Utility of gain/phase margin

⇒ a, γ measure how close polar comes to -1

⇒ If design is nominally stable (Nyquist shows required number of encirclements of -1), then

a, γ measure how much Nyquist plot can <u>change</u> in a pure gain or phase fashion, before -1 would enter a different loop, changing the number of encirclements.

Thus: a, γ are measures of the "tolerance" of the system's stability to gain/phase changes in $L(s)$.

⇒ <u>Relative</u> stability measures.

Robustness (classical)

As measures of the *tolerance* of the control system stability to changes in shape of Nyquist, gain and phase margin are measures of the *robustness* of the design.

That is, the ability of the design to tolerate model errors which would create pure gain or pure phase errors in $L(s)$

Typically caused by errors in model of $G(s)$, since

$$L(s) = G(s)H(s)$$

and there is no uncertainty in $H(s)$.

Classical Robustness Requirements

A "robustly stable" design thus requires:

$\Rightarrow$ Correct number of Nyquist encirclements

AND $\Rightarrow$ Large $|a|$, $|\gamma|$

Typical professional requirements

$\Rightarrow |a_{dB}| \geq 6$ (i.e. $a > 6\,dB$ or $a < -6\,dB$)

$\Rightarrow |\gamma| \geq 30°$

Requirement on a is physically equivalent to no more than a factor of 2 uncertainty on gain of $G(s)$

Recall: a, γ formally measure only how much Nyquist can change before encirclements change

(Assuming design is nominally stable, such changes would usually be bad!)

<u>By themselves</u> (separate from Nyquist) they are not reliable indicators of stability.

i.e. $a > \emptyset$ dB means Nyquist plot crosses neg. real axis to right of -1; $a < \emptyset$ dB means it crosses left of -1

Which is "better" (necessary for stability) depends on full Nyquist analysis.

However:

For a great many physical systems with:
a.) $L(s)$ stable; b.) unique ω_γ; c.) $\gamma(L) > 0°$, the shape of Nyquist plot ensures $T(s)$ stable.

(True even for many $L(s)$ which satisfy c) but violate a) or b); however need to check actual Nyquist shape carefully here).

Common enough to be a major design guideline:

$\Rightarrow$ Design $H(s)$ to ensure that $L(s)$ has positive phase margin

$$\Rightarrow \measuredangle L(j\omega_\gamma) > -180°$$

Constraints for Stability

For most simple (and common) systems (and many not so simple systems) Nyquist will show stability if phase margin of $L(j\omega)$ is positive.

Design prescription: Add LHP zeros in $H(s)$ to increase phase at magnitude xover.

Indeed, we will show using different techniques that it is rare that such a strategy would fail to stabilize.

$\Rightarrow$ Theoretically interesting counter-example: if $G(s)$ has both a zero and a pole in RHP. Such a system may actually require a RHP pole in $H(s)$ to stabilize!

Always check the Nyquist diagram when using simple Guidelines to design $H(s)$!

How much phase margin is "good"

Again, $\gamma > 30°$ is a typical minimum, and would ensure stability in common cases.

Why 30°? Is more better? Unfortunately, there is no simple correlation between freq. domain properties of $L(j\omega)$ and the <u>exact</u> location of poles of $T(s)$.

Nyquist tells us only $Re\{p_k\} < \phi$ for each pole p_k of $T(s)$ when the stability condition is satisfied

However, we can develop some useful intuition correlating (γ, ω_γ) with transient properties of $T(s)$ by looking at some typical simple examples.

Simple Example

$$L(s) = \frac{K}{s(s+\alpha)} \Rightarrow T(s) = \frac{K}{s^2+\alpha s+K}$$

$(\alpha > \emptyset)$

If $K=\alpha^2\sqrt{2}$, then $\omega_\gamma = |\alpha|$ and $\gamma = 45°$

(prove this to yourself if not obvious!)

Closed-loop poles are complex since $\alpha^2-4K<0$

$(\alpha^2-4\sqrt{2}\alpha^2)<0$

and in fact closed-loop damping ratio is $\zeta=0.42$ here.

$\Rightarrow$ Increasing K decreases γ here, and also decreases damping ratio of closed-loop poles.

$\Rightarrow$ Decreasing K increases γ here, and also increases DR of closed-loop poles

In fact, <u>for this system</u> we can show

$$\xi_{CL} \approx \frac{\gamma\,(\text{deg})}{100} \qquad (\text{for } 0 \lesssim \gamma \lesssim 70^\circ)$$

i.e. closed-loop damping ratio ξ_{CL} is directly proportional to the phase margin of L.

What about settling times for a step response of $T(s)$?
$\Rightarrow$ controlled by real parts of closed-loop poles.

Here the real parts are at $-\alpha/2 < 0$

$$t_s = \frac{4}{|\alpha/2|} = \frac{8}{|\alpha|} = \frac{8}{\omega_\gamma} \quad (\text{when } \gamma = 45^\circ \text{ as above})$$

i.e. t_s <u>inversely</u> prop. to ω_γ in this example.

Freq. Domain Constraints for Performance

When simple $\gamma(L) > \phi$ constraint works for stabilization, then typically:

$\Rightarrow$ larger γ gives higher damping for poles of $T(s)$

$\Rightarrow$ larger ω_γ gives faster settling time for $T(s)$ transients

Except for very simple systems, there are no direct mathematical connections between the <u>freq. domain</u> properties of $L(j\omega)$ (like γ and ω_γ) and the corresponding <u>time domain</u> properties of $T(s)$.

However, certain general trends have been found to hold:

For more complex systems, above observations do not hold precisely, but general trends do:

Given 2 possible OL TF: $L_1(s)$, $L_2(s)$

a.) If L_1, L_2 have same ω_γ but suppose $\gamma(L_1) > \gamma(L_2)$ then the closed-loop TFs T_1 and T_2 will have comperable settling times, but T_1 will have a higher Damping ratio

b.) If L_1 and L_2 have same phase margin but different ω_γ, $\omega_{\gamma_1} > \omega_{\gamma_2}$, then T_1 and T_2 will have comperable damping, but T_1 will settle faster than T_2

$\Rightarrow$ Design guideline: make γ, ω_γ as large as possible.

Intro. to Controller Design

Stability and healthy margins are just the first of many different constraints for a good design

Often the constraints conflict, and we must use our judgement to achieve an acceptable trade-off

The general design process is typically:

1.) Look at Bode/Nyquist of $G(s)$.

2.) Determine how plots in 1.) must be changed to achieve desired design goals.

3.) From required changes in 2.), determine the ZPK structure $H(s)$ must have.

Controller Implementation, I

But can we actually have *any* H(s) we want?

Unfortunately no. There are *implementation* constraints:

i.e., can we actually calculate u(t) from e(t) *in real time*

Note that we *do not* calculate u(t) from

$$u(t) = \mathcal{L}^{-1}\{H(s)E(s)\}$$

Why not?

- $y_d(t)$ not always known ahead of time (may come from pilot inputs)

- $y(t)$ cannot be predictedly exactly due to inaccurate model or external "disturbances"

Controller Implementation, II

So how *do* we implement the controller? By solving in real time the differential equation relating $u(t)$ to $e(t)$.

There are mathematical constraints under which this is possible, and these in turn constrain the "allowable" $H(s)$.

$$U(s) = H(s)\,E(s) = H(s)\left[Y_d(s) - Y(s)\right]$$

Suppose

$$H(s) = \frac{a(s)}{b(s)}$$

$a(s), b(s)$ polynomials in s. Then

$$b(s)\,U(s) = a(s)\,E(s)$$

Controller Implementation, II

$$U(s) = H(s)E(s)$$

Suppose
$$H(s) = \frac{a(s)}{b(s)}$$

$a(s), b(s)$ polynomials in s. Then

$$\mathcal{L}^{-1}\{b(s)U(s) = a(s)E(s)\}$$

Gives a differential equation relating $u(t)$ ("output") to $e(t)$ ("input")

This diff'l equation must be solvable in real time using only measurements of

$$e(t) = y_d(t) - y(t)$$

Example

$$H(s) = \frac{6(s+1)^2}{(s+3)(s+5)} = \frac{\overbrace{(6s^2+12s+6)}^{a(s)}}{\underbrace{(s^2+8s+15)}_{b(s)}}$$

$$\Rightarrow (s^2+8s+15)\,U(s) = (6s^2+12s+6)\,E(s)$$

$$\Rightarrow \ddot{u}(t) + 8\dot{u}(t) + 15u(t) = 6\ddot{e}(t) + 12\dot{e}(t) + 6e(t)$$

DE which must be solved during operation of controller on vehicle.

Must be soluable using <u>only</u> measured $e(t)$.

$\dot{e}(t)$, $\ddot{e}(t)$ terms <u>Not</u> assumed to be available!

Note these terms come from <u>zeros</u> of $H(s)$...

Real-time implementation constraint

Computation of $u(t)$ must require only knowledge of $e(t)$, (Not $\dot{e}(t)$, $\ddot{e}(t)$, etc.)

But note the DE from $\mathcal{L}^{-1}\{b(s)U(s) = a(s)E(s)\}$ will have derivatives of $e(t)$ on RHS.

If you think about it, this would seem to suggest $H(s)$ could never have any zeros [i.e. $a(s)$ must be a constant]

Fortunately this is not *the case, if we think a little more deeply:*

$$H(s)E(s) = \boxed{\left[\frac{a(s)}{b(s)}\right]} E(s) = \boxed{\left[d(s) + \frac{a'(s)}{b(s)}\right]} E(s)$$

where $d(s)$ is the quotient polynomial of $\frac{a(s)}{b(s)}$ and $a'(s)$ is the remainder polynomial.

If $\deg\{a(s)\} \geq \deg\{b(s)\}$, then

$$H(s) = \frac{a(s)}{b(s)} = d(s) + \frac{a'(s)}{b(s)}$$

where $\deg\{d(s)\} = \deg\{a(s)\} - \deg\{b(s)\}$

and $\deg\{a'(s)\} = \deg\{b(s)\} - 1$

Since $\deg\{a'(s)\} < \deg\{b(s)\}$ we can expand

$$\boxed{\frac{a'(s)}{b(s)} = \sum_{k=1}^{M} \frac{c_k}{(s-\ell_k)}}$$

(M = # poles of $H(s)$ here!)

where ℓ_k are roots of $b(s)$ (poles of $H(s)$)

and:

$$c_k = \left\{ (s-\ell_k) \left[\frac{a'(s)}{b(s)} \right] \right\}_{s=\ell_k}$$

If instead $\deg\{a(s)\} < \deg\{b(s)\}$ then

$$H(s) = \frac{a(s)}{b(s)} = \sum_{k=1}^{M} \frac{c_k}{(s-\ell_k)} \quad \text{directly}$$

so that $d(s) = \emptyset$ and $a'(s) = a(s)$

in the above.

Thus generally:

PFE of $H(s)$!

$$H(s)E(s) = \boxed{\left[d(s) + \sum_{k=1}^{M} \frac{c_k}{s-\ell_k} \right]} E(s)$$

or $$H(s)E(s) = d(s)E(s) + \sum_{k=1}^{M} c_k \left[\frac{1}{s-\ell_k} \right] E(s)$$

Look at each of the terms individually

$$U(s) = H(s)E(s) = d(s)E(s) + \sum_{k=1}^{M} c_k \left[\frac{1}{s-\ell_k}\right]E(s)$$

Introduce:

$$X_k(s) = \left(\frac{1}{s-\ell_k}\right)E(s)$$

So that

$$U(s) = d(s)E(s) + \sum_{k=1}^{M} c_k X_k(s)$$

and

$$u(t) = \mathcal{L}^{-1}\{d(s)E(s)\} + \sum_{k=1}^{M} c_k x_k(t)$$

with $x_k(t) = \mathcal{L}^{-1}\{X_k(s)\}$

But note from above: $(s-\ell_k)X_k(s) = E(s)$

$$\Rightarrow \dot{x}_k(t) - \ell_k x_k(t) = e(t)$$

} DE which only involves e(t)!

Thus, generally the control calculations required by $H(s)$ can be implemented using:

$$\boxed{u(t) = \mathcal{L}^{-1}\{d(s)E(s)\} + \sum_{k=1}^{M} c_k x_k(t)}$$

where

$$\boxed{\dot{x}_k(t) = \ell_k x_k(t) + e(t)}$$

← M different 1^{st} order DEs. for $x_k(t)$

ℓ_k are poles of $H(s)$, and c_k are the residues:

$$c_k = \left\{(s-\ell_k)\left[\frac{a'(s)}{b(s)}\right]\right\}_{s=\ell_k}$$

What about $\mathcal{L}^{-1}\{d(s)E(s)\}$? Recall $d(s)$ is a polynomial with degree $\deg\{a(s)\} - \deg\{b(s)\}$

If $\deg\{d(s)\} > 1$ $\left(\deg\{a(s)\} > \deg\{b(s)\}\right)$

i.e.

$$d(s) = d_0 + d_1 s + d_2 s^2 + \cdots$$

then $\mathcal{L}^{-1}\{d(s)E(s)\} = d_0 e(t) + \underbrace{d_1 \dot{e}(t) + d_2 \ddot{e}(t) + \cdots}_{\text{Cannot be implemented with assumed measurements.}}$

Thus, these add'l terms can only be implemented if

$$\deg\{d(s)\} = \emptyset$$ (i.e. $d(s)$ is just a constant)

Or equivalently $\deg\{\underline{a(s)}\} \leq \deg\{\underline{b(s)}\}$

numerator of $H(s)$ Denom $H(s)$

Relative Degree

The relative degree of a transfer function $G(s)$ is:

$$\rho(G) = \text{Degree of Denom poly} - \text{Degree of num poly}$$
$$= \#\text{poles of } G - \#\text{zeros of } G$$

From the above, the constraint for real-time implementation of compensator $H(s)$ is:

$$\rho(H) \geq \emptyset$$

$$\boxed{\text{i.e. } H(s) \text{ must have No more zeros than it has poles.}}$$

$\Rightarrow$ Will be a significant constraint on our designs!