1

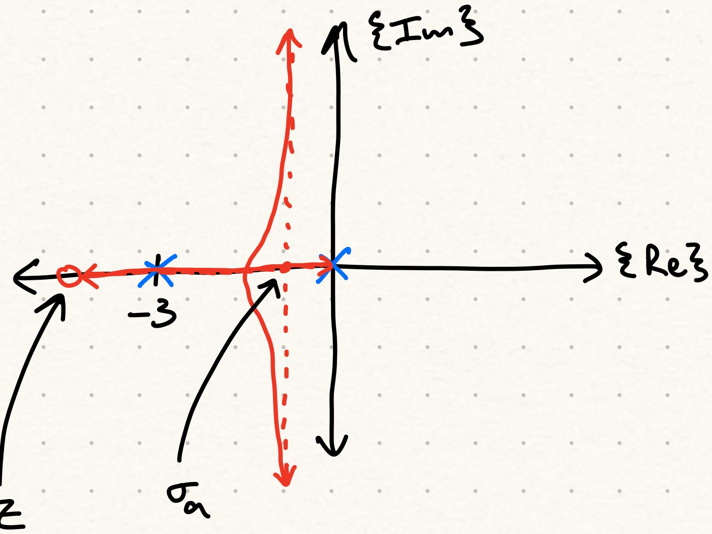
$$G(s)=\frac{6}{(s+3)^2}, \quad H(s)=K_p+\frac{K_i}{s}$$
$$= K_p\left(\frac{s+\frac{K_i}{K_p}}{s}\right)$$

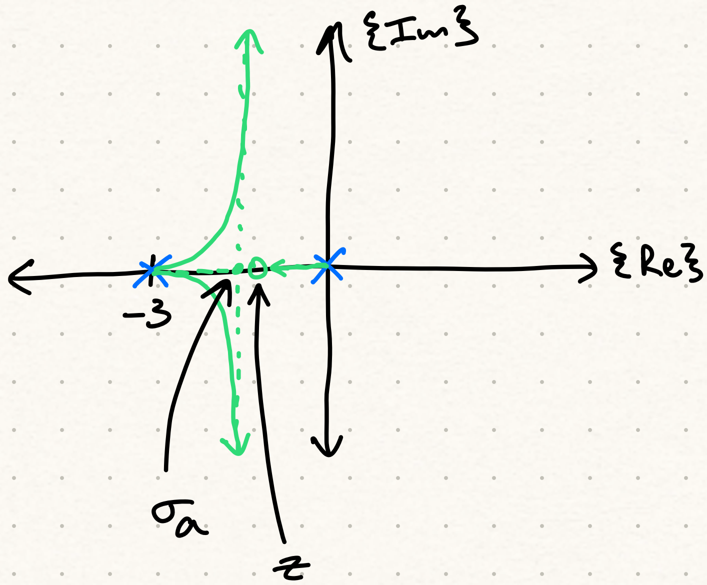
a) i: $(-\infty,-3)$
ii: $(-3,0)$
iii: $(0,+\infty)$

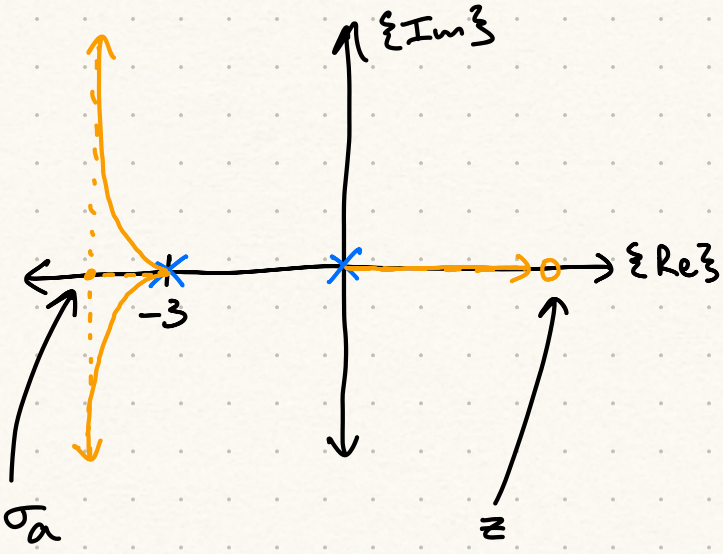
$$L(s)=K\left(\frac{s+z}{(s+3)^2 s}\right), \quad K=6K_p, \quad z=\frac{K_i}{K_p}$$
$$D(s)+KN(s)=0$$
$$s^3+6s^2+9s+Ks+Kz=0$$

$$n=3,\ m=1 \Rightarrow \sigma_a=\frac{-3-3+0-\left(-\frac{K_i}{K_p}\right)}{2}=-3+\frac{K_i}{2K_p}$$
$$\alpha_l=\pm 90^\circ$$

OL poles
case i
case ii
case iii







b) case iii is unstable, as it mandates a singular RHP zero and its LHP branch is unstable

c) Want: $\alpha < 0$

$$\alpha = \frac{-6+z}{2} \Rightarrow z < 6$$

$z > 0$ to avoid case iii

$\hookrightarrow 0 < z < 6$

$\hookrightarrow$ $\boxed{\text{zero must lie between } (-6, 0)}$

Need: $\measuredangle L(s) \neq -180°$

Let: $\measuredangle L(s) = 180°$, $\operatorname{atan}\left(\frac{\omega}{z}\right) - \frac{\pi}{2} - 2\operatorname{atan}\left(\frac{\omega}{3}\right) = -\pi$

$$-\frac{\pi}{2} = \operatorname{atan}\left(\frac{\omega}{z}\right) - 2\operatorname{atan}\left(\frac{\omega}{3}\right)$$

$$\Rightarrow \frac{\omega}{z} = \frac{9-\omega^2}{-6\omega} \Rightarrow \underbrace{\omega^2 = -\frac{9z}{6-z} > 0}_{\text{want this to be false}}$$

$\hookrightarrow 0 < z < 6$

$\hookrightarrow$ zero must lie between $(-6, 0)$

d) $\zeta = \frac{1}{\sqrt{2}}$, cancel a pole : $z = 3$

$\omega_n = \frac{z}{2\zeta}$, $K = \omega_n^2 / 6$

$K_p = K \rightarrow \boxed{K_p = 0.75}$

$K_i = K_p z \rightarrow \boxed{K_i = 2.25}$

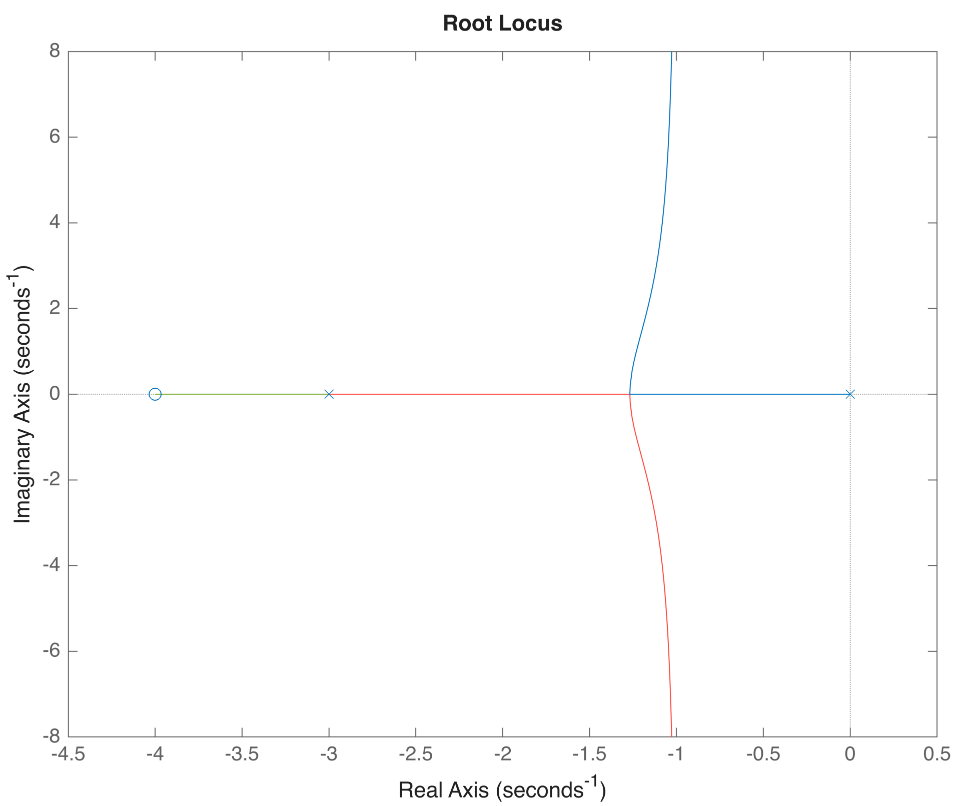
problem 1

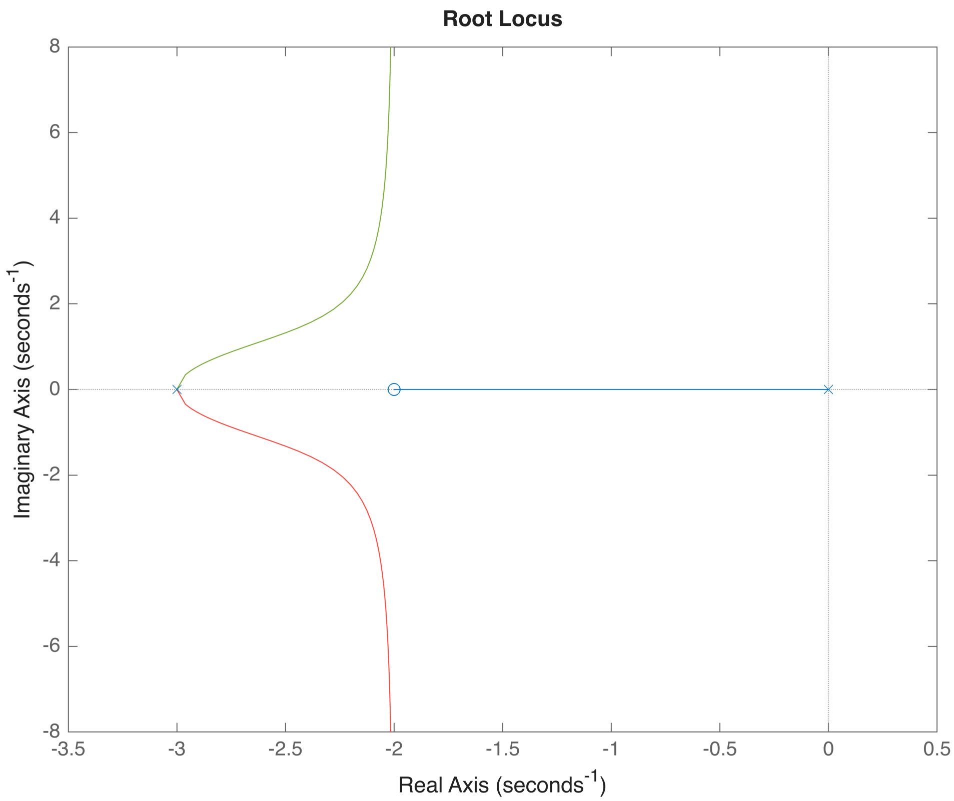
```
% givens
s  = zpk('s');
G = 6/(s+3)^2;

% display
G
```

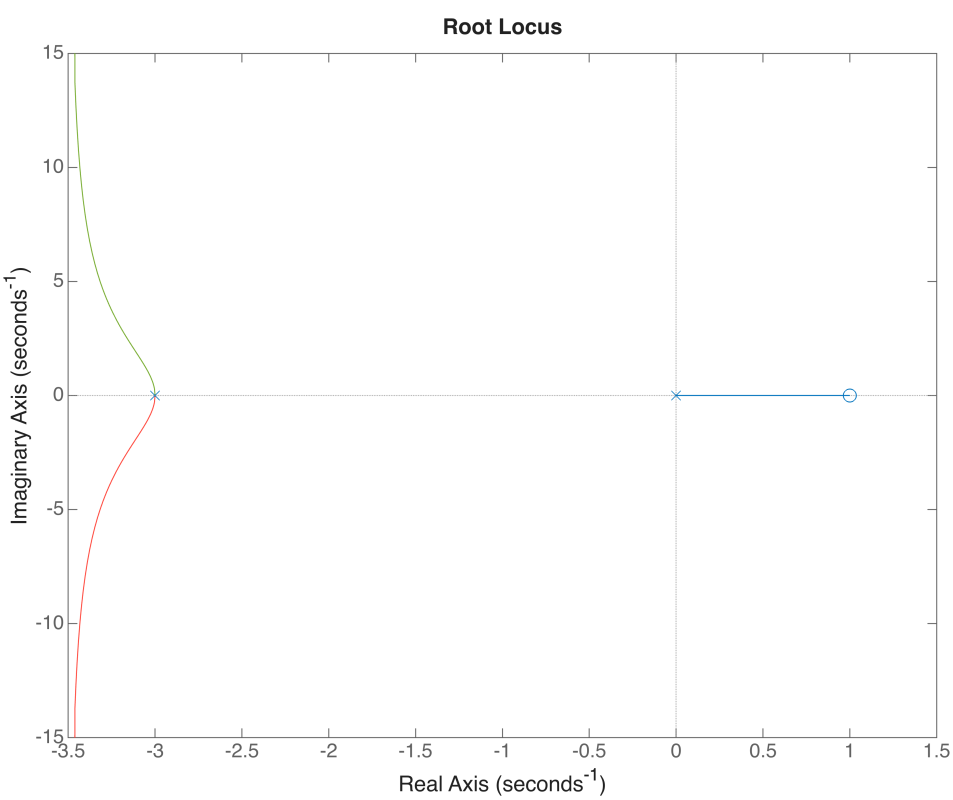
```
G =

     6
  -------
  (s+3)^2

Continuous-time zero/pole/gain model.
Model Properties
```

part a

```
z = [-4, -2, 1];
for i=1:length(z)
        H0 = (s-z(i))/s;
        L0 = H0*G;
        fig = figure;
        rlocusplot(L0);
        saveas(fig, string('./images/s01a' + string(i) + '.png'));
end
```




Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
8
6
4
2
0
-2
-4
-6
-8
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
15
10
5
0
-5
-10
-15
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5

part d

```
zeta = 1/sqrt(2);
z = 3;
wn = 3/(2*zeta);
K  = wn^2/6;
Kp = K;
Ki = K*z;
H  = pid(Kp, Ki);
T  = feedback(G*H,1);

Kp, Ki, H, T
```

```
Kp =
0.7500
Ki =
2.2500
H =

             1
  Kp + Ki * ---
             s

  with Kp = 0.75, Ki = 2.25

Continuous-time PI controller in parallel form.
Model Properties

T =

        4.5 (s+3)
  ----------------------
  (s+3) (s^2 + 3s + 4.5)

Continuous-time zero/pole/gain model.
Model Properties
```

2

$$G(s) = \frac{5(s-1)}{s-6}$$

a) $H(s) = \dfrac{K}{s-p}$, with $p>0$

$$L(s) = \frac{5K(s-1)}{(s-6)(s-p)} \Rightarrow 1 + L(s) = 0$$

$$(s-p)(s-6) + 5K(s-1) = 0 \Rightarrow s^2 + (-p-6)s + 6p + 5Ks - 5K = 0$$

$\hookrightarrow$ $5K - (p+6) > 0$ & $6p - 5K > 0$

$$\frac{p+6}{5} < K < \frac{6}{5}p \Rightarrow \boxed{p > \frac{6}{5}}$$

b) Want: 2 real poles @ -2

CL: $s^2 + (5K - (p+6))s + (6p - 5K)$

T_{des}: $(s+2)^2 = s^2 + 4s + 4$

$$\left[\begin{array}{ccc|c} 1, & 5K-(p+6), & 6p-5K & 0 \\ 1, & 4, & 4 & 0 \end{array}\right]$$

$$p = \frac{5K+4}{6}$$

$$5K - \left(\frac{5K+4}{6} + 6\right) = 4$$

$$5K = 10 + \frac{5K+4}{6}$$

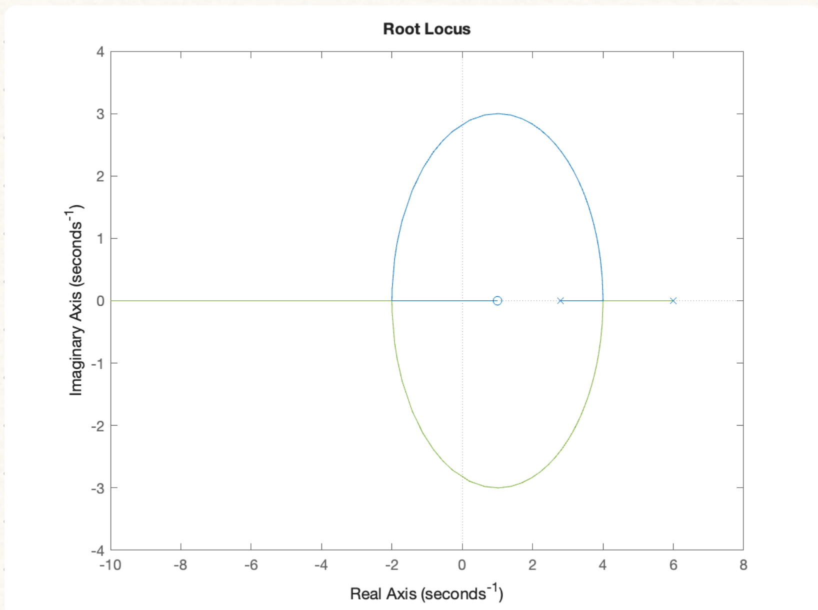
$$30K = 60 + 5K + 4$$

$$25K = 64$$

$$K = \left(\frac{8}{5}\right)^2 = \frac{64}{25}$$

$$p = \frac{\frac{64}{5} + 4}{6} = \frac{84}{30} = \frac{14}{5}$$

$$\boxed{H(s) = \frac{64}{25}\left(\frac{1}{s - \frac{14}{5}}\right)}$$

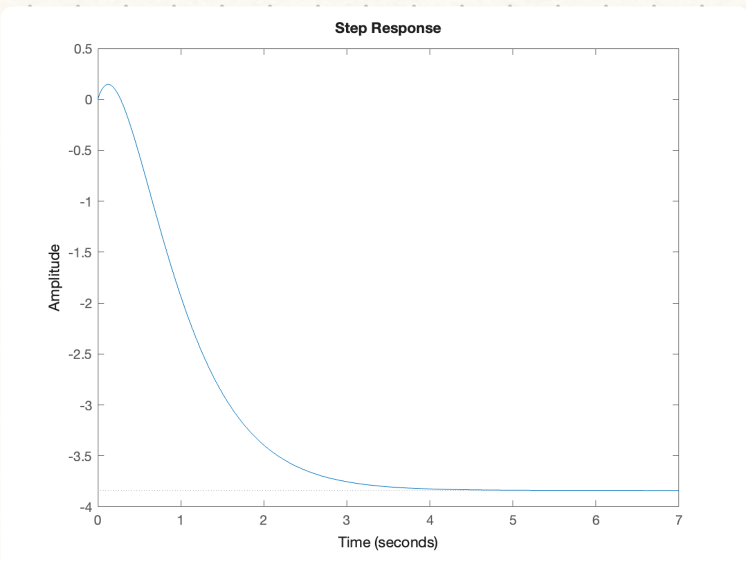


c) $y_d(t) = \mathbb{1}(t)$, $R(s) = \dfrac{H(s)}{1+L(s)}$, $U(s) = R(s)\,Y_d(s)$

$$Y_d(s) = \frac{1}{s} \Rightarrow U(s) = \frac{R(s)}{s} = \frac{H(s)}{s(1+L(s))} = \frac{2.56(s-6)}{s(s+2)^2}$$

$$\boxed{u(t) = \frac{96}{25}\left(e^{-2t} + \frac{8}{3}te^{-2t} - 1\right)}$$

u(t) is bounded, as all poles of R(s) are in LHP




```
R_info = struct with fields:
        RiseTime: 1.6515
   TransientTime: 3.0421
    SettlingTime: 3.0640
     SettlingMin: -3.8400
     SettlingMax: -3.4571
       Overshoot: 0
      Undershoot: 3.8196
            Peak: 3.8400
        PeakTime: 7.8058
```

d)

```
H_zoh = struct with fields:
    Ad: 1.1185
    Bd: 0.0847
    Cd: 1.2800
    Dd: 0
H_tustin = struct with fields:
    Ad: 1.1186
    Bd: 0.3639
    Cd: 0.3158
    Dd: 0.0542
```

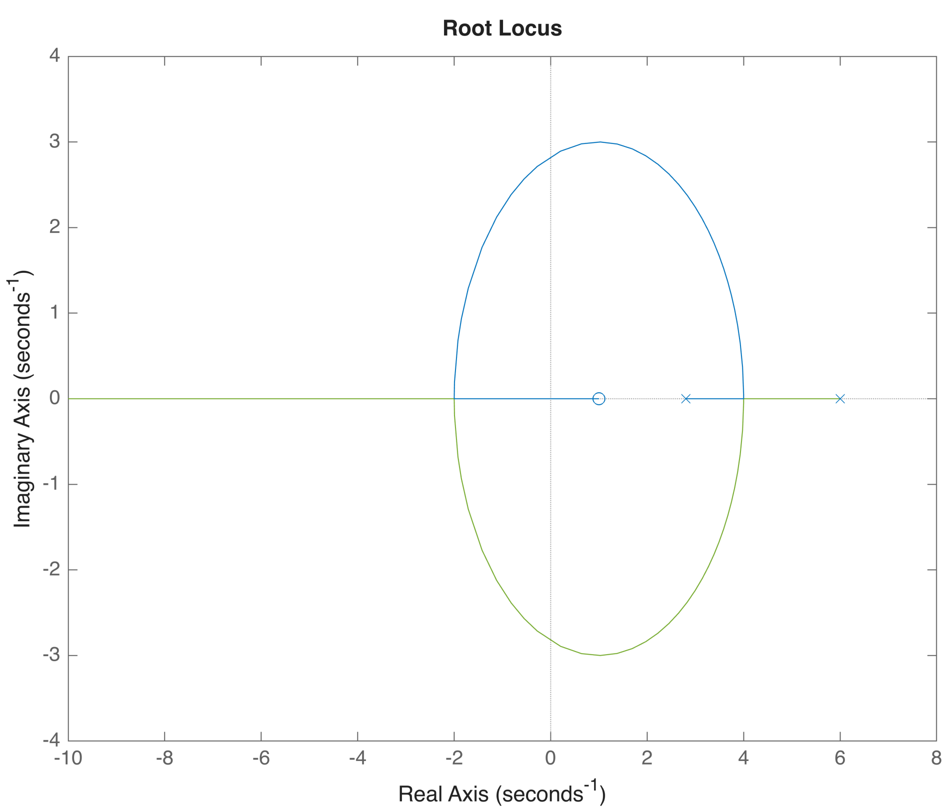
problem 2

```
% givens
s = zpk('s');
G = 5*(s-1)/(s-6);

% display
G
```

```
G =

  5 (s-1)
  -------
   (s-6)

Continuous-time zero/pole/gain model.
Model Properties
```

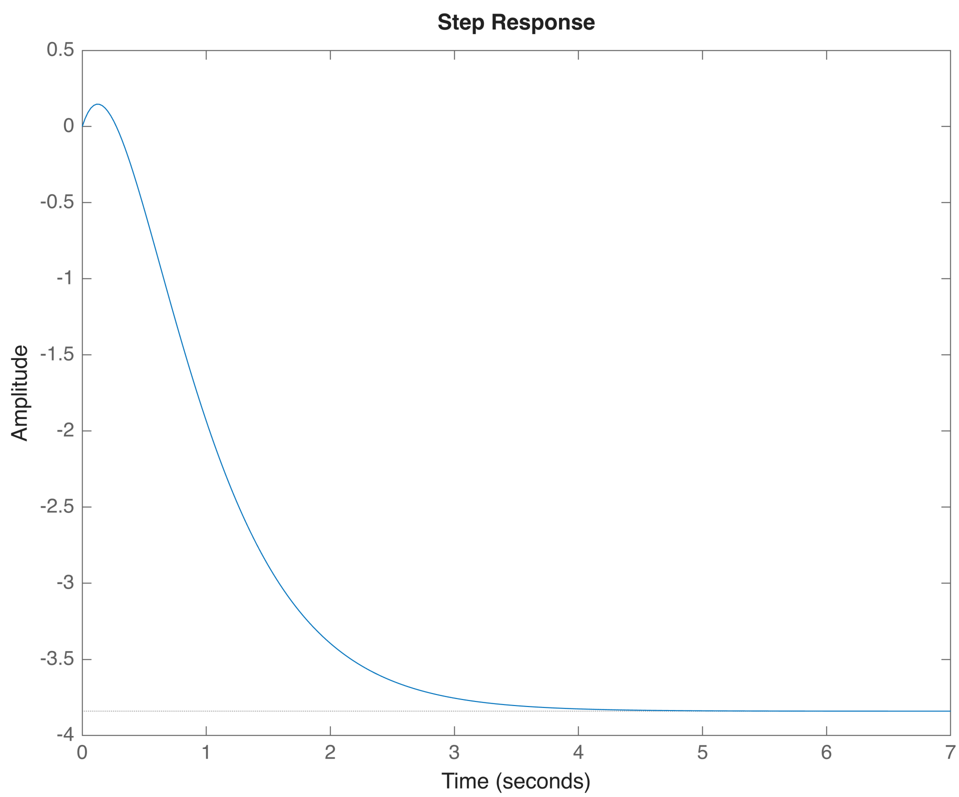
part b

```
K = 64/25;
p = 14/5;
H = K/(s-p);
L = H*G;

fig = figure;
rlocusplot(L);
saveas(fig, './images/s02b.png');
```



part c

```
syms s_sym

R = feedback(H, G);
R_info = stepinfo(R);

fig = figure;
stepplot(R);
saveas(fig, './images/s02c.png');
```




```
[R_num, R_den] = tfdata(R);
RN = poly2sym(R_num, s_sym);
RD = poly2sym(R_den, s_sym);
R_sym = RN/RD;
U_sym = R_sym/s_sym;
ut_sym = ilaplace(U_sym)
```

ut_sym =

$$\frac{96\,e^{-2t}}{25} + \frac{256\,t\,e^{-2t}}{25} - \frac{96}{25}$$

```
% display
R, R_info, ut_sym
```

```
R =

  2.56 (s-6)
  ----------
   (s+2)^2

Continuous-time zero/pole/gain model.
Model Properties
R_info = struct with fields:
         RiseTime: 1.6515
    TransientTime: 3.0421
     SettlingTime: 3.0640
      SettlingMin: -3.8400
      SettlingMax: -3.4571
        Overshoot: 0
       Undershoot: 3.8196
             Peak: 3.8400
         PeakTime: 7.8058
ut_sym =
```

$$\frac{96\,e^{-2\,t}}{25}+\frac{256\,t\,e^{-2\,t}}{25}-\frac{96}{25}$$

part d

```
% assuming sample time Ts = 1 second
Ts = 1/25;

% solution
[Ah, Bh, Ch, Dh] = ssdata(canon(H));
H_zoh.Ad = expm(Ah*Ts);
H_zoh.Bd = Ah \ (H_zoh.Ad-eye(size(Ah)))*Bh;
H_zoh.Cd = Ch;
H_zoh.Dd = Dh;

[Ad, Bd, Cd, Dd] = ssdata(c2d(H, Ts, 'tustin'));
H_tustin = struct('Ad', Ad, 'Bd', Bd, 'Cd', Cd, 'Dd', Dd);

% display tustin
H_zoh, H_tustin
```

```
H_zoh = struct with fields:
    Ad: 1.1185
    Bd: 0.0847
    Cd: 1.2800
    Dd: 0
H_tustin = struct with fields:
    Ad: 1.1186
    Bd: 0.3639
    Cd: 0.3158
    Dd: 0.0542
```

problem 3

```
% givens
s = zpk('s');
H = 15*(3*s+1)^5/((s+1)^3*(s^2+2*s+10));
Ts = 1/25;

% display
H, Ts
```

```
H =

     3645 (s+0.3333)^5
  -----------------------
  (s+1)^3 (s^2 + 2s + 10)

Continuous-time zero/pole/gain model.
Model Properties
Ts =
0.0400
```

part a

```
% solution
[Ah, Bh, Ch, Dh] = ssdata(canon(H));
Ad = expm(Ah*Ts);
Bd = Ah \ (Ad-eye(size(Ah)))*Bh;
Cd = Ch;
Dd = Dh;

% display
Ad, Bd, Cd, Dd
```

```
Ad = 5×5
    0.9608   -0.0256   -0.0253         0         0
         0    0.9608   -0.0256         0         0
         0         0    0.9608         0         0
         0         0         0    0.9539    0.1150
         0         0         0   -0.1150    0.9539
Bd = 5×1
    0.5968
    0.3022
    0.1012
    0.5486
    9.1941
Cd = 1×5
  -46.5020  -46.5020  -46.5020  -24.1168  -46.6136
Dd =
3.6450e+03
```

part b

```
% solution
[Ad, Bd, Cd, Dd] = ssdata(c2d(H, Ts, 'tustin'));

% display
```

```
Ad, Bd, Cd, Dd
```

```
Ad = 5×5
    0.9608   -0.0260   -0.0260   -0.0076   -0.0409
         0    0.9608   -0.0260   -0.0076   -0.0409
         0         0    0.9608   -0.0076   -0.0409
         0         0         0    0.9540    1.0000
         0         0         0   -0.0132    0.9540
Bd = 5×1
   10.3139
   10.3139
   10.3139
         0
   16.5140
Cd = 1×5
   -8.5642   -8.5642   -8.5642   -2.5004  -13.4826
Dd =
3.4011e+03
```

4 $\quad G(s) = \dfrac{2}{s^2(s^2+3)}$

a) $H(s) = K$?

$$L(s) = \frac{2K}{s^2(s^2+3)}, \quad L'(s) = [(s-p)L(s)]$$

$$n-m=4 \Rightarrow \alpha_\ell = \pm 45°, \pm 135° \ \& \ \sigma_a = 0$$

$$\measuredangle L(s) = (1+2\ell)180° = \measuredangle L'(s) - \measuredangle(s-p)$$

$$\delta = \measuredangle L'(p) - (1+2\ell)180°$$

$$\measuredangle L'(s)\Big|_{\varepsilon\text{-circle}} = \measuredangle L'(p)$$

for pole at $s = +\sqrt{3}j$:

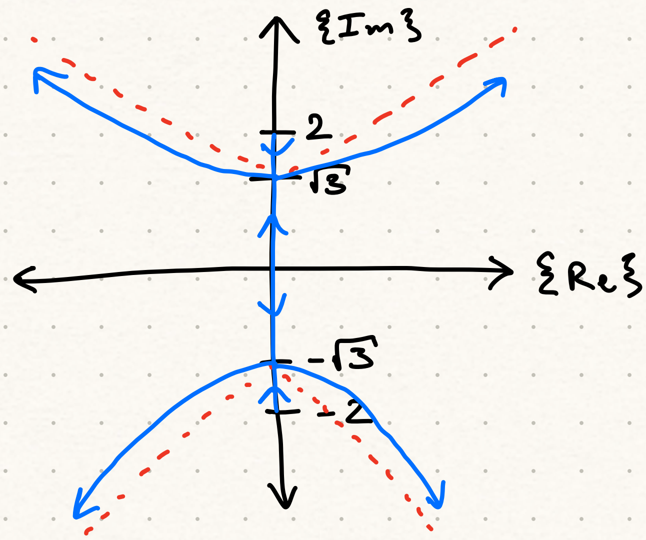
$$L'(s) = \frac{2K}{s^2(s+\sqrt{3}j)} \Rightarrow \measuredangle L'(p) = -180° - 90°$$
$$\delta = -90°$$

by symmetry, for $s = -\sqrt{3}j$: $\delta = 90°$

for pole at $s = 0$ (repeated):

$$L'(s) = \frac{2K}{s^2+3} \Rightarrow \measuredangle L'(p) = -180°$$
$$\delta = \pm 180°$$
$$\hookrightarrow \delta = \pm 90°$$

$\boxed{\text{Will be unstable at all poles for sufficiently large } K}$



b) Want: CL @ $-1 \pm j$ & all else CL @ -4

$$H(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

find: zpk(H(s)) & zpk(T(s))

Char Polys

$$CL: (s^3 + a_2 s^2 + a_1 s + a_0)(s^2(s^2+3)) + (b_3 s^3 + b_2 s^2 + b_1 s + b_0)(2)$$

$$T_{des}: (s-(-1+j))(s-(-1-j))(s+4)^5$$

$\hookrightarrow$ $a_2 = 22$ $\quad b_3 = 1141.5$
$a_1 = 199$ $\quad b_2 = 1031$
$a_0 = 934$ $\quad b_1 = 2304$
$b_0 = 1024$

```
H =

  1141.5 (s+0.4939) (s^2 + 0.4093s + 1.816)
  -----------------------------------------
      (s+11.86) (s^2 + 10.14s + 78.74)

Continuous-time zero/pole/gain model.
Model Properties

T =

  2283 (s+0.4939) (s^2 + 0.4093s + 1.816)
  ---------------------------------------
         (s+4)^5 (s^2 + 2s + 2)

Continuous-time zero/pole/gain model.
Model Properties
poles_T = 7x1 complex
   -4.0075 + 0.0000i
   -4.0023 + 0.0071i
   -4.0023 - 0.0071i
   -3.9939 + 0.0044i
   -3.9939 - 0.0044i
   -1.0000 + 1.0000i
   -1.0000 - 1.0000i

zeros_T = 3x1 complex
   -0.2046 + 1.3321i
   -0.2046 - 1.3321i
   -0.4939 + 0.0000i
```

c) $$H(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

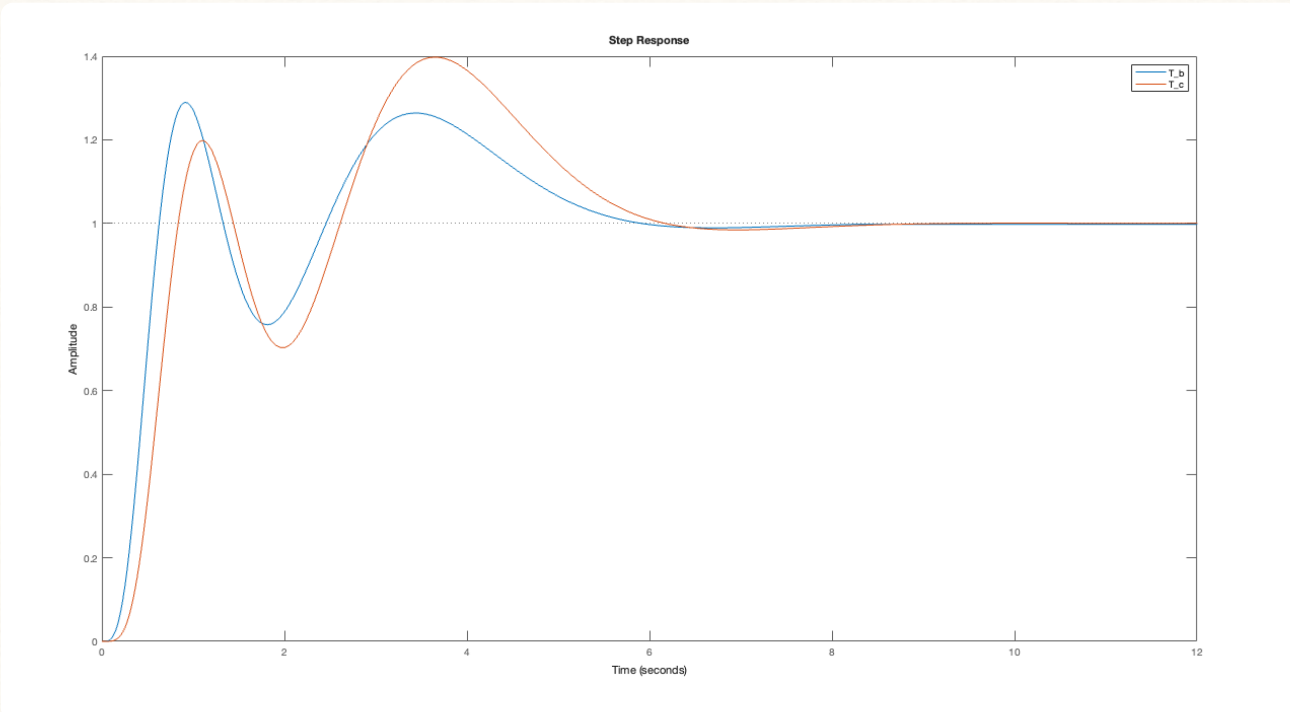
Char Polys

$$CL: (s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)(s^2(s^2+3)) + (b_3 s^3 + b_2 s^2 + b_1 s + b_0)(2)$$

$$T_{des}: (s-(-1+j))(s-(-1-j))(s+4)^6$$

```
H =

    5597 (s+0.4116) (s^2 + 0.125s + 1.778)
  -------------------------------------------
  (s^2 + 20.06s + 115.9) (s^2 + 5.939s + 51.91)

Continuous-time zero/pole/gain model.
Model Properties

T =

  11194 (s+0.4116) (s^2 + 0.125s + 1.778)
  ---------------------------------------
         (s+4)^6 (s^2 + 2s + 2)

Continuous-time zero/pole/gain model.
Model Properties
poles_T = 8x1 complex
   -4.0247 + 0.0000i
   -4.0123 + 0.0214i
   -4.0123 - 0.0214i
   -3.9877 + 0.0213i
   -3.9877 - 0.0213i
   -3.9753 + 0.0000i
   -1.0000 + 1.0000i
   -1.0000 - 1.0000i

zeros_T = 3x1 complex
   -0.0625 + 1.3320i
   -0.0625 - 1.3320i
   -0.4116 + 0.0000i
```

d) Neither controllers are good, as they both have about (or over) 5.5s settling times, and significant overshoot. This is due to the introduction of extra zeroes and their compensatory poles "spoiling" our intended behavior of the designed poles



e) Otherwise, the benefit of C is that it has an extra pole, ensuring it has noise stability, which B does not

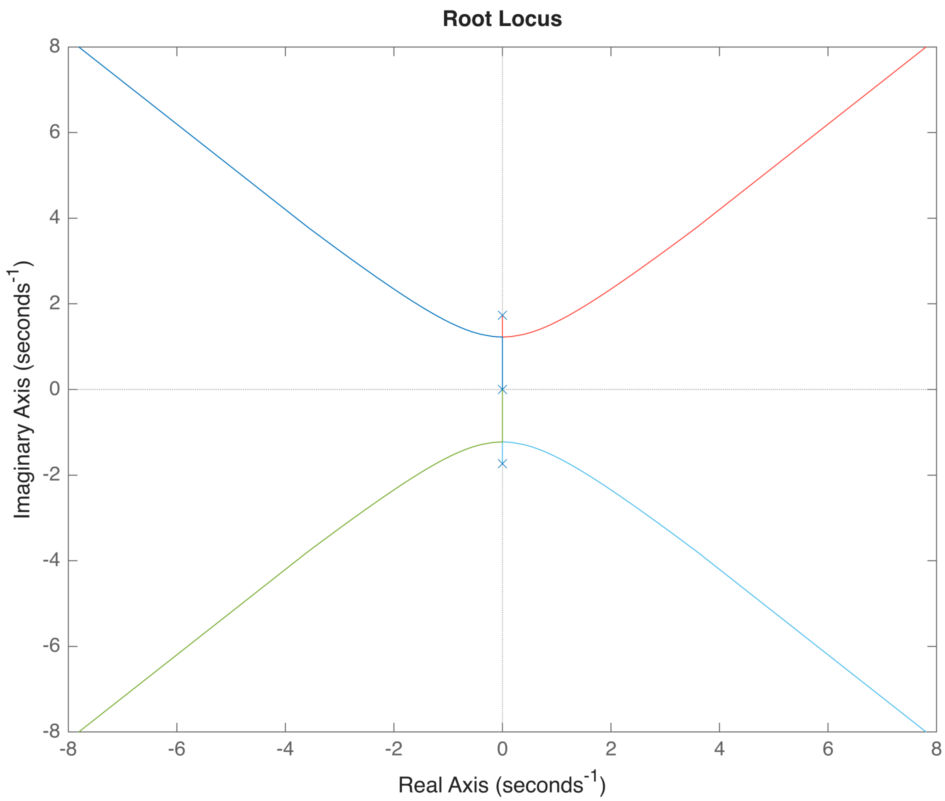
problem 4

```
% givens
s = zpk('s');
G = 2/(s^2*(s^2+3));

% display
G
```

```
G =

        2
  -------------
  s^2 (s^2 + 3)

Continuous-time zero/pole/gain model.
Model Properties
```

part a

```
fig = figure;
rlocusplot(G);
saveas(fig, './images/s04a.png');
```



part b

```
% givens
syms a2 a1 a0 b3 b2 b1 b0 s_sym
```

```
HD = s_sym^3+a2*s_sym^2+a1*s_sym+a0;
HN = b3*s_sym^3+b2*s_sym^2+b1*s_sym+b0;
desired = (s_sym-(-1+1j))*(s_sym-(-1-1j))*(s_sym+4)^5;

% solution
[G_num, G_den] = tfdata(G);
GN = poly2sym(G_num, s_sym);
GD = poly2sym(G_den, s_sym);

CL_den = expand(GD*HD+GN*HN);
coeffCL = coeffs(CL_den, s_sym, 'All');
coeffT  = coeffs(desired, s_sym, 'All');

eqs = coeffCL(2:end) == coeffT(2:end);
sol_sym = solve(eqs, [a2 a1 a0 b3 b2 b1 b0]);
sol = structfun(@double, sol_sym);

H_num = transpose(sol(4:7));
H_den = [1, transpose(sol(1:3))];

Hb = zpk(tf(H_num, H_den));
Lb = G*Hb;
Tb = feedback(Lb, 1);
poles_Tb = pole(Tb);
zeros_Tb = zero(Tb);

% display
Hb, Tb, poles_Tb, zeros_Tb
```

```
Hb =

  1141.5 (s+0.4939) (s^2 + 0.4093s + 1.816)
  -----------------------------------------
      (s+11.86) (s^2 + 10.14s + 78.74)

Continuous-time zero/pole/gain model.
Model Properties

Tb =

  2283 (s+0.4939) (s^2 + 0.4093s + 1.816)
  ---------------------------------------
          (s+4)^5 (s^2 + 2s + 2)

Continuous-time zero/pole/gain model.
Model Properties
poles_Tb = 7×1 complex
  -4.0075 + 0.0000i
  -4.0023 + 0.0071i
  -4.0023 - 0.0071i
  -3.9939 + 0.0044i
  -3.9939 - 0.0044i
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
zeros_Tb = 3×1 complex
  -0.2046 + 1.3321i
```

```
  -0.2046 - 1.3321i
  -0.4939 + 0.0000i
```

part c

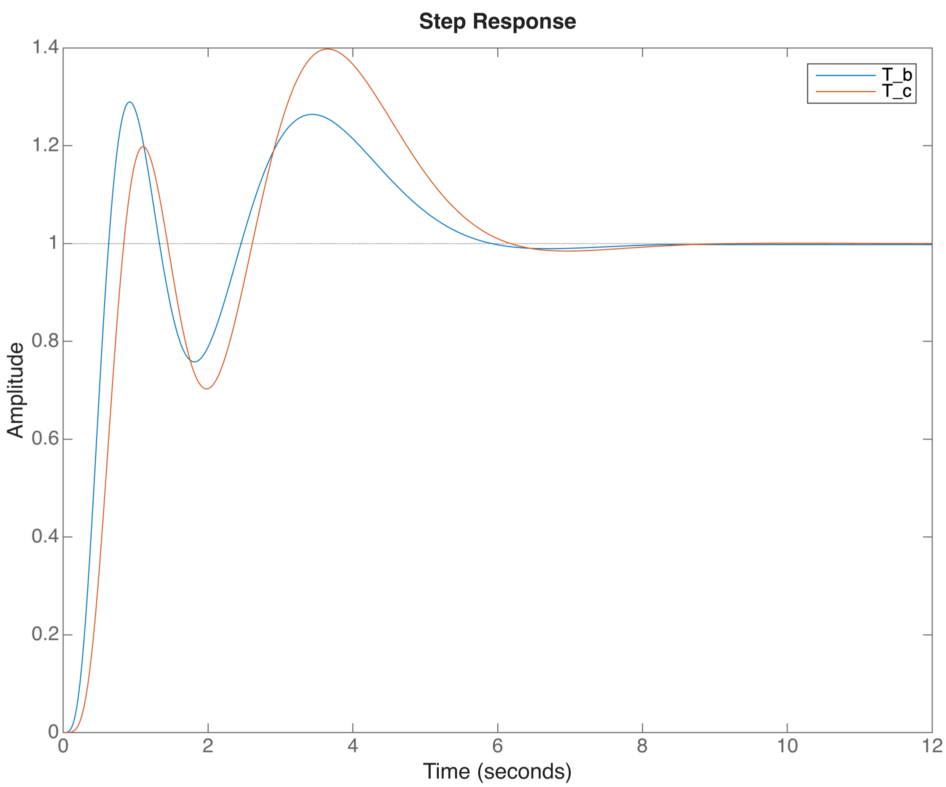
```
% givens
syms a3 a2 a1 a0 b3 b2 b1 b0 s_sym
HD = s_sym^4+a3*s_sym^3+a2*s_sym^2+a1*s_sym+a0;
HN = b3*s_sym^3+b2*s_sym^2+b1*s_sym+b0;
desired = (s_sym-(-1+1j))*(s_sym-(-1-1j))*(s_sym+4)^6;

% solution
[G_num, G_den] = tfdata(G);
GN = poly2sym(G_num, s_sym);
GD = poly2sym(G_den, s_sym);

CL_den = expand(GD*HD+GN*HN);
coeffCL = coeffs(CL_den, s_sym, 'All');
coeffT  = coeffs(desired, s_sym, 'All');

eqs = coeffCL(2:end) == coeffT(2:end);
sol_sym = solve(eqs, [a3 a2 a1 a0 b3 b2 b1 b0]);
sol = structfun(@double, sol_sym);

H_num = transpose(sol(5:8));
H_den = [1, transpose(sol(1:4))];

Hc = zpk(tf(H_num, H_den));
Lc = G*Hc;
Tc = feedback(Lc, 1);
poles_Tc = pole(Tc);
zeros_Tc = zero(Tc);

% display
Hc, Tc, poles_Tc, zeros_Tc
```

```
Hc =

      5597 (s+0.4116) (s^2 + 0.125s + 1.778)
  ---------------------------------------------
  (s^2 + 20.06s + 115.9) (s^2 + 5.939s + 51.91)

Continuous-time zero/pole/gain model.
Model Properties

Tc =

  11194 (s+0.4116) (s^2 + 0.125s + 1.778)
  ---------------------------------------
          (s+4)^6 (s^2 + 2s + 2)

Continuous-time zero/pole/gain model.
Model Properties
poles_Tc = 8×1 complex
```

```
  -4.0247 + 0.0000i
  -4.0123 + 0.0214i
  -4.0123 - 0.0214i
  -3.9877 + 0.0213i
  -3.9877 - 0.0213i
  -3.9753 + 0.0000i
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
zeros_Tc = 3×1 complex
  -0.0625 + 1.3320i
  -0.0625 - 1.3320i
  -0.4116 + 0.0000i
```

part d

```
% solution
fig = figure;
hold on;
stepplot(Tb);
stepplot(Tc);
legend(["T_b" "T_c"]);
hold off;
saveas(fig, './images/s04d.png');
```




```
Tb_info = stepinfo(Tb);
Tc_info = stepinfo(Tc);

% display
```

```
Tb_info, Tc_info
```

```
Tb_info = struct with fields:
         RiseTime: 0.3510
    TransientTime: 5.5002
     SettlingTime: 5.5002
      SettlingMin: 0.7579
      SettlingMax: 1.2897
        Overshoot: 28.9658
       Undershoot: 0
             Peak: 1.2897
         PeakTime: 0.9193
Tc_info = struct with fields:
         RiseTime: 0.4342
    TransientTime: 5.8635
     SettlingTime: 5.8635
      SettlingMin: 0.7028
      SettlingMax: 1.3977
        Overshoot: 39.7657
       Undershoot: 0
             Peak: 1.3977
         PeakTime: 3.6616
```