Special Relativity (The Big Idea)

① Physics is the same in all inertial reference frames.

Inertial reference frame = no external forces = constant velocity

⇒ No local experiment can tell if you're moving at a constant velocity or not.

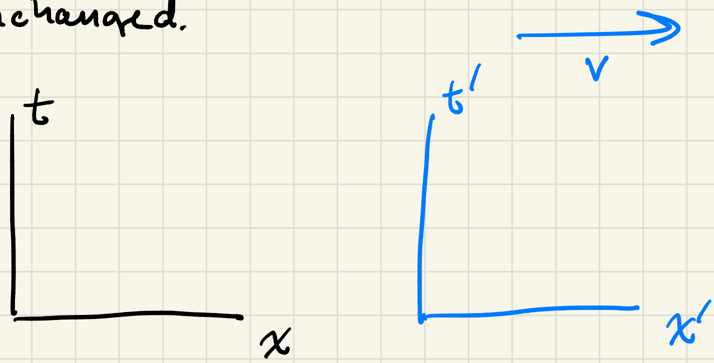
⇒ If you wake up locked in a train car with no windows, there is nothing you can do to tell if you're already moving.

② The speed of light is the same in all inertial frames.

↳ Why? Because experimentally, it works.
Einstein saw that Maxwell's equations only make sense if the speed of light is the same in all inertial reference frames.

① can also be rephrased as a statement about <u>coordinates</u>.

If you apply a coordinate transformation between two inertial frames, physics is unchanged.



Such coordinate transformations are known as <u>Lorentz</u> transformations.

matrix notation: $\overleftrightarrow{\Lambda}\,\vec{x} = \vec{x}'$

component notation: $\Lambda^{\mu'}{}_{\nu}\, x^{\nu} = x^{\mu'}$

code notation: [Sum(Λ[muprime, i] x[i]) for muprime in 1..4]

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & & 0 \\ -\gamma v & \gamma & & \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}$$

also 3 rotations

for "boost" in x-direction by v

$$\gamma \equiv \left(\sqrt{1-v^2}\right)^{-1}$$

We can also write this as

$$\Lambda^{\mu'}{}_{\nu} = \frac{dx^{\mu'}}{dx^{\nu}}$$

new coordinates

old coordinates

for x^{μ}, x^{ν} inertial frames

Lorentz transformations ensure c is the same before and after coordinate change.

General Relativity (The Big Idea)

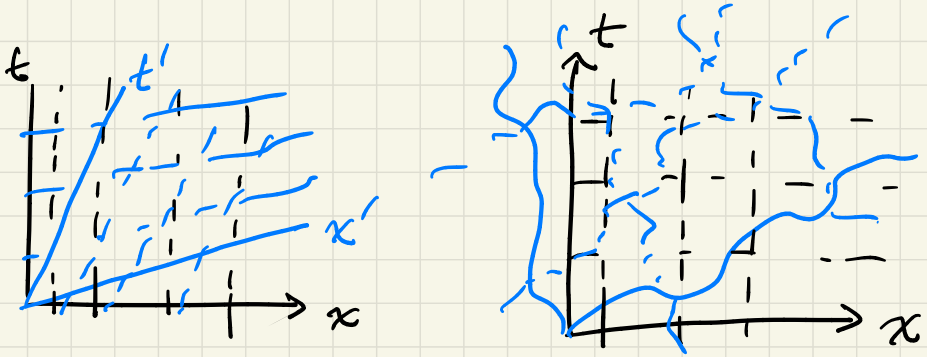
We saw that in SR, the laws of physics should not change under Lorentz transformations. (Rotations + boosts)

Let's elevate this statement...

① **Principle of General Covariance**:

The laws of physics should not change under <u>ANY</u> coordinate transformation.

⇒ * COORDINATES ARE NOT PHYSICAL *



② Equivalence Principle

Many ways to state it...

- All things fall at the same rate

- Gravitational and inertial mass are equal (We saw there are two eq.s with mass, $F_G = \frac{GMm}{r^2}$ and $F=ma$ → no reason these "m"s need to be the same!)

- No local experiment can tell you if you're in a gravitational field or are being accelerated.

- If you wake up in an elevator with no windows, there is no way to tell if it's accelerating upwards or if you're at rest in a gravitational field.

(Much like in SR, Einstein had one pretty obvious and believable postulate and one that was... inspired.)

We like SR and want it to stay true.

Non-accelerating observers should just see SR.

$\Rightarrow$ There is always a <u>local</u> frame that looks inertial in the SR sense. (though it will change between x and $x+dx$)

$\Rightarrow$ On small enough scales, things look like a Minkowski metric.

$\Rightarrow$ There is no way to detect a gravitational field by a local experiment.

$\Rightarrow$ If you wake up floating in an elevator with no windows, you can't tell if you're about to hit the ground.

Gedankenexperiments

1. You are in an elevator in free-fall and drop a ping-pong ball.

2. You are standing on Earth and drop a ping-pong ball.

3. You are in an elevator accelerating up and drop a ping-pong ball.

4. You are in an elevator accelerating up and hold a ping-pong ball in water, then release.

5. You are in a falling elevator and hold a ping-pong ball in water, then release.

6. You are orbiting the Earth on the ISS and hold a ping-pong ball in water, then release.

7. You are on a spinning amusement park ride and hold a ping-pong ball in water, then release.

Manifolds

N.B. We're playing fast and loose here with the math.

How do we build a mathematical structure that

1. Doesn't depend on coordinates
2. Looks locally flat
3. We can do physics on (AKA take derivatives)

Such a structure is called a manifold.

We're going to make space-time a manifold.

Manifold: n-dimensional manifold looks like $\mathbb{R}^n$ at any individual point

Permits a notion of differentiation.

Dimensionality n is the same at all points.

Easiest to understand by example.

Is it a manifold?

$\mathbb{R}^n$?	Yes
Sphere ?	Yes
Torus?	Yes
Cone?	Sort of (not smooth)
Two cones?	No!
Line segment?	Sort of (has boundary)

Ex. Which alphabet letters are 1-manifolds?

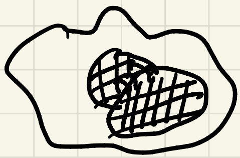
C, D, G, L, M, N, O, S, U, V, W, Z

COLUMNS is the longest word I can find

Coordinates on Manifolds

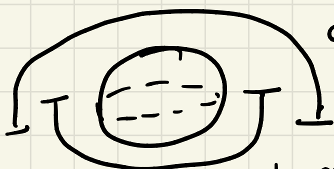
Covered by atlas of charts → coordinate systems in various patches

↳ coordinate charts must overlap smoothly



Many manifolds require more than one chart to cover.

Ex. S^2 sphere



The existence of world maps $\Rightarrow$ the Earth is flat???

Ex. Which letters require more than one chart?

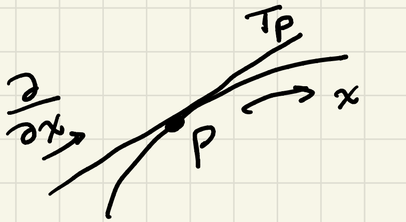
D, O

DODO is the longest word I can find

Vectors

At any point p in the manifold, there exists a tangent space T_p spanned by the partial derivative operators.

i.e. if you choose some coordinates x^μ, then you can express any vector in T_p as $V = V^\mu \partial_\mu$ where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$



Often, people just write V^μ as a vector but that is sloppy.

$$\boxed{V^\mu \text{ are JUST THE COMPONENTS OF } V \text{ for some choice of coordinates } x^\mu}$$

Same as the difference between

$\vec{v} = (1, 3, -4)$ and $\vec{v} = 1\hat{i} + 3\hat{j} + (-4)\hat{k}$

Vector components transform under coordinate change like

$$\boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}} \quad \text{for } x^{\mu} \to x^{\mu'}$$

Follows from chain rule since $\partial_\mu = \frac{\partial}{\partial x^\mu}$

$$\to \quad \partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}}.$$

V DOESN'T CHANGE – IT IS A COORDINATE INDEPENDENT OBJECT.

$$V = V^{\mu}\partial_{\mu} \longrightarrow V^{\mu'}\partial_{\mu'} = \cancel{\frac{\partial x^{\mu'}}{\partial x^{\mu}}} V^{\mu} \cancel{\frac{\partial x^{\mu}}{\partial x^{\mu'}}} \partial_{\mu}$$

$$= V^{\mu}\partial_{\mu} = V$$

You are going to become very familiar with this transformation rule.

Dual vectors

You might be wondering what's up with the indices up top versus down below.

When the index is up top, it is called <u>contravariant</u> and transforms LIKE A VECTOR, i.e. $V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$.

When the index is down below, it is called <u>covariant</u> and transforms like a "dual vector", i.e. $V_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V_{\mu}$

Dual vectors live in dual tangent space, hence we can define a basis for it via

$$dx^{\mu}\left(\frac{\partial}{\partial x^{\nu}}\right) = \frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\ \nu}$$

d denotes gradient operator: $df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}$

The grown-up version of $\vec{\nabla}$ from multivariable.

Here, we're treating x^{μ} like a <u>function</u> $x^{\mu}(x^{\sigma}) = x^{\mu}$

kind of weird.

$x(\vec{x}) = x, \ y(\vec{x}) = y \ldots$

Dual vectors map vectors to real numbers.

$$W_\mu V^\mu \in \mathbb{R}.$$

This number doesn't change under coordinate transformations.

Proof: $$\frac{\partial x^\mu}{\partial x^{\mu'}} W_\mu \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu = W_{\mu'} V^{\mu'}$$
$$= W_\mu V^\mu$$

Quantities that don't change under coordinate transformations are called SCALARS. Just a $\mathbb{R}$ number at a point p.

Scalar field: $\mathbb{R}$ at all points $p \in M$

Vector field: $V \in T_p$ for all $p \in M$

Tensors

Map k vectors and l dual vectors to $\mathbb{R}$.

$$T^{\mu_1 \mu_2 \mu_3 \dots \mu_k}{}_{\nu_1 \nu_2 \dots \nu_l}$$

(again, these are the *COMPONENTS*.

Tensors, like vectors, are invariant, but their componendts and bases change)

e.g. $T = T^{\mu\nu}{}_{\sigma\rho} \left(\partial_\mu \otimes \partial_\nu \otimes dx^\sigma \otimes dx^\rho \right)$

Contraction: sum over matching co and contra $T^{\mu}{}_{\mu}$

Tensor components transform exactly as you'd expect...

$$T^{\mu'}{}_{\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} T^{\mu}{}_{\nu}$$

Ex. Do the transformation of $S_{\mu\nu} = \begin{pmatrix} 1 & \\ & x^2 \end{pmatrix}$

$x' = \frac{2x}{y}$, $y' = \frac{y}{2}$ two ways.

1. Use above formula
2. $S = 1\, dx^2 + x^2 dy^2$, $\quad dx = y'dx' + x'dy'$, $\quad dy = 2\,dy'$