Kepler's Prod

Kepler's Prediction Problem:

3 major concepts related to TOF:

1. How long does it take to get between 2 locations on an orbit?

   $\Rightarrow$ TOF eqn

   Known: OE, $\vec{r}_0$, $\vec{r}_1$ Unknown: TOF

2. Given an orbit & an initial state, where will the s/c be at some later time?

   $\Rightarrow$ Kepler's Pred Prob

   Known: OE, $\vec{r}_0$, TOF Unknown: $\vec{r}_1$

3. Given 2 positions & a TOF, what is the orbit?

   $\Rightarrow$ Lambert's Prob (also relevant to maneuver design)

   Known: $\vec{r}_0$, $\vec{r}_1$, TOF Unknown: OE's

Kepler's Prediction Problem:

Know: OE, $\vec{r}_0$, TOF $\rightarrow$ Find $\vec{r}_1$

Unfortunately, there is no analytical solution to this problem. Must iterate.

Given $\nu_0 \rightarrow E_0$ (Eccentric Anomaly)

M = mean anomaly (rad) = angular location of the s/c if it's angular velocity were constant

$= n(t-T)$

n = mean motion $= \sqrt{\frac{\mu}{a^3}}$

$t-T$ = time since last periapsis passage.

$M = E - e\sin E$

Know: OE's $\Rightarrow n$

TOF is known

Thus: ΔM is known: $M_1 = M_0 + n(TOF)$

If M_1 is known, solve for E_1 (E_1 = location of the s/c $\vec{r}_1$)

<u>But</u>: $M = E - e\sin E \leftarrow$ Cannot be solved analytically for E. Must solve iteratively

Will not go thru the full derivation.

Algorithm for solving Kepler's Prediction Prob:

1. Guess χ_n (a function of s/c position)

2. Calculate $t_n = f(\chi_n)$

3. If $(t_1 - t_n) >$ tolerance

   Calculate χ_{n+1}

4. Repeat steps 2 & 3 until $(t_1 - t_n) \leq$ tolerance

5. Calculate $\vec{r}_1$ & $\vec{v}_1$ using Lagrange Coefficients (which are a $f(\chi_n)$)

Define $\dot{\chi} = \sqrt{\mu}/r$

Pseudo-Code:

Input: $\vec{r}_0, \vec{v}_0, \Delta t$

$\mathcal{E} = \frac{v_0^2}{2} - \frac{\mu}{r_0}$ → tells us the type of Conic

Solve for a: $\mathcal{E} = -\frac{\mu}{2a}$

If circle or Ellipse:

$$\chi_n = \frac{\sqrt{\mu}}{a} \Delta t$$

If parabola:

$$\vec{h} = \vec{r}_0 \times \vec{v}_0$$

$$p = h^2/\mu$$

$$\cot(2s) = 3\sqrt{\frac{\mu}{p^3}}\,\Delta t$$

s & w here are just intermediate variables (not OE's)

$$\tan^3(w) = \tan(s)$$

$$\chi_n = \sqrt{p}\, 2\cot(2w)$$

If hyperbola:

$$\chi_n = \text{sign}(\Delta t)\sqrt{-a}\,\ln\left[\frac{-2\mu\Delta t/a}{(\vec{r}_0 \cdot \vec{v}_0) + \text{sign}(\Delta t)\sqrt{-\mu a}\,(1 - r_0/a)}\right]$$

$$\psi = \chi^2/a$$

Calculate C & S:

if $\psi > 1E\text{-}6$

$$C = \frac{1 - \cos(\sqrt{\psi})}{\psi}$$

$$S = \frac{\sqrt{\psi} - \sin(\sqrt{\psi})}{\sqrt{\psi^3}}$$

elseif $\psi < -1E\text{-}6$

$$C = \frac{1 - \cosh(\sqrt{-\psi})}{\psi}$$

$$S = \frac{\sinh(\sqrt{-\psi}) - \sqrt{-\psi}}{\sqrt{(-\psi)^3}}$$

else $C = 1/2$

$S = 1/6$

while $|t - \Delta t| > 10$ sec (You can play with this tolerance)

(Need to update χ)

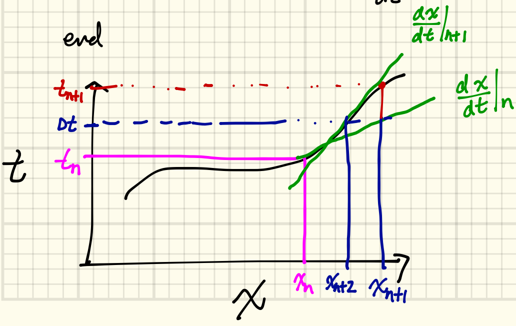
$$r = \chi^2 C + \frac{\vec{r}_0 \cdot \vec{v}_0}{\sqrt{\mu}} \chi (1 - \psi S) + r_0 (1 - \psi C)$$

$$\frac{d\chi}{dt} = \sqrt{\mu}/r$$

$$t = \frac{1}{\sqrt{\mu}} \left[\chi^3 S + \frac{\vec{r}_0 \cdot \vec{v}_0}{\sqrt{\mu}} \chi^2 C + r_0 \chi (1 - \psi S) \right]$$

$$\chi_{n+1} = \chi_n + (\Delta t - t) \frac{d\chi}{dt} \quad \text{(updates } \chi\text{)}$$

end



$$\chi_{n+1} = \chi_n + \left.\frac{d\chi}{dt}\right|_n (\Delta t - t_n)$$

Newton-Raphson Method to update χ_n.

Lagrange Coefficients:

$$f = 1 - \frac{\chi^2}{r_0} C$$

$$g = t - \frac{\chi^3}{\sqrt{\mu}} S$$

$$\dot{f} = \frac{\sqrt{\mu}}{r_0 r} \chi (\psi S - 1)$$

$$\dot{g} = 1 - \frac{\chi^2}{r} C$$

Output:

$$\vec{r} = f \vec{r}_0 + g \vec{v}_0$$

$$\vec{v} = \dot{f} \vec{r}_0 + \dot{g} \vec{v}_0$$