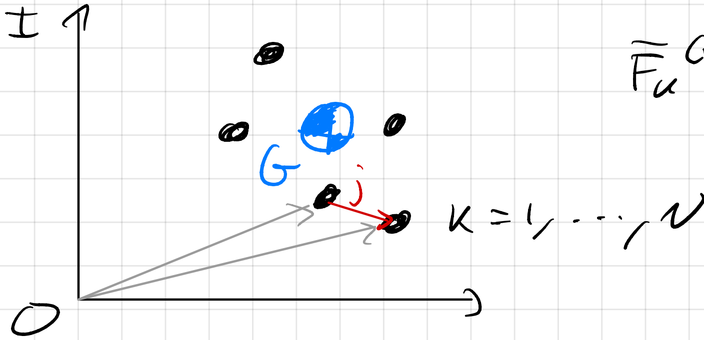
Chapter 7

AM & Energy of MPS

10/8/24

Recall Ch 6



$$\bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j} = m_k \, {}^I\bar{a}_{k/O}$$

$${}^I\bar{P}_O = {}^I\bar{P}_{G/O}$$

$$\bar{F}_G^{(ext)} = m_G \, {}^I\bar{a}_{G/O}$$

Recall Ch 4

$${}^I\bar{h}_{k/O} = \bar{r}_{k/O} \times m_k \, {}^I\bar{v}_{k/O}, \quad k = 1, \dots, N$$

$$\frac{{}^I d}{dt}\left({}^I\bar{h}_{k/O}\right) = \bar{M}_{k/O} = \bar{r}_{k/O} \times \bar{F}_k$$

Dfn <u>total angular momentum</u>

$${}^I\bar{h}_O = \sum_{k=1}^{N} {}^I\bar{h}_{k/O} = {}^I\bar{h}_{1/O} + {}^I\bar{h}_{2/O} + \dots + {}^I\bar{h}_{N/O}$$

Consider

$$\frac{{}^I d}{dt}\left({}^I h_O\right) = \frac{{}^I d}{dt}\left(\sum_{k=1}^{N} \bar{r}_{k/O} \times m_k \, {}^I\bar{v}_{k/O}\right)$$

$$= \sum_{k=1}^{N}\left(\underbrace{{}^I\bar{v}_{k/O} \times m_k \, {}^I\bar{v}_{k/O}}_{=0} + \bar{r}_{k/O} \times \underbrace{m_k \, {}^I\bar{a}_{k/O}}_{=\bar{F}_k \ (N2L)}\right)$$

$$= \sum_{k=1}^{N} \bar{r}_{k/O} \times \left(\bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j}\right)$$

$$= \underbrace{\sum_{k=1}^{N} \bar{r}_{k/O} \times \bar{F}_k^{(ext)}}_{\bar{M}_O^{(ext)}} + \underbrace{\sum_{k=1}^{N}\sum_{j=1}^{N} \bar{r}_{k/O} \times \bar{F}_{k,j}}_{=0 \text{ by assumption}}$$

$$\boxed{\frac{{}^I d}{dt}\left({}^I\bar{h}_O\right) = \bar{M}_O^{(ext)}}$$

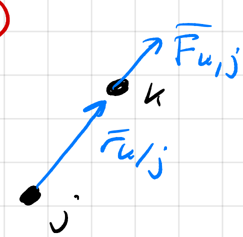
where

$$\bar{M}_O^{(ext)} = \sum_{k=1}^{N} \bar{M}_{k/O}^{(ext)} = \sum_{k=1}^{N} \bar{r}_{k/O} \times \bar{F}_k^{(ext)}$$

★ ${}^I\bar{h}_O$ is conserved if $\bar{M}_O^{(ext)} = 0$

$$\bar r_{k/O} = \bar r_{j/O} + \bar r_{k/j}$$

$$\sum_{k=1}^{N}\sum_{j=1}^{N} \bar r_{k/O}\times \bar F_{k,j} = \frac{1}{2}\left(\sum_{k=1}^{N}\sum_{j=1}^{N} \underbrace{\bar r_{k/O}}_{\substack{= \bar r_{j/O} + \bar r_{k/j} \\ \text{(vector triad)}}}\times \bar F_{k,j} + \sum_{k=1}^{N}\sum_{j=1}^{N} \bar r_{j/O}\times \underbrace{\bar F_{j,k}}_{\substack{= -\bar F_{k,j} \\ \text{(N3)}}}\right)$$

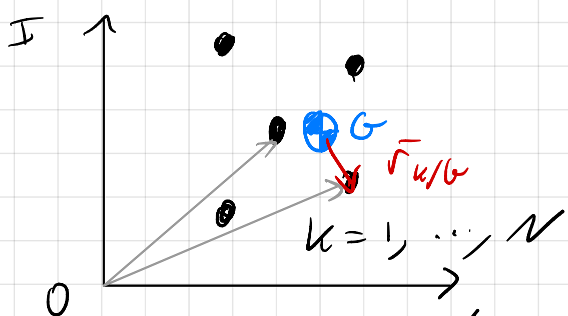


$$= \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N} \underbrace{\bar r_{k/j}\times \bar F_{k,j}}_{=\bar M_{k/j}}$$

$$= 0 \text{ by assumption}$$

<u>Internal Moment Assumption</u>

Separation Angular Momentum



$$\bar r_{k/O} = \bar r_{G/O} + \bar r_{k/G}$$

$${}^{I}\bar v_{k/O} = {}^{I}\bar v_{G/O} + {}^{I}\bar v_{k/G}$$

$${}^{I}\bar h_O = \sum_{k=1}^{N} \bar r_{k/O}\times m_k\, {}^{I}\bar v_{k/O}$$

$$= \sum_{k=1}^{N} \left(\bar r_{G/O} + \bar r_{k/G}\right)\times m_k\left({}^{I}\bar v_{G/O} + {}^{I}\bar v_{k/G}\right) \quad \text{F.O.I.L.}$$

$$m_G = \sum_{k=1}^{N} m_k$$

$$= \overset{F}{\underbrace{\sum_{k=1}^{N} \bar r_{G/O}\times m_k\, {}^{I}\bar v_{G/O}}_{= \bar r_{G/O}\times m_G\, {}^{I}\bar v_{G/O} = {}^{I}\bar h_{G/O}}} + \overset{O}{\underbrace{\sum_{k=1}^{N} \bar r_{G/O}\times m_k\, {}^{I}\bar v_{k/G}}_{= \bar r_{G/O}\times\left(\underbrace{\sum_{k=1}^{N} m_k\, {}^{I}\bar v_{k/G}}_{=0 \text{ COMC}}\right)}} +$$

$$\overset{I}{\underbrace{\sum_{k=1}^{N} \bar r_{k/G}\times m_k\, {}^{I}\bar v_{G/O}}_{= \left(\underbrace{\sum_{k=1}^{N} m_k \bar r_{k/G}}_{=0 \text{ COMC}}\right)\times {}^{I}\bar v_{G/O}}} + \overset{L}{\underbrace{\sum_{k=1}^{N} \bar r_{k/G}\times m_k\, {}^{I}\bar v_{k/G}}_{= \sum_{k=1}^{N} {}^{I}\bar h_{k/G} = {}^{I}\bar h_G \neq 0 !!}}$$

Am relative to G!

$$\boxed{{}^{\mathcal{I}}\bar{h}_O = {}^{\mathcal{I}}\bar{h}_{G/O} + {}^{\mathcal{I}}\bar{h}_G}$$

Separation of A.M.

describes rotation about the COM

Recall $\boxed{{}^{\mathcal{I}}\bar{p}_O = {}^{\mathcal{I}}\bar{p}_{G/O}}$

$${}^{\mathcal{I}}\bar{h}_{G/O} = \bar{r}_{G/O} \times m_G\, {}^{\mathcal{I}}\bar{v}_{G/O}$$ AM of G wrt O

$${}^{\mathcal{I}}\bar{h}_G = \sum_{k=1}^{N} {}^{\mathcal{I}}\bar{h}_{k/G}$$ AM of MPS wrt G

$$\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\bar{h}_O\right) = \bar{M}_O^{(ext)}$$ total AM

$$\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}h_{G/O}\right) = \bar{M}_{G/O}^{(ext)}$$ translation of G wrt O

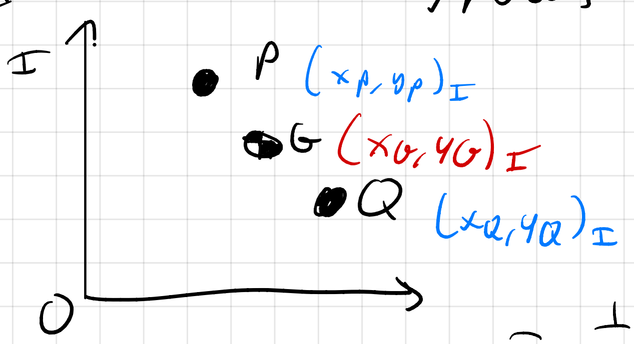
$$\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\bar{h}_G\right) = \bar{M}_G^{(ext)}$$ rotation wrt to G

$$\bar{M}_{G/O}^{(ext)} = \bar{r}_{G/O} \times \bar{F}^{(ext)}$$ total external moment

$$\bar{M}_G^{(ext)} = \sum_{k=1}^{N} \bar{M}_{k/G}^{(ext)} = \sum_{k=1}^{N} \bar{r}_{k/G} \times \bar{F}_k^{(ext)}$$

$\star$ ${}^{\mathcal{I}}\bar{h}_{G/O}$ (resp. ${}^{\mathcal{I}}\bar{h}_G$) is conserved if $\bar{M}_{G/O}^{(ext)} = 0$ (resp. $\bar{M}_G^{(ext)} = 0$)

Ex 7.4 two hockey pucks



$$\bar{r}_{G/O} = \frac{1}{m_G}\sum_{k=1}^{N} m_k \bar{r}_{k/O}$$
$$= \frac{1}{m_P+m_Q}\left(m_P \bar{r}_{P/O} + m_Q \bar{r}_{Q/O}\right)$$
$$= \frac{1}{m_P+m_Q}\Big(\underbrace{(m_P x_P + m_Q x_Q)}_{=x_G}\hat{e}_1 + \underbrace{(m_P y_P + m_Q y_Q)}_{=y_G}\hat{e}_2\Big)$$

$$x_G = \frac{m_P x_P + m_Q x_Q}{m_P + m_Q} \qquad y_G = \frac{m_P y_P + m_Q y_Q}{m_P + m_Q}$$

$$^I\bar{v}_{G/O} = \frac{^I d}{dt}\left(\bar{r}_{G/O}\right) = \frac{^I d}{dt}\left(x_G \hat{e}_1 + y_G \hat{e}_2\right)$$

$$= \dot{x}_G \hat{e}_1 + \dot{y}_G \hat{e}_2$$

$$^I\bar{p}_O = {}^I\bar{p}_{P/O} + {}^I\bar{p}_{Q/O}$$

$$^I\bar{v}_{\underset{Q}{P}/O} = \dot{x}_{\underset{Q}{P}} \hat{e}_1 + \dot{y}_{\underset{Q}{P}} \hat{e}_2$$

$$= m_P\, {}^I\bar{v}_{P/O} + m_Q\, {}^I\bar{v}_{G/O}$$

$$= \underbrace{(m_P \dot{x}_P + m_Q \dot{x}_Q)}_{\stackrel{\checkmark}{=}\ \dot{x}_G m_G}\hat{e}_1 + \underbrace{(m_P \dot{y}_P + m_Q \dot{y}_Q)}_{\stackrel{\checkmark}{=}\ \dot{y}_G m_G}\hat{e}_2$$

$$\stackrel{\checkmark}{=} m_G\, {}^I\bar{v}_{G/O} = {}^I\bar{p}_{G/O}$$

$$^I\bar{h}_O = \bar{r}_{P/O} \times m_P\, {}^I\bar{v}_{P/O} + \bar{r}_{Q/O} \times m_Q\, {}^I\bar{v}_{Q/O}$$

$$= (x_P \hat{e}_1 + y_P \hat{e}_2) \times m_P (\dot{x}_P \hat{e}_1 + \dot{y}_P \hat{e}_2) + \ldots$$

$$= \left[m_P(x_P \dot{y}_P - y_P \dot{x}_P) + m_Q(x_Q \dot{y}_Q - y_Q \dot{x}_Q)\right]\hat{e}_3$$

$$^I\bar{h}_{G/O} = \bar{r}_{G/O} \times m_G\, {}^I\bar{v}_{G/O}$$

$$= (x_G \hat{e}_1 + y_G \hat{e}_2) \times m_G (\dot{x}_G \hat{e}_1 + \dot{y}_G \hat{e}_2)$$

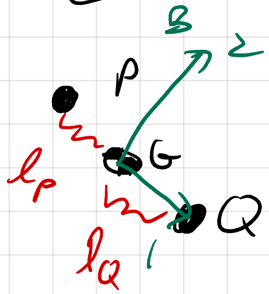
$$= m_G(x_G \dot{y}_G - y_G \dot{x}_G)\hat{e}_3$$

$$^I\bar{h}_G = {}^I\bar{h}_{P/G} + {}^I\bar{h}_{Q/G}$$

$$^I\bar{h}_{P/G} = \bar{r}_{P/G} \times m_P\, {}^I\bar{v}_{P/G}$$

$$= \left[(x_P - x_G)\hat{e}_1 + (y_P - y_G)\hat{e}_2\right] \times$$

$$\left[m_P(\dot{x}_P - \dot{x}_G)\hat{e}_1 + (\dot{y}_P - \dot{y}_G)\hat{e}_2\right] = \cdots$$



$$\bar{r}_{P/G} = -l_P \hat{b}_1 \longrightarrow {}^I\bar{v}_{P/G} = \ldots$$

$$\bar{r}_{Q/G} = l_Q \hat{b}_1 \longrightarrow {}^I\bar{v}_{Q/G} = \ldots$$

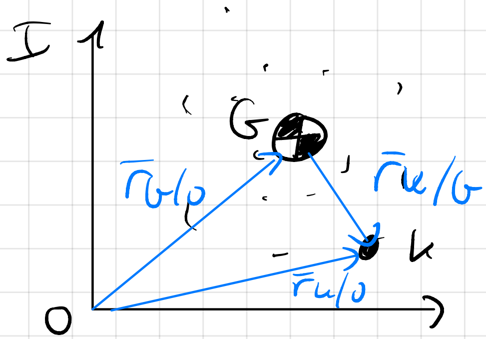
Work & Energy for MPS

Dfn total kinetic energy

$$T_O = \sum_{k=1}^{N} T_{k/O}$$

$$= \frac{1}{2}\sum_{k=1}^{N} m_k \|{}^I\bar{v}_{k/O}\|^2$$

Separation of total K.E.



$$\bar{r}_{k/O} = \bar{r}_{G/O} + \bar{r}_{k/G}$$

$${}^I\bar{v}_{k/O} = {}^I\bar{v}_{G/O} + {}^I\bar{v}_{k/G}$$

$$T_O = \frac{1}{2}\sum_{k=1}^{N} m_k \|{}^I\bar{v}_{G/O} + {}^I\bar{v}_{k/G}\|^2$$

F.O.I.L

$$\left({}^I\bar{v}_{G/O} + {}^I\bar{v}_{k/G}\right)\cdot\left({}^I\bar{v}_{G/O} + {}^I\bar{v}_{k/G}\right)$$

$$T_O = \frac{1}{2}\left(\underbrace{\sum_{k=1}^{N} m_k}_{=m_G}\|{}^I\bar{v}_{G/O}\|^2 + 2\underbrace{\sum_{k=1}^{N} m_k\, {}^I\bar{v}_{G/O}\cdot {}^I\bar{v}_{k/G}}_{=0} + \underbrace{\sum_{k=1}^{N} m_k\|{}^I\bar{v}_{k/G}\|^2}_{=T_G}\right)$$

$$\boxed{T_O = T_{G/O} + T_G} = \text{trans KE} + \text{rot. KE}$$

where
$$T_{G/O} = \frac{1}{2} m_G \left\| {}^I\bar{v}_{G/O} \right\|^2$$
$$T_G = \sum_{k=1}^{N} T_{k/G} = \frac{1}{2} \sum_{k=1}^{N} m_k \left\| {}^I\bar{v}_{k/G} \right\|^2$$

$$+2\sum_{k=1}^{N} m_k\, {}^I\bar{v}_{G/O} \cdot {}^I\bar{v}_{k/G} = 2\, {}^I\bar{v}_{G/O} \cdot \underbrace{\left(\sum_{k=1}^{N} m_k\, {}^I\bar{v}_{k/G} \right)}_{\frac{{}^I d}{dt}\underbrace{\left(\sum_{k=1}^{N} m_k \bar{r}_{k/G} \right)}_{= 0 \text{ by COMC}}}$$

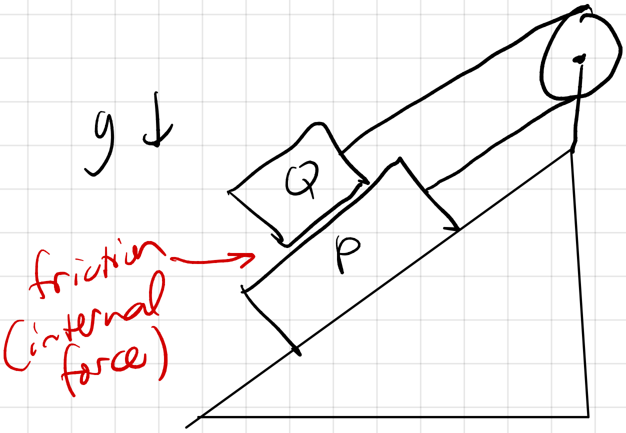
Recall WE#1
$$T_{k/O}(t_2) = T_{k/O}(t_1) + W_k^{(tot)}, \quad k = 1, \dots, N$$
$+$
$$\boxed{T_O(t_2) = T_O(t_1) + \underbrace{W^{(tot)}}_{= \sum_{k=1}^{N} W_k^{(tot)}}}$$

Recall
$$W_k^{(tot)} = \int_{\gamma_k} \bar{F}_k \cdot {}^I d\bar{r}_{k/O},$$
where $\bar{F}_k$ is total force

$$\bar{F}_k = \bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j}$$

total work
$$W^{(tot)} = \sum_{k=1}^{N} \int_{\gamma_k} \left(\bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j} \right) \cdot {}^I d\bar{r}_{k/O}$$

$$= \underbrace{\sum_{k=1}^{N} \int_{\gamma_k} \bar{F}_k^{(ext)} \cdot {}^I d\bar{r}_{k/O}}_{W_k^{(ext)}} + \underbrace{\sum_{k=1}^{N} \sum_{j=1}^{N}}_{\text{internal work} \neq 0} \int_{\gamma_k} \bar{F}_{k,j} \cdot {}^I d\bar{r}_{k/O}$$



$$\boxed{W^{(tot)} = W^{(ext)} + W^{(int)}}$$

$$W^{(ext)} = \sum_{k=1}^{N} W_k^{(ext)}$$

$$W^{(int)} = \frac{1}{2}\left(\sum_{k=1}^{N}\sum_{j=1}^{N} \int_{\gamma_k} \bar{F}_{k,j} \cdot {}^{\mathcal{F}}d\bar{r}_{k/O} + \sum_{k=1}^{N}\sum_{j=1}^{N} \int_{\gamma_j} \underbrace{\bar{F}_{j,k}}_{= -\bar{F}_{k,j}} \cdot {}^{\mathcal{F}}d\bar{r}_{j/O} \right)$$

$$= \frac{1}{2} \sum_{k=1}^{N}\sum_{j=1}^{N} \int_{\gamma_k} \bar{F}_{k,j} \cdot \underbrace{\left({}^{\mathcal{F}}d\bar{r}_{k/O} - {}^{\mathcal{F}}d\bar{r}_{j/O} \right)}_{= {}^{\mathcal{F}}d\bar{r}_{k/j}}$$

$$\boxed{W^{(int)} = \frac{1}{2} \sum_{k=1}^{N}\sum_{j=1}^{N} \int_{\gamma_k} \bar{F}_{k,j} \cdot \underbrace{{}^{\mathcal{F}}d\bar{r}_{k/j}}_{= {}^{\mathcal{F}}\bar{v}_{k/j}\, dt}}$$

Dfn <u>total potential energy</u>

$$U_O = \sum_{k=1}^{N} U_{k/O}$$

$$= \underbrace{\sum_{k=1}^{N} U_{k/O}^{(ext)}}_{U_O^{(ext)}} + \underbrace{\sum_{k=1}^{N} U_{k/O}^{(int)}}_{U_O^{(int)}}$$

$$U_O = U_O^{(ext)} + U_O^{(int)}$$

Recall WE #2

$$
\begin{array}{l}
+\ U_{k/0}(t_2) = U_{k/0}(t_1) - W_k^{(c)}, \quad k = 1, \dots, N \\
\hline
U_0(t_2) = U_0(t_1) - W^{(c)}
\end{array}
$$

$$U_0^{(ext)}(t_2) = U_0^{(ext)}(t_1) - W^{(c,ext)}$$

$$U_0^{(int)}(t_2) = U_0^{(int)}(t_1) - W^{(c,int)}$$

therefore

$$
\begin{array}{l}
T_0(t_2) = T_0(t_1) + W^{(tot)} \\
+\ U_0(t_2) = U_0(t_1) - W^{(c)} \\
\hline
\end{array}
$$

WE #3 $$\boxed{E_0(t_2) = E_0(t_1) + W^{(nc)}}$$

total total energy

$$E_0 = \sum_{k=1}^{N} E_{k/0} = \sum_{k=1}^{N} T_{k/0} + U_{k/0}$$

$$W^{(nc)} = W^{(nc,ext)} + W^{(nc,int)}$$

☆ E_0 is conserved if $W^{(nc,ext)} + W^{(nc,int)} = 0$

$$W^{(nc,int)} = \frac{1}{2} \sum_{k=1}^{N} \sum_{j=1}^{N} \int_{\gamma_k} \bar{F}_{k,j} \cdot d\bar{r}_{k/j}$$

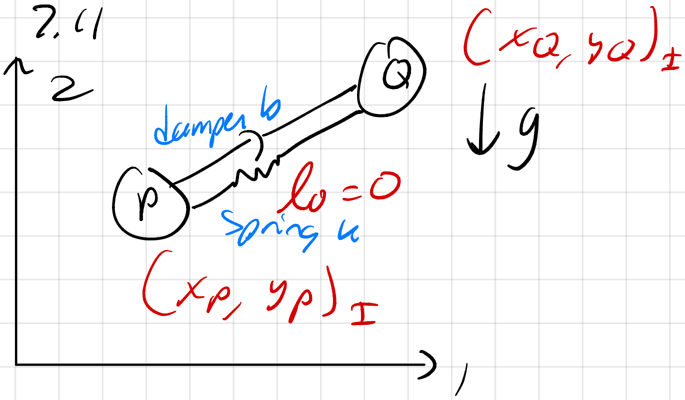
$$= \sum_{k=1}^{N} \sum_{j=1}^{N} \int_{\gamma_k} \bar{F}_{k,j} \cdot d\bar{r}_{k/0}$$

${}^I\bar{p}_O$ is conserved if $\bar{F}^{(ext)} = 0$

${}^I\bar{h}_O$ " " " $\bar{M}^{(ext)} = 0$

E_O " " " $W^{(nc,ext)} + W^{(nc,int)} = 0$

Ex 2.11



$$T_O = T_{P/O} + T_{Q/O}$$
$$= \frac{1}{2} m_P \| {}^I\bar{v}_{P/O} \|^2 + \frac{1}{2} m_Q \| {}^I\bar{v}_{Q/O} \|^2$$

$$T_{P/O} = \frac{1}{2} m_P \| \dot{x}_P \hat{e}_1 + \dot{y}_P \hat{e}_2 \|^2$$
$$= \frac{1}{2} m_P (\dot{x}_P^2 + \dot{y}_P^2)$$

$$T_{Q/O} = \frac{1}{2} m_Q (\dot{x}_Q^2 + \dot{y}_Q^2)$$

$$U_O = U_{P/O} + U_{Q/O}$$
$$= U_O^{(ext)} + U_O^{(int)}$$
$$= U_{P/O}^{(ext)} + U_{Q/O}^{(ext)} + U_O^{(int)}$$
$$= m_P g y_P + m_Q g y_Q + \frac{1}{2} k \Big[\underbrace{(x_P - x_Q)^2}_{\Delta x} + \underbrace{(y_P - y_Q)^2}_{\Delta y} \Big]$$

$r_{k/j}$

$$\left(\sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2} - l_0 \right)^2$$

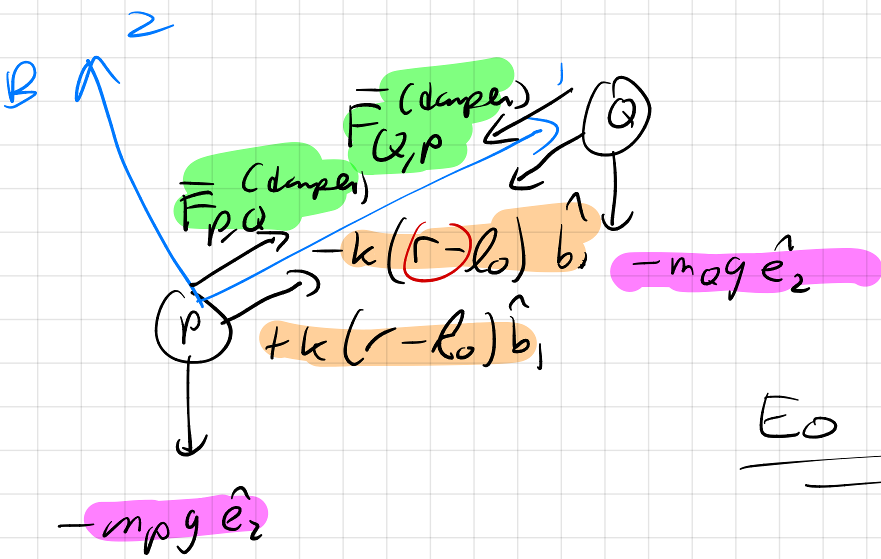
$$E_O = T_O + U_O$$

$$= \tfrac{1}{2} m_P(\dot{x}_P^2 + \dot{y}_P^2) + \tfrac{1}{2} m_Q(\dot{x}_Q^2 + \dot{y}_Q^2)$$

$$+ m_P g y_P + m_Q g y_Q + \tfrac{1}{2} k(\Delta x^2 + \Delta y^2)$$

WE#3 $$E_O(t_2) = E_O(t_1) + \cancel{W^{(nc,ext)}}^{\,0} + \underbrace{W^{(nc,int)}}_{\neq 0}$$

because of damper



$$r = (\Delta x^2 + \Delta y^2)^{1/2}$$

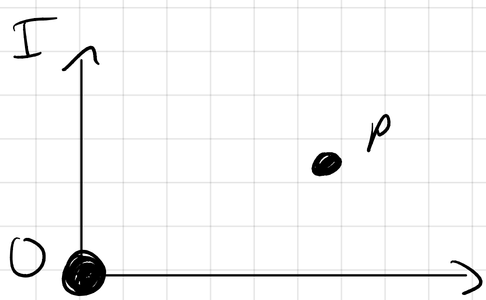
E_O not conserved!!

due to damper

Tutorial 7.1 Two-body problem

Recall

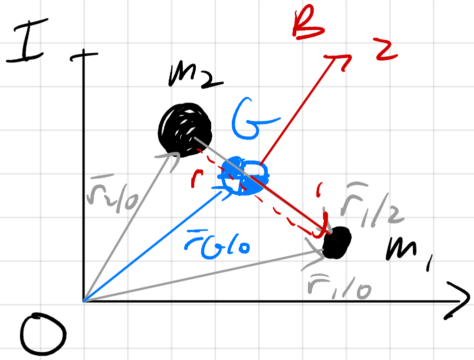
Simple Satellite



$$M = 3 - 1 = 2$$

$$m_O >> m_P$$

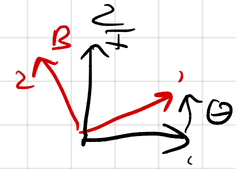
Two-body problem



$$M = 3 \cdot 2 - 2 = 4$$

$$m_2 > m_1$$

$$r = \|\bar{r}_{1/2}\|$$



$$\cos\theta = \hat{b}_1 \cdot \hat{e}_1$$

$$\bar{F}_1 = m_1 \, {}^I\bar{a}_{1/O} \qquad \bar{F}_2 = m_2 \, {}^I\bar{a}_{2/O} \qquad \underbrace{\bar{F}_G^{(ext)}}_{=0} = m_G \, {}^I\bar{a}_{G/O}$$

$$= \cancel{\bar{F}_1^{(ext)}}^{\,0} + \bar{F}_{1,2} \qquad = \cancel{\bar{F}_2^{(ext)}}^{\,0} + \bar{F}_{2,1}$$

$$\frac{{}^I d}{dt}\left({}^I\bar{P}_{G/O}\right) = \bar{F}_G^{(ext)} = 0 \Rightarrow {}^I\bar{P}_{G/O} = \bar{P}_O \text{ is conserved}$$

$$\frac{{}^I d}{dt}\left({}^I\bar{h}_{G/O}\right) = \bar{M}_{G/O}^{(ext)} = 0 \Rightarrow {}^I\bar{h}_{G/O} \text{ is conserved}$$

$$\frac{{}^I d}{dt}\left({}^I\bar{h}_G\right) = \bar{M}_G^{(ext)} = 0 \Rightarrow {}^I\bar{h}_G \text{ is conserved}$$

$$E_O(t_2) = E_O(t_1) + \cancel{W^{(nc,ext)}}_{0} + \underbrace{W^{(nc,int)}}_{=0}$$

$$\Rightarrow E_O \text{ is conserved}$$

$$\bar{F}_1 = \bar{F}_{1,2} = -\frac{G m_1 m_2}{r^2} \hat{b}_1$$

$$\bar{r}_{1/O} = \bar{r}_{G/O} + \bar{r}_{1/G}$$

$$\bar{r}_{2/O} = \bar{r}_{G/O} + \bar{r}_{2/G}$$

$$\vdots$$

$$ {}^I\bar{a}_{1/O} = \mu_2 \, {}^I\bar{a}_{1/2}, \quad \mu_2 = \frac{m_2}{m_1 + m_2}$$

$$\bar{r}_{1/2} = r\hat{b}_1$$

$$ {}^I\bar{v}_{1/2} = \dot{r}\hat{b}_1 + r\dot{\theta}\hat{b}_2$$

$$ {}^I\bar{a}_{1/2} = \dots$$

$$\boxed{\begin{aligned} \ddot{r} &= \dots \\ \ddot{\theta} &= \dots \end{aligned}}$$

same eq. mo. as simple satellite

$\Rightarrow$ sol'ns are conic sections.