PHYS 313

HW 02: Assignment 2

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Problem 1.39:

1. Check the divergence theorem for the function $\vec{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.
2. Do the same for $\vec{v}_2 = \frac{1}{r^2}\hat{\mathbf{r}}$.

Solution

Part A

$$\vec{v}_1 = r^2\,\hat{r}$$

$$\nabla \cdot \vec{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\,(r^2)\Big) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) = \frac{4r^3}{r^2} = 4r.$$

$$\begin{aligned}\int_V (\nabla \cdot \vec{v}_1) d\tau &= 4\int_0^R r\,(r^2 \sin\theta\, dr\, d\theta\, d\phi) \\ &= 4\int_0^R r^3\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \\ &= 4\left(\frac{R^4}{4}\right)(2)(2\pi) = 4\pi R^4.\end{aligned}$$

On the surface: $\vec{v}_1 \cdot \hat{n} = R^2, \quad dA = R^2 \sin\theta\, d\theta\, d\phi.$

$$\Phi = \int_S R^2\,(R^2 \sin\theta\, d\theta\, d\phi) = R^4 \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi R^4.$$

Part B

$$\vec{v}_2 = \frac{1}{r^2}\,\hat{r}$$

$$\nabla \cdot \vec{v}_2 = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2 \frac{1}{r^2}\Big) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$$

$$\int_V (\nabla \cdot \vec{v}_2) d\tau = 0.$$

However, on the surface: $v_r = \frac{1}{R^2}, \quad \Phi = \frac{1}{R^2}(4\pi R^2) = 4\pi.$

The discrepancy is due to the singularity at $r = 0$.

Problem 1.43:

1. Find the divergence of the function

$$\vec{v} = s\left(2 + \sin^2\phi\right)\hat{\mathbf{s}} + s\sin\phi\cos\phi\hat{\boldsymbol{\phi}} + 3z\hat{\mathbf{z}}.$$

2. Test the divergence theorem for this function, using a quarter-cylinder with radius $r = 2, h = 5$.
3. Find the curl of $\vec{v}$.

Solution

Part A

$$\vec{v} = s\left(2 + \sin^2\phi\right)\hat{\mathbf{s}} + s\sin\phi\cos\phi\hat{\boldsymbol{\phi}} + 3z\hat{\mathbf{z}}.$$

$$\nabla \cdot \vec{v} = \frac{1}{s}\frac{\partial}{\partial s}\Big(s\,v_s\Big) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

$$s\,v_s = s^2(2 + \sin^2\phi), \quad \frac{\partial}{\partial s}\Big(s^2(2 + \sin^2\phi)\Big) = 2s(2 + \sin^2\phi).$$

$$\frac{1}{s}\frac{\partial}{\partial s}\Big(s\,v_s\Big) = 2(2 + \sin^2\phi).$$

$$\frac{\partial v_\phi}{\partial \phi} = s(\cos^2\phi - \sin^2\phi) = s\cos 2\phi, \quad \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} = \cos 2\phi.$$

$$\frac{\partial v_z}{\partial z} = 3.$$

$$\nabla \cdot \vec{v} = 2(2 + \sin^2\phi) + \cos 2\phi + 3.$$

$$\text{Using } \cos 2\phi = 1 - 2\sin^2\phi, \quad 2(2 + \sin^2\phi) = 4 + 2\sin^2\phi.$$

$$\nabla \cdot \vec{v} = 4 + 2\sin^2\phi + 1 - 2\sin^2\phi + 3 = 8.$$

Part B

$$\text{Volume of quarter-cylinder: } V = \frac{1}{4}\pi(2)^2(5) = 5\pi.$$

$$\int_V (\nabla \cdot \vec{v})d\tau = 8(5\pi) = 40\pi.$$

Thus, the net flux over the surface is 40π.

Part C

$$\nabla \times \vec{v} = \begin{pmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{1}{s}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ v_s & v_\phi & v_z \end{pmatrix}.$$

$$(\nabla \times \vec{v})_z = \frac{1}{s}\left[\frac{\partial}{\partial s}(s\,v_\phi) - \frac{\partial v_s}{\partial \phi}\right].$$

$$s\,v_\phi = s^2\sin\phi\cos\phi, \quad \frac{\partial}{\partial s}(s^2\sin\phi\cos\phi) = 2s\sin\phi\cos\phi.$$

$$\frac{\partial v_s}{\partial \phi} = \frac{\partial}{\partial \phi}\Big(s(2 + \sin^2\phi)\Big) = 2s\sin\phi\cos\phi.$$

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$$(\nabla \times \vec{v})_z = \frac{1}{s}(2s \sin\phi \cos\phi - 2s \sin\phi \cos\phi) = 0.$$

$$(\nabla \times \vec{v})_s = \frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} = 0, \quad (\nabla \times \vec{v})_\phi = \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} = 0.$$

$$\therefore \nabla \times \vec{v} = \vec{0}.$$

Problem 1.47:

1. Write an expression for the volume charge density of $\rho(\vec{r})$ of a point charge q at $\vec{r}'$. Make sure that the volume integral of ρ equals q.
2. What is the volume charge desnity of an electric dipole, consisting of a point charge $-q$ at the origin at a point charge $+q$ at $\vec{a}$?
3. What is the volume charge density (in spherical coordinates) of a uniform, infinitesimally thin spherical shell of radius R and total charge Q, centered at the origin?

Solution Part A

$$\rho(\vec{r}) = q\,\delta^3(\vec{r} - \vec{r}\,').$$

$$\int \rho(\vec{r})\,d\tau = q.$$

Part B

$$\rho(\vec{r}) = -q\,\delta^3(\vec{r}) + q\,\delta^3(\vec{r} - \vec{a}).$$

Part C

$$\rho(r, \theta, \phi) = \frac{Q}{4\pi R^2}\,\delta(r - R).$$

Problem 1.48:

Evaluate the following integrals:

1. $\int \left(r^2 + \mathbf{r}\cdot\mathbf{a} + a^2\right)\delta^3(\mathbf{r}-\mathbf{a})d\tau$, where $\mathbf{a}$ is a fixed vector, a is its magnitude, and the integral is over all space.
2. $\int_{\mathcal{V}} |\mathbf{r}-\mathbf{b}|^2\delta^3(5\mathbf{r})d\tau$, where $\mathcal{V}$ is a cube of side 2 , centered on the origin, and $\mathbf{b} = 4\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$.
3. $\int_{\mathcal{V}} \left[r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4\right]\delta^3(\mathbf{r}-\mathbf{c})d\tau$, where $\mathcal{V}$ is a sphere of radius 6 about the origin, $\mathbf{c} = 5\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$, and c is its magnitude.
4. $\int_{\mathcal{V}} \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})d\tau$, where $\mathbf{d} = (1,2,3), \mathbf{e} = (3,2,1)$, and $\mathcal{V}$ is a sphere of radius 1.5 centered at $(2,2,2)$.

Solution Part A

$$I = \int \left(r^2 + \mathbf{r}\cdot\mathbf{a} + a^2\right)\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = \left(a^2 + \mathbf{a}\cdot\mathbf{a} + a^2\right) = 3a^2.$$

Part B

$$\delta^3(5\mathbf{r}) = \frac{1}{5^3}\delta^3(\mathbf{r}) = \frac{1}{125}\delta^3(\mathbf{r}).$$

$$I = \int_{\mathcal{V}} |\mathbf{r}-\mathbf{b}|^2\,\delta^3(5\mathbf{r})\,d\tau = \frac{1}{125}\,|\mathbf{0}-\mathbf{b}|^2 = \frac{b^2}{125}.$$

$$b^2 = 4^2 + 3^2 = 16 + 9 = 25, \quad I = \frac{25}{125} = \frac{1}{5}.$$

Part C

$$I = \int_{\mathcal{V}} \left[r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4\right]\delta^3(\mathbf{r}-\mathbf{c})\,d\tau = \left[c^4 + c^2(\mathbf{c}\cdot\mathbf{c}) + c^4\right] = 3c^4.$$

Part D

$$I = \int_{\mathcal{V}} \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})\,d\tau = \mathbf{e}\cdot(\mathbf{d}-\mathbf{e}).$$

$$\mathbf{d} = (1,2,3), \quad \mathbf{e} = (3,2,1),$$

$$\mathbf{e}\cdot\mathbf{d} = 3\cdot 1 + 2\cdot 2 + 1\cdot 3 = 10,$$

$$\mathbf{e}\cdot\mathbf{e} = 3^2 + 2^2 + 1^2 = 14,$$

$$I = 10 - 14 = -4.$$

Problem 2.1:

1. Twelve equal charges, q, are situated at the corners of a regular 12-sided polygon (for instance, on on each numeral of a clock face). What is the net force on a test charge Q at the center?
2. Suppose *one* of the 12 qs is removed (the one at "6 o'clock"). What is the force on Q? Explain your reasoning.
3. Now 13 equal charges, q, are situated at the corners of a regular 13-sided polygon. What is the net force on a test charge Q at the center?
4. If one of the 13 qs is removed, what is the force on Q? Explain your reasoning.

Solution

Part A

$$\text{For 12 charges symmetrically arranged: } \vec{F} = \vec{0}.$$

Part B

$$\text{Removing one charge: } F = \frac{k\,Qq}{R^2} \quad \text{(direction opposite to the missing charge).}$$

Part C

$$\text{For 13 charges symmetrically arranged: } \vec{F} = \vec{0}.$$

Part D

$$\text{Removing one charge: } F = \frac{k\,Qq}{R^2} \quad \text{(direction opposite to the missing charge).}$$

Problem 2.2:

Find the electric field (magnitude and direction) a distance z above the midpoint between equal and opposite charges ($\pm q$), a distance d apart.

Solution

$$\text{Let the charges be at } \left(\pm\frac{d}{2}, 0, 0\right).$$

$$E_z = \frac{1}{4\pi\epsilon_0}\left[\frac{q\,z}{\left(\left(\frac{d}{2}\right)^2 + z^2\right)^{3/2}} - \frac{(-q)\,z}{\left(\left(\frac{d}{2}\right)^2 + z^2\right)^{3/2}}\right] = \frac{1}{4\pi\epsilon_0}\frac{2q\,z}{\left(\left(\frac{d}{2}\right)^2 + z^2\right)^{3/2}}.$$

$$\vec{E} = \frac{qz}{2\pi\epsilon_0\left[\left(\frac{d}{2}\right)^2 + z^2\right]^{3/2}}\hat{z}.$$