Lecture 12: Isentropic Processes and the Speed of Sound

ENAE311H Aerodynamics I

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Entropy change for a perfect gas

To calculate the entropy change (in terms of other variables) for a perfect gas, assume we are undergoing a reversible process. In this case, the first law can be written:

$$de = Tds - pdv, \quad \Longrightarrow \quad Tds = de + pdv.$$

Dividing through by T, and using the ideal gas law and $de = c_v dT$, we can write this as

$$ds = c_v \frac{dT}{T} + R\frac{dv}{v}.$$

Since R and (for a perfect gas) c_v are constant, we can integrate from state 1 to state 2:

$$\int_{s_1}^{s_2} ds = c_v \int_{T_1}^{T_2} \frac{dT}{T} + R\int_{v_1}^{v_2} \frac{dv}{v},$$

to immediately obtain

$$s_2 - s_1 = c_v \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1}. \quad \Longrightarrow \quad s = s(T, v)$$

Alternatively, starting from $dh = Tds + vdp$,

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}. \quad \Longrightarrow \quad s = s(T, p)$$

All variables of state, so true for a general process!

Isentropic processes

In gas dynamics, isentropic processes (both adiabatic and reversible) are particularly important. For such processes, $s_2 - s_1 = 0$, and from our equation for $s = s(T, v)$ on the last slide we have

$$\begin{aligned} R \ln \frac{v_2}{v_1} &= -c_v \ln \frac{T_2}{T_1} \\ \Longrightarrow \frac{v_2}{v_1} &= \left(\frac{T_2}{T_1}\right)^{-c_v/R}, \end{aligned}$$

or equivalently

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{c_v/R}.$$

Identifying $c_v/R = 1/(\gamma - 1)$, we thus have

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}.$$

Similarly, setting $s_2 - s_1 = 0$, and from our equation for $s = s(T, p)$ and using $c_p/R = \gamma/(\gamma - 1)$, we obtain

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}.$$

So, for isentropic flow

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}.$$

General compressible flow theory

- Earlier we derived the three conservation laws for fluid flow. These give us five equations in six variables: ρ, u, v, w, p and e (or h).

- To close the set, we require additional constitutive relations relating the thermodynamic variables
  - For a perfect gas, these are the ideal gas equation (thermal equation of state), $p = \rho R T$, and the caloric equation of state $e = c_v T$ or $h = c_p T$.

- With this complete set, it is possible to solve for a number of canonical problems in compressible flow.

The speed of sound

The propagation of sound in a gas is brought about by collisions between molecules as they undergo their random thermal motion. Since the speed of the molecules increases with T, we might expect the sound speed to do so, too.

Consider a quiescent gas at conditions $\rho = \rho_0, p = p_0,\ u = 0$, and suppose a sound wave (planar, propagating in the x direction) of infinitesimal strength passes through it, causing small fluctuations in these properties of $\rho'(x,t)$, $p'(x,t)$, and $u'(x,t)$. Note the following:

1. No heat is added or taken away (adiabatic)
2. Infinitesimal wave strength means the process is reversible

isentropic! ($s' = 0$)

From the differential form of conservation of mass, we can write

$$\frac{\partial}{\partial t}(\rho_0 + \rho') + \frac{\partial}{\partial x}[(\rho_0 + \rho')u'] = 0.$$

Expanding and discarding the second-order fluctuation term (i.e., $\frac{\partial}{\partial x}(\rho' u')$), we obtain

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0.$$

The speed of sound

We can similarly write the differential form of the one-dimensional momentum equation as

$$(\rho_0 + \rho')\left(\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x}\right) = -\frac{\partial}{\partial x}(p_0 + p').$$

Again dropping second-order terms, we obtain simply

$$\rho_0\frac{\partial u'}{\partial t} = \frac{\partial p'}{\partial x}.$$

Differentiating w.r.t. x, we have

$$\rho_0\frac{\partial^2 u'}{\partial x \partial t} = -\frac{\partial^2 p'}{\partial x^2}.$$

Note, however, that if we differentiate our linearized continuity equation w.r.t. t, we obtain

$$\frac{\partial^2 \rho'}{\partial t^2} + \rho_0\frac{\partial^2 u'}{\partial t \partial x} = 0$$

Then from equality of mixed derivatives, we can write

$$\frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0.$$

The speed of sound

Now, since p, ρ, and s are variables of state, we can write $p = p(\rho, s)$, and thus

$$p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s'$$

As we have noted already though, $s' = 0$, and so

$$p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho'$$

We write $c = \sqrt{(\partial p/\partial \rho)_s}$ and note that the derivative can be evaluated at reference conditions (p_0, ρ_0) and thus be treated as constant. Our differential equation then becomes:

$$\frac{\partial^2 p'}{\partial t^2} - c^2 \frac{\partial^2 p'}{\partial x^2} = 0.$$

This is the one-dimensional wave equation, which has the general solution

$$p'(x,t) = f(x - ct) + g(x + ct)$$

i.e., a travelling wave with speed c travelling in either the $+x$ or $-x$ direction.

The speed of sound

The speed of sound, i.e., the speed of propagation of sound waves, is typically denoted by a. We then see that

$$\boxed{a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}.}$$

To see what form this takes for a perfect gas, recall that for an isentropic process involving a perfect gas we have

$$p = k\rho^\gamma,$$

where k is constant $(= p_0/\rho_0^\gamma)$.

Therefore

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma k \rho^{\gamma-1} = \gamma \frac{p}{\rho},$$

and thus

$$a = \sqrt{\gamma \frac{p}{\rho}}.$$

From the ideal gas law, we then have

$$\boxed{a = \sqrt{\gamma R T}}$$

(= 341 m/s for air at room temperature)

The Mach number

Earlier we defined the Mach number as $M = V/a$. We now see that, for a perfect gas,

$$M^2 = \frac{V^2}{a^2} = \frac{V^2/2}{\gamma RT/2} = \frac{V^2/2}{\gamma(\gamma-1)c_v T/2}$$

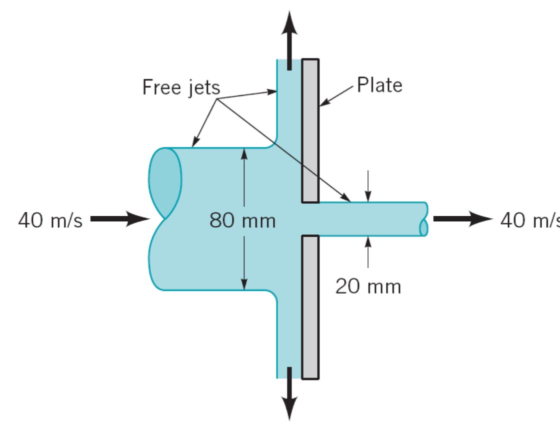
$$= \frac{2}{\gamma(\gamma-1)} \frac{\boxed{V^2/2}}{\boxed{c_v T}}$$

specific kinetic energy

specific internal energy

So the square of the Mach number is a measure of the ratio of the directed kinetic energy to the thermal internal energy of the gas.

5.38 A circular plate having a diameter of 300 mm is held perpendicular to an axisymmetric horizontal jet of air having a velocity of 40 m/s and a diameter of 80 mm as shown in Fig. P5.38. A hole at the center of the plate results in a discharge jet of air having a velocity of 40 m/s and a diameter of 20 mm. Determine the horizontal component of force required to hold the plate stationary.



$$\int_{\text{cs}} -p$$

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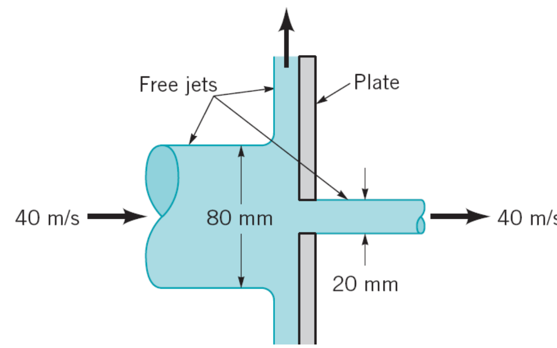

The control volume contains the plate and flowing air as indicated in the sketch above. Application of the horizontal or x direction component of the linear momentum equation yields

$$-u_1 \rho u_1 A_1 + u_2 \rho u_2 A_2 = -F_{A,x}$$

or

$$F_{A,x} = u_1^2 \rho \frac{\pi D_1^2}{4} - u_2^2 \rho \frac{\pi D_2^2}{4} = u_1^2 \rho \frac{\pi}{4}\left(D_1^2 - D_2^2\right)$$

Thus

$$F_{A,x} = \left(40\,\frac{m}{s}\right)^2 \left(1.23\,\frac{kg}{m^3}\right) \frac{\pi}{4} \left[\frac{(80\,mm)^2 - (20\,mm)^2}{\left(1000\,\frac{mm}{m}\right)^2}\right] \left(\frac{1\,N}{kg\cdot\frac{m}{s^2}}\right)$$

and

$$F_{A,x} = \underline{\underline{9.27}}\ N$$