Lecture 4

Example: A S/C is orbiting Earth on an orbit with an eccentricity of 0.2. The radius of periapsis is 1000 km altitude. What is the speed at periapsis, the radius of apoapsis & the speed @ apoapsis?

$r_p = 6378 + 1000 = 7378$ km

Speed at r_p: $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$

$r_p = a(1-e) \Rightarrow a = r_p/(1-e) = 9222.5$ km

$\frac{V_p^2}{2} - \frac{\mu}{r_p} = -\frac{\mu}{2a} \Rightarrow V_p = \sqrt{\frac{2\mu}{r_p} - \frac{\mu}{a}}$

Earth $\Rightarrow \mu = 3.986 \times 10^5$ km^3/s^2 $\Rightarrow \boxed{V_p = 8.052 \text{ km/s}}$

Radius of Apoapsis: $r_a = a(1+e) \Rightarrow \boxed{r_a = 11067 \text{ km}}$

Speed at apoapsis: from energy eqn: $V_a = \sqrt{\frac{2\mu}{r_a} - \frac{\mu}{a}} \Rightarrow \boxed{V_a = 5.368 \text{ km/s}}$

Example: A S/C is orbiting Earth. At periapsis, it has an altitude of 1500 km and a velocity of 8.5 km/s. What is the eccentricity of the orbit? What is the flight path angle & speed of the S/C when its altitude is 3000 km and $\nu < 180°$?

Eccentricity: $r_p = a(1-e)$

$r_p = 1500 + 6378 = 7878$ km

$V_p = 8.5$ km/s

$\mathcal{E} = \frac{V_p^2}{2} - \frac{\mu}{r_p} = -\frac{\mu}{2a}$

$\Rightarrow a = -\frac{\mu}{2}\left[\frac{V_p^2}{2} - \frac{\mu}{r_p}\right]^{-1} \Rightarrow a = 27{,}543.6$ km

$1 - e = r_p/a \Rightarrow e = 1 - r_p/a \Rightarrow \boxed{e = 0.714}$

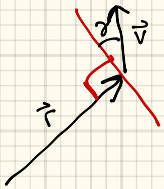
Velocity @ $r = 3000 + 6378 = 9378$ km

$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$

$V = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \Rightarrow \boxed{V = 8.399 \text{ km/s}}$

Flight Path Angle: γ

$h = rv\cos\gamma$



From the previous part, we know r, v at this location

$r = 9378\text{ km}$

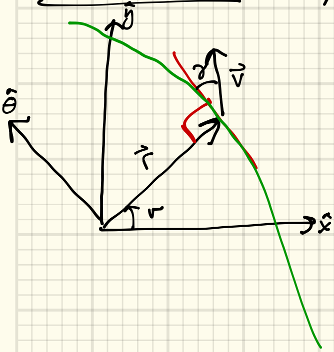
$v = 8.399\text{ km/s}$

$h = rv\cos\gamma = r_p v_p \cos\gamma_p \,, \quad \gamma_p = 0$

$= r_p v_p$

$\cos\gamma = \dfrac{r_p v_p}{rv} \Rightarrow \boxed{\gamma = 31.77^\circ}$ b/c $\nu < 180^\circ$, $\gamma > 0$

Period of an Ellipse: time required to traverse the orbit



From Kinematics:

$\vec{r} = r\hat{r}$

$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$

$\hat{r} = \cos\nu\,\hat{x} + \sin\nu\,\hat{y}$

$\hat{\theta} = -\sin\nu\,\hat{x} + \cos\nu\,\hat{y}$

$\dot{\hat{r}} = -\dot{\nu}\sin\nu\,\hat{x} + \dot{\nu}\cos\nu\,\hat{y}$

$\dot{\hat{r}} = \dot{\nu}\hat{\theta}$

$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\nu}\hat{\theta}$

$h = rv\cos\gamma$

$v\cos\gamma = r\dot{\nu}$ (component of the velocity in the $\hat{\theta}$ direction)

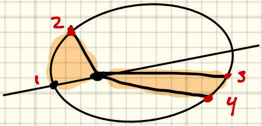
$\Rightarrow h = r^2\dot{\nu} = r^2\dfrac{d\nu}{dt} \quad \Rightarrow dt = \dfrac{r^2}{h}d\nu \qquad r = f(\nu)$ thru the trajectory eqn $r = \dfrac{p}{1+e\cos\nu}$

Area of a circle $= \pi r^2$

Portion of a circle $= \dfrac{\nu}{2\pi}\pi r^2$ (ν in rad here)

$dA = \frac{1}{2}r^2 d\nu$

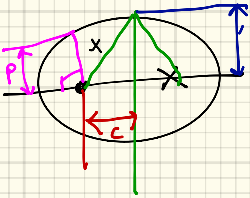
$dt = \dfrac{2}{h}dA \leftarrow$ Proves Kepler's 2nd law: equal areas in equal times



Suppose the 2 shaded areas are the same size, then it will take the spacecraft the same amount of time to go from $r_1 \to r_2$ and from $r_3 \to r_4$

The area of an ellipse is: $A = \pi a b$ (b = semi-minor axis)

Integrating $dt = \frac{2}{h} dA \Rightarrow TP = \frac{2\pi a b}{h}$



$e = \frac{c}{a}$

Length of string $= 2r_a = 2a(1+e)$

$= 2x + 2c = 2x + 2ea$

$\cancel{2}a(1+e) = \cancel{2}x + \cancel{2}ea$

$a + ea = x + ea \Rightarrow x = a$

$$a^2 = b^2 + c^2 \Rightarrow b = \sqrt{a^2 - c^2} = \sqrt{a^2 - e^2 a^2} = \sqrt{a^2(1-e^2)} = \sqrt{ap}$$

$h = \sqrt{\mu p}$

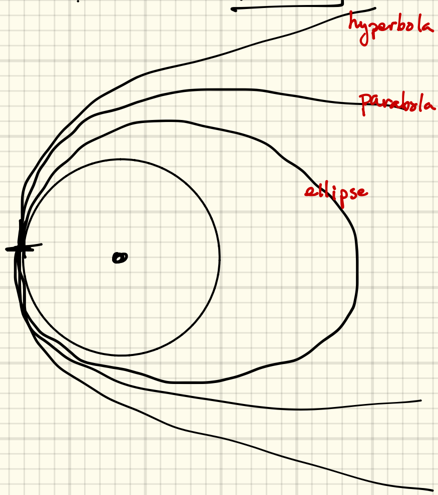
$$TP = \frac{2\pi a \sqrt{ap}}{\sqrt{\mu p}} \Rightarrow \boxed{TP = 2\pi \sqrt{\frac{a^3}{\mu}}}$$

Velocity of a circular orbit:

r = constant $\Rightarrow V$ = constant

$\varepsilon = \frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$ $\qquad a = r$ (for circular orbit)

$$\frac{V_c^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2r} \Rightarrow \boxed{V_c = \sqrt{\frac{\mu}{r}}}$$



Parabola: $\varepsilon = 0$

The escape speed is the speed of a parabolic orbit at a given radius (r).

$$\varepsilon = \frac{V_{esc}^2}{2} - \frac{\mu}{r} = \cancelto{0}{\frac{V_\infty^2}{2}} - \cancelto{0}{\frac{\mu}{r_\infty}} = 0$$

$$\boxed{V_{esc} = \sqrt{\frac{2\mu}{r}}}$$

Hyperbola: The hyperbola has some non-zero velocity at infinity. The hyperbolic excess speed is this velocity.

hyperbolic excess speed = Speed at $r = \infty$

$$\varepsilon = \frac{V_\infty^2}{2} - \cancelto{0,\ b/c\ r_\infty = \infty}{\frac{\mu}{r_\infty}} = -\frac{\mu}{2a} \quad \Rightarrow \boxed{V_\infty = \sqrt{-\frac{\mu}{a}}}$$

Note $a < 0$ for hyperbola