^{\ell}(s-p_{\ell+1})\cdots(s-p_n)$$

i.e. p_1 is repeated ℓ times. Then:

$$y_h(t) = (c_1 + c_2 t + c_3 t^2 + \cdots + c_\ell t^{\ell-1})e^{p_1 t} + \sum_{k=\ell+1}^{n} c_k e^{p_k t}$$

(will prove later)

(Natural) Modes

$r(p_k)=0$

$y_h(t)$ is a linear combination of $e^{p_k t}$ (or $t^i e^{p_k t}$). These describe solutions which are possible without any input

They are "Natural" motions which are intrinsic to the dynamics of the system.

We call them the "MODES".

$\boxed{\text{MODES:}}$ Terms in sol'n for $y(t)$ of form e^{pt}, where $r(p) = \emptyset$

Two cases (non-repeated, to start)

① p real : e^{pt} is a real exponential function

"1st order mode"

(2) <u>p complex</u>: e^{pt} and $e^{\bar{p}t}$ both present in solution, and will combine to form the "2nd order mode"

$$Ae^{\sigma t}\cos(\omega t+\varphi)$$

where $\sigma = \mathrm{Re}\{p\}$, $\omega = \mathrm{Im}\{p\}$

and A, φ depend on the initial conditions