PHYS 313

HW 03: Assignment 3

Due on February 20th, 2025 at 11:59 PM

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Problem 2.4:

Find the electric field a distance z above the center of a square loop (side a) carring a uniform line charge λ.

Solution

$$\text{For one side:} \quad d\vec{E} = \frac{1}{4\pi\epsilon_0}\,\lambda\,dx\,\frac{-x\,\hat{x} - \frac{a}{2}\,\hat{y} + z\,\hat{z}}{\left[x^2 + \left(\frac{a}{2}\right)^2 + z^2\right]^{3/2}},$$

$$E_z^{(\text{side})} = \frac{1}{4\pi\epsilon_0}\,\lambda\,z\int_{-a/2}^{a/2}\frac{dx}{\left[x^2 + \left(\frac{a}{2}\right)^2 + z^2\right]^{3/2}}.$$

$$\text{Let } \beta^2 = \left(\frac{a}{2}\right)^2 + z^2, \quad \int_{-a/2}^{a/2}\frac{dx}{(x^2+\beta^2)^{3/2}} = \frac{a}{\beta^2\sqrt{\left(\frac{a}{2}\right)^2 + \beta^2}}.$$

$$\text{Noting } \left(\frac{a}{2}\right)^2 + \beta^2 = \frac{a^2}{2} + z^2, \quad E_z^{(\text{side})} = \frac{\lambda\,z\,a}{4\pi\epsilon_0\left(\frac{a^2}{4} + z^2\right)\sqrt{\frac{a^2}{2} + z^2}}.$$

$$\text{By symmetry, the four sides contribute only in } \hat{z}: \quad E_z = \frac{\lambda\,a\,z}{\pi\epsilon_0\left(\frac{a^2}{4} + z^2\right)\sqrt{\frac{a^2}{2} + z^2}}.$$

Problem 2.6:

Find the electric field a distance z above the center of a flat circular disk of radius R that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

Solution

$$E_z = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right).$$

$$\lim_{R\to\infty} E_z = \frac{\sigma}{2\epsilon_0}, \qquad z \gg R: \quad E_z \approx \frac{\sigma R^2}{4\epsilon_0\, z^2}.$$

Problem 2.9:

Suppose the electric field in some region is found to $\mathbf{E} = kr^3\hat{\mathbf{r}}$, in spherical coordinates (k is some constant).

1. Find the charge density ρ.
2. Find the total charge contained in a sphere of radius R, centered at the origin. (Do it two different ways)

Solution

Part A

$$\vec{E} = k\,r^3\,\hat{r} \quad \Longrightarrow \quad \nabla\cdot\vec{E} = \frac{1}{r^2}\frac{d}{dr}\Big(r^2\,(kr^3)\Big) = \frac{1}{r^2}\frac{d}{dr}(kr^5) = \frac{5kr^4}{r^2} = 5kr^2.$$

$$\rho = \epsilon_0\,\nabla\cdot\vec{E} = 5\epsilon_0\,k\,r^2.$$

Part B

$$Q = \int_0^R \rho\,d\tau = \int_0^R 5\epsilon_0\,k\,r^2\,(4\pi r^2 dr) = 20\pi\epsilon_0\,k\int_0^R r^4 dr$$

$$= 20\pi\epsilon_0\,k\,\frac{R^5}{5} = 4\pi\epsilon_0\,k\,R^5.$$

Alternatively, using Gauss' law: $4\pi R^2 E(R) = 4\pi R^2(kR^3) = 4\pi kR^5, \quad Q = \epsilon_0\,(4\pi kR^5).$

Problem 2.15:

A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2}\,(a \leq r \leq b)$$

Find the electric field in the three regions:

1. $r < a$
2. $a < r < b$
3. $r > b$

Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

Solution

Part A

$$r < a: \quad Q_{\text{enc}} = 0 \quad \Longrightarrow \quad E = 0.$$

Part B

$$a < r < b: \quad Q_{\text{enc}} = \int_a^r \frac{k}{r'^2}(4\pi r'^2 dr') = 4\pi k\,(r-a),$$
$$4\pi r^2 E = \frac{4\pi k\,(r-a)}{\epsilon_0} \quad \Longrightarrow \quad E = \frac{k\,(r-a)}{\epsilon_0\, r^2}.$$

Part C

$$r > b: \quad Q_{\text{tot}} = 4\pi k\,(b-a), \qquad 4\pi r^2 E = \frac{4\pi k\,(b-a)}{\epsilon_0},$$
$$E = \frac{k\,(b-a)}{\epsilon_0\, r^2}.$$

Part D

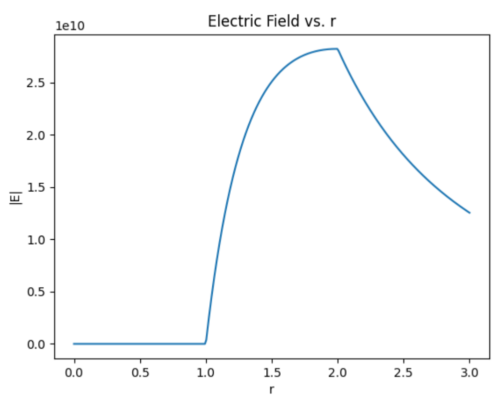

Figure 1: Electric Field $|\mathbf{E}|$ vs r

Problem 2.17:

An infinite plane slab, of thickness $2d$, carries a uniform volume charge density ρ. Find the electric field, as a function of y, where $y = 0$ at the center. Plot E versus y, calling E positive when it points in the $+y$ direction and negative when it points in the $-y$ direction.

Solution

$$\text{For } |y| \leq d: \quad E(y) = \frac{\rho\, y}{\epsilon_0}.$$

$$\text{For } |y| \geq d: \quad E(y) = \frac{\rho\, d}{\epsilon_0}\, \text{sgn}(y).$$

Part A

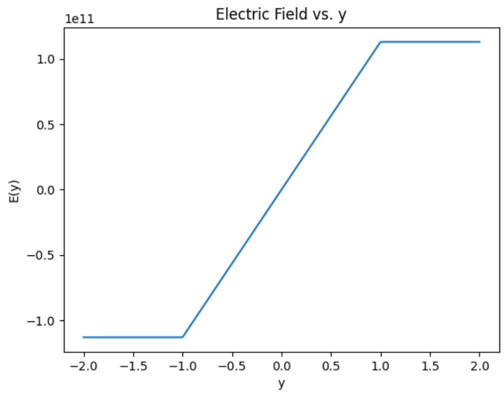

Figure 2: Electric Field $E(y)$ vs y

Problem 2.21:

Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch $V(r)$.

Solution

Part A

$$\text{For } r \geq R: \quad V(r) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}.$$

$$\text{For } r \leq R: \quad V(r) = \frac{q}{4\pi\epsilon_0}\frac{3R^2 - r^2}{2R^3}.$$

Part B

$$-\frac{dV}{dr}\bigg|_{r\geq R} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2} = E(r), \qquad -\frac{dV}{dr}\bigg|_{r\leq R} = \frac{q}{4\pi\epsilon_0}\frac{r}{R^3} = E(r).$$

Part C

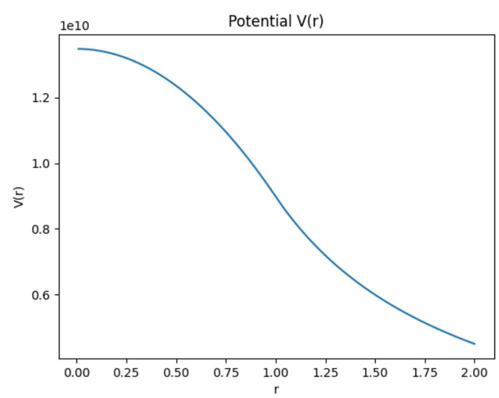

Figure 3: Potential $V(r)$