Lecture 19: Quartornions

Note: $\hat{e} = e_{b1}\hat{b}_1 + e_{b2}\hat{b}_2 + e_{b3}\hat{b}_3$

$\hat{e} = e_{n1}\hat{x} + e_{n2}\hat{y} + e_{n3}\hat{z}$

b/c the vector is fixed in both frames: $e_{bi} = e_{ni} = e_i$

${}^B\vec{e} = [C]^I\vec{e}$

$\vec{e}$ is the eigenvector of $[C]$ corresponding to the eigenvalue of $+1$

We would prefer not to find the eigenvectors of a 3x3 matrix → time consuming

Instead, we can derive this principal axis directly from the rotation matrix.

Full derivation is in Schaub & Junkins (Sec 3.4)

Note: the eigenvector is normalized ⇒ length $= 1 = e_1^2 + e_2^2 + e_3^2$

$\cos\theta = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1)$, C_{ij} is the i^{th} row, j^{th} column of the rotation matrix

θ is the required rotation angle about the principal axis.

$$\vec{e} = \frac{1}{2\sin\theta}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix}$$

$\vec{e}$ = principal axis, unit length

$\pm\theta, \pm\vec{e}$, but we choose θ such that the rotation vector $= \theta\hat{e}$

A single s/c orientation can be described with 2 possible combinations of θ & $\hat{e}$:

$\theta, \hat{e}$ or $-\theta, -\hat{e}$

Quarternions: Euler Parameters

Key idea: Quarternions describe s/c attitude by describing the rotation matrix between the body-fixed & inertial frames.

$$\begin{aligned} q_1 &= e_1 \sin(\theta/2) \\ q_2 &= e_2 \sin(\theta/2) \\ q_3 &= e_3 \sin(\theta/2) \\ q_4 &= \cos(\theta/2) \end{aligned}$$

Note: $\boxed{q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1}$

Benefit of quarternions is that they are non-singular: they do not have a divide by zero.

-also work well for large & small rotations.

Can also map directly between rotation matrix & quaternions

$$[C] = \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_2q_1-q_3q_4) & 1-2(q_1^2+q_3^2) & 2(q_2q_3+q_1q_4) \\ 2(q_3q_1+q_2q_4) & 2(q_3q_2-q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

$$\beta_0 = q_4 = \pm\frac{1}{2}(1+C_{11}+C_{22}+C_{33})^{1/2}$$

$$\beta_1 = q_1 = \frac{1}{4q_4}(C_{23}-C_{32})$$

$$\beta_2 = q_2 = \frac{1}{4q_4}(C_{31}-C_{13})$$

$$\beta_3 = q_3 = \frac{1}{4q_4}(C_{12}-C_{21})$$

$\pm$ term gives 2 sets of quaternions that describe a single orientation (ie a single rotation)

Note: quaternions are not singular, but this mapping has a singularity (when $q_4=0$)

If $\theta = 180°$ ⇒ $q_4 = 0$. ⇒ body-fixed coordinate system is opposite the inertial system ⇒ S/C is upside down

If $q_4 < 0$, $\theta > 180°$ → S/C has traveled thru "upside down" to get to the current orientation

"long-way" rotation

⇒ quaternions have some <u>path dependence</u>

If $\beta_0 = q_4 = \cos(360/2) = -1$ ⇒ S/C has executed a complete rotation

$\vec{\beta}$ & $-\vec{\beta}$ describe the same orientation

$\beta_0 = \cos(\theta/2) \qquad \beta_i = e_i \sin(\theta/2)$

$\beta_0' = \cos(-\theta/2) = \cos(\theta/2) = \beta_0$

$\beta_i' = -e_i \sin(-\theta/2) = -e_i(-\sin(\theta/2)) = e_i \sin(\theta/2) = \beta_i$

For a long-way rotation: $\hat{e}$, $\theta' = \theta - 2\pi$

$$\beta_0' = \cos(\theta'/2) = \cos(\theta/2 - \pi) = -\cos(\theta/2) = -\beta_0$$

$$\beta_i' = e_i \sin(\theta'/2) = e_i \sin(\theta/2 - \pi) = -e_i \sin(\theta/2) = -\beta_i$$

If β_0 is close to 1, you are describing a small rotation.

Another benefit of quaternions is that there is a linear mapping to angular momentum:

$$\begin{bmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

this is the angular velocity of the s/c's rotation

The first column is multiplied by zero, but we use it because it makes the matrix orthogonal.

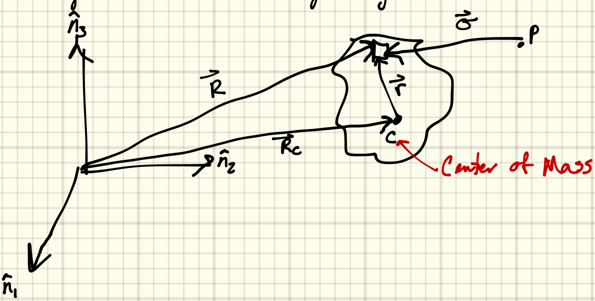
$\vec\beta$ is a unit vector $\Rightarrow$ matrix is orthonormal.

Why do we like orthonormal matrices?

B/c the inverse is the transpose. Use this to get $\vec\omega$ as $f(\dot{\vec\beta})$

<u>Rigid body Dynamics:</u>

Angular Momentum of the Rigid body:



Find the angular momentum about some arbitrary point P.

Note: $\vec\sigma = \vec R - \vec R_P$

Angular momentum: $\vec H_P = \int_B \vec\sigma \times \dot{\vec\sigma}\, dm$

Take the derivative of $\vec H_P$:

$$\dot{\vec H}_P = \cancelto{0}{\int_B \dot{\vec\sigma} \times \dot{\vec\sigma}\, dm} + \int_B \vec\sigma \times \ddot{\vec\sigma}\, dm$$

$$= \int_B \vec\sigma \times \ddot{\vec R}\, dm - \int_B \vec\sigma\, dm \times \ddot{\vec R}_P$$

Note: $\int_B \vec R\, dm = M \vec R_c$

$$\int_B \vec\sigma\, dm = \int_B \vec R\, dm - \int_B dm\, \vec R_P \longrightarrow \Rightarrow \int_B \vec\sigma\, dm = M(\vec R_c - \vec R_P)$$

External torque applied to the system:

$$\vec{L}_P = \int_B \vec{\sigma} \times \ddot{\vec{R}}\, dm$$

$$\Rightarrow \boxed{\dot{\vec{H}}_P = \vec{L}_P - M(\vec{R}_c - \vec{R}_P) \times \ddot{\vec{R}}_P}$$

If $\vec{R}_c = \vec{R}_P$ -or- $\ddot{\vec{R}}_P = 0$, then $\dot{\vec{H}}_P = \vec{L}_P$.