Lecture 14: Normal Shock Waves

ENAE311H Aerodynamics I

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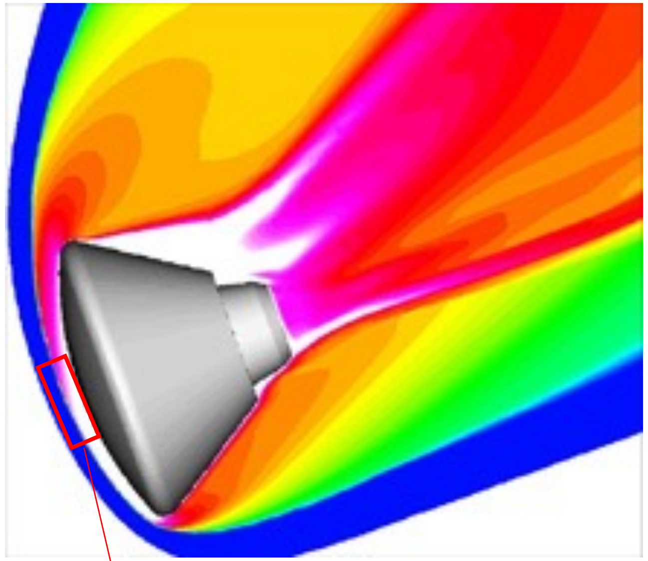
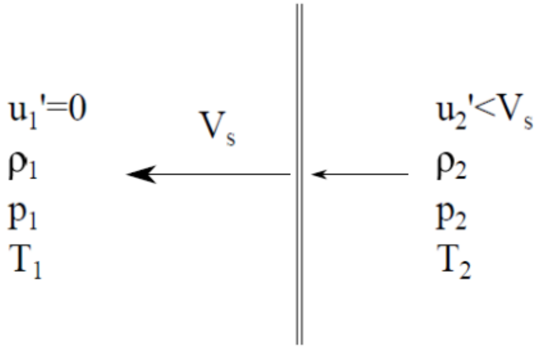
Preliminaries

Imagine we have a normal shock propagating through the atmosphere at speed V_s:

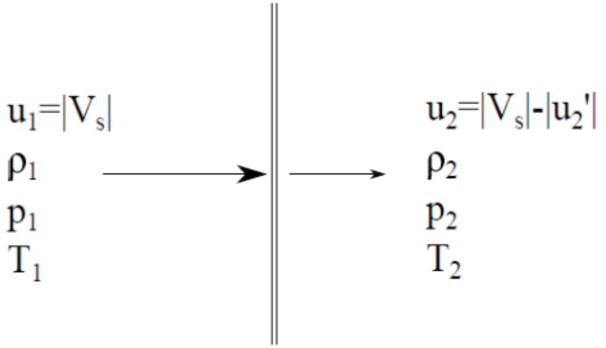
- The shock will see ambient conditions ahead of it, which will change to give the post-shock state
- These changes occur on very small length scales (a few mean free paths) so can treat the shock as a mathematical discontinuity



Now imagine we shift our reference frame from one fixed with the atmosphere to one fixed to the shock. We wish to determine the conditions downstream of the shock, given the upstream conditions and the shock Mach number, $M_1 = V_s/a_1$.



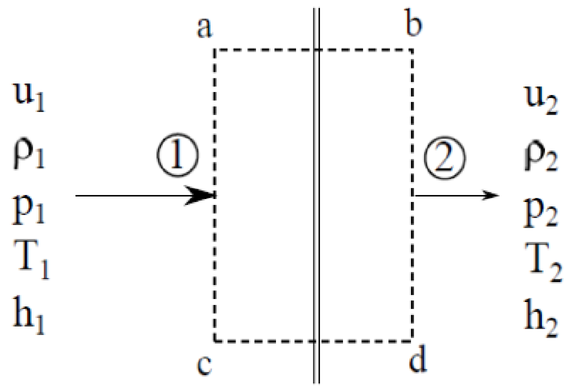
Atmosphere-fixed



Shock-fixed

Conservation laws for normal shocks

Consider a control volume around the shock, as shown to the right. We note or assume the following:



- The surfaces ab and cd are streamlines (no flow crosses them)
- The flow is steady ($\partial/\partial t = 0$)
- There are no viscous effects on surfaces ab and cd (shock infinitely thin)
- Body forces are negligible
- There is no heat addition, so flow is adiabatic

Under these conditions, we can derive particularly simple versions of the conservation equations.

Continuity:

$$\iint_{CS} \rho \mathbf{v} \cdot \mathbf{dA} = 0$$

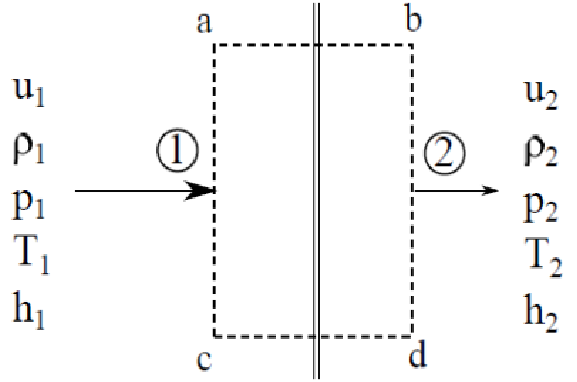
This becomes

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

or, since $A_1 = A_2$

$$\boxed{\rho_1 u_1 = \rho_2 u_2.}$$

Conservation laws for normal shocks



Momentum:

$$\iint_{CS} \mathbf{v}(\rho \mathbf{v} \cdot \mathbf{dA}) = -\iint_{CS} p\mathbf{dA}$$

This becomes

$$-\rho_1 u_1^2 A_1 + \rho_2 u_2^2 A_2 = p_1 A_1 - p_2 A_2,$$

or alternatively

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + +\rho_2 u_2^2.}$$

Energy:

Consider our simplified energy equation from earlier:

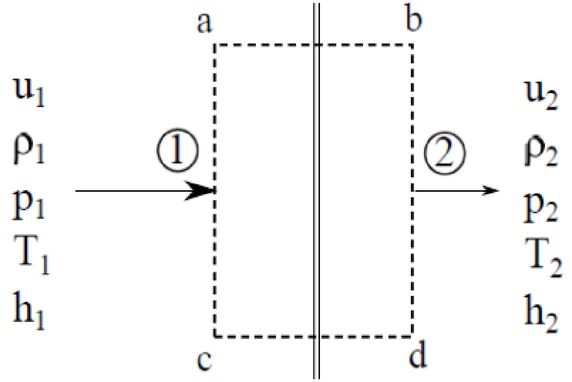
$$\dot{m}\left[h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.$$

In this case, it becomes again

$$\boxed{h_1 + \frac{1}{2}u_1^2 = h_2 + +\frac{1}{2}u_2^2.}$$

Conservation laws for normal shocks

Our conservation laws are thus:



$$\rho_1 u_1 = \rho_2 u_2.$$

$$p_1 + \rho_1 u_1^2 = p_2 + +\rho_2 u_2^2.$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + +\frac{1}{2}u_2^2.$$

To close this set, we additionally require the thermal and caloric equations of state:

$$\begin{aligned} p_2 &= \rho_2 R T_2, \\ h_2 &= c_p T_2. \end{aligned}$$

The Prandtl relation

Dividing the momentum equation by the continuity equation, we have

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1.$$

Noting that $\gamma\, p/\rho = a^2$, this can be expressed as

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1.$$

Since flow is adiabatic, a^* is constant, and we can write a_1 and a_2 in terms of a^* & u_1 and a^* & u_2, respectively (see equation 7.39), to obtain

$$\frac{\gamma+1}{2\gamma u_1 u_2}(u_2 - u_1)a^{*2} + \frac{\gamma-1}{2\gamma}(u_2 - u_1) = u_2 - u_1.$$

Dividing through by $u_2 - u_1$ and simplifying, we arrive at

$$a^{*2} = u_1 u_2.$$

This simple expression is known as the *Prandtl relation.*

The post-shock Mach number

Note that we can rewrite the Prandtl relation

$$a^{*2} = u_1 u_2.$$

as

$$M_2^* = \frac{1}{M_1^*}.$$

Squaring and using our expression relating the characteristic and regular Mach numbers from the previous lecture, we have

$$\frac{(\gamma+1)M_2^2}{2+(\gamma-1)M_2^2} = \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2},$$

which can be rearranged to yield

$$\boxed{M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)}.}$$

This gives the post-shock Mach number (in the shock-fixed frame) in terms of the pre-shock Mach number (in the lab frame, the post-shock Mach number is V_s/a_2 minus this value).

For $M_1 = 1$, $M_2 = 1$; for $M_1 > 1$, $M_2 < 1$. As $M_1 \to \infty$, $M_2 \to \sqrt{(\gamma-1)/2\gamma} = 0.378$ for $\gamma = 1.4$.

Post-shock thermodynamic variables

We now wish to derive ratios of post-shock and pre-shock quantities. For the density, using the continuity equation, we can write

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^{*2}} \\ &= M_1^{*2},\end{aligned}$$

where we have used the Prandtl relation. This can be written as

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}.}$$

Post-shock thermodynamic variables

For the pressure, rearranging the momentum conservation equation, we have

$$\begin{aligned} p_2 - p_1 &= \rho_1 u_1^2 - \rho_2 u_2^2 \\ &= \rho_1 u_1 (u_1 - u_2) \\ &= \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right) \end{aligned}$$

Dividing through by p_1 and using $\gamma\, p/\rho = a^2$, we can write this as

$$\frac{p_2 - p_1}{p_1} = \gamma \frac{u_1^2}{a_1^2}\left(1 - \frac{u_2}{u_1}\right) = \gamma M_1^2 \left(1 - \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2}\right)$$

which can be simplified to

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1).}$$

Meanwhile, for the temperature

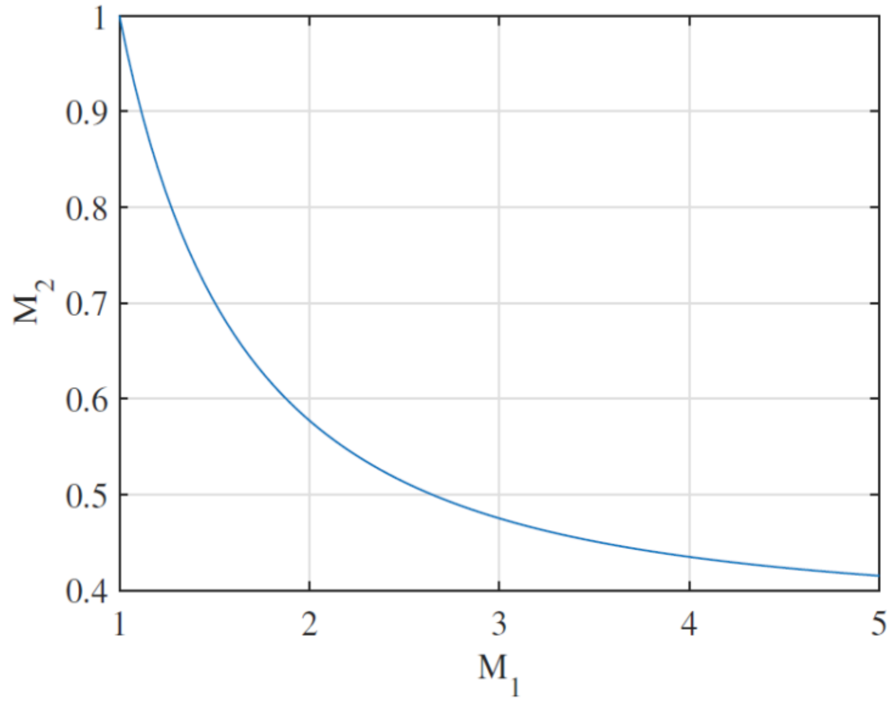
$$\boxed{\begin{aligned} \frac{T_2}{T_1} &= \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \\ &= \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2}. \end{aligned}}$$

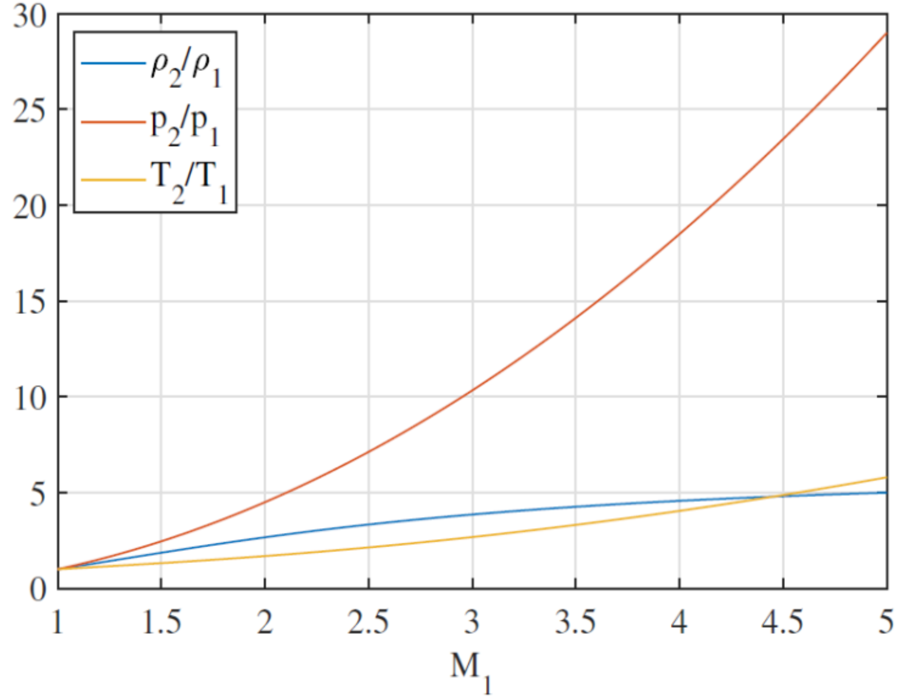
(also holds for h_2/h_1 and a_2^2/a_1^2)

Limiting values

- For $M_1 = 1,\ \rho_2/\rho_1 = p_2/p_1 = T_2/T_1 = 1$.
- For $M_1 > 1$, the ratios are all above 1.
- As $M_1 \to \infty$, we have the following behavior:

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1} = 6 \quad \text{for } \gamma\text{=}1.4$$
$$\frac{p_2}{p_1} \to \frac{2\gamma}{\gamma+1} M_1^2 \to \infty$$
$$\frac{T_2}{T_1} \to \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2 \to \infty.$$





Change in entropy across shock

We can apply our equation for entropy change from earlier to obtain:

$$\begin{aligned}\frac{s_2 - s_1}{R} &= \frac{c_p}{R}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1}\\ &= \frac{\gamma}{\gamma-1}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1}\\ &= \ln\left[\left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}\frac{p_2}{p_1}^{-1}\right].\end{aligned}$$

Using our equations for T_2/T_1 and p_2/p_1, this can be written

$$\frac{s_2 - s_1}{R} = \ln\left[\left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right)^{1/(\gamma-1)}\left(\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right)^{\gamma/(\gamma-1)}\right].$$

For M close to unity, this can be approximated as

$$\frac{s_2 - s_1}{R} \approx \frac{2\gamma}{3(\gamma+1)^2}(M_1^2 - 1)^3.$$

Now, the alternative form of the Prandtl relation tells us that

$$M_2^* = \frac{1}{M_1^*}.$$

Note also that if $M^* < 1$, then $M < 1$, and similarly if $M^* > 1, M > 1$. Thus we must have either supersonic flow upstream and subsonic flow downstream, or subsonic flow upstream and supersonic flow downstream.

However, from the entropy relation to the left, we see that the entropy change would be negative in the latter case.

Therefore, shocks are only possible with $M_1 > 1$ and $M_2 < 1$.