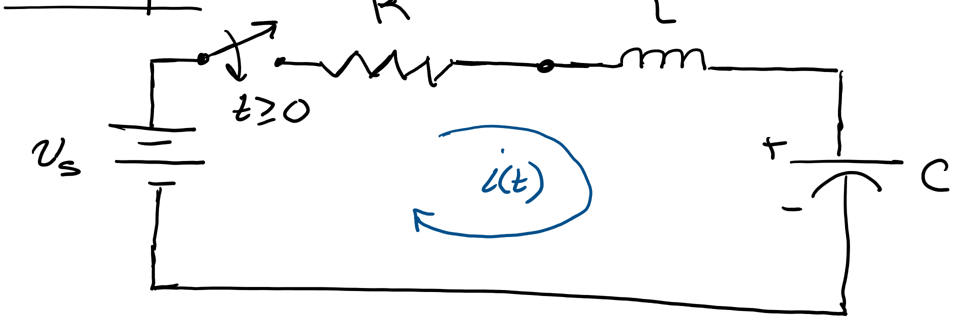
2^{nd} Order Circuits

When two different types of energy storage elements are present in a circuit, the circuit will exhibit a 2^{nd} order transient response when there is a change to one of its inputs.

Example



Find current flow as a function of time.

Apply KVL,

(1) $$v_s - iR - v_L - v_c = 0$$

Recall

(2) $$v_L = L\frac{di_L}{dt}$$

(3) $$i_c = C\frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} = \frac{1}{C}i_c$$

Differentiate (1),

$$\cancel{\frac{dv_s}{dt}} - \frac{di}{dt}R - \frac{dv_L}{dt} - \frac{dv_c}{dt} = 0$$

0^{ι}

$$\Rightarrow \frac{di}{dt}R + L\frac{d^2 i}{dt^2} + \frac{1}{C}i = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

This is a linear, 2nd order, homogeneous ODE.

$$\frac{d^2y}{dt^2} + y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} = -y$$

Fundamental solution

$$i(t) = Ke^{st}$$

Substitute into ODE,

$$s^2 Ke^{st} + \frac{R}{L}sKe^{st} + \frac{1}{LC}Ke^{st} = 0$$

$$\Rightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)Ke^{st} = 0$$

$$\Rightarrow s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$ for nontrivial solution!

$\therefore$

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Two fundamental solutions.
The general solution is the linear combination of the fundamental solutions!

$$i(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

If $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$: $s_{1,2}$ are real and distinct

$$i(t) = K_1 e^{\left(\frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)t} + K_2 e^{\left(\frac{-R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)t}$$

Apply initial conditions to find K_1, K_2

Response is an exponential decay from some initial state

We call this an "overdamped" system

If $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$: $s_{1,2}$ are complex and distinct

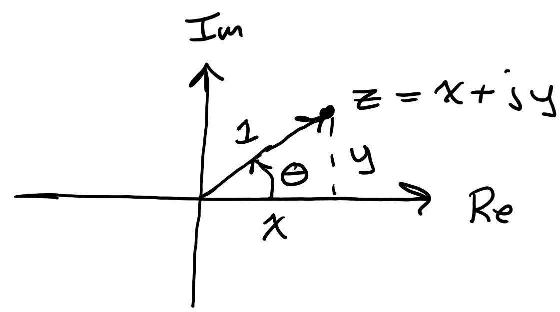
$$s_{1,2} = \frac{-R}{2L} \pm j \underbrace{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}_{\omega} \quad ; \quad j = \sqrt{-1}$$

$\therefore$

$$i(t) = e^{\frac{-R}{2L}t}\left[K_1 e^{j\omega t} + K_2 e^{-j\omega t}\right]$$

Recall Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

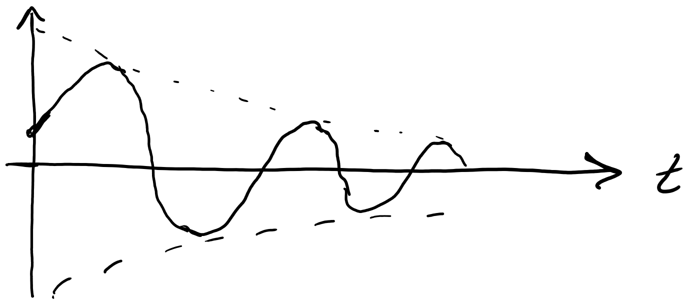


$\therefore$

$$i(t) = e^{\frac{-R}{2L}t}\left[A_1\cos(\omega t) + A_2\sin(\omega t)\right]$$

or rewrite in amplitude and phase form

$$i(t) = e^{\frac{-R}{2L}t} A\cos(\omega t + \phi)$$



Decaying oscillation

We call this an "underdamped" system

If $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$: $S_{1,2}$ are real & repeated

$$S_1 = S_2 = \frac{-R}{2L}$$

$$i(t) = K e^{-\frac{R}{2L}t} + K_2 t e^{-\frac{R}{2L}t}$$

We call this a "critically damped" system