Lecture 21: Fanno Flow

ENAE311H Aerodynamics I

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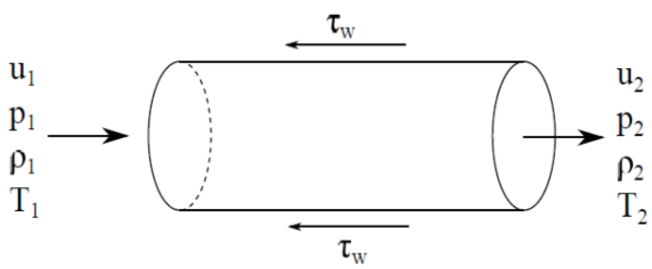
Governing equations – mass and energy

Now consider the one-dimensional flow inside a constant-area duct, but instead of heat addition or subtraction, we have a nonzero shear stress acting at the walls. We again assume this process to be somewhat idealized in that the frictional effects are experienced immediately across the duct area at any location downstream.



The conservation of mass and energy equations can be written

$$\text{Mass:} \qquad \rho_1 u_1 = \rho_2 u_2$$

$$\text{Energy:} \qquad h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2},$$

or simply, $\rho u = \text{const.}$, and $T_0 = \text{const.}$

Note that these are again the same equations as for a normal shock.

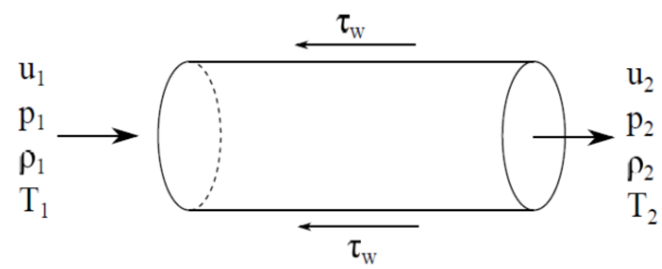
Conservation of momentum

If we draw a control volume around the geometry to the right and apply conservation of momentum, we have

$$\rho_2 u_2^2 A - \rho_1 u_1^2 A = p_1 A - p_2 A - \iint_{wall} \tau_w dA$$

Using continuity, we can write this as

$$\rho_1 u_1 (u_2 - u_1) = p_1 - p_2 - \frac{1}{A} \iint_{wall} \tau_w dA$$



If we now let the distance between 1 and 2 become small, we can write this in differential form:

$$\rho u \, du = -dp - \frac{1}{A} \tau_w dA_w,$$

where dA_w is the wetted area of the duct between 1 and 2.

Now we introduce the following definitions

friction coefficient

$$D \equiv \frac{4A}{dA_w/dx} = \frac{4A}{dAw} dx \qquad \text{and} \qquad f \equiv \frac{2\tau_w}{\rho u^2}.$$

hydraulic diameter

Conservation of momentum

Our momentum equation can then be written

$$\rho u^2 \frac{du}{u} + dp = -4f \frac{\rho u^2}{2} \frac{dx}{D}.$$

Dividing through by p and using $\rho u^2 = \gamma p M^2$, we can write this as

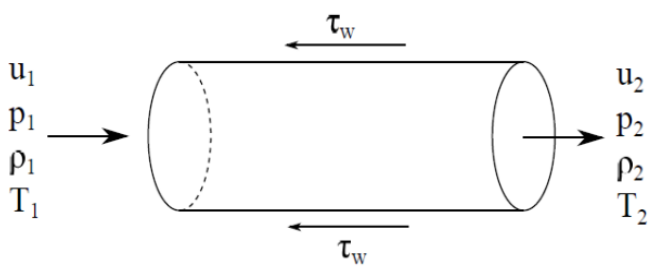
$$\gamma M^2 \frac{du}{u} + \frac{dp}{p} = -4f \frac{\gamma M^2}{2} \frac{dx}{D}.$$

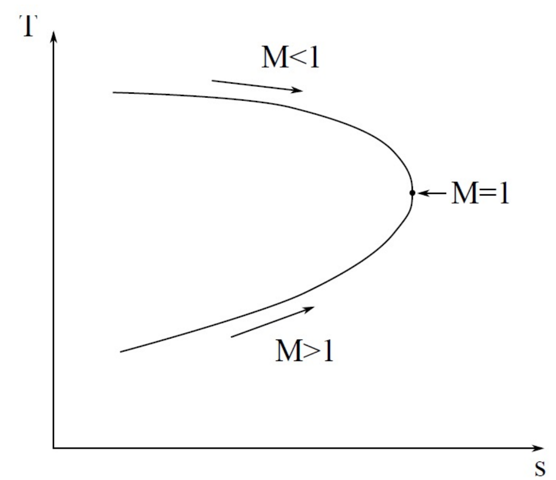
After much manipulation and use of the other conservation equations, we arrive at

$$4f \frac{dx}{D} = \frac{1}{\gamma M^2} \frac{1 - M^2}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM^2}{M^2}.$$

From this equation we see that friction always acts to push the flow towards sonic conditions (whether it is initially subsonic or supersonic).

This we also see in the $T - s$ diagram for Fanno flow, which also shows that the sonic point is the point of maximum entropy.





Frictional choking

Since the sonic point is the maximum-entropy point, no solution can exist beyond it. If it is reached, we say the flow is *frictionally choked*.

Therefore, given initial conditions and a friction coefficient, f, there is a maximum length the duct can have before the sonic point is reached. This can be determined by integrating

$$\int_0^{L_{max}} 4f\frac{dx}{D} = \int_{M^2}^{1} \frac{1}{\gamma M^4}\frac{1-M^2}{1+\frac{\gamma-1}{2}M^2}dM^2$$

to give

$$\boxed{4\overline{f}\frac{L_{max}}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)},}\qquad \text{with } \overline{f} = \frac{1}{L_{max}}\int_0^{L_{max}} f\,dx.$$

If no better estimate of f is available, e.g., from a Moody chart, a reasonable value for a smooth pipe is $f = 0.0025$. With this value and $\gamma = 1.4$, we calculate:

M	0	0.25	0.5	0.75	1	1.5	2	3	∞
L_{max}/D	∞	850	110	12	0	14	31	52	82

So frictional effects are particularly severe for supersonic flows.

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with $\overline{f} = \frac{1}{L_{max}}\int_0^{L_{max}} f\,dx.$

We can also use this equation to solve for the Mach number at a second point down the duct with our frictional flow. In particular,

$$\boxed{\frac{4\overline{f}}{D}\Delta x = -\frac{4\overline{f}}{D}(L_{max,2} - L_{max,1}) = g(M_1) - g(M_2),}$$

with

$$g(M) = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)}$$

Flow conditions in Fanno flow

Assuming we now know the Mach number at some downstream station (together with upstream conditions) we can determine the remaining downstream flow properties as follows.

We know that the total temperature is constant, so can write

$$\boxed{\frac{T_2}{T_1} = \frac{T_2}{T_0}\frac{T_0}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}.}$$

Now, using continuity, we have

$$\frac{p_1}{RT_1}u_1 = \frac{p_2}{RT_2}u_2.$$

which we can write as

$$\sqrt{\frac{\gamma}{RT_1}}p_1\frac{u_1}{\sqrt{\gamma RT_1}} = \sqrt{\frac{\gamma}{RT_2}}p_2\frac{u_2}{\sqrt{\gamma RT_2}},$$

and rearrange to give

$$\frac{p_2}{p_1} = \sqrt{\frac{T_2}{T_1}}\frac{M_1}{M_2}.$$

Substituting in our temperature ratio we have

$$\boxed{\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{1/2}.}$$

Again using the ideal gas equation, we can derive the density ratio from the pressure and temperature:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right)^{1/2}.}$$

Finally, for the stagnation-pressure ratio:

$$\boxed{\frac{p_{02}}{p_{01}} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}.}$$

Sonic-referenced conditions

As with Rayleigh flow, it is customary to take the sonic conditions as reference flow values for normalization, in which case we have:

$$
\begin{aligned}
\frac{p}{p^*} &= \frac{1}{M}\left(\frac{\gamma+1}{2+(\gamma-1)M^2}\right)^{1/2} \\
\frac{\rho}{\rho^*} &= \frac{1}{M}\left(\frac{2+(\gamma-1)M^2}{\gamma+1}\right)^{1/2} = \frac{u^*}{u} \\
\frac{T}{T^*} &= \frac{\gamma+1}{2+(\gamma-1)M^2} = \frac{a^2}{a^{*2}} \\
\frac{p_0}{p_0^*} &= \frac{1}{M}\left(\frac{2+(\gamma-1)M^2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} .
\end{aligned}
$$

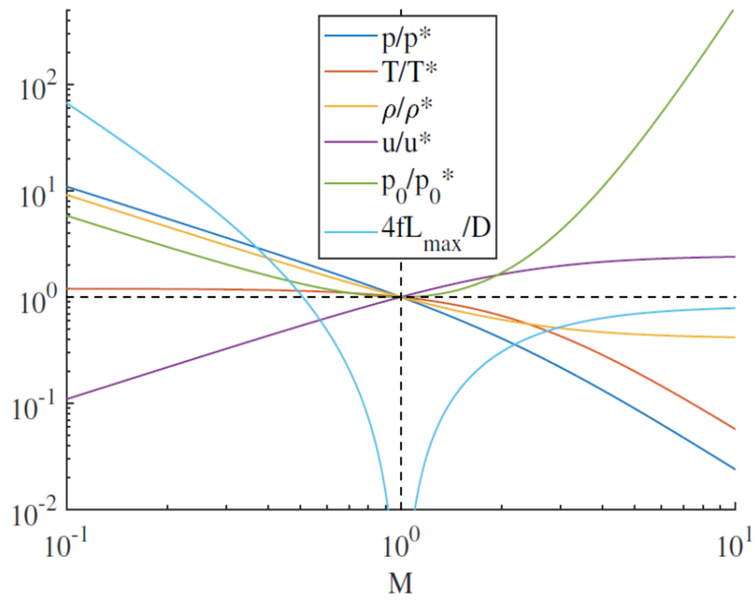
We can then again relate conditions at any two points in the flow using, for example,

$$
\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1}.
$$

Behavior or flow variables in Fanno flow

The trends in the flow properties for increasing distance downstream can be summarized as follows:

	$M < 1$	$M > 1$
M	increases	decreases
T	decreases	increases
p	decreases	increases
ρ	decreases	increases
p_0	decreases	decreases
u	increases !	decreases



	Flow	
Parameter	**Subsonic Flow**	**Supersonic Flow**
Stagnation temperature	Constant	Constant
Ma	Increases (maximum is 1)	Decreases (minimum is 1)
Friction	Accelerates flow	Decelerates flow
Pressure	Decreases	Increases
Temperature	Decreases	Increases