Inverse Transform

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= \frac{1}{2\pi j}\int Y(s)e^{st}\,ds$$

$\Rightarrow$ contour integral over ROC in complex plane

$\Rightarrow$ ugly! Math 463

$\Rightarrow$ We can sidestep this in many cases

General form of Y(s)

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

all polynomials

Suppose $U(s)$ is rational in s
(ratio of polynomials)

i.e. $U(s) = \dfrac{a(s)}{h(s)}$, $a(s)$ $h(s)$ polys

Note: ① Not true for every $u(t)$
② True for many "useful" $u(t)$

Then . . .

$U(s)$

$$Y(s) = \left[\frac{q(s)}{r(s)}\right]\left(\frac{a(s)}{h(s)}\right) + \frac{c(s)-b(s)}{r(s)}$$

$$= \frac{q(s)a(s) + h(s)\left[c(s)-b(s)\right]}{r(s)h(s)}$$

or

$$Y(s) = \frac{N(s)}{D(s)}$$

where both $N(s)$ and $D(s)$ are polynomials

(i.e. $Y(s)$ is rational)

$$Y(s) = \frac{N(s)}{D(s)}$$

Suppose $\deg\{N(s)\} < \deg\{D(s)\} = L$

Let d_ℓ be the roots of $D(s)$: $D(d_\ell) = \phi$

Then:

$$Y(s) = \frac{A_1}{s-d_1} + \frac{A_2}{s-d_2} + \cdots + \frac{A_L}{s-d_L}$$

$$= \sum_{\ell=1}^{L} \frac{A_\ell}{s-d_\ell}$$

"Partial fraction expansion"

and

$$y(t) = \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

How to find expansion coefficients

"Residue formula":

$$A_\ell = \left[(s-d_\ell) Y(s) \right]_{s=d_\ell}$$

(also called "cover up" rule).

Example:

$$Y(s) = \frac{2s+3}{(s+2)(s+3)}$$

$$Y(s) = \frac{A_1}{s+2} + \frac{A_2}{s+3}$$

$$A_1 = \left[\frac{2s+3}{s+3} \right]_{s=-2} = -1, \quad A_2 = \left[\frac{2s+3}{s+2} \right]_{s=-3} = 3$$

$$y(t) = 3e^{-3t} - e^{-2t}$$

Complex d_ℓ

Note if d_ℓ is a complex root of $D(s)$, then its conjugate $\bar{d}_\ell$ will also be a root.

The residue formula then tells us that

for d_ℓ: $A_\ell = \left[(s-d_\ell)Y(s)\right]_{s=d_\ell}$

and for $\bar{d}_\ell$ we instead have

$$\left[(s-\bar{d}_\ell)Y(s)\right]_{s=\bar{d}_\ell} = \bar{A}_\ell$$

i.e. the PFE coefficients are also conjugates

Complex d_ℓ (cont)

Thus, the expression for $y(t)$ will contain

$$A_\ell e^{d_\ell t} + \bar{A}_\ell e^{\bar{d}_\ell t}$$

$$= 2|A_\ell| e^{\sigma t} \cos(\omega t + \measuredangle A_\ell)$$

where $\sigma = \mathrm{Re}\{d_\ell\}$ $\omega = \mathrm{Im}\{d_\ell\}$

$\boxed{\text{Example:}}$

$$Y(s) = \frac{4(s^2+2s+6)}{(s+1)(s^2+4s+13)}$$

$$d_1 = -1; \quad d_2 = -2+3j; \quad d_3 = -2-3j = \bar{d}_2$$

Then:

$$A_1 = \left[(s+1)Y(s)\right]_{s=-1} = 2$$

$$A_2 = \left[(s+2-3j)Y(s)\right]_{s=-2+3j} = 1+j = \boxed{\sqrt{2}\angle \pi/4 = A_2}$$

$$A_3 = \left[(s+2+3j)Y(s)\right]_{s=-2-3j} = 1-j = \bar{A}_2$$

Hence:

$$y(t) = 2e^{-t} + \boxed{(1+j)e^{(-2+3j)t} + (1-j)e^{(-2-3j)t}}$$

or:

$$y(t) = 2e^{-t} + \boxed{2\sqrt{2}\,e^{-2t}\cos(3t+\pi/4)}$$

Recap

$$Y(s) = \underbrace{\left[\frac{q(s)}{r(s)}\right]}_{G(s)} U(s) + \underbrace{\left[\frac{c(s)-b(s)}{r(s)}\right]}_{\text{IC terms}}$$

If $U(s)$ rational, $U(s) = \dfrac{a(s)}{h(s)}$

Then $Y(s) = \dfrac{N(s)}{D(s)}$ (also rational)

$$= \sum_{\ell=1}^{L} \frac{A_\ell}{(s-d_\ell)} \quad \text{where} \quad D(d_\ell) = \emptyset$$

and $A_\ell = \left[(s-d_\ell)Y(s)\right]_{s=d_\ell}$

Inverse transform:

$$y(t) = \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

Assumptions

Above assumes:

1. $\deg\{N(s)\} < \deg\{D(s)\}$
2. No repeated roots of $D(s)$

Simplest, most common case

Both can be relaxed:

1. Suppose $\text{Deg}\{N(s)\} = \text{Deg}\{D(s)\}$

Then do polynomial long division:

$$Y(s) = \frac{N(s)}{D(s)} = A_0 + \frac{N_1(s)}{D(s)}, \quad \text{Deg}\{N_1(s)\} < \text{Deg}\{D(s)\}$$

and $\frac{N_1(s)}{D(s)}$ can be expanded using above

So:

$$Y(s) = \frac{N(s)}{D(s)} = A_0 + \frac{N_1(s)}{D(s)}$$

$$= A_0 + \sum_{\ell=1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

PFE

Where:

$$A_\ell = \left[(s-d_\ell)\frac{N_1(s)}{D(s)}\right]_{s=d_\ell}$$

Inverse transforming:

$$y(t) = \mathcal{L}^{-1}\{A_0\} + \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

What is this?? We'll see later...

Note: $Deg\{N(s)\} > Deg\{D(s)\}$ nonphysical + won't be seen

Repeated Roots

Now suppose:

$$D(s) = (s-d_1)^K (s-d_{K+1}) \cdots (s-d_L)$$

i.e. d_1 is repeated K times, then:

$$Y(s) = \sum_{\ell=1}^{K} \frac{A_\ell}{(s-d_1)^\ell} + \sum_{\ell=K+1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

<u>for $\ell = K+1, \ldots, L$</u> :

$$A_\ell = \left[(s-d_\ell)Y(s)\right]_{s=d_\ell}$$

(unchanged)

<u>for $\ell = 1, \ldots, K$</u> :

$$A_\ell = \frac{1}{(K-\ell)!} \left\{ \frac{d^{K-\ell}}{ds^{K-\ell}} \left[(s-d_1)^K Y(s)\right] \right\}_{s=d_1}$$

(ugh!)

Inverse Transform (Repeated Roots)

$$Y(s) = \sum_{\ell=1}^{K} \frac{A_\ell}{(s-d_1)^\ell} + \sum_{\ell=K+1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

$$\Rightarrow \quad y(t) = \sum_{\ell=1}^{K} \left(\frac{A_\ell t^{\ell-1}}{(\ell-1)!} \right) e^{d_1 t} + \sum_{\ell=K+1}^{L} A_\ell e^{d_\ell t}$$

Example:

$$Y(s) = \frac{2s+1}{(s+1)^3 (s+2)}$$

$d_1 = -1$, $K=3$

$d_4 = -2$

$$\Rightarrow y(t) = \left[A_1 + A_2 t + \frac{A_3}{2} t^2\right] e^{-t} + A_4 e^{-2t}$$

$$A_3 = \left[(s+1)^3 Y(s)\right]_{s=-1} = -1$$

$$A_2 = \left(\frac{1}{1}\right) \left\{ \frac{d}{ds} \left[(s+1)^3 Y(s)\right] \right\}_{s=-1} = \left[\frac{3}{(s+2)^2}\right]_{s=-1} = 3$$

$$A_1 = \left(\frac{1}{2}\right)\left\{\frac{d^2}{ds^2}\left[(s+1)^3 Y(s)\right]\right\}_{s=-1}$$

$$= \left(\frac{1}{2}\right)\left\{\frac{d}{ds}\left[\frac{3}{(s+2)^2}\right]\right\}_{s=-1} = -3$$

And

$$A_4 = \left[(s+2)Y(s)\right]_{s=-2} = 3$$

So finally:

$$y(t) = \left[-3 + 3t - \frac{1}{2}t^2\right]e^{-t} + 3e^{-2t}$$

Note: You aren't responsible for repeated root residue formula. However you should know the general pattern for repeated root solutions.

Alternate System Models

- A dynamical analysis does not always result in a high-order DE Directly connecting $u(t)$ and $y(t)$
- Sometimes the analysis (initially) results in a system of 1st order DEs describing the evolution of the Dynamics
- Each first order equation describes the rate of change of a single physical variable (like airspeed, pitch angle, and angle of attack)
- Generically label these $x_k(t)$ $(k=1 \ldots n)$ Known as the $\boxed{\text{state variables}}$ for the system.

"State variable" form of Dynamics

System of 1^{st} order DEs describing how rate of change in each state depends on other states and forcing input

rate of change of each state

Linear combination of states

effect of input

$$\left\{\begin{aligned}
\dot{x}_1(t) &= \overbrace{a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t)}^{\text{Linear combination of states}} + \overbrace{b_1 u(t)}^{\text{effect of input}} \\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2 u(t) \\
&\vdots \\
\dot{x}_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_n u(t)
\end{aligned}\right.$$

$\rightarrow$ n 1^{st} order DEs

$\Rightarrow$ easier to represent in matrix/vector form

"State-space" Dynamical model

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B u(t)$$ ← "state equation"

with:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (n \times 1)$$

"state vector"

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (n \times n)$$

What about output?

Output $y(t)$ can be any 1 of the states, or any weighted combination of states (and input) as appropriate.

i.e. $y(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t) + D u(t)$

or $y(t) = C \underline{x}(t) + D u(t)$ "output equation"

where $C = [C_1 \; C_2 \; \ldots \; C_n]$ $(1 \times n)$

So complete model is

Standard "state-space" model of dynamics

$$\boxed{\begin{aligned} \underline{\dot{x}}(t) &= A \underline{x}(t) + B u(t) \\ y(t) &= C \underline{x}(t) + D u(t) \end{aligned}}$$

Example from HW #1

Fan: $I\dot{\omega} = K_m i_m - D\omega$

motor: $L\frac{di_m}{dt} = V_m - Ri_m - E\omega$

vehicle: $m\ddot{y} = K_f\omega \Rightarrow m\dot{v} = K_f\omega,\ \dot{y} = \overset{\text{velocity}}{v}$

So:

$$\frac{d}{dt}\overbrace{\begin{bmatrix}\omega\\ i_m\\ v\\ y\end{bmatrix}}^{\underline{x}(t)} = \overbrace{\begin{bmatrix}-D/I & K_m/I & 0 & 0\\ -E/L & -R/L & 0 & 0\\ K_f/m & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}}^{A}\begin{bmatrix}\omega\\ i_m\\ v\\ y\end{bmatrix} + \overbrace{\begin{bmatrix}0\\ 1/L\\ 0\\ 0\end{bmatrix}}^{B}\overset{u}{V_m}$$

$$y(t) = \underbrace{\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}}_{C}\underline{x}(t) + \underbrace{0}_{D}u(t)$$

Where is G(s) for this model?

Not as easy to see transfer function by inspection.

But, we can still use laplace.

Laplace can be applied to vectors too, just apply it to each component of the vector

$$\mathcal{L}\{\underline{x}(t)\} = \underline{X}(s) = \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix} = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$

Linearazity: $\mathcal{L}\{A\underline{x}_1(t) + B\underline{x}_2(t)\} = A\underline{x}_1(s) + B\underline{x}_2(s)$

Derivative rule

$$\mathcal{L}\{\dot{\underline{x}}(t)\} = \begin{bmatrix} sX_1(s) - X_1(0) \\ sX_2(s) - X_2(0) \\ \vdots \\ sX_n(s) - X_n(0) \end{bmatrix} = s\underline{X}(s) - \underline{X}_0$$

Apply Laplace to state space model

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B u(t)$$
$$y(t) = C\underline{x}(t) + D u(t)$$

$$\underline{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

Initial state values

$$\Rightarrow \quad s\underline{x}(s) - \underline{x}_0 = A\underline{x}(s) + B u(s)$$
$$y(s) = C\underline{x}(s) + D u(s)$$

1st eq'n is equivalent to:

$$(sI - A)\underline{x}(s) = \underline{x}_0 + B u(s)$$

(I = n×n identity)

$$\Rightarrow \quad \underline{x}(s) = [sI - A]^{-1}[\underline{x}_0 + B u(s)]$$

Substitute into 2nd eq'n:

$$\boxed{y(s) = C[sI - A]^{-1}\underline{x}_0 + [C(sI-A)^{-1}B + D]\,U(s)}$$

$$y(s) = \overbrace{\underbrace{C[sI-A]^{-1}\underline{x}_0}_{\text{effect of ICs}}}^{} + \overbrace{\underbrace{[C(sI-A)^{-1}B+D]U(s)}_{\text{effect of input}}}^{}$$

Recall: TF derived assuming ICs $=0 \Rightarrow \underline{x}_0 = \underline{0}$

Then

$$y(s) = \overbrace{\boxed{[C(sI-A)^{-1}B+D]}}^{G(s)}U(s)$$

Hence, for any (A,B,C,D) state space representation the corresponding transfer function is:

$$G(s) = [C\overbrace{(sI-A)^{-1}}^{n\times n\text{ matrix inverse}}B+D]$$

Now recall for arbitrary matrix M

$$M^{-1} = \frac{\mathrm{Adj}(M)}{\mathrm{Det}(M)}$$

Adj = $n\times n$ Matrix of cofactors

Det = <u>Scalar</u> Determinant

Thus

$$(s\mathbb{I}-A)^{-1} = \frac{Q(s)}{r(s)}$$

where

$$\left[\begin{array}{l} Q(s) = \mathrm{Adj}(s\mathbb{I}-A) \quad (n\times n \ \text{matrix.}) \\ r(s) = \mathrm{Det}(s\mathbb{I}-A) \quad \underline{\text{polynomial}} \text{ in } s. \end{array}\right.$$

and

$$G(s) = \frac{CQ(s)B}{r(s)} + D = \frac{CQ(s)B + Dr(s)}{r(s)}$$

where both $CQ(s)B$ and $r(s)$ are polynomials

$\Rightarrow$ zeros where $\underbrace{\boxed{CQ(s)B + Dr(s)}}_{q(s)} = 0$

$\Rightarrow$ poles where $r(s) = 0$.

So the poles of $G(s)$ will satisfy

$$r(s) = \emptyset = \mathrm{Det}(s\mathbb{I} - A)$$

$\Rightarrow (s\mathbb{I} - A)$ is singular, at these values of s i.e. there exists nonzero $\underline{v}$

so that

$$(s\mathbb{I} - A)\underline{v} = \underline{0}$$

(singular matrices have nontrivial nullspace)

or:

$$A\underline{v} = s\underline{v} \quad \text{for any } s \text{ with } r(s) = 0$$

$$\boxed{\Rightarrow \text{poles of } G(s) \text{ are } \boxed{\underline{\text{eigenvalues}}} \text{ of } A!!}$$