Counting Encirclements

⇒ Count the number of complete loops the diagram makes around -1.

⇒ A clockwise loop counts as $+1$ encirclement
A counter-clockwise loop counts as -1 encirclement

⇒ Diagrams may have both CW or CCW loops around -1

⇒ Let $N_{cw}(L)$ be the net number of CW encirclements for Nyquist diagram of L
(i.e. result of adding contribution of each loop using the ± 1 convention above).

Easy Way to Count Encirclements

"Ray trick"

=> Draw a ray radially outward from -1 in any direction

=> Looking along the ray, away from -1

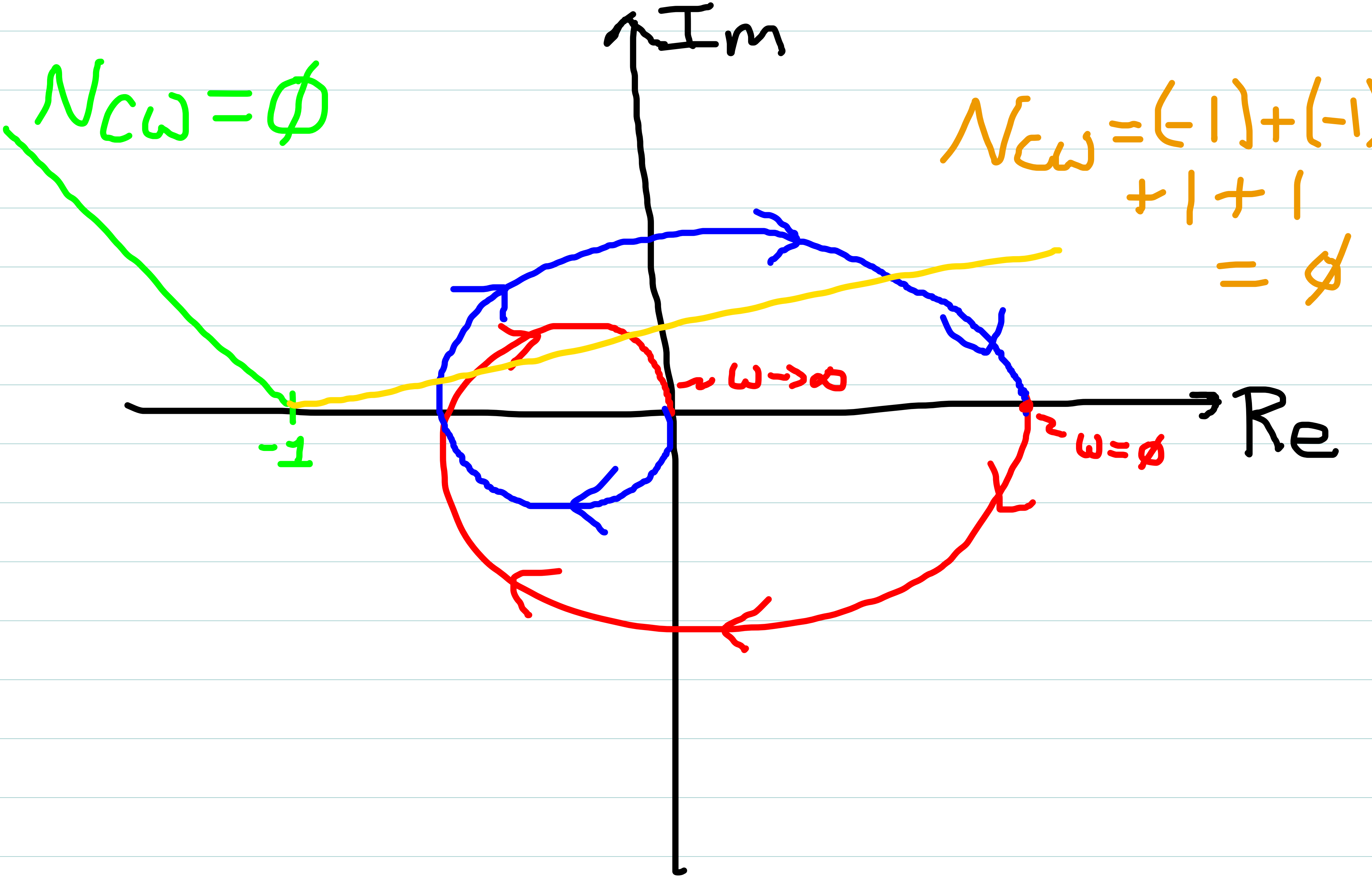
=> Count $+1$ each time diagram crosses ray from left to right.

=> Count -1 each time ray is crossed right to left.

=> Same result regardless of ray direction

=> Choose direction with least number of intersections for easiest counting.

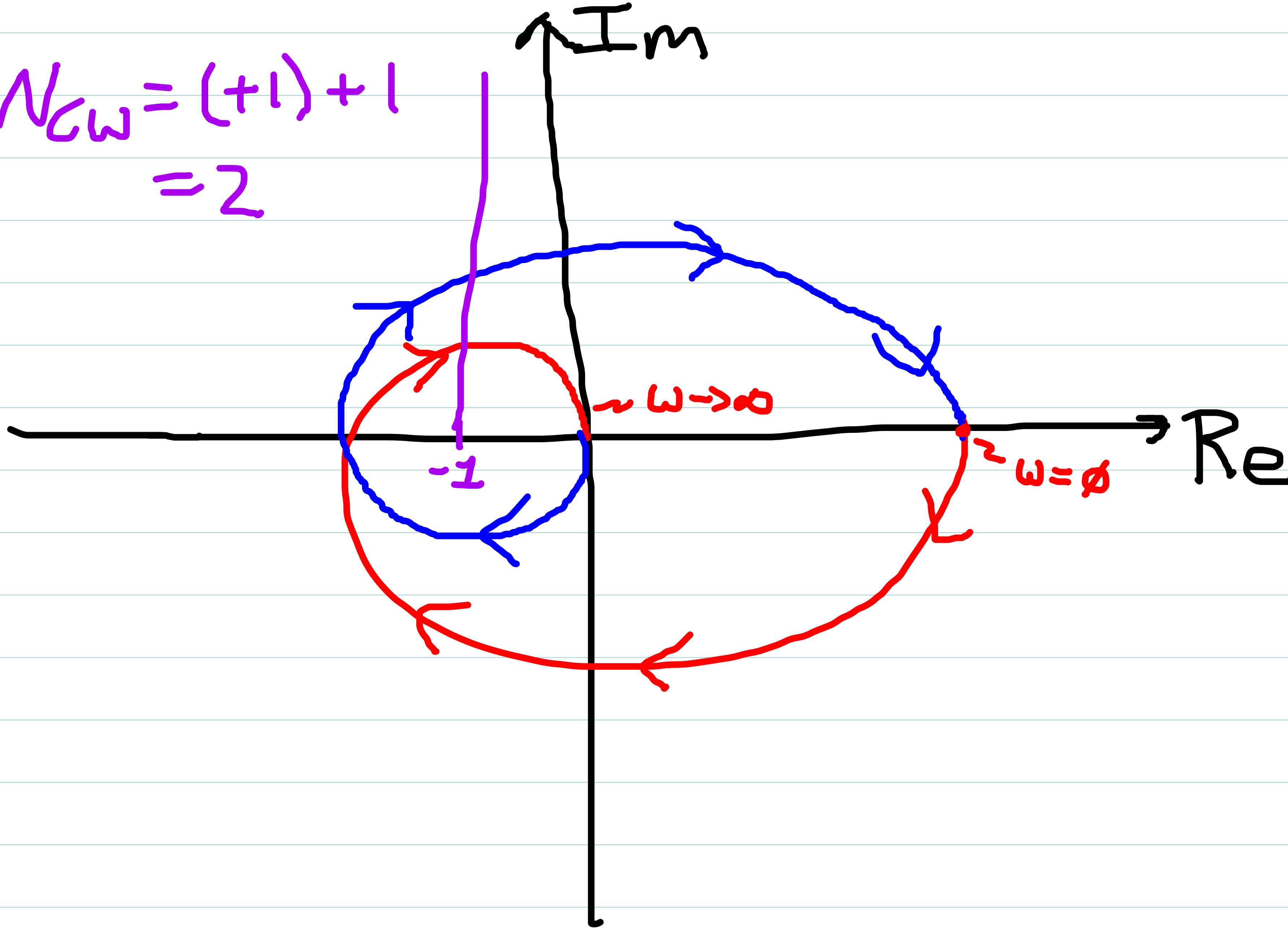
Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$



Example: $L(s) = \frac{K_B}{(\tau s+1)^3}$

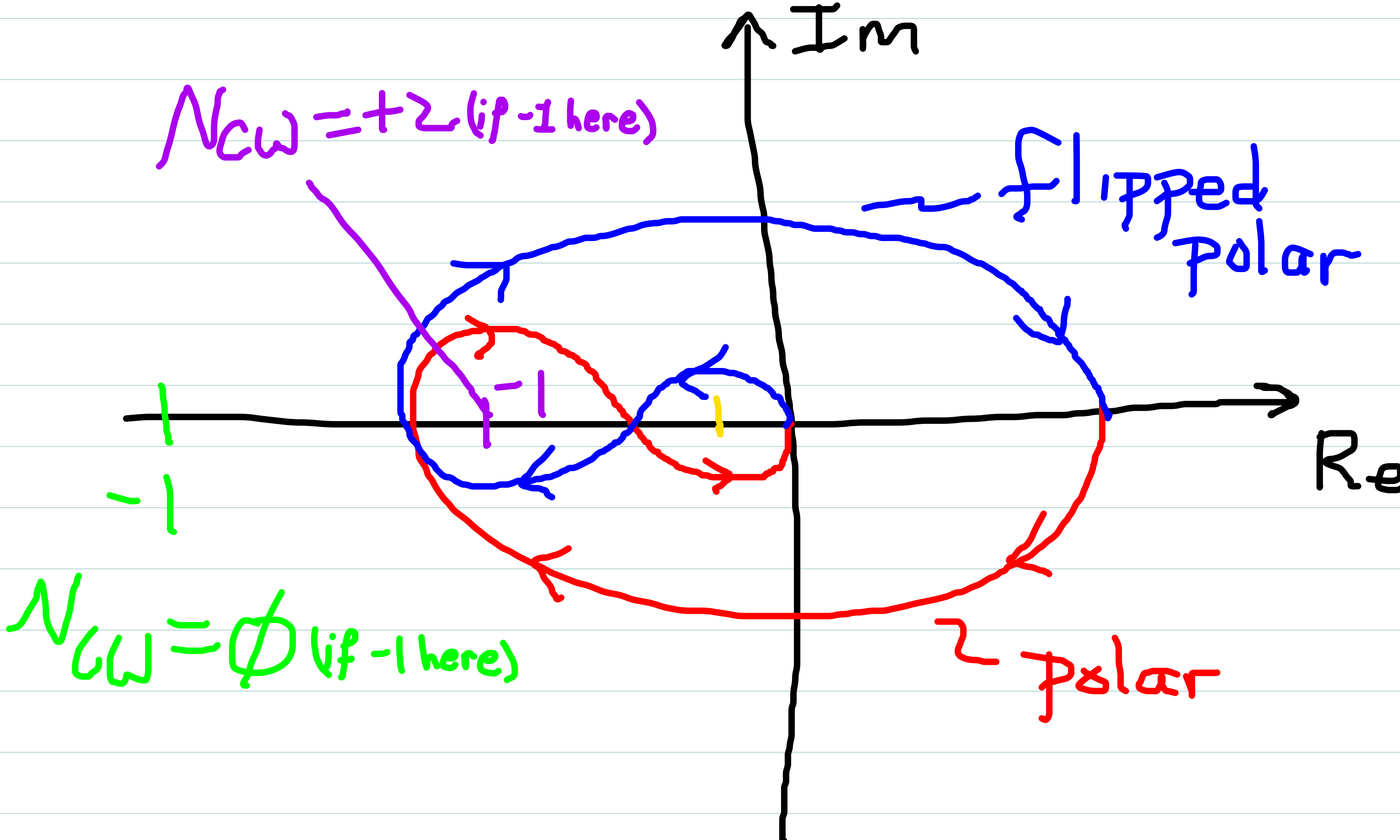
$N_{CW} = (+1) + 1$

$= 2$



A more complicated Example:

$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_2 \gg \tau_1 > \emptyset$$



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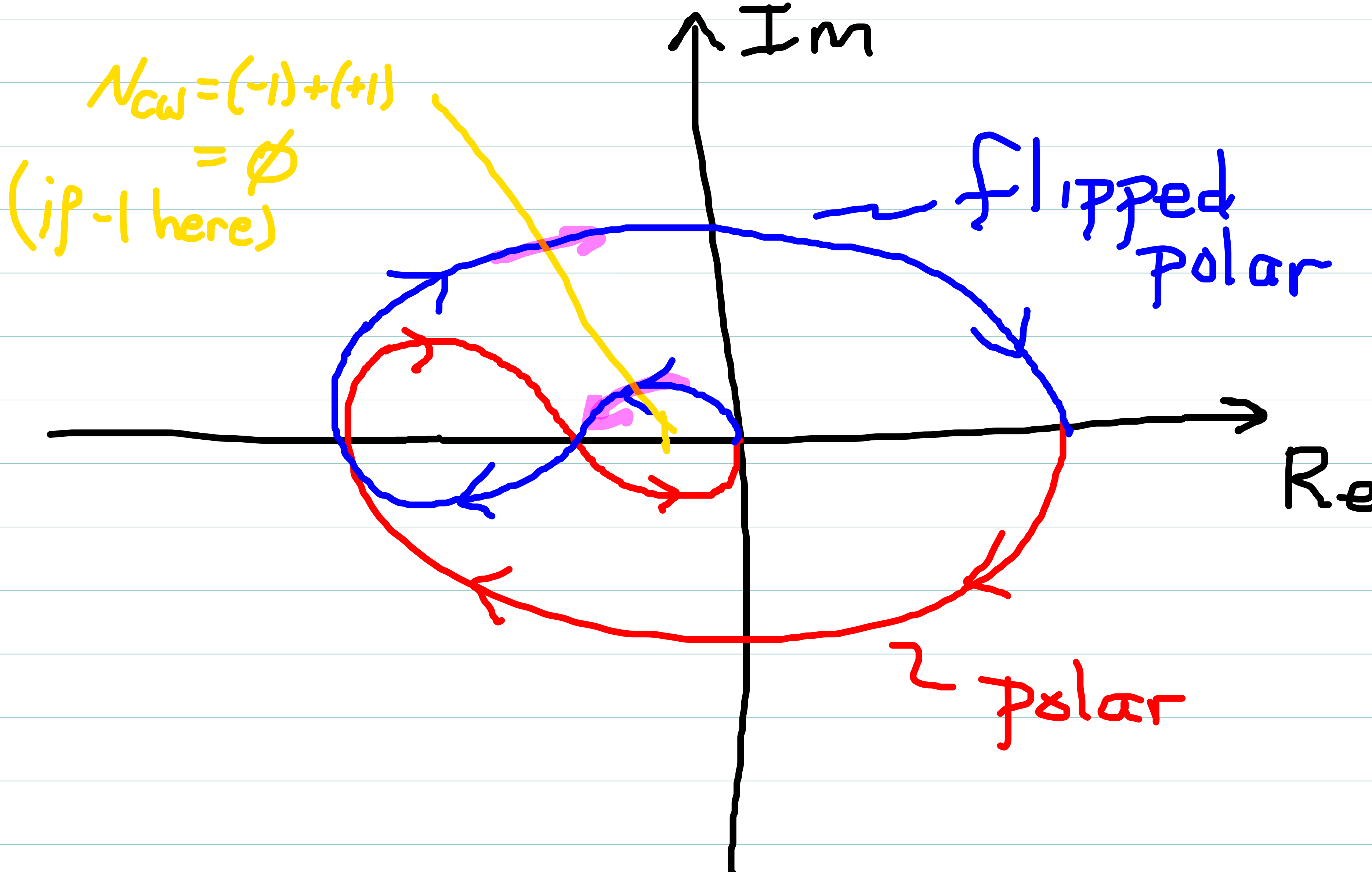


Nyquist Stability Theorem

For an arbitrary transfer function $G(s)$, define

$$P_R(G) = \#\text{RHP (unstable) poles of } G(s)$$

Nyquist showed:

$$\boxed{N_{cw}(L) = P_R(T) - P_R(L)}$$

Re-arranging:

$$\underbrace{P_R(T)}_{\text{want to predict}} = \underbrace{P_R(L) + N_{cw}(L)}_{\text{Known}}$$

Note: $P_R(T) \geq \emptyset$ <u>always</u>. If you compute $P_R(T) < \emptyset$

$\Rightarrow$ you have drawn the diagram incorrectly, or

$\Rightarrow$ you have counted encirclements incorrectly.

Implication

$\Rightarrow$ We <u>must</u> have $P_R(T) = \emptyset$ (stable closed-loop system)

$\Rightarrow$ $\boxed{N_{cw}(L) = -P_R(L)}$ (Stability condition)

i.e. Nyquist diagram must show a net negative number of encirclements, equal to number of unstable poles of L(s).

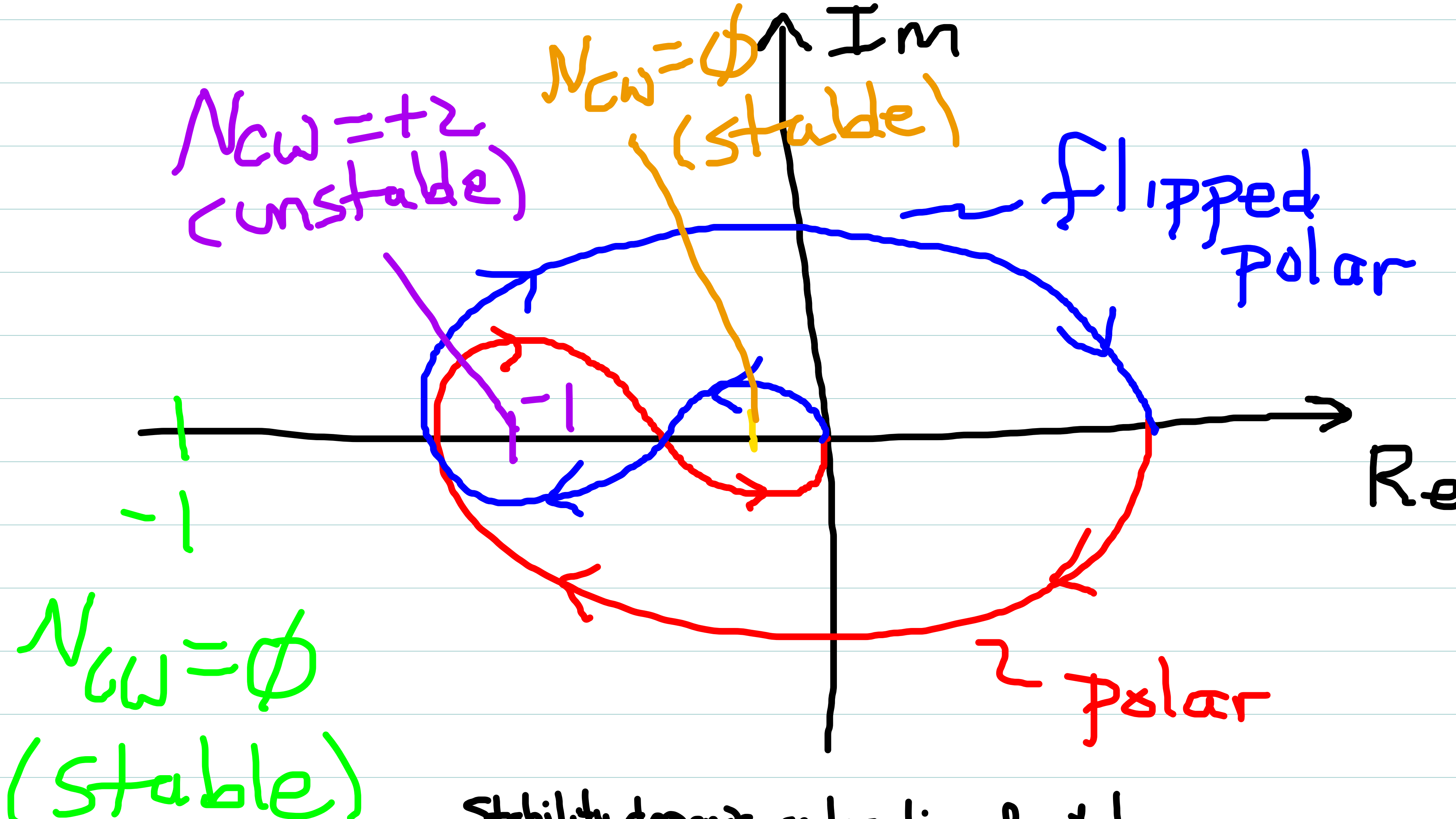
Recall negative CW encirclements are CCW encirclements

$\Rightarrow$ Nyquist diagram must show a net number of CCW encirclements equal to # unstable poles of L(s)

Note: if $P_R(L) = \emptyset$ (L(s) is stable) then the diagram must show NO ($\emptyset$) net encirclements

A more complicated Example:

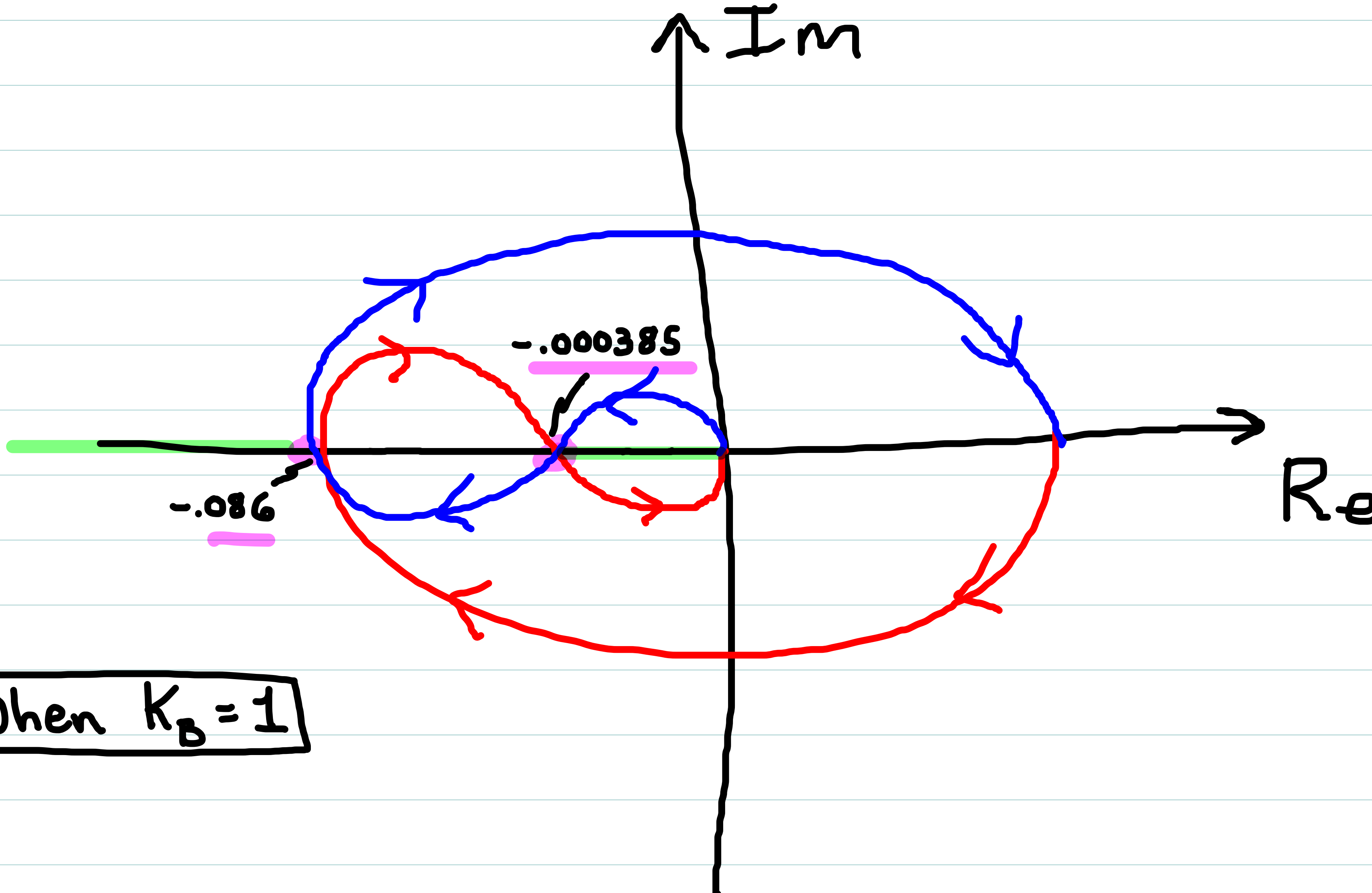
$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_2 \gg \tau_1 > \emptyset \quad \Bigg] \; P_R(L) = \emptyset$$



Stability depends on location of -1!

Effect of gain changes

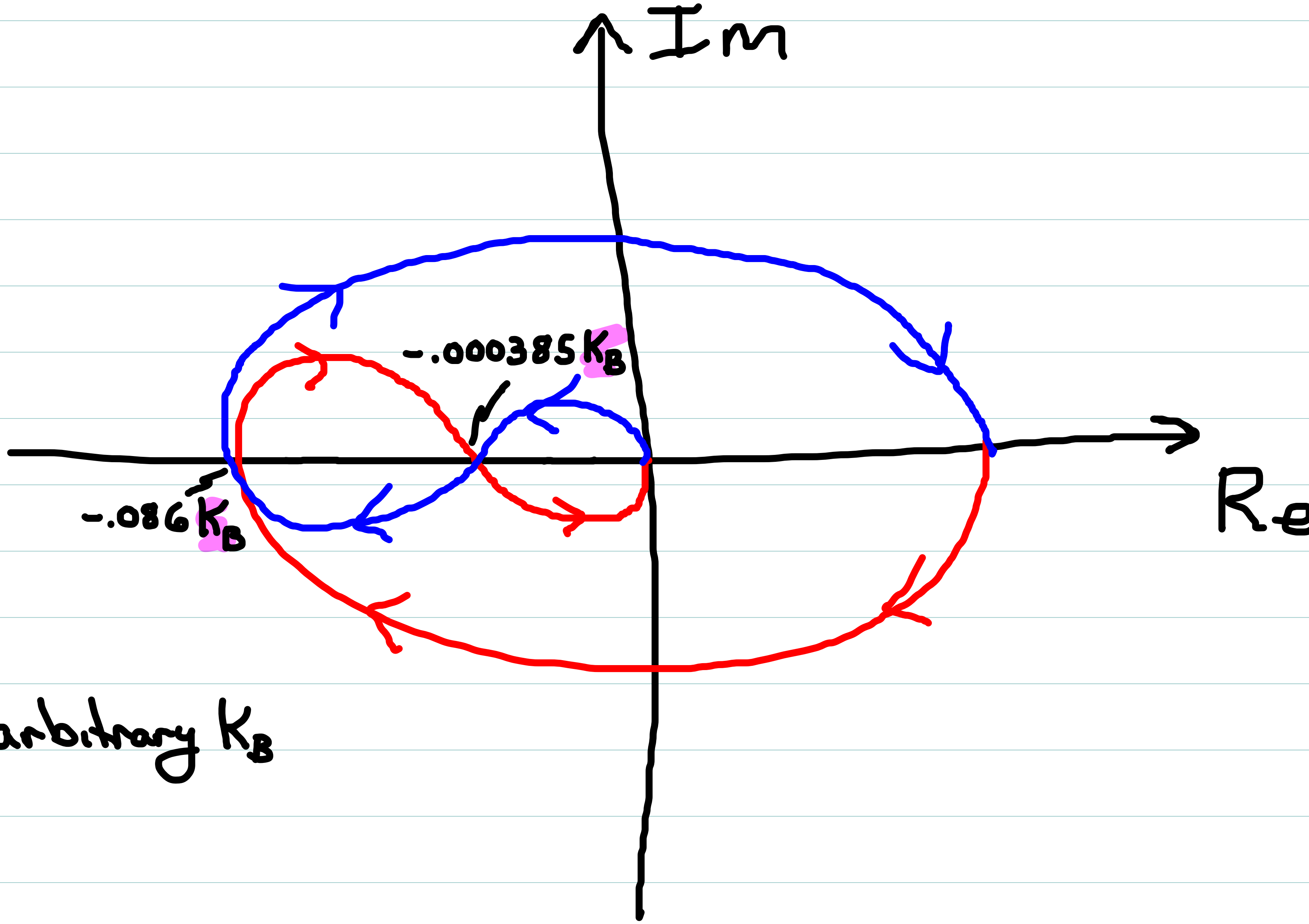
$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_1 = 10,\ \tau_2 = 1$$



$\boxed{\text{When } K_B = 1}$

Effect of gain Changes

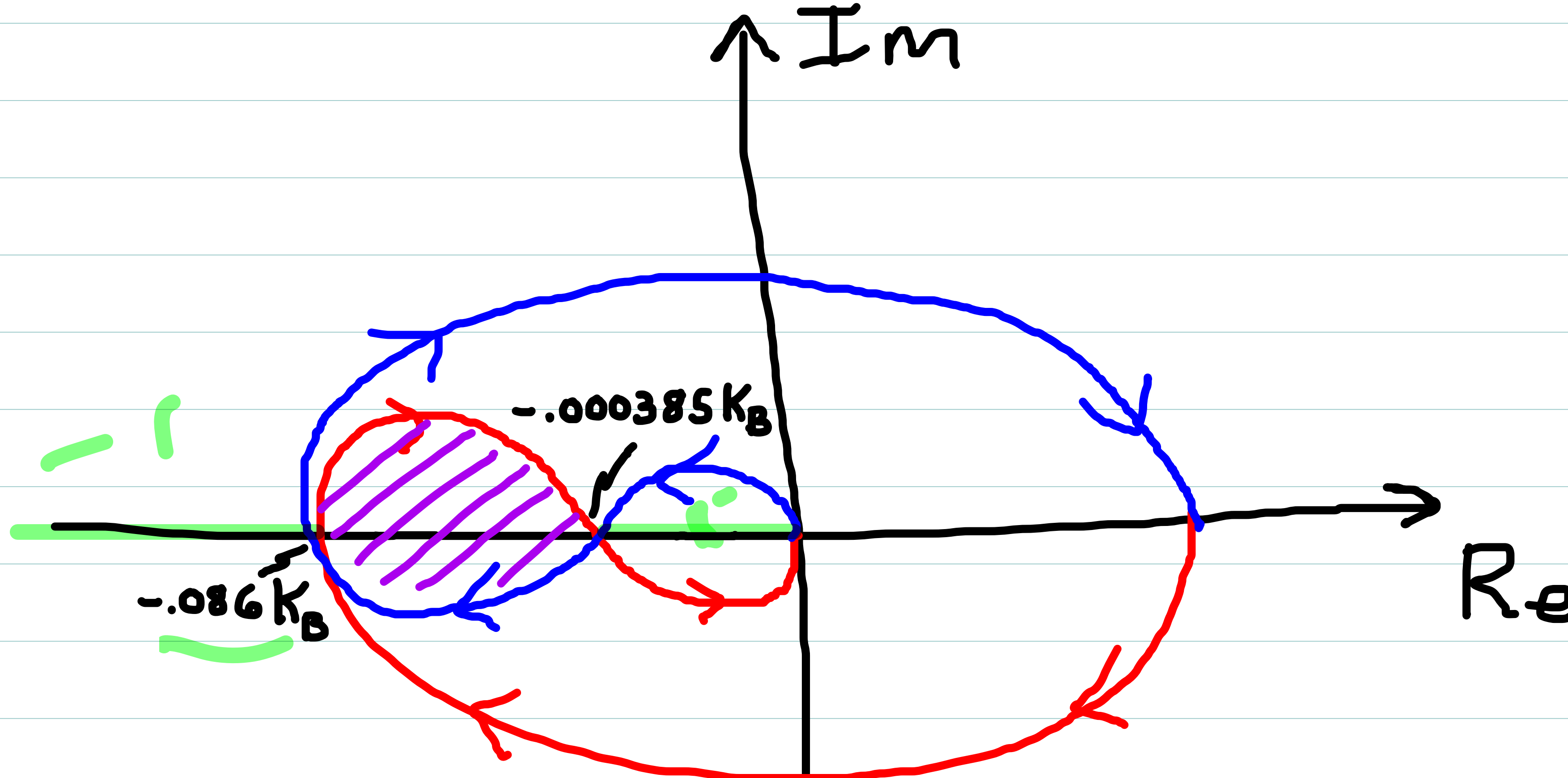
$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_1 = 10,\ \tau_2 = 1$$



for arbitrary K_B

Effect of gain changes

$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_1 = 10,\ \tau_2 = 1$$



for this system, $T(s)$ stable unless -1 is in hashed area

Need: $-1 < -.086 K_B$ or $-.0004 K_B < -1$

Thus: Stable for $K_B < 1/.086 \cong 11.63$
or $K_B > 1/.000385 \cong 2597$

Note: Gain change is easy to accomplish with compensator:

$H(s) = K$ ($\Rightarrow u(t) = Ke(t)$ "proportional" control)

$L(s) = H(s)G(s) = KG(s)$ here

$\Rightarrow (K_B)_L = K(K_B)_G$

However, gain change only affects "size" of polar (hence location of -1 relative to loops in Nyquist).

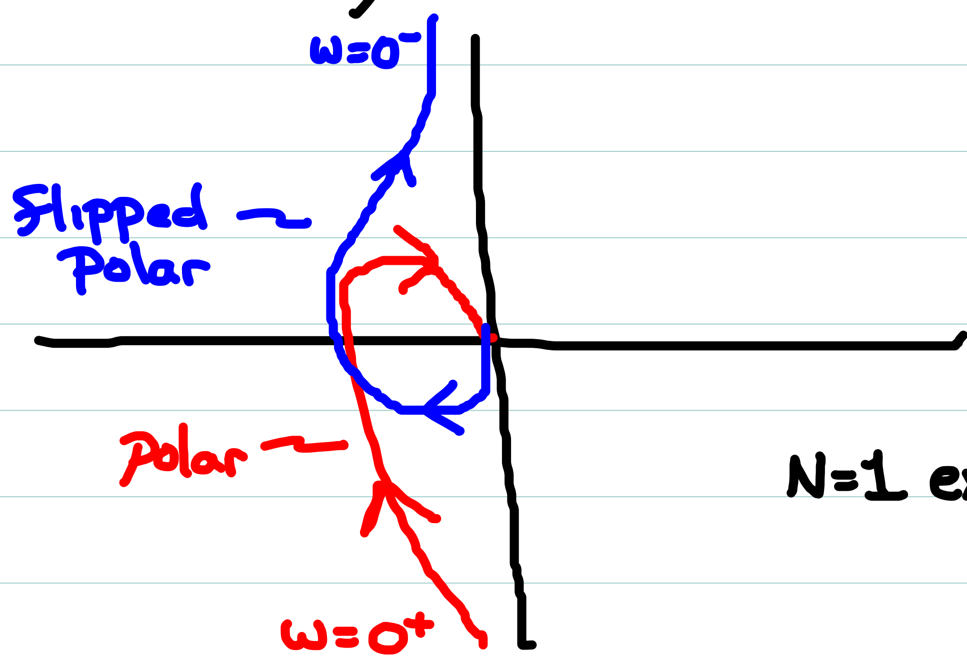
More substantial changes to polar/Nyquist diagram (changes to number and/or location of its loops) require also zeros/poles in $H(s)$.

Nyquist Diagram for N>Ø systems

When L(s) has type N>Ø (one or more poles at origin) the first step to creating Nyquist diagram is same:

⇒ Draw polar of $L(j\omega)$
⇒ Flip polar about real Axis

However, the resulting diagram is Not connected; both halfs have "tails" parallel to coordinate AXES



N=1 example

Completing the diagram, $N > \emptyset$

$\Rightarrow$ Connect the $\omega = 0^-$ tail of flipped polar to $\omega = 0^+$ tail of original polar with a clockwise circular arc of total rotation $N\pi$

(i.e. ½ circle for every pole at origin in $L(s)$)

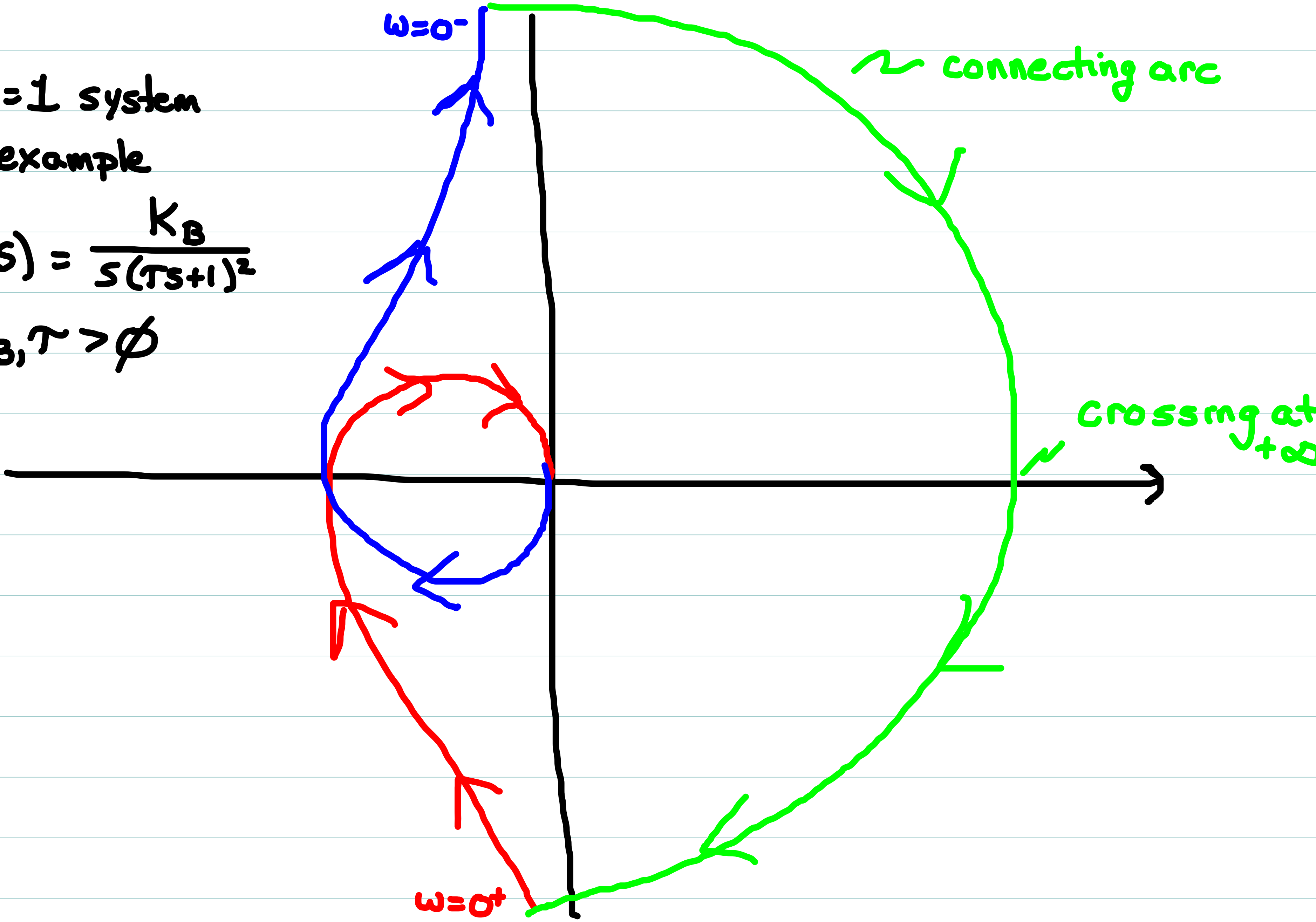
Note: Connecting arc has infinite radius, although we draw it as finite.

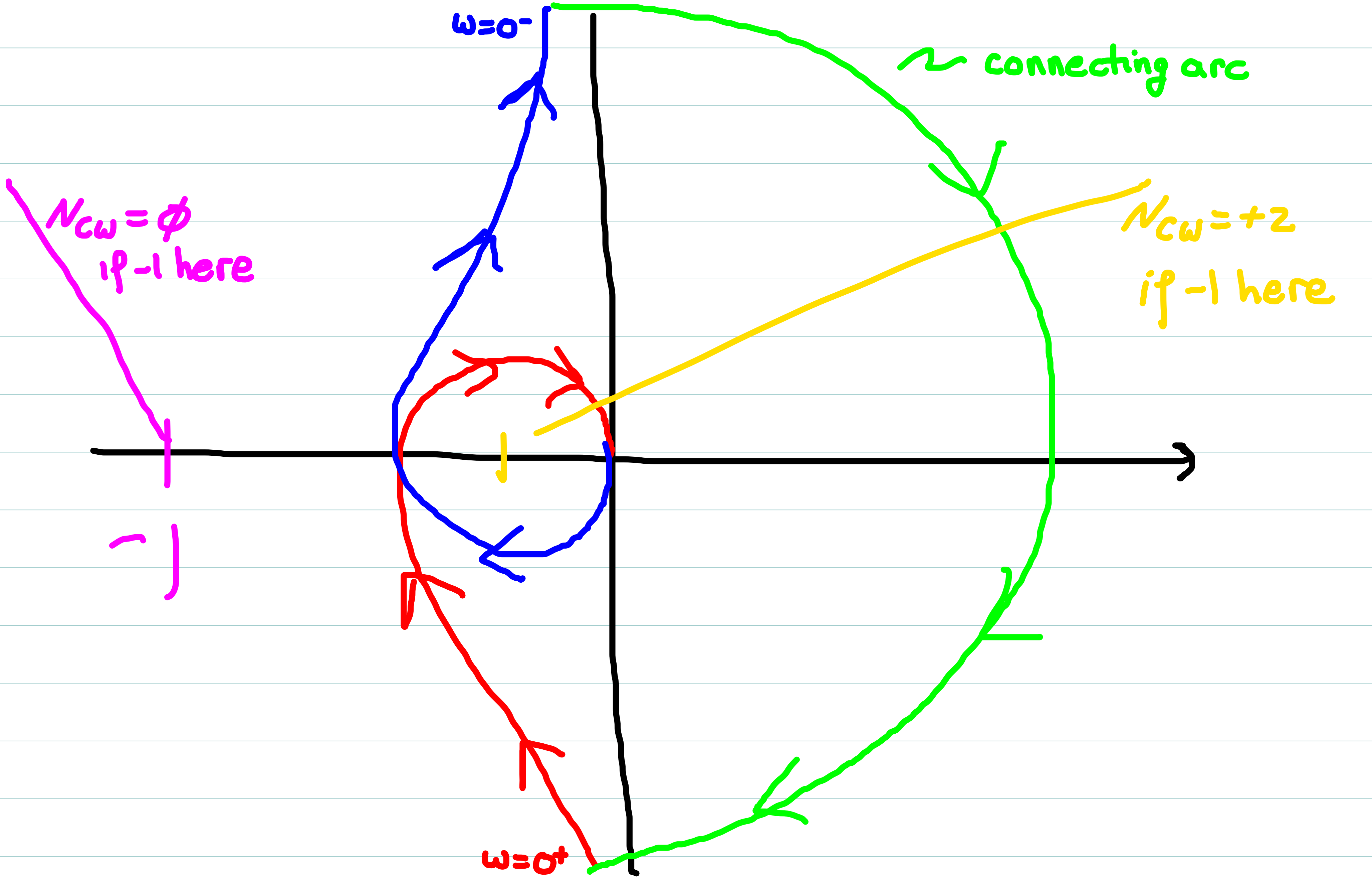
$\Rightarrow$ After connecting tails, compute $N_{cw}(L)$ as before.

N=1 system
for example

$$L(s) = \frac{K_B}{s(\tau s+1)^2}$$

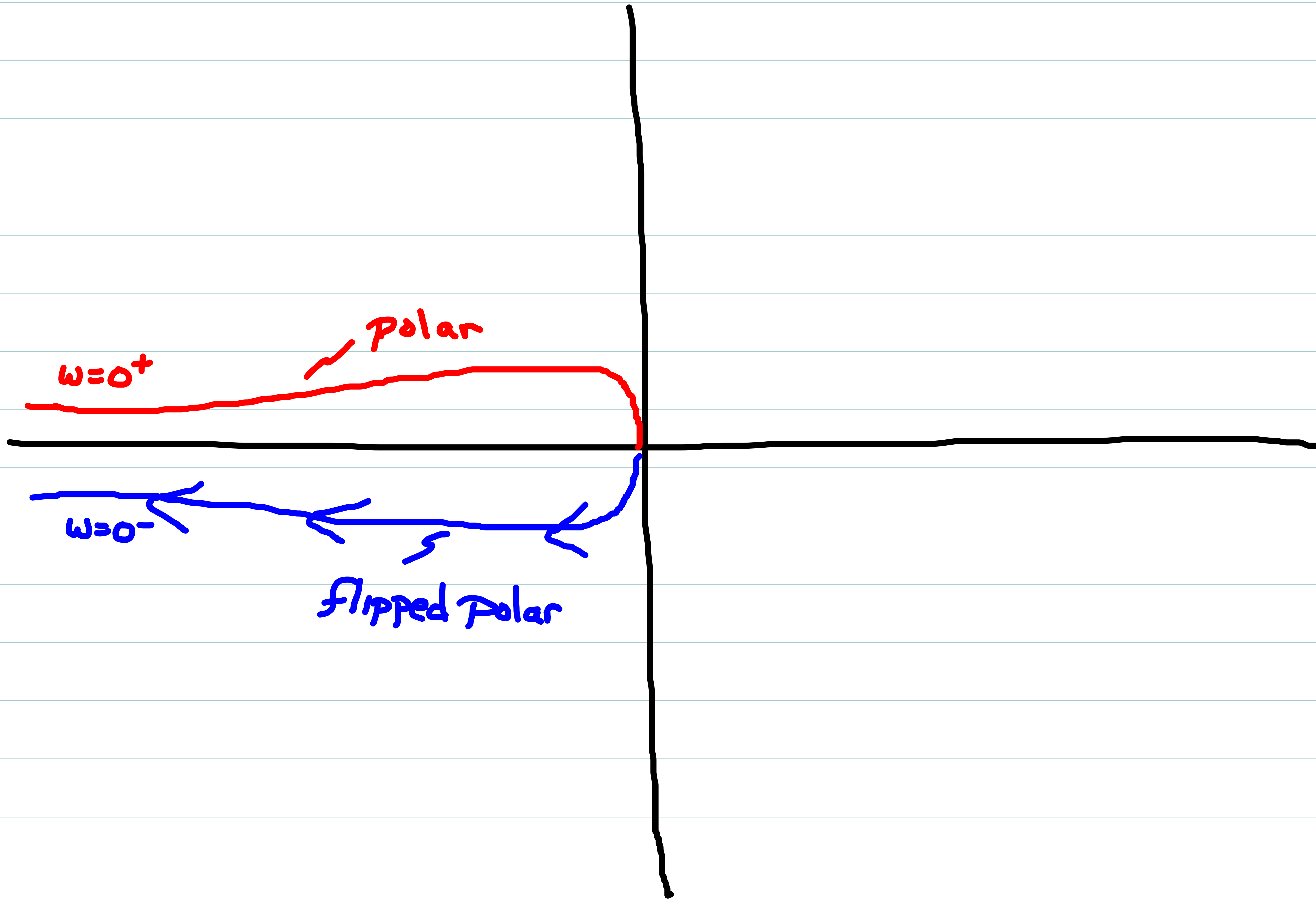
$K_B, \tau > \phi$





ω=0⁻
connecting arc
$N_{cw} = \phi$
if -1 here
$N_{cw} = +2$
if -1 here
-1
ω=0⁺

Be careful with N=2 systems!



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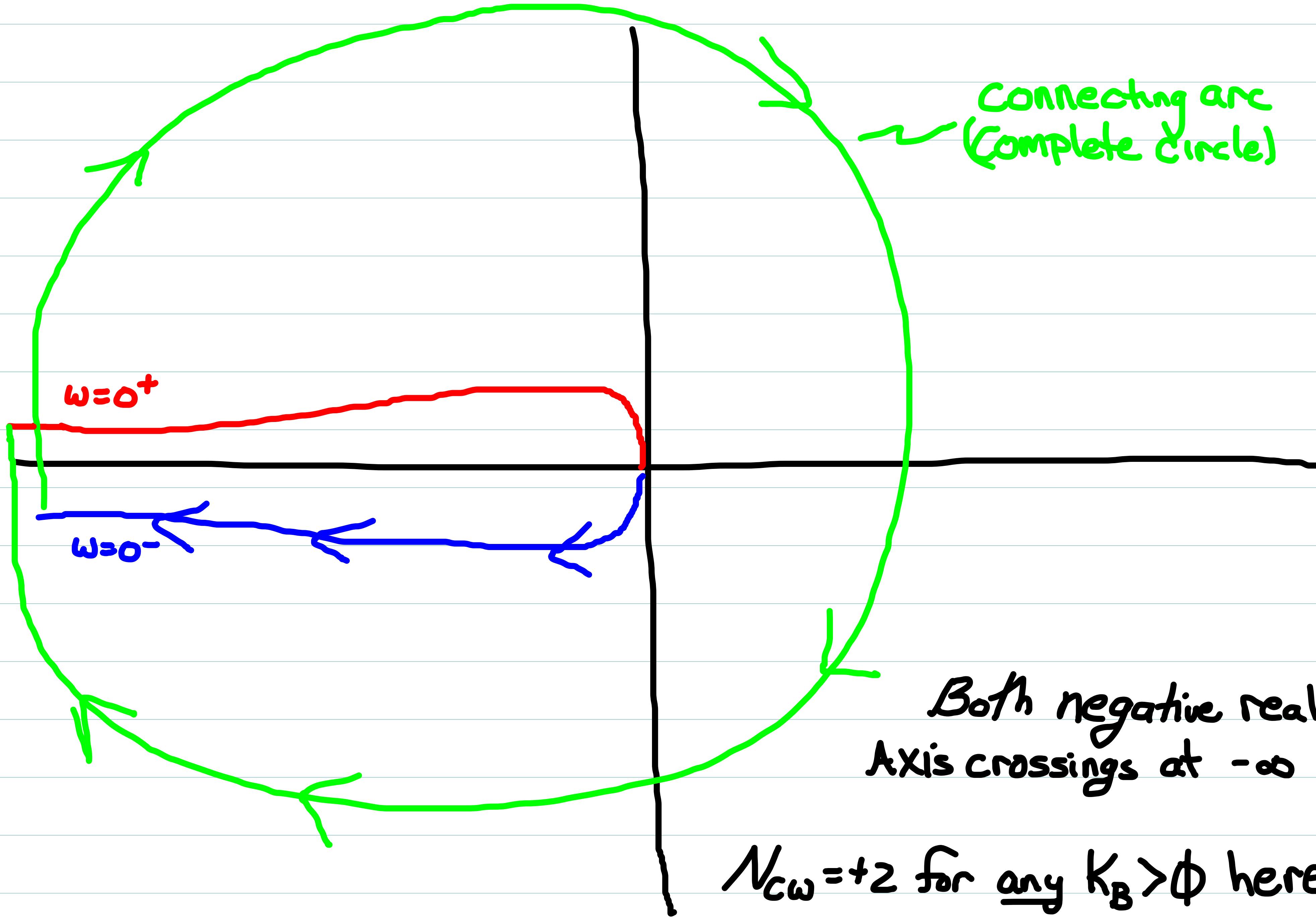
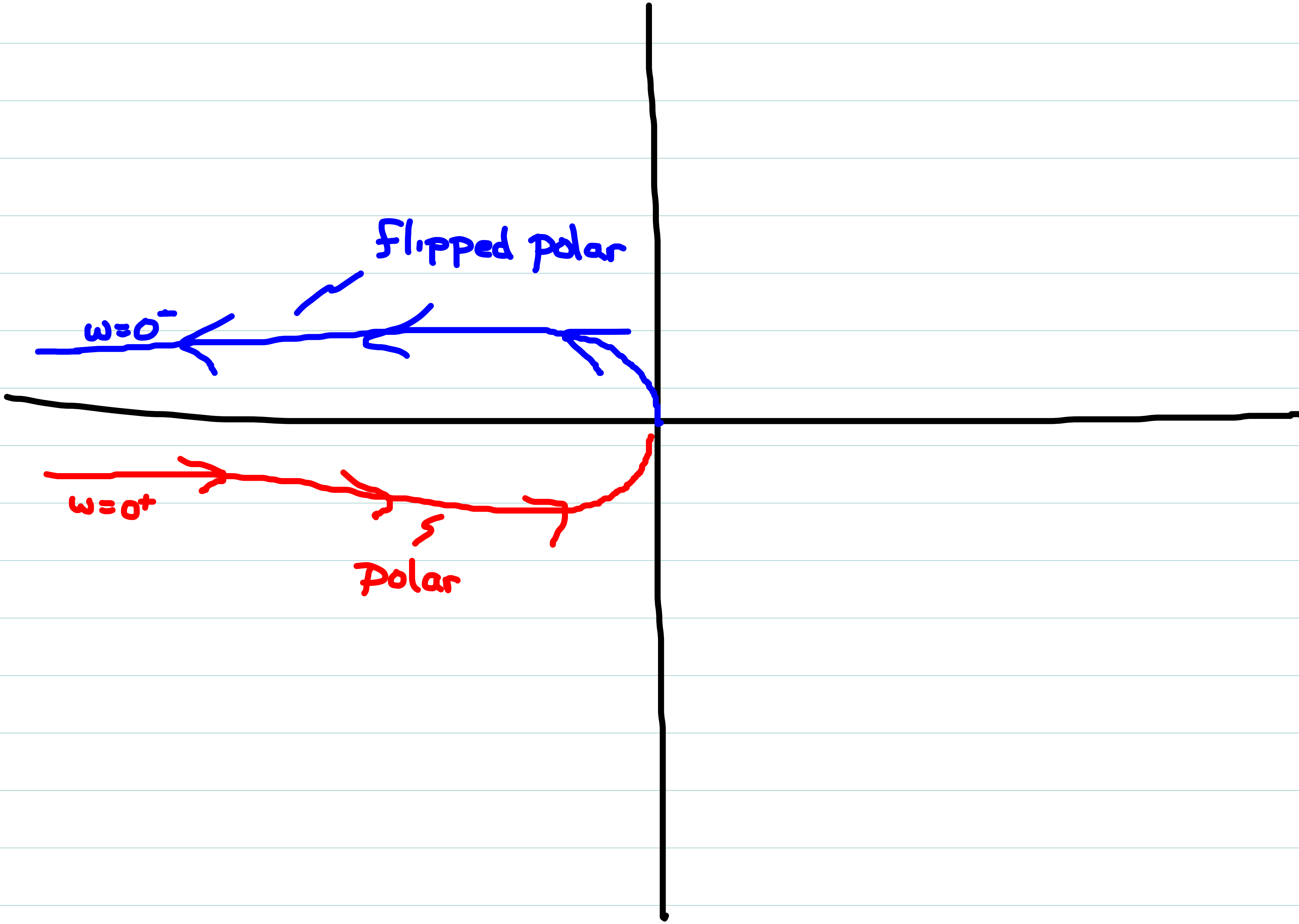
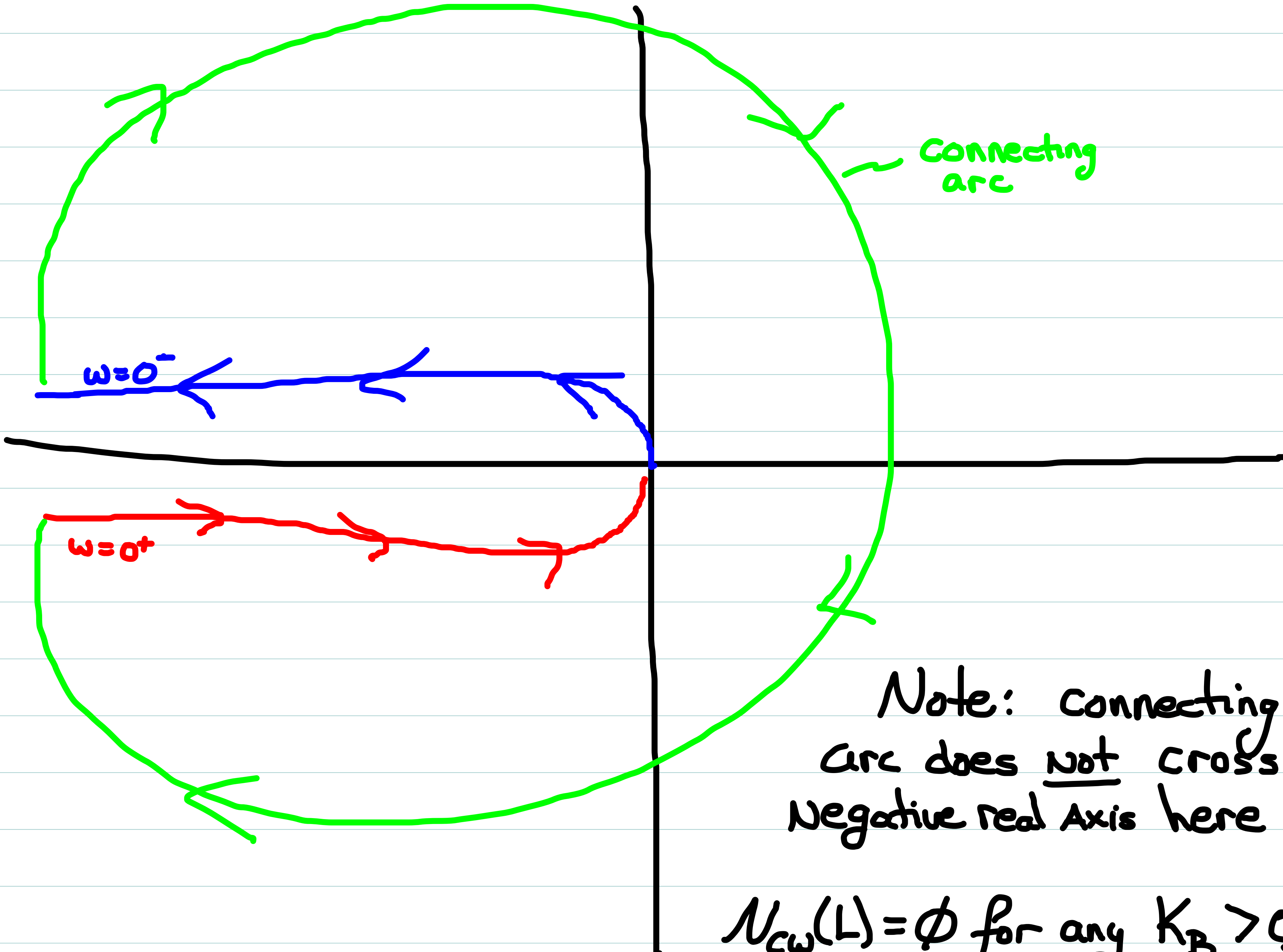


Both negative real
Axis crossings at $-\infty$

$N_{cw} = +2$ for <u>any</u> $K_B > \phi$ here

An apparently similar system (N=2 still)





Note: connecting arc does <u>not</u> cross negative real axis here

$N_{cw}(L)=\emptyset$ for any $K_B>\emptyset$ here

Utility of gain/phase margin

⇒ a, γ measure how close polar comes to -1

⇒ If design is nominally stable (Nyquist shows required number of encirclements of -1), then

a, γ measure how much Nyquist plot can **Change** in a pure gain or phase fashion, before -1 would enter a different loop, changing the number of encirclements.

Thus: a, γ are measures of the "tolerance" of the system's stability to gain/phase changes in $L(s)$.

⇒ **Relative** stability measures.

Robustness (classical)

As measures of the tolerance of the control system stability to changes in shape of Nyquist, gain and phase margin are measures of the robustness of the design.

That is, the ability of the design to tolerate model errors which would create pure gain or pure phase errors in $L(s)$

Typically caused by errors in model of $G(s)$, since

$$L(s) = G(s)H(s)$$

and there is no uncertainty in $H(s)$.

Classical Robustness Requirements

A "robustly stable" design thus requires:

$\Rightarrow$ Correct number of Nyquist encirclements

AND $\Rightarrow$ Large $|\alpha|$, $|\gamma|$

Typical professional requirements

$\Rightarrow$ $|\alpha_{dB}| \geq 6$ (i.e. $\alpha > 6\,dB$ or $\alpha < -6\,dB$)

$\Rightarrow$ $|\gamma| \geq 30^\circ$

Requirement on α is physically equivalent to no more than a factor of 2 uncertainty on gain of $G(s)$