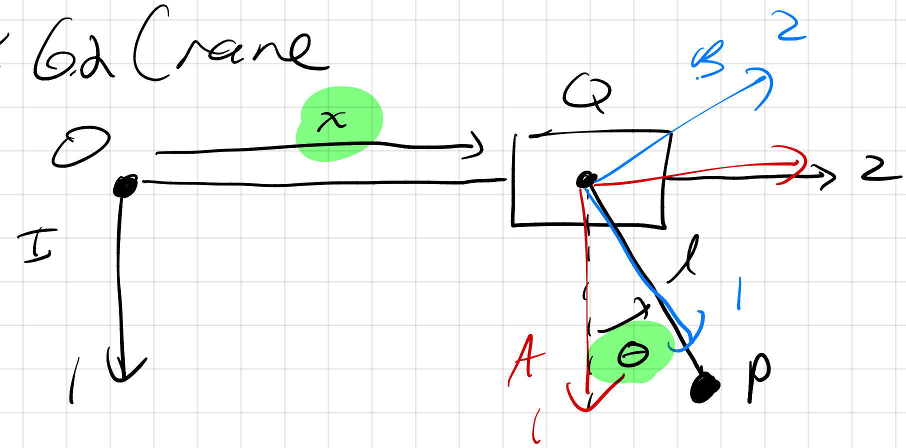
Chapter 6

Multi-Particle Systems

9/26/24

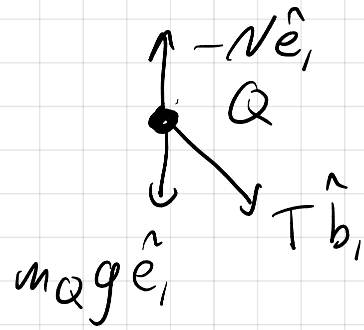
Ex 6a Crane



$$\left.\begin{aligned} z_Q &= 0 \\ z_P &= 0 \\ \|\bar{r}_{P/Q}\| &= \ell \\ y_Q &= 0 \end{aligned}\right\} K = 4$$

$$M = 3N - K = 6 - 4 = 2$$

$$\bar{F}_Q = m_Q\, {}^{I}\bar{a}_{Q/O}$$

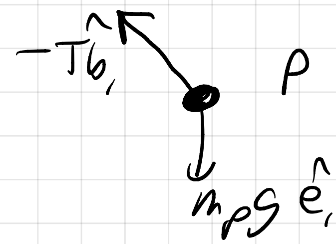


$$\bar{r}_{Q/O} = x\hat{e}_2$$

$${}^{I}\bar{v}_{Q/O} = \dot{x}\hat{e}_2$$

$${}^{I}\bar{a}_{Q/O} = \ddot{x}\hat{e}_2$$

$$\bar{F}_P = m_P\, {}^{I}\bar{a}_{P/O}$$



$${}^{I}\frac{d}{dt}(\hat{b}_1) = {}^{I}\bar{\omega}^{B} \times \hat{b}_1 = \dot{\theta}\hat{b}_3 \times \hat{b}_1$$

$$\bar{r}_{P/O} = x\hat{e}_2 + \ell\hat{b}_1$$

$${}^{I}\bar{v}_{P/O} = \dot{x}\hat{e}_2 + \ell\dot{\theta}\hat{b}_2$$

$${}^{I}\bar{a}_{P/O} = \ddot{x}\hat{e}_2 + \ell\ddot{\theta}\hat{b}_2 - \ell\dot{\theta}^2\hat{b}_1$$

$$(m_Q g - N)\hat{e}_1 + T\hat{b}_1 = m_Q\ddot{x}\hat{e}_2$$

$$-T\hat{b}_1 + m_P g\hat{e}_1 = m_P\left(\ddot{x}\hat{e}_2 + \ell\ddot{\theta}\hat{b}_2 - \ell\dot{\theta}^2\hat{b}_1\right)$$

	$\hat{b}_1$	$\hat{b}_2$
$\hat{a}_1 = \hat{e}_1$	$C\theta$	$-S\theta$
$\hat{a}_2 = \hat{e}_2$	$S\theta$	$C\theta$

#3 $\hat{b}_1$: $-T + m_P g C\theta = m_P(\ddot{x}S\theta - \ell\dot{\theta}^2)$

#4 $\hat{b}_2$: $-m_P g S\theta = m_P(\ddot{x}C\theta + \ell\ddot{\theta})$

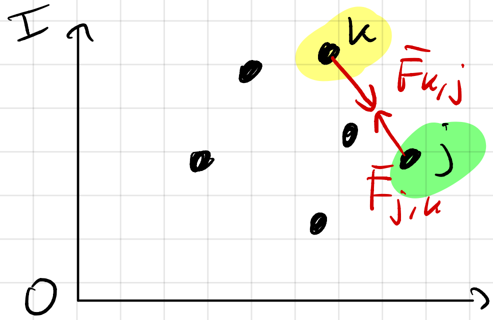
#1 $\hat{e}_1$: $m_Q g - N + TC\theta = 0$

#2 $\hat{e}_2$: $0 + TS\theta = m_Q\ddot{x}$

$\cancel{3}$ 4 unknowns, $\cancel{2}$ 4 equations

$$\boxed{\begin{aligned} \ddot{x} &= \dots \\ \ddot{\theta} &= \dots \\ T &= \dots \\ N &= \dots \end{aligned}}\ \Big\}\text{ eq. mo.}$$

good to go



N # particles

$k = 1, \ldots, N$

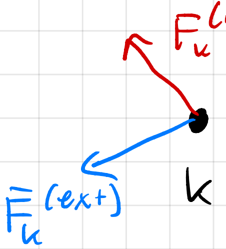
$j = 1, \ldots, N$

N2L: $\bar{F}_k = m_k \, {}^{I}\bar{a}_{k/O}, \quad k = 1, \ldots, N$

N3L: $\bar{F}_{k,j} = -\bar{F}_{j,k}$ } internal forces

$\bar{F}_{k,j}$: force on k, due to j

Assumption $\bar{F}_{k,k} = 0$



$$\bar{F}_k^{(int)} = \sum_{j=1}^{N} \bar{F}_{k,j}$$

$$\bar{F}_k = \bar{F}_k^{(ext)} + \underbrace{\bar{F}_k^{(int)}}_{\bar{F}_{k,1} + \bar{F}_{k,2} + \cdots + \bar{F}_{k,N}}$$

$j = 1, \ldots, N$

$$\boxed{\bar{F}_k = \bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j}}$$

$$\Rightarrow \bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j} = m_k \, {}^{I}\bar{a}_{k/O}, \quad k = 1, \ldots, N$$

Dfn total linear momentum

$$\boxed{{}^{I}\bar{P}_O = \sum_{k=1}^{N} {}^{I}\bar{p}_{k/O}}, \quad \text{where } {}^{I}\bar{p}_{k/O} = m_k \, {}^{I}\bar{v}_{k/O}$$

total: sum over all particles

Recall $\dfrac{{}^{I}d}{dt}\left({}^{I}\bar{p}_{k/O}\right) = \bar{F}_k, \quad k = 1, \ldots, N$

Ch 4 $\quad {}^{I}\bar{p}_{k/O}(t_2) = {}^{I}\bar{p}_{k/O}(t_1) + \underbrace{\bar{F}_k(t_1, t_2)}_{= \int_{t_1}^{t_2} \bar{F}_k \, dt}$

* key idea: sum over all particles

$$
\begin{aligned}
{}^I\bar{p}_{1/0}(t_2) &= {}^I\bar{p}_{1/0}(t_1) + \bar{\bar{F}}_1(t_1, t_2) \\
{}^I\bar{p}_{2/0}(t_2) &= {}^I\bar{p}_{2/0}(t_1) + \bar{\bar{F}}_2(t_1, t_2) \\
&\vdots \\
+ \quad {}^I\bar{p}_{N/0}(t_2) &= {}^I\bar{p}_{N/0}(t_1) + \bar{\bar{F}}_N(t_1, t_2)
\end{aligned}
$$

$$
{}^I\bar{p}_0(t_2) = {}^I\bar{p}_0(t_1) + \sum_{k=1}^{N} \bar{\bar{F}}_k
$$

what is this?

$$
\sum_{k=1}^{N} \int_{t_1}^{t_2} \bar{F}_k \, dt = \int_{t_1}^{t_2} \sum_{k=1}^{N} \left(\bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j} \right) dt
$$

$$
= \int_{t_1}^{t_2} \underbrace{\sum_{k=1}^{N} \bar{F}_k^{(ext)}}_{\text{total external force}} dt + \int_{t_1}^{t_2} \underbrace{\sum_{k=1}^{N} \sum_{j=1}^{N} \bar{F}_{k,j}}_{=0} dt
$$

$$
= \bar{\bar{F}}^{(ext)}
$$

$$
\begin{matrix}
\bar{F}_{1,1} & + & \bar{F}_{1,2} & + & \cdots & + & \bar{F}_{1,N} & + \\
\bar{F}_{2,1} & + & \bar{F}_{2,2} & + & \cdots & + & \bar{F}_{2,N} & + \\
\vdots & & & & & & & \\
\bar{F}_{N,1} & + & \bar{F}_{N,2} & + & \cdots & + & \bar{F}_{N,N} &
\end{matrix}
$$

sum to zero N3L

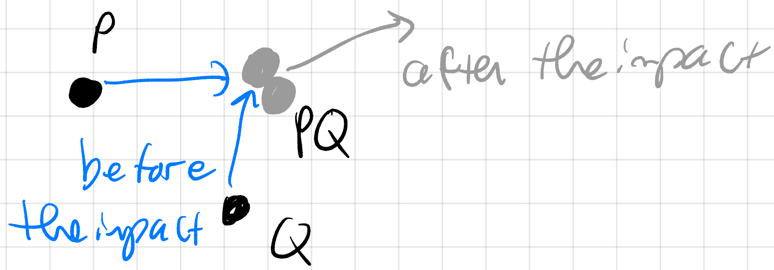
$$
\bar{F}_{k,j} + \bar{F}_{j,k} = \bar{F}_{k,j} - \bar{F}_{k,j} = 0
$$

$= 0$ by assumption

$$
\boxed{{}^I\bar{p}_0(t_2) = {}^I\bar{p}_0(t_1) + \bar{\bar{F}}^{(ext)}(t_1, t_2)}
$$

* key idea: the change in the total linear momentum depends only on external forces!
* ${}^I\bar{p}_0$ is conserved if $\bar{\bar{F}}^{(ext)}(t_1, t_2) = 0$

Ex Sticky impact (with no external forces)



$$ {}^{I}\bar{p}_O(t_2) = {}^{I}\bar{p}_O(t_1) + \cancel{\bar{F}^{(ext)}(t_1,t_2)} \rightarrow 0 $$

$$ \underbrace{(m_P + m_Q)\,{}^{I}\bar{v}_{PQ/O}(t_2)}_{\text{after}} = \underbrace{m_Q\,{}^{I}\bar{v}_{Q/O}(t_1) + m_P\,{}^{I}\bar{v}_{P/O}(t_1)}_{\text{before}} $$

(See Section 6.2 for bouncy collisions, not required)

* one component of ${}^{I}\bar{p}_O$ might be conserved, even if the other component is not!

Ex

$$ \hat{e}_1: \quad {}^{I}\bar{p}_O(t_2)\cdot\hat{e}_1 = {}^{I}\bar{p}_O(t_1)\cdot\hat{e}_1 + \overbrace{\bar{F}^{(ext)}(t_1,t_2)\cdot\hat{e}_1}^{\neq 0} $$

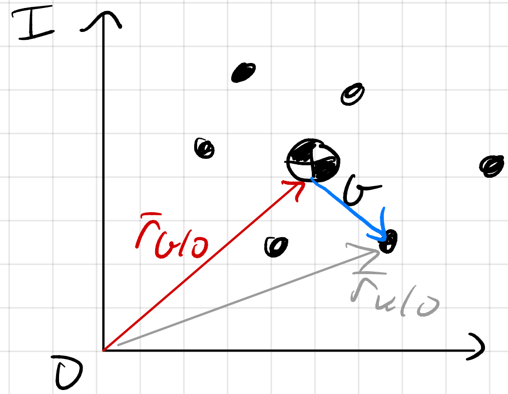
$$ \hat{e}_2: \quad {}^{I}\bar{p}_O(t_2)\cdot\hat{e}_2 = {}^{I}\bar{p}_O(t_1)\cdot\hat{e}_2 + \underbrace{\bar{F}^{(ext)}(t_1,t_2)\cdot\hat{e}_2}_{\text{Suppose } = 0} $$

$$ \Rightarrow {}^{I}\bar{p}_O(t_2)\cdot\hat{e}_2 = {}^{I}\bar{p}_O(t_1)\cdot\hat{e}_2 $$

* $\hat{e}_2$ component is conserved; $\hat{e}_1$ is not

Dfn the position of the center of mass G of a MPS (multi-particle system) is

$$ \boxed{\bar{r}_{G/O} = \frac{1}{m_G}\sum_{k=1}^{N} m_k\,\bar{r}_{k/O},} \quad \underbrace{m_G}_{\text{total mass}} = \sum_{k=1}^{N} m_k $$



Center of Mass Corollary (COMC)

$$\bar r_{k/O} = \bar r_{G/O} + \bar r_{k/G}$$

$$\Rightarrow m_G \bar r_{G/O} = \sum_{k=1}^{N} m_k (\bar r_{G/O} + \bar r_{k/G})$$

$$\cancel{m_G \bar r_{G/O}} = \cancel{m_G \bar r_{G/O}} + \sum_{k=1}^{N} m_k \bar r_{k/G}$$

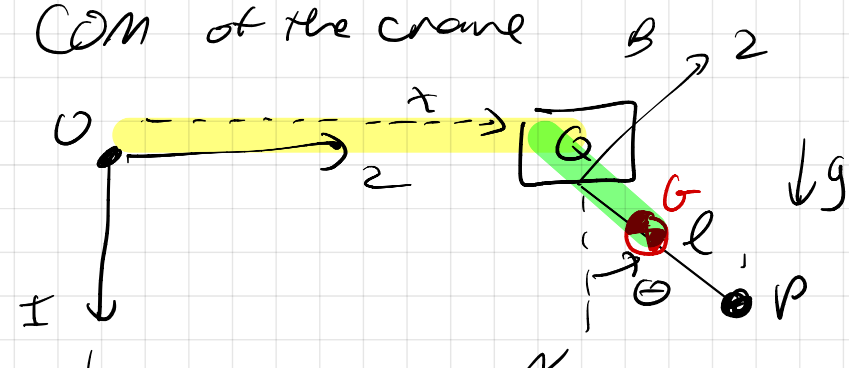
$$\Rightarrow \boxed{\sum_{k=1}^{N} m_k \bar r_{k/G} = 0} \quad \text{COMC}$$

(video: balancing on the pin)

* COMC can be differentiated:

$$\sum_{k=1}^{N} m_k \, {}^{I}\bar v_{k/G} = 0 \quad \text{and} \quad \sum_{k=1}^{N} m_k \, {}^{I}\bar a_{k/G} = 0 \;!!$$

Ex COM of the crane



$$\bar r_{G/O} = \frac{1}{m_G} \sum_{k=1}^{N} m_k \bar r_{k/O}, \quad N = 2$$

$$= \frac{1}{m_P + m_Q} \left(m_P \bar r_{P/O} + m_Q \bar r_{Q/O} \right)$$

$\bar r_{Q/O} = x\hat e_2$ and $\bar r_{P/O} = x\hat e_2 + l\hat b_1$

$$\bar r_{G/O} = \frac{1}{m_P + m_Q} \left(m_Q x \hat e_2 + m_P x \hat e_2 + m_P l \hat b_1 \right)$$

$$= x\hat e_2 + \frac{m_P}{m_P + m_Q} l \hat b_1$$

closer to Q if $m_Q >> m_P$
" " P " $m_P >> m_Q$

Dynamics of COM

$\bar{r}_{k/O} = \bar{r}_{G/O} + \bar{r}_{k/G}$

$^I\bar{v}_{k/O} = {}^I\bar{v}_{G/O} + {}^I\bar{v}_{k/G}$

N2L $\quad \bar{F}_k = m_k \, {}^I\bar{a}_{k/O}, \quad k = 1, \dots, N$

$= \bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j} \qquad = {}^I\bar{a}_{G/O} + {}^I\bar{a}_{k/G}$

$$\sum_{k=1}^{N}\left(\bar{F}_k^{(ext)} + \sum_{j=1}^{N} \bar{F}_{k,j}\right) = \sum_{k=1}^{N} m_k \left({}^I\bar{a}_{G/O} + {}^I\bar{a}_{k/G}\right)$$

$$\bar{F}^{(ext)} + \underbrace{\sum_{k=1}^{N}\sum_{j=1}^{N} \bar{F}_{k,j}}_{=0 \text{ by N3L}} = m_G \, {}^I\bar{a}_{G/O} + \underbrace{\sum_{k=1}^{N} m_k \, {}^I\bar{a}_{k/G}}_{=0 \text{ COMC}}$$

$$\boxed{\bar{F}_G^{(ext)} = m_G \, {}^I\bar{a}_{G/O}}$$

Not N2L! (but it looks just like it!)

* the center of mass G of MPS behaves just like a particle!!!

Dfn inertial kinematics of COM: $\bar{r}_{G/O}, {}^I\bar{v}_{G/O}, {}^I\bar{a}_{G/O}$

Dfn linear momentum of COM: ${}^I\bar{p}_{G/O} = m_G \, {}^I\bar{v}_{G/O}$

Q: what is the relationship between ${}^I\bar{p}_{G/O}$ and ${}^I\bar{p}_O$?

$$^I\bar{p}_O = \sum_{k=1}^{N} {}^I\bar{p}_{k/O} = \sum_{k=1}^{N} m_k \, {}^I\bar{v}_{k/O}$$

$$= \sum_{k=1}^{N} m_k \left({}^I\bar{v}_{G/O} + {}^I\bar{v}_{k/G}\right)$$

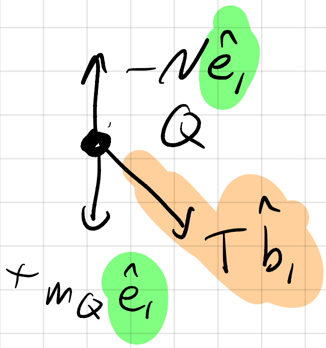
$$= \underbrace{m_G \, {}^I\bar{v}_{G/O}}_{= {}^I\bar{p}_{G/O}} + \underbrace{\sum_{k=1}^{N} m_k \, {}^I\bar{v}_{k/G}}_{=0 \text{ by COMC}}$$

$$\boxed{{}^I\bar{p}_O = {}^I\bar{p}_{G/O}}$$

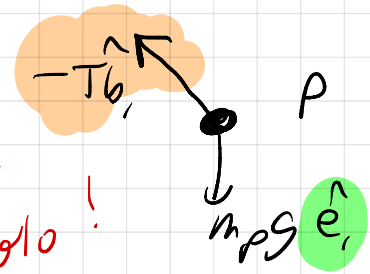
* key idea: the total linear momentum is equal to the linear momentum of the COM G.
(not true for angular momentum!! Stay tuned for Ch 7)

Recall ${}^I\bar{p}_O$ is conserved if $\bar{F}^{(ext)} = 0 \Rightarrow$ also true for ${}^I\bar{p}_{G/O}$!

Ex crane: what is the behavior of the com?



internal forces don't affect $^{I}\bar{p}_O$ or $^{I}\bar{p}_{G/O}$!



* all external forces are in the $\hat{e}_1$ direction...

$$^{I}\bar{p}_O(t_2) = {}^{I}\bar{p}_O(t_1) + \bar{\bar{F}}^{(ext)}$$

$$\hat{e}_2 \cdot \left({}^{I}\bar{p}_{G/O}(t_2) = {}^{I}\bar{p}_G(t_1) + \bar{\bar{F}}^{(ext)} \right)$$

$^{I}\bar{p}_{G/O} \cdot \hat{e}_2$ is conserved!

$\Rightarrow$ $m_G \, {}^{I}\bar{v}_{G/O} \cdot \hat{e}_2$ is conserved

$$\underbrace{^{I}\bar{v}_{G/O}}_{} \cdot \hat{e}_2(t) = {}^{I}\bar{v}_{G/O} \cdot \hat{e}_2(0)$$

horizontal component is constant