Robustness

Robustness is range of inaccuracy in our Nominal model $G(s)$ that we can tolerate before feedback loop might become unstable.

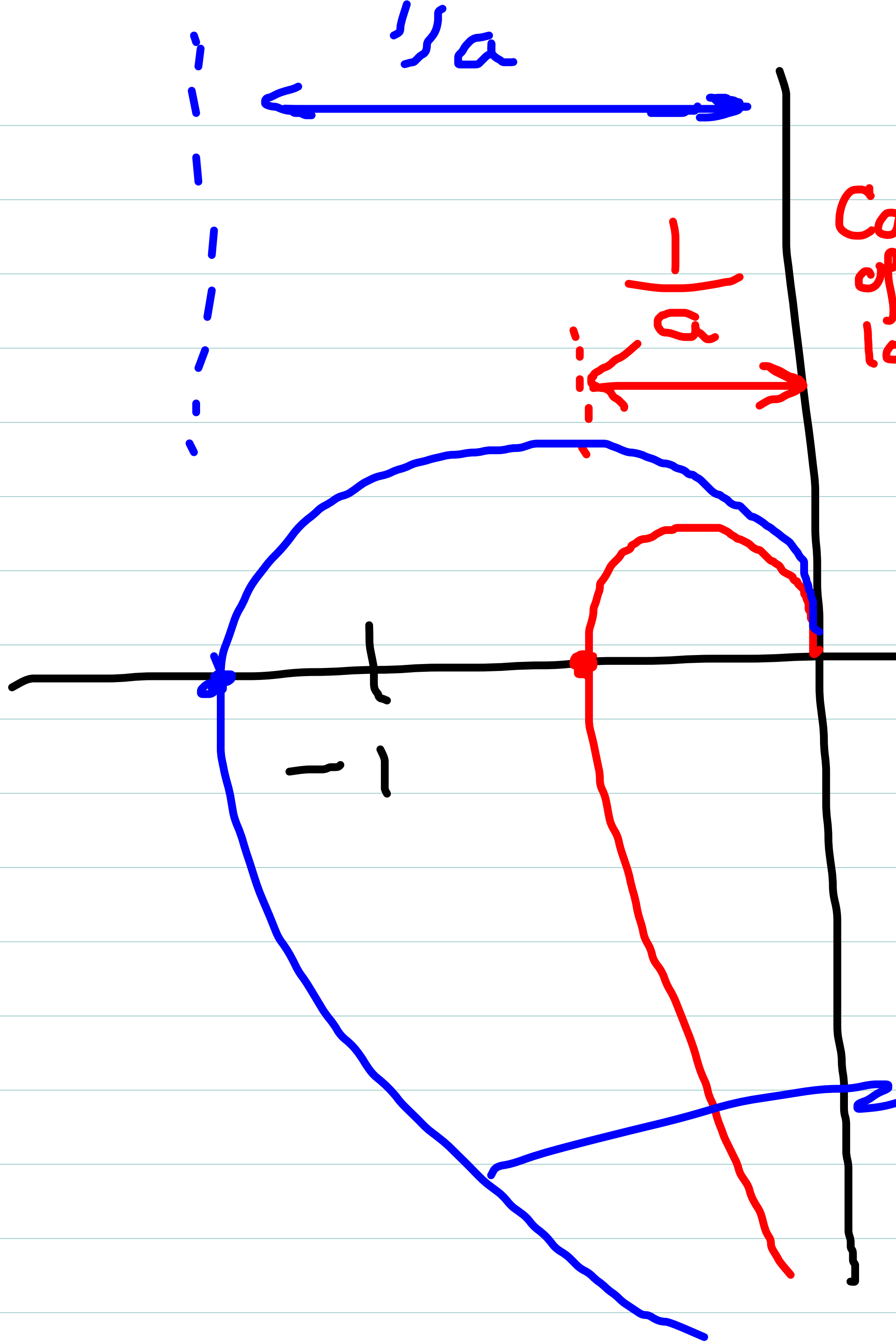
"Perturbations" to Nyquist analysis: how much can polar plot of $L(j\omega)$ be changed without changing the number of -1 encirclements.

Simple measures:

1. gain margin: Measures tolerance to pure gain uncertainty

Common requirement: $|a|_{dB} \geq 6 \Rightarrow a \geq 2$ or $a \leq 1/2$

$\Rightarrow$ Plant gain could be a factor of 2 larger or smaller and -1 encirclements will not change.



$a > 2 \Rightarrow \frac{1}{a} < 1/2$

Can increase gain by factor of 2 w/o putting a new loop around -1

$a < 1/2 \Rightarrow 1/a > 2$

Can <u>decrease</u> gain by factor of 2 w/o moving -1 out of loop

Simple Robustness measure #2: phase margin, γ

Measures pure <u>phase</u> uncertainty tolerable before -1 encirclements change

If $\measuredangle L(j\omega_\gamma)$ is changed by more than $-\gamma$, the new plot will have a loop around -1

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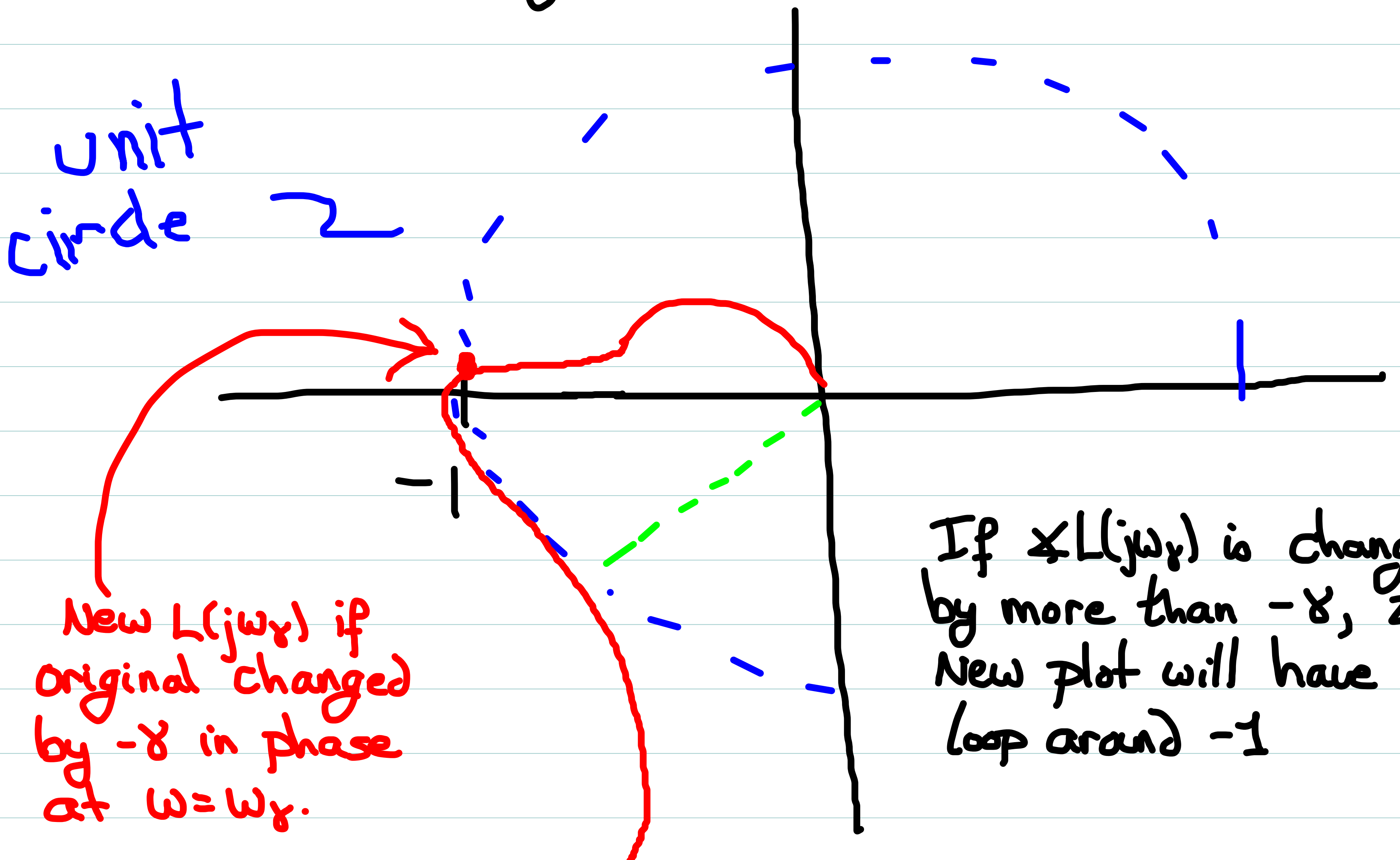


If $\measuredangle L(j\omega_\gamma)$ is changed by more than $-\gamma$, the new plot will have a loop around -1

Physical Sources of <u>Pure</u> Phase Change

Phase margin is an important metric, so there must be an important, common physical mechanism which can introduce pure phase changes. What is it?

$\boxed{\text{Time Delay}}$

We've been modeling our controller as <u>continuously</u> evolving, just like the physical system being controlled.

But the controller is <u>different</u> than a physical system with dynamics governed by continuous diff'l equations.

Models of these differences will create pure phase changes to $L(j\omega)$.

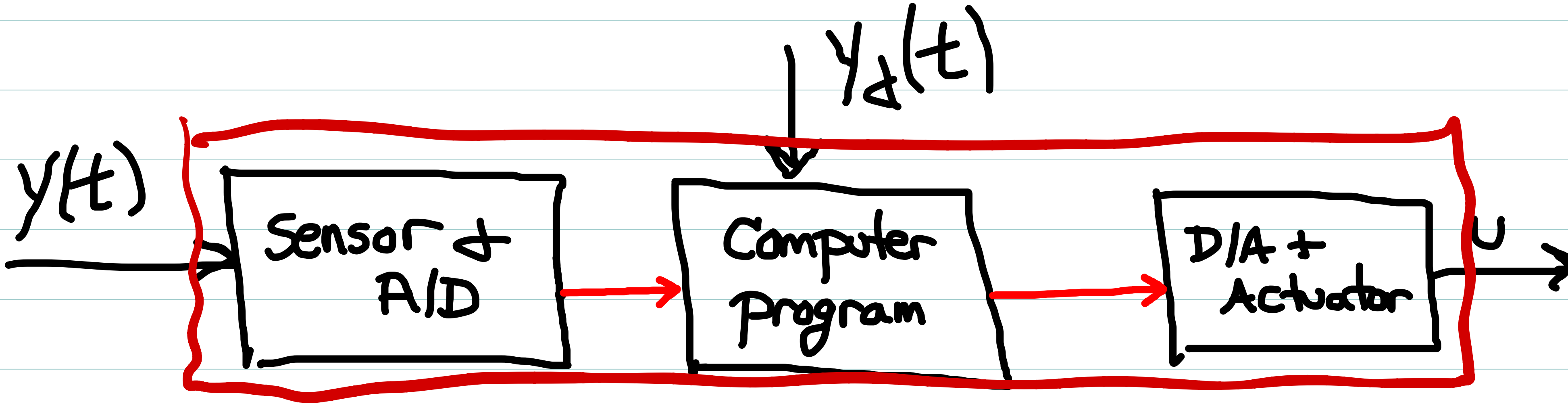
Time Delay

Three typical steps in controller implementation

1. Measure output $y(t)$, and input to computer
2. Compute $u(t)$ via computer program
3. Output $u(t)$ from computer to physical actuator

Each of these steps requires nonzero amount of time!

1. A/D conversion and transmission/read time
2. Time to execute program
3. D/A conversion and transmission time



Each block, and each red arrow, requires nonzero time to operate. Call total required time T_s

T_s may be small (msec), but is always > 0!

The implication is that the $u(t)$ which actually gets applied to the plant depends on the measurement taken T_s seconds ago, i.e. $y(t - T_s)$

We haven't modeled this!

Laplace analysis of ideal delay

$$f(t) \longrightarrow \boxed{\text{delay}} \longrightarrow f_1(t) = f(t-\tau_s)$$

$\tau > 0$, constant

By def'n:

$$F_1(s) = \int_{0^-}^{\infty} f_1(t)\, e^{-st}\, dt = \int_{0^-}^{\infty} f(t-\tau_s)\, e^{-st}\, dt$$

Let $\sigma = t - \tau_s$, $d\sigma = dt$

$$F_1(s) = \int_{-\tau_s}^{\infty} f(\sigma)\, e^{-s(\sigma+\tau_s)} = e^{-s\tau_s} \int_{0^-}^{\infty} f(\sigma)\, e^{-s\sigma}\, d\sigma$$

Since τ_s is constant, and Laplace assumes $f(t) = 0$ for $t < 0$

Thus:

$$F_1(s) = e^{-s\tau_s} F(s)$$

$$F \longrightarrow \boxed{e^{-s\tau_s}} \xrightarrow{F_1}$$

$\boxed{\text{transfer function!}}$

So really:

$$\underbrace{L(s)}_{} = \underbrace{e^{-sT_s}}_{\text{transfer function of delay}} \underbrace{\left[G(s)H(s)\right]}_{L_o(s):\ \text{"ideal" (no delay) open-loop TF.}}$$

Now, e^{-sT_s} is difficult to deal with in standard TF manipulations, because it is not rational. Cannot be described with a finite number of poles and zeros.

It's impact on freq. Domain properties of $L(j\omega)$ are easy to determine, however.

$$|L(j\omega)| = \boxed{|e^{-j\omega T_s}|}\,|L_o(j\omega)|$$

$$\angle L(j\omega) = \boxed{\angle e^{-j\omega T_s}} + \angle L_o(j\omega)$$

What are these?

Recall for complex number in polar form:

$$z = re^{j\theta} \iff |z| = r, \ \measuredangle z = \theta$$

$$e^{-j\omega T_s} = 1 \cdot e^{j(-\omega T_s)} \Rightarrow r = 1, \ \theta = -\omega T_s$$

So $|e^{-j\omega T_s}| = 1$ for all ω, and

$$\measuredangle e^{-j\omega T_s} = -\omega T_s \quad \text{for all } \omega$$

Hence:

$$|L(j\omega)| = |L_o(j\omega)|$$ (unaffected by delay)

$$\measuredangle L(j\omega) = \measuredangle L_o(j\omega) - \omega T_s$$

Effect of delay is pure phase change in $L(j\omega)$!

Delay thus acts to reduce phase margin:

$$\gamma = 180^\circ + \angle L(j\omega) = \underbrace{180^\circ + \angle L_o(j\omega_\gamma)}_{\gamma_o:\ \text{expected phase margin w/o delay}} \underbrace{- \omega_\gamma T_s}_{\text{reduction in actual phase margin due to delay}}$$

i.e. $\boxed{\gamma = \gamma_o - \omega_\gamma T_s}$ ⇐ Key equation!

Note: ω_γ in rad/sec, T_s in sec ⇒ $\omega_\gamma T_s$ in rad

γ_o, γ expressed in deg, so must convert $\omega_\gamma T_s$ to deg here

Recall we typically need $\gamma > 0^\circ$ for Nyquist to show stability

$$\Rightarrow \gamma_o - \omega_\gamma T_s > 0 \quad \text{or} \quad T_s < \frac{\gamma_o(\text{rad})}{\omega_\gamma}$$

$\boxed{T_{max} = \frac{\gamma_o(\text{rad})}{\omega_\gamma}}$ is the maximum tolerable delay, or the "delay margin"

Now, typically T_s is fixed by available hardware.

Then $\gamma_o - \omega_\gamma T_s > 0$ Becomes a design constraint

$\Rightarrow$ Cannot have $\omega_\gamma T_s$ "too big" or it will be impossible to design $H(s)$ to provide necessary positive phase for γ_o.

Typical guideline: Keep $\boxed{\omega_\gamma T_s < 0.1}$ $\left(\omega_\gamma < \frac{1}{10 T_s}\right)$

Then $\gamma = \gamma_o - \omega_\gamma T_s \geq \gamma_o - 5.7^\circ$ $(0.1\,\text{rad} = 5.7^\circ)$

Can design $H(s)$ to provide additional $+5.7^\circ$ of phase margin in γ_o to offset (or, just tolerate the small reduction)

$\Rightarrow$ Note this constrains ω_γ in a manner which works against guideline for good performance (big ω_γ)

$\Longrightarrow$ **Sample rate fundamentally restricts performance!**

Different uses of delay eqn

① Delay margin => $T_{max} = \gamma / \omega_\gamma$. Max tolerable delay w/o creating instability. ("common figure of merit")

② If T_s fixed, rule of thumb $\omega_\gamma T_s < .1$ restricts ω_γ, i.e. $\omega_\gamma < 1/(10 T_s)$ (common)

③ If T_s can be changed (hardware upgrade) then $T_s < \frac{1}{10\omega_\gamma}$ needed to keep delay effect "small" (uncommon, except in early design phase)

④ $\gamma = \gamma_0 - \omega_\gamma T_s$.

Given fixed ω_γ, T_s, target $\gamma_0 = \gamma_{des} + \omega_\gamma T_s$

so $\gamma = \gamma_{des}$

i.e. design $H(s)$ so $L_0(s) = G(s)H(s)$ (OL TF w/o delay) has PM $\gamma_{des} + \omega_\gamma T_s$ at desired ω_γ (rare, but possible sometimes)

Gain and Phase margins are measures of <u>robustness</u>

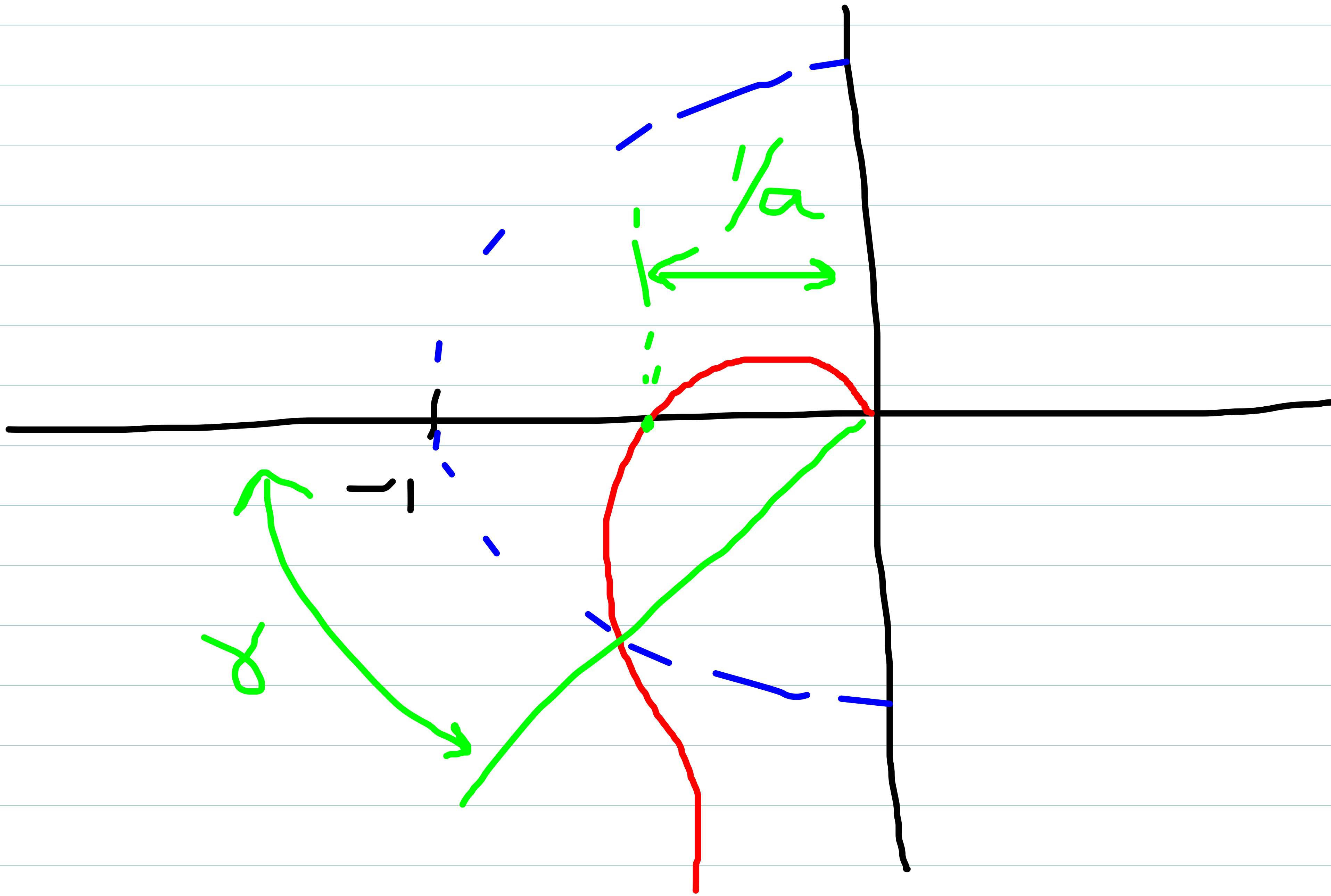
They quantify how close Nyquist diagram comes to -1 in two <u>simple</u> senses

Very common and popular since each corresponds to a <u>physical</u> source of possible model inaccuracy

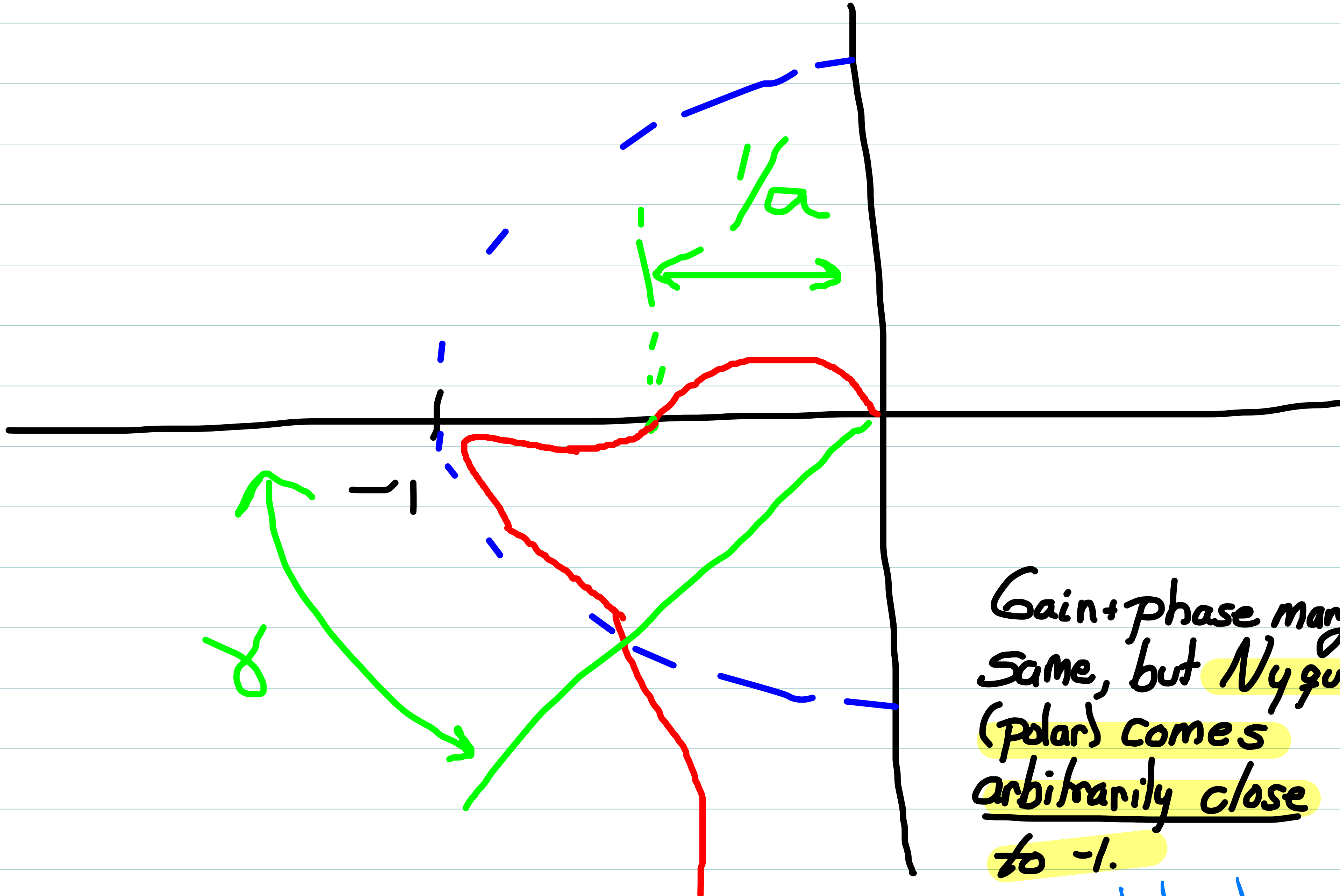
$\Rightarrow$ <u>gain margin</u>: tolerance to variations in overall gain of plant (typically overall mass or inertia)

$\Rightarrow$ <u>phase margin</u>: tolerance to time delays associated with computer implementation of controller

<u>However</u>: mathematically they are poor indicators of the tolerance of the Nyquist diagram to small perturbations



A typical case



Gain+phase margin same, but Nyquist (polar) comes arbitrarily close to -1.

⇒ arbitrarily small change could destabilize!

But this is possible too!

Gain and phase margin are useful, intuitive measures but cannot capture the effects of simultaneous gain and phase changes to $L(j\omega)$

Such changes would occur due to:

$\Rightarrow$ mismodeling of pole/zero locations in $G(s)$

$\Rightarrow$ Incompleteness of $G(s)$ model, i.e. physics has additional dynamics which are too uncertain, or too difficult, to model accurately

$\Rightarrow$ "real" $G(s)$ has additional poles/zeros which aren't present in model we use for design!

$\Rightarrow$ Want a robustness test which can also handle these!