PHYS 313

HW 06: Assignment 6

Due on March 13th, 2025 at 11:59 PM

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Problem 3.2:

In one sentence, justify Earnshaw's Theorem.

Solution

Earnshaw's Theorem is justified because in regions free of charges, the electrostatic potential must satisfy Laplace's equation, which prohibits any local minimum or maximum, thereby ensuring that any static configuration of charges is unstable.

Problem 3.3:

Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r. Do the same for cylindrical coordinates, assuming V depends only on s.

Solution

Part A

$$\begin{aligned}
\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}(V)\right) &= 0 \\
r^2\frac{\mathrm{d}}{\mathrm{d}r}(V) &= C_1, \\
\mathrm{V}(r) &= \frac{C_1}{r} + C_2 \quad \square
\end{aligned}$$

Part B

$$\begin{aligned}
\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\left(s\frac{\mathrm{d}}{\mathrm{d}s}(V)\right) &= 0 \\
s\frac{\mathrm{d}}{\mathrm{d}s}(V) &= C_3, \\
\mathrm{V}(s) &= C_3\log(s) + C_4 \quad \square
\end{aligned}$$

Problem 3.4:

1. Show that the average electric field over a spherical surface, due to charges outside the sphere, is the same as the field at the center.

2. What is the average due to charges inside the sphere?

Solution

Part A

$$
\begin{aligned}
\nabla^2 \phi &= 0 \\
\phi(\mathbf{0}) &= \frac{1}{4\pi R^2} \int_{S_R} \phi(\mathbf{r}) \mathrm{d}A \\
\mathbf{E}(\mathbf{r}) &= -\nabla \phi(\mathbf{r}) \\
\frac{\partial}{\partial x} (\phi(\mathbf{0})) &= \frac{1}{4\pi R^2} \int_{S_R} \frac{\partial}{\partial x} (\phi(\mathbf{r}))\, \mathrm{d}A \\
\mathrm{E_x}(\mathbf{0}) &= \frac{1}{4\pi R^2} \int_{S_R} \mathrm{E_x}(\mathbf{r}) \mathrm{d}A \\
\langle \mathbf{E} \rangle &\equiv \frac{1}{4\pi R^2} \int_{S_R} \mathbf{E}(\mathbf{r}) \mathrm{d}A \\
\langle \mathbf{E} \rangle_{\text{ext}} &= \mathbf{E}(\mathbf{0}) \quad \square
\end{aligned}
$$

Thus, the average field over the sphere (due only to external charges) equals the field at the center.

Part B

$$
\begin{aligned}
\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{r}}}{r^3} + \cdots \right) \\
Q &= \sum_i q_i \quad \text{and} \quad \mathbf{p} = \sum_i q_i\, \mathbf{r}_i \\
\phi_{\text{mon}}(r) &= \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \\
\mathbf{E}_{\text{mon}}(r) &= -\nabla \phi_{\text{mon}} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2} \\
\langle \hat{\mathbf{r}} \rangle &= \frac{1}{4\pi} \int \hat{\mathbf{r}}\, d\Omega = \mathbf{0} \\
\phi_{\text{dip}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \\
\mathbf{E}_{\text{dip}}(\mathbf{r}) &= -\nabla \phi_{\text{dip}}(\mathbf{r}) \\
\langle \mathbf{E} \rangle_{\text{int}} &= -\frac{\mathbf{p}}{3\epsilon_0} \quad \square \\
\langle r_i r_j \rangle &= \frac{r^2}{3} \delta_{ij} \quad \square
\end{aligned}
$$

By integration over the spherical surface, the net average is proportional to $\mathbf{p}$ with the constant of proportionality $\frac{-1}{3\epsilon_0}$.

Problem 3.7:

Find the force on the charge $+q$ in the below image, noting that the xy plane is a grounded conductor.

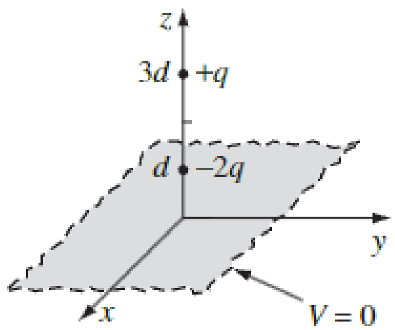

Figure 1: Diagram for Problem 3.7

Solution

The net force on the real $+q$ is given by the Coulomb forces due to the other charges. Since all charges lie on the z–axis, the force on $+q$ will be directed along the z–axis.

1. Force due to the real charge $-2q$ at $z = d$:

$$r_1 = 3d - d = 2d.$$
$$F_1 = \frac{1}{4\pi\epsilon_0}\frac{q\,(2q)}{(2d)^2} = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{4d^2} = \frac{q^2}{8\pi\epsilon_0 d^2}.$$
$$\mathbf{F}_1 = -\frac{q^2}{8\pi\epsilon_0 d^2}\,\hat{\mathbf{z}}.$$

2. Force due to the image of $+q$, namely $-q$ at $z = -3d$:

$$r_2 = 3d - (-3d) = 6d.$$
$$F_2 = \frac{1}{4\pi\epsilon_0}\frac{q\,(q)}{(6d)^2} = \frac{q^2}{4\pi\epsilon_0\,(36d^2)} = \frac{q^2}{144\pi\epsilon_0 d^2}.$$
$$\mathbf{F}_2 = -\frac{q^2}{144\pi\epsilon_0 d^2}\,\hat{\mathbf{z}}.$$

3. Force due to the image of $-2q$, namely $+2q$ at $z = -d$:

$$r_3 = 3d - (-d) = 4d.$$
$$F_3 = \frac{1}{4\pi\epsilon_0}\frac{q\,(2q)}{(4d)^2} = \frac{2q^2}{4\pi\epsilon_0\,(16d^2)} = \frac{q^2}{32\pi\epsilon_0 d^2}.$$

Since both charges are positive, the force is repulsive. The vector from the image charge at $z = -d$ to the real $+q$ at $z = 3d$ points upward; thus, the force on $+q$ is upward:

$$\mathbf{F}_3 = +\frac{q^2}{32\pi\epsilon_0 d^2}\,\hat{\mathbf{z}}.$$

Net Force:

Summing the three contributions we have

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \left[-\frac{q^2}{8\pi\epsilon_0 d^2} - \frac{q^2}{144\pi\epsilon_0 d^2} + \frac{q^2}{32\pi\epsilon_0 d^2}\right]\hat{\mathbf{z}}.$$

$$\frac{1}{8} = \frac{18}{144}, \quad \frac{1}{32} = \frac{4.5}{144}.$$

Thus,

$$-\frac{q^2}{8\pi\epsilon_0 d^2} = -\frac{18q^2}{144\pi\epsilon_0 d^2}, \quad -\frac{q^2}{144\pi\epsilon_0 d^2} = -\frac{q^2}{144\pi\epsilon_0 d^2}, \quad \frac{q^2}{32\pi\epsilon_0 d^2} = \frac{4.5q^2}{144\pi\epsilon_0 d^2}.$$

$$\mathbf{F} = -\frac{18q^2}{144\pi\epsilon_0 d^2}\,\hat{\mathbf{z}} - \frac{q^2}{144\pi\epsilon_0 d^2}\,\hat{\mathbf{z}} + \frac{4.5q^2}{144\pi\epsilon_0 d^2}\,\hat{\mathbf{z}} = -\frac{(18+1-4.5)q^2}{144\pi\epsilon_0 d^2}\,\hat{\mathbf{z}}.$$

$$\mathbf{F} = -\frac{(19-4.5)q^2}{144\pi\epsilon_0 d^2}\,\hat{\mathbf{z}} = -\frac{14.5\,q^2}{144\pi\epsilon_0 d^2}\,\hat{\mathbf{z}}.$$

$$\mathbf{F} = -\frac{29\,q^2}{288\pi\epsilon_0 d^2}\,\hat{\mathbf{z}} \quad \square$$

which indicates that the net force on the charge $+q$ is directed downward (toward the xy–plane).

Problem 3.8:

1. Using the law of consines, show that the following equations are equivalent:

$$\mathrm{V}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) \tag{1}$$

$$\mathrm{V}(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos()(\theta)}} - \frac{q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos()(\theta)}}\right] \tag{2}$$

   Where r and θ are the usual spherical polar coordinates, with the z axis along the line through q. In this form, it is obvious that $V = 0$ on the sphere $r = R$.

2. Find the induced surface charge on the sphere, as a function of θ. Integrate this to get the total induced charge. (What *should* it be?)

3. Calculate the energy of this configuration.

Solution

Part A

$$\begin{aligned}
\mathrm{V}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{r}_q|} + \frac{q'}{|\mathbf{r}-\mathbf{r}_{q'}|}\right)\\
|\mathbf{r}-\mathbf{r}_q| &= \sqrt{r^2+a^2-2ra\cos(\theta)}\\
\mathbf{r}_{q'} &= \frac{R^2}{a}\,\hat{\mathbf{z}} \quad \text{with} \quad q' = -\frac{qR}{a}\\
|\mathbf{r}-\mathbf{r}_{q'}| &= \sqrt{r^2+\left(\frac{R^2}{a}\right)^2-2r\frac{R^2}{a}\cos(\theta)}\\
\mathrm{V}(r,\theta) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos(\theta)}} - \frac{q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos(\theta)}}\right]
\end{aligned}$$

In this form, when $r = R$, the two terms cancel (since the geometry is chosen so that the potential on the sphere vanishes), proving the two expressions are equivalent.

Part B

$$\begin{aligned}
\sigma(\theta) &= -\epsilon_0\,\frac{\partial}{\partial r}(V)\bigg|_{r=R}\\
\sigma(\theta) &= -\frac{q}{4\pi R^2}\,\frac{R^2-a^2}{(R-a\cos(\theta))^3}\\
Q_{\text{ind}} &= \int_0^{2\pi}\int_0^{\pi}\sigma(\theta)\,R^2\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\\
Q_{\text{ind}} &= -q \quad \square
\end{aligned}$$

Part C

Problem 3.8 continued on next page...

$$
\begin{aligned}
U &= \frac{1}{2}\,\frac{q\,q'}{4\pi\epsilon_0}\,\frac{1}{\left|a - \frac{R^2}{a}\right|} \\
\left|a - \frac{R^2}{a}\right| &= \frac{a^2 - R^2}{a} \\
U &= -\frac{q^2 R}{8\pi\epsilon_0}\,\frac{1}{a^2 - R^2} \quad \square
\end{aligned}
$$

Problem 3.13:

Find the potential in the infinite slot of Ex3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = \frac{a}{2}$, is held at a constant potential V_0, and the other, from $y = \frac{a}{2}$ to $y = a$, is at potential $-V_0$.

Solution

Similar to the answer in Ex3.3, the configuration retains its independence from z. We again have to solve Laplace's equation but subjected to different boundary conditions:

$$\frac{\partial^2}{\partial x^2}(V) + \frac{\partial^2}{\partial y^2}(V) = 0, \begin{cases} V = 0 & y = 0 \\ V = 0 & y = a \\ V = V_0 & 0 < y < \frac{a}{2}, x = 0 \\ V = -V_0 & \frac{a}{2} < y < a, x = 0 \\ V \to 0 & x \to \infty \end{cases}.$$

This can be accomplished using a similar technique as Griffiths, as follows:

$$\begin{aligned} Y\frac{\mathrm{d}^2}{\mathrm{d}x^2}(X) + X\frac{\mathrm{d}^2}{\mathrm{d}y^2}(Y) &= 0 \\ \frac{1}{X}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(X) + \frac{1}{Y}\frac{\mathrm{d}^2}{\mathrm{d}y^2}(Y) &= 0 \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}(X) = k^2 X, \quad \frac{\mathrm{d}^2}{\mathrm{d}y^2}(Y) &= -k^2 Y \\ \mathrm{X}(x) = Ae^{kx} + Be^{-kx}, \quad \mathrm{Y}(y) &= C\sin(ky) + D\cos(ky) \\ \mathrm{V}(x, y) &= \left(Ae^{kx} + Be^{-kx}\right)(C\sin(ky) + D\cos(ky)) \\ \text{condition (v)} &\implies A = 0 \\ \therefore \mathrm{V}(x, y) &= e^{-ky}(C\sin(ky) + D\cos(ky)) \\ \text{condition (i)} &\implies D = 0 \\ \therefore \mathrm{V}(x, y) &= Ce^{-ky}\sin(ky) \\ \text{condition (ii)} &\implies \sin(ka) = 0 \\ \therefore k &= \frac{n\pi}{a}, \quad n = \{1, 2, 3, \dots\} \\ \mathrm{V}(x, y) &= \sum_{n=1}^{\infty} C_n e^{-n\pi\frac{x}{a}}\sin(n\pi\frac{y}{a}) \end{aligned}$$

Here is where we diverge from Griffiths' Ex3.3. We want to fulfill our conditions (iii) and (iv) as follows:

$$\begin{aligned} \mathrm{V}(0, 0 < y < \frac{a}{2}) &= \sum_{n=1}^{\infty} C_n e^{-n\pi\frac{x}{a}}\sin(n\pi\frac{y}{a}) = V_0, \\ \mathrm{V}(0, \frac{a}{2} < y < a) &= \sum_{n=1}^{\infty} C_n e^{-n\pi\frac{x}{a}}\sin(n\pi\frac{y}{a}) = -V_0. \end{aligned}$$

The Fourier sine coefficients are given by:

$$\begin{aligned} C_n &= \frac{2}{a}\int_0^a \mathrm{f}(y)\sin(n\pi\frac{y}{a})\mathrm{d}y \\ C_n &= \frac{2}{a}\left[\int_0^{a/2} V_0 \sin(n\pi\frac{y}{a})\mathrm{d}y + \int_{a/2}^{a}(-V_0)\sin(n\pi\frac{y}{a})\mathrm{d}y\right] \end{aligned}$$

$$\begin{aligned}
\int_0^{a/2} \sin(n\pi \frac{y}{a}) \mathrm{d}y &= \Big[-\frac{a}{n\pi}\cos(n\pi\frac{y}{a})\Big]_0^{a/2} \\
&= \frac{a}{n\pi}\Big[1-\cos(\frac{n\pi}{2})\Big] \\
\int_{a/2}^{a} \sin(\frac{n\pi y}{a}) \mathrm{d}y &= \Big[-\frac{a}{n\pi}\cos(n\pi\frac{y}{a})\Big]_{a/2}^{a} \\
&= \frac{a}{n\pi}\Big[\cos(\frac{n\pi}{2})-\cos(n\pi)\Big] \\
C_n &= \frac{2V_0}{n\pi}\Big[\Big(1-\cos(\frac{n\pi}{2})\Big)-\Big(\cos(\frac{n\pi}{2})-\cos(n\pi)\Big)\Big]\Big(1-\cos(\frac{n\pi}{2})\Big)-\Big(\cos(\frac{n\pi}{2})-\cos(n\pi)\Big) \\
&= 1-\cos(\frac{n\pi}{2})-\cos(\frac{n\pi}{2})+\cos(n\pi) \\
&= 1-2\cos(\frac{n\pi}{2})+\cos(n\pi)
\end{aligned}$$

Thus, the solution becomes:

$$\mathrm{V}(x,y) = \sum_{n\,\mathrm{odd}} \frac{4V_0}{n\pi}\, e^{-n\pi\frac{x}{a}} \sin(n\pi\frac{y}{a}) \quad \square$$