PHYS 313: HW 01

Assignment 1

Due on February 6th, 2025 at 11:59 PM

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Problem 1.6:

Prove that

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = \mathbf{0}.$$

Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?

Solution

Part A

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B}) && (1)\\
\mathbf{B} \times (\mathbf{C} \times \mathbf{A}) &= \mathbf{C}\,(\mathbf{B} \cdot \mathbf{A}) - \mathbf{A}\,(\mathbf{A} \cdot \mathbf{C}) && (2)\\
\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}\,(\mathbf{C} \cdot \mathbf{B}) - \mathbf{B}\,(\mathbf{C} \cdot \mathbf{A}) && (3)
\end{aligned}$$

$$\begin{aligned}
1 + 2 + 3 = &\ \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})\\
&+ \mathbf{C}\,(\mathbf{B} \cdot \mathbf{A}) - \mathbf{A}\,(\mathbf{A} \cdot \mathbf{C})\\
&+ \mathbf{A}\,(\mathbf{C} \cdot \mathbf{B}) - \mathbf{B}\,(\mathbf{C} \cdot \mathbf{A})
\end{aligned}$$

The dot product is commuttative, so $\mathbf{A}\,(\mathbf{B} \cdot \mathbf{C}) = \mathbf{A}\,(\mathbf{C} \cdot \mathbf{B})$.

$$\therefore 1 + 2 + 3 = \mathbf{0} \quad \square$$

Part B

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})\\
(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}\,(\mathbf{B} \cdot \mathbf{C})\\
\therefore \mathbf{A}\,(\mathbf{B} \cdot \mathbf{C}) = \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B}) \implies \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \quad \square
\end{aligned}$$

Problem 1.13:

Let $\boldsymbol{r}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let r be its length. Show that

1. $\nabla\left(r^2\right) = 2r$.
2. $\nabla(1/r) = -\hat{r}/r^2$.
3. What is the general formula for $\nabla\left(\tau^n\right)$?

Solution

Part A

To show that

$$\nabla\left(r^2\right) = 2\boldsymbol{r},$$

first note that

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

Taking the gradient with respect to (x, y, z) gives:

$$\frac{\partial(r^2)}{\partial x} = 2(x - x'), \quad \frac{\partial(r^2)}{\partial y} = 2(y - y'), \quad \frac{\partial(r^2)}{\partial z} = 2(z - z').$$

Thus, the gradient is

$$\nabla(r^2) = (2(x - x'),\, 2(y - y'),\, 2(z - z')) = 2\boldsymbol{r} \quad \square$$

Part B

To show that

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2},$$

we first apply the chain rule:

$$\nabla\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\nabla r = -\frac{1}{r^2}\nabla r.$$

Since

$$r = \sqrt{r^2} \quad \Longrightarrow \quad \nabla r = \frac{1}{2r}\nabla(r^2) = \frac{1}{2r}(2\boldsymbol{r}) = \frac{\boldsymbol{r}}{r} = \hat{r},$$

it follows that

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r} \quad \square$$

Part C

That is, we wish to find the gradient of τ^n for a general exponent n. Again, by the chain rule,

$$\nabla(\tau^n) = \frac{d}{d\tau}(\tau^n)\nabla\tau = n\,\tau^{n-1}\nabla\tau.$$

Since $\nabla\tau = \hat{\tau} = \frac{\boldsymbol{\tau}}{\tau}$, we have

$$\nabla(\tau^n) = n\,\tau^{n-1}\left(\frac{\boldsymbol{\tau}}{\tau}\right) = n\,\tau^{n-2}\boldsymbol{\tau} \quad \square$$

Problem 1.15:

Calculate the divergence of the following vector functions:

1. $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$.
2. $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$.
3. $\mathbf{v}_c = y^2\hat{\mathbf{x}} + \left(2xy + z^2\right)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$.

Solution

Part A

For $\mathbf{v}_a$, we have

$$\mathbf{v}_a = \left(x^2,\ 3xz^2,\ -2xz\right).$$

The divergence is given by

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz).$$

Evaluating each term:

$$\frac{\partial}{\partial x}(x^2) = 2x, \quad \frac{\partial}{\partial y}(3xz^2) = 0, \quad \frac{\partial}{\partial z}(-2xz) = -2x.$$

Thus,

$$\nabla \cdot \mathbf{v}_a = 2x + 0 - 2x = 0.$$

Part B

For $\mathbf{v}_b$, we have

$$\mathbf{v}_b = (xy,\ 2yz,\ 3zx).$$

The divergence is

$$\nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx).$$

Computing the derivatives:

$$\frac{\partial}{\partial x}(xy) = y, \quad \frac{\partial}{\partial y}(2yz) = 2z, \quad \frac{\partial}{\partial z}(3zx) = 3x.$$

Hence,

$$\nabla \cdot \mathbf{v}_b = y + 2z + 3x.$$

Part C

For $\mathbf{v}_c$, we have

$$\mathbf{v}_c = \left(y^2,\ 2xy + z^2,\ 2yz\right).$$

The divergence is

$$\nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz).$$

Calculating each derivative:

$$\frac{\partial}{\partial x}(y^2) = 0, \quad \frac{\partial}{\partial y}(2xy + z^2) = 2x, \quad \frac{\partial}{\partial z}(2yz) = 2y.$$

Thus,

$$\nabla \cdot \mathbf{v}_c = 0 + 2x + 2y = 2x + 2y.$$

Problem 1.25:

1. Check product rule (iv) (by calculating each term separately) for the functions $\mathbf{A} = x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}$; $\quad \mathbf{B} = 3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}$
2. Do the same for product rule (ii): $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}$
3. Do the same for rule (vi): $\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Solution

Part A [Product Rule (iv): Divergence of a Cross Product]

We wish to check that

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}).$$

Step 1. Compute $\mathbf{A}\times\mathbf{B}$.
Using the determinant formula,

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} = \Big(2y\cdot 0 - 3z\cdot(-2x),\ -\Big(x\cdot 0 - 3z\cdot 3y\Big),\ x\cdot(-2x) - 2y\cdot(3y)\Big).$$

Thus,

$$\mathbf{A}\times\mathbf{B} = \big(6xz,\ 9yz,\ -2x^2 - 6y^2\big).$$

Step 2. Compute the left-hand side (LHS):

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9yz) + \frac{\partial}{\partial z}(-2x^2 - 6y^2).$$

We have:

$$\frac{\partial}{\partial x}(6xz) = 6z, \quad \frac{\partial}{\partial y}(9yz) = 9z, \quad \frac{\partial}{\partial z}(-2x^2 - 6y^2) = 0.$$

Therefore,

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = 6z + 9z = 15z.$$

Step 3. Compute the right-hand side (RHS).
First, compute $\nabla\times\mathbf{A}$. Since

$$\mathbf{A} = (x,\, 2y,\, 3z),$$

its curl is

$$\nabla\times\mathbf{A} = \Big(\frac{\partial(3z)}{\partial y} - \frac{\partial(2y)}{\partial z},\ \frac{\partial(x)}{\partial z} - \frac{\partial(3z)}{\partial x},\ \frac{\partial(2y)}{\partial x} - \frac{\partial(x)}{\partial y}\Big) = (0-0,\ 0-0,\ 0-0) = (0,0,0).$$

Next, compute $\nabla\times\mathbf{B}$ for

$$\mathbf{B} = (3y,\, -2x,\, 0).$$

We find

$$\nabla\times\mathbf{B} = \Big(\frac{\partial 0}{\partial y} - \frac{\partial(-2x)}{\partial z},\ \frac{\partial(3y)}{\partial z} - \frac{\partial 0}{\partial x},\ \frac{\partial(-2x)}{\partial x} - \frac{\partial(3y)}{\partial y}\Big) = (0-0,\ 0-0,\ -2-3) = (0,0,-5).$$

Thus,

$$\mathbf{B}\cdot(\nabla\times\mathbf{A}) = \mathbf{B}\cdot(0,0,0) = 0,$$

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and

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = (x,\, 2y,\, 3z) \cdot (0, 0, -5) = -15z.$$

Therefore, the RHS is

$$\mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) = 0 - (-15z) = 15z.$$

Since LHS $= 15z$ equals RHS $= 15z$, the product rule (iv) is verified.

Part B [Product Rule (ii): Gradient of a Dot Product]

The identity to verify is

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}.$$

Step 1. Compute the scalar product $\mathbf{A} \cdot \mathbf{B}$.

$$\mathbf{A} \cdot \mathbf{B} = x\,(3y) + 2y\,(-2x) + 3z\,(0) = 3xy - 4xy = -xy.$$

Then,

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \nabla(-xy) = \left(\frac{\partial(-xy)}{\partial x},\ \frac{\partial(-xy)}{\partial y},\ \frac{\partial(-xy)}{\partial z}\right) = (-y,\, -x,\, 0).$$

Step 2. Evaluate the four terms on the RHS.

(a) Term 1: $\mathbf{A} \times (\nabla \times \mathbf{B})$.

We already found $\nabla \times \mathbf{B} = (0, 0, -5)$. Thus,

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = (x,\, 2y,\, 3z) \times (0, 0, -5).$$

Using the determinant formula,

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} = \Big(2y(-5) - 3z(0),\ -\big(x(-5) - 3z(0)\big),\ x(0) - 2y(0)\Big) = (-10y,\ 5x,\ 0).$$

(b) Term 2: $\mathbf{B} \times (\nabla \times \mathbf{A})$.

We have $\nabla \times \mathbf{A} = (0, 0, 0)$, so

$$\mathbf{B} \times (\nabla \times \mathbf{A}) = \mathbf{B} \times (0, 0, 0) = (0, 0, 0).$$

(c) Term 3: $(\mathbf{A} \cdot \nabla)\mathbf{B}$.

This denotes the directional derivative of $\mathbf{B}$ along $\mathbf{A}$. With $\mathbf{B} = (3y,\, -2x,\, 0)$,

$$(\mathbf{A} \cdot \nabla)(3y) = x\,\partial_x(3y) + 2y\,\partial_y(3y) + 3z\,\partial_z(3y) = x \cdot 0 + 2y \cdot 3 + 3z \cdot 0 = 6y,$$

$$(\mathbf{A} \cdot \nabla)(-2x) = x\,\partial_x(-2x) + 2y\,\partial_y(-2x) + 3z\,\partial_z(-2x) = x\,(-2) + 0 + 0 = -2x,$$

$$(\mathbf{A} \cdot \nabla)(0) = 0.$$

Thus,

$$(\mathbf{A} \cdot \nabla)\mathbf{B} = (6y,\, -2x,\, 0).$$

(d) Term 4: $(\mathbf{B} \cdot \nabla)\mathbf{A}$.

For $\mathbf{A} = (x,\, 2y,\, 3z)$,

$$(\mathbf{B} \cdot \nabla)(x) = 3y\,\partial_x(x) + (-2x)\,\partial_y(x) + 0\,\partial_z(x) = 3y \cdot 1 + (-2x) \cdot 0 = 3y,$$

$$(\mathbf{B} \cdot \nabla)(2y) = 3y\,\partial_x(2y) + (-2x)\,\partial_y(2y) + 0\,\partial_z(2y) = 3y \cdot 0 + (-2x) \cdot 2 = -4x,$$

$$(\mathbf{B} \cdot \nabla)(3z) = 3y\,\partial_x(3z) + (-2x)\,\partial_y(3z) + 0\,\partial_z(3z) = 0.$$

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Thus,

$$(\mathbf{B}\cdot\nabla)\mathbf{A} = (3y,\ -4x,\ 0).$$

Step 3. Sum the four terms:

$$\begin{aligned}
&\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \\
&\qquad = (-10y,\ 5x,\ 0) + (0,0,0) + (6y,\ -2x,\ 0) + (3y,\ -4x,\ 0) \\
&\qquad = \big((-10y+6y+3y),\ (5x-2x-4x),\ 0\big) \\
&\qquad = (-y,\ -x,\ 0).
\end{aligned}$$

This agrees with the left-hand side,

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (-y,\ -x,\ 0).$$

Thus, product rule (ii) is verified.

Part C [Product Rule (vi): Curl of a Cross Product]

The identity to verify is

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}\,(\nabla\cdot\mathbf{B}) - \mathbf{B}\,(\nabla\cdot\mathbf{A}).$$

Step 1. Compute $\nabla\times(\mathbf{A}\times\mathbf{B})$.
We already obtained

$$\mathbf{A}\times\mathbf{B} = (6xz,\ 9yz,\ -2x^2-6y^2).$$

Now, taking the curl,

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \left(\frac{\partial}{\partial y}(-2x^2-6y^2) - \frac{\partial}{\partial z}(9yz),\ \frac{\partial}{\partial z}(6xz) - \frac{\partial}{\partial x}(-2x^2-6y^2),\ \frac{\partial}{\partial x}(9yz) - \frac{\partial}{\partial y}(6xz)\right).$$

Calculating each component:

$$\begin{aligned}
&\text{(i)}\ \frac{\partial}{\partial y}(-2x^2-6y^2) = -12y, \quad \frac{\partial}{\partial z}(9yz) = 9y, \\
&\quad \Rightarrow (\nabla\times(\mathbf{A}\times\mathbf{B}))_x = -12y - 9y = -21y; \\
&\text{(ii)}\ \frac{\partial}{\partial z}(6xz) = 6x, \quad \frac{\partial}{\partial x}(-2x^2-6y^2) = -4x, \\
&\quad \Rightarrow (\nabla\times(\mathbf{A}\times\mathbf{B}))_y = 6x - (-4x) = 10x; \\
&\text{(iii)}\ \frac{\partial}{\partial x}(9yz) = 0, \quad \frac{\partial}{\partial y}(6xz) = 0, \\
&\quad \Rightarrow (\nabla\times(\mathbf{A}\times\mathbf{B}))_z = 0 - 0 = 0.
\end{aligned}$$

Thus,

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (-21y,\ 10x,\ 0).$$

Step 2. Evaluate the right-hand side (RHS).
(a) We already computed in part (ii):

$$(\mathbf{B}\cdot\nabla)\mathbf{A} = (3y,\ -4x,\ 0), \quad (\mathbf{A}\cdot\nabla)\mathbf{B} = (6y,\ -2x,\ 0).$$

(b) Next, compute the divergences.
For $\mathbf{B} = (3y,\ -2x,\ 0)$,

$$\nabla\cdot\mathbf{B} = \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial y}(-2x) + \frac{\partial}{\partial z}(0) = 0+0+0 = 0.$$

For $\mathbf{A} = (x,\, 2y,\, 3z)$,

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) = 1 + 2 + 3 = 6.$$

Hence,

$$\mathbf{A}\,(\nabla \cdot \mathbf{B}) = (0, 0, 0), \quad \mathbf{B}\,(\nabla \cdot \mathbf{A}) = 6\,(3y,\, -2x,\, 0) = (18y,\, -12x,\, 0).$$

Step 3. Combine the terms:

$$\begin{aligned}
\text{RHS} &= (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) \\
&= (3y,\, -4x,\, 0) - (6y,\, -2x,\, 0) + (0, 0, 0) - (18y,\, -12x,\, 0) \\
&= \big[(3y - 6y - 18y),\, (-4x + 2x + 12x),\, 0\big] \\
&= (-21y,\; 10x,\; 0).
\end{aligned}$$

This matches the left-hand side computed earlier.
Thus, product rule (vi) is verified.

Problem 1.33:

Test the divergence theorem for the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}},$$

taking the volume to be a cube with side length 2.

Solution

We first compute the divergence of $\mathbf{v}$. Since

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx),$$

we have

$$\frac{\partial}{\partial x}(xy) = y, \quad \frac{\partial}{\partial y}(2yz) = 2z, \quad \frac{\partial}{\partial z}(3zx) = 3x.$$

Thus,

$$\nabla \cdot \mathbf{v} = y + 2z + 3x.$$

Volume Integral: We now evaluate the volume integral

$$\int_V (\nabla \cdot \mathbf{v})\, dV = \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x)\, dx\, dy\, dz.$$

This integral splits into three parts:

$$I_1 = \int_0^2 \int_0^2 \int_0^2 y\, dx\, dy\, dz, \quad I_2 = \int_0^2 \int_0^2 \int_0^2 2z\, dx\, dy\, dz, \quad I_3 = \int_0^2 \int_0^2 \int_0^2 3x\, dx\, dy\, dz.$$

Since the integrals are separable, we compute:

$$I_1 = \left(\int_0^2 y\, dy\right)\left(\int_0^2 dx\right)\left(\int_0^2 dz\right) = \left[\frac{y^2}{2}\right]_0^2 \cdot (2)(2) = \left(\frac{4}{2}\right)(4) = 2 \cdot 4 = 8,$$

$$I_2 = 2\left(\int_0^2 z\, dz\right)\left(\int_0^2 dx\right)\left(\int_0^2 dy\right) = 2\left[\frac{z^2}{2}\right]_0^2 \cdot (2)(2) = 2 \cdot \left(\frac{4}{2}\right) \cdot 4 = 2 \cdot 2 \cdot 4 = 16,$$

$$I_3 = 3\left(\int_0^2 x\, dx\right)\left(\int_0^2 dy\right)\left(\int_0^2 dz\right) = 3\left[\frac{x^2}{2}\right]_0^2 \cdot (2)(2) = 3 \cdot \left(\frac{4}{2}\right) \cdot 4 = 3 \cdot 2 \cdot 4 = 24.$$

Thus, the total volume integral is

$$\int_V (\nabla \cdot \mathbf{v})\, dV = I_1 + I_2 + I_3 = 8 + 16 + 24 = 48.$$

Surface Flux: Next, we compute the flux of $\mathbf{v}$ through the surface of the cube. The divergence theorem states that

$$\int_V (\nabla \cdot \mathbf{v})\, dV = \oint_{\partial V} \mathbf{v} \cdot \hat{n}\, dS.$$

The cube has six faces. We calculate the flux for each face.

Face 1 ($x = 0$): The outward normal is $\hat{n} = (-1, 0, 0)$. On this face, $x = 0$ so

$$\mathbf{v} = (0 \cdot y,\, 2yz,\, 3z \cdot 0) = (0,\, 2yz,\, 0).$$

Thus,

$$\mathbf{v} \cdot \hat{n} = 0,$$

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and the flux is zero.
Face 2 ($x = 2$): The outward normal is $\hat{n} = (1, 0, 0)$. On this face, $x = 2$ so

$$\mathbf{v} = (2y,\, 2yz,\, 3z \cdot 2) = (2y,\, 2yz,\, 6z).$$

Thus,

$$\mathbf{v} \cdot \hat{n} = 2y.$$

The flux through this face is

$$\int_{z=0}^{2} \int_{y=0}^{2} 2y\, dy\, dz.$$

Since

$$\int_{y=0}^{2} 2y\, dy = y^2\Big|_0^2 = 4, \quad \int_{z=0}^{2} dz = 2,$$

the flux is $4 \times 2 = 8$.
Face 3 ($y = 0$): The outward normal is $\hat{n} = (0, -1, 0)$. Here, $y = 0$ so

$$\mathbf{v} = (x \cdot 0,\, 2 \cdot 0 \cdot z,\, 3zx) = (0,\, 0,\, 3zx),$$

and hence $\mathbf{v} \cdot \hat{n} = 0$. The flux is zero.
Face 4 ($y = 2$): The outward normal is $\hat{n} = (0, 1, 0)$. On this face, $y = 2$ so

$$\mathbf{v} = (x \cdot 2,\, 2 \cdot 2 \cdot z,\, 3zx) = (2x,\, 4z,\, 3zx).$$

Thus,

$$\mathbf{v} \cdot \hat{n} = 4z.$$

The flux through this face is

$$\int_{z=0}^{2} \int_{x=0}^{2} 4z\, dx\, dz.$$

Since

$$\int_{x=0}^{2} dx = 2, \quad \int_{z=0}^{2} 4z\, dz = 4\left[\frac{z^2}{2}\right]_0^2 = 4 \cdot 2 = 8,$$

the flux is $2 \times 8 = 16$.
Face 5 ($z = 0$): The outward normal is $\hat{n} = (0, 0, -1)$. On this face, $z = 0$ so

$$\mathbf{v} = (xy,\, 2yz,\, 3zx) = (xy,\, 0,\, 0),$$

and therefore $\mathbf{v} \cdot \hat{n} = 0$. The flux is zero.
Face 6 ($z = 2$): The outward normal is $\hat{n} = (0, 0, 1)$. On this face, $z = 2$ so

$$\mathbf{v} = (xy,\, 2y \cdot 2,\, 3x \cdot 2) = (xy,\, 4y,\, 6x).$$

Thus,

$$\mathbf{v} \cdot \hat{n} = 6x.$$

The flux through this face is

$$\int_{y=0}^{2} \int_{x=0}^{2} 6x\, dx\, dy.$$

Here,

$$\int_{x=0}^{2} 6x\, dx = 6\left[\frac{x^2}{2}\right]_0^2 = 6 \cdot 2 = 12, \quad \int_{y=0}^{2} dy = 2,$$

so the flux is $12 \times 2 = 24$.
Total Flux: Summing the fluxes from all six faces yields

$$0 + 8 + 0 + 16 + 0 + 24 = 48.$$

Since the volume integral of the divergence is 48 and the net flux through the surface is also 48, the divergence theorem is verified.

Problem 1.34:

Test Stokes' theorem for the function

$$\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}},$$

using an isosceles right triangle, lying in the yz plane, with side length 2.

Solution

We wish to verify Stokes' theorem,

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{v}) \cdot \hat{n}\, dS,$$

where S is the surface (our triangle) with boundary C and $\hat{n}$ is a unit normal to S. Since the triangle lies in the yz plane ($x = 0$), we choose

$$\hat{n} = \hat{\mathbf{x}},$$

which is consistent with a counterclockwise orientation of C when viewed from the positive x direction.

Step 1. Compute $\nabla \times \mathbf{v}$.
Given

$$\mathbf{v} = (xy,\ 2yz,\ 3zx),$$

its curl is computed by

$$\nabla \times \mathbf{v} = \Big(\frac{\partial}{\partial y}(3zx) - \frac{\partial}{\partial z}(2yz),\ \frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3zx),\ \frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\Big).$$

Evaluating each component:

$$\begin{aligned}
(\nabla \times \mathbf{v})_x &= \frac{\partial(3zx)}{\partial y} - \frac{\partial(2yz)}{\partial z} = 0 - 2y = -2y,\\
(\nabla \times \mathbf{v})_y &= \frac{\partial(xy)}{\partial z} - \frac{\partial(3zx)}{\partial x} = 0 - 3z = -3z,\\
(\nabla \times \mathbf{v})_z &= \frac{\partial(2yz)}{\partial x} - \frac{\partial(xy)}{\partial y} = 0 - x = -x.
\end{aligned}$$

Thus,

$$\nabla \times \mathbf{v} = (-2y,\ -3z,\ -x).$$

On the surface S we have $x = 0$, so the curl reduces to

$$\nabla \times \mathbf{v} = (-2y,\ -3z,\ 0).$$

Step 2. Evaluate the Surface Integral.
The surface integral is

$$\iint_S (\nabla \times \mathbf{v}) \cdot \hat{n}\, dS.$$

Since $\hat{n} = (1, 0, 0)$, we have

$$(\nabla \times \mathbf{v}) \cdot \hat{n} = (-2y,\ -3z,\ 0) \cdot (1, 0, 0) = -2y.$$

Parameterize the triangle in the yz plane using y and z. The region is given by

$$0 \le y \le 2, \quad 0 \le z \le 2 - y.$$

Thus, the surface integral becomes

$$\iint_S (-2y)\, dS = \int_{y=0}^{2} \int_{z=0}^{2-y} (-2y)\, dz\, dy.$$

Integrate with respect to z:

$$\int_{z=0}^{2-y} (-2y)\, dz = -2y\,(2-y).$$

Then,

$$\iint_S (-2y)\, dS = -2 \int_0^2 y(2-y)\, dy.$$

Compute the integral:

$$\int_0^2 y(2-y)\, dy = \int_0^2 (2y - y^2)\, dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = \left(4 - \frac{8}{3}\right) = \frac{4}{3}.$$

Thus,

$$\iint_S (\nabla \times \mathbf{v}) \cdot \hat{n}\, dS = -2 \cdot \frac{4}{3} = -\frac{8}{3}.$$

Step 3. Evaluate the Line Integral.
Next, we compute the line integral

$$\oint_C \mathbf{v} \cdot d\mathbf{r},$$

where C is the boundary of the triangle. Note that every point on C lies in the yz plane ($x = 0$), so on C the vector field becomes

$$\mathbf{v} = (xy,\ 2yz,\ 3zx) = (0,\ 2yz,\ 0).$$

We break the boundary C into three segments:

Segment AB: From $A = (0,0,0)$ to $B = (0,2,0)$. Parameterize by

$$\mathbf{r}_{AB}(t) = (0,\ 2t,\ 0), \quad 0 \le t \le 1.$$

Then,

$$d\mathbf{r}_{AB} = (0,\ 2\,dt,\ 0).$$

On this segment, $y = 2t$ and $z = 0$ so

$$\mathbf{v} = (0,\ 2 \cdot (2t) \cdot 0,\ 0) = (0,0,0).$$

Thus,

$$\int_{AB} \mathbf{v} \cdot d\mathbf{r} = 0.$$

Segment BC: From $B = (0,2,0)$ to $C = (0,0,2)$. A suitable parameterization is

$$\mathbf{r}_{BC}(t) = (0,\ 2(1-t),\ 2t), \quad 0 \le t \le 1.$$

Then,

$$d\mathbf{r}_{BC} = (0,\ -2\,dt,\ 2\,dt)\,.$$

On this segment, $y = 2(1-t)$ and $z = 2t$, so

$$\mathbf{v} = \Big(0,\ 2 \cdot [2(1-t)] \cdot (2t),\ 0\Big) = \big(0,\ 8t(1-t),\ 0\big).$$

Problem 1.34 continued on next page...

Thus, the dot product is

$$\mathbf{v} \cdot d\mathbf{r}_{BC} = (0,\, 8t(1-t),\, 0) \cdot (0, -2, 2)\, dt = -16t(1-t)\, dt.$$

Hence,

$$\int_{BC} \mathbf{v} \cdot d\mathbf{r} = \int_0^1 -16t(1-t)\, dt.$$

Compute the integral:

$$\int_0^1 t(1-t)\, dt = \int_0^1 (t - t^2)\, dt = \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

so that

$$\int_{BC} \mathbf{v} \cdot d\mathbf{r} = -16 \cdot \frac{1}{6} = -\frac{16}{6} = -\frac{8}{3}.$$

Segment CA: From $C = (0, 0, 2)$ to $A = (0, 0, 0)$. Parameterize by

$$\mathbf{r}_{CA}(t) = (0,\, 0,\, 2(1-t)), \quad 0 \le t \le 1.$$

Then,

$$d\mathbf{r}_{CA} = (0,\, 0,\, -2\, dt).$$

Here, $y = 0$ so that

$$\mathbf{v} = (0,\ 0,\ 0).$$

Thus,

$$\int_{CA} \mathbf{v} \cdot d\mathbf{r} = 0.$$

Summing the contributions from all three segments, we obtain

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = 0 + \left(-\frac{8}{3}\right) + 0 = -\frac{8}{3}.$$

Conclusion: Both the surface integral and the line integral yield

$$-\frac{8}{3}.$$

Thus, Stokes' theorem is verified for the given vector field and surface.