Lecture 6: Reynolds' Transport Theorem

ENAE311H Aerodynamics I

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Introduction

Definition: A fluid system is an arbitrary region/quantity of fluid of fixed identity (c.f. a control volume, which identifies the region in space rather than the fluid itself).

The Reynolds' Transport Theorem relates changes of properties in a fluid system (to which basic physical laws are more easily applied) to changes in properties of the corresponding control volume (which are easier to deal with in practical contexts).

To begin, note that any property of a fluid system may fall into one of two classes:

1. *Extensive properties* depend on the amount of material present, e.g., mass, momentum, energy – we denote these by N.
2. *Intensive properties* are independent of the amount of material present, e.g., pressure, temperature, density, velocity – these we denote by η.

Note that for every extensive property there is a corresponding intensive property, defined by

$$N = \iiint_M \eta dm = \iiint_V \eta \rho dV,$$

where η is the amount of N per unit mass.

Extensive property, N	Mass (M)	Momentum ($\boldsymbol{P}$)	Energy (E)
Intensive property, η	1	Velocity ($\boldsymbol{v}$)	Specific energy (e)

Derivation of RTT

Define a stationary CV that exactly bounds our chosen system at time t. At time $t + \Delta t$, the system will have moved while the CV remains fixed.

For the change in N of the system we have

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \frac{N_s(t+\Delta t) - N_s(t)}{\Delta t}$$

Now, at time t we have

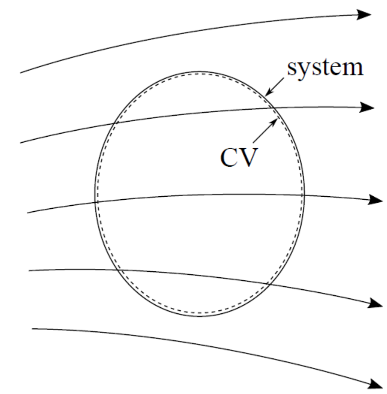
$$N_s = N_{CV}$$

while at time $t + \Delta t$ we have

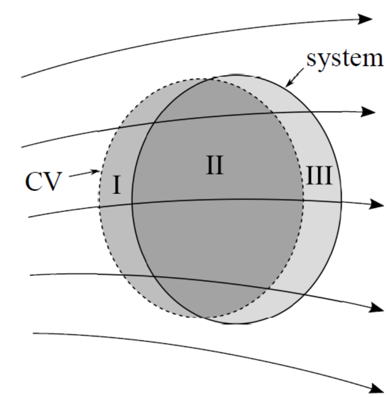
$$N_s = N_{II} + N_{III} = N_{CV} - N_I + N_{III}.$$

We can thus write the above derivative as

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left[\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} - \frac{N_I(t+\Delta t)}{\Delta t} + \frac{N_{III}(t+\Delta t)}{\Delta t} \right]$$



(a) time t



(b) time $t+\Delta t$

Derivation of RTT

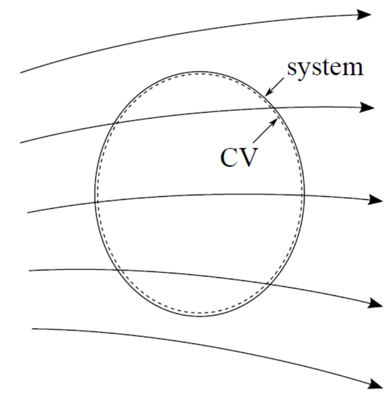
Let us look at the terms on the RHS individually….

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left[\boxed{\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t}} - \frac{N_I(t+\Delta t)}{\Delta t} + \frac{N_{III}(t+\Delta t)}{\Delta t} \right]$$

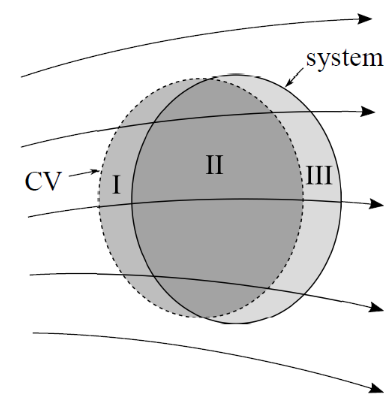
Since the CV is stationary, we have

$$\lim_{\Delta t \to 0} \frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} = \frac{\partial N_{CV}}{\partial t}$$

$$= \frac{\partial}{\partial t} \iiint_{CV} \eta \rho dV.$$



(a) time t



(b) time $t+\Delta t$

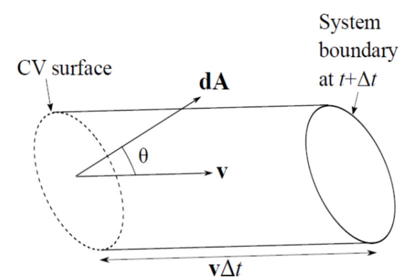
Derivation of RTT

Let us look at the terms on the RHS individually….

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left[\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} - \frac{N_I(t+\Delta t)}{\Delta t} + \boxed{\frac{N_{III}(t+\Delta t)}{\Delta t}} \right]$$

For region III, consider a small subregion dN_{III} at time $t+\Delta t$, as shown to the lower right. The side length is $||\boldsymbol{v}||\Delta t$, and thus the volume of the element is

$$dV = dA||\mathbf{v}||\Delta t \cos\theta = \mathbf{v} \cdot \mathbf{dA}\Delta t$$

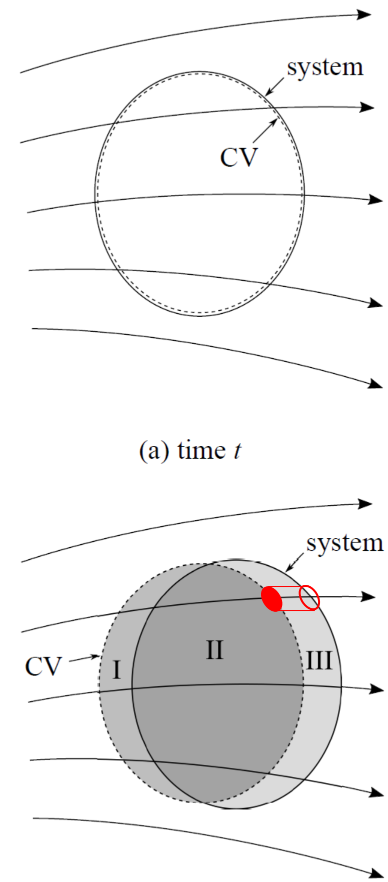


Note that the amount of N in dN_{III} is

$$dN_{III}(t+\Delta t) = \eta\rho dV.$$

We can thus integrate over the entire region III to obtain

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{N_{III}(t+\Delta t)}{\Delta t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint_{CS_{III}} dN_{III}(t+\Delta t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint_{CS_{III}} \eta\rho\mathbf{v} \cdot \mathbf{dA}\Delta t \\ &= \iint_{CS_{III}} \eta\rho\mathbf{v} \cdot \mathbf{dA}.\end{aligned}$$



(a) time t

(b) time t+Δt

Derivation of RTT

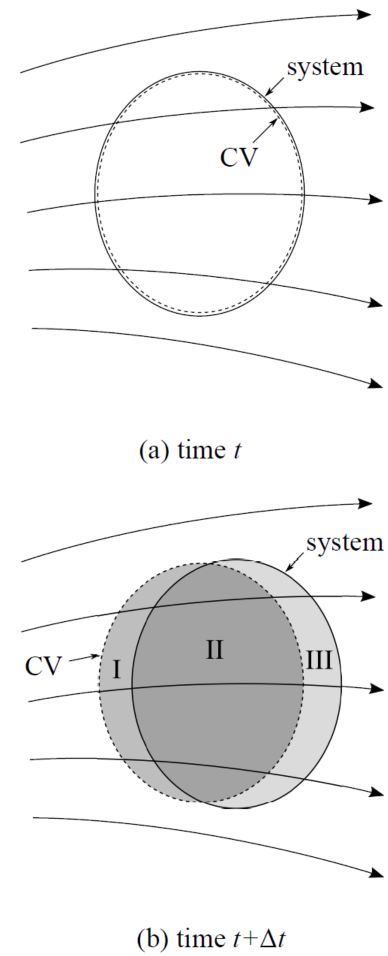
Let us look at the terms on the RHS individually….

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0}\left[\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} - \boxed{\frac{N_I(t+\Delta t)}{\Delta t}} + \frac{N_{III}(t+\Delta t)}{\Delta t}\right]$$

Similarly, for region I we obtain

$$\lim_{\Delta t \to 0}\frac{N_I(t+\Delta t)}{\Delta t} = -\iint_{CS_I} \eta\rho\mathbf{v}\cdot\mathbf{dA}.$$

Here the negative sign is necessary because the velocity vector $\boldsymbol{v}$ and the outward normal vector $\hat{\boldsymbol{n}}$ point in opposite directions (hence the dot product is negative), but we require N_I to be positive.



Derivation of RTT

Combining these results, we have

$$
\begin{aligned}
\boxed{\frac{dN_s}{dt}} &= \frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV + \iint_{CS_I} \eta\rho\mathbf{v}\cdot\mathbf{dA} + \iint_{CS_{III}} \eta\rho\mathbf{v}\cdot\mathbf{dA} \\
&= \boxed{\frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV} + \boxed{\iint_{CS} \eta\rho\mathbf{v}\cdot\mathbf{dA},}
\end{aligned}
$$

since CS_I and CS_{III} constitute the entire control surface.

This is the Reynolds Transport Theorem, which relates the rate of change of an extensive property of a fluid system with variations of the associated intensive property within (and through the boundaries of) the corresponding control volume.

Physical interpretations:

- Rate of change of system extensive property, e.g., mass, momentum
- Rate of change of N inside the control volume
- Rate of flux of N across the control-volume boundaries

Conservation of mass (continuity)

For the case of mass, $N = M$ and $\eta = 1$. Since the mass of a fluid system doesn't change, we have

$$\frac{dM_s}{dt} = 0$$

In this case then, the RTT becomes

$$\underbrace{\frac{\partial}{\partial t}\iiint_{CV} \rho dV}_{\substack{\text{rate of change of} \\ \text{mass within CV}}} + \underbrace{\iint_{CS} \rho \mathbf{v} \cdot \mathbf{dA}}_{\substack{\text{Net mass flux} \\ \text{through CV boundaries}}} = 0.$$

This is the integral form of the conservation of mass equation (sometimes known as the continuity equation). It is often useful for solving the flow through macroscopic configurations.

Conservation of mass – differential form

We can also derive a corresponding differential form of the continuity equation.

Starting from the integral form, for a stationary CV we have

$$\frac{\partial}{\partial t}\iiint_{CV} \rho dV = \iiint_{CV} \frac{\partial \rho}{\partial t} dV.$$

From the divergence theorem, we can write the second term on the RHS of the integral equation as

$$\iint_{CS} (\rho \mathbf{v}) \cdot \mathbf{dA} = \iiint_{CV} \nabla \cdot (\rho \mathbf{v}) dV.$$

Combining these results, we have

$$\iiint_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right] dV = 0.$$

Since the CV is arbitrary, this statement can only hold in general if the integrand in [] is identically zero, i.e.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.}$$

Differential form of the continuity equation

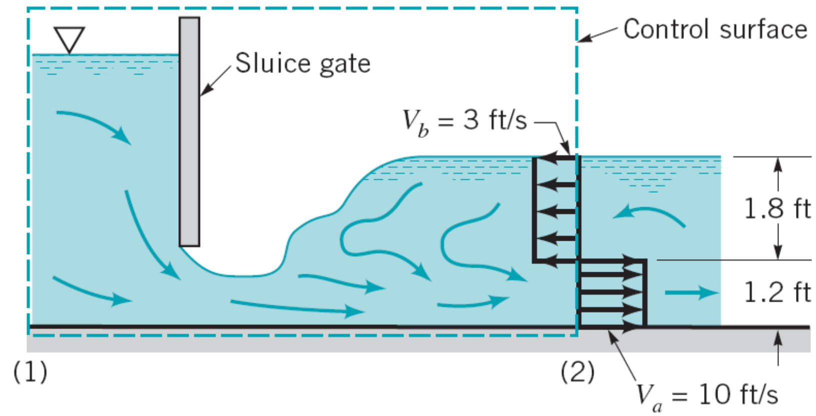
Conservation of mass – differential form

Note that if the flow is steady ($\frac{\partial}{\partial t} = 0$) and incompressible ($\rho =$ const.), the differential form of the continuity equation reduces to simply

$$\nabla \cdot v = 0$$

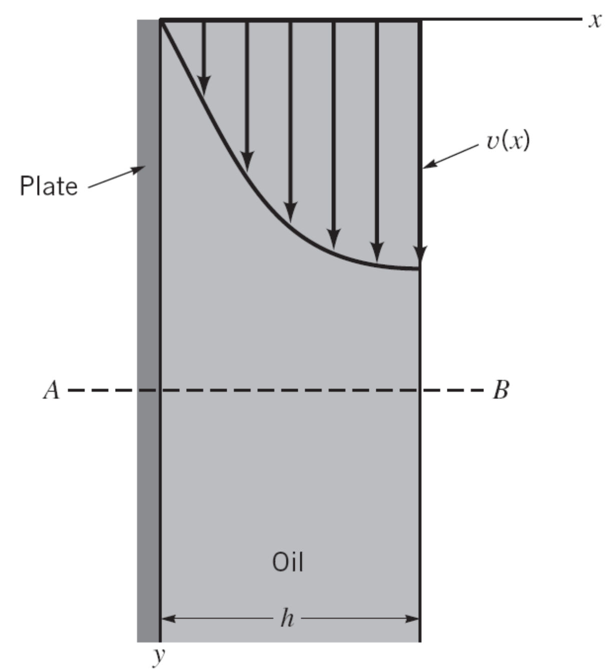
We thus see that, for a steady, incompressible flow, the velocity field is divergence-free.

4.51 In the region just downstream of a sluice gate, the water may develop a reverse flow region as is indicated in Fig. P4.51 and Video V10.5. The velocity profile is assumed to consist of two uniform regions, one with velocity $V_a = 10$ fps and the other with $V_b = 3$ fps. Determine the net flowrate of water across the portion of the control surface at section (2) if the channel is 20 ft wide.



$$Q = V_a A_a - V_b A_b = \left(10\ \frac{\text{ft}}{\text{s}}\right)(1.2\ \text{ft})(20\ \text{ft}) - \left(3\ \frac{\text{ft}}{\text{s}}\right)(1.8\ \text{ft})(20\ \text{ft}) = \underline{\underline{132\ \frac{\text{ft}^3}{\text{s}}}}$$

4.55 A layer of oil flows down a vertical plate as shown in Fig. P4.55 with a velocity of $\mathbf{V} = (V_0/h^2)(2hx - x^2)\,\hat{\mathbf{j}}$ where V_0 and h are constants. (a) Show that the fluid sticks to the plate and that the shear stress at the edge of the layer ($x = h$) is zero. (b) Determine the flowrate across surface AB. Assume the width of the plate is b. (Note: The velocity profile for laminar flow in a pipe has a similar shape. See Video V6.6.)



a) $v = \left(\dfrac{V_0}{h^2}\right)(2hx - x^2)$

Thus,

$$v\big|_{x=0} = \left(\frac{V_0}{h^2}\right)(0-0) = 0 \text{ and}$$

$$\tau\big|_{x=h} = \mu \left.\frac{dv}{dx}\right|_{x=h} = \mu \frac{V_0}{h^2}[2h - 2x]\big|_{x=h} = 0$$

Hence, the fluid sticks to the plate and there is no shear stress at the free surface.

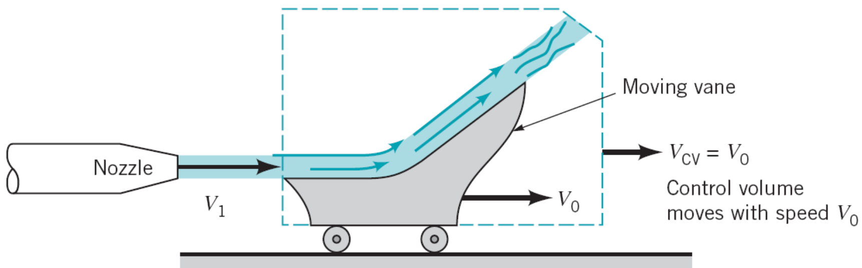
b) $Q_{AB} = \displaystyle\int v\,dA = \int_{x=0}^{x=h} v\,b\,dx = \int_0^h \frac{V_0}{h^2}(2hx - x^2)\,b\,dx$

or

$$Q_{AB} = \frac{V_0 b}{h^2}\left[hx^2 - \frac{1}{3}x^3\right]_0^h = \underline{\underline{\frac{2}{3}V_0 hb}}$$

Deforming Control Volumes

- Reynolds Transport Theorem can be applied to any CV even if it moves



- V is relative velocity
- $= V_1 - V_0$

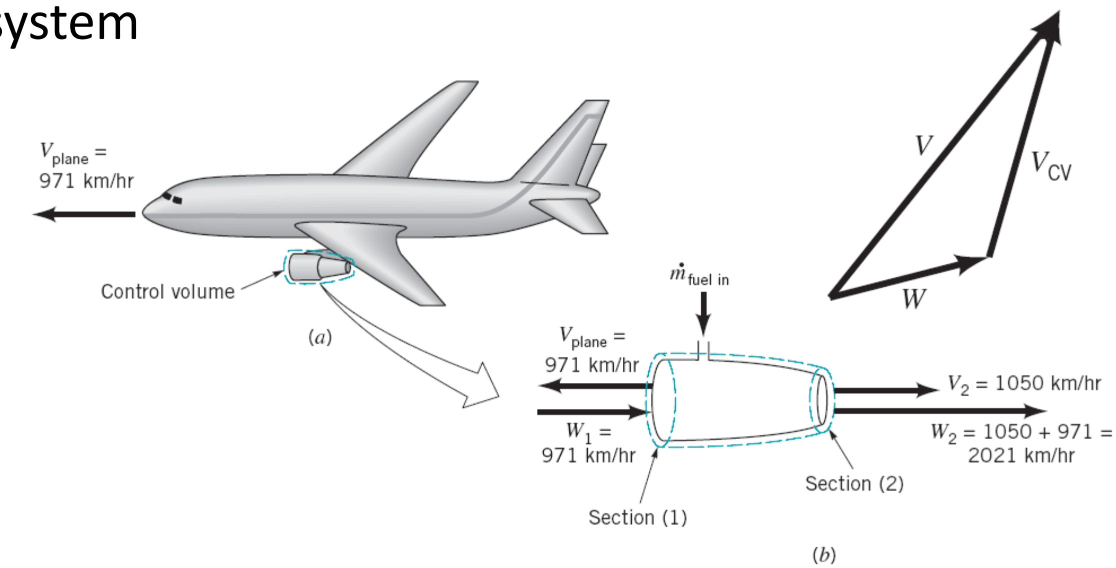
$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b d\forall + \int_{cs} \rho b \vec{V} \cdot \hat{n} dA$$

Moving Control Volumes

- If the control volume is moving, the coordinate system is fixed to the control volume
- The velocity of flow across the control surface is evaluated relative to this coordinate system

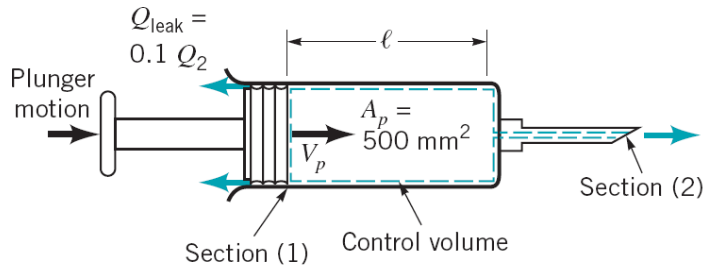
$$\vec{W} = \vec{V} - \vec{V}_{cv}$$

$$\frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{W} \cdot \hat{n} dA = 0$$



Deforming Control Volumes

- Deforming control volumes are both moving and unsteady
- Local relative velocities used at all surfaces



$$\frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{W} \cdot \hat{n} dA = 0$$

Deforming Control Volumes

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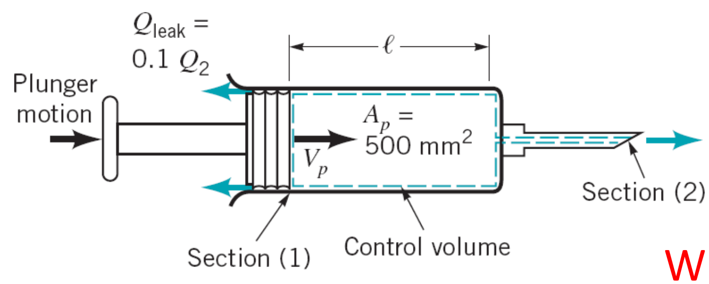
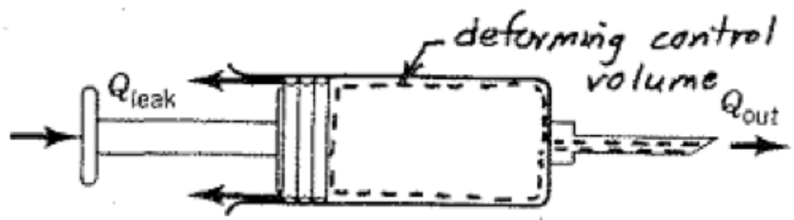


What is the velocity V_2?

- D_1 and D_2 given

$$\frac{\partial}{\partial t}\int_{cv}\rho d\forall+\int_{cs}\rho\vec{W}\cdot\hat{n}dA=0$$



Using a deforming control volume and the conservation of mass principle

$$\frac{DM_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\cancel{V} + \int_{CS} \rho \mathbf{W}\cdot\mathbf{n}dA = 0$$

we obtain $\quad -\rho A_1 V_p + \rho Q_2 + \rho Q_{leak} = 0 \quad (1)$

Since $\rho = \text{constant}$, $Q_{leak} = 0.1 Q_2$ and $Q_2 = A_2 V_2$

we obtain from Eq. 1

$$1.1 A_2 V_2 = A_1 V_p$$

or

$$V_2 = \left(\frac{A_1}{A_2}\right)\frac{V_p}{1.1} = \left(\frac{d_1^2}{d_2^2}\right)\frac{V_p}{1.1} = \frac{(20\text{ mm})^2\left(20\,\frac{\text{mm}}{\text{s}}\right)}{(0.7\text{ mm})^2(1.1)\left(1000\,\frac{\text{mm}}{\text{m}}\right)}$$

and

$$\underline{\underline{V_2 = 14.8\,\frac{\text{m}}{\text{s}}}}$$