Chapter 9

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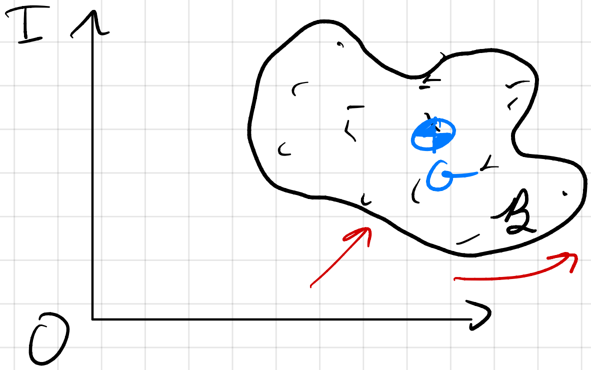


<u>a rigid body is a MPS</u>

$N \to \infty$ "continuum"

body is free to translate & rotate (in the plane)

Recall Ch6

$$\bar{F}^{(ext)} = m_G \, {}^{I}\bar{a}_{G/O} \quad (G \text{ behaves like a particle})$$

$$\boxed{\bar{F}_G = m_G \, {}^{I}\bar{a}_{G/O}}$$

<u>Euler's First Law</u>

* governs translational motion of the COM of a RB

Recall Ch6

$$\bar{r}_{G/O} = \frac{1}{m_G} \sum_{k=1}^{N} m_k \bar{r}_{k/O}$$

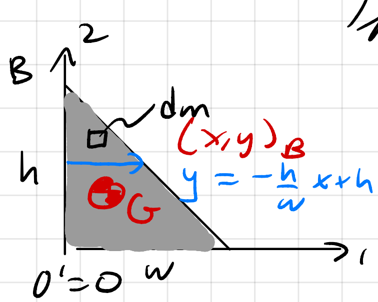
$$\boxed{\bar{r}_{G/O} = \frac{1}{m_G} \int_{\mathcal{B}} dm \, \bar{r}_{dm/O}} \quad (N \to \infty)$$

2D : $\frac{\text{mass}}{\text{area}}$ = density $\quad \frac{dm}{dA} = \rho(\bar{r}_{dm/O})$

3D : $\frac{\text{mass}}{\text{vol}}$ = density $\quad \frac{dm}{dV} = \rho(\bar{r}_{dm/O})$

$$\Rightarrow \bar{r}_{G/O} = \frac{1}{m_G} \int_{\mathcal{B}} \rho \, dA \, \bar{r}_{dm/O} \quad (\text{in 2D})$$

<u>Ex 9.1</u>



$\frac{1}{2} wh = \frac{m}{\rho}$

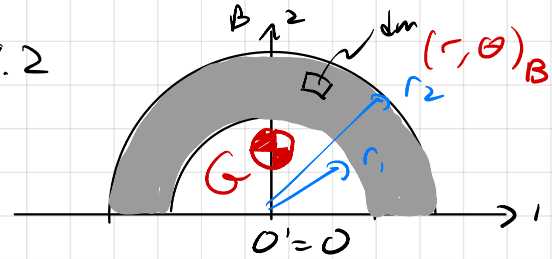
<u>Cartesian Coordinates</u>

$$\bar{r}_{G/O} = \frac{\rho}{m} \int_0^{h} \int_0^{-\frac{w}{h}(y-h)} dx\,dy \,(x \hat{b}_1 + y \hat{b}_2)$$

$\vdots$

$$= \frac{1}{3}(w\hat{b}_1 + h\hat{b}_2)$$

EX 9.2



polar Coords

$$\bar{r}_{G/O} = \frac{\rho}{m_G} \int_{B} dA \; \bar{r}_{dm/O}$$

$$dA = r\,dr\,d\theta$$

$$\bar{r}_{G/O} = \frac{\rho}{m} \int_0^{\pi} \int_{r_1}^{r_2} r\,dr\,d\theta \left(r\cos\theta\, \hat{b}_1 + r\sin\theta\, \hat{b}_2 \right)$$

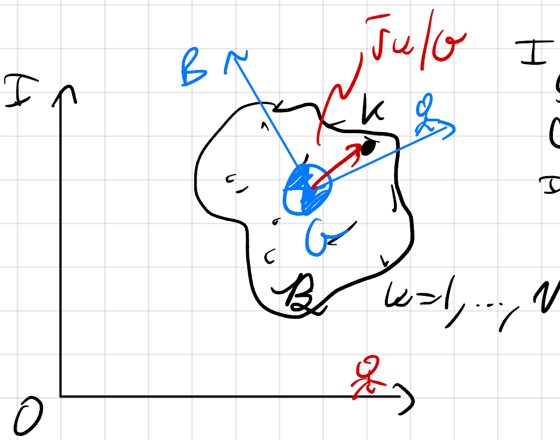
$$\vdots$$

$$= \frac{4(r_2^3 - r_1^3)}{3\pi(r_2^2 - r_1^2)} \hat{b}_2$$

$$\frac{\pi r_2^2 - \pi r_1^2}{2} = \frac{\rho}{m}$$

Rotational dynamics

Recall Ch. 7 $\quad {}^I\bar{h}_O = \underbrace{{}^I\bar{h}_{G/O}}_{\text{translation}} + \underbrace{{}^I\bar{h}_G}_{\text{rotation}}$ Separation of AM



$$\frac{{}^I d}{dt}\left({}^I\bar{h}_G\right) = \bar{M}_G^{(ext)} = \sum_{k=1}^{N} \bar{M}_{k/G}^{(ext)}$$

$$ {}^I\bar{h}_G = \sum_{k=1}^{N} {}^I\bar{h}_{k/G}$$

$$= \sum_{k=1}^{N} \bar{r}_{k/G} \times m_k \, {}^I\bar{v}_{k/G}$$

Recall Ch. 8 $\quad \frac{{}^I d}{dt}(\cdot) = \frac{{}^B d}{dt}(\cdot) + {}^I\bar{\omega}^B \times (\cdot)$

$${}^I\bar{v}_{k/G} = \frac{{}^I d}{dt}\left(\bar{r}_{k/G}\right) \overset{TE}{=} \underbrace{\frac{{}^B d}{dt}\left(\bar{r}_{k/G}\right)}_{=\bar{0}_N} + {}^I\bar{\omega}^B \times \bar{r}_{k/G}$$

transport equation

$\Rightarrow$ For a rigid body $\quad {}^I\bar{h}_G = \sum_{k=1} \underbrace{\bar{r}_{k/G}}_{\bar{a}} \times m_k \left(\underbrace{{}^I\bar{\omega}^B}_{\bar{b}} \times \underbrace{\bar{r}_{k/G}}_{\bar{c}} \right)$

App B p. 640 vector triple product identity

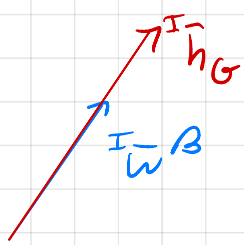
$$\bar{a}\times(\bar{b}\times\bar{c}) = \bar{b}(\bar{c}\cdot\bar{a}) - \bar{c}(\bar{a}\cdot\bar{b})$$

$$\Rightarrow {}^I\bar{h}_G = \sum_{k=1}^{N} m_k \left[{}^I\bar{\omega}^B \underbrace{(\bar{r}_{k/G}\cdot\bar{r}_{k/G})}_{=\|\bar{r}_{k/G}\|^2} - \bar{r}_{k/G}\underbrace{(\bar{r}_{k/G}\cdot{}^I\bar{\omega}^B)}_{=0\ (2D)} \right]$$

$$= \underbrace{\left(\sum_{k=1}^{N} m_k \|\bar{r}_{k/G}\|^2\right)}_{\mathbb{I}_G \text{ Moment of inertia}} {}^I\bar{\omega}^B$$

$$\Rightarrow \boxed{{}^I\bar{h}_G = \mathbb{I}_G\, {}^I\bar{\omega}^B} \quad \text{where} \quad \boxed{\mathbb{I}_G = \sum_{k=1}^{N} m_k \|\bar{r}_{k/G}\|^2}$$

* AM is parallel to AV (2D only)
* MOI is scalar (2D only)



Rotational Dynamics $\boxed{\frac{{}^I d}{dt}\left({}^I\bar{h}_G\right) = \bar{M}_G}$

* Euler's 2nd Law

For a continuum RB $(N \to \infty)$

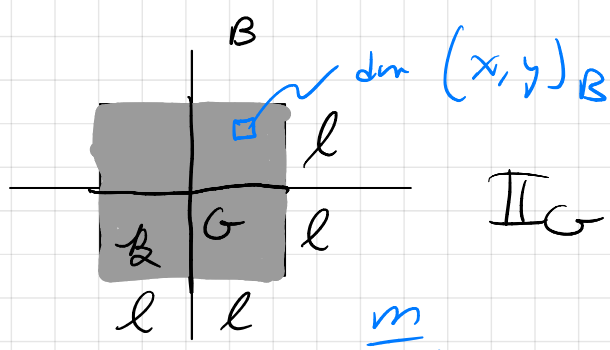
$$\mathbb{I}_G = \int_{\mathcal{B}} dm\, \|\bar{r}_{dm/G}\|^2$$

$dm = \rho\, dA$ (2D)

$dm = \rho\, dV$ (3D)

Ex 9.2

MOI (Cartesian Coords.)



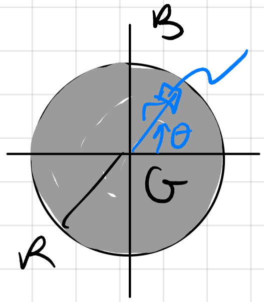
$\rho = \frac{m}{4\ell^2}$

$$\mathbb{I}_G = \rho \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} dx\, dy\, (x^2+y^2)$$

$$= \cdots = 2m\ell^2/3$$

Ex 9.8

MoI
(polar coords)



$$\mathbb{I}_G = \rho \int_0^{2\pi}\int_0^{R} r^3\, dr\, d\theta$$

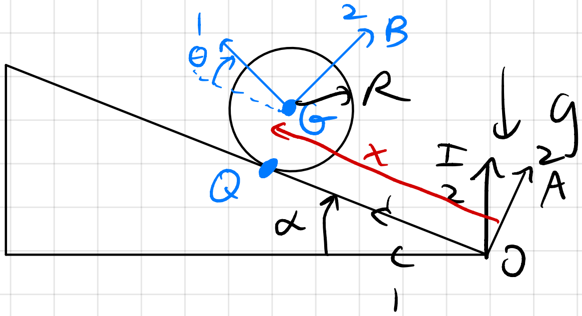
$$= \dots = \frac{1}{2} m R^2$$

$$\rho = \frac{m}{\pi R^2}$$

Ex 9.9 Disc rolling w/o slipping ($\underbrace{\dot{x} = -R\dot{\theta}}_{\text{constraint}}$)

	$\hat{b}_1$	$\hat{b}_2$
$\hat{a}_1$	$C\theta$	$-S\theta$
$\hat{a}_2$	$S\theta$	$C\theta$

	$\hat{a}_1$	$\hat{a}_2$
$\hat{e}_1$	$C\alpha$	$-S\alpha$
$\hat{e}_2$	$S\alpha$	$C\alpha$



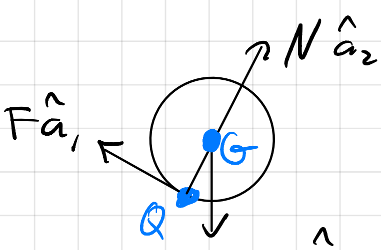
$I = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ inertial
$A = (O, \hat{a}_1, \hat{a}_2, \hat{a}_3)$ ramp
$B = (G, \hat{b}_1, \hat{b}_2, \hat{b}_3)$ body

$$M = 6N - K = 6 - \underbrace{2}_{\text{translation}} - \underbrace{2}_{\text{rotation}} = 2 \text{ DOF}$$

constraints

Euler 1st Law: translation

$$\bar{F}_G = m_G\, {}^{I}\bar{a}_{G/O}$$



$$\bar{r}_{G/O} = x\hat{a}_1 + R\hat{a}_2$$

$${}^{I}\bar{v}_{G/O} = \dot{x}\hat{a}_1$$

$${}^{I}\bar{a}_{G/O} = \ddot{x}\hat{a}_1$$

$$F\hat{a}_1 + N\hat{a}_2 - mg\hat{e}_2 = m\ddot{x}\hat{a}_1$$

$$F - mg S\alpha = m\underbrace{\ddot{x}}_{\text{unknown}}$$

Euler 2nd Law: Rotation

$$\frac{{}^{I}d}{dt}\left({}^{I}\bar{h}_G\right) = \bar{M}_G$$

$$\bar{M}_G = \sum_{k=1}^{N} \bar{M}_{k/G}$$

$$= \sum_{k=1}^{N} \bar{r}_{k/G} \times \bar{F}_k$$

$$= \underbrace{\bar{r}_{Q/G} \times \bar{N}}_{=0} + \bar{r}_{Q/G} \times \bar{F} + \underbrace{\bar{r}_{G/G}}_{=0} \times \bar{W}$$

$$= -R\hat{a}_2 \times F\hat{a}_1 = RF\hat{a}_3$$

$${}^{I}\bar{h}_G = \mathbb{I}_G\, {}^{I}\bar{\omega}^{B} = \mathbb{I}_G \dot{\theta}\hat{b}_3$$

$$\mathbb{I}_G \ddot{\theta}\hat{b}_3 = RF\hat{a}_3 \Rightarrow \underbrace{\mathbb{I}_G\ddot{\theta} = RF}_{\text{2 unknowns}}$$

① $\ddot{x} = -R\ddot{\theta}$ constraint RW/oS

② $I_G \ddot{\theta} = RF$ rot. dyn.

③ $F - mg S\alpha = m\ddot{x}$ trans. dyn

$$\frac{I_G \ddot{\theta}}{R} - mg S\alpha = m\ddot{x}$$

$$\frac{I_G}{R}\frac{\ddot{x}}{-R} - mg S\alpha = m\ddot{x} \Rightarrow \left(m + \frac{I_G}{R^2}\right)\ddot{x} = -mg S\alpha$$

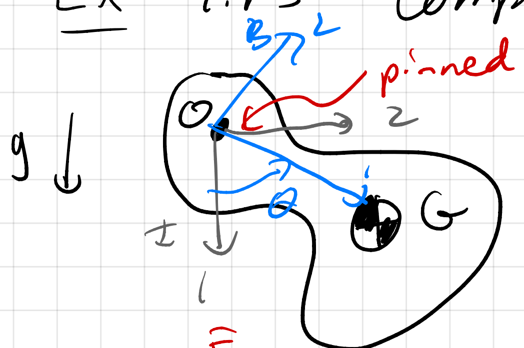
$$\boxed{\ddot{x} = \frac{-mg S\alpha}{m + I_G/R^2}}$$

$$I_G = \tfrac{1}{2} m R^2$$

$$\Rightarrow \ddot{x} = -\frac{2}{3} g S\alpha$$

$$\ddot{\theta} = \frac{2}{3}\frac{g S\alpha}{R}$$

Ex 9.15 Compound Pendulum



$${}^I\bar{h}_O = I_O\, {}^I\bar{\omega}^B = I_O \dot{\theta}\,\hat{b}_3$$

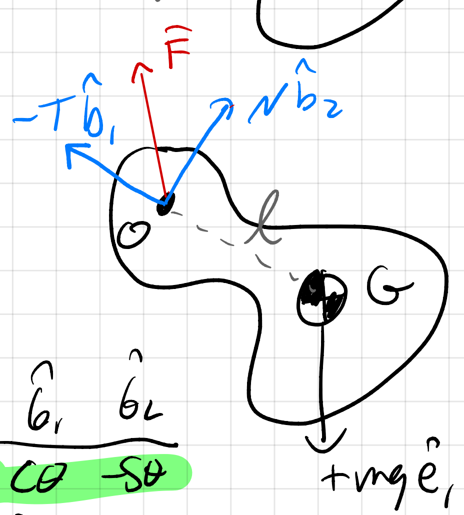
$${}^I\frac{d}{dt}\left({}^I\bar{h}_O\right) = \bar{M}_O$$

$$\bar{M}_O = \bar{r}_{G/O} \times \bar{W} + \bar{r}_{O/O} \times (\bar{T} + \bar{N})$$

$$= -\ell \hat{e}_1 \times N\hat{b}_2 = -\ell N \hat{b}_3$$

$$I_O \ddot{\theta} = -\ell N$$

2 unknowns



	$\hat{b}_1$	$\hat{b}_2$
$\hat{e}_1$	$C\theta$	$-S\theta$
$\hat{e}_2$	$S\theta$	$C\theta$

Option #1 use Euler's 1st law

Option #2 use parallel axis theorem

* key idea: for a pinned RB, write the dynamics relative to the pin and use the parallel axis theorem.

let Q be a point fixed in B and I

$$\boxed{\frac{{}^{I}d}{dt}\left({}^{I}\bar{h}_Q\right) = \bar{M}_Q}$$ pinned rigid body

$${}^{I}\bar{h}_Q = \mathbb{I}_Q \, {}^{I}\bar{\omega}^B$$

$$\boxed{\mathbb{I}_Q = \mathbb{I}_G + m_G \|\bar{r}_{Q/G}\|^2}$$ parallel axis theorem

$$\bar{M}_Q = \sum_{k=1}^{N} \bar{M}_{k/Q} = \sum_{k=1}^{N} \bar{r}_{k/Q} \times \bar{F}_k$$

Ex Compound Pendulum let $Q = O$

$$\frac{{}^{I}d}{dt}\left({}^{I}\bar{h}_Q\right) = \bar{M}_Q$$

$${}^{I}\bar{h}_Q = \left(\mathbb{I}_G + m\ell^2\right)\dot{\theta}\,\hat{b}_3$$

$$\bar{M}_Q = \bar{r}_{G/O} \times \bar{W}$$

$$= \ell\hat{b}_1 \times mg\hat{e}_1 = \ell\hat{b}_1 \times mg\left(c\theta\hat{b}_1 - s\theta\hat{b}_2\right)$$

$$= -mg\ell s\theta\,\hat{b}_3$$

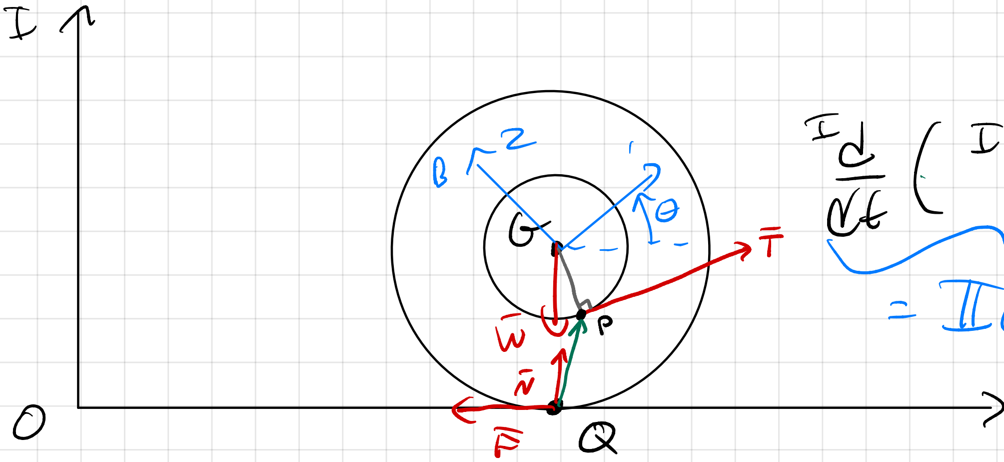
$$\left(\mathbb{I}_G + m\ell^2\right)\ddot{\theta}\,\hat{b}_3 = -mg\ell s\theta\,\hat{b}_3$$

$$\boxed{\ddot{\theta} = \frac{-mg\ell s\theta}{\mathbb{I}_G + m\ell^2}}$$

Comparison Simple Pendulum $\mathbb{I}_G \to 0$

$$\ddot{\theta} \stackrel{\checkmark}{=} -\frac{g}{\ell}\sin\theta$$

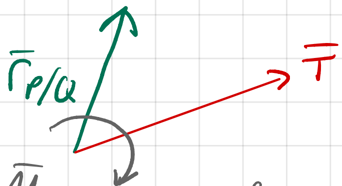
Ex 9.16



$${}^{I}\frac{d}{dt}\left({}^{I}\bar{h}_Q\right) = \bar{M}_Q$$

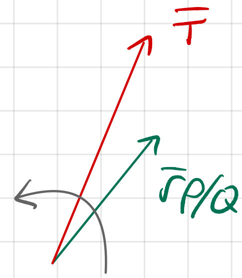
$$= \mathbb{I}_G\,\ddot{\theta}\,\hat{b}_3$$

$$\bar{M}_Q = \bar{r}_{P/Q} \times \bar{T}$$



$\bar{M}_Q$ into the page

$\Rightarrow$ wheel rolls right



$\bar{M}_Q$ out of the page

$\Rightarrow$ wheel rolls left

Work & Energy Rigid Body

Recall $T_O = T_{G/O} + T_G$ Separation of KE

$$T_{G/O} = \frac{1}{2} m_G \| {}^{I}\bar{v}_{G/O} \|^2 \quad \text{translational KE}$$

$$T_G = \frac{1}{2} \sum_{k=1}^{N} m_k \| {}^{I}\bar{v}_{k/G} \|^2$$

$${}^{I}\bar{v}_{k/G} = {}^{I}\frac{d}{dt}\left(\bar{r}_{k/G}\right) = \underbrace{{}^{B}\frac{d}{dt}\left(\bar{r}_{k/G}\right)}_{=0} + {}^{I}\bar{\omega}^{B} \times \bar{r}_{k/G}$$

$$T_G = \frac{1}{2} \sum_{k=1}^{N} m_k \left(\underbrace{{}^{I}\bar{\omega}^{B}}_{\bar{a}} \times \underbrace{\bar{r}_{k/G}}_{\bar{b}}\right) \cdot \left(\underbrace{{}^{I}\bar{\omega}^{B} \times \bar{r}_{k/G}}_{\bar{c}}\right)$$

Scalar triple product $\quad \bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \bar{c}$

$$T_G = \frac{1}{2} \sum_{k=1}^{N} m_k\, {}^{I}\bar{\omega}^{B} \cdot \bar{r}_{k/G} \times {}^{I}\bar{\omega}^{B} \times \bar{r}_{k/G}$$

$$= \frac{1}{2}\, {}^{I}\bar{\omega}^{B} \cdot \underbrace{\sum_{k=1}^{N} m_k\, \bar{r}_{k/G} \times {}^{I}\bar{\omega}^{B} \times \bar{r}_{k/G}}_{= \mathbb{I}_G\, {}^{I}\bar{\omega}^{B}}$$

$$= \frac{1}{2}\, {}^{I}\bar{\omega}^{B} \cdot \mathbb{I}_G\, {}^{I}\bar{\omega}^{B} = \frac{1}{2}\, {}^{I}\bar{\omega}^{B} \cdot {}^{I}\bar{h}_G$$

$$\boxed{T_G = \frac{1}{2} \mathbb{I}_G \| {}^{I}\bar{\omega}^{B} \|^2}$$ rotational KE

$$W = W_{G/O} + W_G$$ separation of work

$$W_{G/O} = \int_{\gamma_G} \bar{F}_G \cdot {}^{I}d\bar{r}_{G/O} = \int_{\gamma_G} \bar{F}_G \cdot {}^{I}\bar{v}_{G/O}\, dt$$

$$W_G = \int \bar{M}_G \cdot {}^{I}\bar{\omega}^{B}\, dt$$

WE #1 $\quad T_O(t_2) = T_O(t_1) + W_{G/O} + W_G$

translation $\quad T_{G/O}(t_2) = T_{G/O}(t_1) + W_{G/O}$

rotation $\quad T_G(t_2) = T_G(t_1) + W_G$

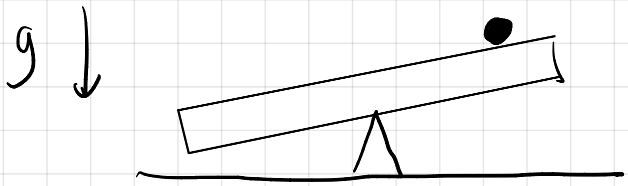
WE #2 $\quad U_O(t_2) = U_O(t_1) - W^{(c)}$

WE#3 $\quad E_O(t_2) = E_O(t_1) + W^{(nc)}$

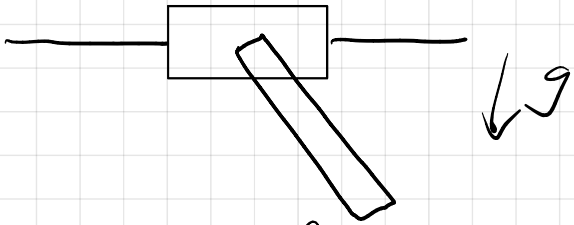
$$\boxed{\begin{aligned} &T_{G/O}(t_2) + T_G(t_2) + U_O(t_2) = \\ &\quad T_{G/O}(t_1) + T_G(t_1) + U_O(t_1) + W_{G/O}^{(nc)} + W_G^{(nc)} \end{aligned}}$$

E_O is conserved if $W^{(nc)} = W_{G/O}^{(nc)} + W_G^{(nc)} = 0$

A Collection of RB & Particles



ball on a beam



Compound crane