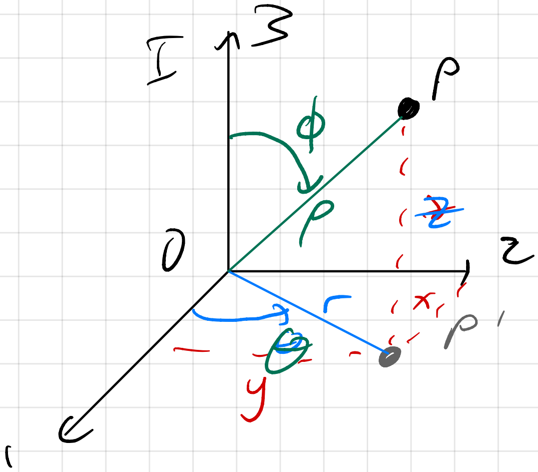
Chapter 10

particle kinematics & kinetics 3D

11/5/24

3D Coordinate Systems



Cartesian Coordinates

$$(x, y, z)_I$$

$$\bar{r}_{P/O} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$$

$${}^I\bar{v}_{P/O} = \dot{x}\hat{e}_1 + \dot{y}\hat{e}_2 + \dot{z}\hat{e}_3$$

$${}^I\bar{a}_{P/O} = \ddot{x}\hat{e}_1 + \ddot{y}\hat{e}_2 + \ddot{z}\hat{e}_3$$

Cylindrical Coordinates

$$(r, \theta, z)_I$$

$$\bar{r}_{P/O} = r\cos\theta\,\hat{e}_1 + r\sin\theta\,\hat{e}_2 + z\hat{e}_3$$

$${}^I\bar{v}_{P/O} = (\dot{r}c\theta - rs\theta\dot{\theta})\hat{e}_1 + (\dot{r}s\theta + rc\theta\dot{\theta})\hat{e}_2 + \dot{z}\hat{e}_3$$

$${}^I\bar{a}_{P/O} = \dots$$

Spherical Coordinates

$$(\rho, \theta, \phi)_I$$

$$\bar{r}_{P/O} = \rho\cos\theta\sin\phi\,\hat{e}_1 + \rho\sin\theta\sin\phi\,\hat{e}_2 + \rho\cos\phi\,\hat{e}_3$$

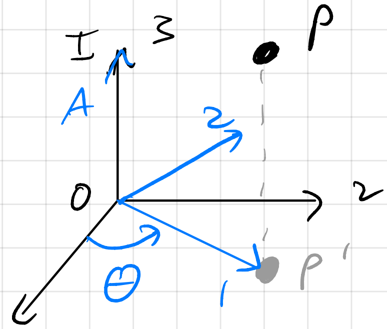
$${}^I\bar{v}_{P/O} = \dots$$

$${}^I\bar{a}_{P/O} = \dots$$

Cylindrical Frame

$$A = (O, \hat{a}_1, \hat{a}_2, \hat{a}_3)$$

$\hat{a}_1 = \hat{r}_{P'/O}$, $\hat{a}_3 = \hat{e}_3$,



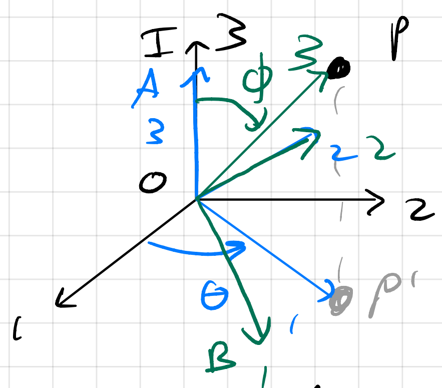
simple rotation about the 3-axis

$$\bar{r}_{P/O} = r\hat{a}_1 + z\hat{a}_3 \qquad {}^I\bar{\omega}^A = \dot{\theta}\hat{a}_3$$

$$^I\bar{v}_{P/O} = \dot{r}\hat{a}_1 + r\dot{\theta}\hat{a}_2 + \dot{z}\hat{a}_3$$

$$^I\bar{a}_{P/O} = \ddot{r}\hat{a}_1 + 2\dot{r}\dot{\theta}\hat{a}_2 + r\ddot{\theta}\hat{a}_2 - r\dot{\theta}^2\hat{a}_1 + \ddot{z}\hat{a}_3$$

Spherical Frame



$I = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ inertial

$A = (O, \hat{a}_1, \hat{a}_2, \hat{a}_3)$ intermediate

$B = (O, \hat{b}_1, \hat{b}_2, \hat{b}_3)$ spherical frame

$$\hat{b}_3 = \hat{r}_{P/O}$$

$$\bar{r}_{P/O} = \rho\hat{b}_3 \qquad {}^I\bar{\omega}^B = {}^I\bar{\omega}^A + {}^A\bar{\omega}^B$$

$$^I\bar{v}_{P/O} = \dot{\rho}\hat{b}_3 + \rho\left({}^I\bar{\omega}^B \times \hat{b}_3\right) \qquad = \dot{\theta}\hat{a}_3 + \dot{\phi}\hat{b}_2$$

$$= \dot{\rho}\hat{b}_3 + \rho(\dot{\theta}\hat{a}_3 + \dot{\phi}\hat{b}_2) \times \hat{b}_3$$

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$
$\hat{e}_1$	$c\theta$	$-s\theta$	0
$\hat{e}_2$	$s\theta$	$c\theta$	0
$\hat{e}_3$	0	0	1

	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{a}_1$	$c\phi$	0	$s\phi$
$\hat{a}_2$	0	1	0
$\hat{a}_3$	$-s\phi$	0	$c\phi$

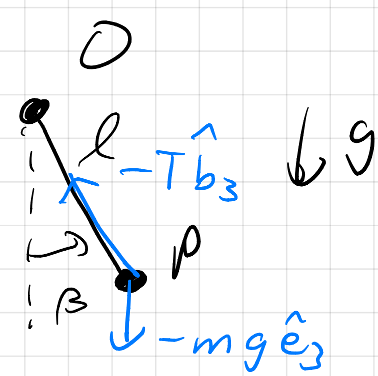
$$^I\bar{v}_{P/O} = \dot{\rho}\hat{b}_3 + \rho\left(\dot{\theta}(-s\phi\hat{b}_1 + c\phi\hat{b}_3) + \dot{\phi}\hat{b}_2\right) \times \hat{b}_3$$

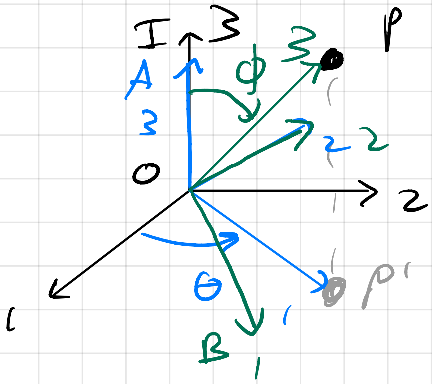
$$= \dot{\rho}\hat{b}_3 + \rho\dot{\theta}s\phi\hat{b}_2 + \rho\dot{\phi}\hat{b}_1$$

$$^I\bar{a}_{P/O} = \ldots$$

Ex 10.3 **Spherical pendulum**

$$\beta + \phi = \pi$$





$$\bar{F}_p = m_p \, {}^{I}\bar{a}_{p/o}$$

$$\bar{r}_{p/o} = \ell \hat{b}_3$$

Q: find $\ddot{\theta}$ & $\ddot{\phi}$

$$\begin{aligned}
{}^{I}\bar{\omega}^{B} &= \dot{\theta}\hat{a}_3 + \dot{\phi}\hat{b}_2 \\
&= \dot{\theta}(-s\phi\,\hat{b}_1 + c\phi\,\hat{b}_3) + \dot{\phi}\hat{b}_2 \\
&= -s\phi\dot{\theta}\hat{b}_1 + \dot{\phi}\hat{b}_2 + c\phi\dot{\theta}\hat{b}_3
\end{aligned}$$

$$\frac{{}^{I}d}{dt}(\hat{b}_1) = -\dot{\phi}\hat{b}_3 + c\phi\dot{\theta}\hat{b}_2$$

$$\frac{{}^{I}d}{dt}(\hat{b}_2) = -s\phi\dot{\theta}\hat{b}_3 - c\phi\dot{\theta}\hat{b}_1$$

$$\frac{{}^{I}d}{dt}(\hat{b}_3) = s\phi\dot{\theta}\hat{b}_2 + \dot{\phi}\hat{b}_1$$

$$\bar{r}_{p/o} = \ell\hat{b}_3$$

$${}^{I}\bar{v}_{p/o} = \ell(s\phi\dot{\theta}\hat{b}_2 + \dot{\phi}\hat{b}_1)$$

$$\begin{aligned}
{}^{I}\bar{a}_{p/o} = {}& \ell c\phi\dot{\phi}\dot{\theta}\hat{b}_2 + \ell s\phi\ddot{\theta}\hat{b}_2 + \ell s\phi\dot{\theta}(-s\phi\dot{\theta}\hat{b}_3 - c\phi\dot{\theta}\hat{b}_1) \\
&+ \ell\ddot{\phi}\hat{b}_1 + \ell\dot{\phi}(-\dot{\phi}\hat{b}_3 + c\phi\dot{\theta}\hat{b}_2)
\end{aligned}$$

$$\begin{aligned}
\bar{F}_p &= -mg\hat{e}_3 - T\hat{b}_3 \\
&= -mg\hat{a}_3 - T\hat{b}_3 \\
&= -mg(-s\phi\,\hat{b}_1 + c\phi\,\hat{b}_3) - T\hat{b}_3
\end{aligned}$$

$\hat{b}_1$: $mg s\phi = m(-\ell s\phi c\phi\dot{\theta}^2 + \ell\ddot{\phi})$ $\Rightarrow \ddot{\phi} = \ldots$

$\hat{b}_2$: $0 = m(\ell c\phi\dot{\phi}\dot{\theta} + \ell s\phi\ddot{\theta} + \ell c\phi\dot{\phi}\dot{\theta})$ $\ddot{\theta} = \ldots$

$\hat{b}_3$: $-mg c\phi - T = m(-\ell s^2\phi\dot{\theta}^2 - \ell\dot{\phi}^2)$ $T = \ldots$

vector notation

$$\bar{r}_{P/O} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$$

$$^{I}\bar{v}_{P/O} = \dot{x}\hat{e}_1 + \dot{y}\hat{e}_2 + \dot{z}\hat{e}_3$$

matrix notation

$$[\bar{r}_{P/O}]_I = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_I$$

$$[{}^{I}\bar{v}_{P/O}]_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_I$$

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$
$\hat{e}_1$	$C\theta$	$-S\theta$	0
$\hat{e}_2$	$S\theta$	$C\theta$	0
$\hat{e}_3$	0	0	1

$${}^{I}R^{A} = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of Rotation Matrices

$$\left({}^{I}R^{A}\right)^{T} = {}^{A}R^{I} = \left({}^{I}R^{A}\right)^{-1}$$

transpose inverse

$$\left({}^{I}R^{A}\right)\left({}^{I}R^{A}\right)^{T} = \left({}^{I}R^{A}\right)\left({}^{I}R^{A}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\theta & S\theta & 0 \\ -S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underbrace{c^2\theta + s^2\theta}_{=1} & 0 & 0 \\ 0 & \underbrace{c^2\theta + s^2\theta}_{=1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider $A = (O, \hat{a}_1, \hat{a}_2, \hat{a}_3)$ cylindrical frame

$$\bar{r}_{P/O} = r\hat{a}_1 + z\hat{a}_3 = r(C\theta\hat{e}_1 + S\theta\hat{e}_2) + z\hat{e}_3$$

Transformation table

$$[\bar{r}_{P/O}]_A = \begin{bmatrix} r \\ 0 \\ z \end{bmatrix}_A$$

rotation matrix

$$[\bar{r}_{P/O}]_I = {}^{I}R^{A}[\bar{r}_{P/O}]_A$$

$$= \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ 0 \\ z \end{bmatrix}_A = \begin{bmatrix} rC\theta \\ rS\theta \\ z \end{bmatrix}_I$$

$$\bar{r}_{P/O} = r c\theta \hat{e}_1 + r s\theta \hat{e}_2 + z \hat{e}_3$$

Consider $B = (O, \hat{b}_1, \hat{b}_2, \hat{b}_3)$ spherical sphere

$$\bar{r}_{P/O} = \rho \hat{b}_3$$

$$[\bar{r}_{P/O}]_B = \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix}_B$$

$$[\bar{r}_{P/O}]_I = {}^I R^A \, {}^A R^B [\bar{r}_{P/O}]_B$$

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} c\theta c\phi & -s\theta & c\theta s\phi \\ s\theta c\phi & c\theta & s\theta s\phi \\ -s\phi & 0 & c\phi \end{bmatrix}}_{= {}^I R^B = {}^I R^A \, {}^A R^B} \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix}$$

$$= \begin{bmatrix} \rho c\theta s\phi \\ \rho s\theta s\phi \\ \rho c\phi \end{bmatrix}_I$$

Linear momentum $\quad {}^I\bar{p}_{Q/O} = m_Q \, {}^I\bar{v}_{Q/O}$

Ang momentum $\quad {}^I\bar{h}_{Q/O} = \bar{r}_{Q/O} \times m_Q \, {}^I\bar{v}_{Q/O}$

$$\left.\begin{aligned} \bar{F}_Q &= \frac{{}^I d}{dt}\left({}^I\bar{p}_{Q/O}\right) \\ \bar{M}_Q &= \frac{{}^I d}{dt}\left({}^I\bar{h}_{Q/O}\right) \end{aligned}\right\}$$

Same in 3D as in 2D!

$$W_P(\bar{r}_{P/O}; \gamma_P) = \int_{\gamma_P} \bar{F}_P \cdot {}^I d\bar{r}_{P/O}$$

$$T_{P/O} = \frac{1}{2} m_P \| {}^I\bar{v}_{P/O} \|^2$$

$$U_{P/O} = -\int \bar{F}_P \cdot {}^I d\bar{r}_{P/O}$$

$$E_O(t_2) = E_O(t_1) + W_P^{(nc)}$$

Same in 3D as in 2D!

Ex 10.5, 10.6 <u>spherical pendulum</u>

Find AM; is it conserved

Find the eq mo. using AM

Find Energy; is it conserved

$${}^I\bar{h}_{P/O} = \bar{r}_{P/O} \times m_P {}^I\bar{v}_{P/O}$$

$$= l\hat{b}_3 \times ml(s\phi\dot{\theta}\hat{b}_2 + \dot{\phi}\hat{b}_1)$$

$$= -ml^2 s\phi\dot{\theta}\hat{b}_1 + ml^2\dot{\phi}\hat{b}_2$$

$$\bar{M}_P = \bar{r}_{P/O} \times F_P$$

$$= l\hat{b}_3 \times (-mg\hat{e}_3 - T\hat{b}_3)$$

$$= l\hat{b}_3 \times (-mg(-s\phi\hat{b}_1 + c\phi\hat{b}_3))$$

$$= mgls\phi\hat{b}_2$$

$$\frac{{}^I d}{dt}({}^I\bar{h}_{P/O}) = \frac{{}^I d}{dt}(ml^2(-s\phi\dot{\theta}\hat{b}_1 + \dot{\phi}\hat{b}_2))$$

$$= ml^2\Big(-c\phi\,\dot\phi\,\dot\theta\,\hat b_1 - s\phi\,\ddot\theta\,\hat b_1 - s\phi\,\dot\theta\,\frac{{}^{I}d}{dt}(\hat b_1) + \ddot\phi\,\hat b_2 + \dot\phi\,\frac{{}^{I}d}{dt}(\hat b_2)\Big)$$

$$\frac{{}^{I}d}{dt}(\hat b_1) = -\dot\phi\,\hat b_3 + c\phi\,\dot\theta\,\hat b_2$$

$$\frac{{}^{I}d}{dt}(\hat b_2) = -s\phi\,\dot\theta\,\hat b_3 - c\phi\,\dot\theta\,\hat b_1$$

$$\hat b_1: \quad ml^2\big(-c\phi\,\dot\phi\,\dot\theta - s\phi\,\ddot\theta - \dot\phi\,c\phi\,\dot\theta\big) = 0$$

$$\hat b_2: \quad ml^2\big(-s\phi\,\dot\theta\,c\phi\,\dot\theta + \ddot\phi\big) = mgl\,s\phi$$

$$\hat b_3: \quad ml^2\big(+s\phi\,\dot\theta\,\dot\phi - \dot\phi\,s\phi\,\dot\theta\big) = 0$$

$$\boxed{\begin{aligned}\ddot\theta &= \dots\\ \ddot\phi &= \dots\end{aligned}}$$

$$E_{P/O} = T_{P/O} + U_{P/O} \sim mgh$$

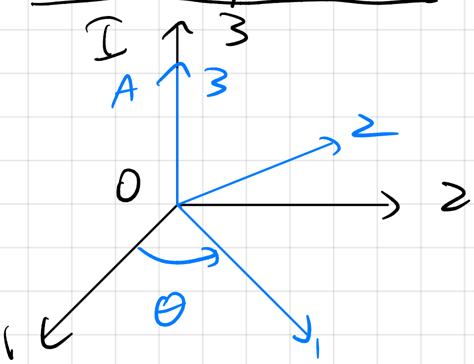
$$= \tfrac{1}{2} m_P \,\| l\,(s\phi\,\dot\theta\,\hat b_2 + \dot\phi\,\hat b_1)\|^2 + mgl\,c\phi$$

$$= \tfrac{1}{2} ml^2\big(s^2\phi\,\dot\theta^2 + \dot\phi^2\big) + mgl\,c\phi$$

$$\underbrace{E_{P/O}(t_2) = E_{P/O}(t_1)}_{\text{Yes, conserved}} + \cancel{W_P^{(nc)}} = 0$$

$$\int_{\gamma_P} \bar F_P \cdot d\bar r_{P/O} = \int_{\gamma_P} \underbrace{\bar T \cdot {}^{I}\bar v_{P/O}}_{=0}\, dt$$

3D Body Frame

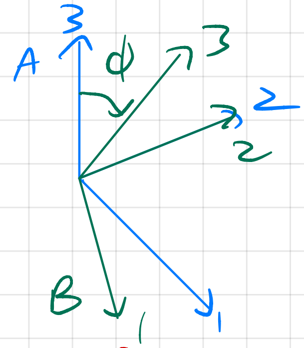


$$I = (O, \hat e_1, \hat e_2, \hat e_3)$$

$$A = (O, \hat a_1, \hat a_2, \hat a_3) \quad \text{intermediate frame}$$

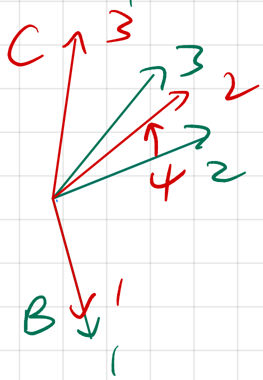
$${}^{I}R^{A} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{I}\bar\omega^{A} = \dot\theta\,\hat a_3$$



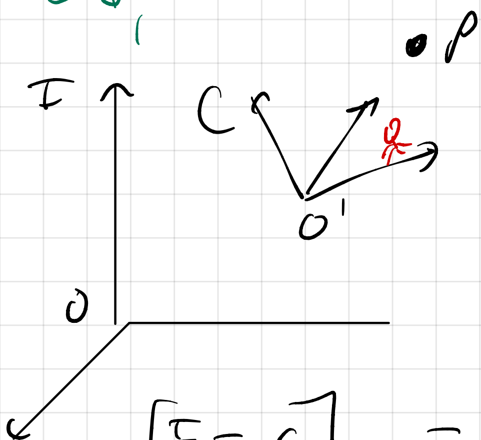
$$B = (O, \hat{b}_1, \hat{b}_2, \hat{b}_3) \text{ intermediate}$$

$$^{A}R^{B} = \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix} \qquad ^{A}\bar{\omega}^{B} = \dot{\phi}\,\hat{b}_2$$



$$C = (O, \hat{c}_1, \hat{c}_2, \hat{c}_3) \text{ body frame}$$

$$^{B}R^{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix} \qquad ^{B}\bar{\omega}^{C} = \dot{\psi}\,\hat{c}_1$$



$$^{I}\bar{\omega}^{C} = {}^{I}\bar{\omega}^{A} + {}^{A}\bar{\omega}^{B} + {}^{B}\bar{\omega}^{C}$$
$$= \dot{\theta}\,\hat{a}_3 + \dot{\phi}\,\hat{b}_2 + \dot{\psi}\,\hat{c}_1$$

$$\left[{}^{I}\bar{\omega}^{C}\right]_C = {}^{C}R^{A}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}_A + {}^{C}R^{B}\begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix}_B + \begin{bmatrix} \dot{\psi} \\ 0 \\ 0 \end{bmatrix}_C$$

$$^{C}R^{A} = \left({}^{A}R^{C}\right)^T$$
$$= \left({}^{A}R^{B}\,{}^{B}R^{C}\right)^T = \left({}^{B}R^{C}\right)^T\left({}^{A}R^{B}\right)^T$$
$$= {}^{C}R^{B}\,{}^{B}R^{A}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & s\psi \\ 0 & -s\psi & c\psi \end{bmatrix}\begin{bmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{bmatrix} = \begin{bmatrix} c\phi & 0 & -s\phi \\ s\psi s\phi & c\psi & s\psi c\phi \\ c\psi s\phi & -s\psi & c\psi c\phi \end{bmatrix}$$

$$\left[{}^{I}\bar{\omega}^{C}\right]_C = \begin{bmatrix} c\phi & 0 & -s\phi \\ s\psi s\phi & c\psi & s\psi c\phi \\ c\psi s\phi & -s\psi & c\psi c\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}_A + \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & s\psi \\ 0 & -s\psi & c\psi \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix}_B + \begin{bmatrix} \dot{\psi} \\ 0 \\ 0 \end{bmatrix}_C$$

$$= \begin{bmatrix} -s\phi\dot{\theta} + \dot{\psi} \\ s\psi c\phi\dot{\theta} + c\psi\dot{\phi} \\ c\psi c\phi\dot{\theta} - s\psi\dot{\phi} \end{bmatrix}_C = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_C$$

$3-2-1 \ (\theta, \phi, \psi)\frac{C}{I}$ Euler Angles

All possible Euler angles

12 possible sets

~~3 3 3~~

~~3 3 2~~

~~3 3 1~~

3 2 3

~~3 2 2~~

3 2 1

3 1 3

3 1 2

~~3 1 1~~

3-2-1 $(\theta, \phi, \psi)_I^C$ Euler angles

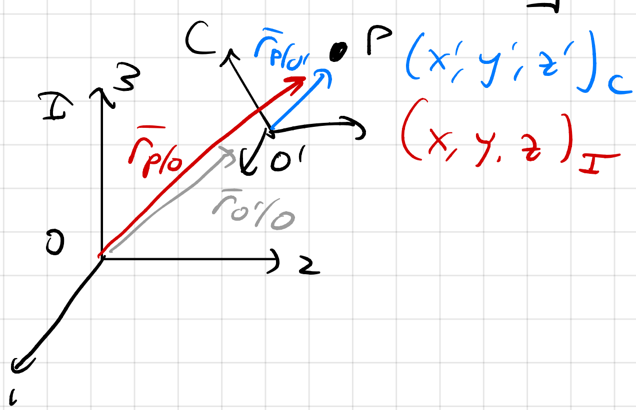
$$
{}^C R^A = \begin{bmatrix} c\phi & 0 & -s\phi \\ s\psi s\phi & c\psi & s\psi c\phi \\ c\psi s\phi & -s\psi & c\psi c\phi \end{bmatrix}
$$

$$
{}^C R^I = {}^C R^A \, {}^A R^I
$$

$$
= \begin{bmatrix} c\phi & 0 & -s\phi \\ s\psi s\phi & c\psi & s\psi c\phi \\ c\psi s\phi & -s\psi & c\psi c\phi \end{bmatrix} \begin{bmatrix} c\theta & +s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

3-2-1 (θ, ϕ, ψ)

$$
= \begin{bmatrix} c\phi c\theta & c\phi s\theta & -s\phi \\ s\psi s\phi c\theta - c\psi s\theta & s\psi s\phi s\theta + c\psi c\theta & s\psi c\phi \\ c\psi s\phi c\theta + s\psi s\theta & c\psi s\phi s\theta - s\psi c\theta & c\psi c\phi \end{bmatrix}
$$



$$
[\bar{r}_{P/O}]_C = {}^C R^I [\bar{r}_{P/O}]_I
$$

$$
\bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}
$$

$$
[\bar{r}_{P/O}]_I = [\bar{r}_{O'/O}]_I + {}^I R^C [\bar{r}_{P/O'}]_C
$$

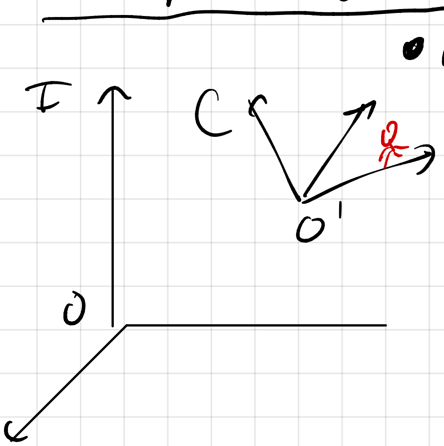
$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix}_I = [\bar{r}_{O'/O}]_I + {}^I R^C \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}_C
$$

$$
{}^I\bar{v}_{P/O} = \dot{x}\hat{e}_1 + \dot{y}\hat{e}_2 + \dot{z}\hat{e}_3
$$

$$
[{}^I v_{P/O}]_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_I
$$

Suppose $\left[{}^I\bar{v}_{P/O}\right]_C = \begin{bmatrix}\dot{x}'\\ \dot{y}'\\ \dot{z}'\end{bmatrix}_C$

<u>transport equation</u>



$${}^I\bar{v}_{P/O} = {}^I\bar{v}_{O'/O} + {}^I\bar{v}_{P/O'}$$

$${}^I\bar{v}_{P/O'} = \frac{{}^I d}{dt}\left(\bar{r}_{P/O'}\right)$$

Ch. 8 transport equation

$$= \frac{{}^C d}{dt}\left(\bar{r}_{P/O'}\right) + {}^I\bar{\omega}^C \times \left(\bar{r}_{P/O'}\right)$$

* TE is the same in 3D

$${}^I\bar{a}_{P/O} = {}^I\bar{a}_{O'/O} + {}^I\bar{a}_{P/O'}$$

$${}^I\bar{a}_{P/O'} = \frac{{}^C d}{dt}\left({}^I\bar{v}_{P/O'}\right) + {}^I\bar{\omega}^C \times \left({}^I\bar{v}_{P/O'}\right)$$

$$= {}^C\bar{a}_{P/O'} + {}^I\bar{\alpha}^C \times \bar{r}_{P/O'} + \underbrace{2\,{}^I\bar{\omega}^C \times {}^C\bar{v}_{P/O'}}_{\text{Coriolis accel.}} + \underbrace{{}^I\bar{\omega}^C \times {}^I\bar{\omega}^C \times \bar{r}_{P/O'}}_{\text{Centripetal accel}}$$

$$\left[{}^I\bar{\omega}^C\right]_C = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}_C = \begin{bmatrix}p\\ q\\ r\end{bmatrix}_C$$

aerospace convention

$$= \begin{bmatrix} -s\phi\dot{\theta} + \dot{\psi} \\ s\psi c\phi\dot{\theta} + c\psi\dot{\phi} \\ c\psi c\phi\dot{\theta} - s\psi\dot{\phi} \end{bmatrix}_C$$

3-2-1 $(\theta, \phi, \psi)_I^C$ kinematic equations of rotation

$$\begin{aligned} p &= -s\phi\dot{\theta} + \dot{\psi} \\ q &= s\psi c\phi\dot{\theta} + c\psi\dot{\phi} \\ r &= c\psi c\phi\dot{\theta} - s\psi\dot{\phi} \end{aligned} \Rightarrow \begin{aligned} \dot{\psi} &= (q s\phi + r c\phi)\sec\theta \\ \dot{\theta} &= q c\phi - r s\phi \\ \dot{\phi} &= (q s\phi + r c\phi)\tan\theta + p \end{aligned}$$

* Euler angle rates $\neq$ body angular rates
* @ $\theta = \pi/2$ singularity "gimbal lock"

$$^{I}\bar{\omega}^{C} \times \bar{r}_{P/O'} = (p\hat{c}_1 + q\hat{c}_2 + r\hat{c}_3) \times (x'\hat{c}_1 + y'\hat{c}_2 + z'\hat{c}_3)$$

$$\begin{aligned} = &\; p y'\hat{c}_3 - p z' c_2 \\ & - q x'\hat{c}_3 + q z'\hat{c}_1 \\ & \; r x'\hat{c}_2 - y'\hat{c}_1 \end{aligned}$$

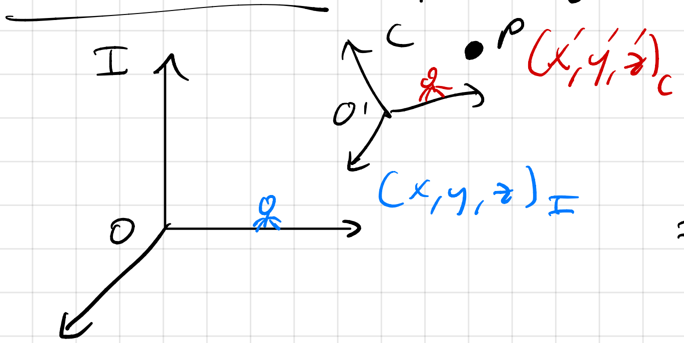
$$= (q z' - r y')\hat{c}_1 + (r x' - p z')\hat{c}_2 + (p y' - q x')\hat{c}_3$$

$$\left[{}^{I}\bar{\omega}^{C} \times \bar{r}_{P/O}\right]_C = \left[{}^{I}\bar{\omega}^{C}\right]_C \times \left[\bar{r}_{P/O}\right]_C$$

$$= \begin{bmatrix} p \\ q \\ r \end{bmatrix}_C \times \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}_C$$

$$= \underbrace{\begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}}_{\text{cross-product matrix}} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}_C \stackrel{\checkmark}{=} \begin{bmatrix} q z' - r y' \\ r x' - p z' \\ p y' - q x' \end{bmatrix}_C$$

Ex 10.13 Toolbag in space



$F_P = 0 \qquad \ddot{x} = 0 \quad \ddot{y} = 0 \quad \ddot{z} = 0$

$$\bar{r}_{P/O'} = x'\hat{c}_1 + y'\hat{c}_2 + z'\hat{c}_3$$

$$^{I}\bar{v}_{P/O'} = \frac{^{I}d}{dt}\left(\bar{r}_{P/O'}\right)$$

$$= \frac{^{C}d}{dt}\left(\bar{r}_{P/O'}\right) + {}^{I}\bar{\omega}^{C} \times \bar{r}_{P/O'}$$

$$ {}^I\frac{d}{dt}\left(\hat{c}_1\right) = {}^I\bar{\omega}^c \times \hat{c}_1 = \left(\omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3\right)\times\hat{c}_1 $$
$$ = -\omega_2\hat{c}_3 + \omega_3\hat{c}_2 $$

$$ {}^I\frac{d}{dt}\left(\hat{c}_2\right) = {}^I\bar{\omega}^c \times \hat{c}_2 = \left(\omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3\right)\times\hat{c}_2 $$
$$ = \omega_1\hat{c}_3 - \omega_3\hat{c}_1 $$

$$ {}^I\frac{d}{dt}\left(\hat{c}_3\right) = {}^I\bar{\omega}^c \times \hat{c}_3 = \left(\omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3\right)\times\hat{c}_3 $$
$$ = -\omega_1\hat{c}_2 + \omega_2\hat{c}_1 $$

$$ {}^I\bar{v}_{p/o'} = {}^I\frac{d}{dt}\left(x'\hat{c}_1 + y'\hat{c}_2 + z'\hat{c}_3\right) $$
$$ = \dot{x}'\hat{c}_1 + x'\left(-\omega_2\hat{c}_3 + \omega_3\hat{c}_2\right) + $$
$$ \dot{y}'\hat{c}_2 + y'\left(\omega_1\hat{c}_3 - \omega_3\hat{c}_1\right) + $$
$$ \dot{z}'\hat{c}_3 + z'\left(-\omega_1\hat{c}_2 + \omega_2\hat{c}_1\right) $$

$$ = \left(\dot{x}' - y'\omega_3 + z'\omega_2\right)\hat{c}_1 + $$
$$ \left(x'\omega_3 + \dot{y}' - z'\omega_1\right)\hat{c}_2 + $$
$$ \left(-x'\omega_2 + y'\omega_1 + \dot{z}'\right)\hat{c}_3 $$

$$ {}^I\bar{a}_{p/o'} = {}^I\frac{d}{dt}\left({}^I\bar{v}_{p/o'}\right) = \ldots $$

let $\omega_2 = 0$ and $\omega_3 = 0$ and $\dot{\omega}_1 = 0$, ${}^I\bar{a}_{o'/o} = 0$

$$ {}^I\bar{v}_{p/o'} = \dot{x}'\hat{c}_1 + \left(\dot{y}' - z'\omega_1\right)\hat{c}_2 + \left(y'\omega_1 + \dot{z}'\right)\hat{c}_3 $$
$$ {}^I\bar{a}_{p/o'} = \ddot{x}'\hat{c}_1 + \left(\ddot{y}' - \dot{z}'\omega_1\right)\hat{c}_2 + \left(\dot{y}' - z'\omega_1\right)\omega_1\hat{c}_3 $$
$$ + \left(\dot{y}'\omega_1 + \ddot{z}'\right)\hat{c}_3 - \left(y'\omega_1 + \dot{z}'\right)\omega_1\hat{c}_2 $$

$$ {}^I\bar{a}_{p/o} = \cancelto{0}{{}^I\bar{a}_{o'/o}} + {}^I\bar{a}_{p/o'} \Rightarrow \bar{F}_p = m_p\,{}^I\bar{a}_{p/o'} = 0 $$

$\hat{c}_1 : \ddot{x}' = 0$

$\hat{c}_2 : \ddot{y}' - \dot{z}'\omega_1 - (y'\omega_1 - \dot{z}')\omega_1 = 0 \Rightarrow \ddot{y}' = \ldots$

$\hat{c}_3 : (\dot{y}' - z'\omega_1)\omega_1 + \dot{y}'\omega_1 + \ddot{z}' = 0 \Rightarrow \ddot{z}' = \ldots$

10.14 Translational dynamics of an airplane

3-2-1

$(\theta, \phi, \psi)^C_I$

$$\bar{F}_G = m_G \, {}^I\bar{a}_{G/O}$$

$$[F_G]_C = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}_C$$

position $(x, y, z)_I$

velocity $(u, v, w)_C$

trans. kinematics

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_I = {}^I R^C \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C}$$

$$[{}^I\bar{v}_{G/O}]_C = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C$$

aerospace convention

$$[F_G]_C = m_G [{}^I\bar{a}_{G/O}]_C$$

$$
\begin{aligned}
{}^I\bar{a}_{G/O} &= \frac{{}^I d}{dt}\left({}^I\bar{v}_{G/O}\right) \\
&= \frac{{}^C d}{dt}\left({}^I\bar{v}_{G/O}\right) + {}^I\bar{\omega}^C \times \left({}^I\bar{v}_{G/O}\right)
\end{aligned}
$$

$$
\begin{aligned}
[{}^I\bar{a}_{G/O}]_C &= \frac{{}^C d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix}_C + \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
&= \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix}
\end{aligned}
$$

$$\hat{c}_1: \quad F_1 = m(\dot{u} + qw - rv)$$

$$\hat{c}_2: \quad F_2 = m(\dot{v} + ru - pw)$$

$$\hat{c}_3: \quad F_3 = m(\dot{w} + pv - qu)$$

$$\Rightarrow \boxed{\begin{aligned} \dot{u} &= \ldots \\ \dot{v} &= \ldots \\ \dot{w} &= \ldots \end{aligned}}$$

dynamics