Lecture 5: Review of Vectors, Fields, and Operations

ENAE311H Aerodynamics I

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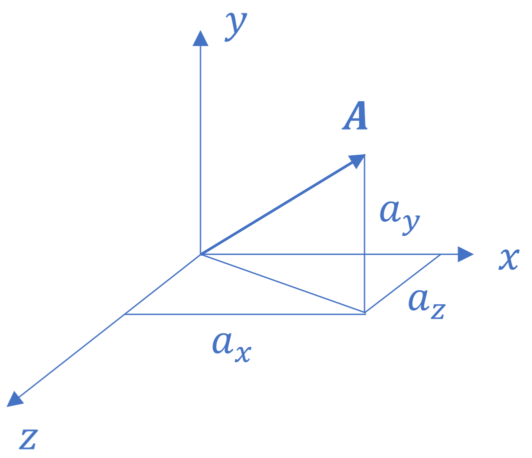
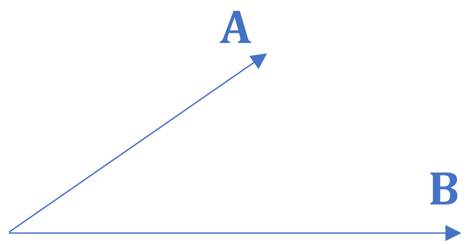
Vectors

- *Definition:* A vector is a directed line segment with both a magnitude (length) and direction
- The vectors of interest to us in this course are velocity, $\boldsymbol{v}$, acceleration, $\boldsymbol{a}$, and force, $\boldsymbol{F}$ (we denote vectors with boldface throughout this course).
- A *unit vector* is a vector of magnitude unity (denote with a hat); we call the unit vectors aligned with the coordinate axes *versors* (for a Cartesian coordinate system, the versors are $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\widehat{\boldsymbol{k}}$.
- If a vector is given by $\boldsymbol{A} = (a_x, a_y, a_z)$= $a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\widehat{\boldsymbol{k}}$, the magnitude of $\boldsymbol{A}$ is

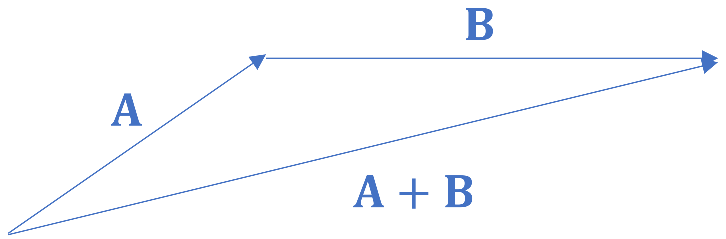
$$||\boldsymbol{A}|| = \sqrt{\boldsymbol{a}_{\boldsymbol{x}}^{2} + \boldsymbol{a}_{\boldsymbol{y}}^{2} + \boldsymbol{a}_{\boldsymbol{z}}^{2}}$$



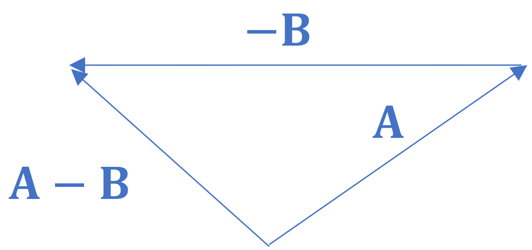
Vector operations



- Vector addition/subtraction:

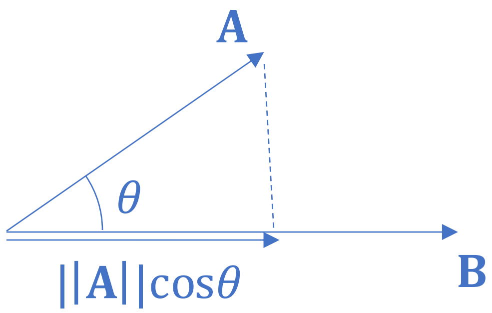


$$\begin{aligned}\mathbf{A}+\mathbf{B} &= (a_x+b_x)\hat{\mathbf{i}}+(a_y+b_y)\hat{\mathbf{j}}+(a_z+b_z)\hat{\mathbf{k}}\\ \mathbf{A}-\mathbf{B} &= \mathbf{A}+(-\mathbf{B})=(a_x-b_x)\hat{\mathbf{i}}+(a_y-b_y)\hat{\mathbf{j}}+(a_z-b_z)\hat{\mathbf{k}}\end{aligned}$$



- Multiplication – scalar/dot product:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= a_xb_x+a_yb_y+c_xc_y\\ &= ||\mathbf{A}||\,||\mathbf{B}||\cos\theta\end{aligned}$$



- Multiplication - vector/cross product:

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}\\ &= (a_yb_z-a_zb_y)\hat{\mathbf{i}}-(a_xb_z-a_zb_x)\hat{\mathbf{j}}+(a_xb_y-a_yb_x)\hat{\mathbf{k}}\\ &= ||\mathbf{A}||\,||\mathbf{B}||\sin\theta\hat{\mathbf{n}},\end{aligned}$$

Gradient of a scalar field

- *Scalar field:* a scalar quantity given as a pointwise function of space and time, e.g., $p(x, y, z, t)$
- The gradient of a scalar field, p, is given by

$$\nabla p \quad = \quad \frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}}$$

- The magnitude of the gradient, $||\nabla p||$, is the maximum rate of change of p per unit length of the coordinate system
- To calculate the component of the gradient in a particular direction, $\boldsymbol{s}$, can use

$$\frac{\partial p}{\partial s} = \nabla p \cdot \hat{\mathbf{s}} = ||\nabla p||\ ||\hat{\mathbf{s}}|| \cos\theta = ||\nabla p|| \cos\theta$$

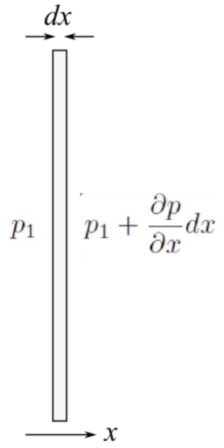
where θ is the angle between ∇p and $\boldsymbol{s}$.

- Note that ∇p provides the driving force to accelerate the flow:

Consider flat plate to the right (x-normal faces) – force acting in x-direction is given by

$$\begin{aligned} F &= p_1 A - p_2 A \approx p_1 A - \left(p_1 + \frac{\partial p}{\partial x}dx\right) A \\ &= -\frac{\partial p}{\partial x} dx A, \end{aligned}$$

and so $\boxed{F \propto \frac{\partial p}{\partial x}}$



Divergence and curl of a vector field

- *Vector field:* a vector quantity given as a pointwise function of space and time, e.g., $\mathbf{v}(x, y, z, t)$
- The divergence of a vector field, $\mathbf{v}$, is given by

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial x_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

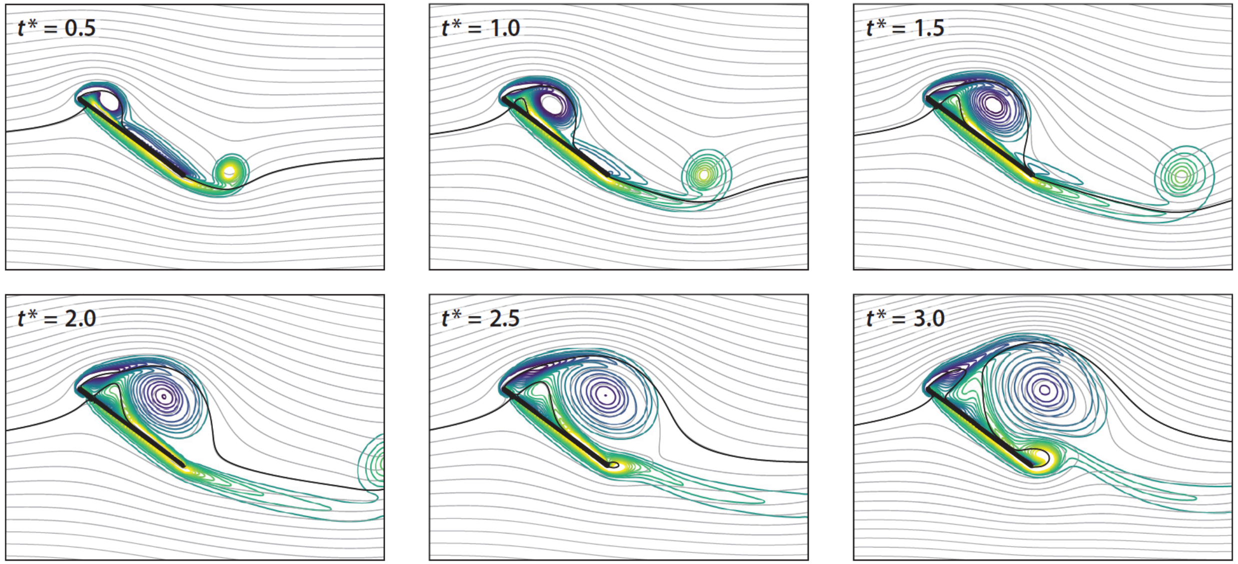
- It is a scalar quantity describing the tendency of the vector field to behave as a source or a sink.
- The divergence of the velocity field is particularly important in the context of conservation of mass.

- The curl of a vector field, $\mathbf{v}$, is given by

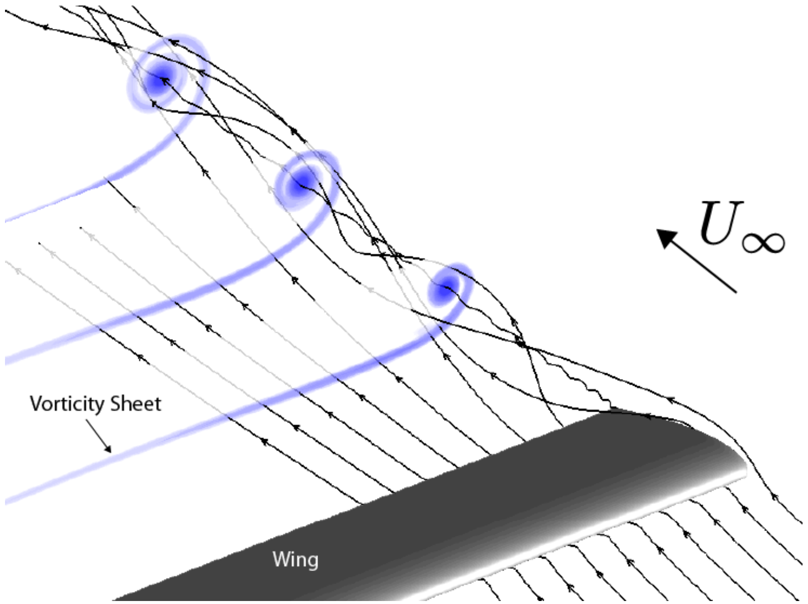
$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

- It is a vector quantity describing the rotational tendency of the underlying vector field.
- The curl of the velocity field is known as the "vorticity" and is a very important quantity in many branches of fluid mechanics.

Vortices



Leading-edge vortex



Wingtip vortices

Line, surface, and volume integrals

- Line integrals:

$$\int_a^b \mathbf{v} \cdot \mathbf{ds} = \int_a^b \mathbf{v} \cdot \hat{\mathbf{s}} ds$$

  - If the curve is closed, then $\int_a^b \rightarrow \oint_C$

- Surface integrals:

$$\begin{aligned}
\iint_S p\mathbf{dA} &= \iint_S p\hat{\mathbf{n}}dA \rightarrow \text{vector (force)} \\
\iint_S \mathbf{v} \cdot \mathbf{dA} &= \iint_S \mathbf{v} \cdot \hat{\mathbf{n}}dA \rightarrow \text{scalar (volumetric flow rate)}
\end{aligned}$$

rate of change of a volume

- Volume integrals:

$$\begin{aligned}
\iiint_V \rho dV &\rightarrow \text{scalar (mass)} \\
\iiint_V \mathbf{v} dV &\rightarrow \text{vector}
\end{aligned}$$

Integral theorems of vector calculus

- Stokes' theorem:

$$\oint_C \mathbf{v} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{dA}$$

- Gauss' (divergence) theorem:

$$\iint_S \mathbf{v} \cdot \mathbf{dA} = \iiint_V \nabla \cdot \mathbf{v} dV$$

where *S* is the closed surface bounding *V*.

- Gradient theorem:

$$\iint_S p\mathbf{dA} = \iiint_V \nabla p dV$$

where *S* is again the closed surface bounding *V*.

$$\Delta \mathcal{V} = [(\mathbf{V}\Delta t) \cdot \mathbf{n}]dS = (\mathbf{V}\Delta t) \cdot \mathbf{dS} \tag{2.28}$$

Over the time increment Δt, the total change in volume of the whole control volume is equal to the summation of Equation (2.28) over the total control surface. In the limit as $dS \to 0$, the sum becomes the surface integral

$$\oint\!\!\!\!\int_S (\mathbf{V}\Delta t) \cdot \mathbf{dS}$$

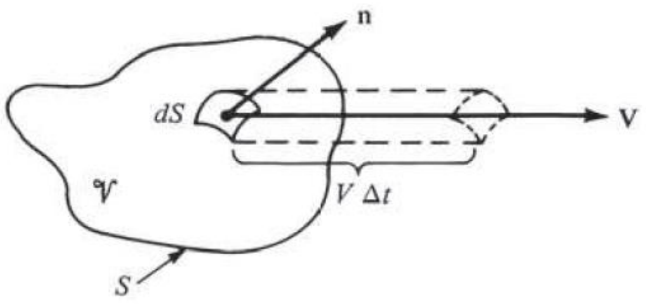


Figure 2.15 Moving control volume used for the physical interpretation of the divergence of velocity.

If this integral is divided by Δt, the result is physically the time rate of change of the control volume, denoted by $D\mathcal{V}/Dt$; that is,

$$\frac{D\mathcal{V}}{Dt} = \frac{1}{\Delta t}\oint\!\!\!\!\int_S (\mathbf{V}\Delta t) \cdot \mathbf{dS} = \oint\!\!\!\!\int_S \mathbf{V} \cdot \mathbf{dS} \tag{2.29}$$

(The significance of the notation D/Dt is revealed in Section 2.9.) Applying the divergence theorem, Equation (2.26), to the right side of Equation (2.29), we have

$$\frac{D\mathcal{V}}{Dt} = \oiiint_{\mathcal{V}} (\nabla \cdot \mathbf{V})d\mathcal{V} \tag{2.30}$$

Now let us imagine that the moving control volume in Figure 2.15 is shrunk to a very small volume $\delta\mathcal{V}$, essentially becoming an infinitesimal moving fluid element as sketched on the right of Figure 2.14. Then Equation (2.30) can be written as

$$\frac{D(\delta\mathcal{V})}{Dt} = \oiiint_{\delta\mathcal{V}} (\nabla \cdot \mathbf{V})d\mathcal{V} \tag{2.31}$$

Assume that $\delta\mathcal{V}$ is small enough such that $\nabla \cdot \mathbf{V}$ is essentially the same value throughout $\delta\mathcal{V}$. Then the integral in Equation (2.31) can be approximated as $(\nabla \cdot \mathbf{V})\delta\mathcal{V}$. From Equation (2.31), we have

$$\frac{D(\delta\mathcal{V})}{Dt} = (\nabla \cdot \mathbf{V})\delta\mathcal{V}$$

or

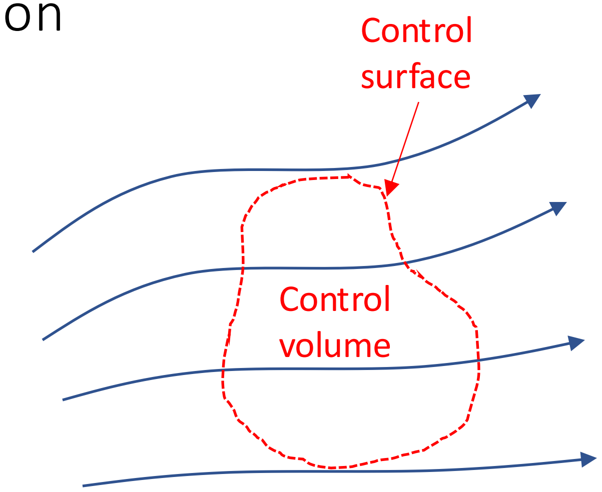
$$\boxed{\nabla \cdot \mathbf{V} = \frac{1}{\delta\mathcal{V}}\frac{D(\delta\mathcal{V})}{Dt}} \tag{2.32}$$

Examine Equation (2.32). It states that $\nabla \cdot \mathbf{V}$ is physically the *time rate of change of the volume of a moving fluid element, per unit volume*. Hence, the interpretation of $\nabla \cdot \mathbf{V}$, first given in Section 2.2.6, Divergence of a Vector Field, is now proved.

Control volumes

- Fluid dynamics problems typically involve solving one or more of the conservation equations:
  - Mass is conserved
  - The rate of change of momentum is equal to the net force applied (Newton's second law)
  - Energy is conserved (but can change form)

- For these equations to be applicable, however, they must be applied to a certain finite region of the flow.
- A "control volume (CV)" is any closed volume bounding a finite region of the flow; its boundary is called a "control surface" (CS).



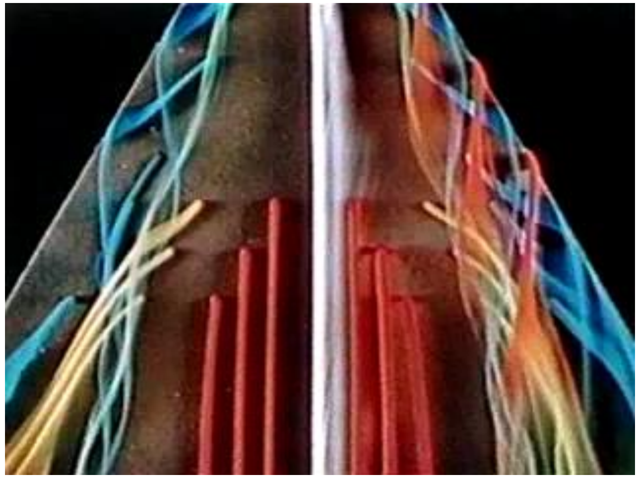
- For specific problems, can choose CV to make application of conservation law particularly easy; alternatively can use arbitrary CV to prove a general relation.

Flow Visualization (1)

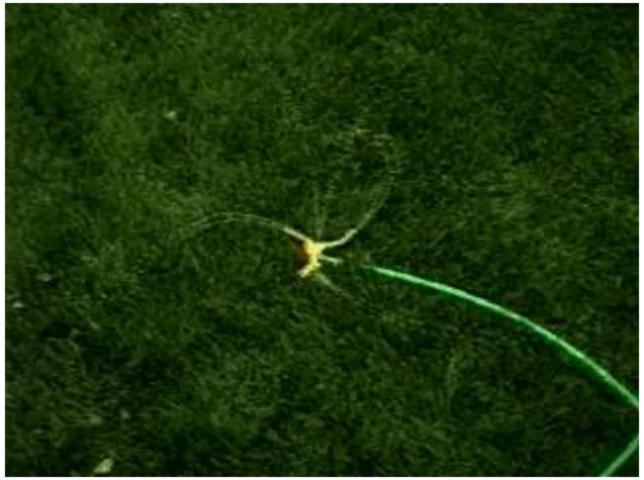
- Dye injection in liquid can be used to “see” flow – dye follows the flow
- Movie show streaks representing path of fluid particles

Flow Visualization (2)

- Streaklines, Pathlines, and Streamlines
  - Streaklines = instantaneous location of fluid particles that once passed through a specified point
    - inject dye continuously at fixed points and take snapshot at later time

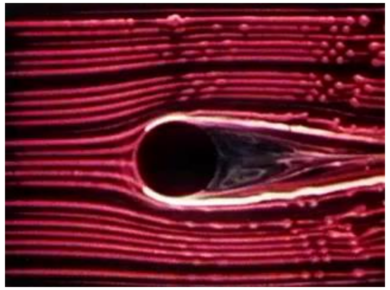


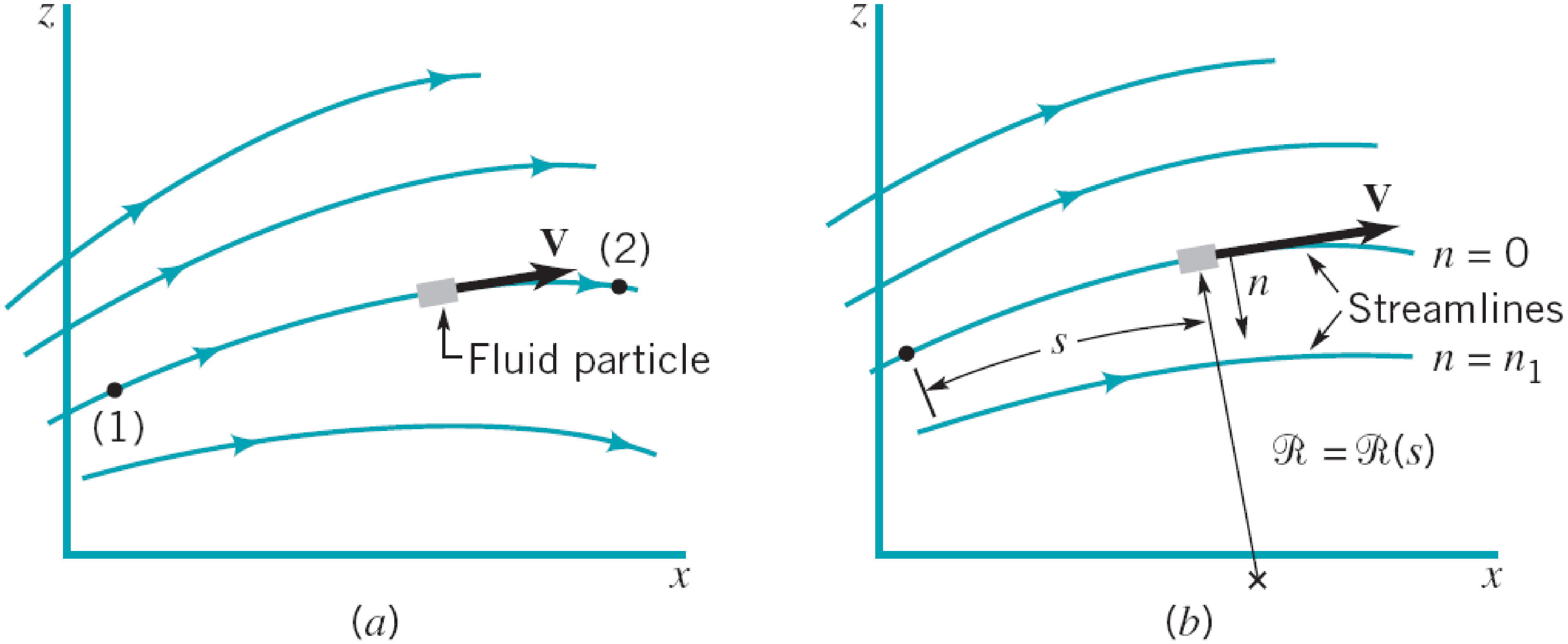
  - Pathlines = path that particles follow
    - inject dye briefly at fixed points and take time-lapsed photo for a period of time



Streamlines

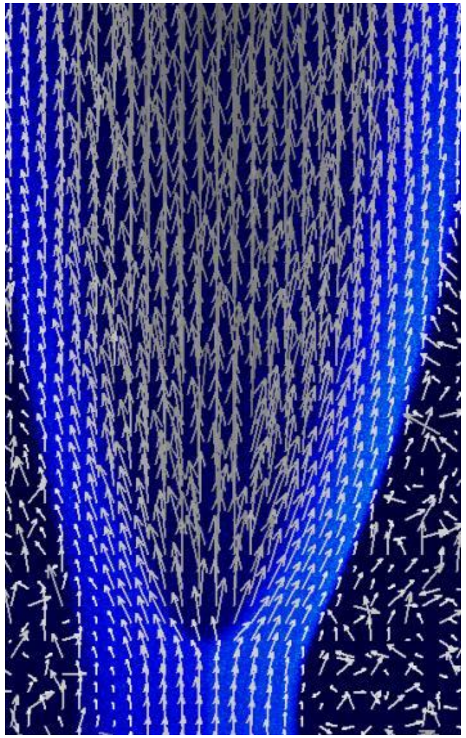
- Streamlines = lines in the flow that are locally tangent to the velocity of the fluid

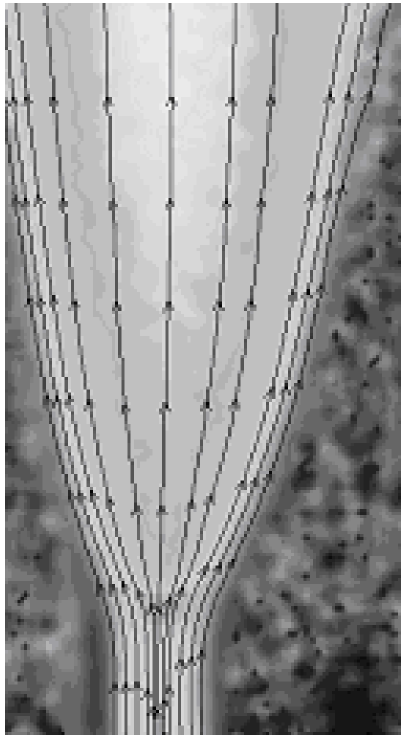




Streamline (2)

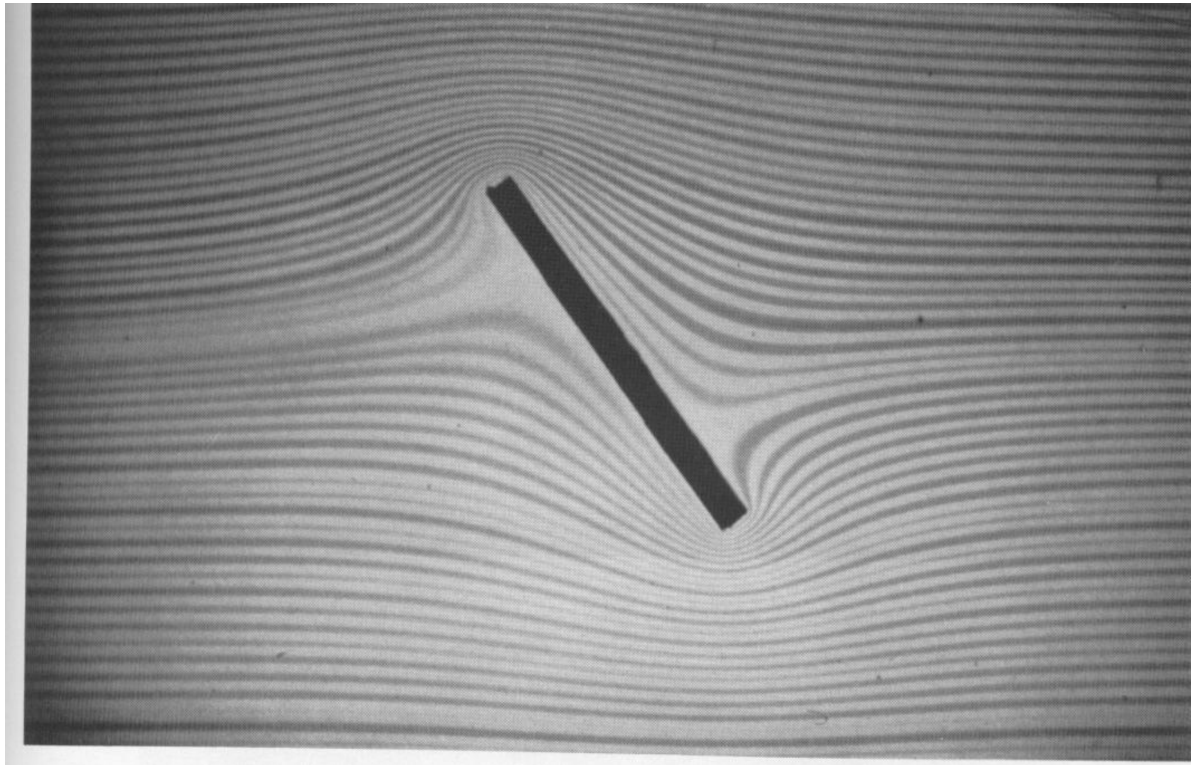
- Streamlines determined by measuring instantaneous velocity and integrating to find tangent lines
- Harder to measure than streaklines
- Most useful to mathematically describe flow





Streamlines (3)

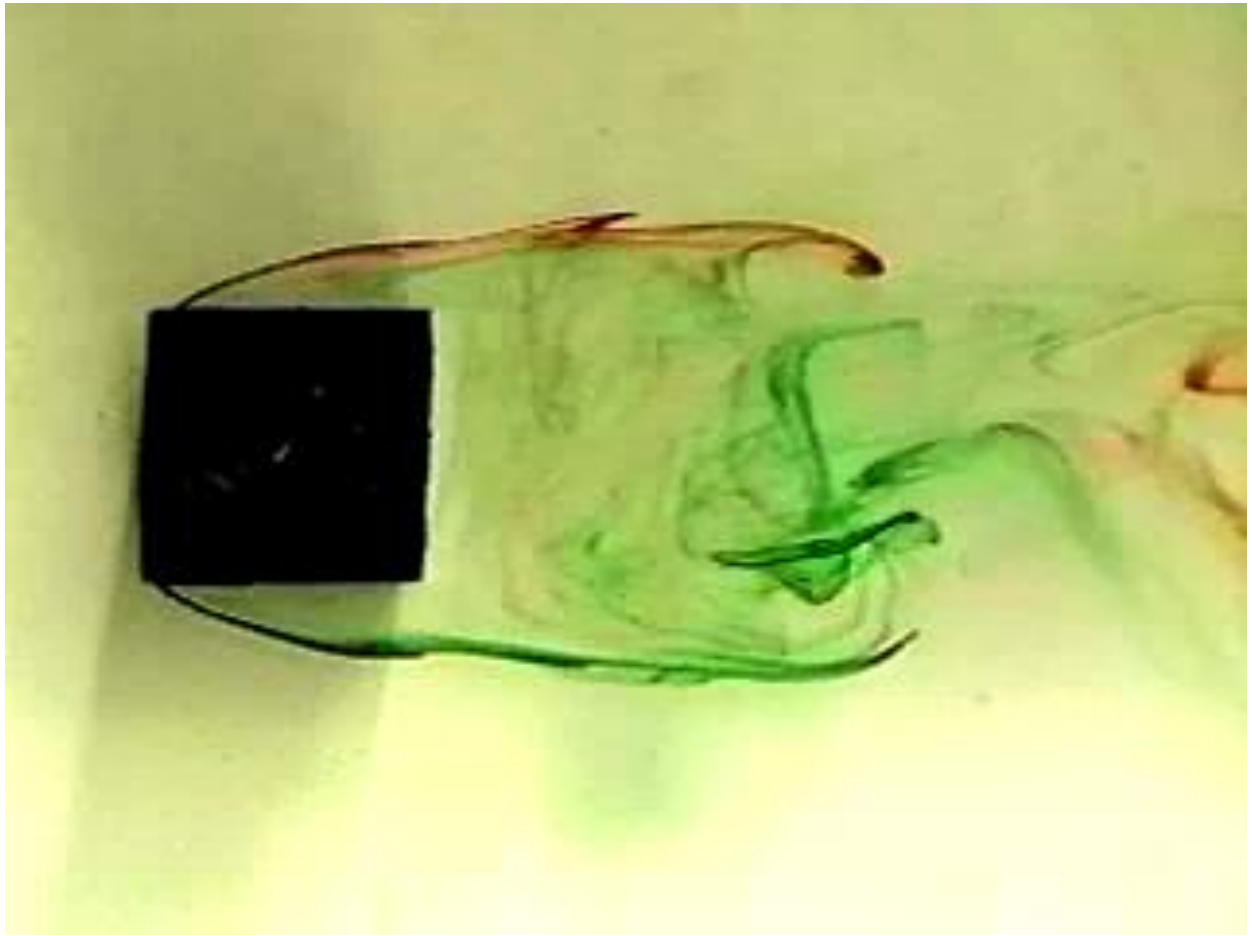
- For steady flows – pathlines, streaklines, and streamlines are identical



3. Hele-Shaw flow past an inclined plate. The Hele-Shaw analogy cannot represent a flow with circulation. It therefore shows the streamlines of potential flow past an inclined plate with zero lift. Dye flows in water between glass plates spaced 1 mm apart. *Photograph by D. H. Peregrine*

Streamlines (3)

- For steady flows – pathlines, streaklines, and streamlines are identical



NOT FOR UNSTEADY!!!

EXAMPLE 4.2 Streamlines for a Given Velocity Field

GIVEN Consider the two-dimensional steady flow discussed in Example 4.1, $\mathbf{V} = (V_0/\ell)(-x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$.

FIND Determine the streamlines for this flow.

SOLUTION

Since

$$u = (-V_0/\ell)x \text{ and } v = (V_0/\ell)y \qquad (1)$$

it follows that streamlines are given by solution of the equation

$$\frac{dy}{dx} = \frac{v}{u} = \frac{(V_0/\ell)y}{-(V_0/\ell)x} = -\frac{y}{x}$$

in which variables can be separated and the equation integrated to give

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

or

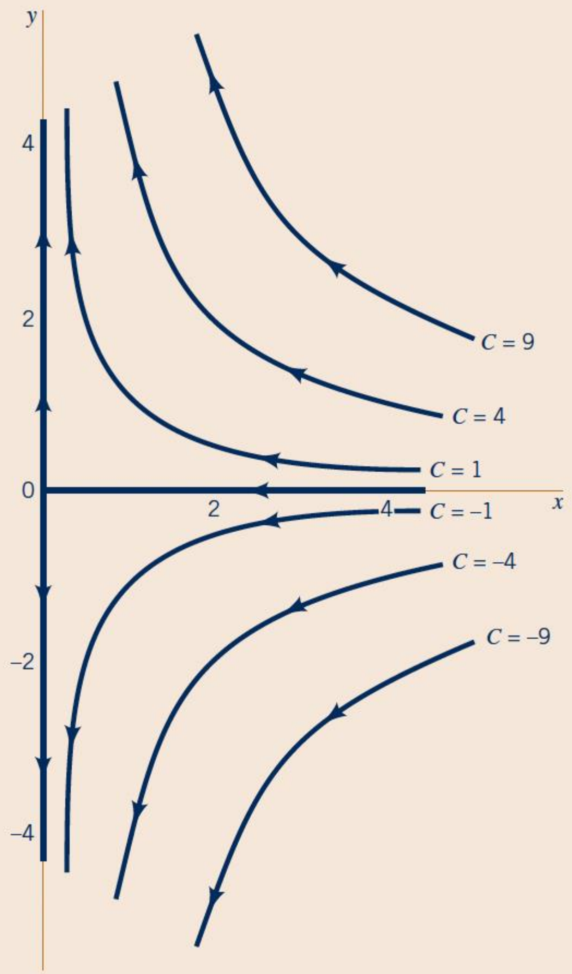
$$\ln y = -\ln x + \text{constant}$$

Thus, along the streamline

$$xy = C, \qquad \text{where } C \text{ is a constant} \qquad \textbf{(Ans)}$$

By using different values of the constant C, we can plot various lines in the x–y plane—the streamlines. The streamlines for $x \geq 0$ are plotted in Fig. E4.2. A comparison of this figure with Fig. E4.1*a* illustrates the fact that streamlines are lines tangent to the velocity field.

COMMENT Note that a flow is not completely specified by the shape of the streamlines alone. For example, the streamlines for the flow with $V_0/\ell = 10$ have the same shape as those for the flow with $V_0/\ell = -10$. However, the direction of the flow is opposite for these two cases. The arrows in Fig. E4.2 representing the flow direction are correct for $V_0/\ell = 10$ since, from Eq. 1, $u = -10x$ and $v = 10y$. That is, the flow is from right to left. For $V_0/\ell = -10$ the arrows are reversed. The flow is from left to right.



■ Figure E4.2

EXAMPLE 4.3 Comparison of Streamlines, Pathlines, and Streaklines

GIVEN Water flowing from the oscillating slit shown in Fig. E4.3*a* produces a velocity field given by $\mathbf{V} = u_0 \sin[\omega(t - y/v_0)]\hat{\mathbf{i}} + v_0\hat{\mathbf{j}}$, where u_0, v_0, and ω are constants. Thus, the *y* component of velocity remains constant ($v = v_0$), and the *x* component of velocity at $y = 0$ coincides with the velocity of the oscillating sprinkler head $[u = u_0 \sin(\omega t)$ at $y = 0]$.

FIND **(a)** Determine the streamline that passes through the origin at $t = 0$; at $t = \pi/2\omega$.
(b) Determine the pathline of the particle that was at the origin at $t = 0$; at $t = \pi/2$.
(c) Discuss the shape of the streakline that passes through the origin.

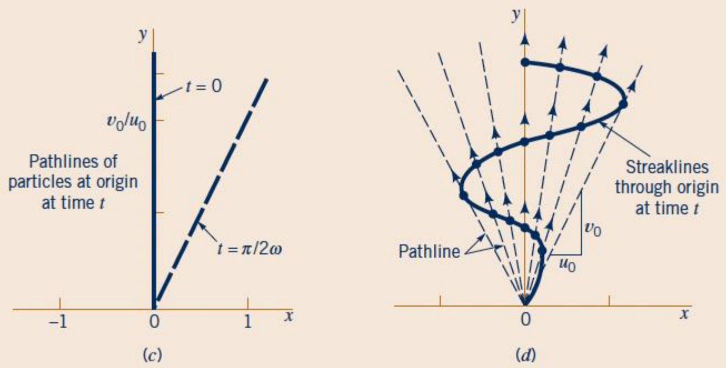
SOLUTION

(a) Since $u = u_0 \sin[\omega(t - y/v_0)]$ and $v = v_0$, it follows from Eq. 4.1 that streamlines are given by the solution of

$$\frac{dy}{dx} = \frac{v}{u} = \frac{v_0}{u_0 \sin[\omega(t - y/v_0)]}$$

in which the variables can be separated and the equation integrated (for any given time *t*) to give

$$u_0 \int \sin\left[\omega\left(t - \frac{y}{v_0}\right)\right] dy = v_0 \int dx$$



■ Figure E4.3(*c*), (*d*)

or

$$u_0(v_0/\omega)\cos\left[\omega\left(t - \frac{y}{v_0}\right)\right] = v_0 x + C \qquad (1)$$

where *C* is a constant. For the streamline at $t = 0$ that passes through the origin ($x = y = 0$), the value of *C* is obtained from Eq. 1 as $C = u_0 v_0/\omega$. Hence, the equation for this streamline is

$$x = \frac{u_0}{\omega}\left[\cos\left(\frac{\omega y}{v_0}\right) - 1\right] \qquad (2) \quad \text{(Ans)}$$

Similarly, for the streamline at $t = \pi/2\omega$ that passes through the origin, Eq. 1 gives $C = 0$. Thus, the equation for this streamline is

$$x = \frac{u_0}{\omega}\cos\left[\omega\left(\frac{\pi}{2\omega} - \frac{y}{v_0}\right)\right] = \frac{u_0}{\omega}\cos\left(\frac{\pi}{2} - \frac{\omega y}{v_0}\right)$$

or

$$x = \frac{u_0}{\omega}\sin\left(\frac{\omega y}{v_0}\right) \qquad (3) \quad \text{(Ans)}$$

COMMENT These two streamlines, plotted in Fig. E4.3*b*, are not the same because the flow is unsteady. For example, at the origin ($x = y = 0$) the velocity is $\mathbf{V} = v_0\hat{\mathbf{j}}$ at $t = 0$ and $\mathbf{V} = u_0\hat{\mathbf{i}} + v_0\hat{\mathbf{j}}$ at $t = \pi/2\omega$. Thus, the angle of the streamline passing through the origin changes with time. Similarly, the shape of the entire streamline is a function of time.

(b) The pathline of a particle (the location of the particle as a function of time) can be obtained from the velocity field and the definition of the velocity. Since $u = dx/dt$ and $v = dy/dt$ we obtain

$$\frac{dx}{dt} = u_0 \sin\left[\omega\left(t - \frac{y}{v_0}\right)\right] \quad \text{and} \quad \frac{dy}{dt} = v_0$$

The *y* equation can be integrated (since v_0 = constant) to give the *y* coordinate of the pathline as

$$y = v_0 t + C_1 \qquad (4)$$

where C_1 is a constant. With this known $y = y(t)$ dependence, the *x* equation for the pathline becomes

$$\frac{dx}{dt} = u_0 \sin\left[\omega\left(t - \frac{v_0 t + C_1}{v_0}\right)\right] = -u_0 \sin\left(\frac{C_1 \omega}{v_0}\right)$$

This can be integrated to give the *x* component of the pathline as

$$x = -\left[u_0 \sin\left(\frac{C_1\omega}{v_0}\right)\right]t + C_2 \qquad (5)$$

where C_2 is a constant. For the particle that was at the origin ($x = y = 0$) at time $t = 0$, Eqs. 4 and 5 give $C_1 = C_2 = 0$. Thus, the pathline is

$$x = 0 \quad \text{and} \quad y = v_0 t \qquad (6) \quad \text{(Ans)}$$

Similarly, for the particle that was at the origin at $t = \pi/2\omega$, Eqs. 4 and 5 give $C_1 = -\pi v_0/2\omega$ and $C_2 = -\pi u_0/2\omega$. Thus, the pathline for this particle is

$$x = u_0\left(t - \frac{\pi}{2\omega}\right) \quad \text{and} \quad y = v_0\left(t - \frac{\pi}{2\omega}\right) \qquad (7)$$

The pathline can be drawn by plotting the locus of $x(t)$, $y(t)$ values for $t \geq 0$ or by eliminating the parameter *t* from Eq. 7 to give

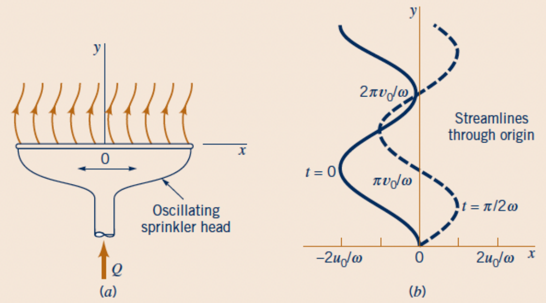
$$y = \frac{v_0}{u_0}x \qquad (8) \quad \text{(Ans)}$$

COMMENT The pathlines given by Eqs. 6 and 8, shown in Fig. E4.3*c*, are straight lines from the origin (rays). The pathlines and streamlines do not coincide because the flow is unsteady.

(c) The streakline through the origin at time $t = 0$ is the locus of particles at $t = 0$ that previously ($t < 0$) passed through the origin. The general shape of the streaklines can be seen as follows. Each particle that flows through the origin travels in a straight line (pathlines are rays from the origin), the slope of which lies between $\pm v_0/u_0$ as shown in Fig. E4.3*d*. Particles passing through the origin at different times are located on different rays from the origin and at different distances from the origin. The net result is that a stream of dye continually injected at the origin (a streakline) would have the shape shown in Fig. E4.3*d*. Because of the unsteadiness, the streakline will vary with time, although it will always have the oscillating, sinuous character shown.

COMMENT Similar streaklines are given by the stream of water from a garden hose nozzle that oscillates back and forth in a direction normal to the axis of the nozzle.

In this example neither the streamlines, pathlines, nor streaklines coincide. If the flow were steady, all of these lines would be the same.



■ Figure E4.3(*a*), (*b*)

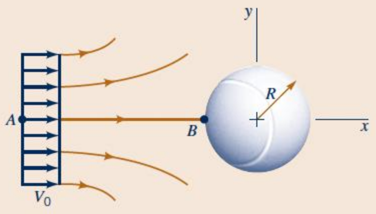
EXAMPLE 4.4 Acceleration along a Streamline

GIVEN An incompressible, inviscid fluid flows steadily past a tennis ball of radius R, as shown in Fig. E4.4*a*. According to a more advanced analysis of the flow, the fluid velocity along streamline A–B is given by

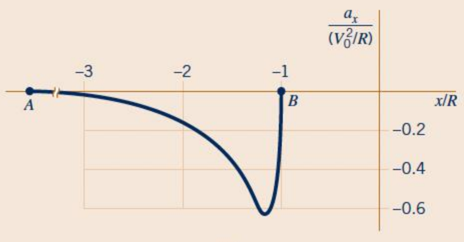
$$\mathbf{V} = u(x)\hat{\mathbf{i}} = V_0\left(1 + \frac{R^3}{x^3}\right)\hat{\mathbf{i}}$$

where V_0 is the upstream velocity far ahead of the sphere.

FIND Determine the acceleration experienced by fluid particles as they flow along this streamline.



(*a*)



(*b*)

■ **Figure E4.4**

SOLUTION

Along streamline A–B there is only one component of velocity ($v = w = 0$) so that from Eq. 4.3

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\hat{\mathbf{i}}$$

or

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \qquad a_y = 0, \qquad a_z = 0$$

Since the flow is steady, the velocity at a given point in space does not change with time. Thus, $\partial u/\partial t = 0$. With the given velocity distribution along the streamline, the acceleration becomes

$$a_x = u\frac{\partial u}{\partial x} = V_0\left(1 + \frac{R^3}{x^3}\right)V_0[R^3(-3x^{-4})]$$

or

$$a_x = -3(V_0^2/R)\frac{1 + (R/x)^3}{(x/R)^4} \qquad \textbf{(Ans)}$$

COMMENTS Along streamline A–B ($-\infty \le x \le -R$ and $y = 0$) the acceleration has only an x component, and it is negative (a deceleration). Thus, the fluid slows down from its upstream velocity of $\mathbf{V} = V_0\hat{\mathbf{i}}$ at $x = -\infty$ to its stagnation point velocity of $\mathbf{V} = 0$ at $x = -R$, the "nose" of the ball. The variation of a_x along streamline A–B is shown in Fig. E4.4*b*. It is the same result as is obtained in Example 3.1 by using the streamwise component of the acceleration, $a_x = V\,\partial V/\partial s$. The maximum deceleration occurs at $x = -1.205R$ and has a value of $a_{x,max} = -0.610\ V_0^2/R$. Note that this maximum deceleration increases with increasing velocity and decreasing size. As indicated in the following table, typical values of this deceleration can be quite large. For example, the $a_{x,max} = -4.08 \times 10^4\ \text{ft/s}^2$ value for a pitched baseball is a deceleration approximately 1500 times that of gravity.

Object	V_0 (ft/s)	R (ft)	$a_{x,max}$ (ft/s^2)
Rising weather balloon	1	4.0	-0.153
Soccer ball	20	0.80	-305
Baseball	90	0.121	-4.08×10^4
Tennis ball	100	0.104	-5.87×10^4
Golf ball	200	0.070	-3.49×10^5

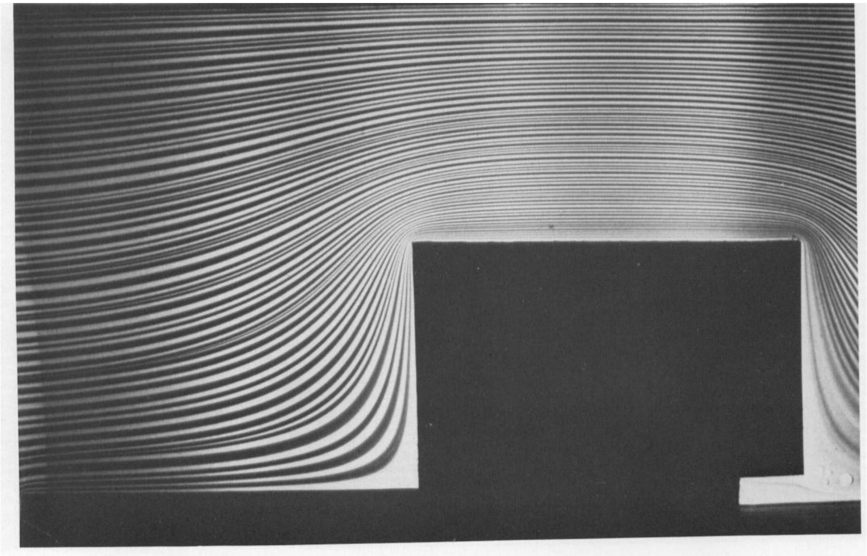
In general, for fluid particles on streamlines other than A–B, all three components of the acceleration (a_x, a_y, and a_z) will be nonzero.

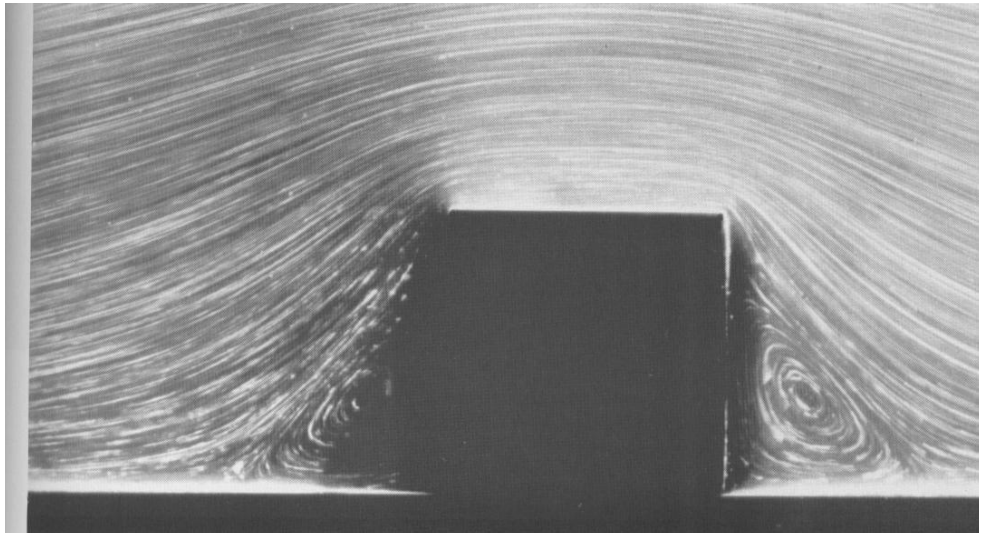
Inviscid Flow (1)

- In real fluids, if there is fluid motion with non-uniform velocity then there will be strain and shear forces
- However, it is often true that these shear forces are much smaller than forces due to pressure gradients or gravity
- In these cases the fluid is assumed to be inviscid (μ=0)

Inviscid Flow (2)

- Inviscid flows are not strongly affected by drag at surfaces and can flow around sharp corners
- Viscid flows are slowed by drag at the surface much more strongly





Inviscid Flow (3)

- Changes in overall velocity or geometry of a problem can change the importance of viscous forces
- Some regions of a flow may be inviscid while others show strong viscous effects