Lecture 24: Linearized Supersonic Flow

ENAE311H Aerodynamics I

Christoph Brehm

Boundary conditions for the linearized equation

Recall our linearized potential equation:

$$(1 - M_1^2)\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0.$$

To solve this for a particular geometry, we also require boundary conditions for $\hat{\phi}$.

- Far from any solid boundaries, we require $\hat{\phi}$ to tend to a constant (so that the perturbation velocities are zero).
- At any solid boundaries we have a no-throughflow condition (can't enforce no-slip for an inviscid wall), so if θ is angle of wall relative to freestream direction, we have

$$\frac{v}{u} = \frac{\hat{v}}{V_1 + \hat{u}} = \tan\theta.$$

Since $\hat{u} \ll V_1$, however, an appropriate form of this equation (under the present assumptions) is

$$\boxed{\frac{\partial \hat{\phi}}{\partial y} = V_1 \tan\theta.}$$

Pressure coefficient in linearized flow

We can also derive a simple relation for the pressure coefficient under our linearized assumptions. Recall the pressure coefficient is defined as

$$C_p \equiv \frac{p - p_1}{q_1}.$$

For a perfect gas, we can write

$$q_1 = \frac{1}{2}\rho_1 V_1^2 = \frac{\gamma}{2} p_1 M_1^2,$$

and thus

$$C_p = \frac{2}{\gamma M_1^2}\left(\frac{p}{p_1} - 1\right).$$

Now, according to our isentropic assumption, we can write

$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{\gamma/(\gamma-1)}.$$

Now recall the form of the energy equation we used in deriving the linearized potential equation:

$$a^2 = a_1^2 - \frac{\gamma - 1}{2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2)$$

Since $a \propto \sqrt{T}$, this can be written as

$$\frac{T}{T_1} = 1 - \frac{\gamma - 1}{2a_1^2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2).$$

Substituting into our isentropic relation, we have

$$\frac{p}{p_1} = \left[1 - \frac{\gamma - 1}{2a_1^2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2)\right]^{\gamma/(\gamma-1)},$$

or

$$\frac{p}{p_1} = \left[1 - \frac{\gamma - 1}{2}M_1^2\left(\frac{2\hat{u}}{V_1} + \frac{\hat{u}^2 + \hat{v}^2}{V_1^2}\right)\right]^{\gamma/(\gamma-1)}.$$

Pressure coefficient in linearized flow

Now, since $\hat{u}, \hat{v} \ll V_1$, we can discard second-order (i.e., squared) terms in favor of first-order ones. Then, provided M_1 is not too large, we can write

$$\frac{p}{p_1} = (1-\epsilon)^{\gamma/(\gamma-1)},$$

with

$$\epsilon = \frac{\gamma-1}{2} M_1^2 \frac{2\hat{u}}{V_1} \ll 1.$$

Also,

$$(1-\epsilon)^a \approx 1 - a\epsilon$$

so we can write

$$\frac{p}{p_1} = 1 - \frac{\gamma}{2} M_1^2 \frac{2\hat{u}}{V_1}.$$

Our expression for the linearized pressure coefficient is then

$$\boxed{C_p = -\frac{2\hat{u}}{V_1}.}$$

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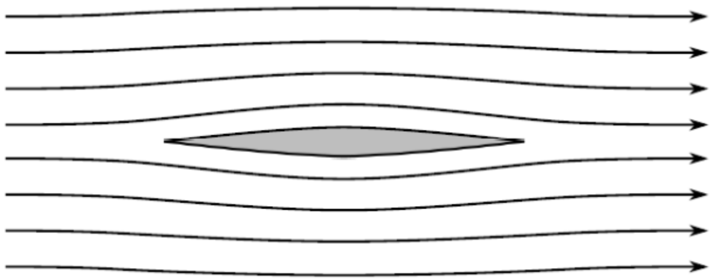
or

$$\frac{p}{p_1} = \left[1 - \frac{\gamma-1}{2} M_1^2 \left(\frac{2\hat{u}}{V_1} + \frac{\hat{u}^2 + \hat{v}^2}{V_1^2}\right)\right]^{\gamma/(\gamma-1)}.$$

Supersonic linearized flow

Now we concentrate on the case of supersonic flow, e.g., over a thin airfoil.



The linearized velocity potential equation can then be written as

$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0$$

with

$$\lambda = \sqrt{M_1^2 - 1}.$$

Now, recall our discussion of the sound speed. The equation we derived governing the propagation of sound was

$$\frac{\partial^2 p'}{\partial t^2} - c^2 \frac{\partial^2 p'}{\partial x^2} = 0.$$

which had the general solution

$$p'(x,t) = f(x - ct) + g(x + ct)$$

Comparing to our linearized potential equation above, we might expect

$$\hat{\phi} = f(x - \lambda y) + g(x + \lambda y)$$

However, we can neglect the g solution (disturbances can only propagate downstream in supersonic flow), so

$$\boxed{\hat{\phi} = f(x - \lambda y).}$$

Supersonic linearized flow

Consider our general solution:

$$\hat{\phi} = f(x - \lambda y).$$

It does not tell us anything about the precise form of f (this will depend on the boundary conditions), but we do see that $\hat{\phi}$, and thus $\hat{u}, \hat{v}$, and all other flow properties will be constant along lines for which $x - \lambda y$ is constant.

The slope of these lines will be

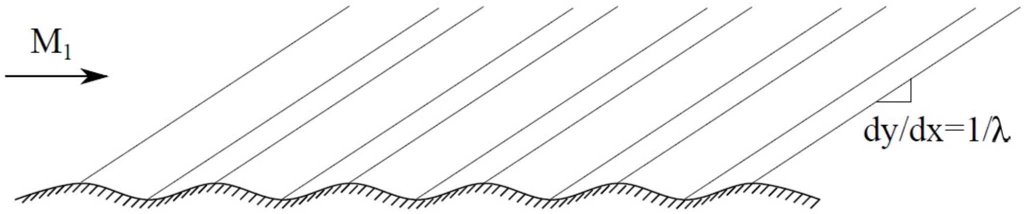
$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_1^2 - 1}}.$$

Recall that Mach lines in supersonic flow have an angle given by

$$\sin\mu = 1/M_1$$

and thus (from trig. identities)

$$\tan\mu = 1/\sqrt{M_1^2 - 1}.$$



We thus see that the flow properties are constant along Mach lines in linearized supersonic flow (note that the Mach number is, to first order, constant throughout the flow field).

Pressure coefficient in supersonic linearized flow

Recall that the pressure coefficient in linearized flow is

$$C_p = -\frac{2\hat{u}}{V_1}.$$

Also, we have at a solid boundary with angle θ to the freestream flow

$$\hat{v} = \frac{\partial \hat{\phi}}{\partial y} = V_1 \tan\theta \approx V_1\theta,$$

where we have assumed that θ is small.

Now from our general solution to the linearized equation,

$$\hat{u} = \frac{\partial \hat{\phi}}{\partial x} = f', \quad \text{and} \quad \hat{v} = \frac{\partial \hat{\phi}}{\partial y} = -\lambda f'$$

Equating f' in these two equations:

$$\hat{u} = -\frac{\hat{v}}{\lambda} = \frac{V_1\theta}{\lambda}$$

Substituting into our expression for C_p above, we arrive at

$$\boxed{C_p = \frac{2\theta}{\sqrt{M_1^2 - 1}}.}$$