PHYS499G: Problems

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Problem 1. Carroll 1.6

Problem 2. Carroll 1.7

Problem 3. On the Euclidean manifold $\mathbb{R}^2$, take two sets of coordinates: Cartesian $x^\mu = (x, y)$ and polar $x^{\mu'} = (r, \phi)$.

(a) Write down the change of coordinate formulae and the transformation matrices $\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)$ and $\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)$, both expressed in terms of primed coordinates.

(b) If the vector U has components $U^\mu = (3, 4)$ at the point $x^\mu = (2, 0)$, then find $U^{\mu'}$.

(c) If the vector U has components $U^\mu = (3, 4)$ at the point $x^\mu = (3, 4)$, then find $U^{\mu'}$.

Problem 4. Use the same manifold and coordinate as in Exercise 3.

(a) If the vector field U has components $U^\mu = (1, 0)$, find $U^{\mu'}$. Then write the operator U in both coordinate bases.

(b) Given vector fields $V^\mu = (0, 1)$ and $W^\mu = (-y, x)$, find $V^{\mu'}, W^{\mu'}, V$, and W in both coordinate bases.

Problem 5. Consider this metric:

$$ds^2 = \frac{1}{2}\,dx^2 + \frac{1}{2}\,dy^2 + \frac{1}{z^2+1}\,dz^2$$

(a) Compute all nonzero Christoffel symbols.

(b) Compute all nonzero components of $R^\mu{}_{\nu\rho\sigma}$. What does this result tell you?

Now let's consider the arbitrary coordinate transformation given by $x' = \frac{1}{2}(x+y)$, $y' = \frac{1}{2}(x-y)$, and $z' = \sinh^{-1}(z)$.

(a) Compute $(\frac{\partial x^{\mu'}}{\partial x^\mu})$ and $(\frac{\partial x^\mu}{\partial x^{\mu'}})$.

(b) Use the previous result to transform $g_{\mu\nu}$ into $g_{\mu'\nu'}$.

(c) Why does this result make sense?

Problem 6. The metric for the 3-sphere in the coordinate system $x^\mu = (\psi, \theta, \phi)$ is

$$ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)$$

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Prove the 3-sphere is not flat.

Problem 7. Caroll 3.13a

Problem 8. Carroll 3.6a, b

Problem 9.

(a) In the Newtonian limit, consider a source with energy density ρ and negligible pressure p ("dust"). Add the cosmological constant to Einstein's equation rederive the Poisson equation to see how it is modified.

(b) When there is no cosmological constant, we know that $\Phi = -GM/r$. Derive the equivalent expression in the case with a cosmological constant and see how it is modified. Can you how the Λ term gives rise to a *repulsive* gravity term? How does it scale with r?

Problem 10. Carroll 5.3