Step Responses

The (unit) <u>step response</u> of a system is the output $y(t)$ when $u(t) = \mathbb{1}(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[\cancel{c}^{\,0}(s) - \cancel{b}^{\,0}(s)]}{r(s)}$$

$U(s) = \frac{1}{s}$ here, so

$$Y(s) = \left(\frac{1}{s}\right)G(s) = \frac{q(s)}{s\,r(s)}$$

General Thoughts about step responses

① Every system has a unit step response:

$$Y(s) = \left[\left(\frac{1}{s}\right) G(s)\right]$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} G(s)\right\} \triangleq y_{us}(t)$$

Find $y_{us}(t)$ as usual by partial fraction expansion and inverse transform of each term

However, we want to be able to predict main features of $y_{us}(t)$ by inspection for 1^{st} and 2^{nd} order systems

$\Rightarrow$ Very common special cases

$\Rightarrow$ "Building blocks" for more complex systems

② (Use of linearity, I)

$$u(t) = c\,\mathbb{1}(t) \implies y(t) = c\, y_{us}(t)$$

All y(t) values are the unit step values multiplied by c.

Equivalent to "rescaling" vertical axis on plot of y(t), however horizontal (time) axis is unaffected

$\Rightarrow$ Characteristic times (t_s, t_r, t_p) are unaffected

We'll encounter these shortly.

$\Rightarrow$ Corresponding y(t) values scaled by c:

$$y_{ss} = c\,G(0), \quad y_p = c\,G(0)[1 + M_p]$$

$\Rightarrow$ True for any c, positive or negative

③ (Use of Linearity, II)

By definition, unit step response assumes all ICs are zero.

However, can easily "Add on" effects of nonzero ICs.

Nonzero Now

$$Y(s) = \left[\frac{1}{s}G(s)\right] + \left[\frac{c(s)}{r(s)}\right]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right)G(s)\right\} + \mathcal{L}^{-1}\left\{\frac{c(s)}{r(s)}\right\}$$

$$= y_{us}(t) + \mathcal{L}^{-1}\left\{\frac{c(s)}{r(s)}\right\}$$

Added terms from ICs

Solve for last term by PFE

Effect of added terms on t_s, t_p, y_p etc depends on specific ICs. No simple formulae to quantify their effects.

"1st Order" Responses

$$\dot{y}(t) + \alpha_0 y(t) = \beta_0 u(t) \implies G(s) = \frac{\beta_0}{s+\alpha_0}$$

Single real pole at $P_1 = -\alpha_0$ (stable if $\alpha_0 > 0$)

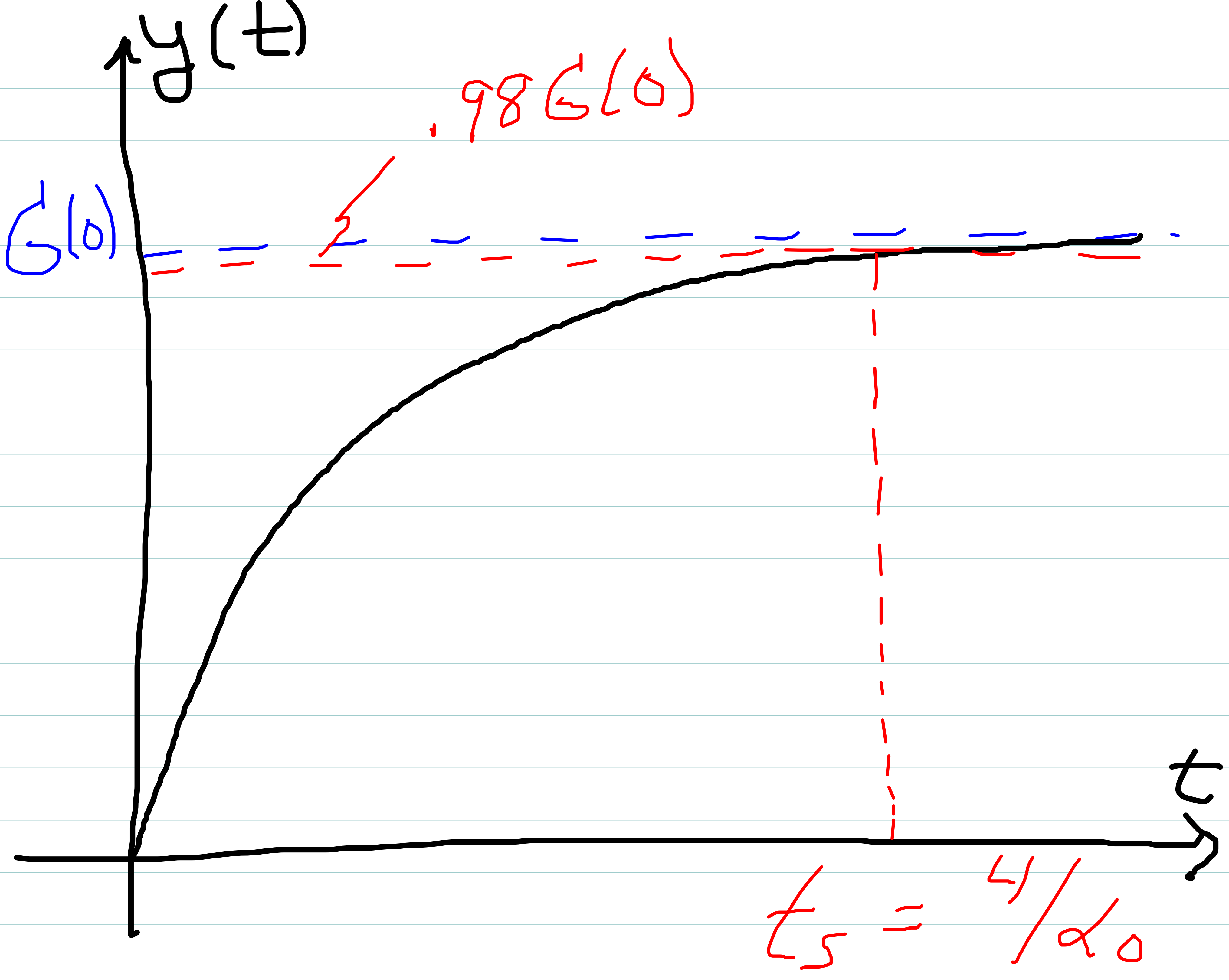
$$Y(s) = \frac{\beta_0}{s(s+\alpha_0)} = \frac{A_1}{s} + \frac{A_2}{s+\alpha_0}$$

$$A_1 = [sY(s)]_{s=0} = \frac{\beta_0}{\alpha_0} = G(0)$$

$$A_2 = [(s+\alpha_0)Y(s)]_{s=-\alpha_0} = \frac{-\beta_0}{\alpha_0} = -G(0)$$

Thus:

$$\boxed{y(t) = G(0)\left[1 - e^{-\alpha_0 t}\right]}$$



y(t)
.98G(0)
G(0)
t
$t_s = 4/\alpha_0$

Notes

1. Response asymptotically approaches steady-state

$$y_{ss}(t) = G(0)$$ (as expected)

2. Response **never** crosses its steady-state

3. Response **settles** within 2% of its steady-state in

$$t_s = \frac{4}{|\mathrm{Re}\{p\}|} = \frac{4}{\alpha_0}$$

4. "Shape" of graph is same for any 1st order system

Responses only differ by:

- steady-state level, $G(0)$
- settling time, t_s

"2^{nd} Order" Step Responses

$$\ddot{y}(t) + \alpha_1 \dot{y}(t) + \alpha_0 y(t) = \beta_0 u(t) \implies G(s) = \frac{\beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

2 poles, both stable if $\alpha_1 > 0, \alpha_0 > 0$.

3 possibilities for poles:

1. $\alpha_1^2 < 4\alpha_0 \implies p_1, p_2$ complex conjugates
2. $\alpha_1^2 = 4\alpha_0 \implies p_1 = p_2$ repeated real
3. $\alpha_1^2 > 4\alpha_0 \implies p_1, p_2$ real, non-repeated

Case 1. is most interesting (and complicated)
tackle this after the other two

2^{nd} order response, Case 2

$$G(s) = \frac{\beta_0}{s^2+\alpha_1 s+\alpha_0} \qquad \alpha_1^2 = 4\alpha_0 \quad (\xi = 1)$$

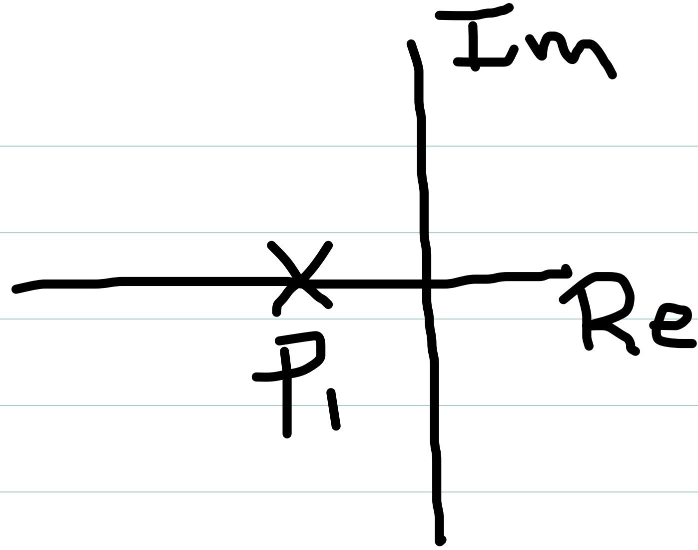
$$= \frac{\beta_0}{(s-p_1)^2} \qquad \text{repeated real pole}$$

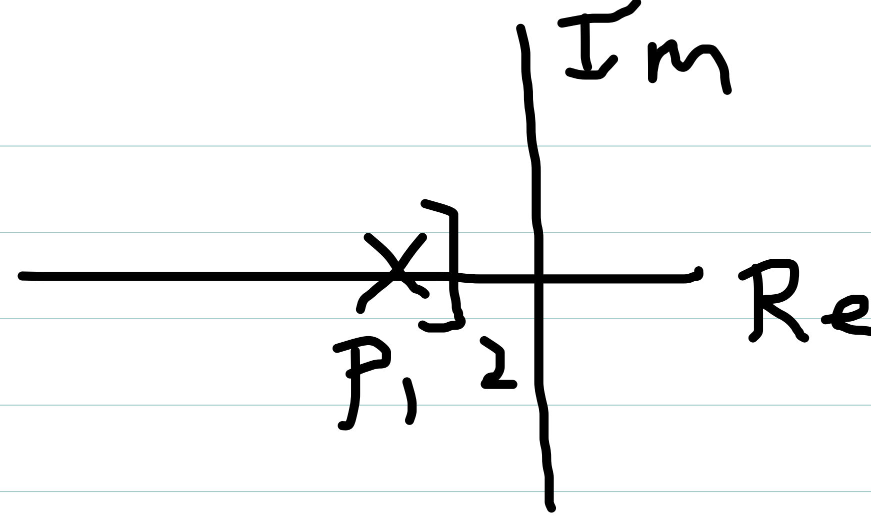
$$Y(s) = \left(\frac{1}{s}\right)G(s) = \frac{A_1}{s} + \frac{A_2}{(s-p_1)} + \frac{A_3}{(s-p_1)^2}$$

$$y(t) = G(\phi) + [A_2 + A_3 t]e^{p_1 t}$$

Non-oscillatory, since poles are real

features resemble 1^{st} order response
(No overshoot, $y_{ss} = G(0)$ approached asymptotically from below), but $\boxed{t_s \text{ 50\% longer } \left(\frac{6}{|p_1|}\right)}$

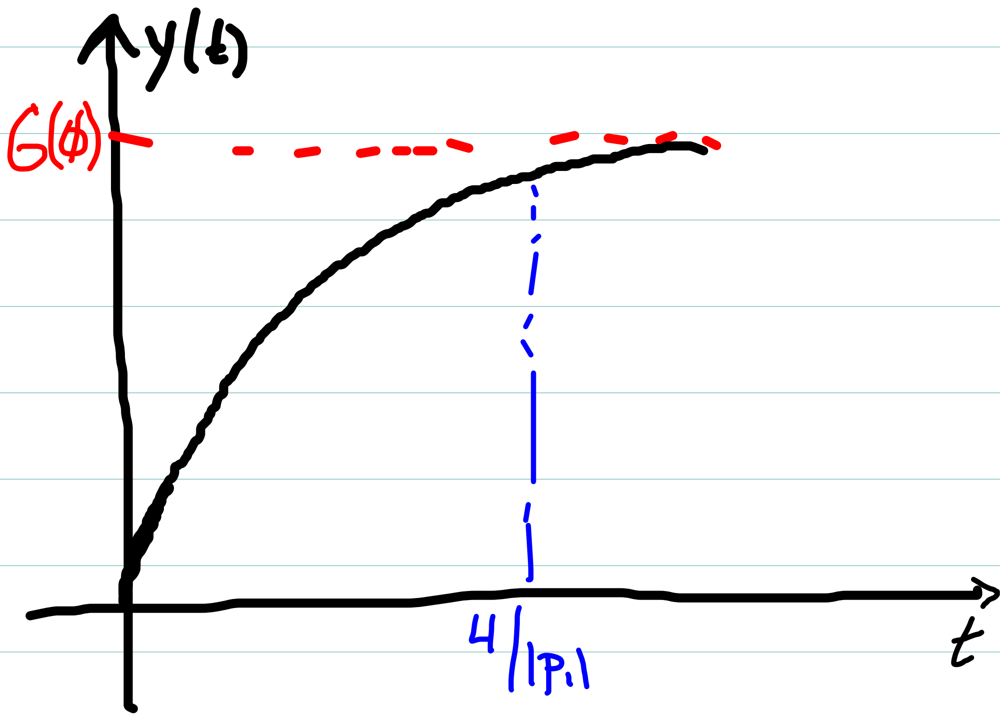


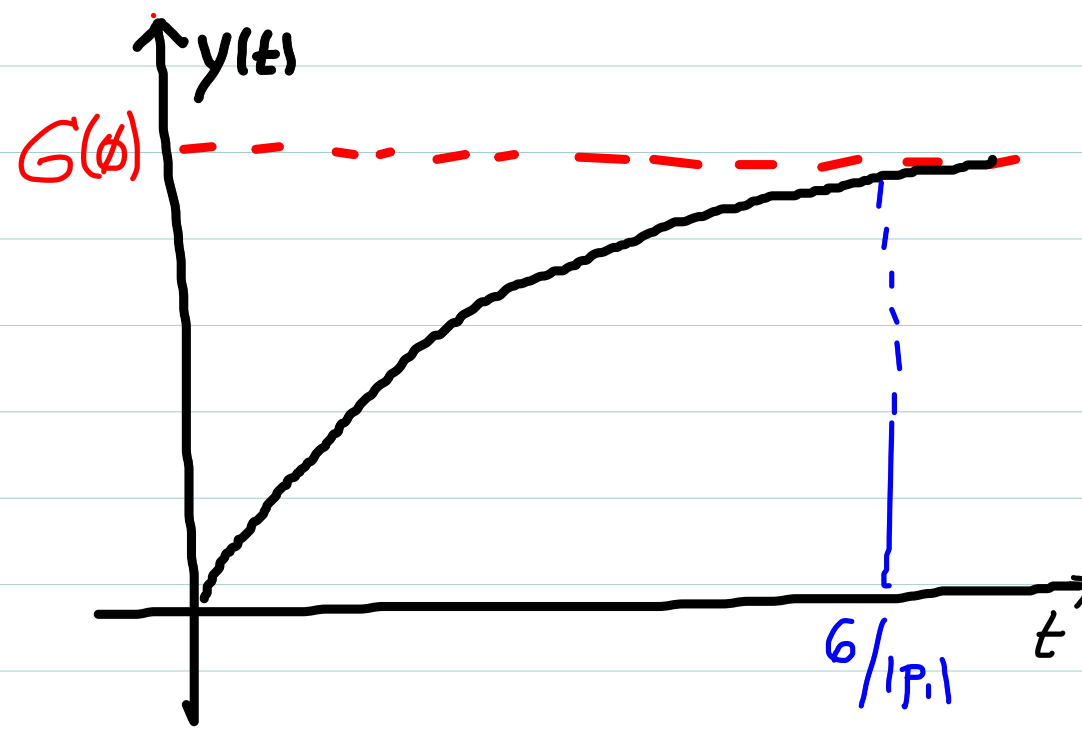


1^{st} order

2^{nd} order, repeated real

Add'l $te^{p_1 t}$ term "slows down" response.





2^{nd} order Response, Case 3

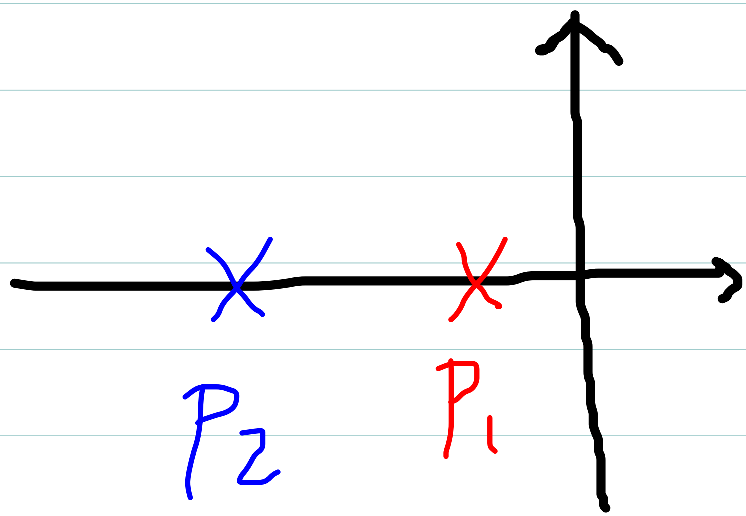
$\alpha_1^2 > 4\alpha_0$

$$Y(s) = \frac{\beta_0}{s(s-p_1)(s-p_2)} \qquad p_1 \neq p_2 .$$

$$\Rightarrow y(t) = G(0) + A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

Assume for notation sake that poles are numbered so that

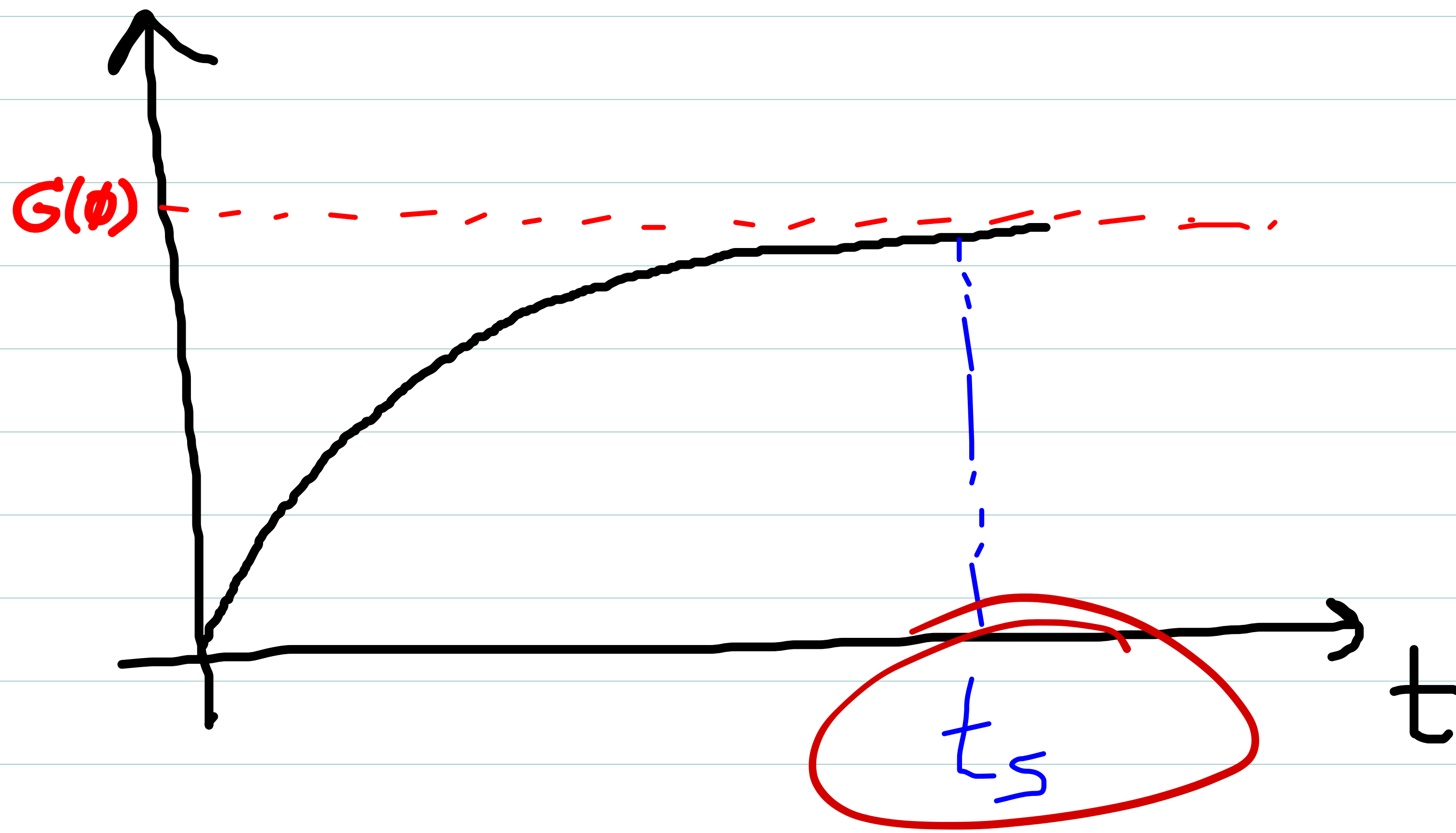
$$p_2 < p_1 \qquad (\Rightarrow |p_2| > |p_1| \text{ since } p_1, p_2 \text{ assumed negative})$$



p_1 is the "slow pole"

p_2 is the "fast pole"

General sol'n again resembles 1^{st} order response



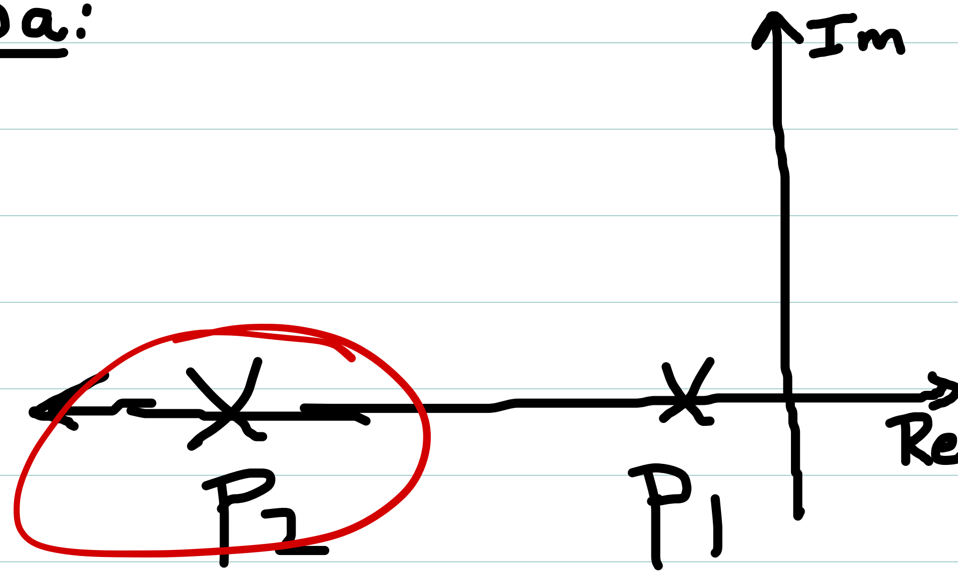
t_s difficult to quantify precisely for arbitrary p_1, p_2

Two Limiting cases:

<u>Case 3a:</u> $|p_2| \gg |p_1|$

<u>Case 3b:</u> $|p_2| \approx |p_1|$

Case 3a:



$y(t) = G(\phi) + A_1 e^{p_1 t} + A_2 e^{p_2 t}$

$|p_2| >> |p_1|$

$\Rightarrow p_2$ much further into LHP than p_1

$\Rightarrow e^{p_2 t} \to \phi$ much faster than $e^{p_1 t}$

$\Rightarrow e^{p_1 t}$ controls settling time ("slow pole")

So $t_s \approx \frac{4}{|p_1|}$ in this case

$\Rightarrow$ Corresponds with previous "1st cut" of approximating system settling time with settling time of slowest mode.

Dominant Modes

When $|p_2| \gg |p_1|$ we say that mode $e^{p_1 t}$ "dominates" transient response, or that $e^{p_1 t}$ ("slow mode") is the

Dominant mode

What is a sufficient separation for a mode to be dominant

Generally, if $\boxed{|p_2| > 5|p_1| \quad \text{or} \quad |p_2| > 10|p_1|}$

i.e. if p_2 is 5-10 times further into LHP

$\Rightarrow$ setting time of $e^{p_2 t}$ 5-10 times faster than that of $e^{p_1 t}$

(5 is usually sufficient. Some authors use 8 or even 10)

Case 3b

$|p_2| \approx |p_1| \implies p_2 \approx p_1$ Poles are "nearly" repeated

Here it is best to approximate the settling time as though the poles were actually repeated

$$t_s \approx \frac{6}{|p_1|}$$

Simple rule of thumb for this:

$$1 \leq \frac{|p_2|}{|p_1|} \leq 1.1$$

Intermediate Case 3 Situations

If $1.1 < \frac{|P_2|}{|P_1|} < 5$ (or 8 or 10)

$$\frac{4}{|P_1|} < t_s < \frac{6}{|P_1|}$$

Unfortunately, there is no simple formula for interpolating between the two limits based on the exact ratio.

"2^{nd} Order" Step Responses

$\ddot{y}(t) + \alpha_1 \dot{y}(t) + \alpha_0 y(t) = \beta_0 u(t) \Rightarrow G(s) = \dfrac{\beta_0}{s^2 + \alpha_1 s + \alpha_0}$

2 poles, both stable if $\alpha_1 > 0, \alpha_0 > 0$.

3 possibilities for poles:

(1.) $\alpha_1^2 < 4\alpha_0 \Rightarrow p_1, p_2$ complex conjugates

(2.) $\alpha_1^2 = 4\alpha_0 \Rightarrow p_1 = p_2$ repeated real

(3.) $\alpha_1^2 > 4\alpha_0 \Rightarrow p_1, p_2$ real, non-repeated

Case (1.) is most interesting (and complicated)
tackle this after the other two

Useful Observation (Case 1)

$p_1 = \sigma + j\omega_d \qquad \omega_d = \mathrm{Im}\{p_1\}$

Note slight change of notation! $\omega \to \omega_d$

$$s^2 + \alpha_1 s + \alpha_0 = (s - p_1)(s - \overline{p}_1)$$

$$= s^2 - (p_1 + \overline{p}_1)s + p_1\overline{p}_1$$

$$= s^2 - 2\mathrm{Re}\{p_1\}s + |p_1|^2$$

$$= s^2 - 2\sigma s + (\sigma^2 + \omega_d^2)$$

Hence:

$$\boxed{\begin{aligned} \alpha_1 &= -2\sigma = -2\mathrm{Re}\{p_1\} \\ \alpha_0 &= \sigma^2 + \omega_d^2 = |p_1|^2 \end{aligned}}$$

Rapidly identify pole location from coefs.

2^{nd} Order Response, Case 1:

$$Y(s) = \frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)} = \frac{A_1}{s} + \frac{A_2}{(s-p_1)} + \frac{\bar{A}_2}{(s-\bar{p}_1)}$$

$$A_1 = [sY(s)]_{s=0} = \frac{\beta_0}{p_1\bar{p}_1} = \frac{\beta_0}{\alpha_0} = G(0)$$

$$A_2 = [(s-p_1)Y(s)]_{s=p_1} = \frac{\beta_0}{p_1(p_1-\bar{p}_1)} = \frac{\beta_0}{(\sigma+j\omega_d)(2j\omega_d)}$$

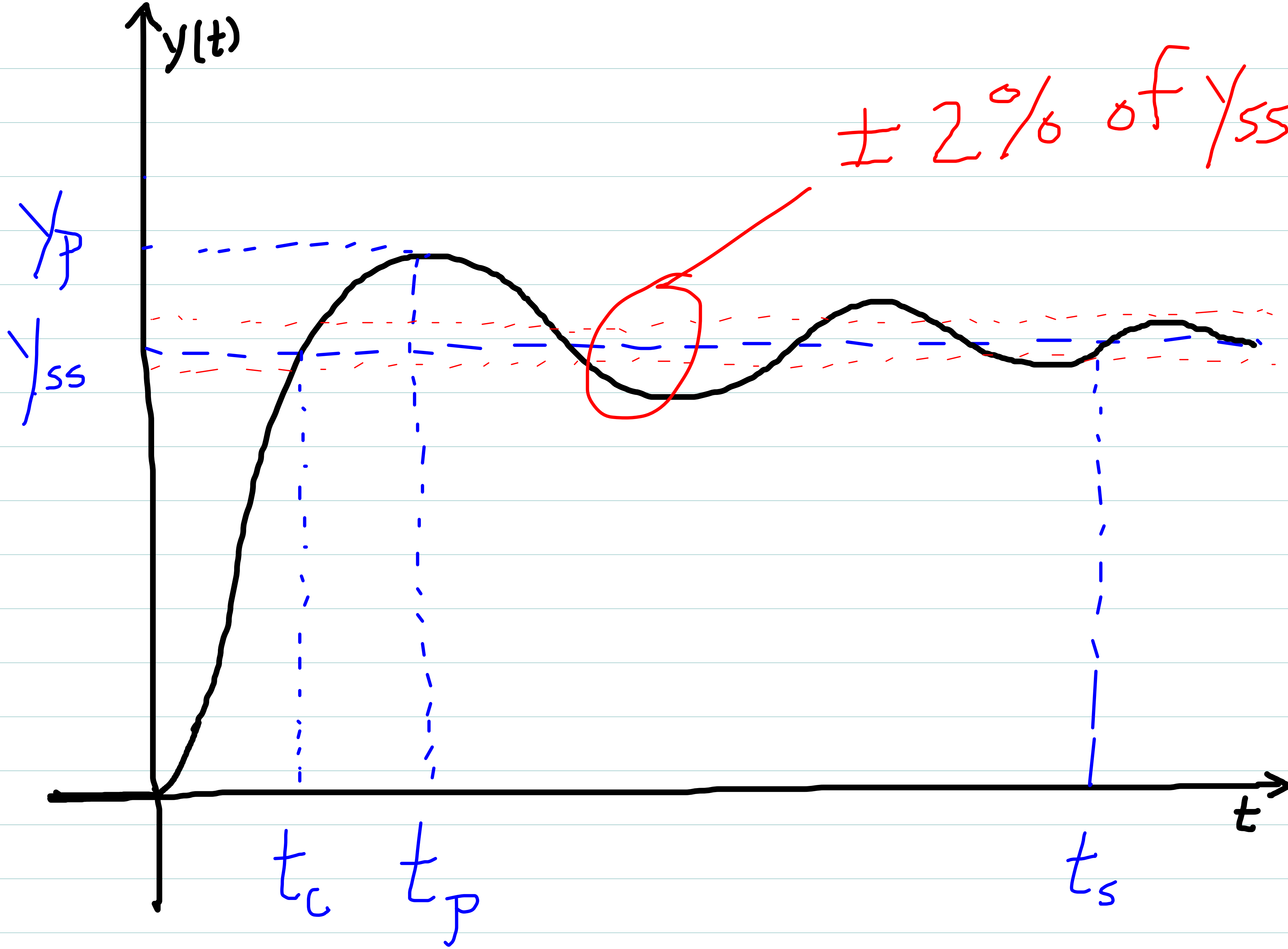
$$= \underbrace{\left(\frac{\beta_0}{2\alpha_0}\right)}_{\frac{1}{2}G(0)} \underbrace{\left(\frac{\alpha_0}{(\sigma+j\omega_d)(j\omega_d)}\right)}_{B}$$

SO:

$$y(t) = G(0) + 2|A_2|e^{\sigma t}\cos(\omega_d t + \measuredangle A_2)$$

OR:

$$\boxed{y(t) = G(0)\left[1 + |B|e^{\sigma t}\cos(\omega_d t + \measuredangle B)\right]}$$



y(t)
± 2% of y_{ss}
y_p
y_{ss}
t_c
t_p
t_s
t

General Observations

(1.) $y(t)$ continually oscillates about its steady-state value

$y_{ss} = G(\phi)$

(2.) t_c = time steady-state is first crossed

(3.) 1^{st} oscillation is largest, and creates an initial overshoot past the steady-state.

(4.) This initial overshoot has peak value y_p, and occurs at time t_p

(5.) Settling time t_s defined where response enters $\pm 2\%$ tolerance band and remains within it for times thereafter

Must learn to rapidly quantify these!!

$$y(t) = G(0)\left[1 + |B|\, e^{\sigma t}\cos(\omega_d t + \measuredangle B)\right]$$

where:
$$B = \frac{\alpha_0}{(j\omega_d)(\sigma + j\omega_d)} = \frac{|p_1|^2}{(j\omega_d)\, p_1}$$

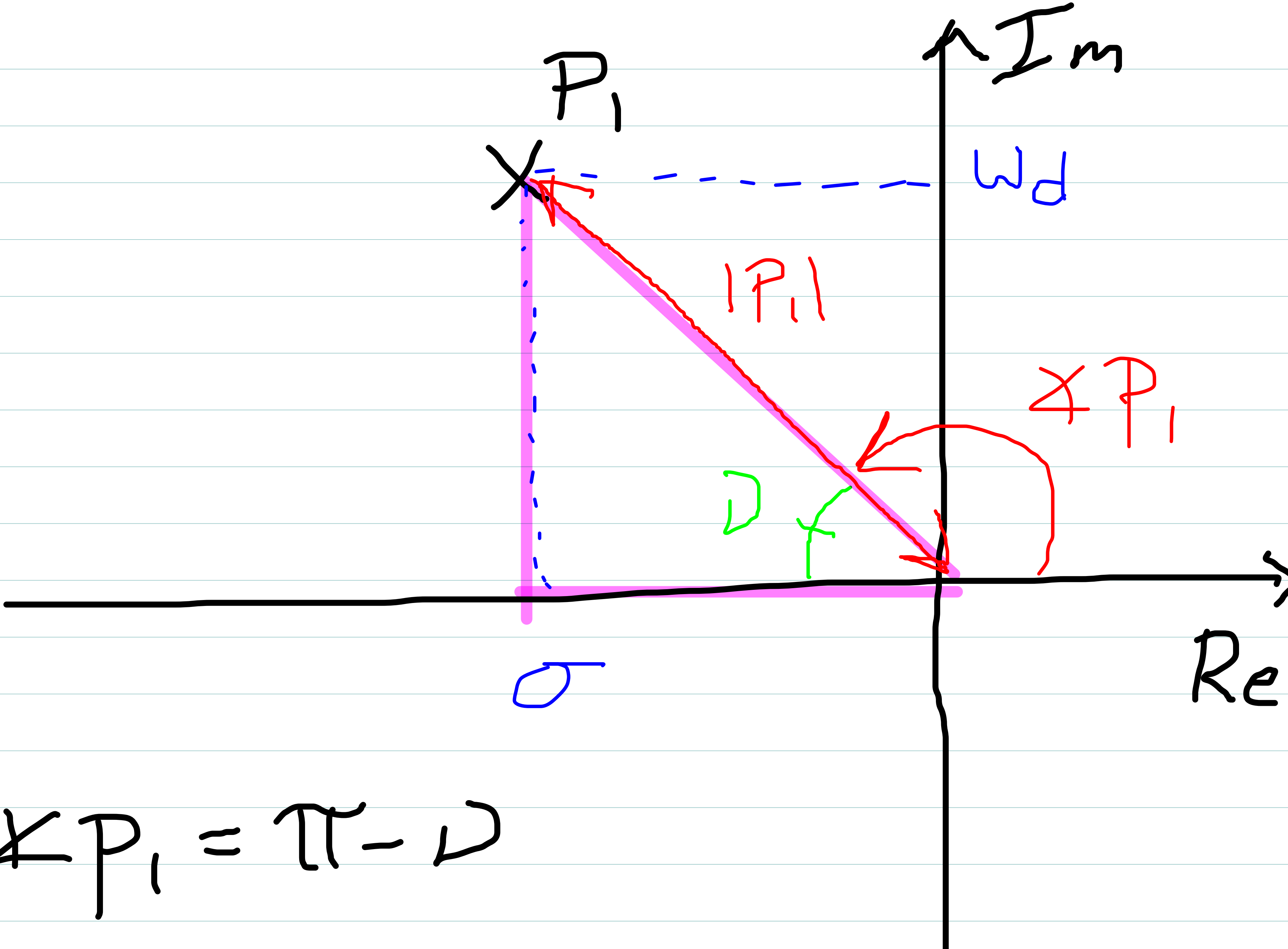
$\Rightarrow$ Transient features completely determined by location of pole $p_1 = \sigma + j\omega_d$ in complex plane

$$|B| = \frac{|p_1|^2}{|j\omega_d| \cdot |p_1|} = \frac{|p_1|}{\omega_d}$$

$$\measuredangle B = \cancelto{0}{\measuredangle |p_1|^2} - \left(\measuredangle (j\omega_d) + \measuredangle p_1\right)$$

$$= -\left(\frac{\pi}{2} + \measuredangle p_1\right)$$

must quantify this!



$$\measuredangle P_1 = \pi - \nu$$

<u>Note</u>: $\nu > 0$ is supplement of $\measuredangle P_1$

So:

$$\measuredangle B = -\left(\frac{\pi}{2} + \measuredangle p_1\right) = -\left(\frac{\pi}{2} + (\pi - \nu)\right)$$

$$= -\frac{3\pi}{2} + \nu$$

and thus:

$$y(t) = G(0)\left[1 + \left(\frac{|p_1|}{\omega_d}\right) e^{\sigma t} \cos\left(\omega_d t - \frac{3\pi}{2} + \nu\right)\right]$$

so:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{|p_1|}{\omega_d}\right) e^{\sigma t} \sin(\omega_d t + \nu)\right]}$$

Need to understand how ν depends on p_1