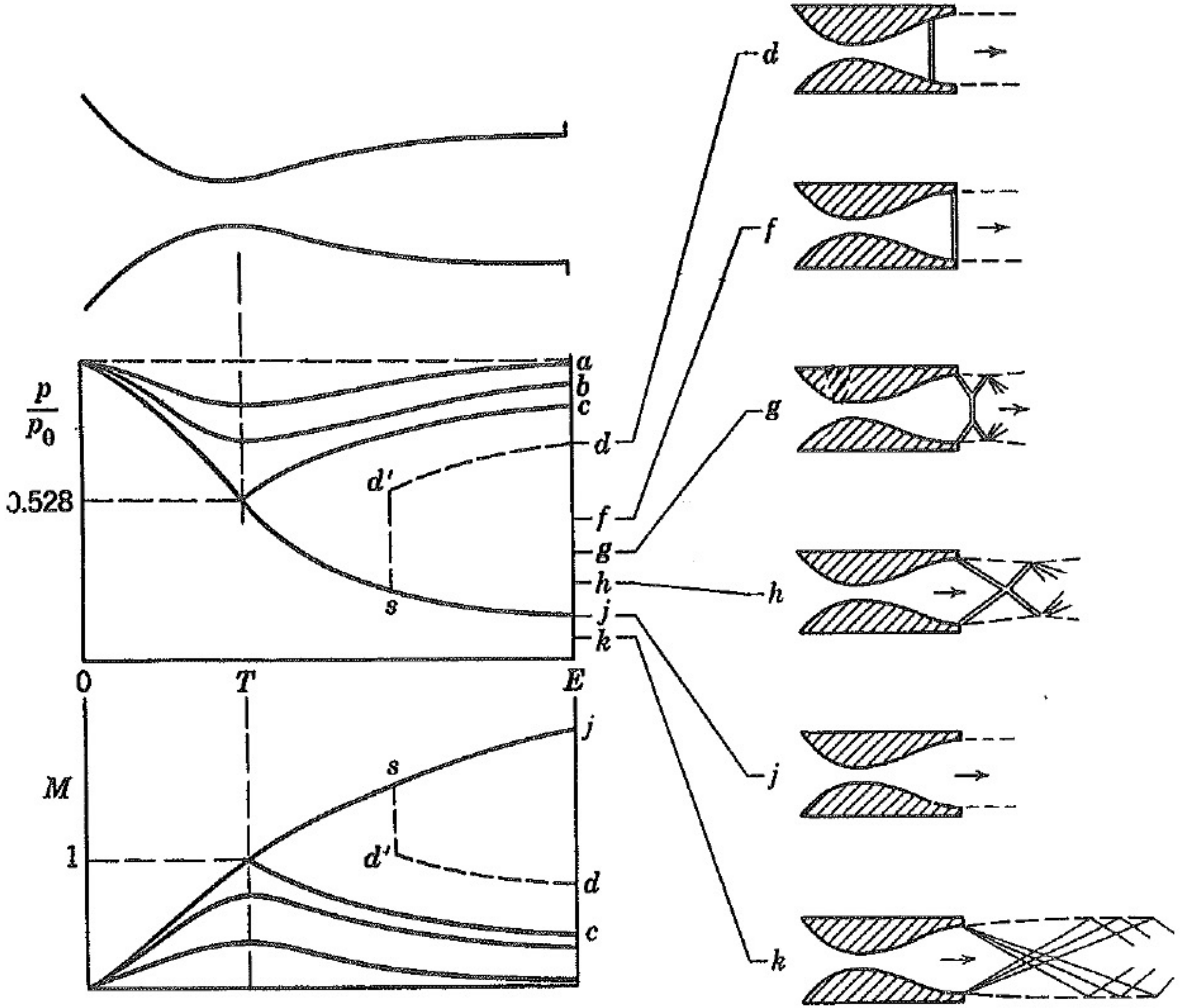
Review Lecture 19



Lecture 20: Rayleigh Flow

ENAE311H Aerodynamics I

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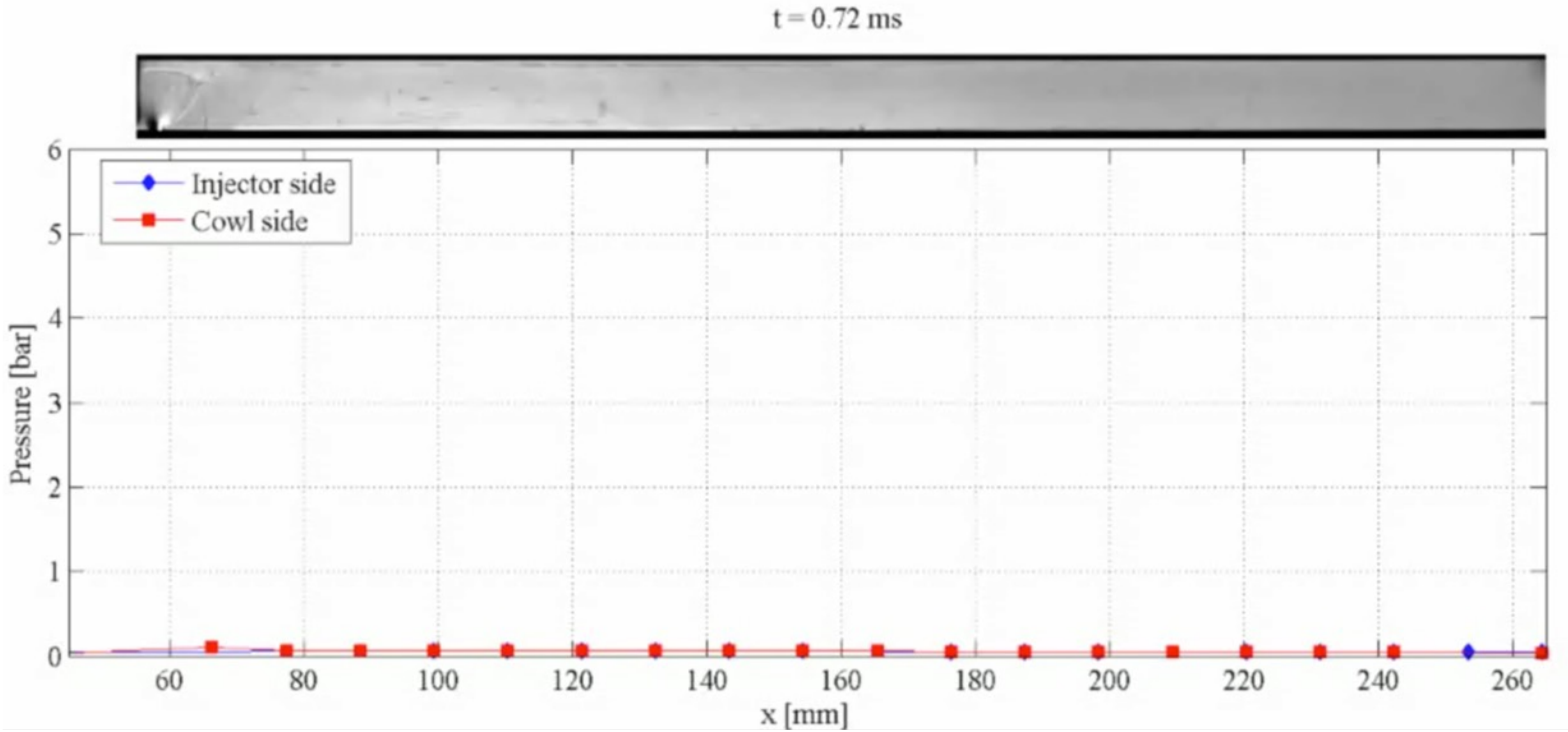
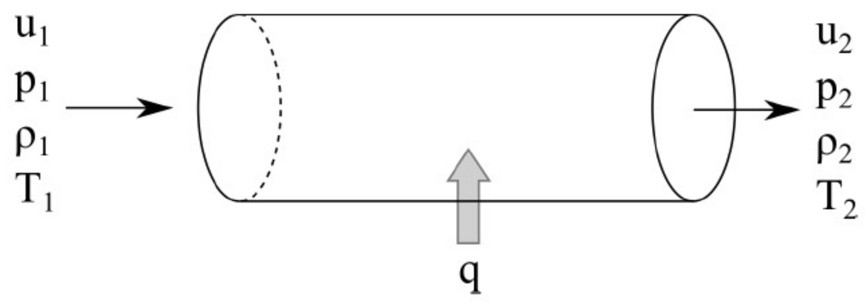
Unstart inside a scramjet combustor



Governing equations

Consider the steady, one-dimensional flow inside a constant-area duct with no frictional effects at the walls. We assume that heat is added or subtracted in an idealized fashion that otherwise leaves the nature of the fluid (e.g., composition) unchanged. We additionally assume a perfect gas.



Using a simple control volume around the pipe, the continuity and momentum equations are:

$$\text{Mass:} \qquad \rho_1 u_1 = \rho_2 u_2$$

$$\text{Momentum:} \qquad p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

(note that these are the same as for a normal shock).

For the energy, we can use our simple CV equation from earlier, i.e.,

$$\dot{m}\left[h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.$$

Heat addition produces an increase in stagnation temperature

Here, this becomes

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2},$$

with $q = \dot{Q}/\dot{m}$.

This can be written as

$$h_{01} + q = h_{02},$$

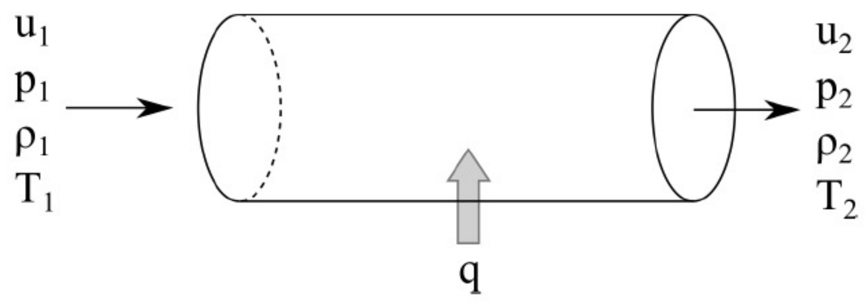
which, for a perfect gas is

$$c_p T_{01} + q = c_p T_{02},$$

or

$$\boxed{T_{02} = T_{01} + \frac{q}{c_p}.}$$

Flow variables



We wish to relate flow variables at the downstream station to those upstream, given the heat addition or subtraction. The Mach number is again a convenient means to do this.

We first note

$$\frac{M_2}{M_1} = \frac{u_2}{u_1}\frac{a_1}{a_2} = \frac{u_2}{u_1}\sqrt{\frac{T_1}{T_2}}.$$

Now, since $\rho u^2 = \gamma p M^2$ for a perfect gas, we can write the momentum equation as

$$p_1 + \gamma p_1 M_1^2 = p_2 + \gamma p_2 M_2^2$$

which can be rearranged as

$$\boxed{\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}}$$

For the temperature ratio, we have

$$\begin{aligned}\frac{T_2}{T_1} &= \frac{\rho_1}{\rho_2}\frac{1+\gamma M_1^2}{1+\gamma M_2^2} \\ &= \frac{u_2}{u_1}\frac{1+\gamma M_1^2}{1+\gamma M_2^2}.\end{aligned}$$

Substituting our expression for u_2/u_1 from left:

$$\frac{T_2}{T_1} = \sqrt{\frac{T_2}{T_1}}\frac{M_2}{M_1}\frac{1+\gamma M_1^2}{1+\gamma M_2^2},$$

And so

$$\boxed{\frac{T_2}{T_1} = \left(\frac{M_2}{M_1}\right)^2\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2.}$$

Flow variables

For the density, we can use the ideal gas equation, $\rho = p/(RT)$, together with our equations for pressure and temperature, to obtain

$$\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{M_1}{M_2}\right)^2 \frac{1+\gamma M_2^2}{1+\gamma M_1^2} = \frac{u_1}{u_2}.}$$

Now, for the ratios of stagnation properties, we can use, for example,

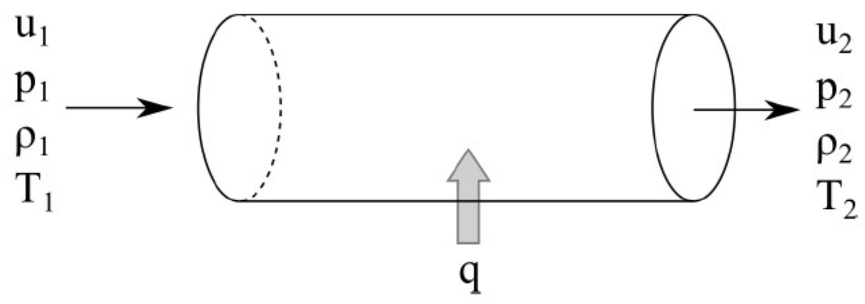
$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

to write

$$\boxed{\frac{p_{02}}{p_{01}} = \frac{p_2}{p_1}\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right)^{\gamma/(\gamma-1)}},$$

and

$$\boxed{\frac{T_{02}}{T_{01}} = \frac{T_2}{T_1}\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}.}$$



But remember that we also have another expression relating T_{01} and T_{02}, i.e.,

$$T_{02} = T_{01} + \frac{q}{c_p}.$$

Thus, given conditions at 1 and q, we can solve this equation for T_{02}, which allows us to solve (implicitly) for M_2 using the equation to the left.

The remaining flow conditions at 2 then follow immediately from the relations we have just derived.

Entropy changes in Rayleigh flow

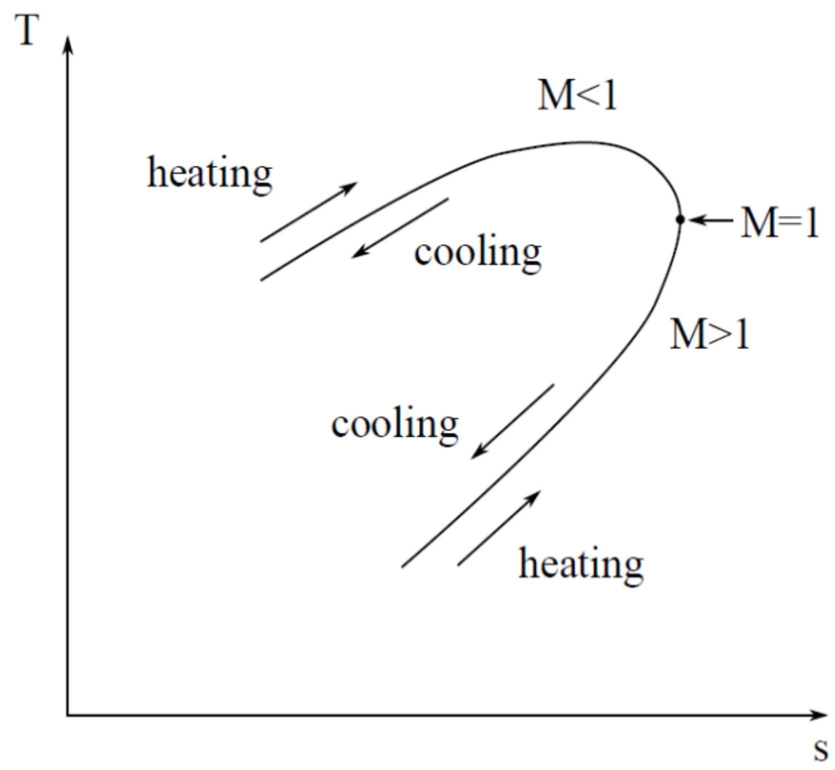
The change in entropy can be written as

$$\frac{s_2 - s_1}{c_p} = \ln \frac{T_2}{T_1} - \ln \left[\left(\frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma} \right].$$

If we plot temperature versus entropy on a $T - s$ diagram, we see that the maximum entropy point occurs when the flow is sonic, i.e., $M = 1$.

Whether the flow is initially subsonic or supersonic, heating will drive the Mach number towards unity (increasing the entropy), while cooling will shift it away from sonic conditions.

Behavior of other flow variables depends on whether the flow is subsonic or supersonic.



Sonic-referenced conditions

Conditions at the sonic point provide convenient reference values for normalization of the flow variables. Setting $M = 1$ in our earlier expressions, we have

$$
\begin{aligned}
\frac{p}{p^*} &= \frac{\gamma+1}{1+\gamma M^2} \\
\frac{T}{T^*} &= \left(\frac{(\gamma+1)M}{1+\gamma M^2}\right)^2 \\
\frac{\rho}{\rho^*} &= \frac{1+\gamma M^2}{(\gamma+1)M^2} = \frac{u^*}{u} \\
\frac{p_0}{p_0^*} &= \frac{\gamma+1}{1+\gamma M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\gamma/(\gamma-1)} \\
\frac{T_0}{T_0^*} &= \frac{2(\gamma+1)M^2}{(1+\gamma M^2)^2}\left(1+\frac{\gamma-1}{2}M^2\right) \\
\frac{s-s^*}{c_p} &= \ln\left[M^2\left(\frac{\gamma+1}{1+\gamma M^2}\right)^{(\gamma+1)/\gamma}\right].
\end{aligned}
$$

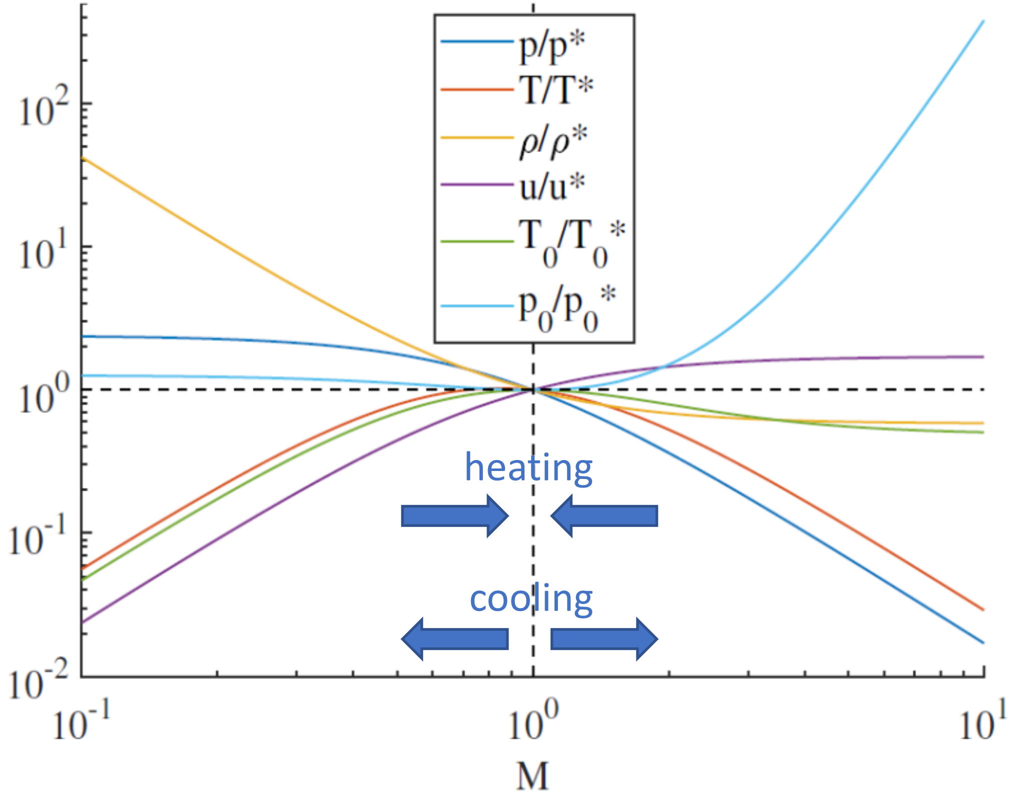
Since the sonic point is constant in Rayleigh flow, conditions at two other points can then be related as, e.g.,

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1}.$$

Behavior of flow variables

The qualitative behavior for positive heat addition ($q > 0$) in Rayleigh flow can be summarized as follows:

	$M < 1$	$M > 1$
T_0	increases	increases
M	increases	decreases
T	increases for $M < 1/\sqrt{\gamma}$ decreases for $M > 1/\sqrt{\gamma}$	increases
p	decreases	increases
ρ	decreases	increases
p_0	decreases	decreases
u	increases	decreases



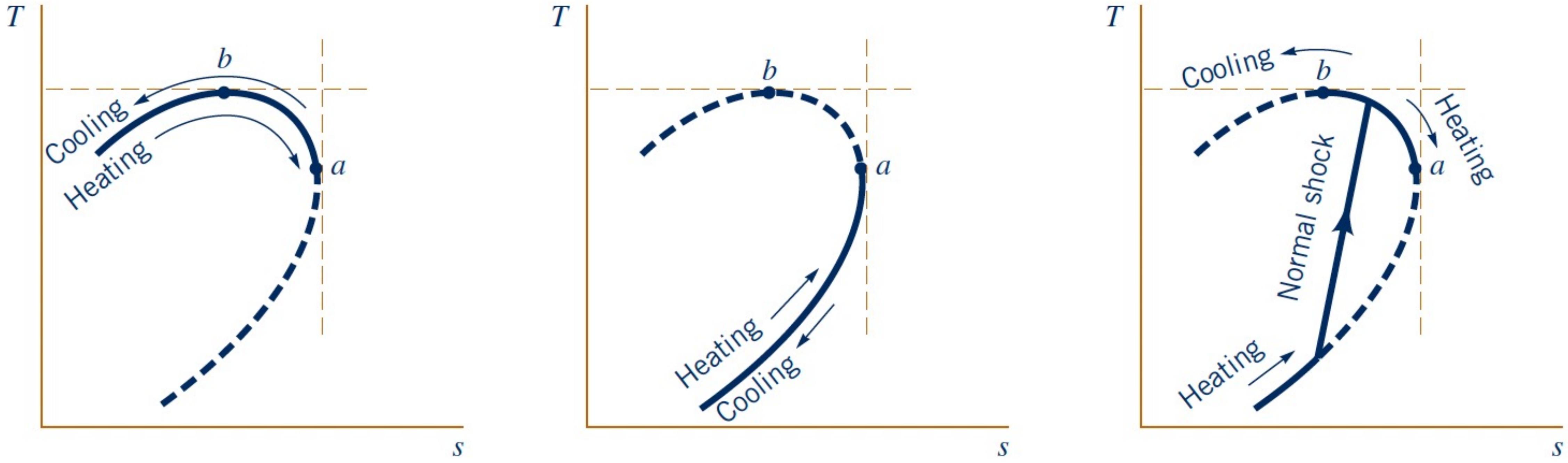
Trends for cooling ($q < 0$) are reversed.

Choking in Rayleigh flow

Note that if the flow reaches $M = 1$ from either above or below, no more heat can be added to the flow while maintaining a steady flow

→ we say the flow is *thermally choked*.

If we attempt to add more heat beyond this point, the flow responds by sending disturbances upstream to modify the inlet flow.



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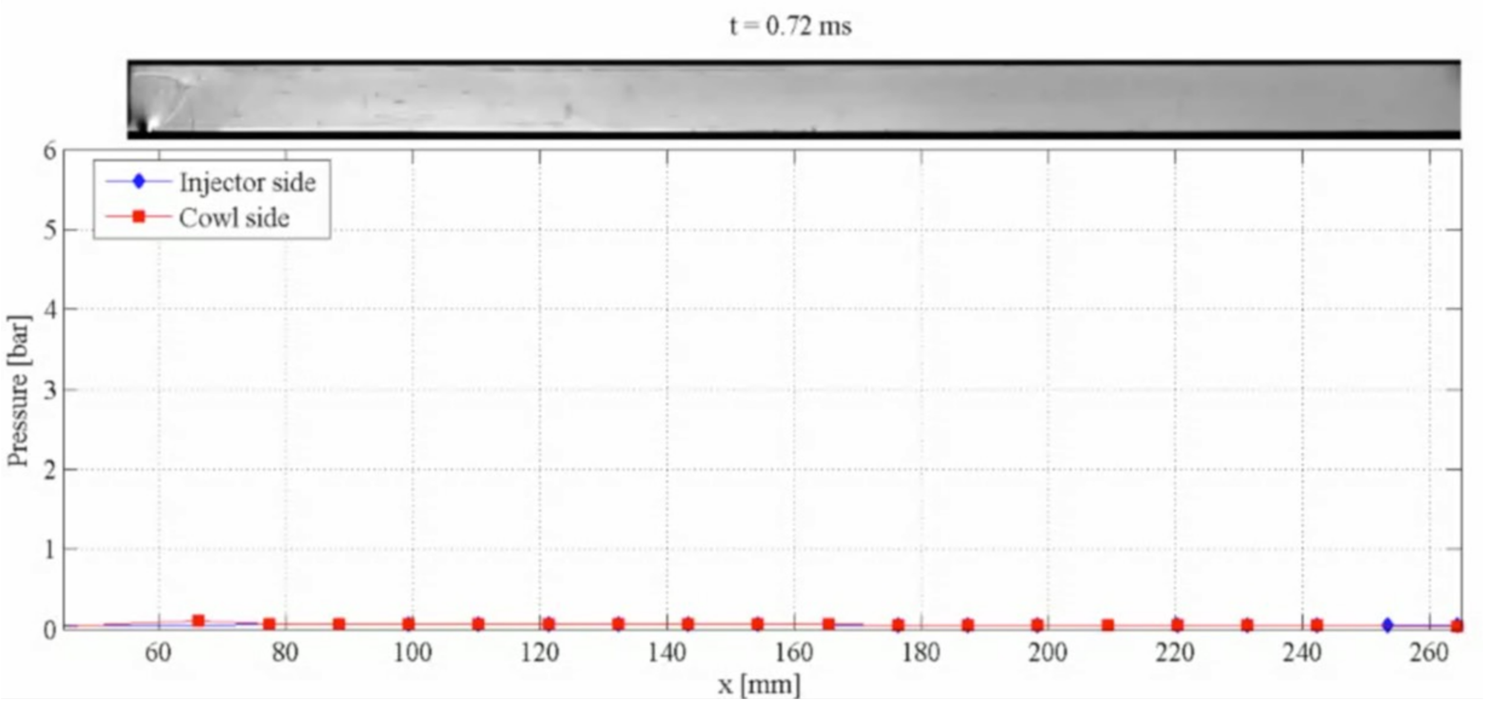
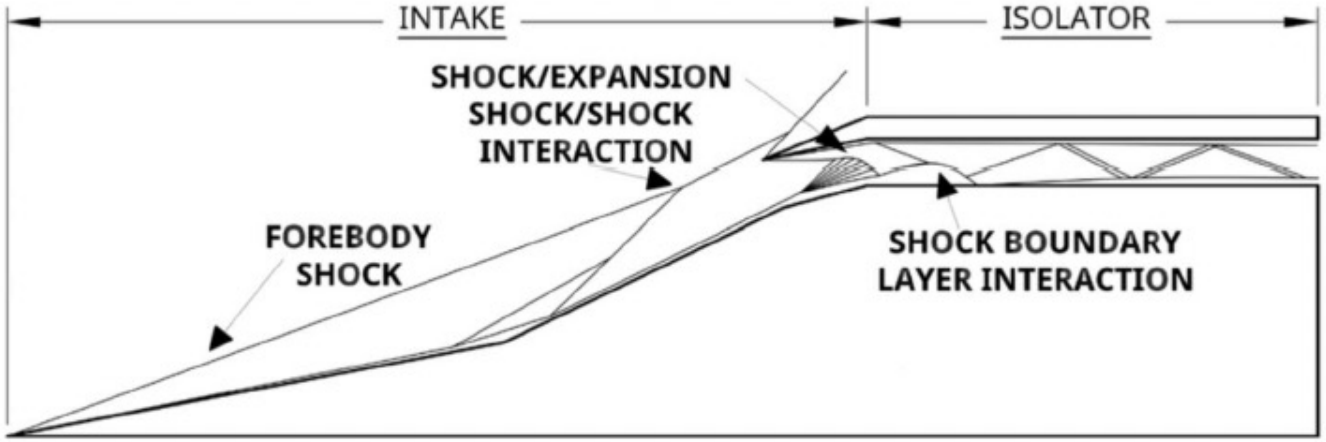
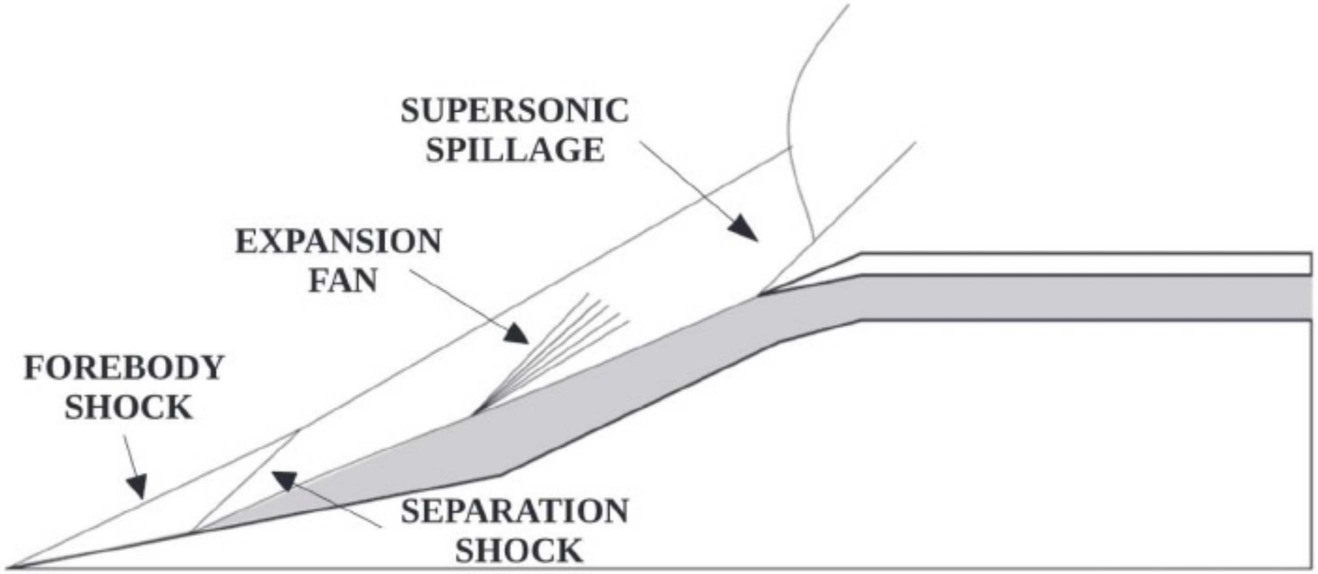
If the flow is supersonic, these disturbances must be shock waves (or a “shock train”) to move upstream.





(a)



(b)

(a) Diagram of a scramjet at nominal conditions and (b) scramjet at unstarted conditions.