Lecture 10: The Velocity Potential and Bernoulli's Equation

ENAE311H Aerodynamics I

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Circulation

A related concept to vorticity is the circulation, Γ, which is defined as

$$\Gamma = -\oint_c \mathbf{v} \cdot \mathbf{ds}$$

i.e., the integral around a closed curve of the dot product of the velocity with the curve element.

Using Stokes' theorem, we can write

$$\begin{aligned} \Gamma &= -\iint_s (\nabla \times \mathbf{v}) \cdot \mathbf{dA} \\ &= -\iint_s \boldsymbol{\xi} \cdot \hat{\mathbf{n}} dA. \end{aligned}$$

The Kutta-Joukowski theorem tells us that the lift produced by an airfoil is directly proportional to the circulation about it.

The velocity potential

For an irrotational flow, we have that

$$\boldsymbol{\xi} = \nabla \times \mathbf{v} = 0.$$

From vector calculus, we know that, if ϕ is a scalar function

$$\nabla \times (\nabla \phi) = 0.$$

Thus, the irrotational flow condition will be satisfied if

$$\mathbf{v} = \nabla \phi.$$

The function ϕ is called the *velocity potential* and any flow that can be described in this way is known as a *potential flow*.

In Cartesian and cylindrical coordinates, we have

$$u = \frac{\partial \phi}{\partial x}, \; v = \frac{\partial \phi}{\partial y}, \; w = \frac{\partial \phi}{\partial z},$$

and

$$v_r = \frac{\partial \phi}{\partial r}, \; v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \; v_z = \frac{\partial \phi}{\partial z}.$$

Note that, for steady flow, the differential continuity equation is

$$\nabla \cdot (\rho \mathbf{v}) = \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \mathbf{v} = 0.$$

Substituting $\boldsymbol{v} = \nabla \boldsymbol{\phi}$,

$$\rho \nabla^2 \phi = -\nabla \rho \cdot \nabla \phi.$$

Thus, for an incompressible flow (ρ constant),

$$\nabla^2 \phi = 0$$

i.e., the velocity potential satisfies Laplace’s equation.

The velocity potential and the stream function

Lines of constant ϕ are called *equipotential lines*.

Lines tangential to $\nabla\phi$ are called *gradient lines* and are streamlines of the flow (i.e., if the stream function exists, are lines of constant Ψ).

Equipotential lines and streamlines are perpendicular to one another.

	Dimensions	**Flow type**
Stream function	Two	Rotational or irrotational
Velocity potential	Two or three	Irrotational

Bernoulli's equation

Bernoulli's equation is probably the most famous equation in fluid mechanics, but is probably also the most widely misused, so we must be very aware of the assumptions used in deriving it.

We by assuming a steady, inviscid flow with negligible gravity. In this case, the x component of the differential momentum conservation equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

We can multiply through by dx to obtain

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx.$$

Let us now assume that we are moving along a streamline. We then have

$$\begin{aligned} vdx &= udy \\ wdx &= udz. \end{aligned}$$

The above equation can thus be written

$$u\underbrace{\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right)}_{=du \text{ for steady flow}} = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx. \Longrightarrow udu = d\left(\frac{u^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx.$$

Bernoulli's equation

We can repeat exactly the same procedure for the y and z components of the momentum conservation equation (multiplying by dy and dz, and using the relevant streamline equations in each case). This results in two further equations:

$$d\left(\frac{v^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy$$

$$d\left(\frac{w^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz.$$

Combining these, we have

$$\frac{1}{2}d(u^2+v^2+w^2) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

Now, since $u^2 + v^2 + w^2 = V^2$ and, for a steady flow,

$$\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = dp$$

we can re-write this as

$$\frac{1}{2}d(V^2) = -\frac{1}{\rho}dp,$$

Or equivalently

$$\boxed{dp = -\rho V\,dV.}$$

Euler's equation

If density is constant, we can integrate immediately:

$$\frac{1}{2}\int_{V_1^2}^{V_2^2} d(V^2) = -\frac{1}{\rho}\int_{p_1}^{p_2} dp,$$

to obtain

$$\boxed{p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2.}$$

Bernoulli's equation

We thus see that, for an inviscid, incompressible flow, $p + \frac{1}{2}\rho V^2$ is constant along a streamline. Different streamlines will, in general, have different values of that constant (but all the same for the special case of irrotational flow).

If gravity is important:

$$p + \frac{1}{2}\rho V^2 + \rho g y = \text{const.}$$

F=ma Normal to Streamline (1)

$$\sum \delta F_n = \delta m a_n = \frac{V^2}{\Re} \rho \delta \forall$$

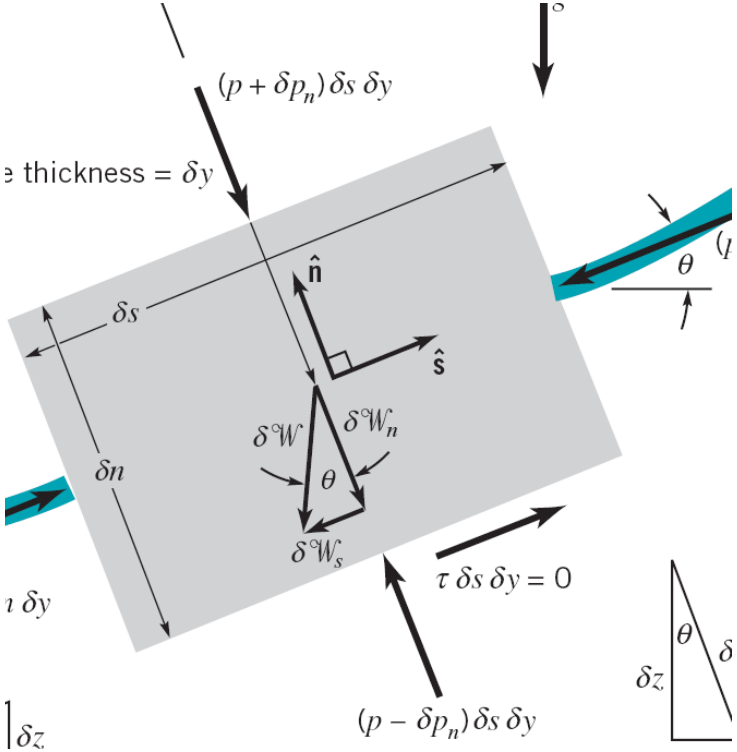
- Force due to gravity across streamline is

$$\delta F_{n,g} = -\delta m g \cos\theta = -\rho g \cos\theta \delta \forall$$

- Force due to pressure

$$\delta F_{n,p} = (p - \delta p_n)\delta s \delta y - (p + \delta p_n)\delta s \delta y$$

$$= -2\delta p_n \delta s \delta y = -\frac{\partial p}{\partial n} \delta \forall$$

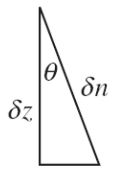


$$\delta p_n = \frac{\partial p}{\partial n} \frac{\delta n}{2}$$

F=ma Normal to Streamline (2)

- Across streamline

$$-\rho g \cos\theta - \frac{\partial p}{\partial n} = \rho \frac{V^2}{\Re}$$



Normal to streamline

$$\cos\theta = \frac{\partial z}{\partial n}$$

$$\frac{\partial p}{\partial n} + \rho \frac{V^2}{\Re} + \rho g \frac{\partial z}{\partial n} = 0$$

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dn = \frac{\partial p}{\partial n} dn$$

- Since normal to streamline ds=0, for any derivative partial and ordinary derivatives in n are the same (Note: analysis is limited to normal to a streamline)

$$dp + \rho \frac{V^2}{\Re} dn + \rho g dz = 0$$

F=ma Normal to Streamline (3)

- Integrating normal to streamline

$$dp + \rho \frac{V^2}{\Re} dn + \rho g dz = 0$$

$$\int \frac{dp}{\rho} + \int \frac{V^2}{\Re} dn + \int g dz = Const.$$

- If we assume fluid is incompressible

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F=ma Normal to Streamline (3)

- Integrating normal to streamline

$$dp + \rho \frac{V^2}{\Re} dn + \rho g dz = 0$$

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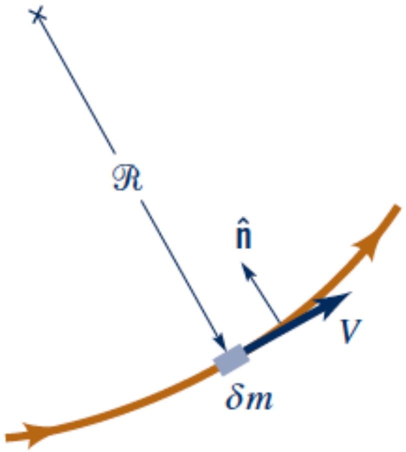
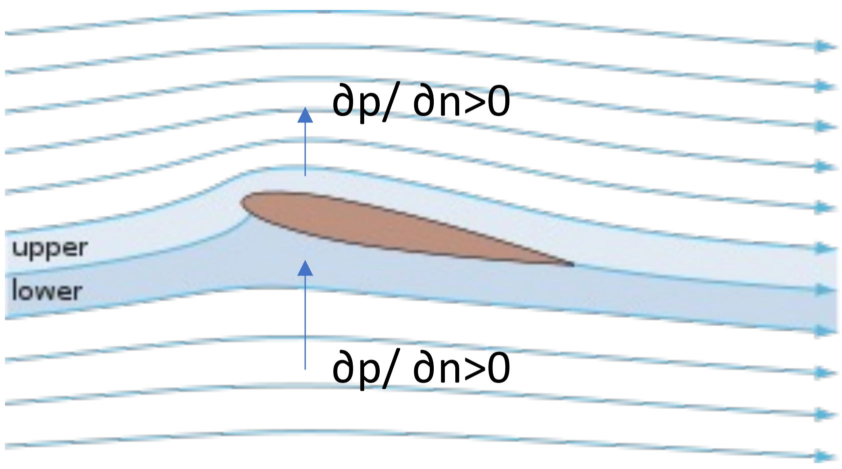
- If we assume fluid is incompressible

$$\frac{\partial p}{\partial n} = -\frac{\rho V^2}{\mathcal{R}}$$





F=ma Normal to Streamline (3)

- Integrating normal to streamline

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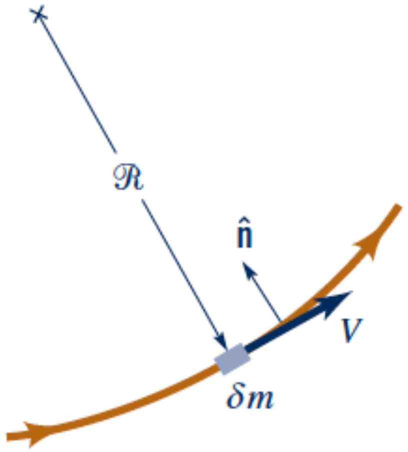

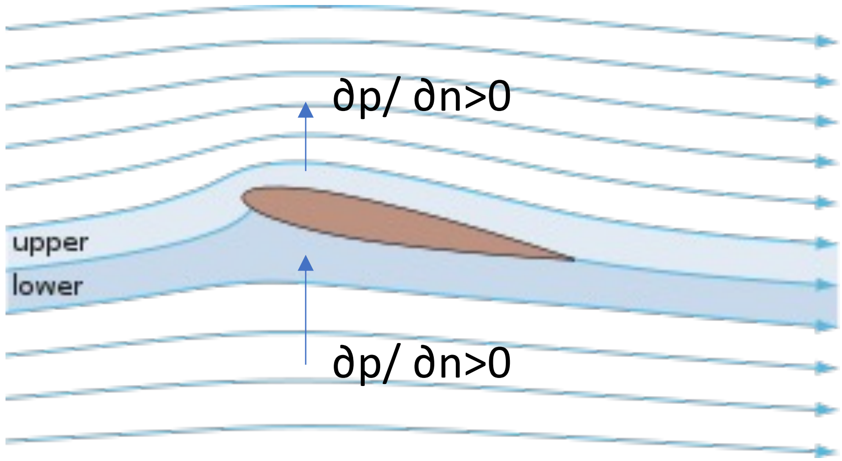

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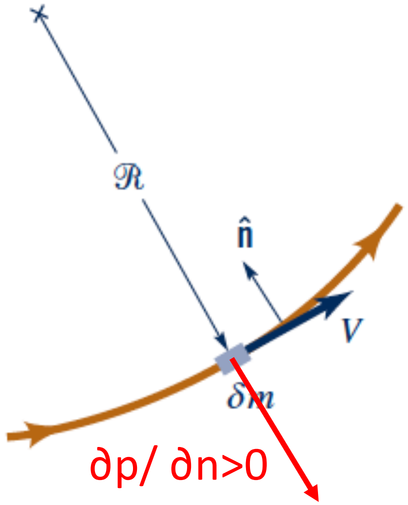
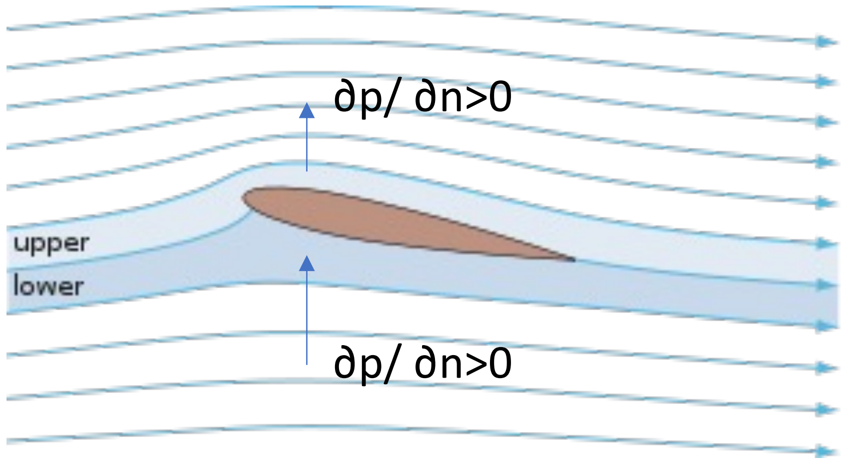




Steady, Inviscid, Incompressible Flow

- Along streamline: $$p + \frac{1}{2}\rho V^2 + \rho g z = Const.$$

- Across streamline: $$p + \rho \int \frac{V^2}{\Re} dn + \rho g z = Const.$$

- Pressure changes along streamline accelerates fluid particles
- Pressure changes normal to streamline turns fluid particles (changes streamline direction)

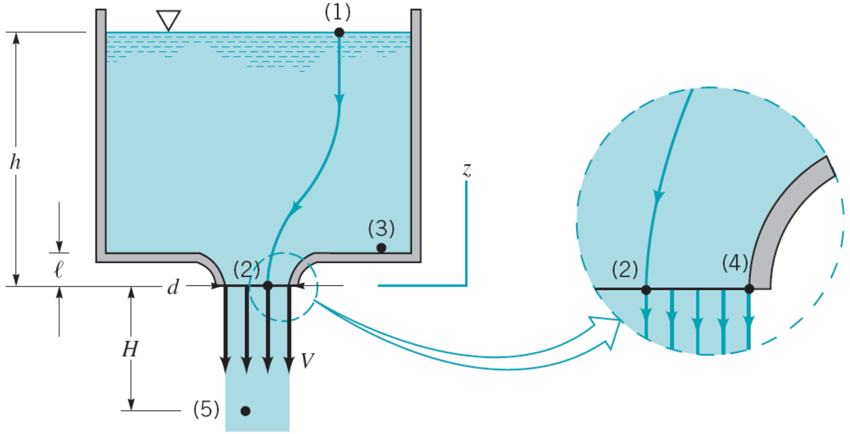
Free Jets

- If streamlines are straight at jet exit (free jet, R=∞) then no pressure gradient across jet, p2=p1
- V1=0

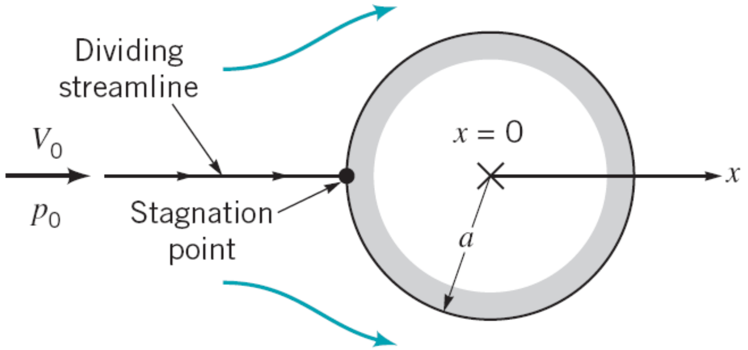
$$p_1 + \frac{1}{2}\rho V_1^{\,2} + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^{\,2} + \rho g z_2$$

$$V_2 = \sqrt{2g(z_1 - z_2)}$$

$$V_2 = \sqrt{2gh}$$



*3.8 A wind of velocity V_0 blows past a smokestack of radius $a = 2.5$ ft as shown in Fig. P3.8. The fluid velocity along the dividing streamline $(-\infty \le x \le -a)$ is found to be $V = V_0(1 - a^2/x^2)$. Plot the pressure distribution from a distance 30 ft ahead of the smokestack to the stagnation point on the smokestack for wind speeds of $V_0 = 0$, 10, 20, 30, 40, and 50 mph.



From the Bernulli eqn. with z = constant:

$$p_0 + \frac{1}{2}\rho V_0^2 = p + \frac{1}{2}\rho V^2 \text{, or with } p_0 = 0:$$

$$p = \frac{1}{2}\rho\left[V_0^2 - V^2\right] = \frac{1}{2}\rho\left[V_0^2 - V_0^2(1 - a^2/x^2)^2\right]$$

$$= \frac{1}{2}\rho V_0^2\left[1 - 1 + 2(a/x)^2 - (a/x)^4\right]$$

or

$$p = \frac{1}{2}\rho V_0^2\left[2(a/x)^2 - (a/x)^4\right]$$

Hence, with the given data:

$$p = \frac{1}{2}(0.00238 \text{ slugs}/\text{ft}^3)\left[2(2.5\text{ ft}/x)^2 - (2.5\text{ ft}/x)^4\right] V_0^2 (\text{mph})^2 (88 \text{ ft/s}/60\text{mph})^2$$

or

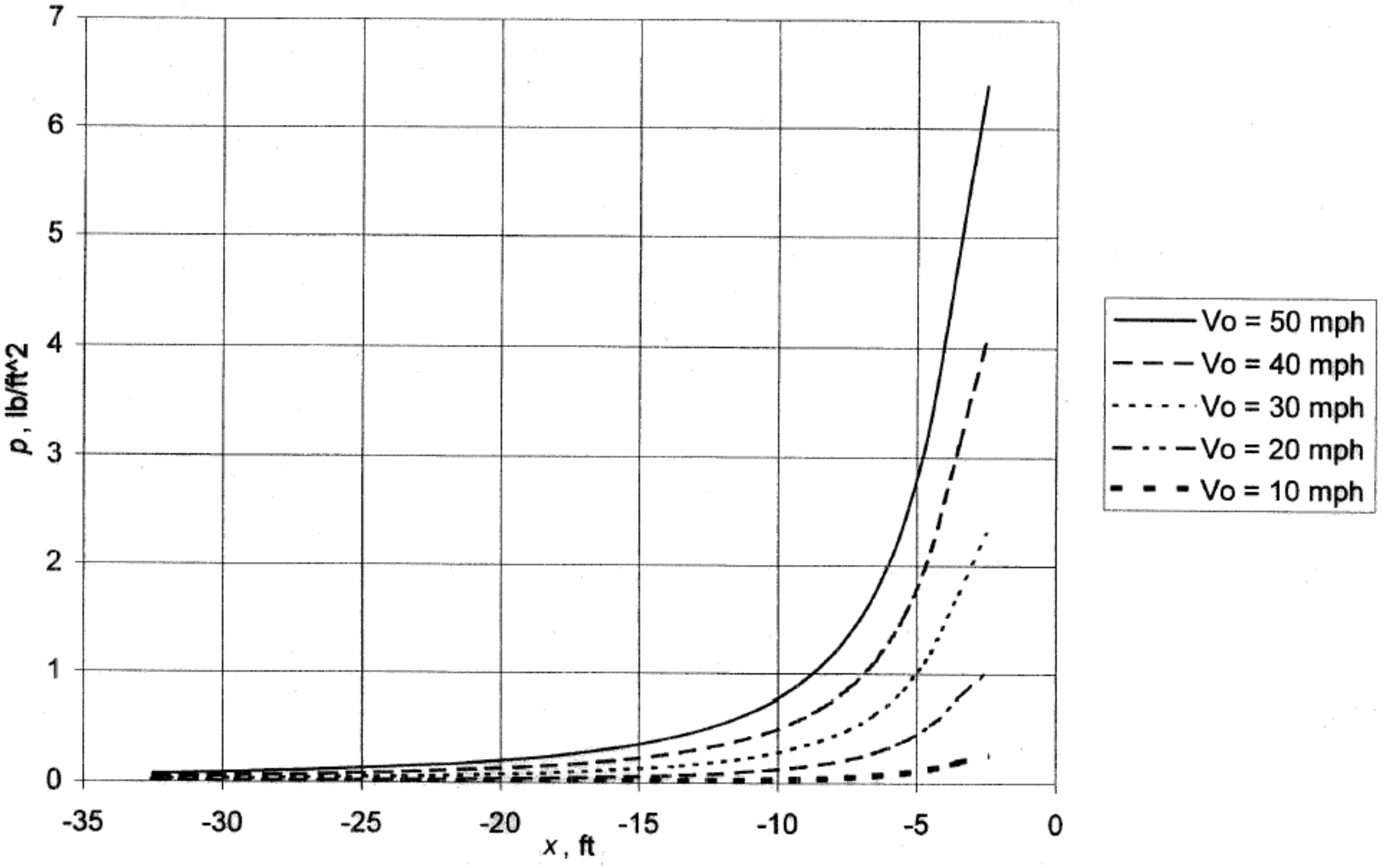
$$p = 0.00256\left[2(2.5/x)^2 - (2.5/x)^4\right] V_0^2 \text{ lb/ft}^2 \text{, where } V_0 \sim \text{mph and } x \sim \text{ft}$$

For example, with $x = -3.5$ ft and $V_0 = 50$ mph,

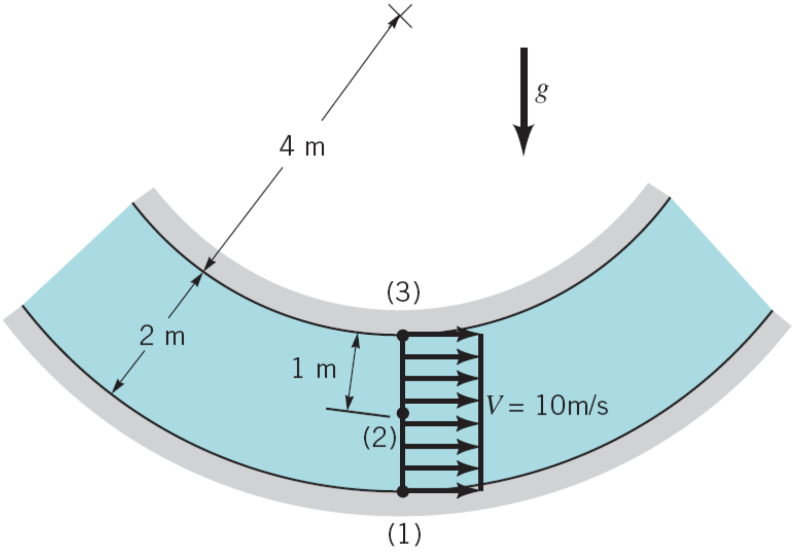
$$p = 0.00256\left[2(2.5/3.5)^2 - (2.5/3.5)^4\right](50)^2 = 4.86 \text{ lb/ft}^2$$

The results for various x and V_0 are plotted below.

Pressure Along Dividing Streamline



3.10 Water flows around the vertical two-dimensional bend with circular streamlines and constant velocity as shown in Fig. P3.10. If the pressure is 40 kPa at point (1), determine the pressures at points (2) and (3). Assume that the velocity profile is uniform as indicated.



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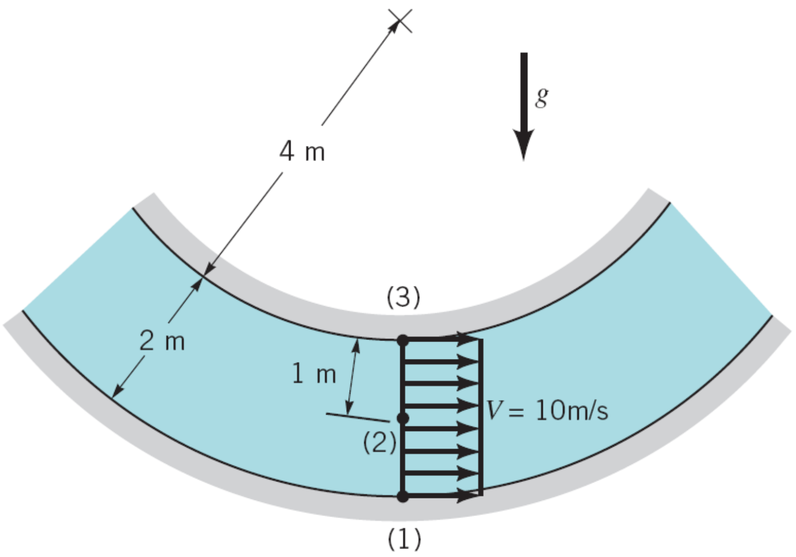
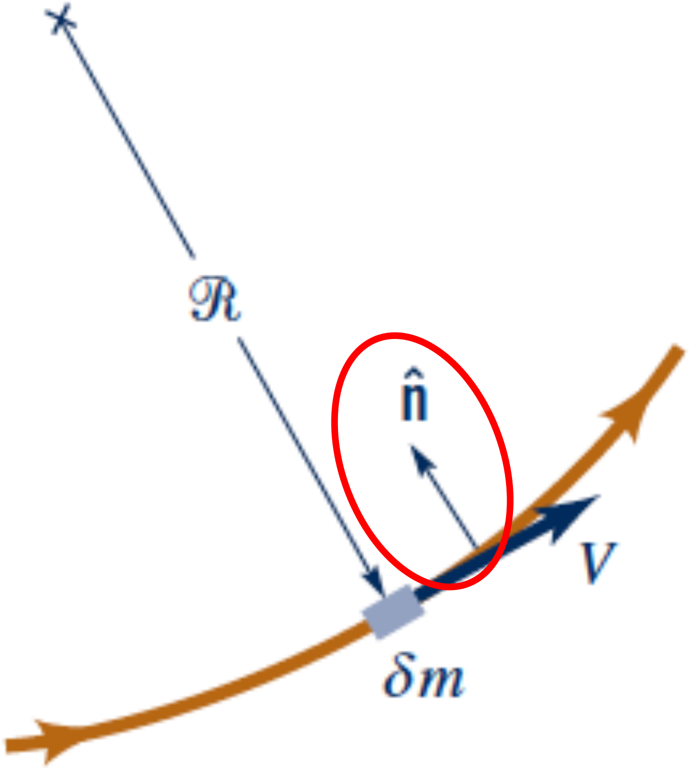




$$-\gamma \frac{dz}{dn} - \frac{\partial p}{\partial n} = \frac{\rho V^2}{\mathcal{R}} \quad \text{with } \frac{dz}{dn} = 1 \text{ and } V = 10\,m/s$$

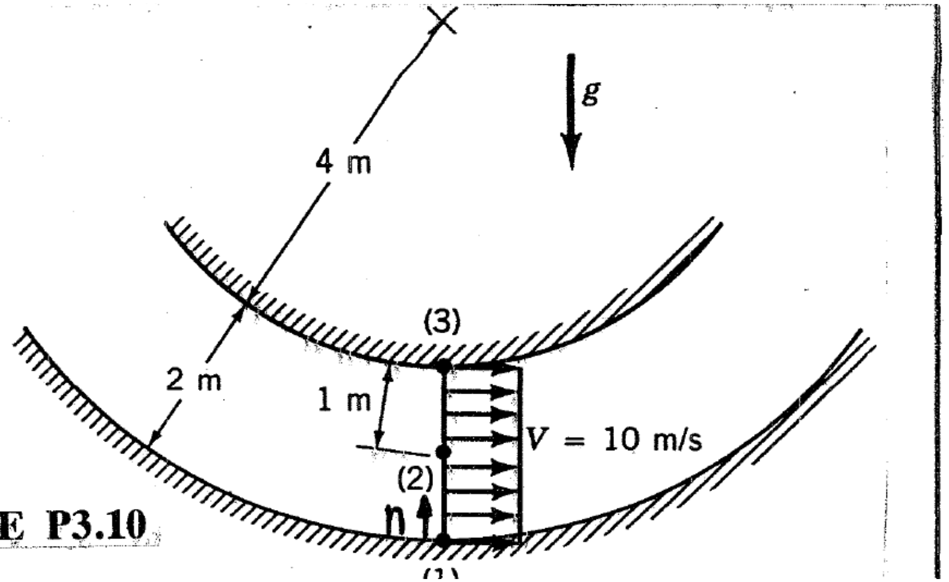
Thus, with $\mathcal{R} = 6 - n$

$$\frac{dp}{dn} = -\gamma - \frac{\rho V^2}{6-n} \quad \text{or}$$

$$\int_{n=0}^{n} \frac{dp}{dn}\,dn = -\int_{n=0}^{n} \gamma\,dn - \int_{n=0}^{n} \frac{\rho V^2 dn}{6-n}$$

so that since γ and V are constants

$$p - p_1 = -\gamma n - \rho V^2 \int_{n=0}^{n} \frac{dn}{6-n}$$



RE P3.10

Thus,

$$p = p_1 - \gamma n - \rho V^2 \ln\left(\frac{6}{6-n}\right)$$

With $p_1 = 40\,kPa$ and $n_2 = 1\,m$: $p_2 = 40\,kPa - 9.8 \times 10^3 \frac{N}{m^3}(1\,m)$

$$- 999 \frac{kg}{m^3}\left(10\frac{m}{s}\right)^2 \ln\left(\frac{6}{5}\right)$$

or

$$p_2 = \underline{\underline{12.0\ kPa}}$$

and

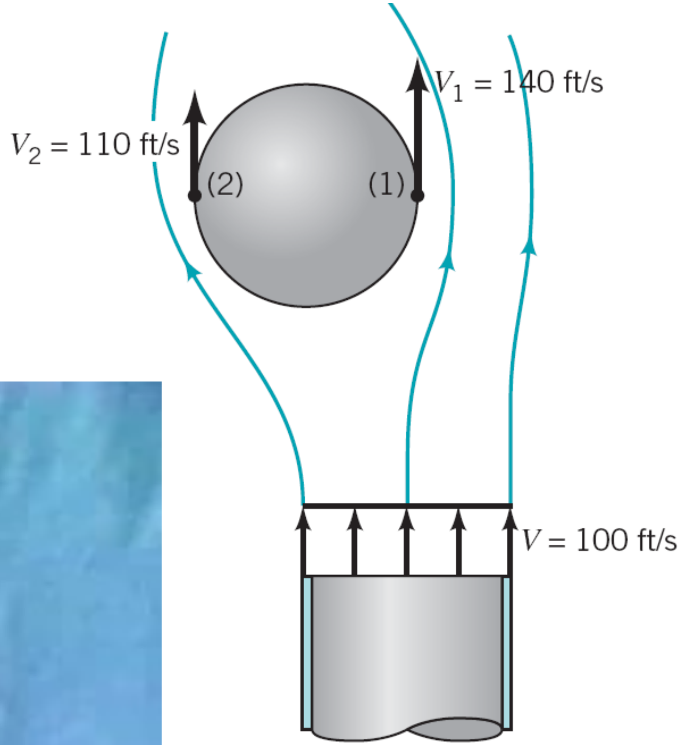
with $p_1 = 40\,kPa$ and $n_3 = 2\,m$: $p_3 = 40\,kPa - 9.80 \times 10^3 \frac{N}{m^3}(2\,m)$

$$- 999 \frac{kg}{m^3}\left(10\frac{m}{s}\right)^2 \ln\left(\frac{6}{4}\right)$$

or

$$p_3 = \underline{\underline{-20.1\ kPa}}$$

3.16 A 100 ft/s jet of air flows past a ball as shown in Video V3.1 and Fig. P3.16. When the ball is not centered in the jet, the air velocity is greater on the side of the ball near the jet center [point (1)] than it is on the other side of the ball [point (2)]. Determine the pressure difference, $p_2 - p_1$, across the ball if $V_1 = 140$ ft/s and $V_2 = 110$ ft/s. Neglect gravity and viscous effects.



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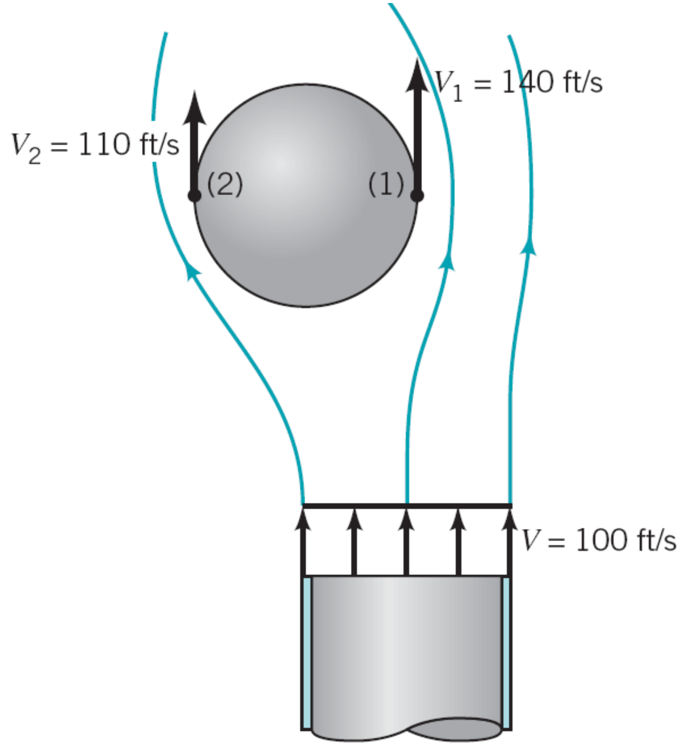
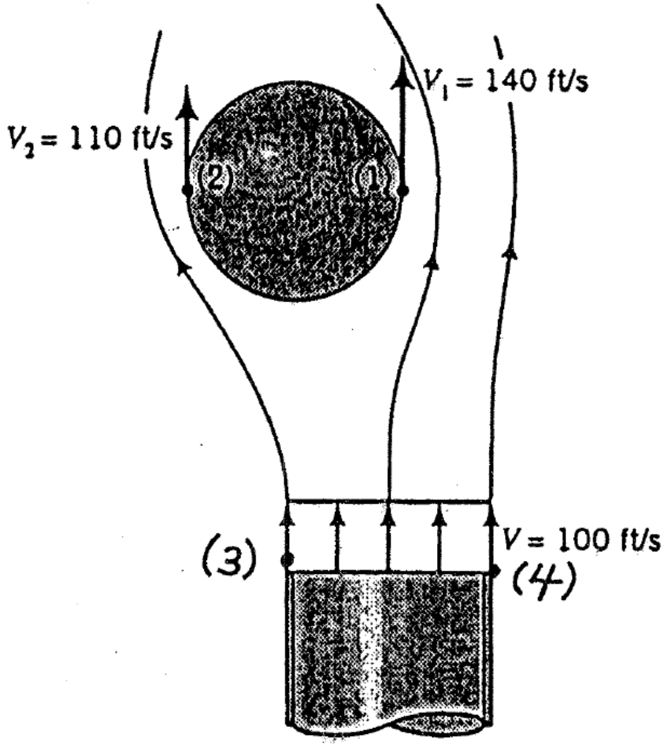




The Bernoulli equation from point (3) to (2) and (4) to (1) with gravity neglected gives

$$p_3 + \frac{1}{2}\rho V_3^2 = p_2 + \frac{1}{2}\rho V_2^2 \quad \text{and} \quad p_4 + \frac{1}{2}\rho V_4^2 = p_1 + \frac{1}{2}\rho V_1^2$$

But $p_3 = p_4 = 0$ and $V_3 = V_4$

Thus, even though points (1) and (2) are not on the same streamline,

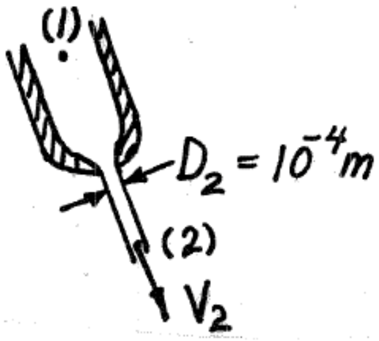
$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

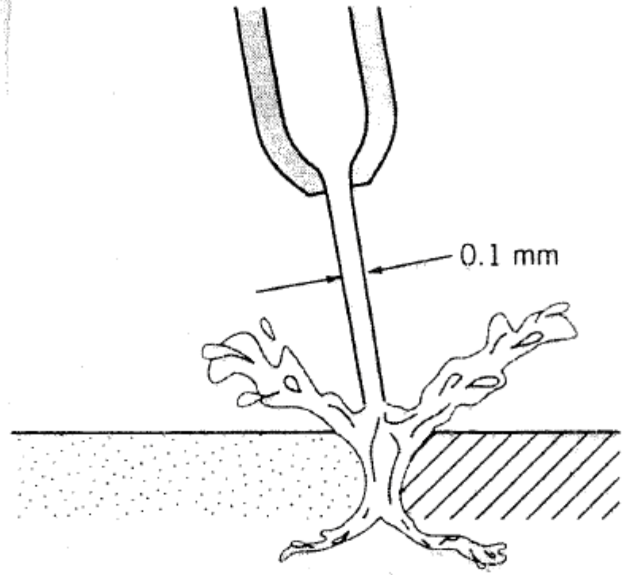
or

$$p_1 - p_2 = \frac{1}{2}\rho\left(V_1^2 - V_2^2\right) = \frac{1}{2}\left(0.00238\,\frac{\text{slugs}}{\text{ft}^3}\right)\left[\left(140\,\frac{\text{ft}}{\text{s}}\right)^2 - \left(110\,\frac{\text{ft}}{\text{s}}\right)^2\right]$$

$$= 8.93\,\frac{\text{slugs}}{\text{ft}\cdot\text{s}^2} = \underline{\underline{8.93\,\frac{\text{lb}}{\text{ft}^2}}}$$

3.26 Small-diameter, high-pressure liquid jets can be used to cut various materials as shown in Fig. P3.26. If viscous effects are negligible, estimate the pressure needed to produce a 0.10-mm-diameter water jet with a speed of 700 m/s. Determine the flowrate.





■ FIGURE P3.26

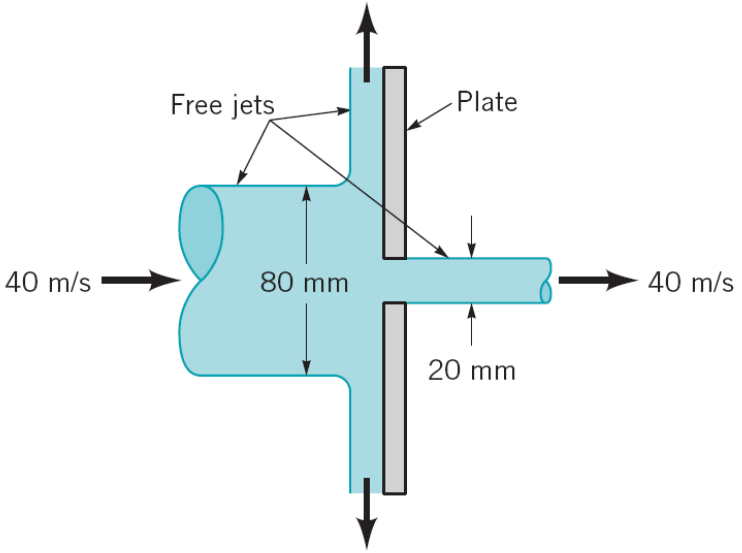
$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$ where $V_1 \approx 0$, $z_1 \approx z_2$, and $p_2 = 0$

Thus $p_1 = \frac{1}{2}\frac{\gamma}{g}V_2^2 = \frac{1}{2}\rho V_2^2 = \frac{1}{2}(999\frac{kg}{m^3})(700\frac{m}{s})^2 = \underline{\underline{2.45 \times 10^5 \frac{kN}{m^2}}}$

Also,

$Q = V_2 A_2 = 700\frac{m}{s}\left[\frac{\pi}{4}(10^{-4}m)^2\right] = \underline{\underline{5.50 \times 10^{-6} \frac{m^3}{s}}}$

5.38 A circular plate having a diameter of 300 mm is held perpendicular to an axisymmetric horizontal jet of air having a velocity of 40 m/s and a diameter of 80 mm as shown in Fig. P5.38. A hole at the center of the plate results in a discharge jet of air having a velocity of 40 m/s and a diameter of 20 mm. Determine the horizontal component of force required to hold the plate stationary.



$$\int_{cs} -p$$

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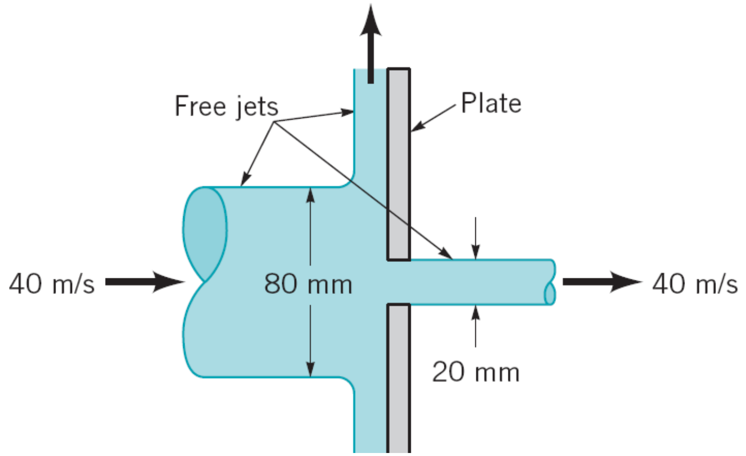

The control volume contains the plate and flowing air as indicated in the sketch above. Application of the horizontal or x direction component of the linear momentum equation yields

$$-u_1 \rho u_1 A_1 + u_2 \rho u_2 A_2 = -F_{A,x}$$

or

$$F_{A,x} = u_1^2 \rho \frac{\pi D_1^2}{4} - u_2^2 \rho \frac{\pi D_2^2}{4} = u_1^2 \rho \frac{\pi}{4}\left(D_1^2 - D_2^2\right)$$

Thus

$$F_{A,x} = \left(40 \frac{m}{s}\right)^2 \left(1.23 \frac{kg}{m^3}\right) \frac{\pi}{4} \left[\frac{(80\ mm)^2 - (20\ mm)^2}{\left(1000 \frac{mm}{m}\right)^2}\right] \left(1 \frac{N}{kg \cdot \frac{m}{s^2}}\right)$$

and

$$F_{A,x} = \underline{\underline{9.27\ N}}$$