Chapter 5

Energy of a Particle

9/24/24

Dfn The **work** on P by $\bar{F}_P$ along γ_P is

$$\boxed{W_P^{(\bar{F}_P)}(\bar{r}_{P/O};\gamma_P) = \int_{\gamma_P} \bar{F}_P \cdot {}^I d\bar{r}_{P/O}}$$

force

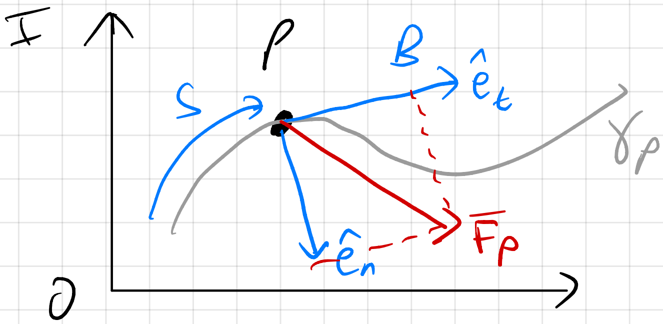
path of the particle

scalar product

Recall ${}^I\bar{v}_{P/O} = \frac{{}^I d}{dt}(\bar{r}_{P/O})$

$$\Rightarrow \quad {}^I d\bar{r}_{P/O} = {}^I\bar{v}_{P/O}\,dt$$

$$\Rightarrow \quad W_P^{(\bar{F}_P)}(\bar{r}_{P/O};\gamma_P) = \int_{\gamma_P} \bar{F}_P \cdot {}^I\bar{v}_{P/O}\,dt$$



B path frame $(P, \hat{e}_t, \hat{e}_n, \hat{e}_3)$

Recall $\hat{e}_t = {}^I\hat{v}_{P/O} = \frac{{}^I\bar{v}_{P/O}}{\|{}^I\bar{v}_{P/O}\|}$

$${}^I\bar{v}_{P/O} = \dot{s}\,\hat{e}_t = \frac{ds}{dt}\hat{e}_t$$

$$\Rightarrow \quad W_P^{(\bar{F}_P)}(\bar{r}_{P/O};\gamma_P) = \int_{\gamma_P} \bar{F}_P \cdot \hat{e}_t\,ds$$

tangential component

$$\bar{F}_P = (\bar{F}_P\cdot\hat{e}_t)\,\hat{e}_t + (\bar{F}_P\cdot\hat{e}_n)\,\hat{e}_n$$

$\hat{e}_t$: $\quad \bar{F}_P\cdot\hat{e}_t = \bar{F}_P\cdot\hat{e}_t$ tangential

$\hat{e}_n$: $\quad \bar{F}_P\cdot\hat{e}_n = \bar{F}_P\cdot\hat{e}_n$ normal

★ key idea: the normal component of $\bar{F}_P$ does no work!

Now suppose $\bar{F}_P$ is the total force on P:

$$W_P^{(tot)}(\bar{r}_{P/O};\gamma_P) = \int_{\gamma_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

$$= \int_{\gamma_P} m_P \frac{{}^{I}d}{\cancel{dt}}\left({}^{I}\bar{v}_{P/O}\right) {}^{I}\bar{v}_{P/O}\, \cancel{dt}$$

Start/end of path γ_P

$$= m_P \int_{t_1}^{t_2} {}^{I}\bar{v}_{P/O}\, {}^{I}d\bar{v}_{P/O}$$

$$\int v\,dv = \frac{1}{2}v^2 + c$$

$$= \frac{1}{2}m_P \underbrace{\|{}^{I}\bar{v}_{P/O}(t_2)\|^2}_{{}^{I}\bar{v}_{P/O}\cdot{}^{I}\bar{v}_{P/O}} - \underbrace{\frac{1}{2}m_P\|{}^{I}\bar{v}_{P/O}(t_1)\|^2}_{=T_{P/O}(t_1)}$$

Dfn the kinetic energy of P wrt O is

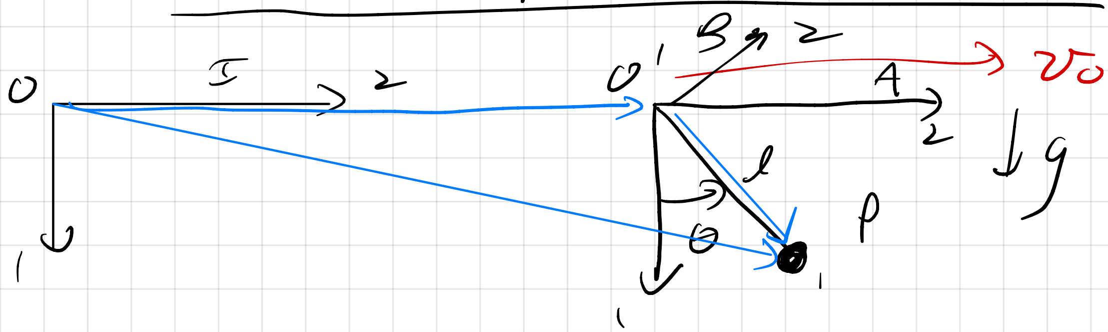
$$\boxed{T_{P/O} = \frac{1}{2}m_P\|{}^{I}\bar{v}_{P/O}\|^2}$$

$$W_P^{(tot)}(\bar{r}_{P/O};\gamma_P) = T_{P/O}(t_2) - T_{P/O}(t_1)$$

Work-energy #1

$$\boxed{T_{P/O}(t_2) = T_{P/O}(t_1) + W_P^{(tot)}(\bar{r}_{P/O};\gamma_P)}$$

Ex 5.2 K.E. of a pendulum in a car



$$T_{P/O} = \frac{1}{2} m_P \| {}^{I}\bar{v}_{P/O} \|^2$$

$$\bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}$$

$$\bar{r}_{P/O'} = \ell \hat{b}_1 \qquad {}^{I}\bar{v}_{P/O'} = {}^{I}\bar{\omega}^{B} \times \bar{r}_{P/O'}$$

	$\hat{b}_1$	$\hat{b}_2$
$\hat{a}_1 = \hat{e}_1$	$c\theta$	$-s\theta$
$\hat{a}_2 = \hat{e}_2$	$s\theta$	$c\theta$

$${}^{I}\bar{\omega}^{B} = \dot{\theta}\, \hat{b}_3$$

$${}^{I}\bar{v}_{P/O} = v_0 \hat{e}_2 + \ell \dot{\theta} \hat{b}_2$$

$$T_{P/O} = \frac{1}{2} m_P \left(v_0 \hat{e}_2 + \ell \dot{\theta} \hat{b}_2 \right) \cdot \left(v_0 \hat{e}_2 + \ell \dot{\theta} \hat{b}_2 \right)$$

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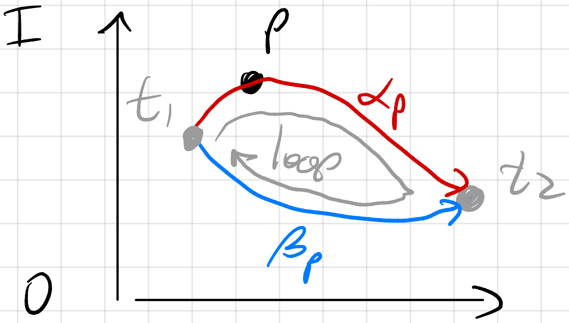
$$= \frac{1}{2} m_P \left(v_0^2 + 2 v_0 \ell \dot{\theta} c\theta + \ell^2 \dot{\theta}^2 \right)$$

$$T_{P/O'} = \frac{1}{2} m_P \| {}^{I}\bar{v}_{P/O'} \|^2$$

$$= \frac{1}{2} m_P \ell^2 \dot{\theta}^2 = T_{P/O} \text{ if } v_0 = 0$$

Dfn the work performed by a <u>conservative</u> force $\bar{F}_P$ depends only on the endpoints of the path

$$W_P^{(\bar{F}_P)}(\bar{r}_{P/O}; \gamma_P) = W_P^{(\bar{F}_P)}(t_1, t_2)$$



$$W_P^{(\bar{F}_P)}(\bar{r}_{P/O}; \alpha_P) = \int_{\alpha_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

$$W_P^{(\bar{F}_P)}(\bar{r}_{P/O}; \beta_P) = \int_{\beta_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

If $\bar{F}_P$ is <u>conservative</u>, consider the <u>loop</u> integral

$$\int_{\alpha_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O} + \int_{-\beta_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O} =$$

$$\underbrace{\int_{\alpha_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O} - \int_{\beta_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}}_{\text{equal (if } \bar{F}_P \text{ is conservative)}} = 0 = \oint \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

$$\boxed{W_P^{(tot)}(\bar{r}_{P/O}; \gamma_P) = W_P^{(c)}(t_1, t_2) + W_P^{(nc)}(\bar{r}_{P/O}; \gamma_P)}$$

conservative — non-conservative

conservative	non-conservative
gravity	damper
spring	air resistance
buoyancy	friction
weight	

Dfn the potential energy of P associated with $\bar{F}_P$ conservative

$$U_{P/O}^{(\bar{F}_P)}(\bar{r}_{P/O}) = -\int \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

Let $\bar{F}_P$ be conservative:

$$W_P^{(\bar{F}_P)}(t_1, t_2) = \int_{\gamma_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

$$= \int_{\bar{r}_{P/O}(t_1)}^{\bar{r}_{P/O}(t_2)} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

$$= U_{P/O}^{(\bar{F}_P)}(\bar{r}_{P/O}(t_1)) - U_{P/O}^{(\bar{F}_P)}(\bar{r}_{P/O}(t_2))$$

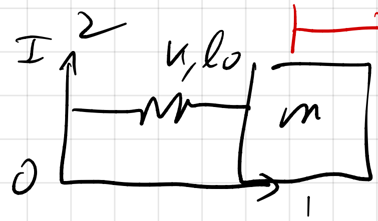
$$U_{P/O}^{(\bar{F}_P)}(\bar{r}_{P/O}(t_2)) = U_{P/O}^{(\bar{F}_P)}(\bar{r}_{P/O}(t_1)) - W_P^{(\bar{F}_P)}(t_1, t_2)$$

Now, let $\bar{F}_P$ be the sum of all cons. forces on P:

W-E #2
$$\boxed{U_{P/O}(t_2) = U_{P/O}(t_1) - W_P^{(c)}(t_1, t_2)}$$

all cons. forces on P

Ex Spring



$\longmapsto x$ displacement from unstretched length

Goal: Find $U_{P/O}$ associated with $\bar{F}_P = -kx\hat{e}_1$

$$U_{P/O}^{(\bar{F}_P)}(\bar{r}_{P/O}) = -\int \bar{F}_P \cdot {}^I d\bar{r}_{P/O}$$

$$\bar{r}_{P/O} = (l_0 + x)\hat{e}_1$$

$$^I d\bar{r}_{P/O} = {}^I\bar{v}_{P/O}\, dt = \dot{x}\hat{e}_1\, dt = dx\, \hat{e}_1$$

$$U_{P/O}^{(\bar{F}_P)}(x) = -\int (-kx\hat{e}_1) \cdot dx\, \hat{e}_1$$

$$= k\int x\, dx = \frac{1}{2}kx^2 + C$$

@ $x = 0$ $\quad U_{P/O}(0) = 0 + C$

$$\boxed{U_{P/O}^{(\bar{F}_P)}(x) = \frac{1}{2}kx^2} + \underbrace{U_{P/O}^{(\bar{F}_P)}(0)}_{=0 \text{ by assumption}}$$

Dfn total energy of P wrt O

$$\boxed{E_{P/O} = T_{P/O} + U_{P/O}}$$

W-E #1 $\quad T_{P/O}(t_2) = T_{P/O}(t_1) + W_P^{(tot)}$

$+$ #2 $\quad U_{P/O}(t_2) = U_{P/O}(t_1) - W_P^{(c)}$

$*$ W-E #3 $\quad \boxed{E_{P/O}(t_2) = E_{P/O}(t_1) + W_P^{(nc)}}$ $*$

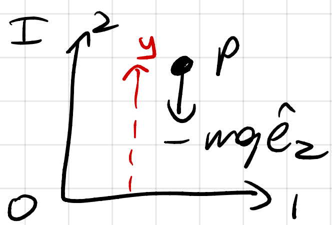
<u>Conservation of total energy</u> occurs if

$$W_P^{(nc)} = 0 \Rightarrow E_{P/O}(t) = E_{P/O}(0)$$

$$W_P^{(nc)}(\bar{r}_{P/O}; \gamma_P) = \int_{\gamma_P} \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

all of non-cons. forces in free-body diagram

<u>Ex</u> Potential energy of Weight



$$U_{P/O}^{(\bar{F}_P)}(\bar{r}_{P/O}) = -\int \bar{F}_P \cdot {}^{I}d\bar{r}_{P/O}$$

$$\bar{F}_P = -mg\hat{e}_2$$

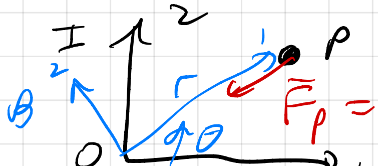
$${}^{I}d\bar{r}_{P/O} = {}^{I}\bar{v}_{P/O}\, dt = \dot{y}e_2\, dt = dy\,\hat{e}_2$$

$$U_{P/O}(y) = -\int -mg\,\hat{e}_2 \cdot dy\,\hat{e}_2 = mgy + C$$

@ $y = 0 \quad U_{P/O}(0) = C$

$$\boxed{U_{P/O}(y) = mgy} + \cancel{U_{P/O}(0)} \rightarrow 0$$

<u>Ex</u> P.E. of gravity



$$\bar{F}_P = -\frac{G m_o m_p}{r^2}\hat{b}_1$$

$$U_{P/O}^{(\bar{F}_P)}(\bar{r}_{P/O}) = -\int \bar{F}_P \cdot d\bar{r}_{P/O}$$

$$\bar{r}_{P/O} = r\hat{b}_1$$

$${}^{I}\bar{v}_{P/O} = \dot{r}\hat{b}_1 + r\dot{\theta}\hat{b}_2$$

$$\Rightarrow \quad {}^{I}d\bar{r}_{P/O} = dr\,\hat{b}_1 + r\,d\theta\,\hat{b}_2$$

$$\Rightarrow \quad U_{P/O}(r) = -\int -\frac{Gm_O m_P}{r^2}\hat{b}_1 \cdot \left(dr\,\hat{b}_1 + r\,d\theta\,\hat{b}_2\right)$$

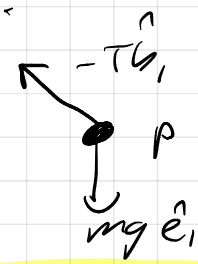
$$= Gm_O m_P \int \frac{dr}{r^2} = -\frac{Gm_O m_P}{r} + C$$

@ $r = \infty$ $\quad U_{P/O}(\infty) = C$

$$\boxed{U_{P/O}(r) = -\frac{Gm_O m_P}{r}} + \cancel{U_{P/O}(\infty)} \quad 0$$

Ex Find total energy of pend. in car.

Q: is it conserved?



normal? Tension

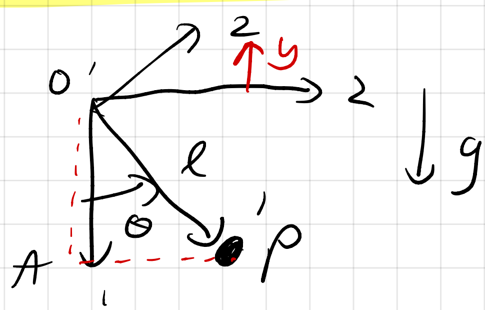
conservative weight

$$E_{P/O} = T_{P/O} + U_{P/O}$$

$$T_{P/O} = \frac{1}{2}m_P\left(v_0^2 + 2v_0 l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right)$$

$$U_{P/O}^{(\bar{W})} = mgy$$

$$= -mgl\cos\theta$$



$$E_{P/O}(t_2) = E_{P/O}(t_1) + W_P^{(nc)}$$

$$W_P^{(nc)}(\bar{r}_{P/O}; \gamma_P) = \int_{\gamma_P} \bar{T} \cdot {}^{I}d\bar{r}_{P/O}$$

$$-T\hat{b}_1 \cdot \left(dx\,\hat{e}_2 + l\,d\theta\,\hat{b}_2\right)$$

$\neq 0 \Rightarrow E_{P/O}$ is not conserved (unless car is stopped)

$$\bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}$$

$$= x\,\hat{e}_2 + l\,\hat{b}_1$$

$${}^{I}\bar{v}_{P/O}\,dt = dx\,\hat{e}_2 + l\,d\theta\,\hat{b}_2$$

Ex ① Find total energy of simple satellite

② Is it conserved?

$$-\frac{G m_O m_P}{r^2}\hat{b}_1$$

conservative

$$E_{P/O} = T_{P/O} + U_{P/O}$$

$$= -\frac{G m_O m_P}{r}$$

$$T_{P/O} = \frac{1}{2} m_P \left\| {}^I\bar{v}_{P/O} \right\|^2$$

$${}^I\bar{v}_{P/O} \cdot {}^I\bar{v}_{P/O}$$

$$\bar{r}_{P/O} = r\hat{b}_1$$

$${}^I\bar{v}_{P/O} = \dot{r}\hat{b}_1 + r\dot{\theta}\hat{b}_2 \Rightarrow \left\|{}^I\bar{v}_{P/O}\right\|^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$E_{P/O} = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{G m_O m_P}{r}$$

$$E_{P/O}(t_2) = E_{P/O}(t_1) + W_P^{(nc)}$$

$\Rightarrow$ $E_{P/O}$ is conserved