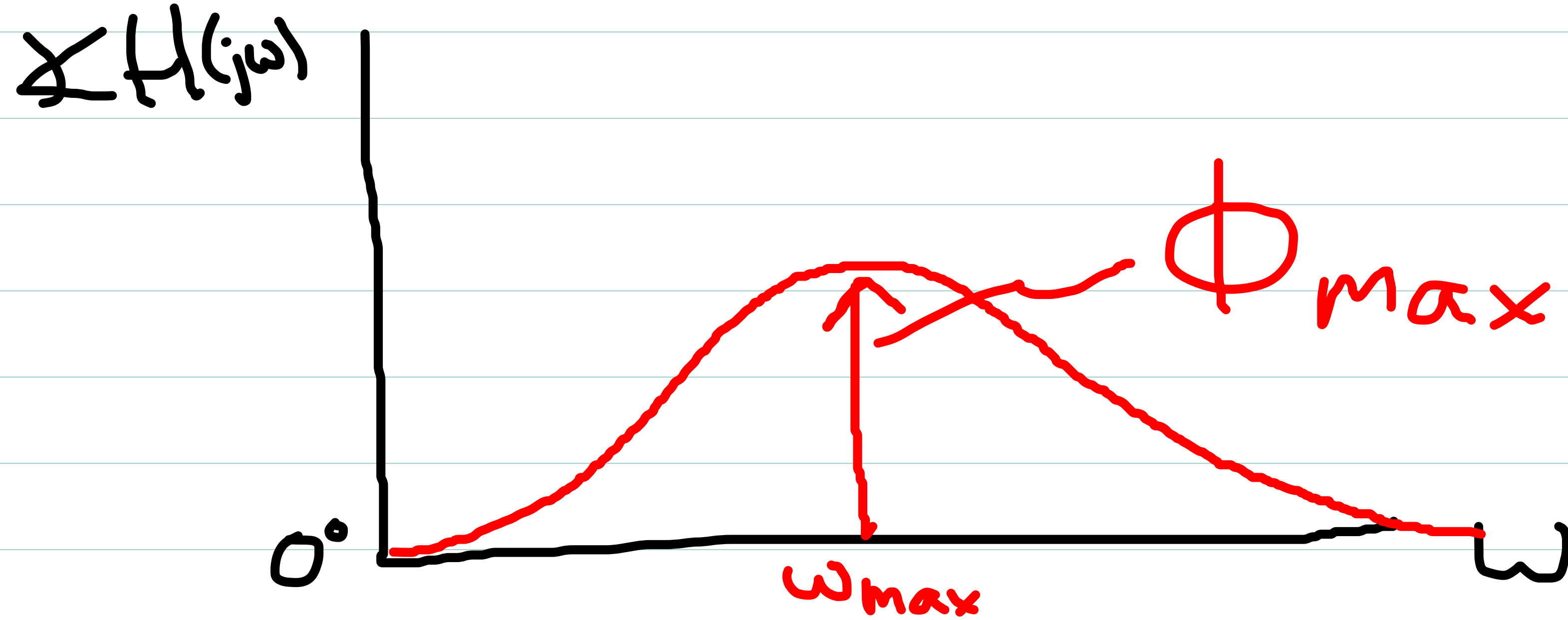
Compensators of general form:

$$H(s) = K\left[\frac{\beta\tau s+1}{\tau s+1}\right] \quad \text{with} \quad \begin{cases}\tau > 0 \\ \beta > 1\end{cases}$$

$\Rightarrow$ $z_c = -1/\beta\tau$, $p_c = -1/\tau$ so $|z_c| < |p_c|$ (zero closer to imag axis)

and $\beta = p_c/z_c$ ("lead ratio")

Are called "lead compensators", since $\angle H(j\omega) > 0$ for all $\omega > 0$ (positive phase is called "lead").



Note here that:

$$U(s) = H(s)E(s) = K\left[\frac{\beta T s+1}{Ts+1}\right]E(s)$$

$$= K\left[\beta - \frac{\beta-1}{Ts+1}\right]E(s)$$

$$\boxed{\begin{aligned} &\Rightarrow u(t) = K\beta e(t) + K(1-\beta)x_1(t) \\ &\tau \dot{x}_1(t) = -x_1(t) + e(t) \end{aligned}}$$

Corresponding implementation equations

So that $|u(t)| \propto \beta$ generally.

Want to find smallest value of $\beta = p_c/z_c$ which ensures $\phi_{max} \geq \phi_{req}$.

We can compute:

$$\phi_{max} = \sin^{-1}\left[\frac{\beta-1}{\beta+1}\right]$$

which is an <u>increasing</u> function of $\beta > 1$

Thus, the minimum required $\beta$ value is obtained when

$$\phi_{max} = \phi_{req}.$$

Selection of $\tau$ can then be achieved using analytical result

$$\omega_{max} = \frac{1}{\tau\sqrt{\beta}}$$

and setting $\omega_{max} = \omega_{Des}$.

Revisit previous example:

$$G(s) = \frac{3}{s(s+2)}$$ want $\gamma_{Des} = 60°$, $\omega_{Des} = 6$

for which $\phi_{req} = 41.56°$

Previous design used $H(s) = 93.4\left[\frac{s+5.54}{s+60}\right]$ $(\beta = 10.8)$

New design:

$$41.56^\circ = \Phi_{req} = \Phi_{max} = \sin^{-1}\left[\frac{\beta-1}{\beta+1}\right]$$

$$\Longrightarrow \beta = 4.94$$

Then $\quad 6 = \omega_{Des} = \omega_{max} = \dfrac{1}{\tau\sqrt{\beta}} \Rightarrow \tau = \dfrac{1}{\omega_{Des}\sqrt{\beta}}$

So $\tau = 0.075$ here.

and thus $\quad H(s) = K\left[\dfrac{0.375s+1}{0.075s+1}\right] \quad \begin{pmatrix} z_c = -2.67 \\ p_c = -13.3 \end{pmatrix}$

Choose $K$ as before: Let $L_0(s) = [L(s)]_{K=1}$

Then take $\quad K = \dfrac{1}{|L_0(j\omega_{Des})|} = 5.63$ here

# Comparison of Designs:

For a unit step $y_d(t)$:

Old design: $t_s \approx 0.96$ sec, $M_p \approx 15\%$, $u_{max} \approx 92$ "far pole"

New design: $t_s \approx 0.86$ sec, $M_p \approx 10\%$, $u_{max} \approx 27$ "lead"

New design is essentially the same (a little better) in transient performance, and requires a factor of 3 less maximum control effort.

$\Rightarrow$ minimizing $\beta$ is very beneficial!

# Another Example

Suppose $G(s) = \dfrac{3}{s(s-2)}$

and we again want $\gamma_{Des} = 45°$ with $\omega_{Des} = 6$, which we know is assured if:

$$L(s) = \frac{6^2\sqrt{2}}{s(s+6)}$$

You might be tempted to try $H(s) = \left(\dfrac{6^2\sqrt{2}}{3}\right)\left[\dfrac{s-2}{s+6}\right]$

Don't do it! Pole-zero cancellation cannot be guaranteed to be exact here, and CL system <u>will</u> have an unstable pole (try it! Use $(s-2.1)$ in numerator and see what happens).

Instead, meet targets using only LHP poles/zeros.

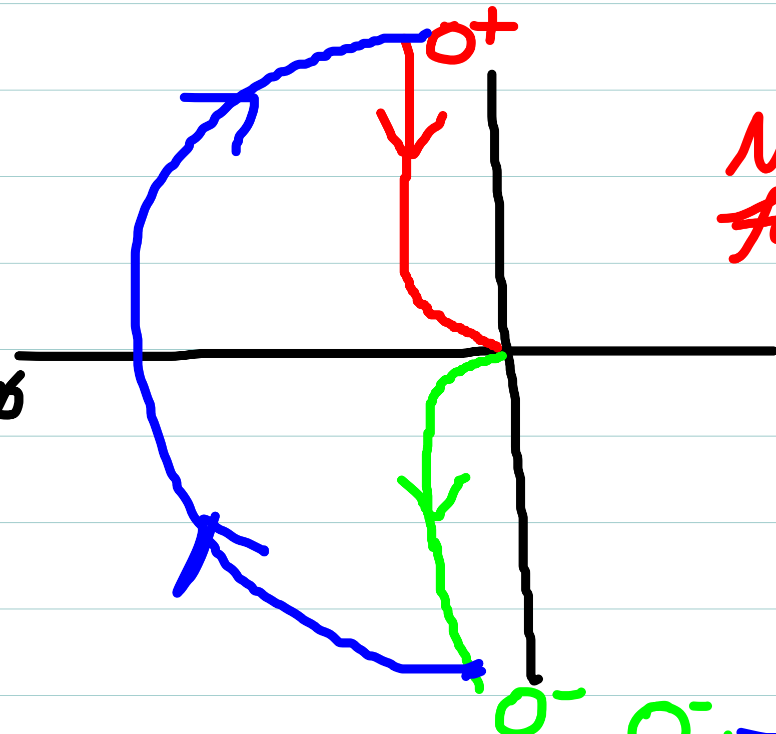
# Check Nyquist: Ensure $\gamma > 0$ still stabilizing

Using $\underline{H(s) = K > \emptyset}$

$P_R(L) = 1$

$N_{cw}(L) = 1$ for any $K > \emptyset$

$\Rightarrow T(s)$ unstable

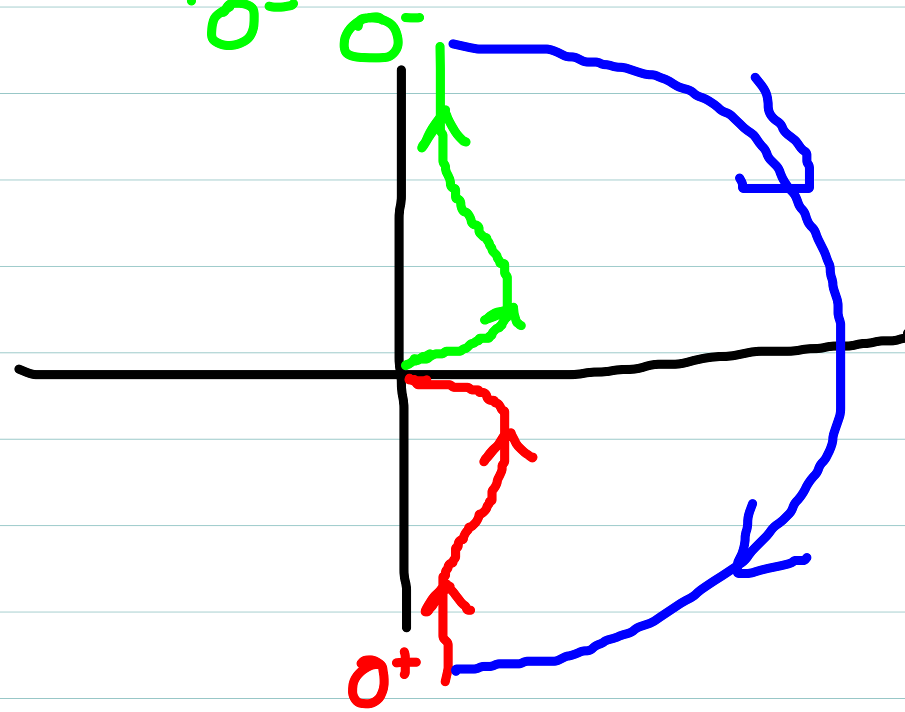



Need $N_{cw}(L) = -1$ for stability!

Using $H(s) = K < \emptyset$

$N_{cw}(L) = \emptyset$ for any $K < \emptyset$

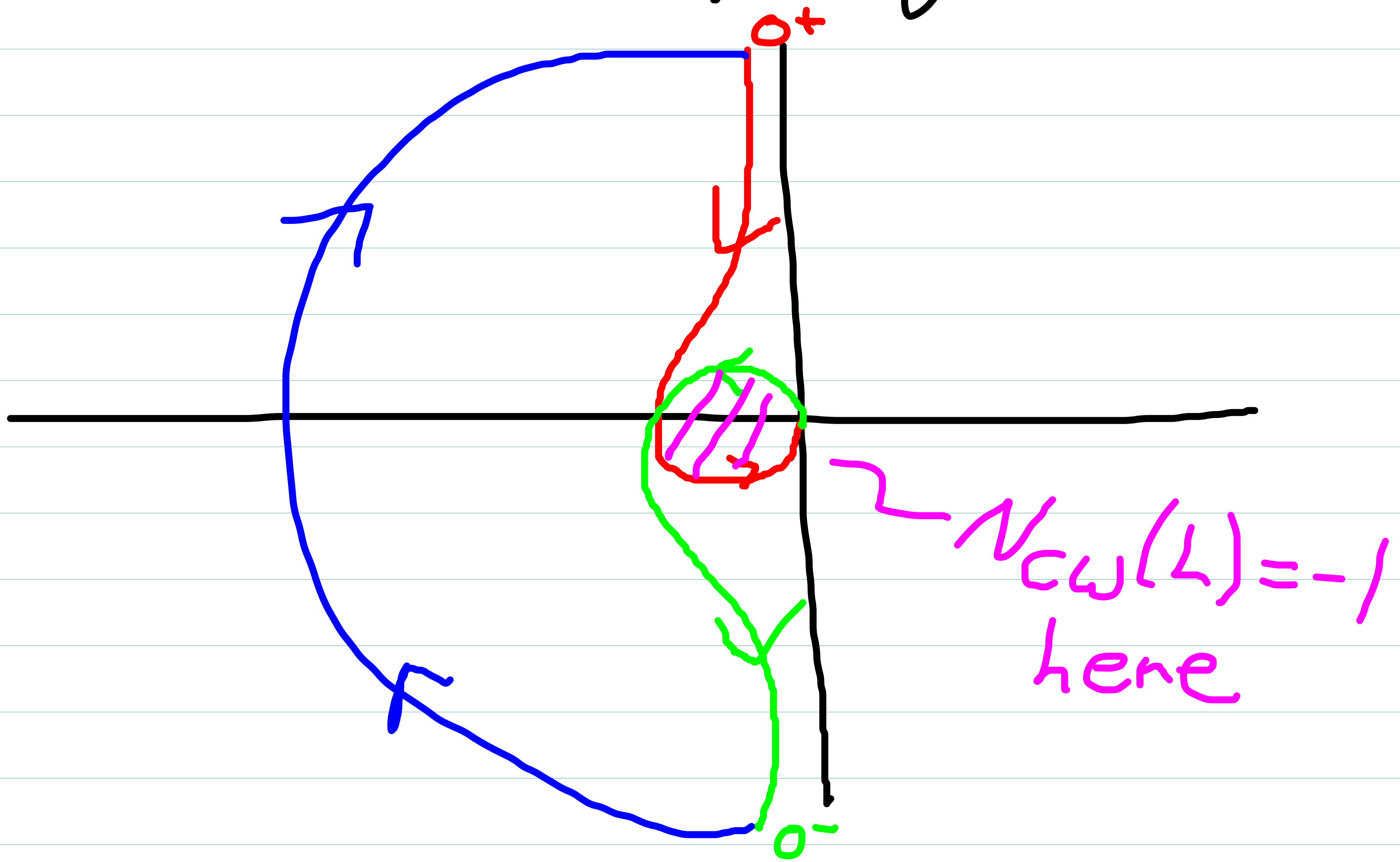
$T(s)$ unstable



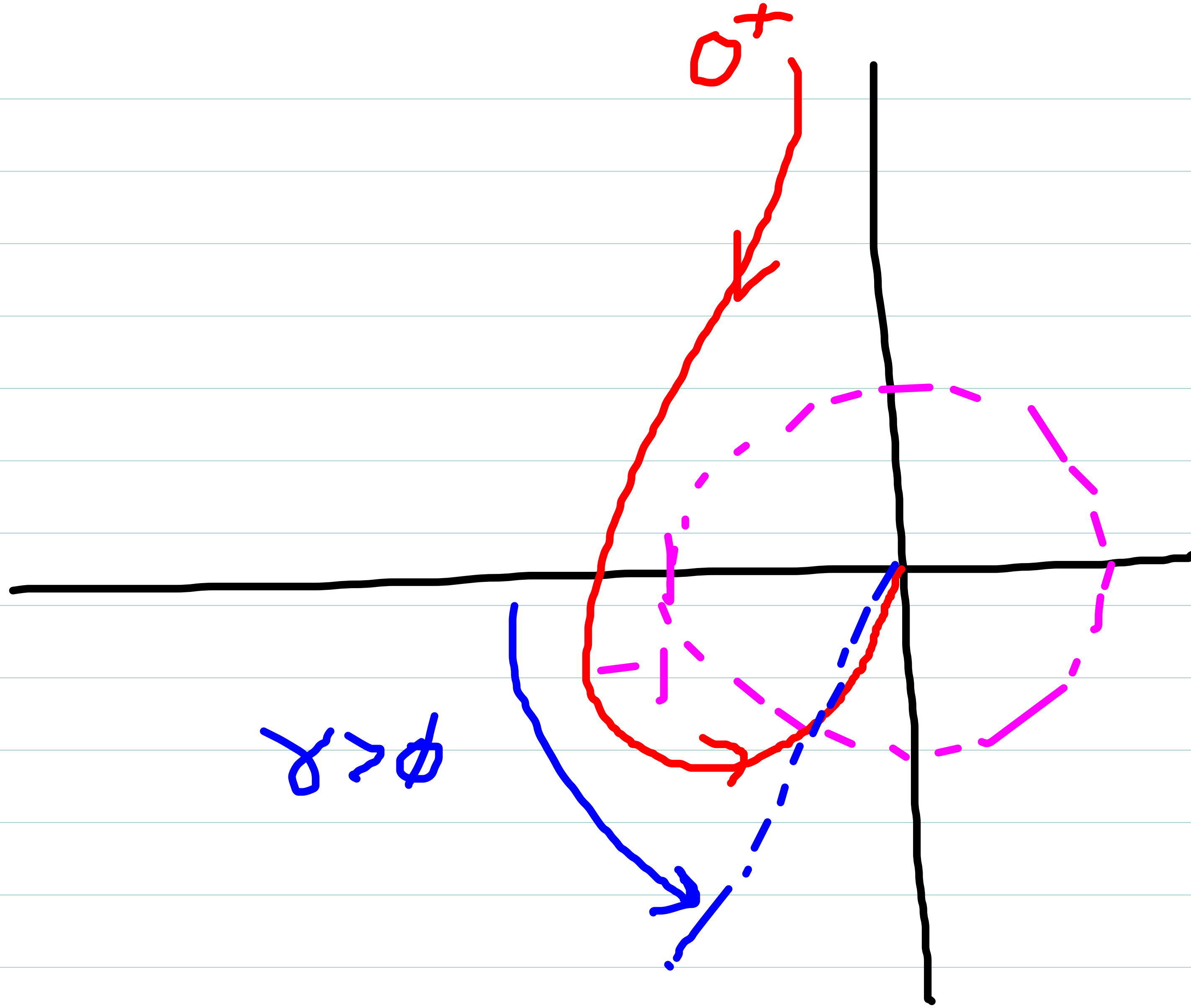

Conclusion: Cannot stabilize with $H(s) = K$

If $H(s) = K(Ts+1)$, $K>0$, $T>0$

(LHP zero, positive gain)




Will stabilize if $-1$ in shaded area!




With -1 in desired region, phase margin will be positive => agrees with our basic design guideline positive $\gamma$ is stabilizing here!

For $G(s) = \frac{3}{s(s-2)}$, $\gamma_{des} = 45°$, $\omega_{des} = 6$ :

$$\Phi_{req} = 45 - 180 - \measuredangle G(6j)$$

$$= 63.43°$$

$$\Rightarrow z_c = -\left[\frac{\omega_{des}}{\tan \Phi_{req}}\right] = -3$$

$\Rightarrow K = \frac{1}{|L_0(6j)|} = 1.89$. So $H(s) = 1.89\,(s+3)$ works

For an implementable design, simple approach is again to put pole at $-10\omega_{des}$, increase $\Phi_{req}$ by $5.7°$, giving

$$H(s) = 118.79\left[\frac{(s+2.29)}{(s+60)}\right]$$

(Or, we could do a lead comp design, if the above requires excessively large $u(t)$.)

$$G(s) = \frac{3}{s(s-2)}$$

Suppose we want instead $\gamma_{Des} = 70°$ at $\omega_{Des} = 6$

$\Phi_{req} = 70 - 180 - \measuredangle G(j\ ) = 88.43°$, Add $+5.7°$ for pole, need
$\measuredangle(j\omega_{Des} - z_c) = 94.13°$

This condition cannot be satisfied with a single LHP zero
Need $\underline{\underline{2}}$ zeros here.

With 2 zeros, can add up to $+180°$ to $\measuredangle L(j\omega)$. Lots of choices for 2 zeros adding up to $94.13°$ at $\omega = 6$. Can simplify design if we assume zeros/poles repeated

$$H(s) = K\left[\frac{(s-z_c)^2}{(s-p_c)^2}\right]$$

So $\measuredangle H(j\omega) = 2\measuredangle(j\omega - z_c) - 2\measuredangle(j\omega - p_c)$

$$\Rightarrow \measuredangle(j\omega_{Des} - z_c) = \frac{\Phi_{req}}{2} + \measuredangle(j\omega_{Des} - p_c)$$

for example, using $p_c = -10\omega_{Des} = -60$ again

We get $z_c = -5.05$, $K = 747.89$

and $H(s) = 747.89 \left[ \frac{(s+5.05)^2}{(s+60)^2} \right]$

(Again, could instead do a lead comp design - See next page).

Note: Above considerations apply any time we would need $\measuredangle H(j\omega_{Des}) > 90°$. Not specific to this example.

We can apply similar thinking for more complicated situations: i.e. if we would need

$$\measuredangle H(j\omega_{Des}) > 180°$$

Then we need 3 LHP zeros in $H(s)$, etc.

# Alternate lead comp design for above

Using again $\gamma_{des} = 70°$, $\omega_{des} = 6$

Set $\Phi_{max} = \frac{\Phi_{req}}{2} \Longrightarrow \beta = 5.61$

$\omega_{max} = 6 \Longrightarrow \tau = .07$

Then:

$$H(s) = K\left[\frac{(.395s+1)^2}{(.07s+1)^2}\right]$$

and

$$K = \frac{1}{|L_0(6j)|} = 2.255$$

# Comparison of different designs

$\boxed{\gamma = 45°}$:

zero/farpole: $t_s = 1.95$ sec, $M_p = 51\%$, $u_{max} = 119$

reduced

Lead: $t_s = 1.4$ sec, $M_p = 53\%$, $u_{max} = 52$

(less overshoot)

$\boxed{\gamma = 70°}$

zero/farpole: $t_s = 3.0$ sec, $M_p = 40\%$, $u_{max} = 700$

much less

Lead: $t_s = 3.9$ sec, $M_p = 47\%$, $u_{max} = 71$

Which is better...? That becomes a judgement call

Also need to consider tracking, bandwidth, robustness, and how "tight" your requirements actually are...

$$\left[\ G(s) = \frac{3}{s(s-2)},\quad \omega_{DES} = 6\ \right]$$

# (Intro:) Effect of Uncertainty

Suppose again we had wanted $\gamma = 45°$, $\omega_\gamma = 6$
and we (naively) ignored my caveat and chose

$$H(s) = \left(\frac{6^2\sqrt{2}}{3}\right)\left(\frac{s-2}{s+6}\right)$$

If the $G(s)$ really is:

$$G(s) = \frac{3}{s(s-1.9)}$$

[5% uncertainty in unstable pole location]

we'd have CL poles at: $-3.04 \pm 6.5j$, $1.97$ unstable!

But...

With the two designs above, we'd instead have CL poles:

zero/far pole: $-2.1 \pm 3.3j$, $-53.9$
Lead: $-2.8 \pm 2.3j$, $-18$

Both still stable!