Theoretical problems in integrals when discontinuities or singularities at one of the end points.

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt$$

0 ⟶ possible problem here

Resolve these by taking lower limit at $t = 0^-$ (the instant before $t = 0$).

⟹ integral "sees" effect of singularities at $t = 0$.

Starting the integral at 0^- instead of 0

- Avoids singularities at endpoints
- Causes transform to "see" singularities and discontinuities at $t=0$, so their effects will be reflected in the solutions for $y(t)$.

Hence:

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

<u>Implications</u>:

Assumed ICs for $y(t)$: just <u>before</u> $t=0$

$$\mathcal{L}\{\dot{y}(t)\} = sY(s) - \boxed{y(0^-)}$$

$$\mathcal{L}\{\ddot{y}(t)\} = s^2Y(s) - \boxed{\dot{y}(0^-) - sy(0^-)}$$

etc.

$$\mathcal{L}\{\dot{u}(t)\} = sU(s) - \boxed{u(0^-)}$$

$$\mathcal{L}\{\ddot{u}(t)\} = s^2U(s) - \boxed{\dot{u}(0^-) - su(0^-)}$$

etc.

Always $=0$ in our analysis!

$\Rightarrow b(s) = 0$ always

Thus:

$$Y(s) = \left[\frac{q(s)}{r(s)}\right]U(s) + \left[\frac{c(s) - \cancel{b(s)}^{\,\emptyset}}{r(s)}\right]$$

$$\boxed{= \left[\frac{q(s)}{r(s)}\right]U(s) + \left[\frac{c(s)}{r(s)}\right]}$$

$\Rightarrow$ IC polynomial $b(s)$ for input vanishes

$\Rightarrow$ specific to controls convention for $u(t)$

$\Rightarrow$ Not a common assumption in regular math classes.

$\Rightarrow$ In controls, want to know effect of discontinuities

Common, discontinuous "test functions"

$u(t) = \mathbb{1}(t)$ (unit step) $\quad \mathcal{L}\{\mathbb{1}(t)\} = \frac{1}{s}$

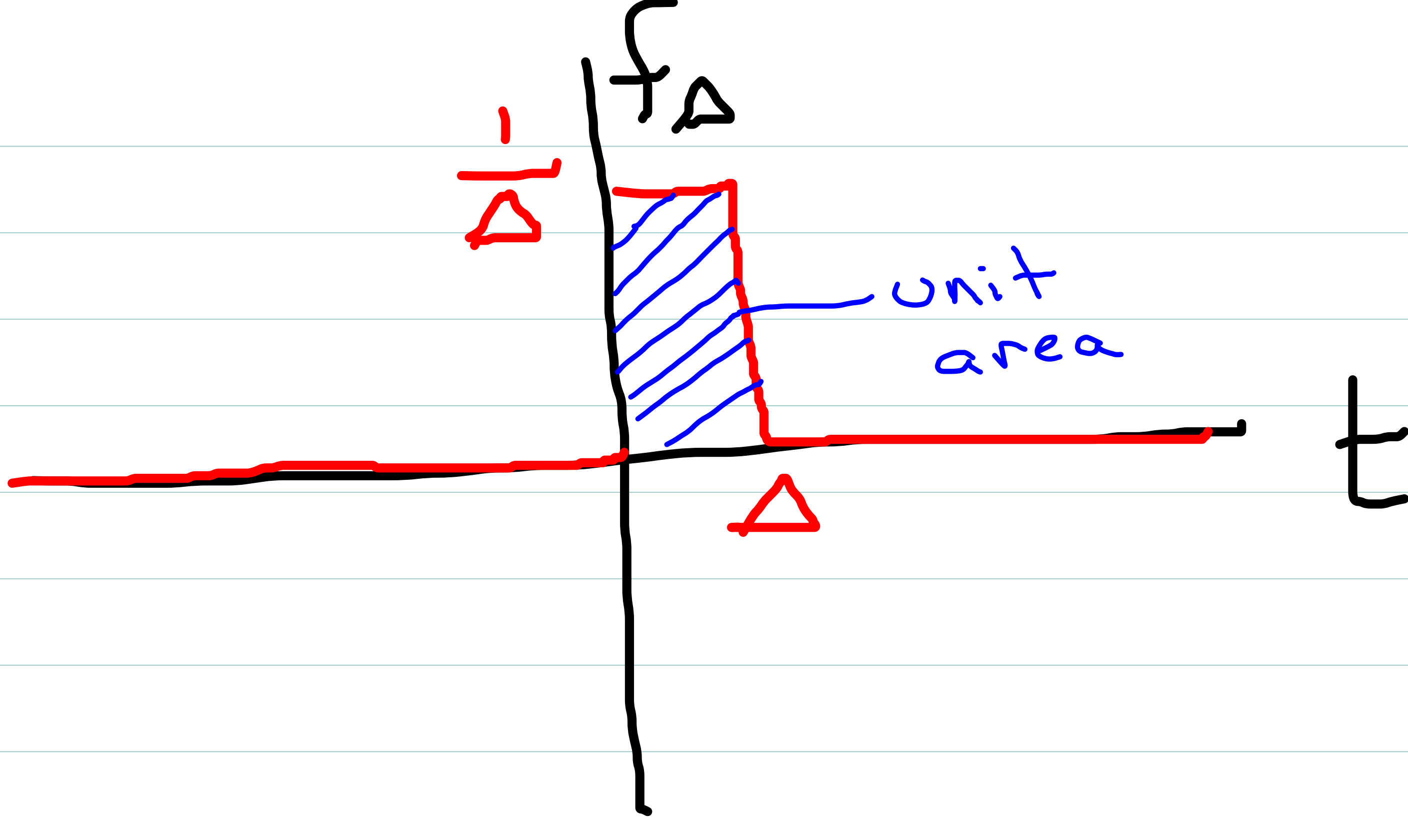
$$u(t) = \cos(\omega t)\, \mathbb{1}(t)$$

$$= \begin{cases} \cos(\omega t) & t \geq \phi \\ \phi & t < \phi \end{cases}$$

$\Longrightarrow$ $u(t) = f_\Delta(t)$

$$= \begin{cases} \frac{1}{\Delta} & \phi \leq t \leq \Delta \\ \phi & \text{otherwise} \end{cases}$$

"Unit pulse function"



Note: for <u>any</u> $\Delta > 0$

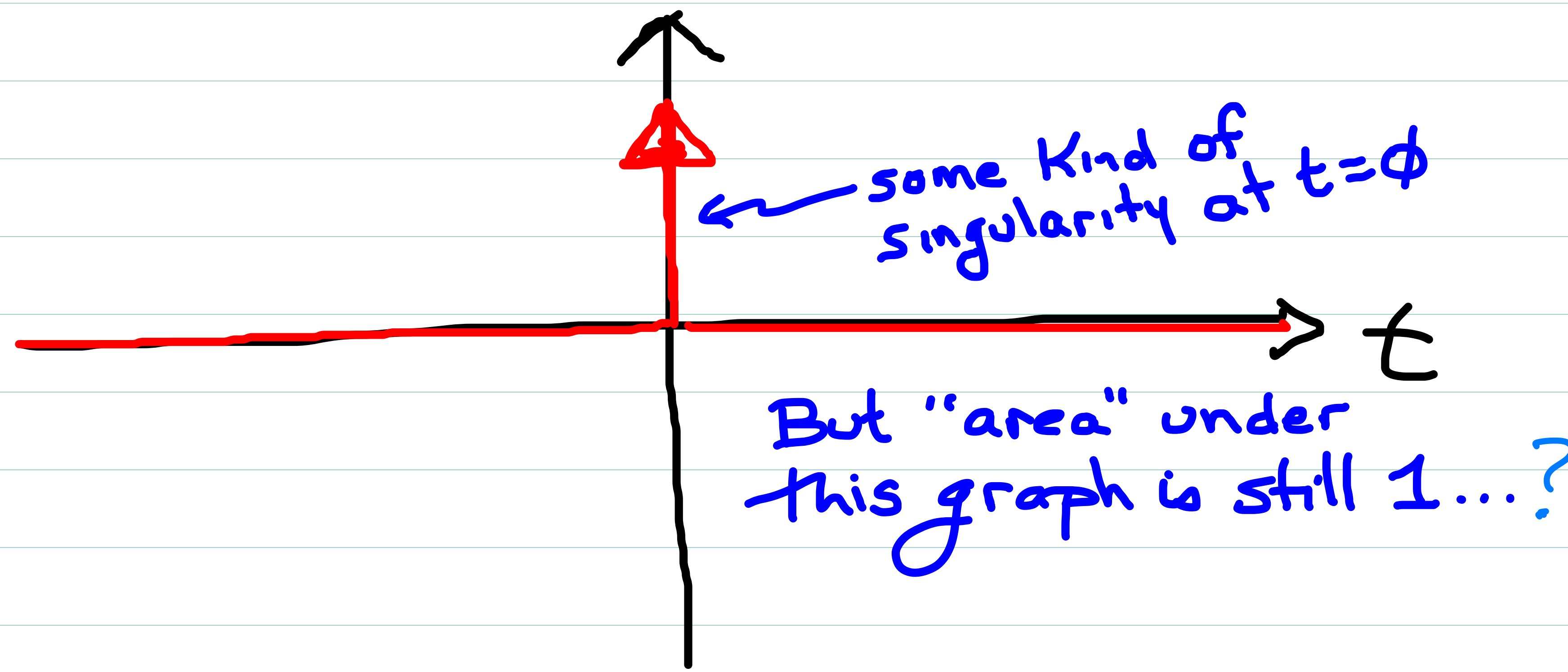
$$\int_{0^-}^{\infty} f_\Delta(t)\,dt = \int_{0^-}^{\Delta} \left(\frac{1}{\Delta}\right) dt$$

$$= 1$$

What is $\lim_{\Delta \to 0} f_\Delta(t)$?

$$= \lim_{\Delta \to 0} \begin{cases} \frac{1}{\Delta} & \phi \le t \le \Delta \\ \phi & \text{otherwise} \end{cases}$$

$$= \begin{cases} \infty & t = \phi \\ \phi & \text{otherwise} \end{cases}$$



Define:

$$\boxed{\delta(t) = \lim_{\Delta \to 0} f_\Delta(t)}$$

"ideal impulse": models delivering a unit of input energy over negligibly small time.

(sharp "kick")

Alternate names:

"delta function"
"impulse function"
"Dirac delta"

Note: Not really a meaningful function at all!

More formally, belongs to a class of mathematical objects called

"distributions" or "generalized functions"

Suppose $\delta(t)$ appears in an integral

$$\int_{-\infty}^{\infty} \delta(t)\, h(t)\, dt \quad , \quad h(t) \text{ arbitrary f'n}$$

$$= \int_{-\infty}^{\infty} \left[\lim_{\Delta \to 0} f_\Delta(t)\right] h(t)\, dt$$

$$= \lim_{\Delta \to 0} \int_{-\infty}^{\infty} f_\Delta(t)\, h(t)\, dt$$

$$= \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \int_{0^-}^{\Delta} h(t)\, dt \right\}$$

$$\cong \left(\frac{1}{\Delta}\right)\left(\Delta\, h(\phi)\right)$$

$$= h(\phi)$$

Note: with $h(t) = 1$ for all t, we get

$$\int_{0^-}^{\infty} \delta(t)\, dt = 1$$

Defining Property of $\delta(t)$

$$\int_a^b \delta(t)h(t)\,dt = \begin{cases} h(\phi) & \text{if } \phi \in (a,b) \\ \phi & \text{otherwise} \end{cases}$$

"sifting property"

$\Rightarrow$ $\delta(t)$ is defined by what it does in an integral

Not as an ordinary function

Now we can compute:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt$$

$$= \left[e^{-st}\right]_{t=\phi} = 1$$

Thus:

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

and by Linearity:

$\Rightarrow \mathcal{L}^{-1}\{c\} = c\,\delta(t)$

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall $\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t=0 \\ 0 & \text{otherwise} \end{cases}$

which looks like $\frac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\left\{\frac{d}{dt}\mathbb{1}(t)\right\} = s\,\mathcal{L}\{\mathbb{1}(t)\} - \mathbb{1}(0^-)$$

Thus:

$$\mathcal{L}\{\delta(t)\} = 1$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall $\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases}$

which looks like $\frac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\left\{\frac{d}{dt}\mathbb{1}(t)\right\} = s\,\overbrace{\mathcal{L}\{\mathbb{1}(t)\}}^{1/s} - \cancelto{0}{\mathbb{1}(0^-)}$$

$$= 1 = \mathcal{L}\{\delta(t)\}$$

YES

Recap: Unit Impulse

$$\delta(t) = \lim_{\Delta \to 0} f_\Delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

"sifting property":

$$\int_a^b \delta(t) h(t)\, dt = \begin{cases} h(0) & \text{if } 0 \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Laplace Transform:

$$\mathcal{L}\{\delta(t)\} = 1$$

Useful property:

$$\frac{d}{dt} \mathbb{1}(t) = \delta(t)$$

Impulse Response

The <u>impulse response</u> of a system is the output $y(t)$ when $u(t) = \delta(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[c(s) - b(s)]}{r(s)}$$

$\Rightarrow u(t) = \delta(t) \Rightarrow b(s) = \emptyset$ and $U(s) = 1$

$\Rightarrow$ all ICs on $y(t)$ zero $\Rightarrow c(s) = \emptyset$

So:

$$Y(s) = G(s)$$

and thus

$$y(t) = \mathcal{L}^{-1}\{G(s)\} \triangleq g(t)$$

$\boxed{\text{The \underline{impulse response} } g(t) \text{ is the inverse transform of the transfer function } G(s)}$

Conversely, knowledge (or <u>measurement</u>) of $g(t)$ tells us what the transfer function is, and hence the governing diff'l eq'ns.

$\Rightarrow$ Foundation of "system identification" theory.

Additional Laplace Property

For any two functions $f_1(t), f_2(t)$ with transforms $F_1(s), F_2(s)$

$$\mathcal{L}^{-1}\{F_1(s)F_2(s)\} = \int_{0^-}^{\infty} f_1(t-\tau)f_2(\tau)\,d\tau$$

"convolution"

Implication: $\mathcal{L}^{-1}\{G(s)U(s)\} = \int_{0^-}^{\infty} g(t-\tau)u(\tau)\,d\tau$

proving generally what we showed specifically for the hovercraft problem.

There we had $\ddot{y}(t) = Ku(t)$

$$\Rightarrow G(s) = \frac{K}{s^2} \Rightarrow g(t) = Kt \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

and thus $g(t-\tau) = K(t-\tau)$.

Note:

Laplace actually let's us "divide out" the effect of any Known input to recover the transfer function (impulse response)

$$Y(s) = G(s)U(s) \qquad \text{(assuming } \emptyset \text{ ICs)}$$

$$\left[Y(s) = G(s)U(s)\right] \times \left(\frac{1}{U(s)}\right)$$

$$\left[\frac{Y(s)}{U(s)}\right] = G(s)\left[\frac{U(s)}{U(s)}\right]$$

$$= G(s) \cdot 1$$

response to ideal impulse.

Structure of Impulse Response

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{q(s)}{r(s)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\sum_{k=1}^{n} \frac{\gamma_k}{(s-p_k)}\right\} \quad p_k \text{ poles of } G(s)$$

or

$$\boxed{\begin{aligned} g(t) &= \sum_{k=1}^{n} \gamma_k e^{p_k t} \\ \gamma_k &= \left[(s-p_k)G(s)\right]_{s=p_k} \end{aligned}}$$

(assuming non-repeated modes for simplicity)

$$g(t) = \sum_{K=1}^{n} \gamma_K e^{p_K t}$$

Note:

$\Rightarrow$ $g(t)$ is a specific linear combination of the modes.

$\Rightarrow$ Like a special homogeneous response

Alternate characterization of system stability

$$\boxed{\lim_{t \to \infty} |g(t)| \to \emptyset}$$

(if system is stable)

Impulse response in state-space

For a state-space model recall: $G(s) = C(sI-A)^{-1}B + D$

Assume $D=0$ for simplicity (most common case)

then $$g(t) = \mathcal{L}^{-1}\{C(sI-A)^{-1}B\}$$
$$= C\,\mathcal{L}^{-1}\{(sI-A)^{-1}\}B \quad \text{(linearity)}$$

Let $\Phi(s) \triangleq (sI-A)^{-1}$ ($n \times n$ matrix)

and $\phi(t) \triangleq \mathcal{L}^{-1}\{\Phi(s)\}$ ($n \times n$ matrix)

Then
$$g(t) = C\phi(t)B$$

$\boxed{\Rightarrow \text{what is this matrix } \phi(t) \text{ ??}}$

Matrix Exponential Function

Note for scalar a,

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = \mathcal{L}\{(s-a)^{-1}\} = e^{at}$$

By analogy

$$\mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}^{-1}\{(s\mathbb{I}-A)^{-1}\} \triangleq e^{At}$$

The "matrix exponential function"

How to calculate it?

Laplace (and its inverse) works on matrices just like it does on vectors:

Apply inverse transform to each entry of $\Phi(s) = (s\mathbb{I}-A)^{-1}$ (so n^2 inverse transforms ugh!)

General Observations

Recall that $\Phi(s) = (sI - A)^{-1} = \dfrac{Q(s)}{r(s)}$

$Q(s) = \mathrm{Adj}(sI - A)$ nxn matrix of polynomials in s

$r(s) = \mathrm{Det}(sI - A)$ ordinary polynomial in s

Hence:

1.) Each entry of $\Phi(s)$ is rational

$\Rightarrow$ can use PFE tricks for inverse xform

2.) Each entry of $\Phi(s)$ has same denom, hence same poles (roots of $r(s)$)

Recall: roots of $r(s)$ same as eigenvalues of A

Thus: Each entry of $e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$ will be a linear combination of $e^{\lambda_k t}$ where λ_k are eigenvalues of A

Example

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \Rightarrow \text{companion form. By inspection}$$

$$r(s) = s^2 + 5s + 6 = (s+2)(s+3)$$

$\Rightarrow$ each entry of e^{At} is a linear comb. of e^{-2t}, e^{-3t}

$$(sI - A) = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix} \qquad Q(s) = \mathrm{Adj}(sI - A) = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\[2ex] \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix}$$

inverse xform each entry separately using PFE

Example, cont

(1,1) $\dfrac{s+5}{(s+2)(s+3)} = \dfrac{A_1}{s+2} + \dfrac{A_2}{s+3}$ $\quad A_1 = \dfrac{-2+5}{1} = 3$

$A_2 = \dfrac{-3+5}{-1} = -2$

(1,2) $\dfrac{1}{(s+2)(s+3)} = \dfrac{A_1}{s+2} + \dfrac{A_2}{s+3}$ $\quad A_1 = \dfrac{1}{1} = 1$

$A_2 = \dfrac{1}{-1} = -1$

(2,1) $\dfrac{-6}{(s+2)(s+3)}$ is just $-6 \times \left(\dfrac{1}{(s+2)(s+3)}\right)$

(2,2) $\dfrac{s}{(s+2)(s+3)} = \dfrac{A_1}{s+2} + \dfrac{A_2}{s+3}$ $\quad A_1 = \dfrac{-2}{1} = -2$

$A_2 = \dfrac{-3}{-1} = 3$

Thus here:

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ 6(e^{-3t} - e^{-2t}) & 3e^{-3t} - 2e^{-2t} \end{bmatrix}$$

Recap

So with a state-space model we equivalently have

$$g(t) = Ce^{At}B$$

where:

- e^{At} is an $n \times n$ matrix
- each entry of e^{At} is a linear combination of MODES
- MODES determined by poles (p_k) $\Longleftrightarrow$ eigenvalues of A (λ_k)

and thus:

$$g(t) = Ce^{At}B = \sum_{k=1}^{n} \gamma_k e^{p_k t}$$

a linear combination of MODES, just like before.

Step Responses

The (unit) step response of a system is the output $y(t)$ when $u(t) = \mathbb{1}(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[\cancelto{\emptyset}{c(s)} - \cancelto{\emptyset}{b(s)}]}{r(s)}$$

$U(s) = \frac{1}{s}$ here, so

$$Y(s) = \left(\frac{1}{s}\right) G(s) = \frac{q(s)}{s\, r(s)}$$

Intermediate Case 3 Situations

If $1.1 < \frac{|P_2|}{|P_1|} < 5$ (or 8 or 10)

$$\frac{4}{|P_1|} < t_s < \frac{6}{|P_1|}$$

Unfortunately, there is no simple formula for interpolating between the two limits based on the exact ratio.