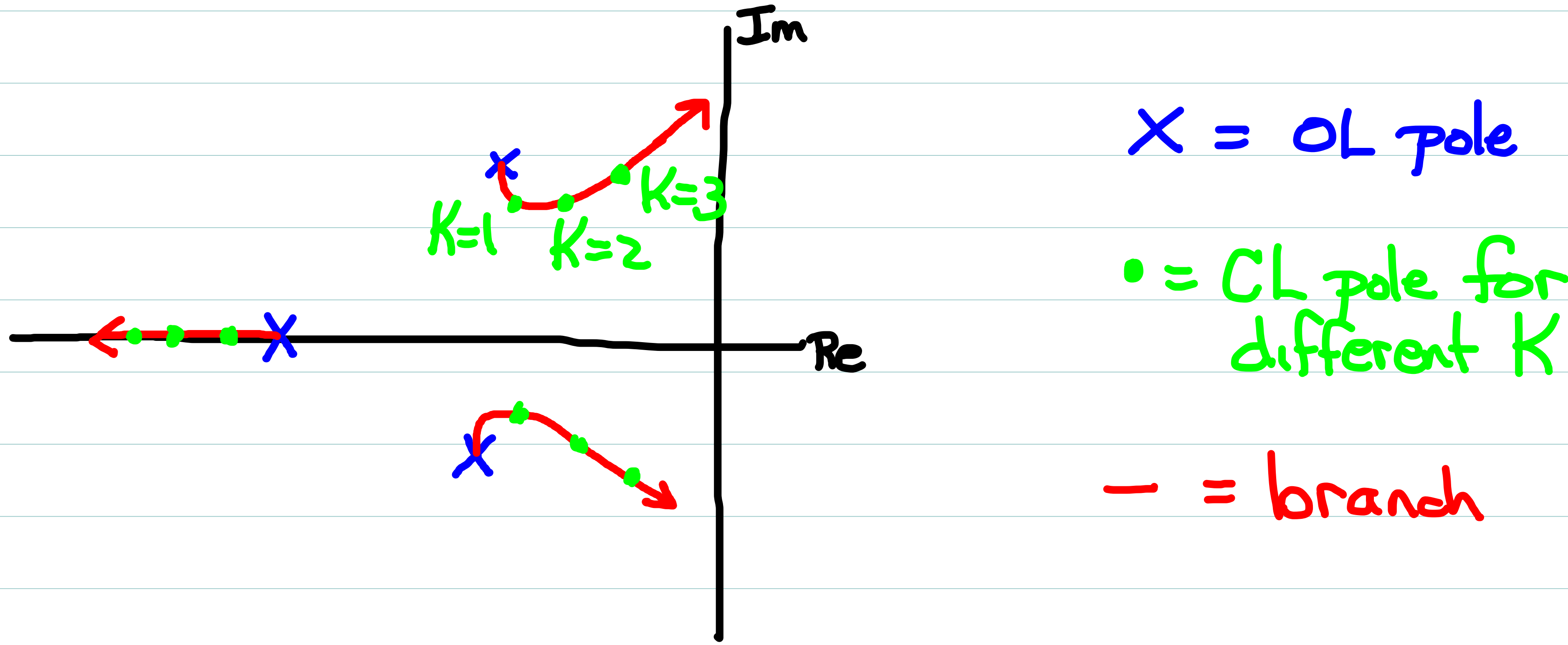
Varying K

⇒ As K changes, the CL pole locations migrate away from OL poles

⇒ Each CL pole location traces out a continuous curve starting at an OL pole. These curves are called **branches**.

⇒ Since there are n CL poles, there are n branches



Symmetry

⇒ Recall that complex roots of polynomial equations occur in conjugate pairs.

⇒ If $s \in \mathbb{C}$ satisfies $1+L(s)=0$, so also $\bar{s}$ satisfies $1+L(\bar{s})=0$.

⇒ CL pole locations are symmetric about real Axis.

⇒ Branches of CL pole loci are symmetric ("mirror image") about real Axis.

⇒ Can we predict branch behavior as $|K|$ increases?

High gain limit : $|K| \to \infty$

Recall CL poles satisfy $D(s) + KN(s) = 0$

Equivalently if $K \neq 0$:

$$N(s) + \left[\frac{1}{K}\right] D(s) = 0$$

and as $|K| \to \infty$ we have : $N(s) = 0$

$\Rightarrow$ As $|K| \to \infty$, the CL poles coincide with OL zeros!

$\Rightarrow$ Branches terminate at OL zeros!

$\Rightarrow$ OL zeros "attract" CL poles to them in high gain limit

$\Rightarrow$ RHP zeros in $L(s)$ are dangerous!

High gain limit, cont

$\Rightarrow$ n CL poles (branches), but only $m \leq n$ OL zeros.

$\Rightarrow$ What happens to other $n-m$ CL poles (branches)?

$\Rightarrow$ The remaining $n-m$ branches asymptote to infinity

$\Rightarrow$ But how? Depends on sign of K. Suppose for simplicity we take $K > 0$.

$\Rightarrow$ Recall "angle condition" for $K > 0$:

if s is a possible CL pole, then

$$\measuredangle L(s) = (1+2\ell)180^\circ \quad \text{(odd multiple of } 180^\circ\text{)}.$$

Interpretation of Angle Condition

Just like in Bode, for any $s \in \mathbb{C}$:

$$\measuredangle L(s) = \sum_{i=1}^{m} \measuredangle (s - z_i) - \sum_{k=1}^{n} \measuredangle (s - p_k)$$

More compactly:

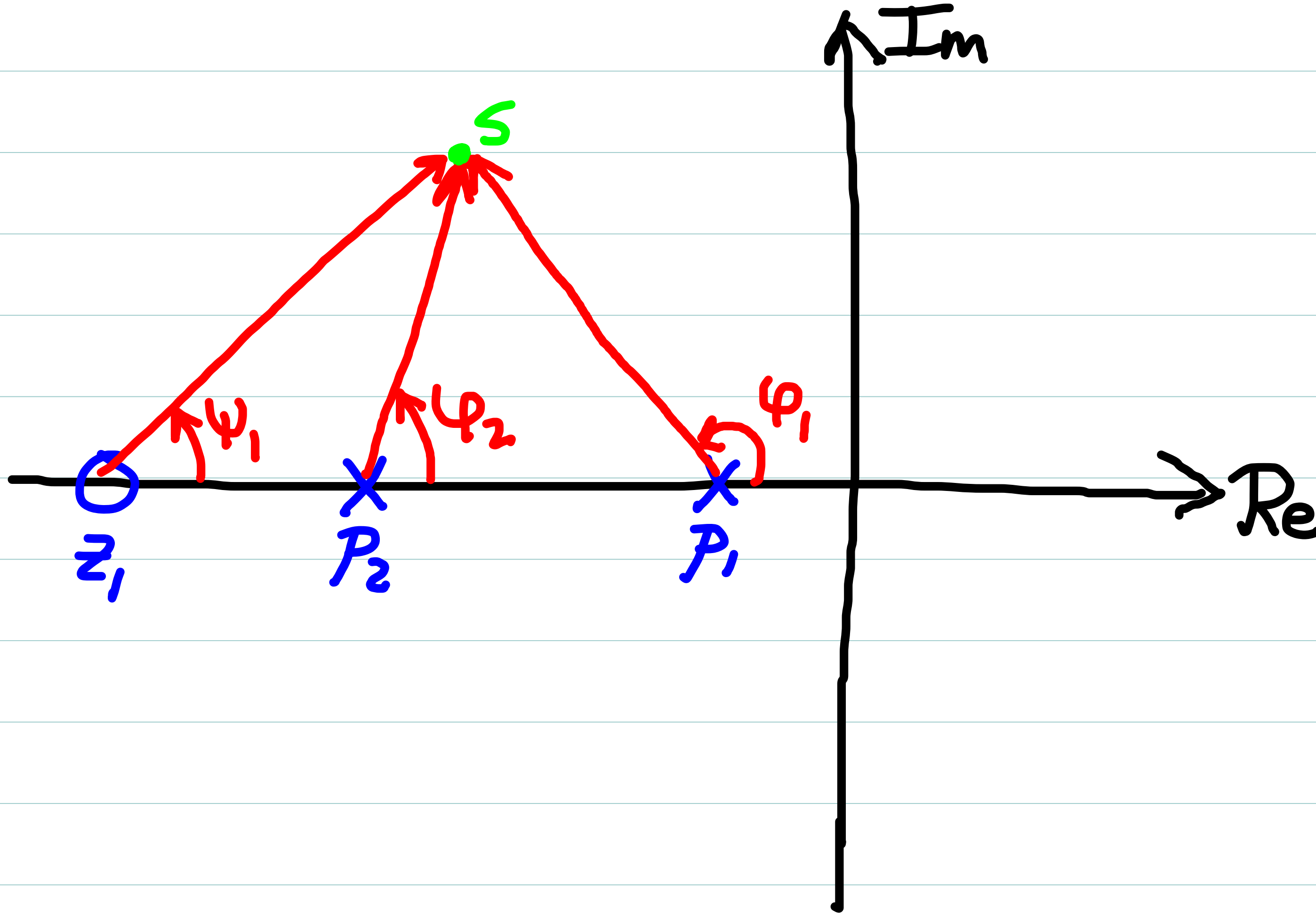
$$\measuredangle L(s) = \sum_{i=1}^{m} \psi_i - \sum_{k=1}^{n} \varphi_k$$

where.

$\psi_i = \measuredangle (s - z_i)$ (Contribution of each zero)

$\varphi_k = \measuredangle (s - p_k)$ (Contribution of each pole)

Graphical (Phasor) Interpretation



$\Rightarrow$ s is a possible CL pole (hence lies on a branch of the locus) if:

$$\psi_1 - \varphi_1 - \varphi_2 = (1+2\ell)180^\circ$$

For high gain limit, look for s with $|s| \gg 1$ which satisfy this

Note that for such s, the phasors become indistinguishable!

$\varphi_1 \approx \varphi_1 \approx \varphi_2 = \alpha$

Thus, for $|s| >> 1$, the angle condition becomes:

$$(1+2\ell)180° = \sum_{i=1}^{m} \psi_i - \sum_{k=1}^{n} \varphi_k$$

$$= m\alpha - n\alpha$$

$$= (m-n)\alpha$$

Where α is common phasor angle from z_i or p_k to s.

We then have

$$\boxed{\alpha = \frac{(1+2\ell)180°}{n-m}}$$

is the angular direction in complex plane for s with $|s| >> 1$ that satisfy angle condition

Asymptotes

$\Rightarrow$ The $n-m$ branches which diverge to infinity do so along asymptotes which make angles of

$$\boxed{\alpha_\ell = \frac{(1+2\ell)180^\circ}{n-m}}$$

with respect to real Axis.

$\Rightarrow$ A slightly messy additional derivation shows these asymptotes intersect at a common point on real Axis, given by

$$\boxed{\sigma_a = \frac{\sum_{k=1}^{n} Re\{p_k\} - \sum_{i=1}^{m} Re\{z_i\}}{n-m}}$$

where again z_i, p_k are zeros and poles of $L(s)$.

"Asymptote rule"

There will be $n-m$ <u>unique</u> asymptotes that result from this equation.

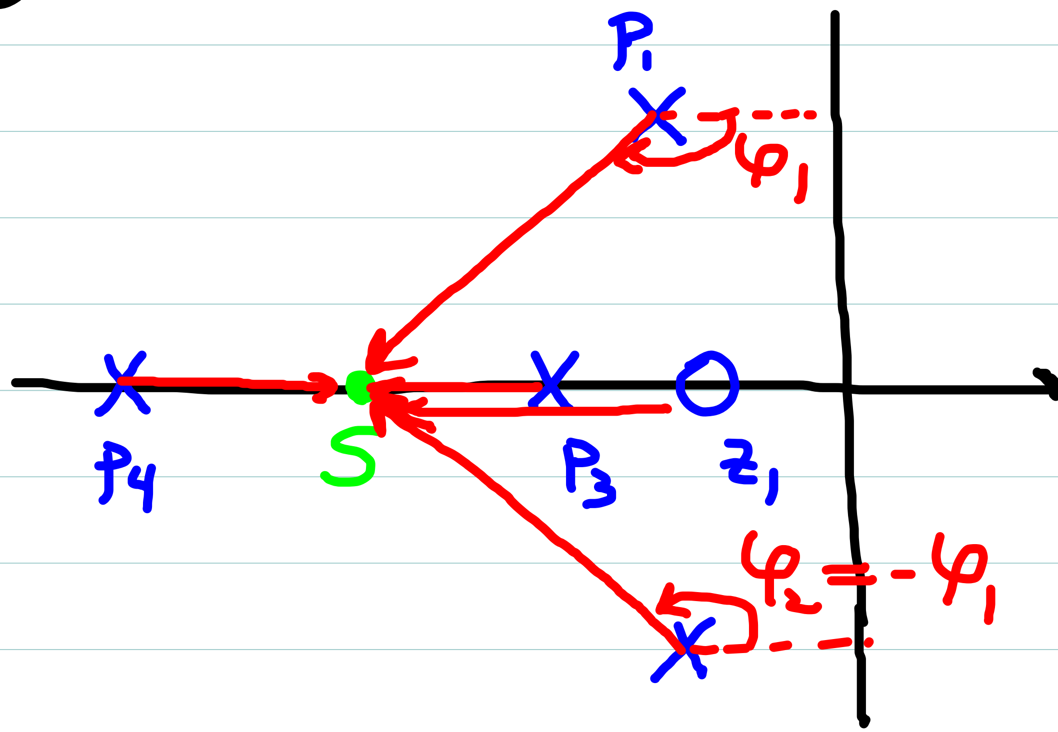
$n-m$	α_e
1	$-180°$
2	$\pm 90°$
3	$\pm 60°, -180°$
4	$\pm 45°, \pm 135°$

$\vdots$

etc

Branches on real Axis

Look at angle condition on real Axis



$\Rightarrow$ Contribution to angle condition from complex conjugate pole or zero pairs will <u>cancel</u>.

$\Rightarrow$ Contribution from any real pole or zero to <u>left</u> of s will be zero

Thus, only the real poles and zeros lying to right of s will contribute to angle condition at a real s.

In particular:

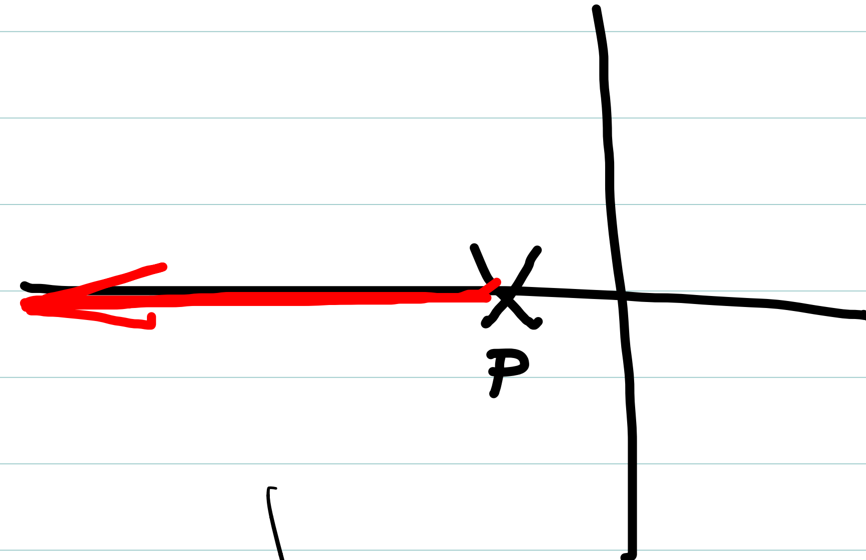
If the total number of real poles and zeros lying to the right of a point s on real Axis is odd, then that point satisfies the angle condition.

Portions of branches of the locus lie on segments of real Axis which satisfy this cond'n:

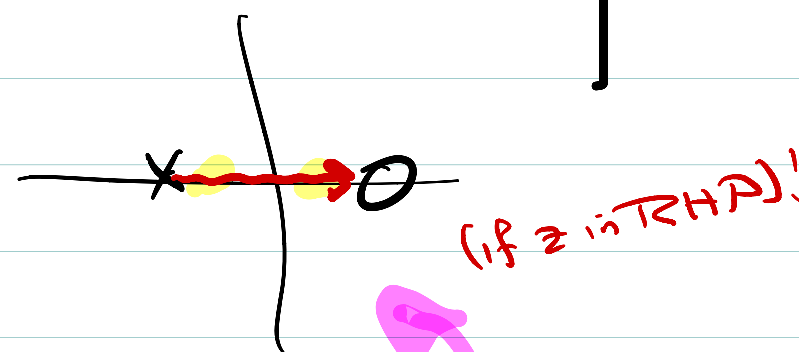
"real Axis rule"

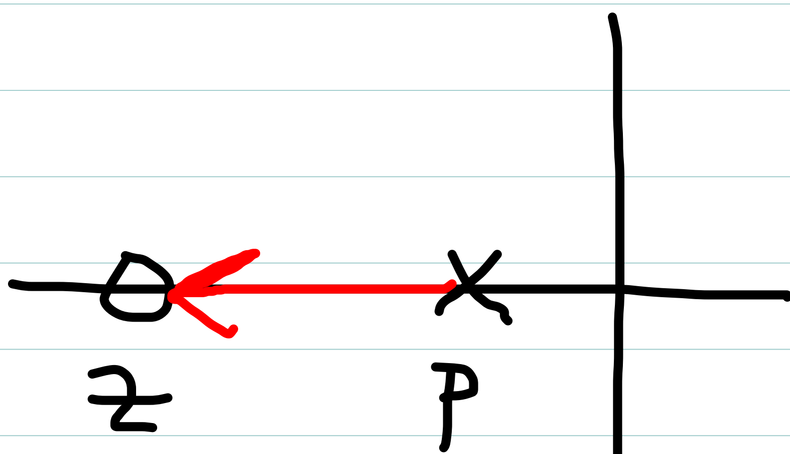
Simple Examples

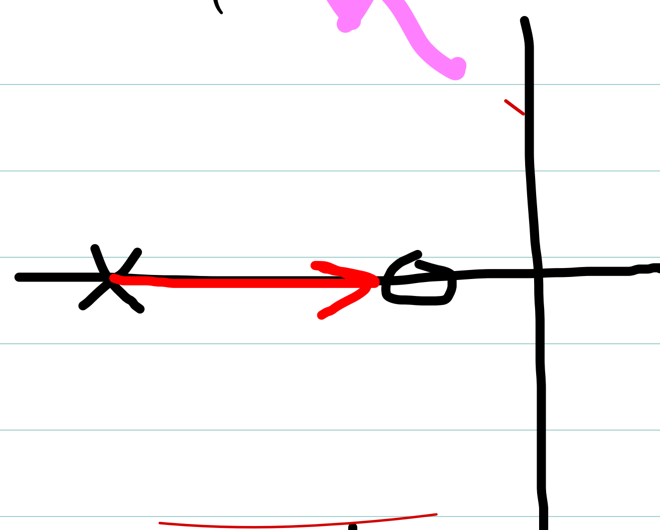
#1) $L(s) = \frac{K}{s-p}$



#2) $L(s) = K\left[\frac{(s-z)}{(s-p)}\right]$







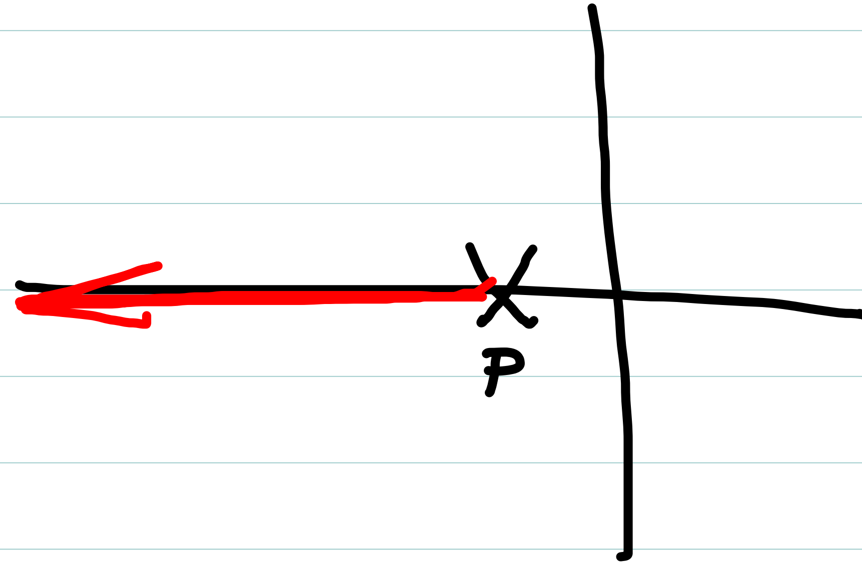
$|z| > |p|$

$|z| < |p|$

Simple Examples

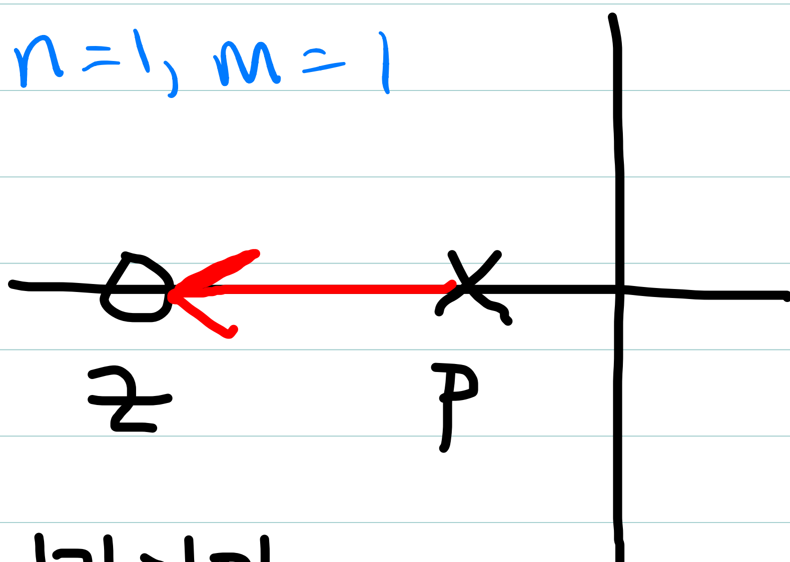
#1) $L(s) = \frac{K}{s-p}$

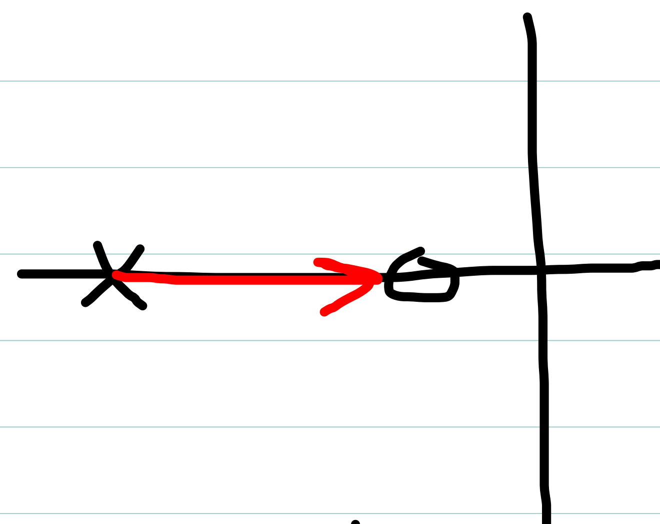
$n=1, m=0$



#2) $L(s) = K\left[\frac{(s-z)}{(s-p)}\right]$

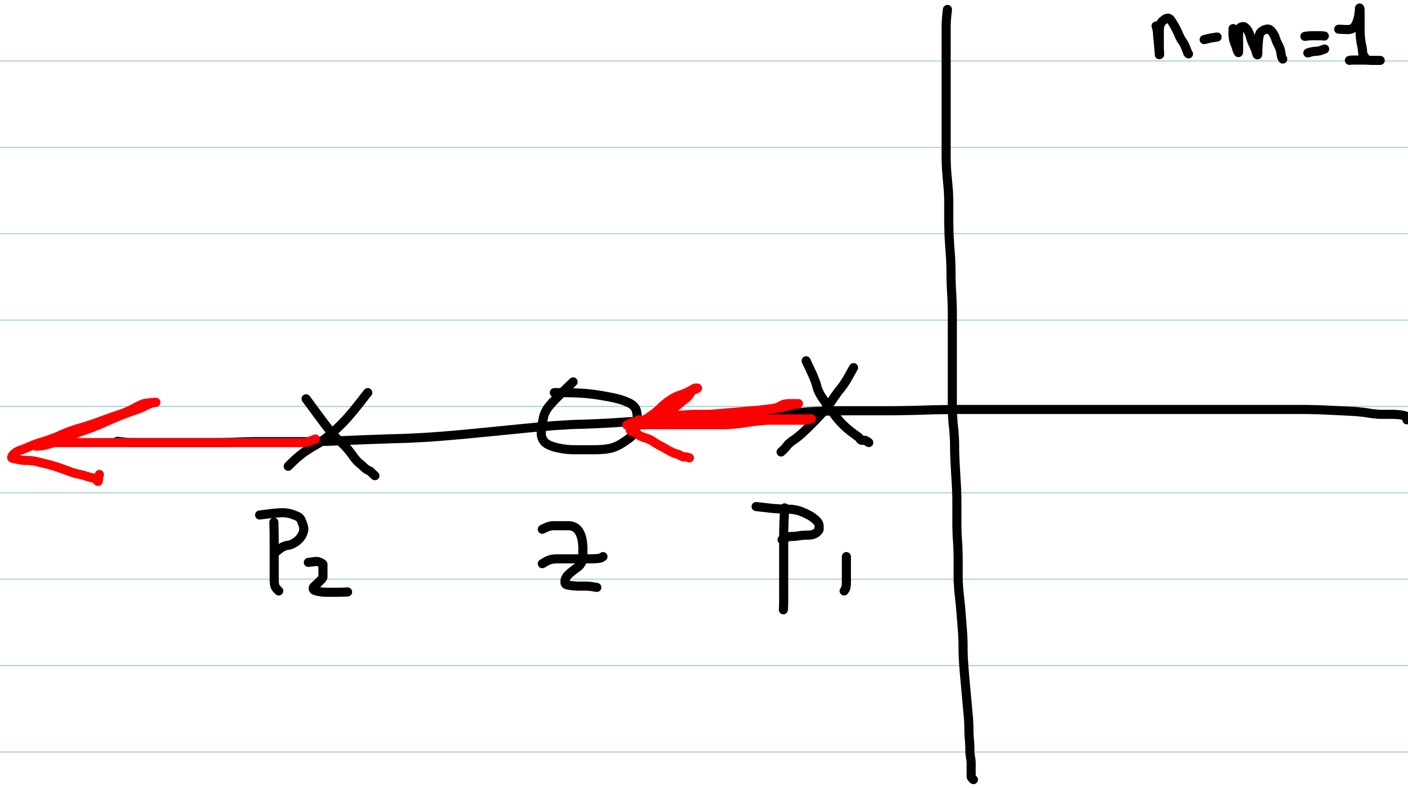
$n=1, m=1$





$|z|>|p|$ $|z|<|p|$

#3] $L(s) = K\left[\frac{(s-z)}{(s-p_1)(s-p_2)}\right]$



$n-m=1$: asymptote along neg real Axis

$|p_2| > |z| > |p_1|$

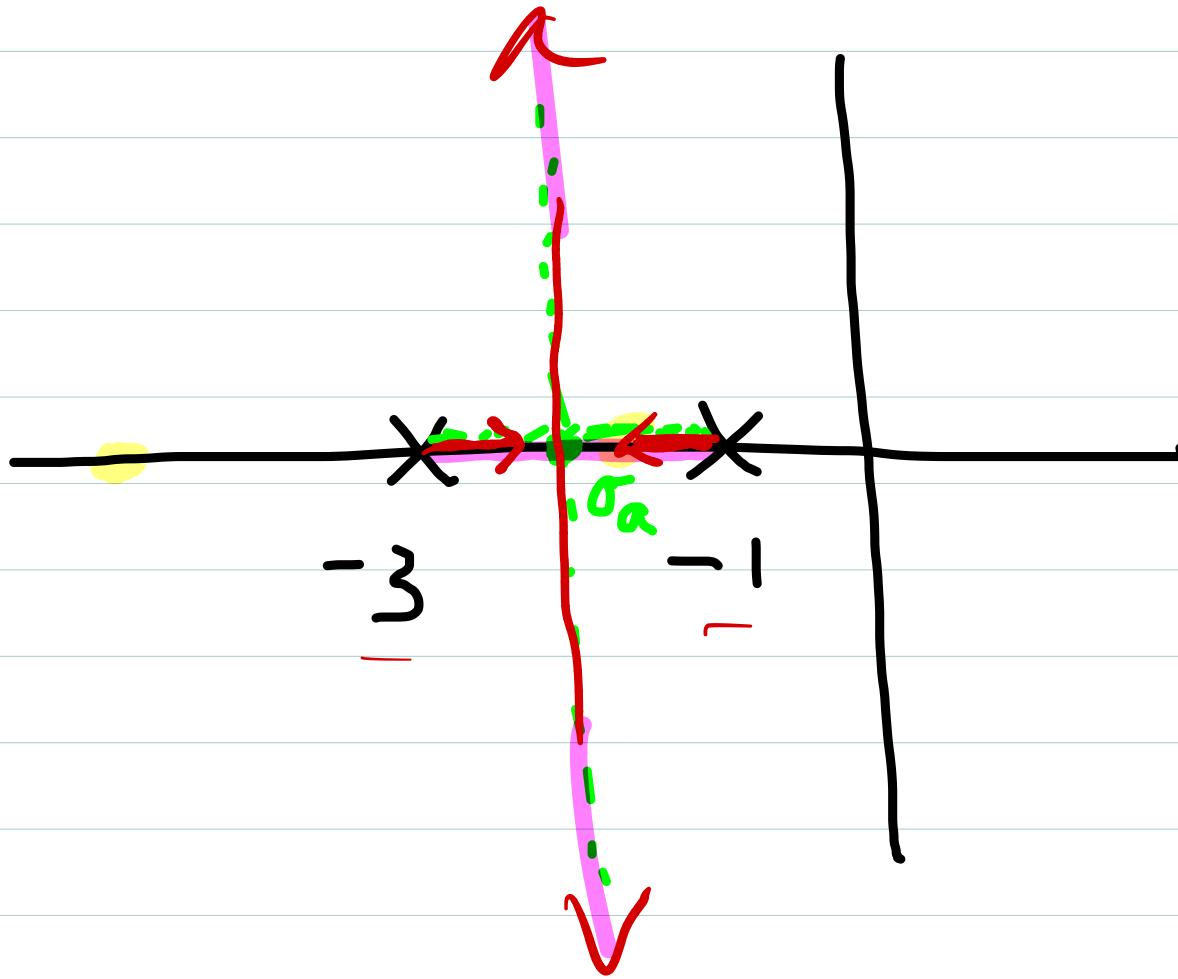
Case where $|z| < |p_1| \leq |p_2|$ is more complicated -- see below.

#4] $L(s) = \frac{K}{(s+1)(s+3)}$

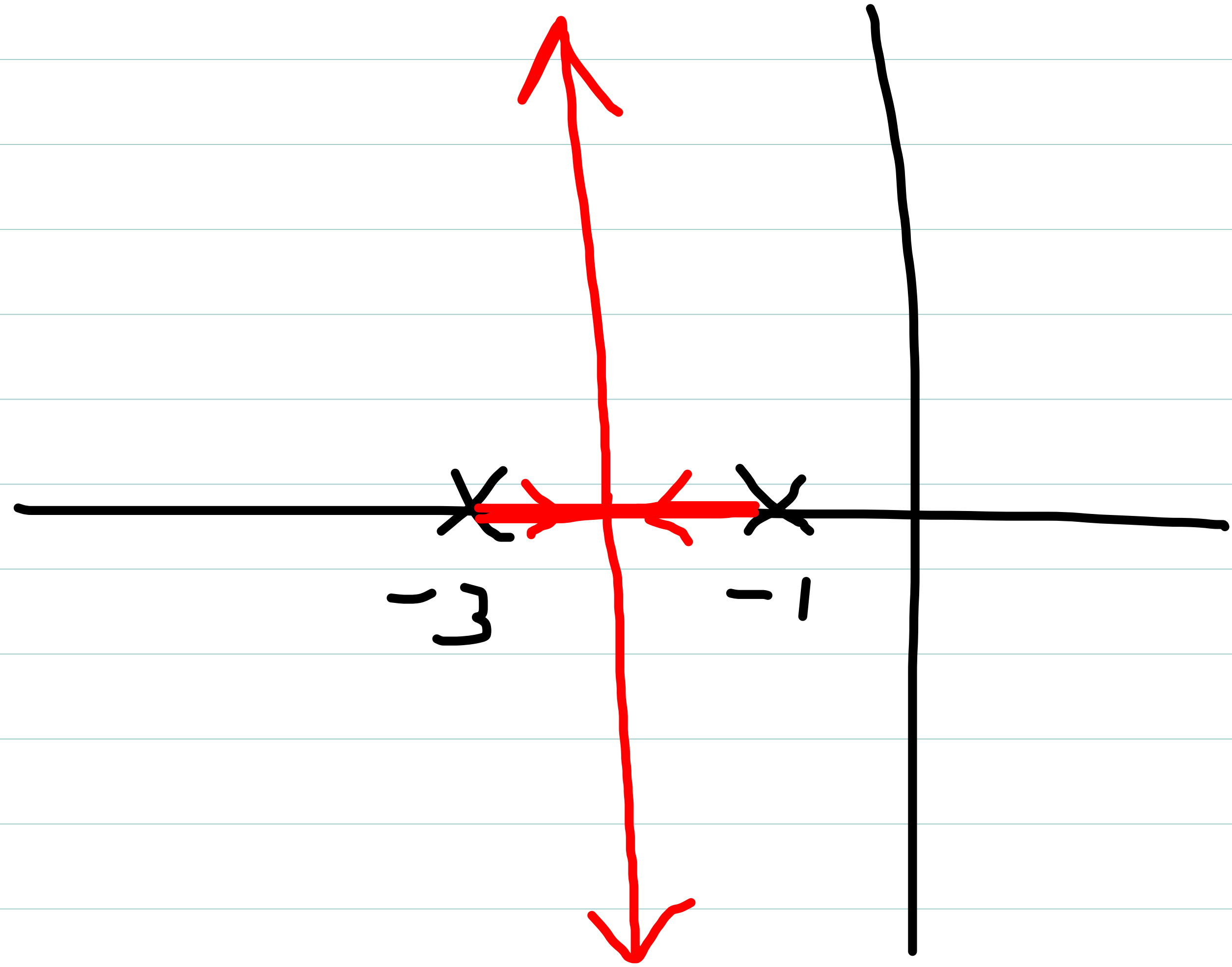
$n-m=2 \Rightarrow \alpha_\ell = \pm 90°$

$\sigma_a = \frac{(-1)+(-3)}{2} = -2$

real Axis locus from $-3 \rightarrow -1$



Actual locus:



Compare w/ exact sol'n for CL poles:

$$s^2+4s+(3+K)=0 \implies s = -2 \pm \frac{\sqrt{16-(3+K)^2}}{2}$$

Example #5

$$L(s) = \frac{K}{s(s+1)(s+2)}$$

$n = 3$, $m = \emptyset$, $n - m = 3 \Rightarrow$ 3 branches go to ∞ along

asymptotes: $\alpha_e = \pm 60°, 180°$

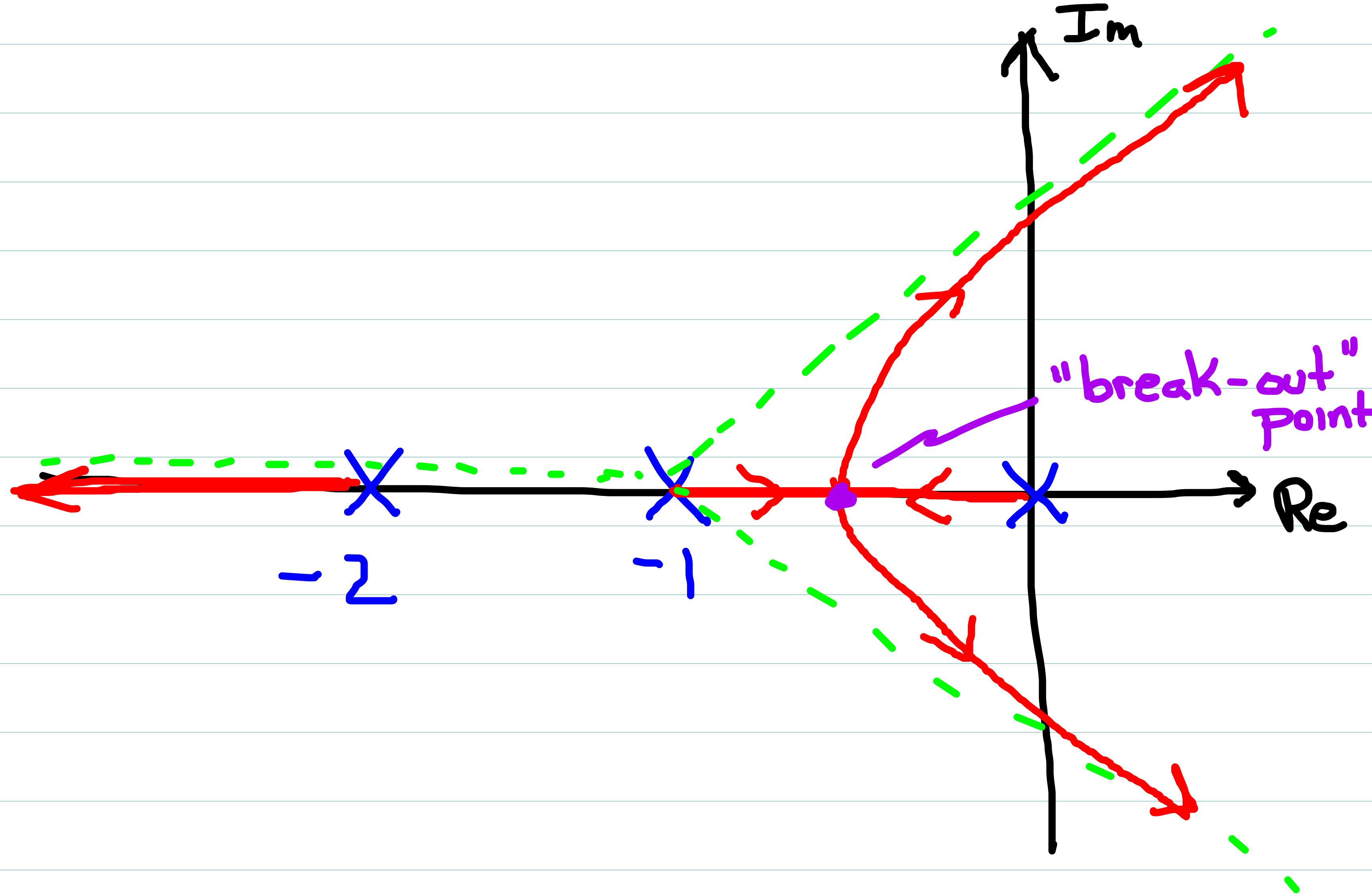
with intercept: $\sigma_a = \frac{0 + (-1) + (-2)}{3} = -1$

Real Axis branch locations.

$\Rightarrow$ between -1 and $\emptyset$

$\Rightarrow$ left of -2

Example #5, cont



Break-out Points

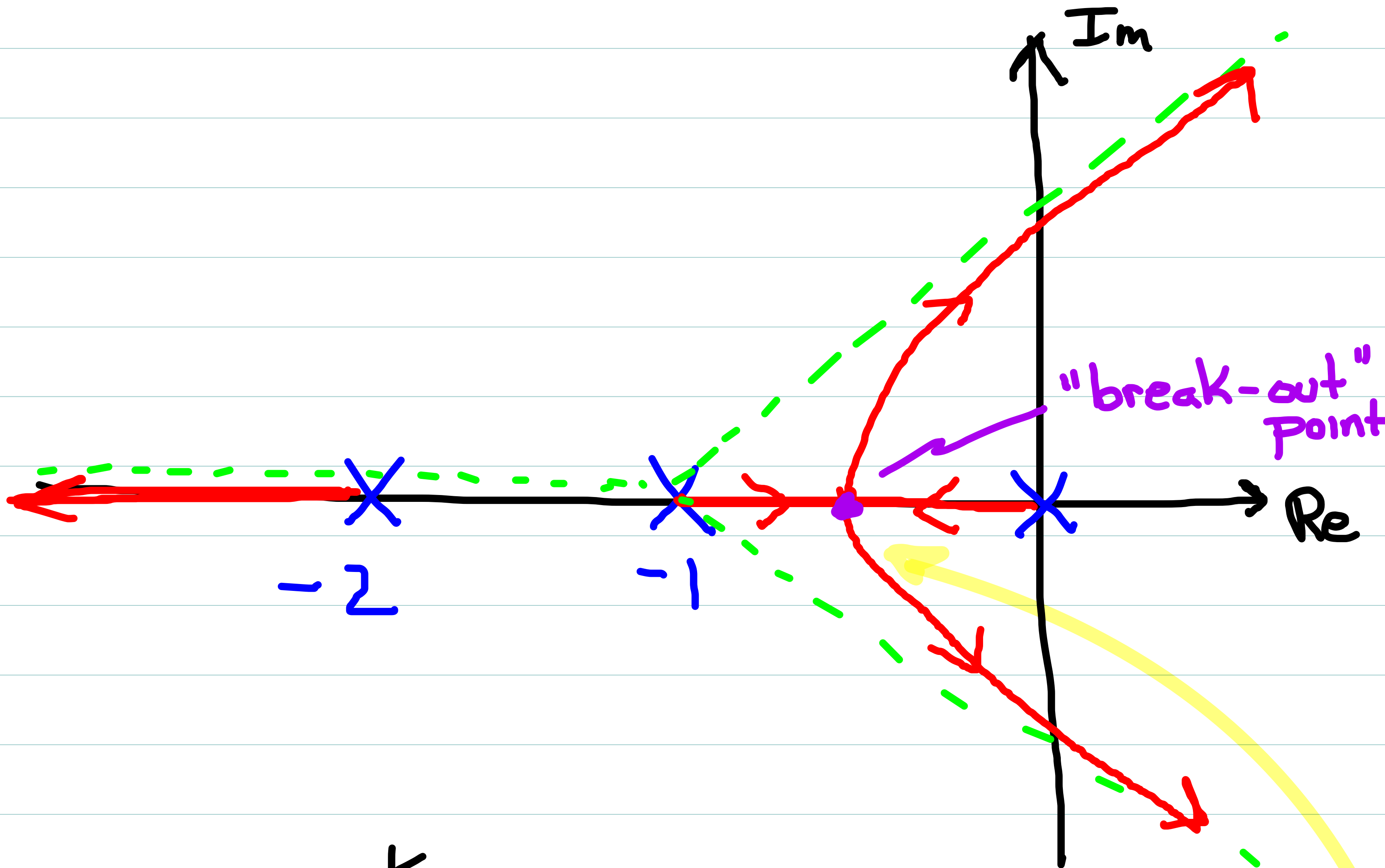
Break-out points occur for values of s satisfying

$$\frac{d}{ds}\left(L(s)\right) = 0$$

Since this (usually) leads to another high-order polynomial to factor, we often just approximate a break-out as occuring half-way along the branch

Use Matlab ("rlocus" command) to nail exact details when needed).

Example #5, cont



$$L(s) = \frac{K}{s(s+1)(s+2)}$$

$$\frac{d}{ds}L(s) = -[(s+1)(s+2) + s(s+2) + s(s+1)]$$
$$= -(3s^2 + 6s + 2)$$

$$\frac{dL(s)}{ds} = 0$$
$$\Rightarrow s = \cancel{-1.57}, -.423$$