University of Maryland at College Park
Dept. of Aerospace Engineering

ENAE 432: Aerospace Control Systems

Problem Set #10

Issued: 19 Apr. 2025
Due By: 25 Apr. 2025

Question 1:

A common model for the dynamics of vehicles with appreciable structural flexibility is

$$\begin{aligned} I\ddot{\theta}(t) + \sum_{i=1}^{n} \delta_i \ddot{\eta}_i(t) &= \tau(t) \\ \ddot{\eta}_i(t) + 2\zeta_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) &= -\delta_i\ddot{\theta}(t), \quad i = 1, \ldots, n \end{aligned}$$

where $\theta(t)$ is the orientation angle of the vehicle, n is the number of structural modes, and ζ_i, ω_i are respectively the damping and natural frequency of these modes. Structural modes tend to have very light damping $\zeta \leq 0.01$ ("1% damping" or less), and relatively high frequencies.

a.) Show that

$$G(s)^{-1} = Is^2 \left[1 - \sum_{i=1}^{n} \rho_i s^2 (s^2 + 2\zeta_i\omega_i s + \omega_i^2)^{-1}\right]$$

where $G(s)$ is the transfer function from $\tau(t)$ to $\theta(t)$, and $\rho_i = \delta_i^2/I$.

b) Now suppose specifically that $n = 2$ with $\zeta_1 = .01$, $\zeta_2 = .005$, $\omega_1 = 4$, $\omega_2 = 15$ for the structural modes, with coupling constants $\rho_1 = \rho_2 = 0.04$ and overall vehicle inertia $I = 50$. Find $G(s)$ in ZPK form and generate its Bode diagrams.

c.) Notice the complicated behavior of these diagrams near the natural frequencies of the two structural modes, due to the very lightly damped second-order poles and zeros in $G(s)$. But focus on the low frequency behavior of $G(s)$, in particular for frequencies an order of magnitude or more below the first structural mode. Show that the behavior of $G(j\omega)$ at these lower frequencies is indistinguishable from the much simpler system model

$$G_0(s) = \frac{1}{Is^2}$$

by sketching the Bode magnitude (only) of this system over the Bode magnitude of $G(s)$ in b)

d.) We desire to develop a control law for this vehicle to ensure: 1) Approximately 20 seconds settling time and moderate overshoot ($\leq 40\%$) to a step input; 2) perfect rejection of constant disturbances and perfect tracking of step and ramp (linearly increasing) $y_d(t)$. The step response requirements can be loosely translated to the frequency domain requirements: magnitude crossover around 0.4 rad/sec and phase margin at least 50°. Let's use these numbers as a starting point. Design a derivative-free compensator $H(s)$ which will provide these specifications, as well as the required tracking and disturbance rejection. However, note that the crossover/margin specifications are solidly in the frequency region where $G(j\omega)$ is behaving like the much simpler $G_0(j\omega)$. So, use the simpler model $G_0(s)$ in designing your $H(s)$. Compute the corresponding $T_0(s)$ and characterize its step response properties.

Question 2:

a.) Of course, your design in Question #1 is going to be used with the real system $G(s)$, not the simplified model $G_0(s)$. Will this make a difference? Compute the actual closed-loop transfer function $T(s)$ that would arise from using your $H(s)$ in Question #1 with the true vehicle dynamics $G(s)$. Is this $T(s)$ still stable? Generate a step response with the actual $T(s)$ and overlay it on top of the response of $T_0(s)$ from Question #1. Are there any appreciable differences?

b.) One of the points of robustness analysis is to formally quantify when we can get away with the kind of "model reduction" exploited for the design in Question #1. Compute the multiplicative uncertainty model $\Delta(s)$ that corresponds to the simplification we used to reduce $G(s)$ to $G_0(s)$ and carry out the multiplicative uncertainty stability test. Does this test agree with the calculated properties of $T(s)$ in a)? That is, does the uncertainty test predict that the actual $T(s)$ will be stable even though we did our design for the much simpler model $G_0(s)$?

c.) Suppose we wanted to make our design in Question #1 more agressive by increasing the magnitude crossover frequency (to decrease the settling time) while keeping the same phase margin (hence the peak in $|T_0(j\omega)|$ should have about the same height.) Looking at the robust stability analysis in b), about how large could we make ω_γ before the discrepancy between the simple model and the actual system dynamics would potentially result in our design being unstable when used on the real vehicle?

d.) Suppose that $d(t) = d_0 + D\sin(\omega t + \phi)$ where d_0 is an unknown constant, and ϕ is an unknown phase. Determine the range of disturbance frequencies ω for which you can ensure the induced additional tracking error created will be no larger than $D/10$. Use the actual $G(s)$ in this analysis.

e.) Suppose the sensor used for the feedback calculations has additive noise $n(t) = N\sin(\omega t + \phi)$ where ϕ is an unknown phase. Determine the range of noise frequencies ω for which you can ensure the induced additional tracking error will be no larger than $N/10$. Use the actual $G(s)$ in this analysis.

Question 3:

We will show in class on Monday that any s in the complex plane where $\angle L(s) = (1 + 2k)180°$ (i.e. any odd multiple of $180°$) will be a closed-loop pole for some positive value of the compensator gain K. Suppose specifically that

$$L(s) = \frac{4K(s - z)}{s(s + 5)^3}$$

a.) Choose a value for z such that the above "angle condition" is satisfied at $-2 \pm 3j$, then choose the corresponding K such that $T(s)$ has these poles. Show all steps of your calculation. "Guess and check" or exhaustive search solutions will not be accepted as an answer here; you should be able to explicitly calculate an exact result.

b.) Identify all the other closed-loop poles that result from the choices in a) Are they all stable?

c.) What properties do you expect in a step response of $T(s)$? Describe your reasoning in detail based on the ZPK structure of $T(s)$. Use Matlab to generate a step response and compare the results to your predictions.

d.) Suppose the gain K found in a) was increased by a factor of 7. Where are the poles of $T(s)$ now? Are they still stable?