Lecture 3: Dimensional Analysis

ENAE311H Aerodynamics I

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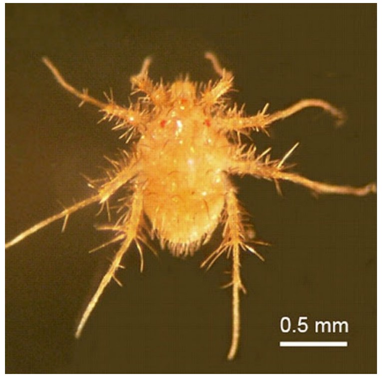
Introduction

- In studying any fluid-dynamical system, we are typically interested in how certain measurable properties of the system respond to changes in other parameters.
    - For an airfoil, for example, we may wish to know how the lift force changes as we vary the flow velocity.
- The dimensional properties and parameters we most commonly deal with consist of a number and a standard measure, e.g., L = 10 N.
- There are certain quantities that are more fundamental than these common dimensional quantities, however, in that:
    - the system can be fully described with fewer such quantities
    - differing (but in some ways similar) systems can be more meaningfully compared
- Most fundamental are dimensionless quantities that consist of numbers alone (e.g., force coefficients).
- It is the job of dimensional analysis to identify such dimensionless parameters.

Cheetah: 70 miles/hour, 16 body lengths per second

Australian tiger beetle: 1.86 m/s, 171 body lengths per second



Paratarsotomus macropalpis: 0.225 m/s, 322 body lengths per second

Buckingham π theorem

The essence of dimensional analysis is contained in the Buckingham π theorem.

Buckingham π theorem: let K be the number of basic dimensions required to describe the relevant physical variables in a system (in mechanics problems, $K = 3$, i.e., mass, length, time). Now, suppose we have a physical relationship between N dimensional variables, p_i:

$$f(p_1, p_2, ..., p_N) = 0$$

It is possible to restate this relationship as

$$F(\pi_1, \pi_2, ..., \pi_{N-K}) = 0$$

where the π_j are dimensionless parameters of the form

$$\pi_j = p_1^{a_1} p_2^{a_2} ... p_N^{a_N} \qquad (a_j \text{ rational})$$

The choice of the π_j is not unique, but should in general follow these guidelines:

- They should be of the form

$$\begin{aligned}
\pi_1 &= g_1(p_1, ..., p_K, p_{K+1}) \\
\pi_2 &= g_2(p_1, ..., p_K, p_{K+2}) \\
&\cdots\cdots \\
\pi_{N-K} &= g_{N-K}(p_1, ..., p_K, p_{N-K})
\end{aligned}$$

  where the repeating set, $p_1, \dots, p_K$, contain all K dimensions.

- If there is a dependent dimensional variable, it shouldn't appear in the repeating set.
- The variables chosen for the repeating set should be *linearly independent* in their dimensions (e.g., if one variable has dimensions l/t, you shouldn't choose another variable with dimensions l^2/t^2).

Drag force on a sphere

As an example of the Buckingham π theorem, let us consider the drag force, D, on a sphere. From experience, we know that this should, at the very least, depend on the following variables:

1. The freestream flow density, ρ_∞
2. The freestream flow velocity, V_∞
3. The freestream coefficient of viscosity, μ_∞
4. The sphere size, which we can characterize through the radius, r
5. The freestream speed of sound, a_∞

Our dimensional relationship linking these variables is thus

$$f(D, \rho_\infty, V_\infty, \mu_\infty, r, a_\infty) = 0$$

And thus, $N = 6$.

Next, we write down the dimensions of these variables (square brackets denote dimensionality):

$$\begin{aligned}
[D] &= mlt^{-2} \\
[\rho_\infty] &= ml^{-3} \\
[V_\infty] &= lt^{-1} \\
[\mu_\infty] &= ml^{-1}t^{-1} \\
[r] &= l \\
[a_\infty] &= lt^{-1}.
\end{aligned}$$

where m, l, t denote mass length and time. We thus have $\mathrm{K} = 3$, and $\mathrm{N} - \mathrm{K} = 3$ dimensionless π groups.

Let us choose ρ_∞, V_∞, and r as our repeating set. Then:

$$\begin{aligned}
&\boxed{\pi_1 = g_1(\rho_\infty, V_\infty, r, D)} \\
&\pi_2 = g_2(\rho_\infty, V_\infty, r, \mu_\infty) \\
&\pi_3 = g_3(\rho_\infty, V_\infty, r, a_\infty)
\end{aligned}$$

Drag force on a sphere

From the Buckingham π theorem we have for π_1:

$$\pi_1 = \rho_\infty^{a_1} V_\infty^{a_2} r^{a_3} D^{a_4}$$

Note that π_1 is dimensionless, i.e., $[\pi_1] = 1$.

For the above expression to be dimensionally consistent then, we must have

$$(ml^{-3})^{a_1}(lt^-1)^{a_2} l^{a_3} (mlt^{-2})^{a_4} = m^{a_1+a_4} l^{-3a_1+a_2+a_3+a_4} t^{-a_2-2a_4} = 1$$

and so each exponent must be zero (e.g., $a_1 + a_4 = 0$).

Note that we have more variables than equations, so choose $a_4 = 1$ (since we are ultimately interested in D). Solving, we obtain $a_1 = -1, a_2 = -2, a_3 = -2$, and thus:

$$\pi_1 = \frac{D}{\rho_\infty V_\infty^2 r^2}$$

Since pure numbers are dimensionless, can rewrite as:

$$\pi_1 = \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 \pi r^2} = \frac{D}{q_\infty S} \quad = C_D!$$

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Drag force on a sphere

Following a similar procedure for π_2 and π_3, we have

$$\pi_2 = \frac{\rho_\infty V_\infty r}{\mu_\infty} \quad \text{= Reynolds number } (Re)$$

$$\pi_3 = \frac{V_\infty}{a_\infty} \quad \text{= Mach number } (M)$$

Then, according to the Buckingham π theorem, we can write:

$$F(C_D, Re, M) = 0$$

or

$$C_D = F_1(Re, M)$$

i.e., have reduced the number of independent variables from five to two.

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For the airfoil studied in the previous lecture, we could go through the same procedure for the lift and pitching moment (in place of the drag). Note we should then also include the angle of attack as a π group (angles are always dimensionless), i.e.,

$$\begin{aligned} C_D &= F_2(Re, M, \alpha) \\ C_L &= F_3(Re, M, \alpha) \\ C_M &= F_4(Re, M, \alpha) \end{aligned}$$

Note that, for flows involving thermodynamics/heat transfer, our analysis would involve additional parameters, such as:

- $\gamma = c_p/c_v$ (ratio of specific heats)
- T_w/T_∞ (wall temperature ratio)
- Pr (Prandtl number – ratio of momentum diffusion to heat conduction)

Limiting cases

Sometimes (often in limiting cases), only a single dimensionless variable can be formed, i.e., $N = K + 1$. Our equation involving the π groups,

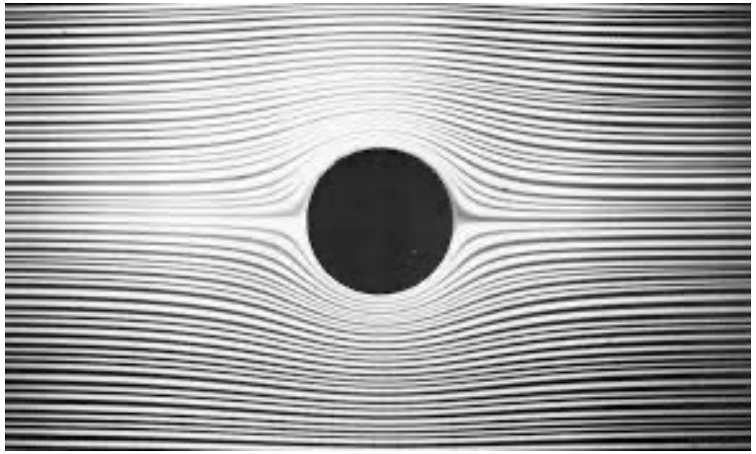
$$F(\pi_1, \pi_2, \ldots, \pi_{N-K}) = 0$$

then becomes simply

$$F(\pi_1) = \text{const.}$$

For F to be nontrivial, we must then have

$$\boxed{\pi_1 = \text{const.}}$$



Take, for example, the case of a sphere moving extremely slowly in a fluid. Then we would expect neither the sound speed nor the fluid inertia (represented by ρ_∞) to be important.

We can thus write

$$f(D, V_\infty, r, \mu_\infty) = 0$$

and so

$$\pi_1 = \frac{D}{\mu_\infty V_\infty r} = \text{const.} = c$$

Rearranging, we then have

$$D = c\mu_\infty V_\infty r$$

Detailed calculations show $c = 6\pi$.

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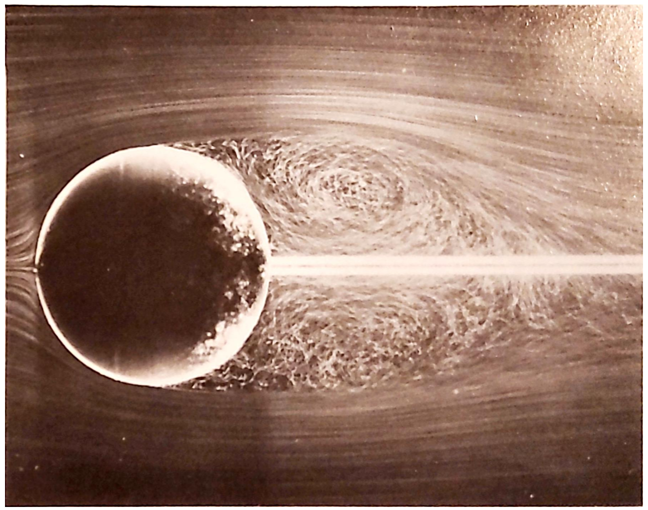


At the other extreme, if the sphere radius or fluid density is very large, we wouldn't expect viscosity to play much of a role.

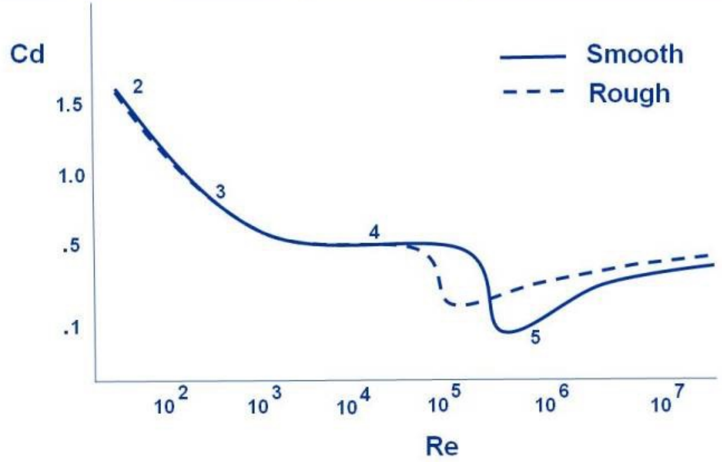
In this case, we can write

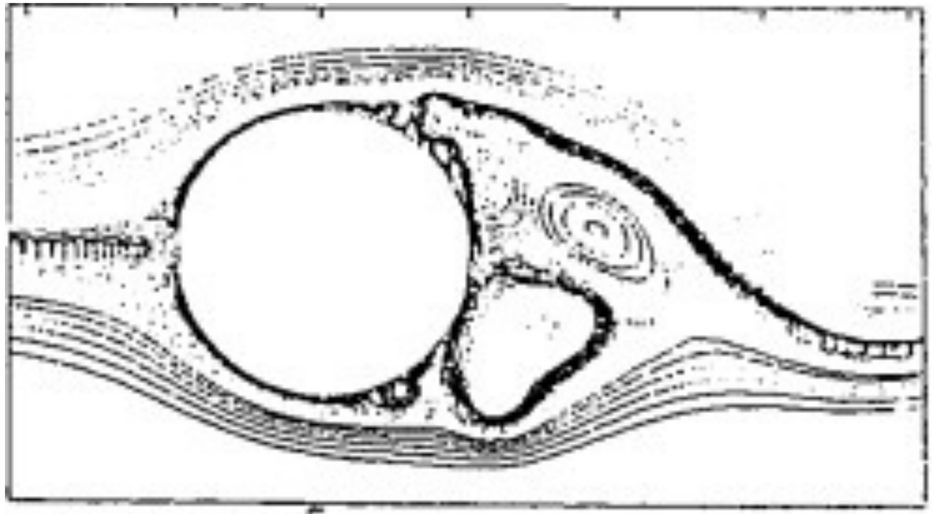
$$f(D, \rho_\infty, V_\infty, r) = 0$$

from which

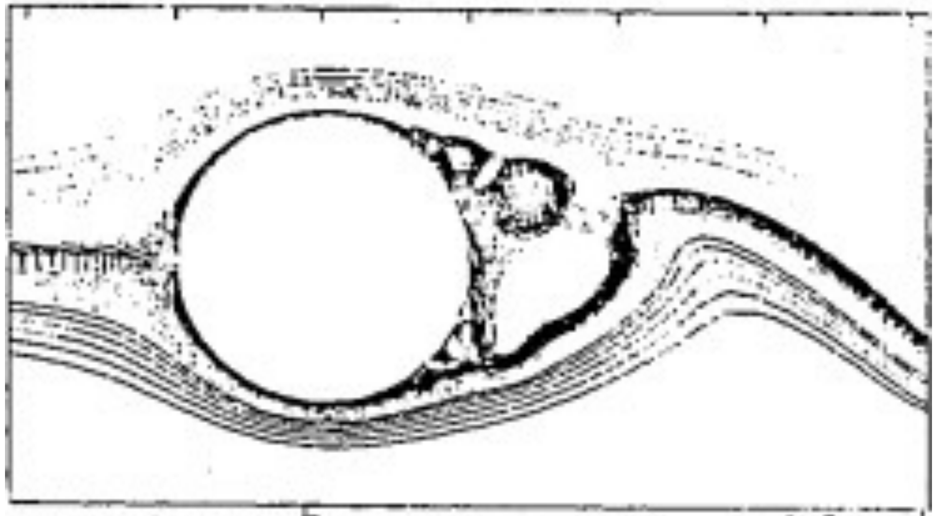
$$\pi_1 = \frac{D}{\rho_\infty V_\infty^2 r^2} = \text{const.}$$

We therefore conclude that C_D should become constant at high Reynolds numbers.



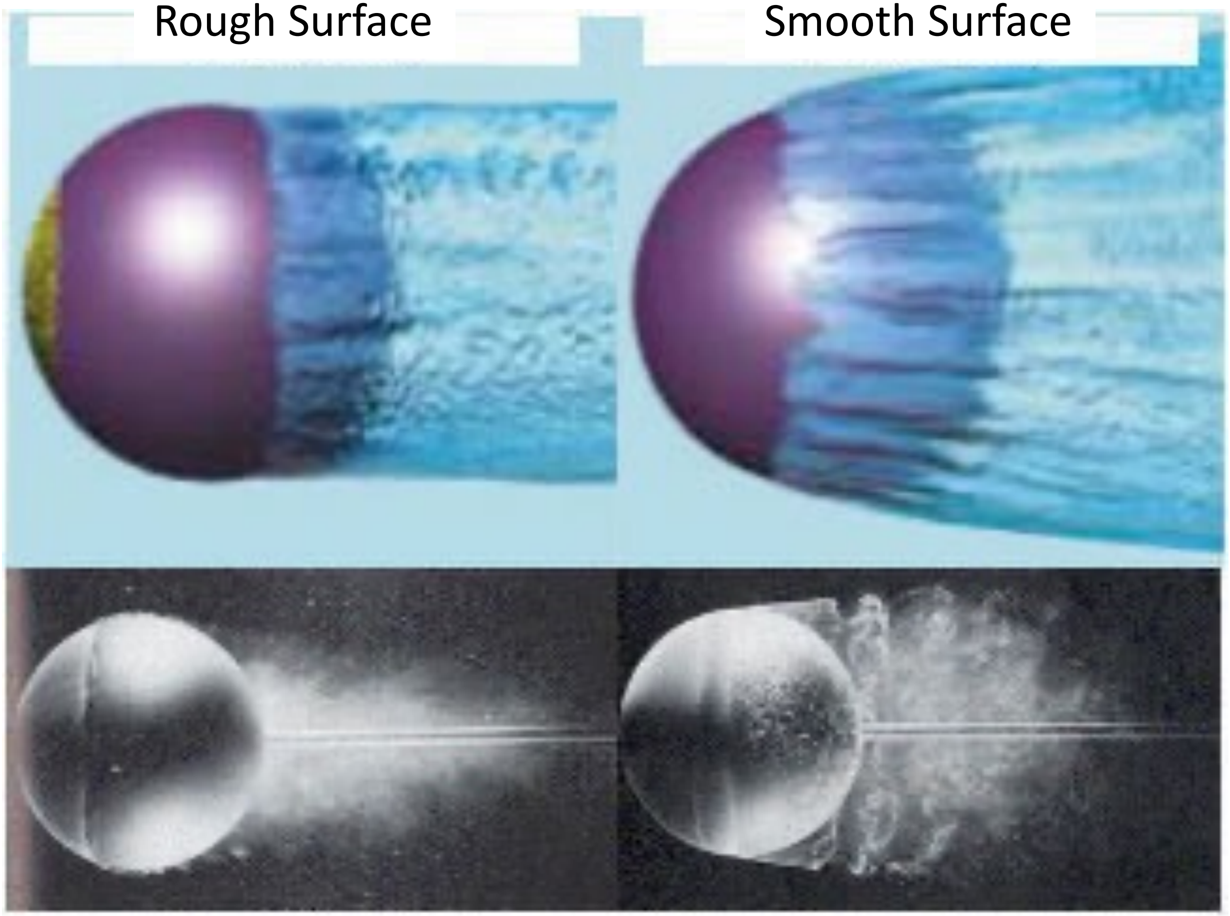


Re = 2 x 10^5

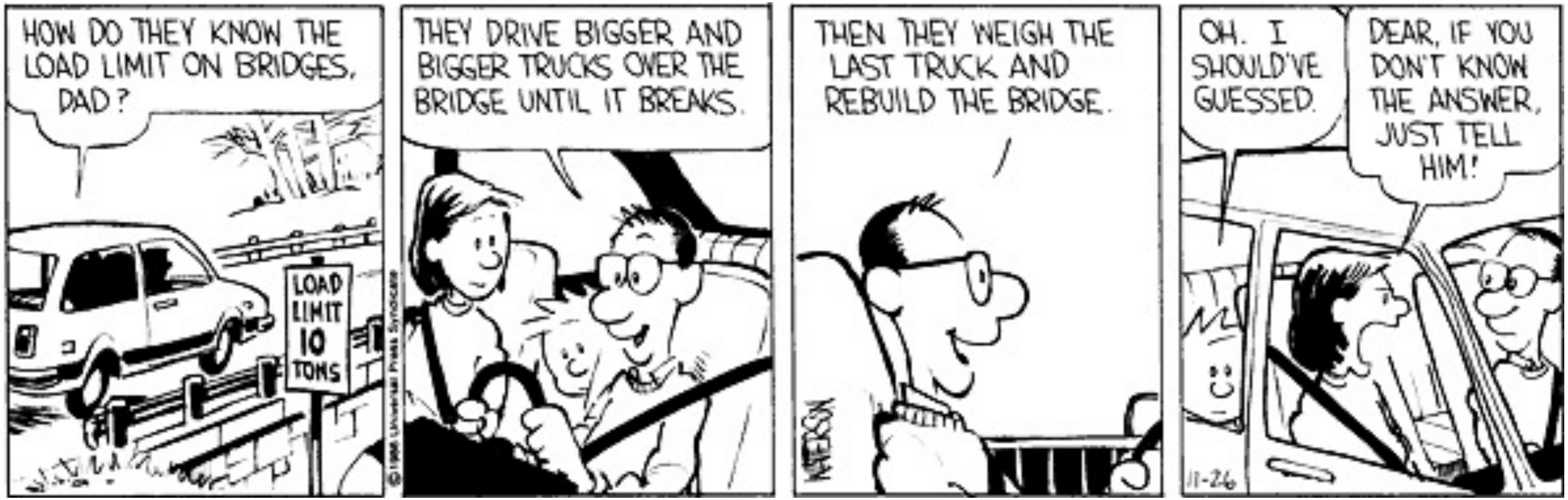


Re = 4 x 10^5 A 3

The drag crisis has been clearly obtained by this scheme.



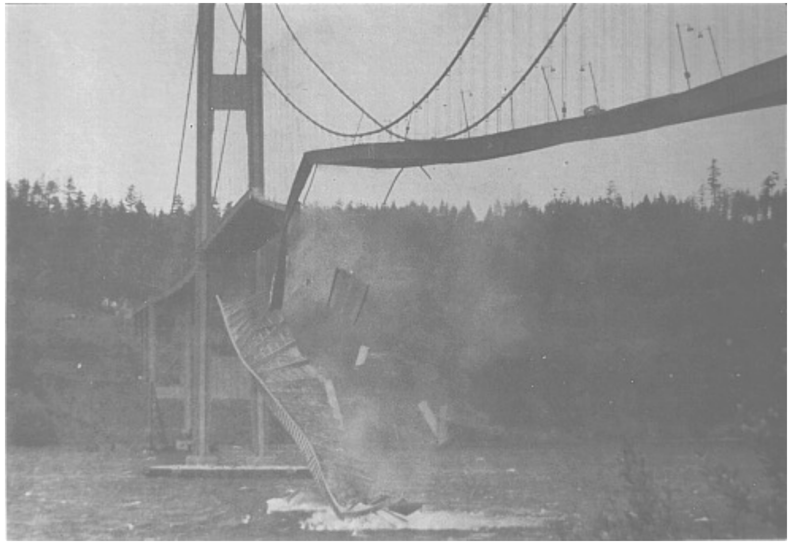
Why Modeling?

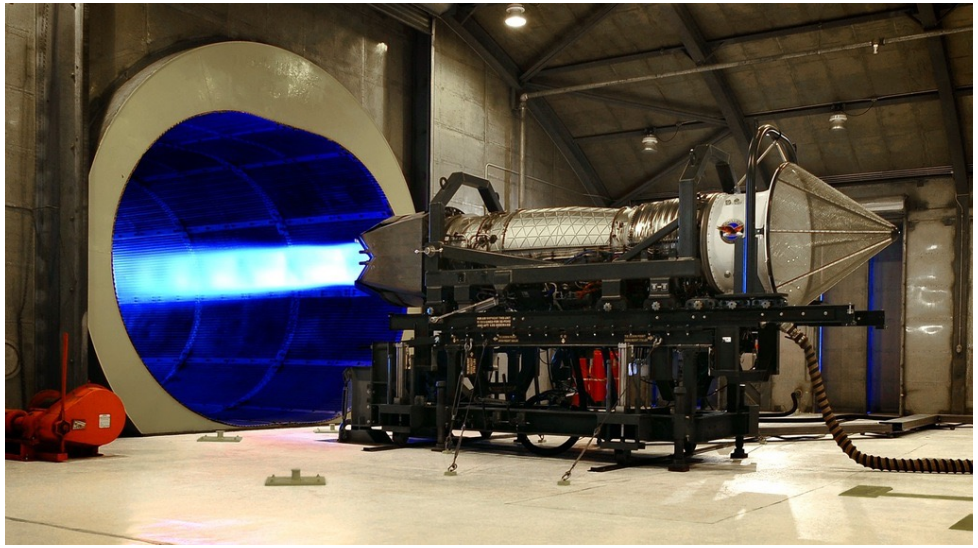


- This is not the way we should design things …

Tacoma Narrows Bridge

- Bridge collapse in WA (1940) due to wind-induced oscillations




Model Similarity (1)

- We are interested in knowing a particular variable for a large design (drag, pressure drop, oscillation frequency, etc.)
- IF we know what other variables affect the variable of interest then we can generalize the relationship in terms of Pi groups

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots)$$

Model Similarity (2)

- If we build a model that has the same value of all dependent PI groups, then the independent Pi group measured will be accurate

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots)$$

- If:

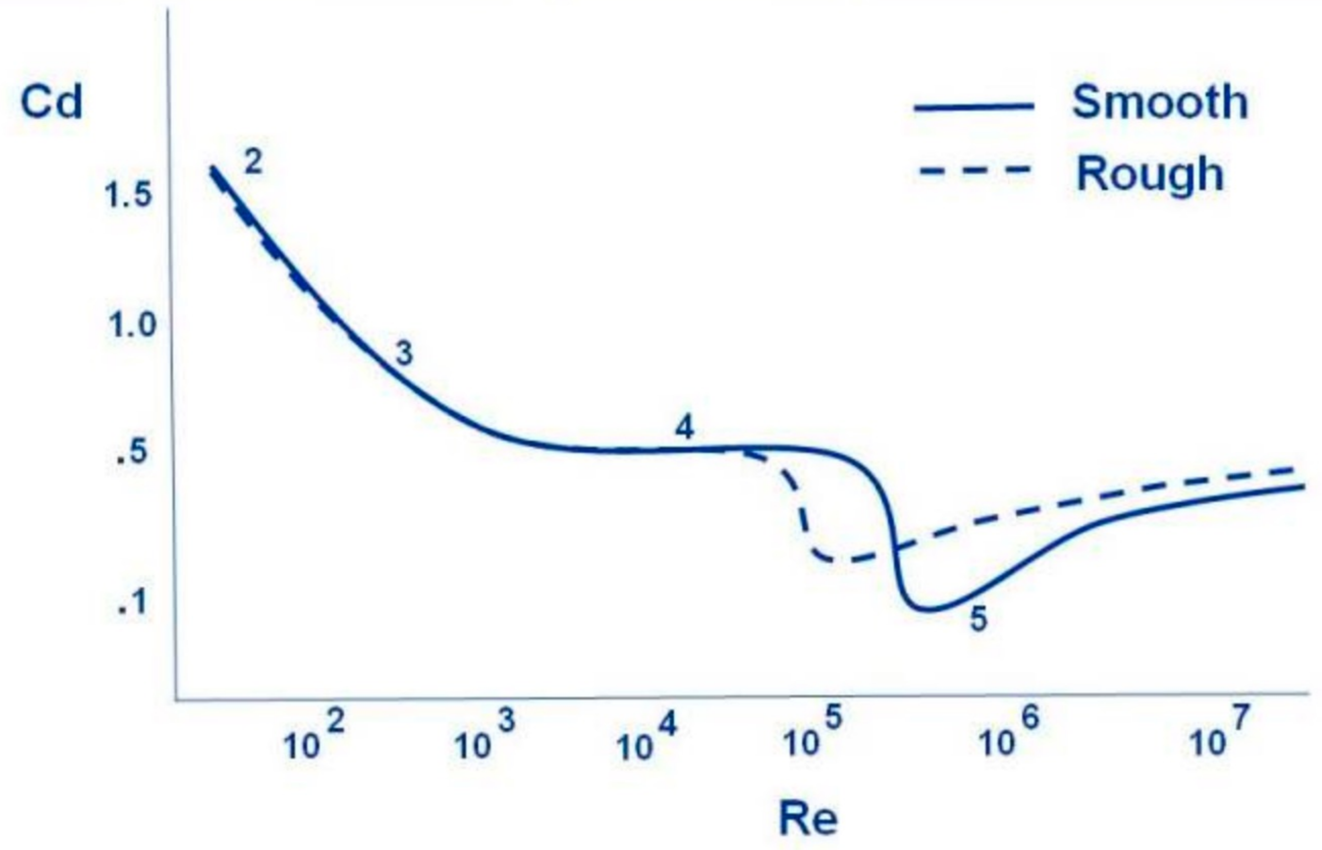
$$\Pi_{2,\mathrm{mod}} = \Pi_{2,real} \qquad \Pi_{3,\mathrm{mod}} = \Pi_{3,real}$$

- Then:

$$\Pi_{1,\mathrm{mod}} = \Pi_{1,real}$$

The drag characteristics of a torpedo are to be studied in a water tunnel using a 1:5 scale model. The tunnel operates with freshwater at 20 degrees Celsius, whereas the prototype torpedo is to be used in seawater at 15.6 degrees Celsius. To correctly simulate the behavior of the prototype moving with a velocity of 30m/s, what velocity is required in the wall tunnel?

Drag: D=f(Re)



For dynamic similarity, the Reynolds number must be the same for model and prototype. Thus,

$$\frac{V_m D_m}{\nu_m} = \frac{VD}{\nu}$$

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Since, ν_m (water @ 20°C) = 1.004×10^{-6} m²/s (Table B.2), ν (seawater @ 15.6°C) = 1.17×10^{-6} m²/s (Table 1.6), and $D/D_m = 5$, it follows that

$$V_m = \frac{\left(1.004 \times 10^{-6} \frac{m^2}{s}\right)}{\left(1.17 \times 10^{-6} \frac{m^2}{s}\right)} (5)\left(30 \frac{m}{s}\right) = \underline{\underline{129 \frac{m}{s}}}$$

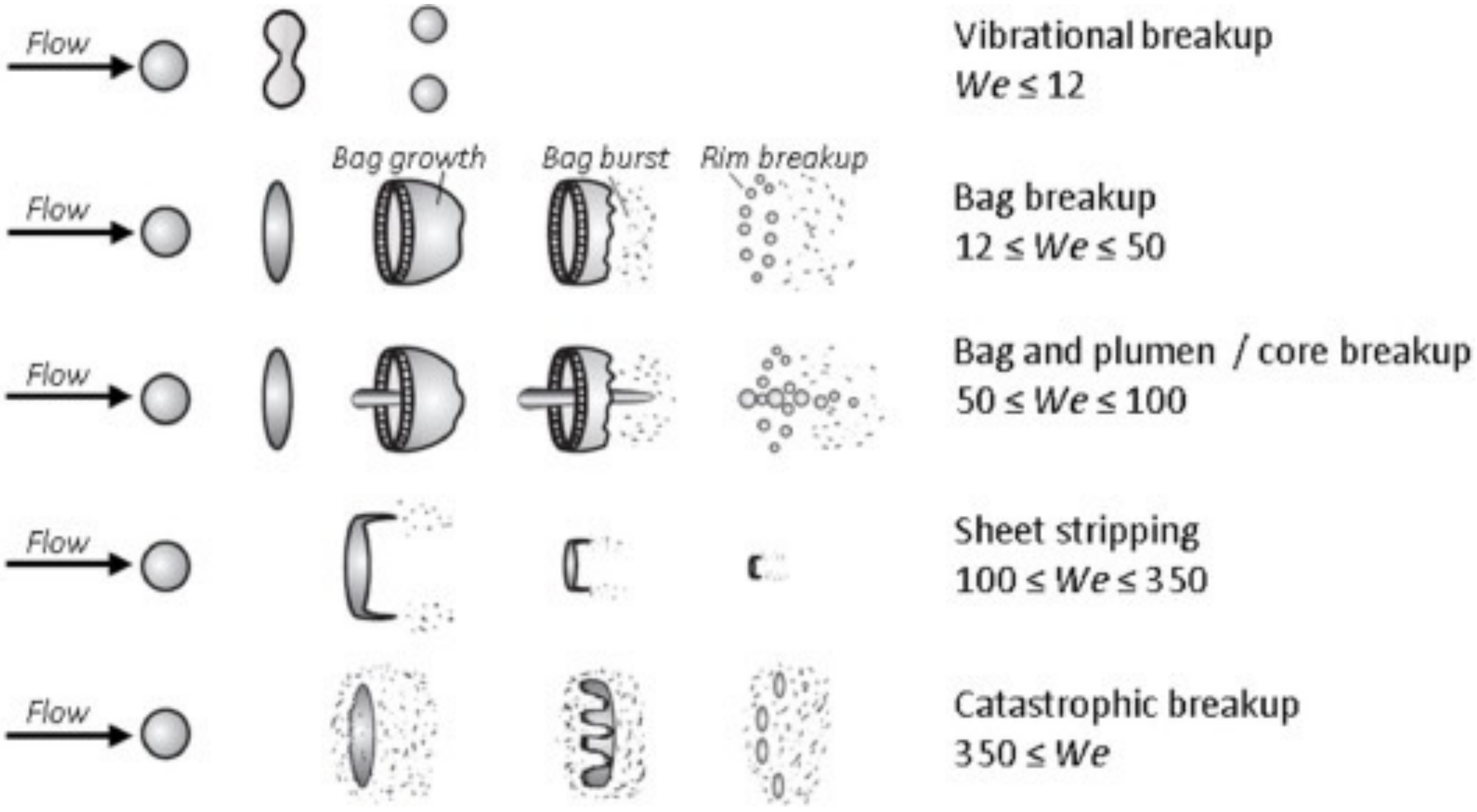
Example:

- I want to know the pressure drop that will occur in a 3-foot diameter, 1 mile long oil pipe while pumping 500 lbm/s of oil (this determines the size of pump required)
- I have a ¼" pipe to experiment with

$$\frac{\Delta p}{\rho V^2} = f\left(\mathrm{Re}, \left(\frac{L}{D}\right)\right)$$

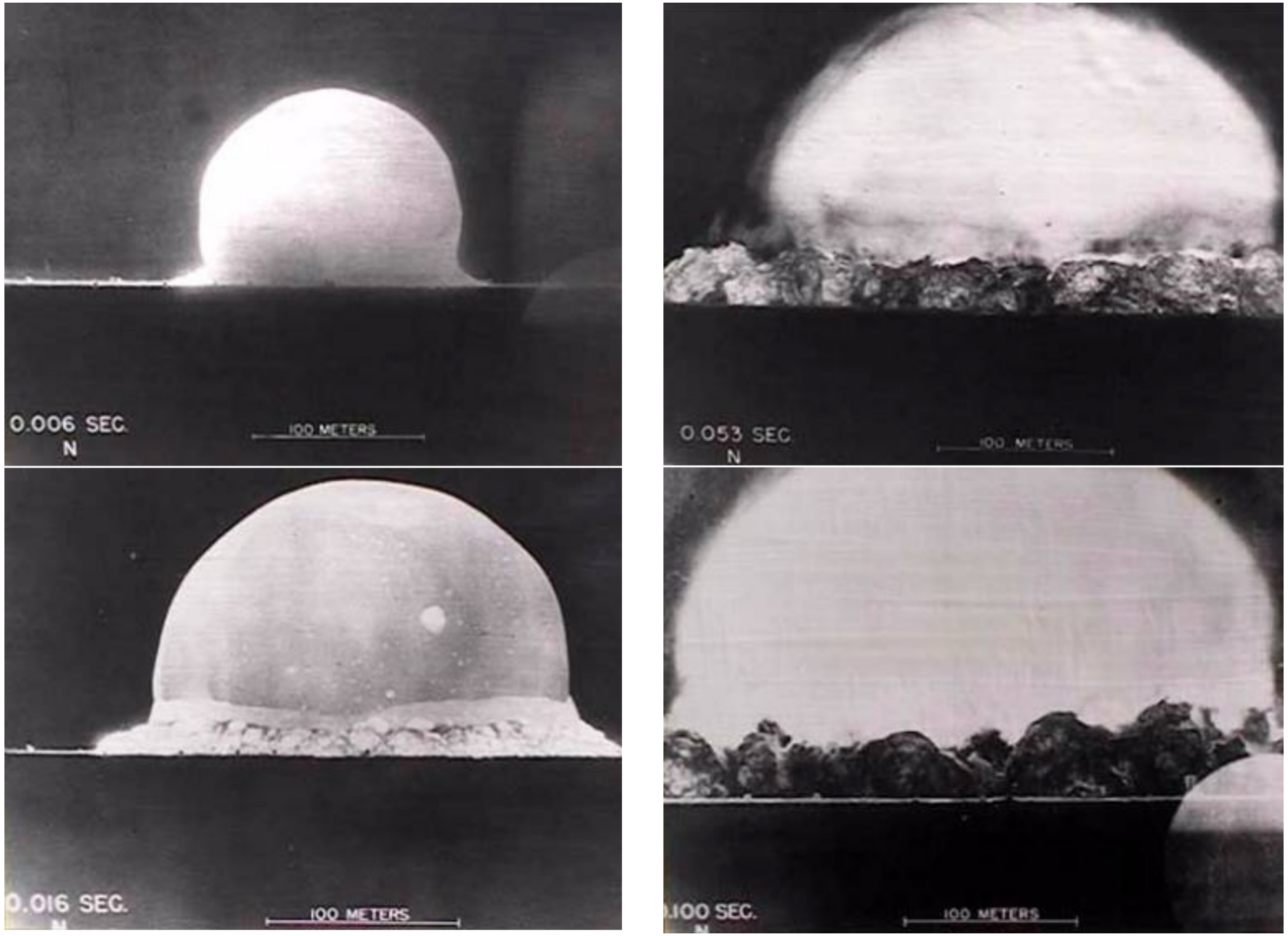
- Choose pipe length so that L/D is the same
- Choose fluid and velocity so that Re is the same

Droplet breakup



$$\mathrm{We} = \frac{\text{Drag Force}}{\text{Cohesion Force}} = \left(\frac{8}{C_\mathrm{D}}\right) \frac{\left(\frac{\rho\, v^2}{2}\, C_\mathrm{D}\, \pi \frac{l^2}{4}\right)}{(\pi\, l\, \sigma)} = \frac{\rho\, v^2\, l}{\sigma}$$

1945 Trinity Test (New Mexico)



Based on these photographs a British physicist named G. I. Taylor was able to estimate the power released by the explosion (which was still a secret at that time).
How can the following pictures be used to make this estimate?

First two assumptions need to be made:

1. The energy (E) was released in a small space.
2. The shock wave was spherical.

We have the size of the fire ball (R as a function of t) at several different times. How does the radius (R) depend on:

- energy (E)
- time (t)
- density of the surrounding medium (ρ – initial density of air)

Let’s perform a dimensional analysis of the problem:

- [R] = L :radius is determined by a distance
- [E] = ML2/T2 :energy is determined by a mass times a distance squared divided by timesquared.
- [t] = T :Time is determined by the time.
- [ρ] = M/L3 :density is determined by a mass divided by a distance cubed.

We can say

$$[R] = L = [E]^x[\rho]^y[t]^z$$

Substituting the units for energy, time and density that we listed above we have:

$$[R] = L = M^{(x+y)}L^{(2x-3y)}T^{(-2x+z)}$$

This provides three simultaneous equations:
$x + y = 0$,
$2x - 3y = 1$,
$-2x + z = 0$,

yielding the results:
$x = 1/5$, $y = -1/5$, $z = 2/5$.

The radius of the shock wave is therefore:
$R = E^{1/5}\rho^{-1/5}t^{2/5} * \text{constant}$

Let's assume the constant is approximately 1.
Solving the equation for E we get:
$E = (R^5\rho)/t^2$.

At $t = .006$ seconds the radius of the shock wave was approximately 80 meters. The density of air is $\rho = 1.2$ kg/m3. Plugging these values into the energy equation gives:
$E = (805) \times 1.2/(.0062)$ kg $*$m2/s2
$= 1 \times 10^{14}$ kg $*$m2/s2

corresponds to E = 25 kilo −tons of TNT