ENAE311H – Aerodynamics I

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Grader: TBD

Class Schedule: Tues./Thurs., 9:30 – 10:45, ITV 1111

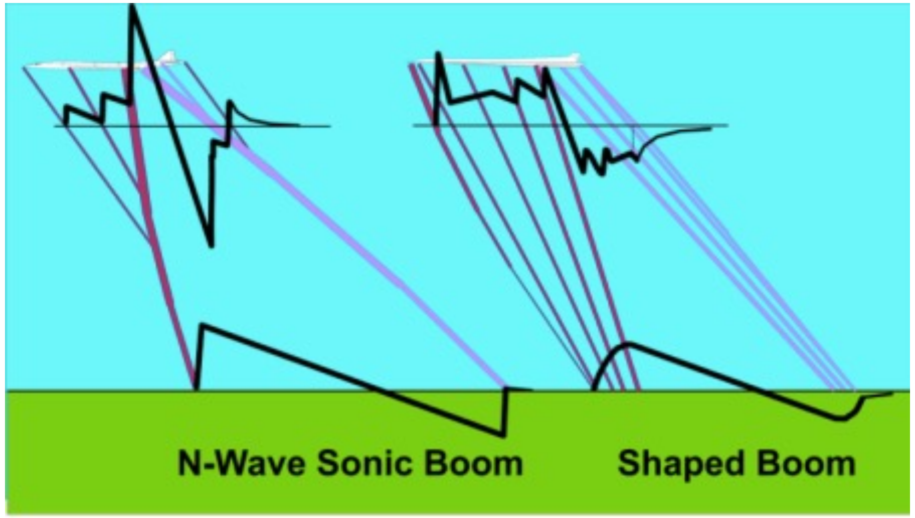
Office Hours: TBD

Textbooks: Fundamentals of Aerodynamics, John D. Anderson (Required)
Introduction to Fluid Mechanics, Fox & McDonald (FM Supplemental)
Elements of Gasdynamics, Liepmann & Roshko (LR Supplemental)
Modern Compressible Flow with Historical Perspective, John D. Anderson (A2 Supplemental)

Syllabus and Course Outline: uploaded on Canvas

Class Project:
Low Noise Supersonic Airplane Design Competition

- Design a supersonic airplane (based on Boom Overture)
  - Speed: Mach 1.7
  - Cruising Altitude: 60,000 ft
  - Passengers: 65 (premium-configured 1-1 cabin)
  - Full-scale length: 200ft (CFD analysis at 1:10 scale)
  - Wingspan: 60ft
  - Max takeoff weight: 170,000lb
  - Powerplant: 3 x Rolls-Royce medium-bypass turbofans without afterburners, 17,500 lbf (78kN) thrust
  - **It's expected that this could cut travel time in half, with New York to London flights taking 3hr15min, Tokyo to Seattle flights taking 4hr30min, etc.**
- Split up the design process into several parts
  - 2-D airfoil design
  - 3-D wing design
  - add streamlined main cabin without vertical stabilizer and no engines
- 28 students are split up into 4 teams
  - Split up team into 1 team captain, 2 CAD designers, 2 meshing and 2 CFD specialists
- Deliverables: 1 report describing the design and analysis are due at the last day of classes (limited to 20 pages) and a final presentation (20mins and 10mins Q&A)
- Evaluation Criteria: noise signature, correctness of analysis, how closely are the design parameters met?, as well as cabin comfort. (33% evaluation of presentation by your peers and 66% instructor)



Introduction

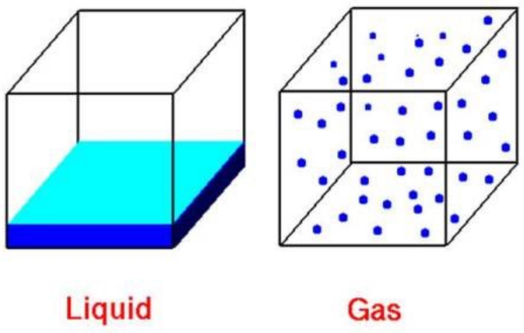
ENAE311H Aerodynamics I

Christoph Brehm

Definitions

- *What is aerodynamics?*
  - A branch of fluid mechanics dealing with the motion of air (or other gases) and its interactions with solid surfaces
  - **External aerodynamics:** forces and moments (also heat transfer at high speeds) acting on bodies moving through air
  - **Internal aerodynamics:** flow of air through ducts, pipes, etc.

- What is a fluid?
  - A fluid is a substance that continuously deforms in response to an applied force (c.f. a solid, which undergoes a finite deformation in response to constant force)
  - **Liquid:** essentially incompressible (constant density) because of strong intermolecular forces
  - **Gas:** compressible because intermolecular forces are very weak



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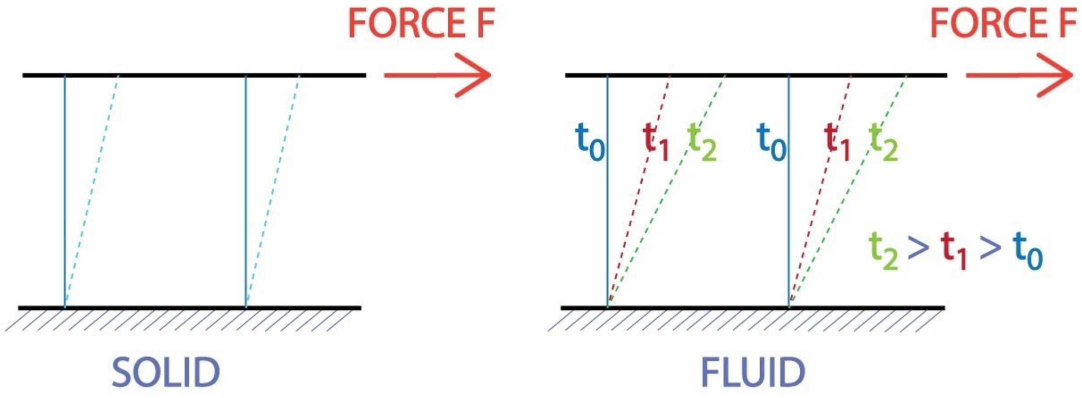

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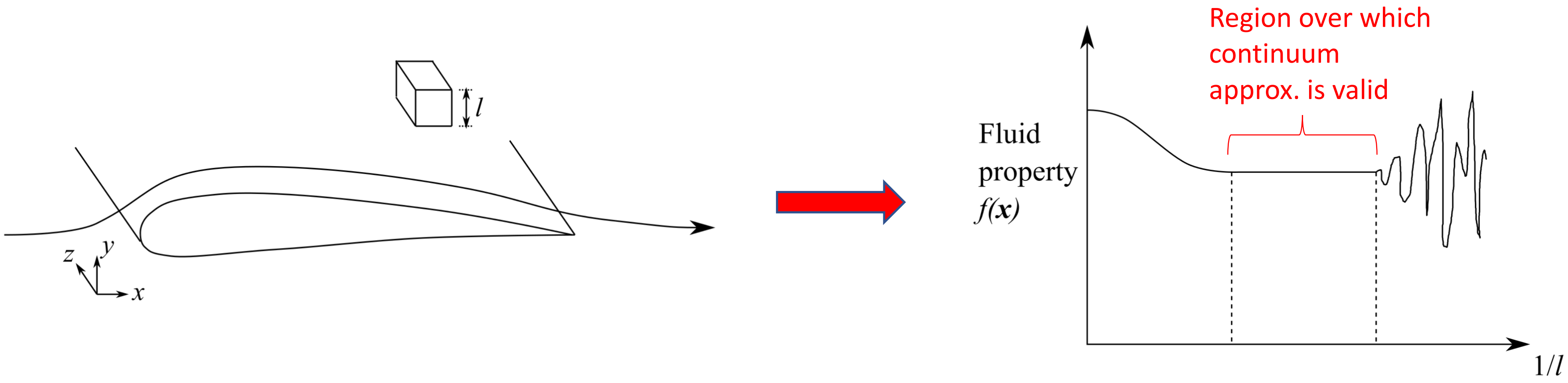
The continuum approximation

- Almost all fluid-dynamical theories treat a gas as a continuum (whereas we know it is composed of discrete molecules)
- Possible because, in macroscopic aerodynamics, typically there will be a huge number of molecules even in a volume corresponding to the smallest length scale of interest
  - For air at STP conditions, there are 2.7×10^{7} particles in box with 1-micron sides
- This means all flow conditions are well-defined and smoothly varying (except, e.g., at shock)

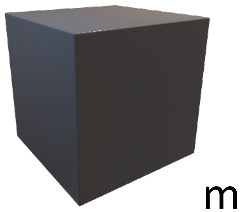


- Note, however, that transport coefficients must be derived from microscopic considerations

Density

- Density is the mass per unit volume of fluid – it is a scalar point property.
- Consider a fluid element at position $\boldsymbol{x}$; the density, $\rho(\boldsymbol{x})$, is defined as

$$\rho(\mathbf{x}) = \lim_{dv \to 0} \frac{dm}{dv}$$

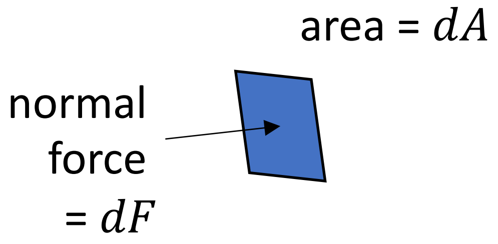


- At low speeds (M<0.3), air can generally be considered a constant density fluid.

Pressure

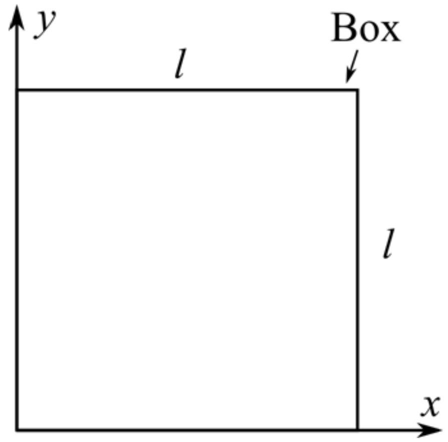
- Pressure is the force per unit area acting on a surface (real or imagined), moving at the bulk velocity of the flow – it is a scalar point property.
- Consider a fluid area element at position $\boldsymbol{x}$ moving with the fluid; the pressure, $p(\boldsymbol{x})$, is defined as

$$p(\mathbf{x}) = \lim_{dA \to 0} \frac{dF}{dA}$$



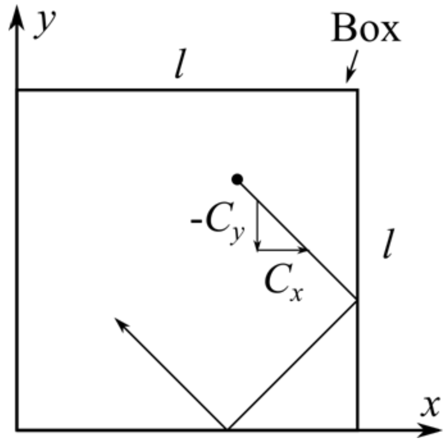
Pressure

- Pressure is related both to the density of the gas and the average random speed of the molecules that make it up. To see this, we consider a particle bouncing around inside a cubical box.



Pressure

- Pressure is related both to the density of the gas and the average random speed of the molecules that make it up. To see this, we consider a particle bouncing around inside a cubical box.
  - For simplicity, consider just a single particle colliding with x-normal wall (assume specular reflection)
  - Momentum transferred per collision: $2m|C_x|$
  - Time between collisions: $2l/|C_x|$
  - ⇒ Momentum deposited per unit time (force): mC_x^2/l
  - ⇒ Pressure from one particle: $mC_x^2/lA = mC_x^2/V$



Pressure

- Pressure is related both to the density of the gas and the average random speed of the molecules that make it up. To see this, we consider a particle bouncing around inside a cubical box.
  - Summing over all particles, we have

$$p(\mathbf{x}) = \frac{1}{V}\sum_i m_i C_{x_i}^2$$

  - Note we can repeat for y- and z-normal walls, sum and divide by 3 to obtain

$$p(\mathbf{x}) = \frac{1}{3V}\sum_i m_i(C_{x_i}^2 + C_{y_i}^2 + C_{z_i}^2) = \frac{1}{3V}\sum_i m_i C_i^2$$

  - From our definition of density

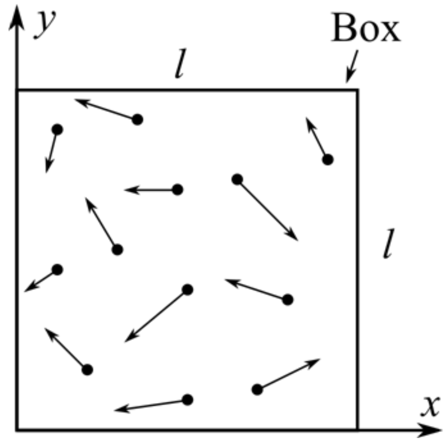
$$\rho(\mathbf{x}) = \frac{1}{V}\sum_i m_i$$

  - And defining the mean-square molecular speed

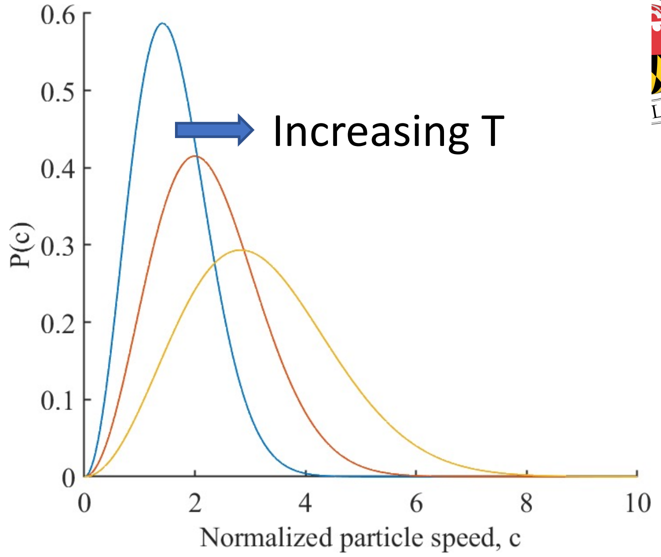
$$\overline{C^2} = \frac{\sum_i m_i C_i^2}{\sum_i m_i}$$

  - We obtain:

$$\boxed{p = \frac{1}{3}\rho\overline{C^2}}$$



Temperature



- Temperature is a thermodynamic property related to the average random kinetic energy of the molecules within the gas. It can be defined as

$$E_{tr} = \frac{3}{2}\mathcal{N}\tilde{R}T$$

where E_{tr} is the total random molecular kinetic energy, $\mathcal{N}$ is the number of moles of gas, and $\tilde{R}$ is the universal gas constant.

- Noting that $E_{tr} = \frac{1}{2}\sum_i m_i C_i^2$ and comparing for our earlier expression for pressure, we can write

$$pV = \mathcal{N}\tilde{R}T \qquad \text{or} \qquad p = \rho RT.$$

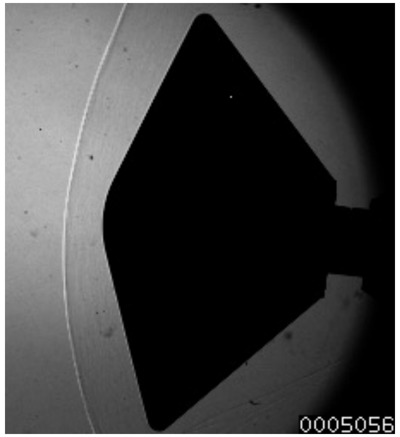
These are alternative expressions of the ideal gas law.

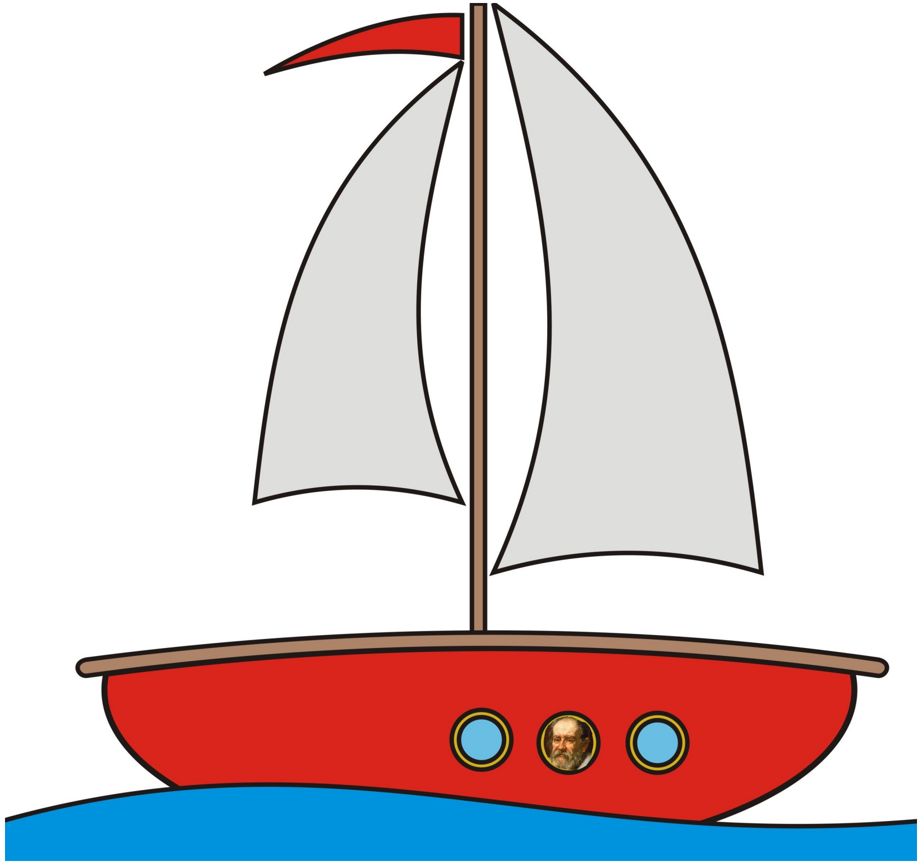
- At low speeds, temperature is essentially a passive scalar quantity (only way it affects the flow is through the viscosity) but for compressible flows it plays a much more important role.

Velocity

- Fluid velocity, $\boldsymbol{v}(\boldsymbol{x}) = (u, v, w)$ or (v_x, v_y, v_z), is the mean velocity of a fluid element at $\boldsymbol{x}$.
- We denote the velocity magnitude V.

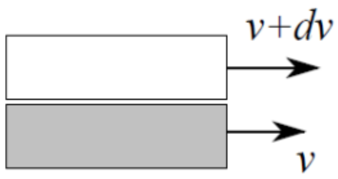
- An important concept in fluid mechanics is Galilean invariance:
  - "The laws of fluid motion are unchanged in any inertial reference frame." (an inertial frame is one moving at a constant velocity)
  - Note that all purely thermodynamic properties (e.g., p, ρ, T, s) are also unchanged by a shift in inertial reference frame
  - Galilean invariance makes possible the testing of flight vehicles in wind tunnels:





Friction/shear stress

- Consider two adjacent elements of fluid, moving parallel but at slightly different speeds; we can imagine them exerting a tangential "rubbing" force on one another (though the physical origin of this force is the diffusional transport of momentum across their common boundary).



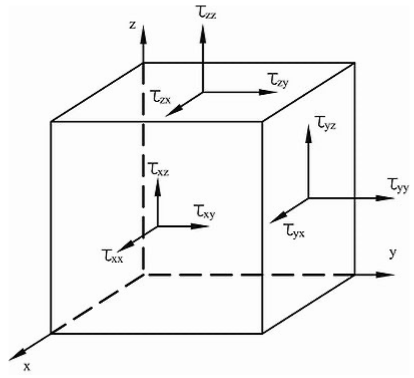
- If this tangential force is dF_f and the area over which it is applied is dA, the local shear stress, τ, is

$$\tau = \lim_{dA \to 0} \frac{dF_f}{dA}$$

- For flow past a solid surface, the shear stress is similarly the tangential force per unit area.
- A *Newtonian* fluid (as are most common fluids) is one for which the shear stress is proportional to the velocity derivative normal to the surface element, e.g., for flow in the *x* direction that changes only in the *y* direction,

$$\tau = \mu \frac{du}{dy}$$

  where μ, the coefficient of viscosity, depends only on temperature.
- Note that, in general, the velocity has three components, each of which may change in every direction, so the shear stress is actually a 2nd-order tensor (9-element matrix).



Units

Lecture 2: Airfoils

ENAE311H Aerodynamics I

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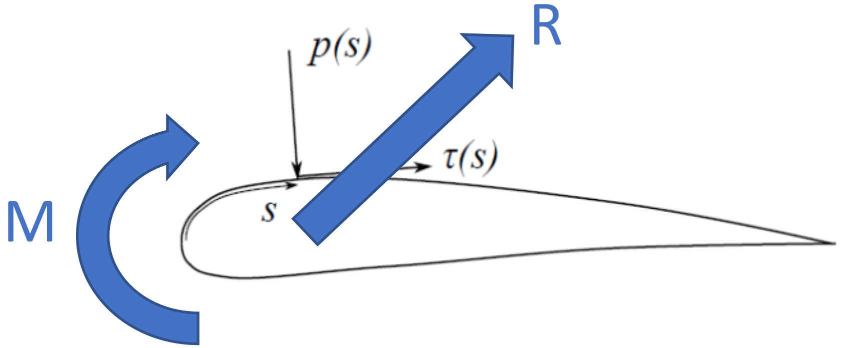
Definitions

- An airfoil is a body designed to produce a desired reaction force/moment when in motion relative to the surrounding air. On an aircraft, airfoils
  - provide lift to oppose gravity
  - provide forces and moments to stabilize and maneuver the aircraft

- Aerodynamic forces and moments derive from two sources:
  1. The pressure distribution on the body, p
  2. The shear-stress distribution, τ

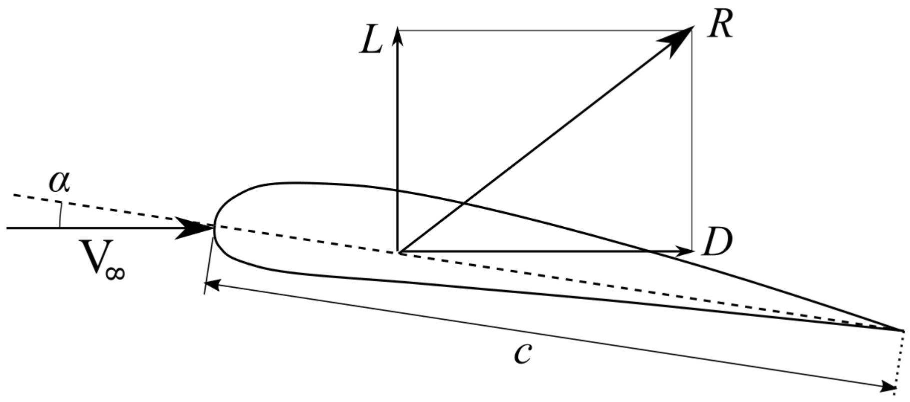
  → together these give the resultant force, R, and moment, M, acting on an airfoil



s is distance from leading edge on 2D airfoil

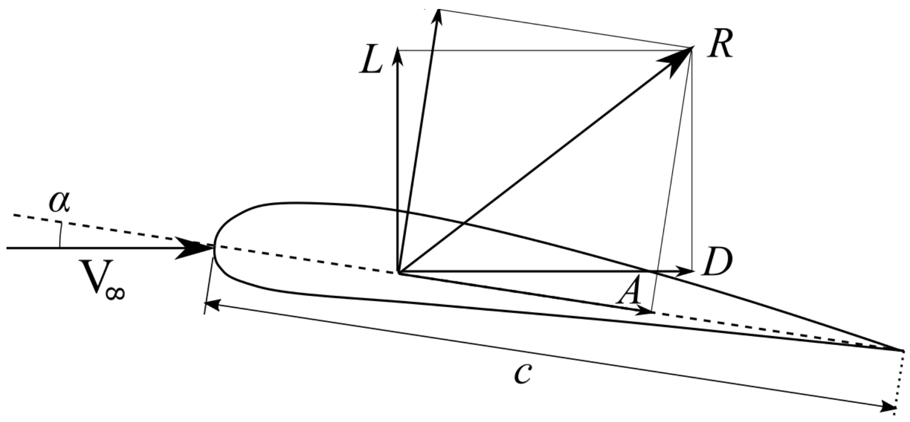
Force components

- It is useful to split the resultant force, $\boldsymbol{R}$, on a 2D airfoil into two components. The coordinate system for this split can be defined relative to either
  - The freestream direction, in which case the components are
    - Lift $(L) \equiv$ force normal to $\boldsymbol{V}_{\infty}$
    - Drag $(D) \equiv$ force tangential to $\boldsymbol{V}_{\infty}$



Force components

- It is useful to split the resultant force, $\boldsymbol{R}$, on a 2D airfoil into two components. The coordinate system for this split can be defined relative to either
  - The freestream direction, in which case the components are
    - Lift $(L) \equiv$ force normal to $\boldsymbol{V}_\infty$
    - Drag $(D) \equiv$ force tangential to $\boldsymbol{V}_\infty$
  - The airfoil chord direction, in which case the components are
    - Normal force component $(N) \equiv$ force normal to $\boldsymbol{c}$
    - Axial force component $(A) \equiv$ force tangential to $\boldsymbol{c}$



- Treating as vector quantities, in either case we have $\boldsymbol{L} + \boldsymbol{D} = \boldsymbol{N} + \boldsymbol{A} = \boldsymbol{R}$

- Given the normal and axial force components, the lift and drag can be calculated according to

$$
\begin{aligned}
L &= N\cos\alpha - A\sin\alpha \\
D &= A\cos\alpha + N\sin\alpha
\end{aligned}
$$

where α is the angle of attack, i.e., the angle between the freestream and chord directions.

Force/moment coefficients

- It is standard practice to cast the force components in nondimensional form – this allows them to be compared to reference quantities in a meaningful way. Let us introduce the dynamic pressure:

$$q_\infty \equiv \frac{1}{2}\rho_\infty V_\infty^2$$

- Since pressure has dimensions of force divided by area, by introducing a reference area S, this allows us to write nondimensional force coefficients as follows:

$$\text{Lift coefficient: } C_L \equiv \frac{L}{q_\infty S}$$

$$\text{Drag coefficient: } C_D \equiv \frac{D}{q_\infty S}$$

$$\text{Normal force coefficient: } C_N \equiv \frac{N}{q_\infty S}$$

$$\text{Axial force coefficient: } C_A \equiv \frac{A}{q_\infty S}$$

- Using a reference length, l, we can also define the nondimensional moment coefficient:

$$\text{(Pitching) Moment coefficient: } C_M \equiv \frac{M}{q_\infty S l}$$

- Note that S and l need to be chosen to be appropriate for the particular geometry (e.g., planform area and mean chord length for an airfoil).

Force/moment coefficients

- For a two-dimensional body, we specify the force and moment coefficients per unit span. For a 2D airfoil then:

$$c_l \equiv \frac{L'}{q_\infty c}, \quad c_d \equiv \frac{D'}{q_\infty c}, \quad c_m \equiv \frac{M'}{q_\infty c^2}$$

Here, a primed variable denotes the force or moment per unit span.

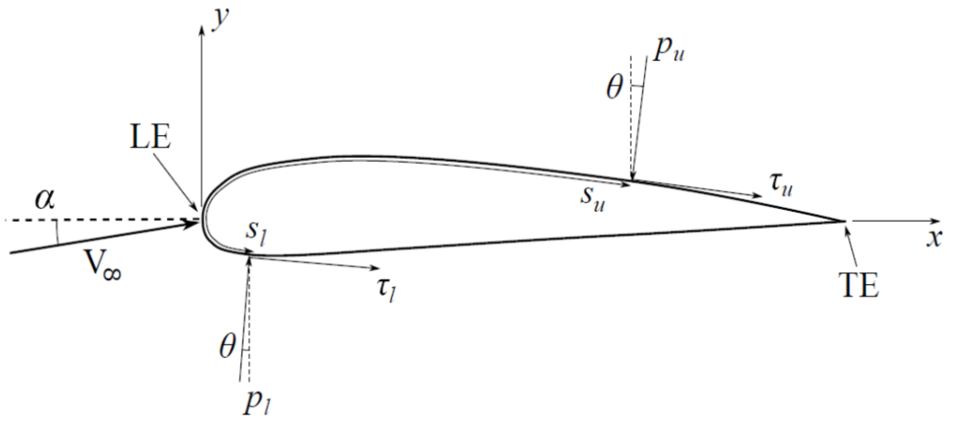
- It is often also useful to express the pressure and surface shear stress in terms of nondimensional pressure and skin-friction coefficients:

$$\text{Pressure coefficient: } C_p \equiv \frac{p - p_\infty}{q_\infty}$$

$$\text{Skin-friction coefficient: } C_f \equiv \frac{\tau}{q_\infty}$$

Calculating forces

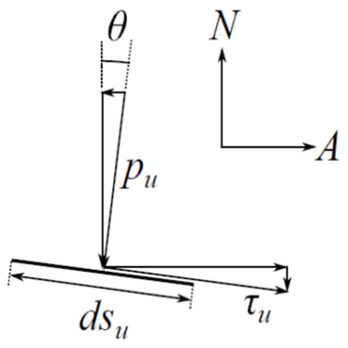
- Assume we are given the pressure and shear-stress distributions on an airfoil (along with its surface profile $y_u(x)$ and $y_l(x)$). We consider a coordinate system aligned with the airfoil chord:



θ is angle of surface normal relative to vertical (positive for clockwise)

- Consider first an element, ds_u, on the upper surface. The pressure and shear stress will contribute to the normal and axial forces as:

$$
\begin{aligned}
dN' &= -p_u ds_u \cos\theta - \tau_u ds_u \sin\theta \\
dA' &= -p_u ds_u \sin\theta + \tau_u ds_u \cos\theta
\end{aligned}
$$



- Similarly, for an element on the lower surface:

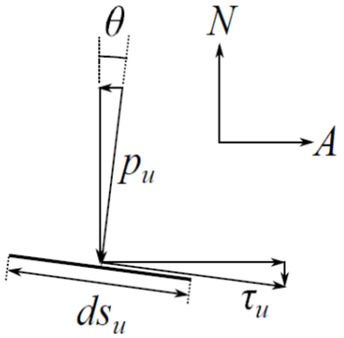
$$
\begin{aligned}
dN' &= p_l ds_l \cos\theta - \tau_l ds_l \sin\theta \\
dA' &= p_l ds_l \sin\theta + \tau_l ds_l \cos\theta
\end{aligned}
$$

Calculating forces

- We can now integrate p and τ along each surface of the airfoil to obtain the normal and axial forces:

$$
\begin{aligned}
N' &= -\int_{LE}^{TE} (p_u \cos\theta + \tau_u \sin\theta)ds_u + \int_{LE}^{TE} (p_l \cos\theta - \tau_l \sin\theta)ds_l \\
A' &= \int_{LE}^{TE} (-p_u \sin\theta + \tau_u \cos\theta)ds_u + \int_{LE}^{TE} (p_l \sin\theta + \tau_l \cos\theta)ds_l
\end{aligned}
$$

- It is generally easier, however, to integrate w.r.t. x than s. Noting from geometry that



$$dx = ds\cos\theta, \quad dy = -ds\sin\theta$$

and also that $dy = \frac{dy}{dx}dx$, we can then write

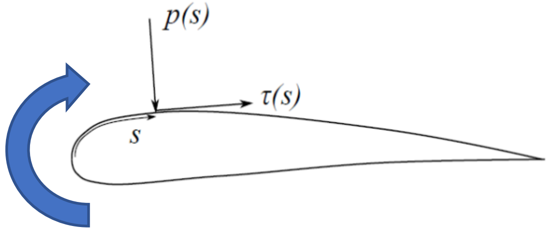
$$
\begin{aligned}
N' &= -\int_0^c \left(p_u - \tau_u \frac{dy_u}{dx}\right) dx + \int_0^c \left(p_l + \tau_l \frac{dy_l}{dx}\right) dx \\
&= \int_0^c (p_l - p_u)dx + \int_0^c \left(\tau_u \frac{dy_u}{dx} + \tau_l \frac{dy_l}{dx}\right) dx \\
A' &= \int_0^c \left(p_u \frac{dy_u}{dx} - p_l \frac{dy_l}{dx}\right) dx + \int_0^c (\tau_u + \tau_l)dx.
\end{aligned}
$$

Convert to L' and D' via our earlier coordinate transformations

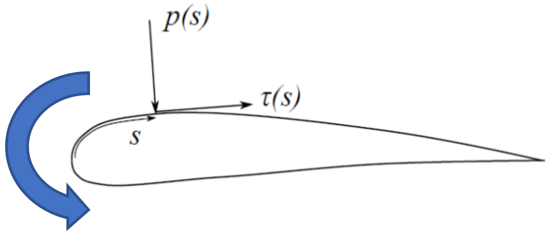
Calculating pitching moment

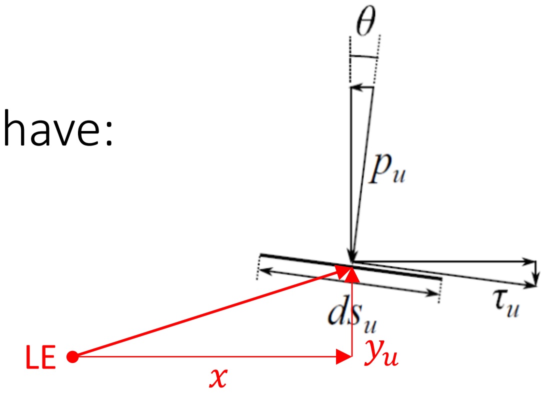
- Unlike the forces, the pitching moment depends on the point about which the moment is taken. Here we consider the moment about the leading edge and choose the convention that pitch up (increasing α) is a positive M:

M positive



M negative





- If we consider contributions to the moment from a small length (area) element, we have:

$$dM'_u = (p_u \cos\theta + \tau_u \sin\theta) x\, ds_u + (-p_u \sin\theta + \tau_u \cos\theta) y_u\, ds_u$$

$$dM'_l = (-p_l \cos\theta + \tau_l \sin\theta) x\, ds_l + (p_l \sin\theta + \tau_l \cos\theta) y_l\, ds_l$$

- Integrating from LE to TE:

$$M'_{LE} = \int_{LE}^{TE} [(p_u \cos\theta + \tau_u \sin\theta)x + (-p_u \sin\theta + \tau_u \cos\theta) y_u] ds_u + \int_{LE}^{TE} [(-p_l \cos\theta + \tau_l \sin\theta)x + (p_l \sin\theta + \tau_l \cos\theta) y_l] ds_l$$

- And with our expressions for dx and dy:

$$M'_{LE} = \int_0^c \left[p_u - p_l - \tau_u \frac{dy_u}{dx} - \tau_l \frac{dy_l}{dx} \right] x\, dx + \int_0^c \left[\left(p_u \frac{dy_u}{dx} + \tau_u \right) y_u + \left(-p_l \frac{dy_l}{dx} + \tau_l \right) y_l \right] dx$$

Formulae for force/moment coefficients

- If we divide our expressions for N', A', and M' through by $q_\infty c$ or $q_\infty c^2$, and make use of our definitions of the pressure and skin-friction coefficients, we can write for the force and moment coefficients:

$$
\begin{aligned}
c_n &= \frac{1}{c}\left[\int_0^c (C_{p_l} - C_{p_u})dx + \int_0^c \left(C_{f_u}\frac{dy_u}{dx} + C_{f_l}\frac{dy_l}{dx}\right)dx\right] \\
c_a &= \frac{1}{c}\left[\int_0^c \left(C_{p_u}\frac{dy_u}{dx} - C_{p_l}\frac{dy_l}{dx}\right)dx + \int_0^c (C_{f_u} + C_{f_l})dx\right] \\
c_{m_{LE}} &= \frac{1}{c^2}\left\{\int_0^c \left[C_{p_u} - C_{p_l} - C_{f_u}\frac{dy_u}{dx} - C_{f_l}\frac{dy_l}{dx}\right] x dx + \right. \\
&\left. \int_0^c \left[\left(C_{p_u}\frac{dy_u}{dx} + C_{f_u}\right) y_u + \left(-C_{p_l}\frac{dy_l}{dx} + C_{f_l}\right) y_l\right] dx\right\}
\end{aligned}
$$

Convert to c_l and c_d via our coordinate transformations

Center of pressure

- Although it is the distributed pressure and shear stress on an airfoil that contribute to the net forces and moment, it is possible to represent the resultant forces as acting through a single point to produce the same moment – this point we call the "center of pressure".
- In fact, in general there are an infinite number of points for which this will hold, but if we restrict ourselves to points along the chord line (so A' doesn't contribute to the moment), this reduces to finding the location x_{cp} such that

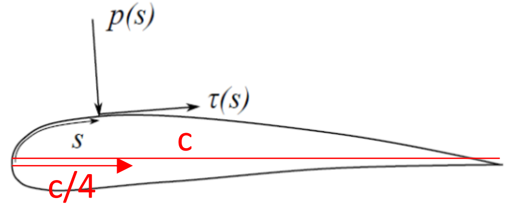
$$\boxed{M'_{LE} = -x_{cp}N'}$$

- If the angle of attack is small, we also have

$$x_{cp} \approx -\frac{M'_{LE}}{L'}$$

- Note that, if the moment were calculated about x_{cp}, the result would be zero.
- For a two-dimensional airfoil, x_{cp} is usually close to the quarter-chord point. For transforming between effective origins, we can use, for example,

$$M'_{LE} = -\frac{c}{4}N' + M'_{c/4} \qquad \text{or} \qquad M'_{LE} \approx -\frac{c}{4}L' + M'_{c/4}$$



1. Consider an infinitely thin flat plate with a 1 m chord at an angle of attack of 15° to an oncoming flow. The pressure distributions on the upper and lower surfaces are given by $p_u = 2 \times 10^4(x-1) + 2.7 \times 10^4$ and $p_l = 1 \times 10^4(x-1) + 1.1 \times 10^5$, where x is the distance from the leading edge along the chord; the shear stress distributions are $\tau_u = 144x^{-0.3}$ and $\tau_l = 360x^{-0.3}$. Here, the units of p and τ are N m^{-2}. Calculate the normal and axial forces, the lift and drag, moments about the leading edge and quarter chord, all per unit span, as well as the center of pressure.

2. A series of experiments is performed on a two-dimensional airfoil in which the lift, drag and moment coefficients (the latter about the quarter chord) are measured over a range of angles of attack from 0 to 10°. The lift coefficient curve is found to be well approximated by the equation

$$c_l = 0.2 + 6\alpha, \tag{1}$$

where α is the angle of attack in radians. The drag is found to be well approximated by

$$c_d = 0.006 + 0.3\alpha^2 \tag{2}$$

while $c_{m,c/4}$ increases linearly from -0.04 for α=0 to -0.03 for α=10°. Make a plot of x_{cp}/c as a function of α for this airfoil.

Lecture 3: Dimensional Analysis

ENAE311H Aerodynamics I

C. Brehm

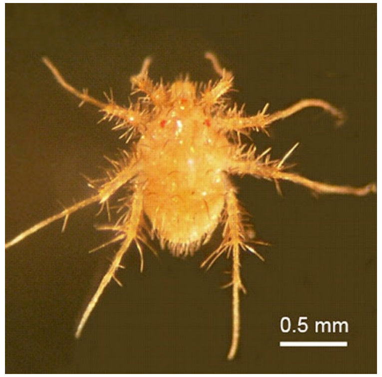
Introduction

- In studying any fluid-dynamical system, we are typically interested in how certain measurable properties of the system respond to changes in other parameters.
    - For an airfoil, for example, we may wish to know how the lift force changes as we vary the flow velocity.
- The dimensional properties and parameters we most commonly deal with consist of a number and a standard measure, e.g., L = 10 N.
- There are certain quantities that are more fundamental than these common dimensional quantities, however, in that:
    - the system can be fully described with fewer such quantities
    - differing (but in some ways similar) systems can be more meaningfully compared
- Most fundamental are dimensionless quantities that consist of numbers alone (e.g., force coefficients).

- It is the job of dimensional analysis to identify such dimensionless parameters.

Cheetah: 70 miles/hour, 16 body lengths per second

Australian tiger beetle: 1.86 m/s, 171 body lengths per second



Paratarsotomus macropalpis: 0.225 m/s, 322 body lengths per second

Buckingham π theorem

The essence of dimensional analysis is contained in the Buckingham π theorem.

Buckingham π theorem: let K be the number of basic dimensions required to describe the relevant physical variables in a system (in mechanics problems, $K = 3$, i.e., mass, length, time). Now, suppose we have a physical relationship between N dimensional variables, p_i:

$$f(p_1, p_2, ..., p_N) = 0$$

It is possible to restate this relationship as

$$F(\pi_1, \pi_2, ..., \pi_{N-K}) = 0$$

where the π_j are dimensionless parameters of the form

$$\pi_j = p_1^{a_1} p_2^{a_2} ... p_N^{a_N} \qquad (a_j \text{ rational})$$

The choice of the π_j is not unique, but should in general follow these guidelines:

- They should be of the form

$$\begin{aligned} \pi_1 &= g_1(p_1, ..., p_K, p_{K+1}) \\ \pi_2 &= g_2(p_1, ..., p_K, p_{K+2}) \\ &\cdots\cdots \\ \pi_{N-K} &= g_{N-K}(p_1, ..., p_K, p_{N-K}) \end{aligned}$$

  where the repeating set, $p_1, \dots, p_K$, contain all K dimensions.

- If there is a dependent dimensional variable, it shouldn't appear in the repeating set.
- The variables chosen for the repeating set should be *linearly independent* in their dimensions (e.g., if one variable has dimensions l/t, you shouldn't choose another variable with dimensions l^2/t^2).

Drag force on a sphere

As an example of the Buckingham π theorem, let us consider the drag force, D, on a sphere. From experience, we know that this should, at the very least, depend on the following variables:

1. The freestream flow density, ρ_∞
2. The freestream flow velocity, V_∞
3. The freestream coefficient of viscosity, μ_∞
4. The sphere size, which we can characterize through the radius, r
5. The freestream speed of sound, a_∞

Our dimensional relationship linking these variables is thus

$$f(D, \rho_\infty, V_\infty, \mu_\infty, r, a_\infty) = 0$$

And thus, $N = 6$.

Next, we write down the dimensions of these variables (square brackets denote dimensionality):

$$\begin{aligned}
[D] &= mlt^{-2} \\
[\rho_\infty] &= ml^{-3} \\
[V_\infty] &= lt^{-1} \\
[\mu_\infty] &= ml^{-1}t^{-1} \\
[r] &= l \\
[a_\infty] &= lt^{-1}.
\end{aligned}$$

where m, l, t denote mass length and time. We thus have $\mathrm{K} = 3$, and $\mathrm{N} - \mathrm{K} = 3$ dimensionless π groups.

Let us choose ρ_∞, V_∞, and r as our repeating set. Then:

$$\begin{aligned}
&\boxed{\pi_1 = g_1(\rho_\infty, V_\infty, r, D)} \\
&\pi_2 = g_2(\rho_\infty, V_\infty, r, \mu_\infty) \\
&\pi_3 = g_3(\rho_\infty, V_\infty, r, a_\infty)
\end{aligned}$$

Drag force on a sphere

From the Buckingham π theorem we have for π_1:

$$\pi_1 = \rho_\infty^{a_1} V_\infty^{a_2} r^{a_3} D^{a_4}$$

Note that π_1 is dimensionless, i.e., $[\pi_1] = 1$.

For the above expression to be dimensionally consistent then, we must have

$$(ml^{-3})^{a_1}(lt^{-}1)^{a_2} l^{a_3}(mlt^{-2})^{a_4} = m^{a_1+a_4} l^{-3a_1+a_2+a_3+a_4} t^{-a_2-2a_4} = 1$$

and so each exponent must be zero (e.g., $a_1 + a_4 = 0$).

Note that we have more variables than equations, so choose $a_4 = 1$ (since we are ultimately interested in D). Solving, we obtain $a_1 = -1, a_2 = -2, a_3 = -2$, and thus:

$$\pi_1 = \frac{D}{\rho_\infty V_\infty^2 r^2}$$

Since pure numbers are dimensionless, can rewrite as:

$$\pi_1 = \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 \pi r^2} = \frac{D}{q_\infty S} \quad = C_D!$$

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\end{aligned}$$

Drag force on a sphere

Following a similar procedure for π_2 and π_3, we have

$$\pi_2 = \frac{\rho_\infty V_\infty r}{\mu_\infty} \quad \text{= Reynolds number } (Re)$$

$$\pi_3 = \frac{V_\infty}{a_\infty} \quad \text{= Mach number } (M)$$

Then, according to the Buckingham π theorem, we can write:

$$F(C_D, Re, M) = 0$$

or

$$C_D = F_1(Re, M)$$

i.e., have reduced the number of independent variables from five to two.

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For the airfoil studied in the previous lecture, we could go through the same procedure for the lift and pitching moment (in place of the drag). Note we should then also include the angle of attack as a π group (angles are always dimensionless), i.e.,

$$\begin{aligned} C_D &= F_2(Re, M, \alpha) \\ C_L &= F_3(Re, M, \alpha) \\ C_M &= F_4(Re, M, \alpha) \end{aligned}$$

Note that, for flows involving thermodynamics/heat transfer, our analysis would involve additional parameters, such as:

- $\gamma = c_p/c_v$ (ratio of specific heats)
- T_w/T_∞ (wall temperature ratio)
- Pr (Prandtl number – ratio of momentum diffusion to heat conduction)

Limiting cases

Sometimes (often in limiting cases), only a single dimensionless variable can be formed, i.e., $N = K + 1$. Our equation involving the π groups,

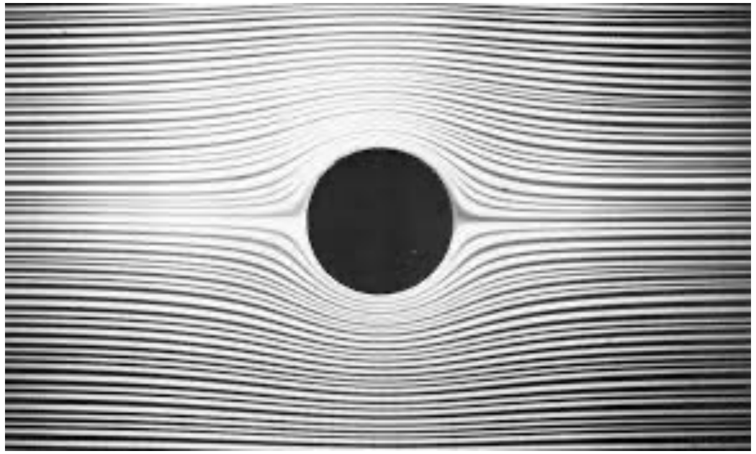
$$F(\pi_1, \pi_2, ..., \pi_{N-K}) = 0$$

then becomes simply

$$F(\pi_1) = \text{const.}$$

For F to be nontrivial, we must then have

$$\boxed{\pi_1 = \text{const.}}$$



Take, for example, the case of a sphere moving extremely slowly in a fluid. Then we would expect neither the sound speed nor the fluid inertia (represented by ρ_∞) to be important.

We can thus write

$$f(D, V_\infty, r, \mu_\infty) = 0$$

and so

$$\pi_1 = \frac{D}{\mu_\infty V_\infty r} = \text{const.} = c$$

Rearranging, we then have

$$D = c\mu_\infty V_\infty r$$

Detailed calculations show $c = 6\pi$.

Limiting cases

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At the other extreme, if the sphere radius or fluid density is very large, we wouldn't expect viscosity to play much of a role.

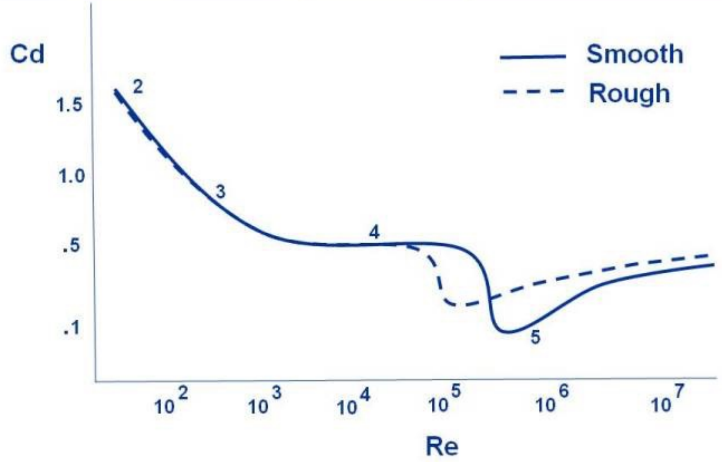
In this case, we can write

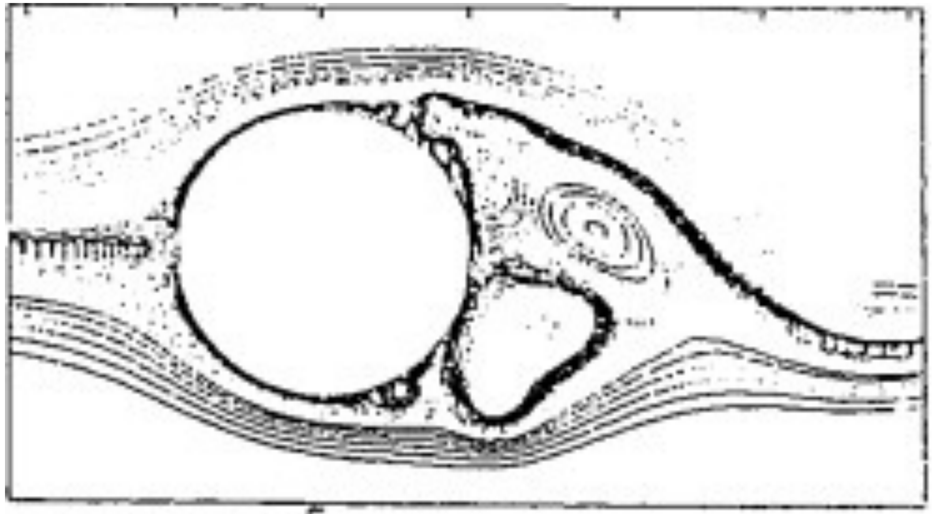
$$f(D, \rho_\infty, V_\infty, r) = 0$$

from which

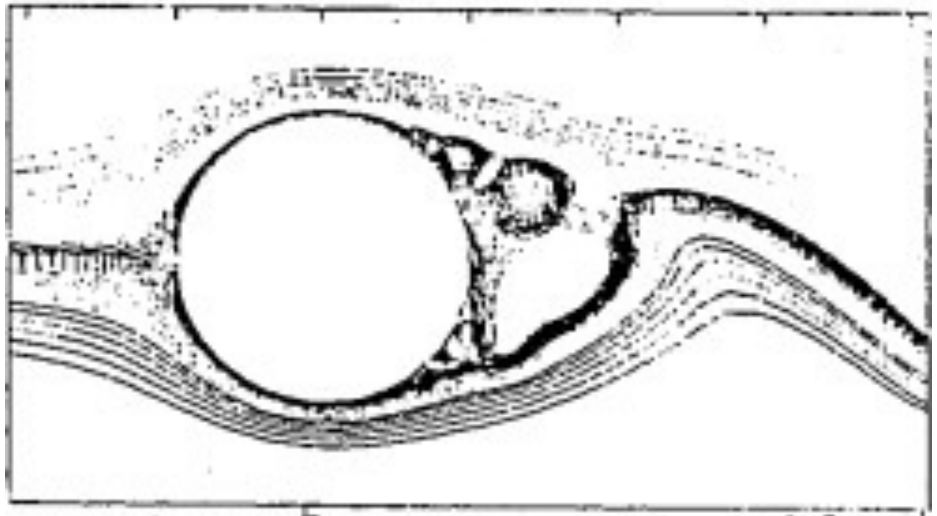
$$\pi_1 = \frac{D}{\rho_\infty V_\infty^2 r^2} = \text{const.}$$

We therefore conclude that C_D should become constant at high Reynolds numbers.



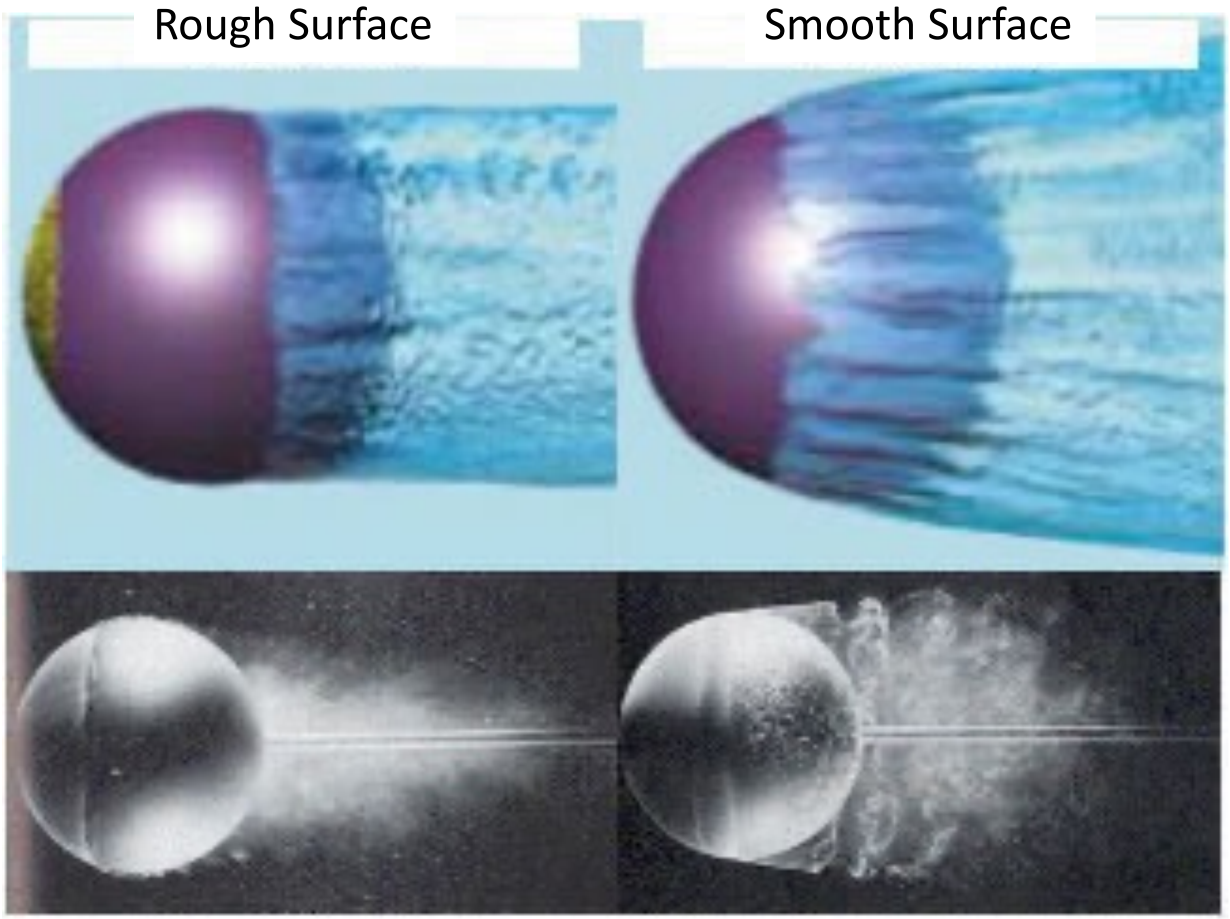


$Re = 2 \times 10^5$

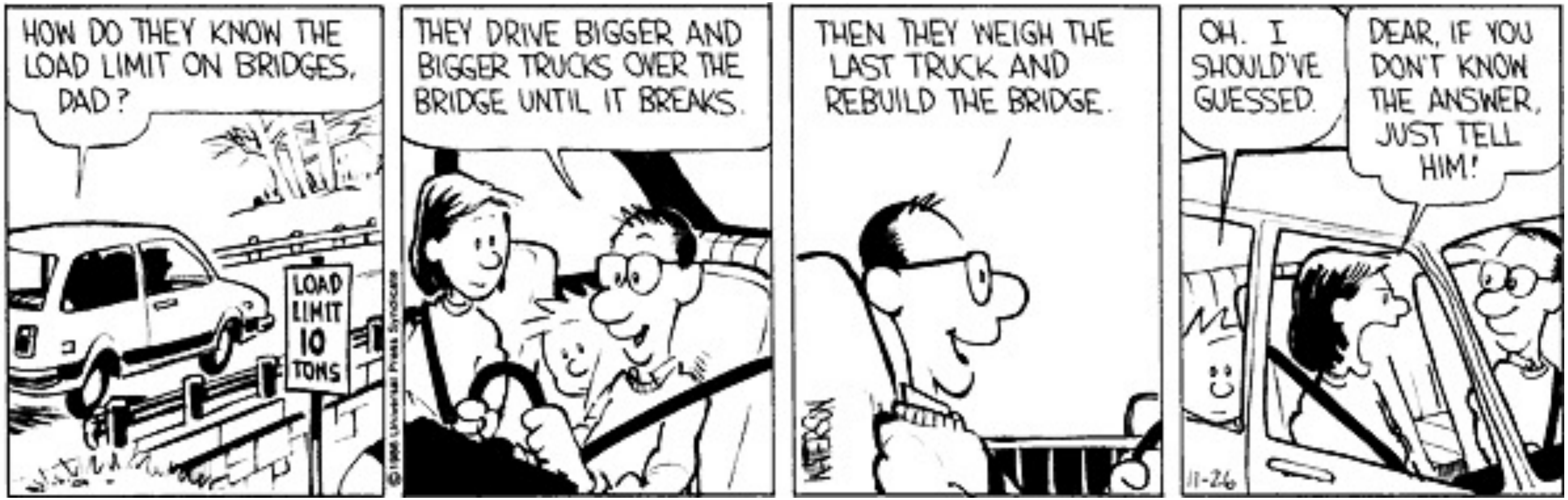


$Re = 4 \times 10^5$ A 3

The drag crisis has been clearly obtained by this scheme.



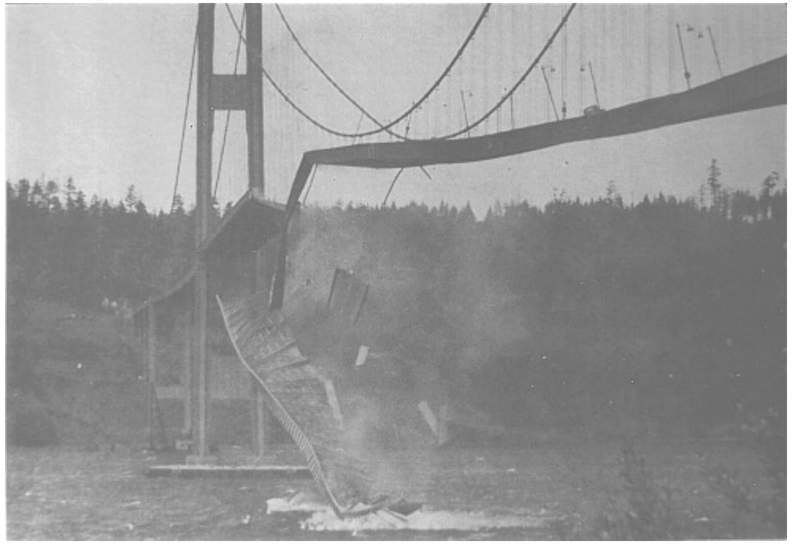
Why Modeling?

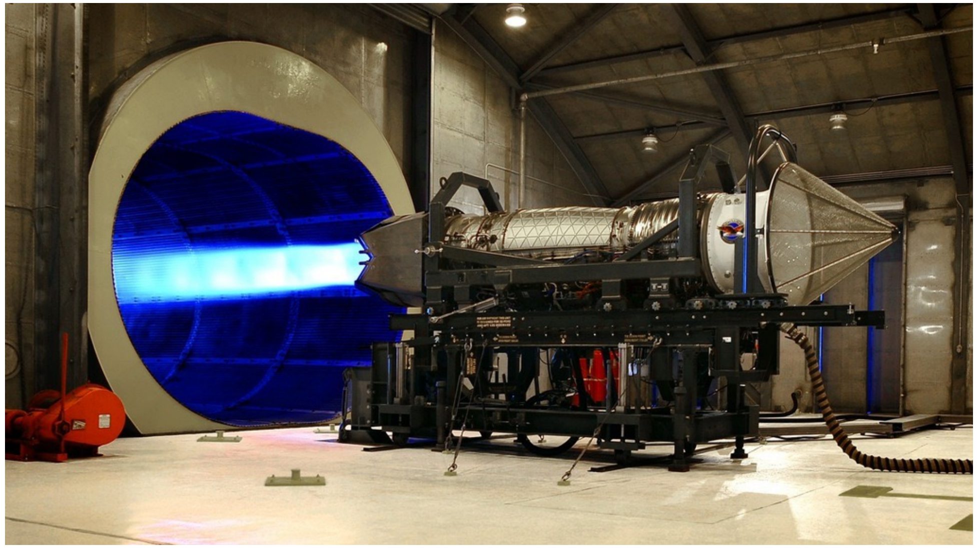


- This is not the way we should design things …

Tacoma Narrows Bridge

- Bridge collapse in WA (1940) due to wind-induced oscillations




Model Similarity (1)

- We are interested in knowing a particular variable for a large design (drag, pressure drop, oscillation frequency, etc.)
- IF we know what other variables affect the variable of interest then we can generalize the relationship in terms of Pi groups

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots)$$

Model Similarity (2)

- If we build a model that has the same value of all dependent PI groups, then the independent Pi group measured will be accurate

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots)$$

- If:

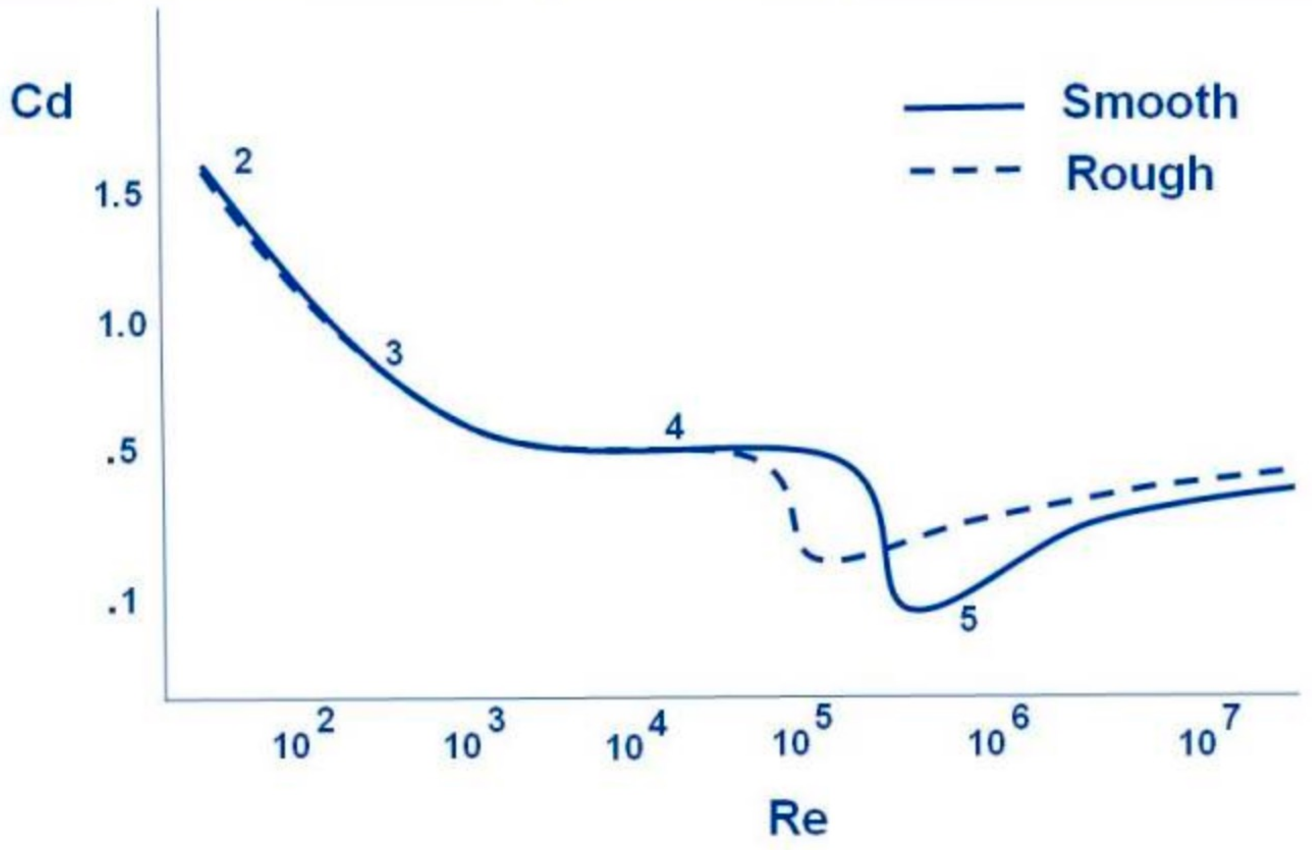
$$\Pi_{2,\mathrm{mod}} = \Pi_{2,real} \qquad \Pi_{3,\mathrm{mod}} = \Pi_{3,real}$$

- Then:

$$\Pi_{1,\mathrm{mod}} = \Pi_{1,real}$$

The drag characteristics of a torpedo are to be studied in a water tunnel using a 1:5 scale model. The tunnel operates with freshwater at 20 degrees Celsius, whereas the prototype torpedo is to be used in seawater at 15.6 degrees Celsius. To correctly simulate the behavior of the prototype moving with a velocity of 30m/s, what velocity is required in the wall tunnel?

Drag: D=f(Re)



For dynamic similarity, the Reynolds number must be the same for model and prototype. Thus,

$$\frac{V_m D_m}{\nu_m} = \frac{VD}{\nu}$$

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so that

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Since, ν_m (water @ 20°C) = $1.004 \times 10^{-6}\ m^2/s$ (Table B.2), ν (seawater @ 15.6°C) = $1.17 \times 10^{-6}\ m^2/s$ (Table 1.6), and $D/D_m = 5$, it follows that

$$V_m = \frac{\left(1.004 \times 10^{-6}\ \frac{m^2}{s}\right)}{\left(1.17 \times 10^{-6}\ \frac{m^2}{s}\right)} (5)\left(30\ \frac{m}{s}\right) = \underline{\underline{129\ \frac{m}{s}}}$$

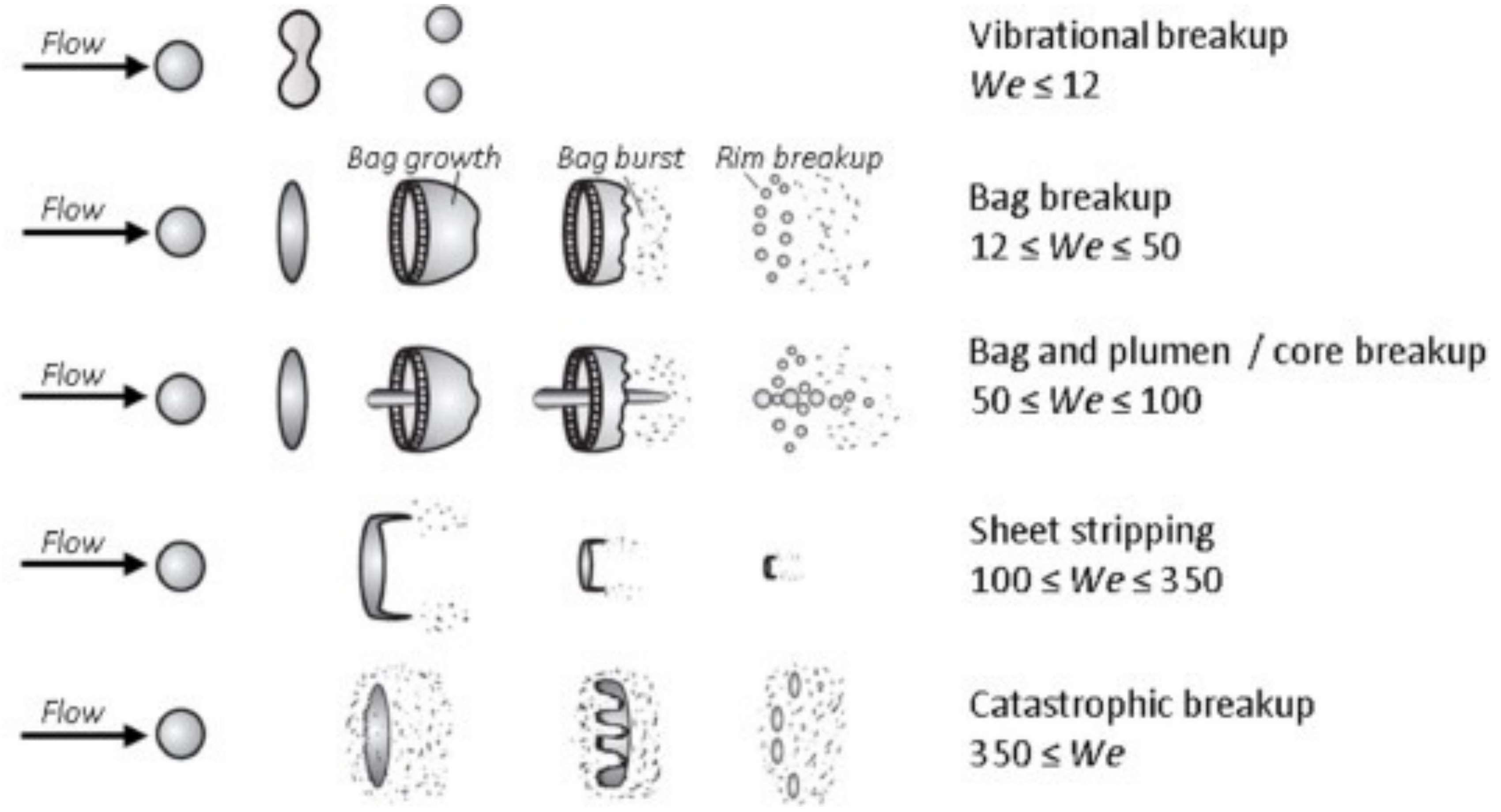
Example:

- I want to know the pressure drop that will occur in a 3-foot diameter, 1 mile long oil pipe while pumping 500 lbm/s of oil (this determines the size of pump required)
- I have a ¼” pipe to experiment with

$$\frac{\Delta p}{\rho V^2} = f\left(\mathrm{Re}, \left(\frac{L}{D}\right)\right)$$

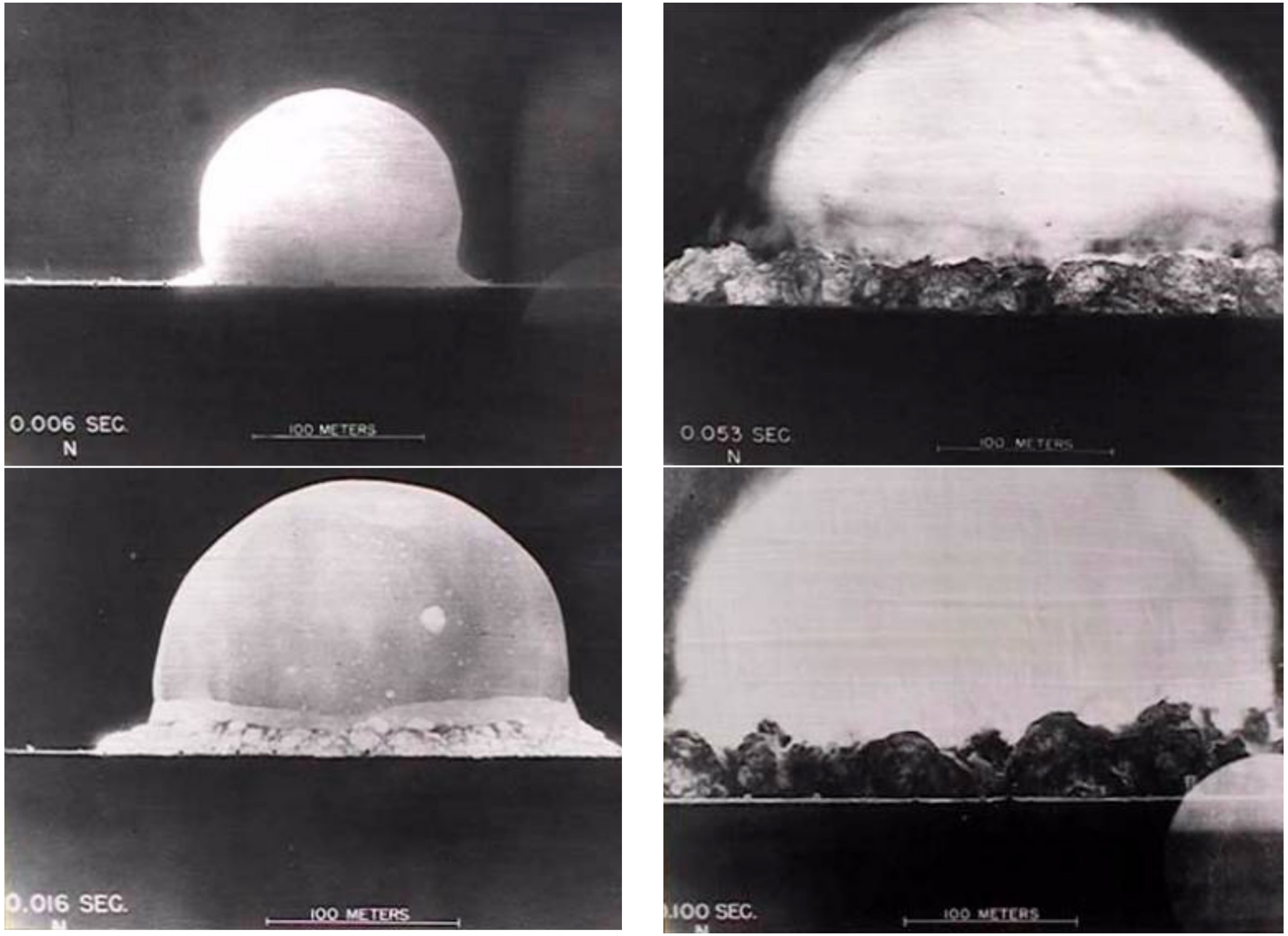
- Choose pipe length so that L/D is the same
- Choose fluid and velocity so that Re is the same

Droplet breakup



$$\mathrm{We} = \frac{\text{Drag Force}}{\text{Cohesion Force}} = \left(\frac{8}{C_\mathrm{D}}\right) \frac{\left(\frac{\rho\, v^2}{2}\, C_\mathrm{D}\, \pi \frac{l^2}{4}\right)}{(\pi\, l\, \sigma)} = \frac{\rho\, v^2\, l}{\sigma}$$

1945 Trinity Test (New Mexico)



Based on these photographs a British physicist named G. I. Taylor was able to estimate the power released by the explosion (which was still a secret at that time).
How can the following pictures be used to make this estimate?

First two assumptions need to be made:

1. The energy (E) was released in a small space.
2. The shock wave was spherical.

We have the size of the fire ball (R as a function of t) at several different times. How does the radius (R) depend on:

- energy (E)
- time (t)
- density of the surrounding medium (ρ – initial density of air)

Let's perform a dimensional analysis of the problem:

- [R] = L :radius is determined by a distance
- [E] = ML2/T2 :energy is determined by a mass times a distance squared divided by timesquared.
- [t] = T :Time is determined by the time.
- [ρ] = M/L3 :density is determined by a mass divided by a distance cubed.

We can say

$$[R] = L = [E]^x[\rho]^y[t]^z$$

Substituting the units for energy, time and density that we listed above we have:

$$[R] = L = M^{(x+y)}L^{(2x-3y)}T^{(-2x+z)}$$

This provides three simultaneous equations:
$x + y = 0$,
$2x - 3y = 1$,
$-2x + z = 0$,

yielding the results:
$x = 1/5$, $y = -1/5$, $z = 2/5$.

The radius of the shock wave is therefore:
$R = E^{1/5}\rho^{-1/5}t^{2/5} * \text{constant}$

Let’s assume the constant is approximately 1.
Solving the equation for E we get:
$E = (R^5\rho)/t^2$.

At $t = .006$ seconds the radius of the shock wave was approximately 80 meters. The density of air is $\rho = 1.2$ kg/m3. Plugging these values into the energy equation gives:
$E = (805) \times 1.2/(.0062)$ kg $*$m2/s2
$= 1 \times 10^{14}$ kg $*$m2/s2

corresponds to E = 25 kilo −tons of TNT

Lecture 4: Dynamical Similarity and Flow Regimes

ENAE311H Aerodynamics I

Christoph Brehm

Flow similarity

Consider the flow fields generated by two different bodies. We define the two flows as *dynamically similar* if:

1. The streamline patterns (i.e., paths in space taken by the flow) are geometrically similar
2. The distributions of $\frac{V}{V_\infty}$, $\frac{p}{p_\infty}$, $\frac{T}{T_\infty}$, etc. are identical throughout if plotted in common nondimensional coordinates.
3. The force and moment coefficients are the same.

To ensure dynamical similarity, it is required that:

1. The bodies (and any other solid boundaries) are geometrically similar
2. All relevant nondimensional similarity parameters (e.g., Re, M) are the same for the two flows.

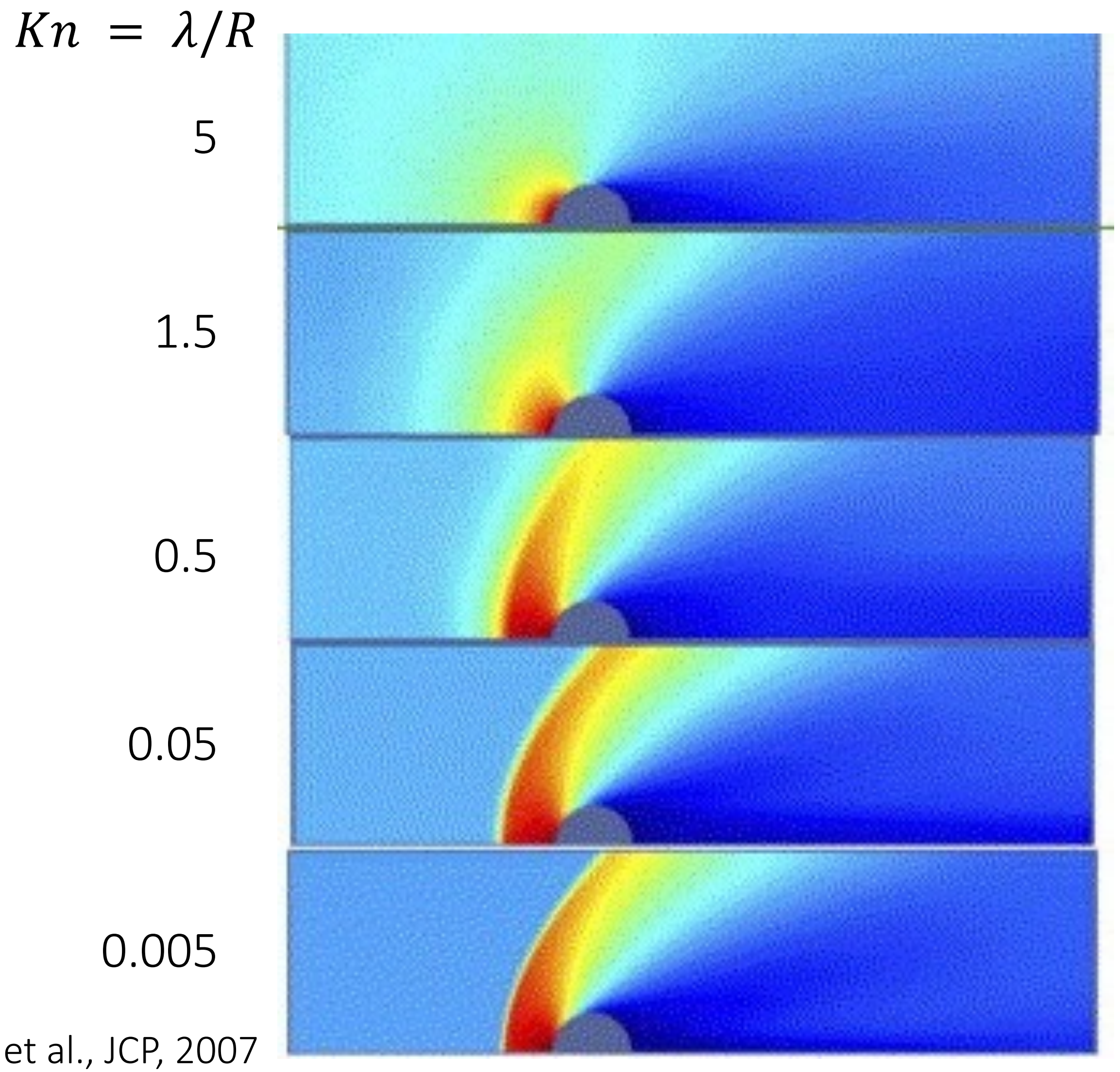
Continuum versus rarefied flows

We have already discussed the continuum approximation, which holds for the flows we will be interested in in this course. If we imagine, however, either shrinking the body in question or decreasing the density of the gas so that the body dimension is comparable to the mean free path (mean distance travelled by molecules between collisions), certain non-continuum effects become important:

- No-slip boundary conditions no longer hold
- Shocks become diffuse
- In free-molecular regime, molecules impact surfaces without interacting with one another.

The parameter that governs the degree of rarefaction of a flow is the Knudsen number, Kn:

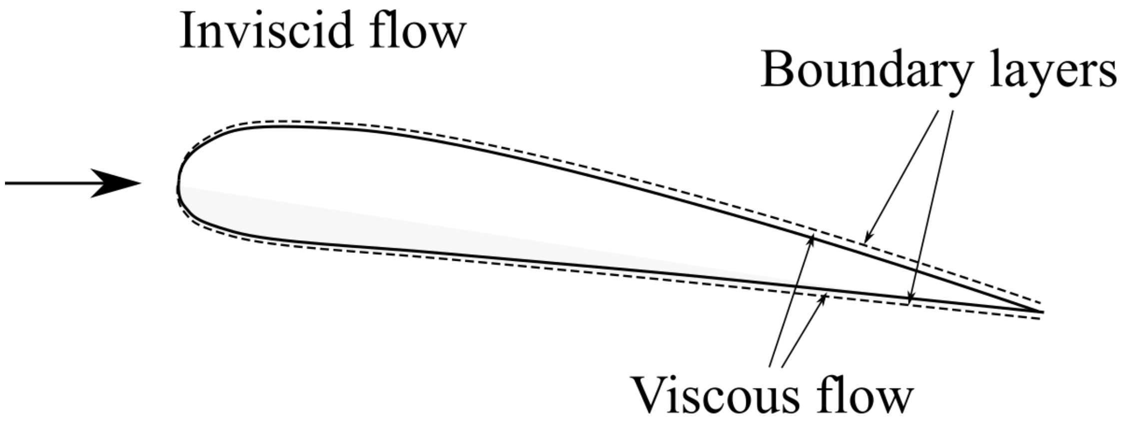
$$Kn = \frac{\lambda}{d} = \frac{mean\ free\ path}{body\ dimension}$$



From Kolobov et al., JCP, 2007

Inviscid versus viscous flows

- Viscosity is brought about by molecular transport of momentum → all flows to some extent viscous
- The degree of influence of flow viscosity depends on the Reynolds number:
  - At low Re, the entire flowfield is essentially viscous
  - As $Re \to \infty$, the flowfield becomes effectively inviscid
  - For large but finite Re, the viscous effects are typically confined to a “boundary layer” close to the surface (pressure forces determined by external inviscid flow; viscous forces determined by viscous boundary-layer flow).
- Certain flowfields (e.g., separated flows)are dominated by viscous effects, even at large Re.



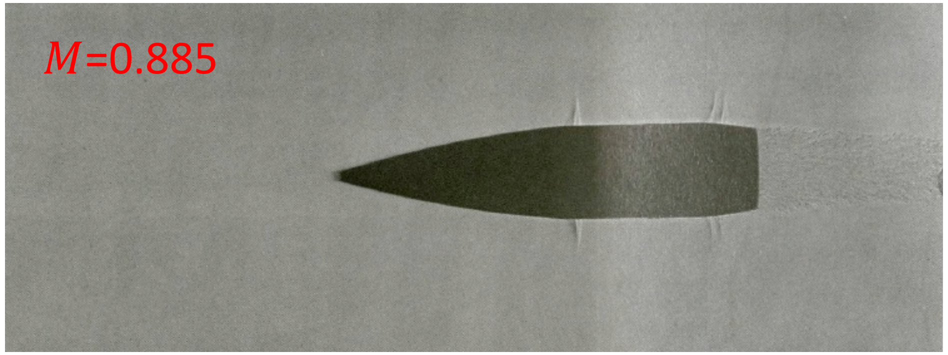
Compressible versus incompressible flow

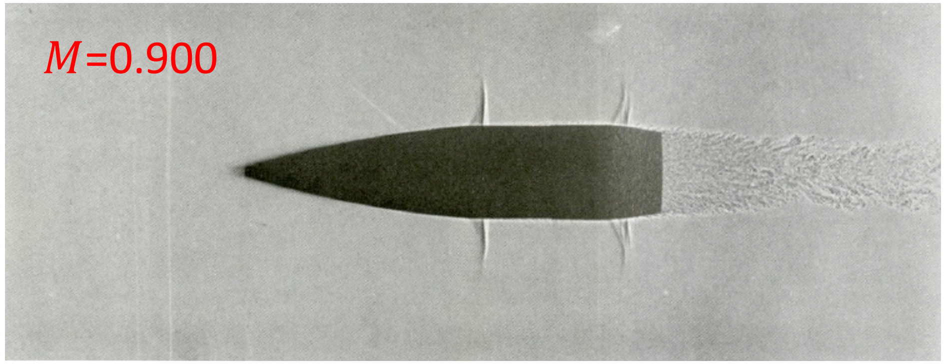
- All fluids are compressible to some extent, in that the density will change, if ever so slightly.
- Liquids are effectively incompressible, except at rather extreme conditions.
- The flow of gases can, to a reasonable approximation, be treated as incompressible for Mach numbers up to approximately 0.3.

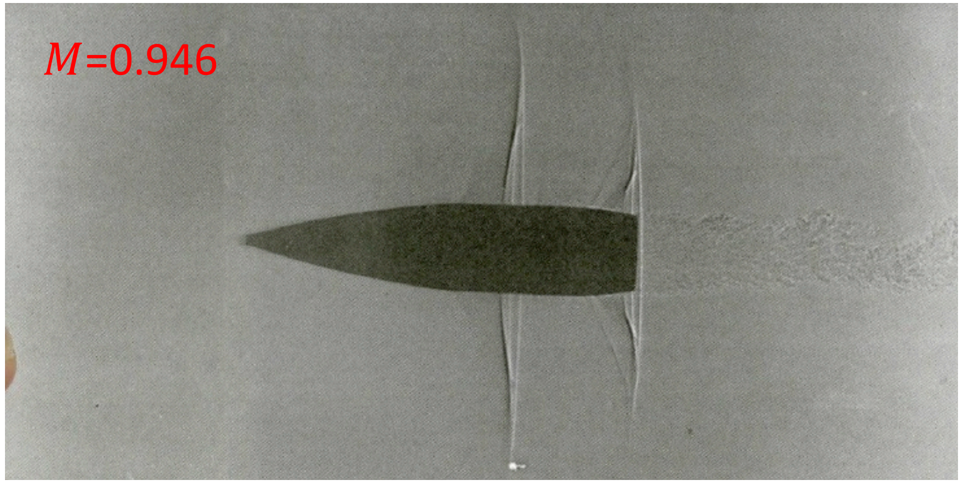
Mach-number regimes

- Subsonic flow ($M < 1$)
  - Mach number is <1 everywhere; information can propagate everywhere within the fluid domain and streamlines smoothly varying

- Transonic flow ($0.8 \lesssim M \lesssim 1.2$)
  - Contains mixed regions of subsonic and supersonic flow; characterized by weak shocks at very steep angles







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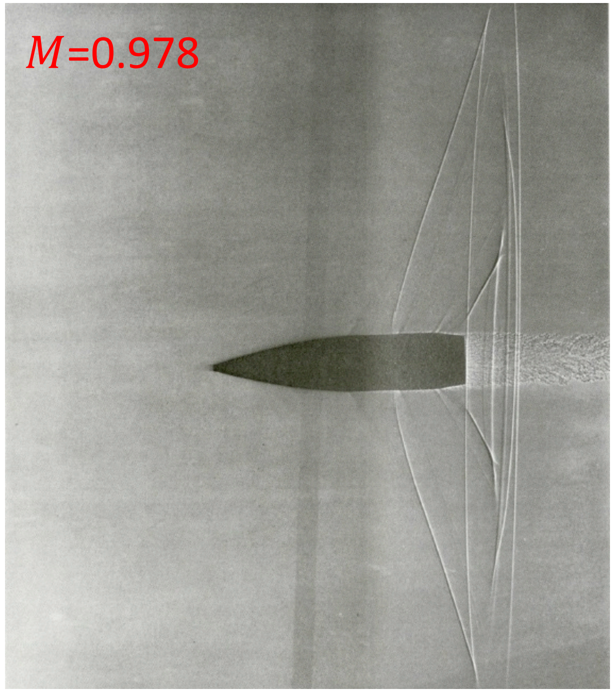

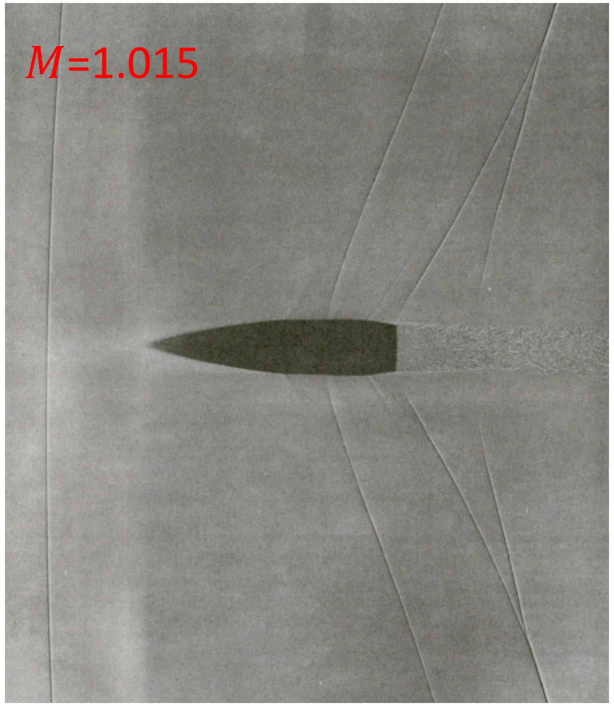

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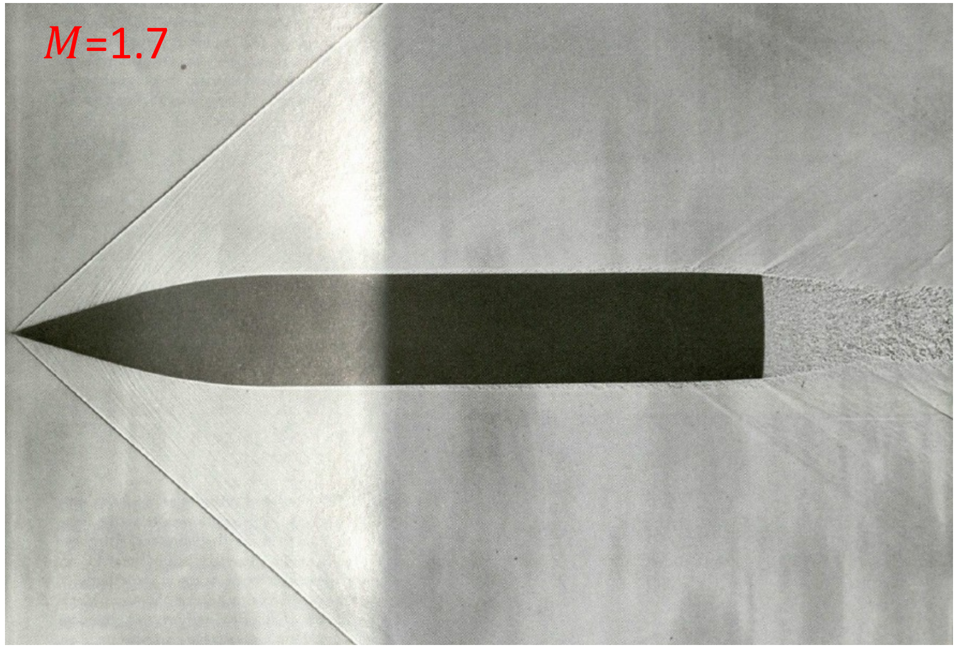

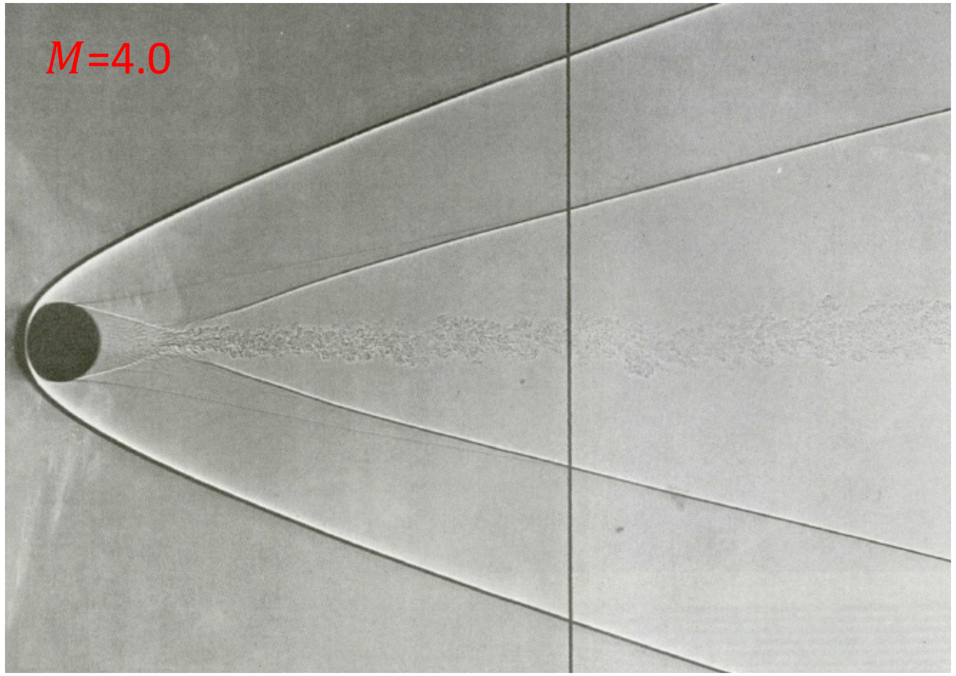

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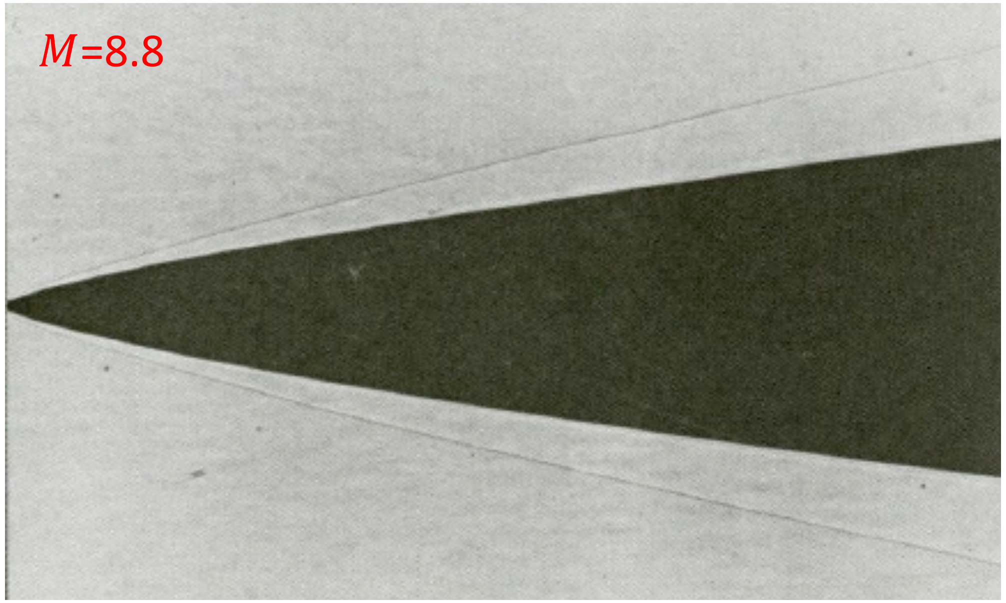
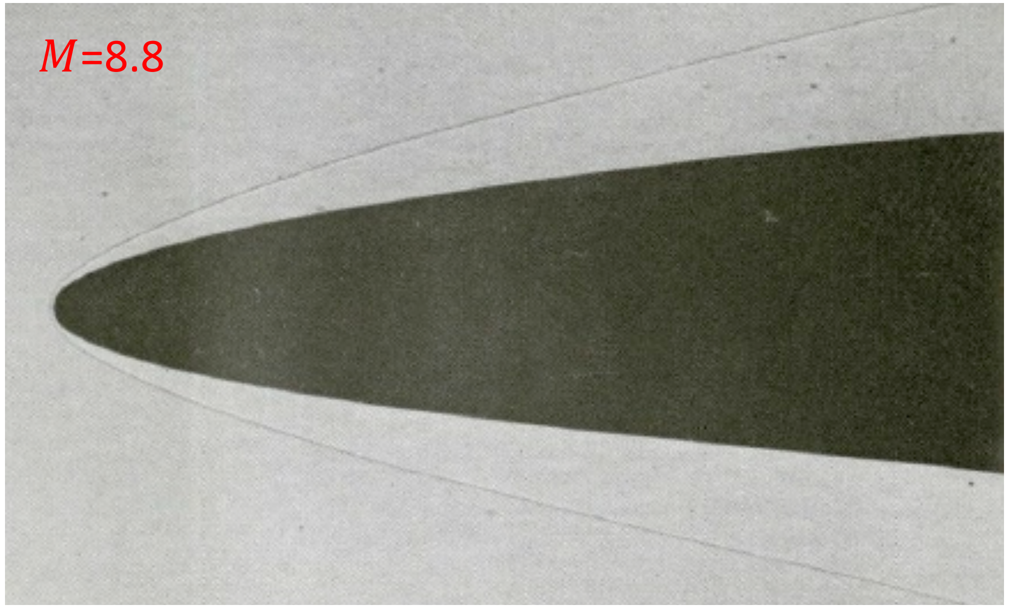
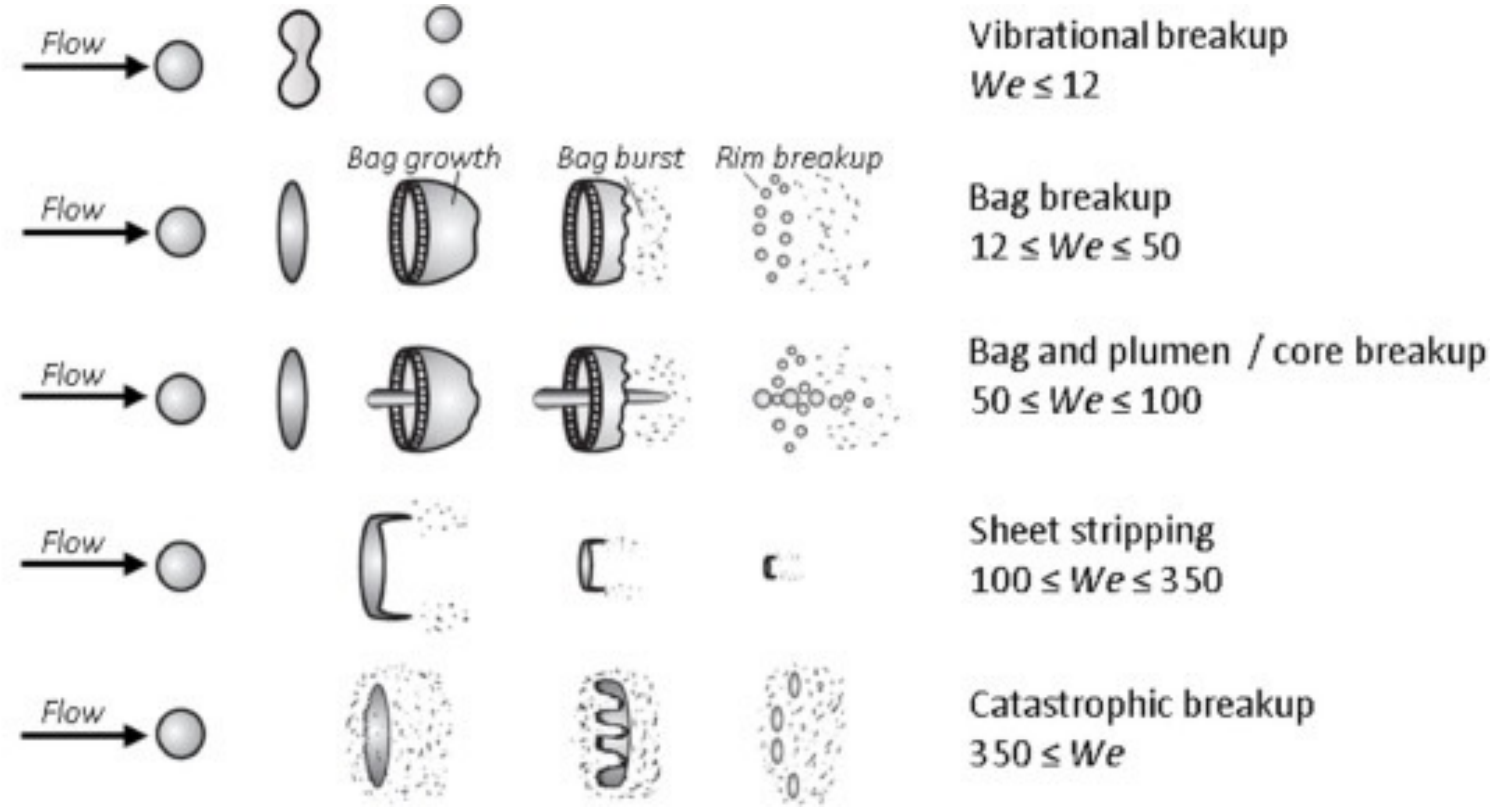
- Supersonic flow ($M > 1$)
  - Characterized by discontinuous flow structures and limited domains of influence
- Hypersonic flow ($M \gg 1$)
  - Characterized by thin shock layers for slender bodies and strongly curved shocks for blunt bodies
  - At very high Mach numbers, real-gas effects such as molecular dissociation become important





Droplet breakup



$$\mathrm{We} = \frac{\text{Drag Force}}{\text{Cohesion Force}} = \left(\frac{8}{C_\mathrm{D}}\right) \frac{\left(\frac{\rho\, v^2}{2}\, C_\mathrm{D}\, \pi \frac{l^2}{4}\right)}{(\pi\, l\, \sigma)} = \frac{\rho\, v^2\, l}{\sigma}$$

Lecture 5: Review of Vectors, Fields, and Operations

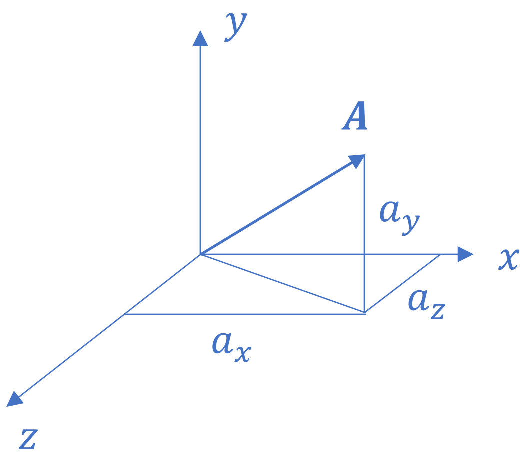
ENAE311H Aerodynamics I

Christoph Brehm

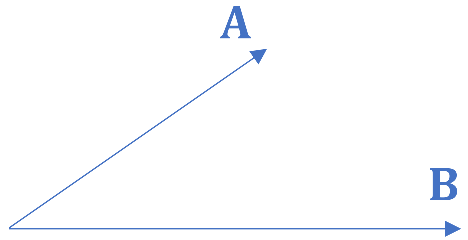
Vectors

- *Definition:* A vector is a directed line segment with both a magnitude (length) and direction
- The vectors of interest to us in this course are velocity, $\boldsymbol{v}$, acceleration, $\boldsymbol{a}$, and force, $\boldsymbol{F}$ (we denote vectors with boldface throughout this course).
- A *unit vector* is a vector of magnitude unity (denote with a hat); we call the unit vectors aligned with the coordinate axes *versors* (for a Cartesian coordinate system, the versors are $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\widehat{\boldsymbol{k}}$.
- If a vector is given by $\boldsymbol{A} = (a_x, a_y, a_z)$= $a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\widehat{\boldsymbol{k}}$, the magnitude of $\boldsymbol{A}$ is

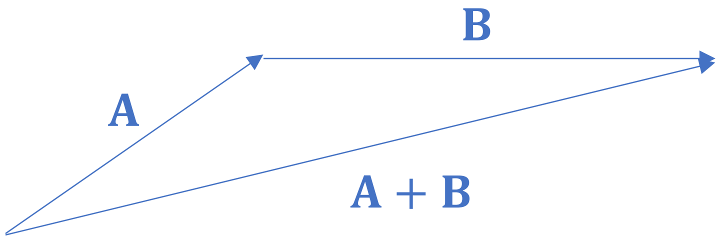
$$||\boldsymbol{A}|| = \sqrt{\boldsymbol{a}_x^2 + \boldsymbol{a}_y^2 + \boldsymbol{a}_z^2}$$



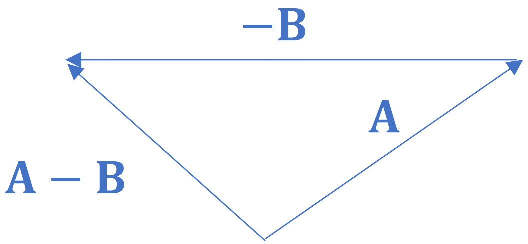
Vector operations



- Vector addition/subtraction:

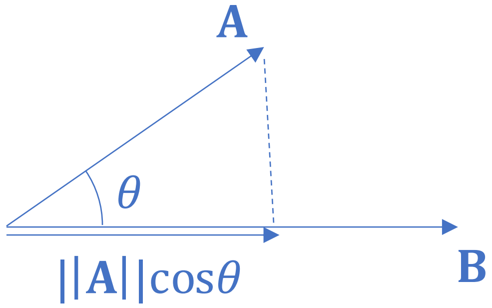


$$\begin{aligned}\mathbf{A}+\mathbf{B} &= (a_x+b_x)\hat{\mathbf{i}}+(a_y+b_y)\hat{\mathbf{j}}+(a_z+b_z)\hat{\mathbf{k}}\\ \mathbf{A}-\mathbf{B} &= \mathbf{A}+(-\mathbf{B})=(a_x-b_x)\hat{\mathbf{i}}+(a_y-b_y)\hat{\mathbf{j}}+(a_z-b_z)\hat{\mathbf{k}}\end{aligned}$$



- Multiplication – scalar/dot product:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= a_xb_x+a_yb_y+c_xc_y\\ &= ||\mathbf{A}||\,||\mathbf{B}||\cos\theta\end{aligned}$$



- Multiplication - vector/cross product:

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}\\ &= (a_yb_z-a_zb_y)\hat{\mathbf{i}}-(a_xb_z-a_zb_x)\hat{\mathbf{j}}+(a_xb_y-a_yb_x)\hat{\mathbf{k}}\\ &= ||\mathbf{A}||\,||\mathbf{B}||\sin\theta\,\hat{\mathbf{n}},\end{aligned}$$

Gradient of a scalar field

- *Scalar field:* a scalar quantity given as a pointwise function of space and time, e.g., $p(x, y, z, t)$
- The gradient of a scalar field, p, is given by

$$\nabla p = \frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}}$$

- The magnitude of the gradient, $||\nabla p||$, is the maximum rate of change of p per unit length of the coordinate system
- To calculate the component of the gradient in a particular direction, $\boldsymbol{s}$, can use

$$\frac{\partial p}{\partial s} = \nabla p \cdot \hat{\mathbf{s}} = ||\nabla p|| \; ||\hat{\mathbf{s}}|| \cos\theta = ||\nabla p|| \cos\theta$$

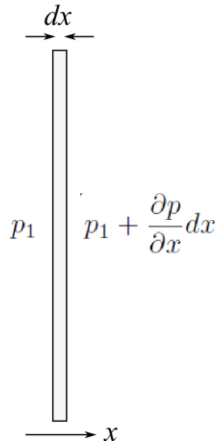
where θ is the angle between ∇p and $\boldsymbol{s}$.

- Note that ∇p provides the driving force to accelerate the flow:

Consider flat plate to the right (x-normal faces) – force acting in x-direction is given by

$$\begin{aligned} F &= p_1 A - p_2 A \approx p_1 A - \left(p_1 + \frac{\partial p}{\partial x}dx\right)A \\ &= -\frac{\partial p}{\partial x}dxA, \end{aligned}$$

and so $\boxed{F \propto \frac{\partial p}{\partial x}}$



Divergence and curl of a vector field

- *Vector field:* a vector quantity given as a pointwise function of space and time, e.g., $\mathbf{v}(x, y, z, t)$
- The divergence of a vector field, $\mathbf{v}$, is given by

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial x_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

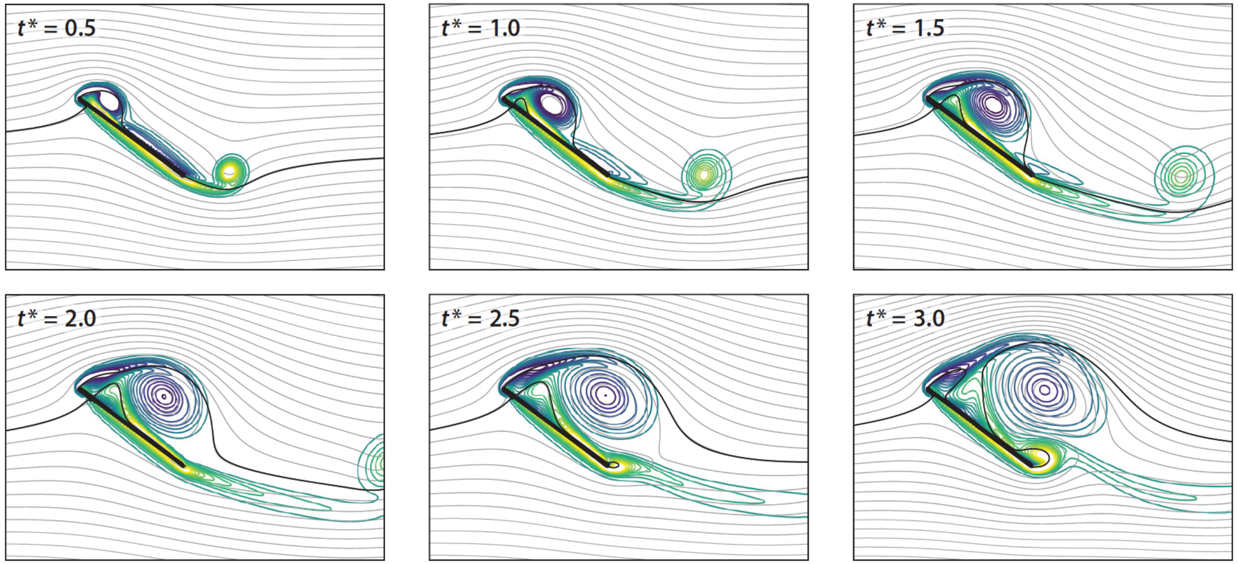
- It is a scalar quantity describing the tendency of the vector field to behave as a source or a sink.
- The divergence of the velocity field is particularly important in the context of conservation of mass.

- The curl of a vector field, $\mathbf{v}$, is given by

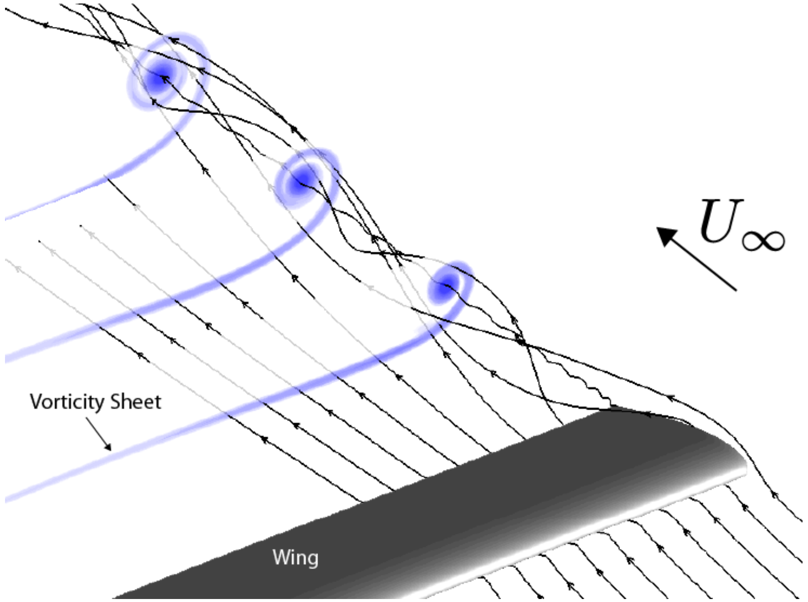
$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

- It is a vector quantity describing the rotational tendency of the underlying vector field.
- The curl of the velocity field is known as the “vorticity” and is a very important quantity in many branches of fluid mechanics.

Vortices



Leading-edge vortex



Wingtip vortices

Line, surface, and volume integrals

- Line integrals:

$$\int_a^b \mathbf{v} \cdot \mathbf{ds} = \int_a^b \mathbf{v} \cdot \hat{\mathbf{s}} ds$$

  - If the curve is closed, then $\int_a^b \to \oint_C$

- Surface integrals:

$$\begin{aligned} \iint_S p\mathbf{dA} &= \iint_S p\hat{\mathbf{n}}dA \to \text{vector (force)} \\ \iint_S \mathbf{v} \cdot \mathbf{dA} &= \iint_S \mathbf{v} \cdot \hat{\mathbf{n}}dA \to \text{scalar (volumetric flow rate)} \end{aligned}$$

rate of change of a volume

- Volume integrals:

$$\begin{aligned} \iiint_V \rho dV &\to \text{scalar (mass)} \\ \iiint_V \mathbf{v} dV &\to \text{vector} \end{aligned}$$

Integral theorems of vector calculus

- Stokes' theorem:

$$\oint_C \mathbf{v} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{dA}$$

- Gauss' (divergence) theorem:

$$\iint_S \mathbf{v} \cdot \mathbf{dA} = \iiint_V \nabla \cdot \mathbf{v} dV$$

where *S* is the closed surface bounding *V*.

- Gradient theorem:

$$\iint_S p\mathbf{dA} = \iiint_V \nabla p dV$$

where *S* is again the closed surface bounding *V*.

$$\Delta\mathcal{V} = [(\mathbf{V}\Delta t)\cdot\mathbf{n}]dS = (\mathbf{V}\Delta t)\cdot\mathbf{dS} \tag{2.28}$$

Over the time increment Δt, the total change in volume of the whole control volume is equal to the summation of Equation (2.28) over the total control surface. In the limit as $dS \rightarrow 0$, the sum becomes the surface integral

$$\oiint_S (\mathbf{V}\Delta t)\cdot\mathbf{dS}$$

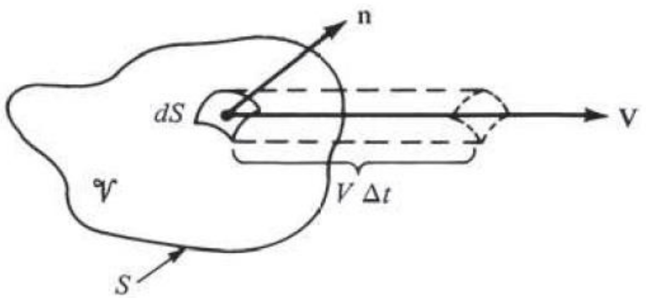


Figure 2.15 Moving control volume used for the physical interpretation of the divergence of velocity.

If this integral is divided by Δt, the result is physically the time rate of change of the control volume, denoted by $D\mathcal{V}/Dt$; that is,

$$\frac{D\mathcal{V}}{Dt} = \frac{1}{\Delta t}\oiint_S (\mathbf{V}\Delta t)\cdot\mathbf{dS} = \oiint_S \mathbf{V}\cdot\mathbf{dS} \tag{2.29}$$

(The significance of the notation D/Dt is revealed in Section 2.9.) Applying the divergence theorem, Equation (2.26), to the right side of Equation (2.29), we have

$$\frac{D\mathcal{V}}{Dt} = \iiint_{\mathcal{V}} (\nabla\cdot\mathbf{V})d\mathcal{V} \tag{2.30}$$

Now let us imagine that the moving control volume in Figure 2.15 is shrunk to a very small volume $\delta\mathcal{V}$, essentially becoming an infinitesimal moving fluid element as sketched on the right of Figure 2.14. Then Equation (2.30) can be written as

$$\frac{D(\delta\mathcal{V})}{Dt} = \iiint_{\delta\mathcal{V}} (\nabla\cdot\mathbf{V})d\mathcal{V} \tag{2.31}$$

Assume that $\delta\mathcal{V}$ is small enough such that $\nabla\cdot\mathbf{V}$ is essentially the same value throughout $\delta\mathcal{V}$. Then the integral in Equation (2.31) can be approximated as $(\nabla\cdot\mathbf{V})\delta\mathcal{V}$. From Equation (2.31), we have

$$\frac{D(\delta\mathcal{V})}{Dt} = (\nabla\cdot\mathbf{V})\delta\mathcal{V}$$

or

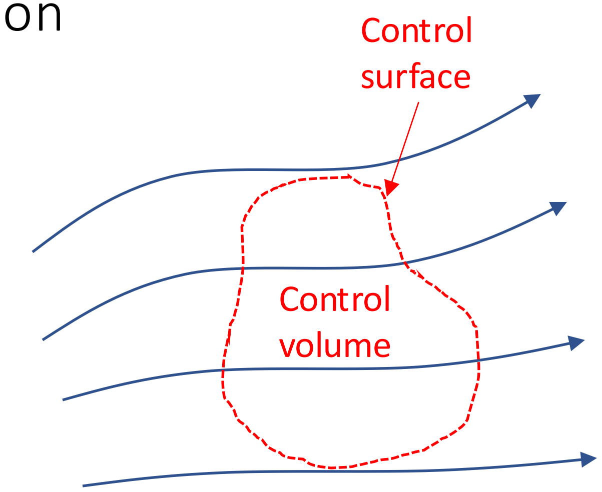
$$\boxed{\nabla\cdot\mathbf{V} = \frac{1}{\delta\mathcal{V}}\frac{D(\delta\mathcal{V})}{Dt}} \tag{2.32}$$

Examine Equation (2.32). It states that $\nabla\cdot\mathbf{V}$ is physically the *time rate of change of the volume of a moving fluid element, per unit volume*. Hence, the interpretation of $\nabla\cdot\mathbf{V}$, first given in Section 2.2.6, Divergence of a Vector Field, is now proved.

Control volumes

- Fluid dynamics problems typically involve solving one or more of the conservation equations:
  - Mass is conserved
  - The rate of change of momentum is equal to the net force applied (Newton’s second law)
  - Energy is conserved (but can change form)

- For these equations to be applicable, however, they must be applied to a certain finite region of the flow.
- A “control volume (CV)” is any closed volume bounding a finite region of the flow; its boundary is called a “control surface” (CS).



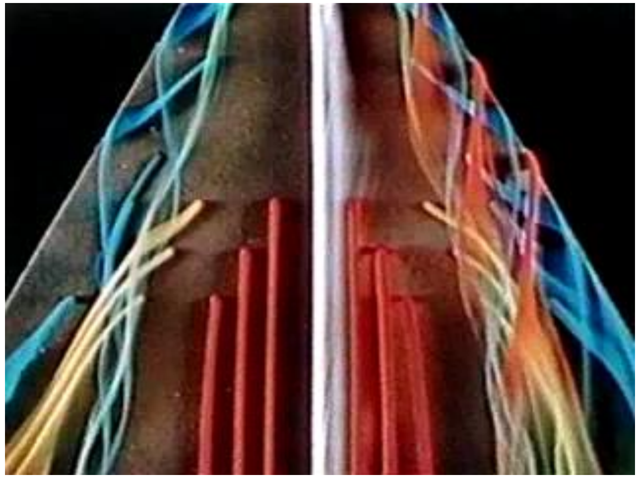
- For specific problems, can choose CV to make application of conservation law particularly easy; alternatively can use arbitrary CV to prove a general relation.

Flow Visualization (1)

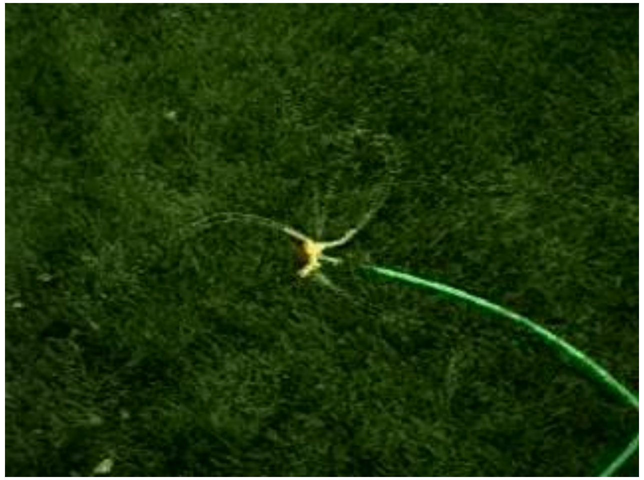
- Dye injection in liquid can be used to “see” flow – dye follows the flow
- Movie show streaks representing path of fluid particles

Flow Visualization (2)

- Streaklines, Pathlines, and Streamlines
  - Streaklines = instantaneous location of fluid particles that once passed through a specified point
    - inject dye continuously at fixed points and take snapshot at later time

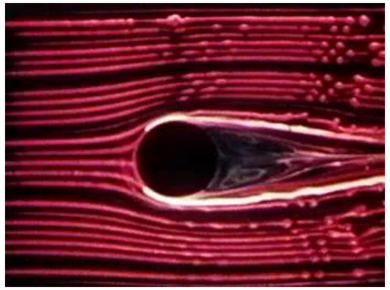


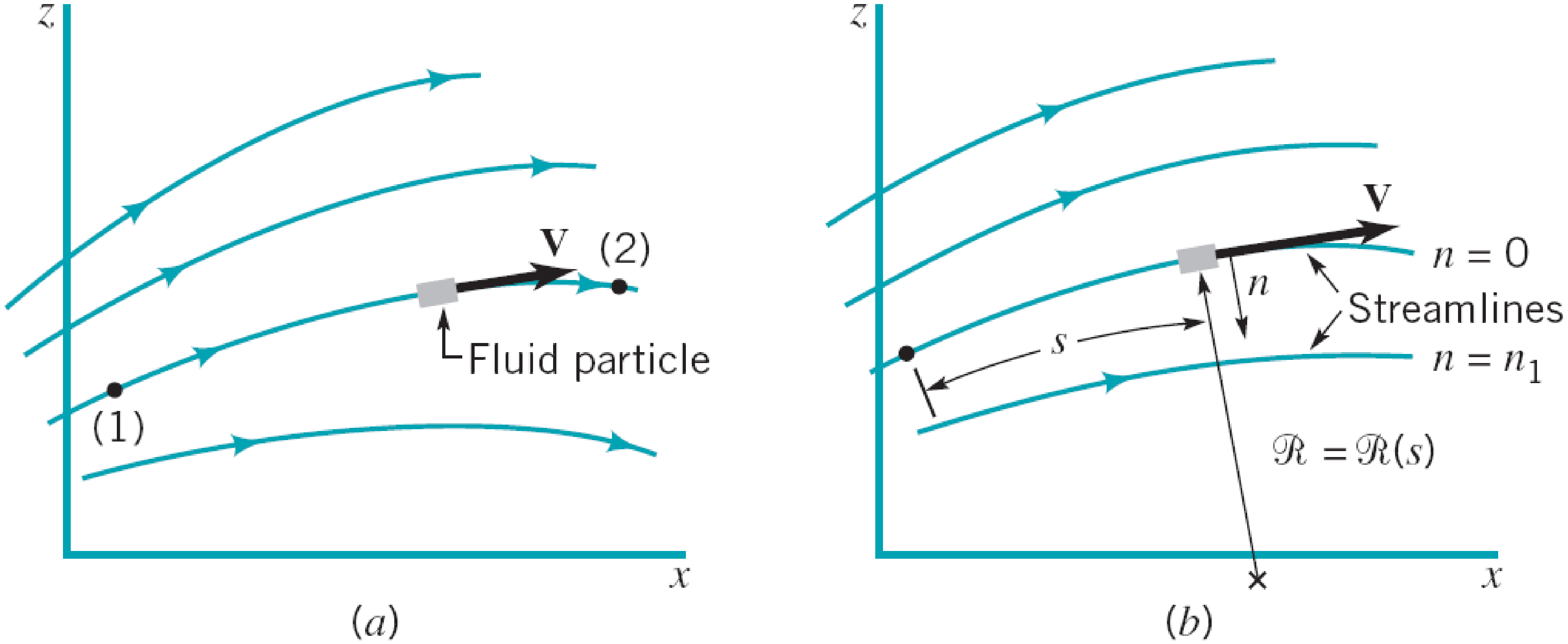
  - Pathlines = path that particles follow
    - inject dye briefly at fixed points and take time-lapsed photo for a period of time



Streamlines

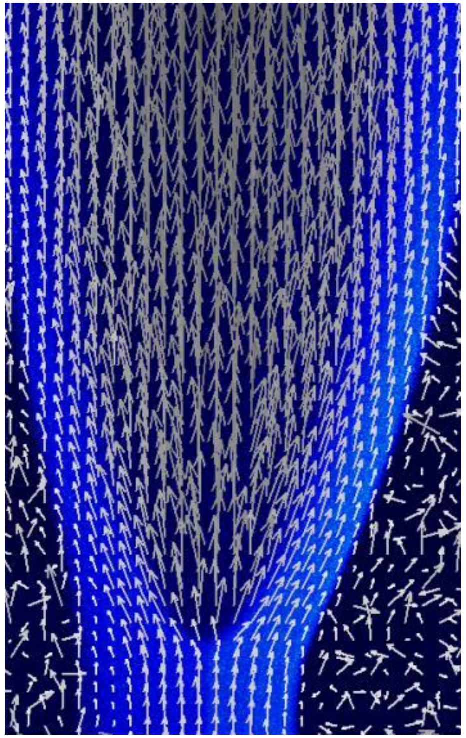
- Streamlines = lines in the flow that are locally tangent to the velocity of the fluid

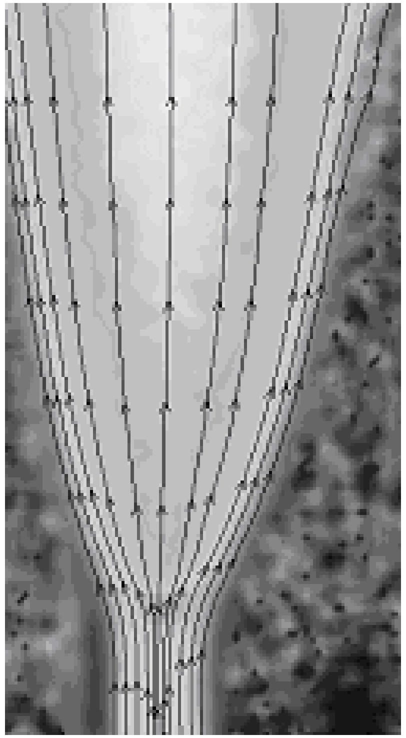




Streamline (2)

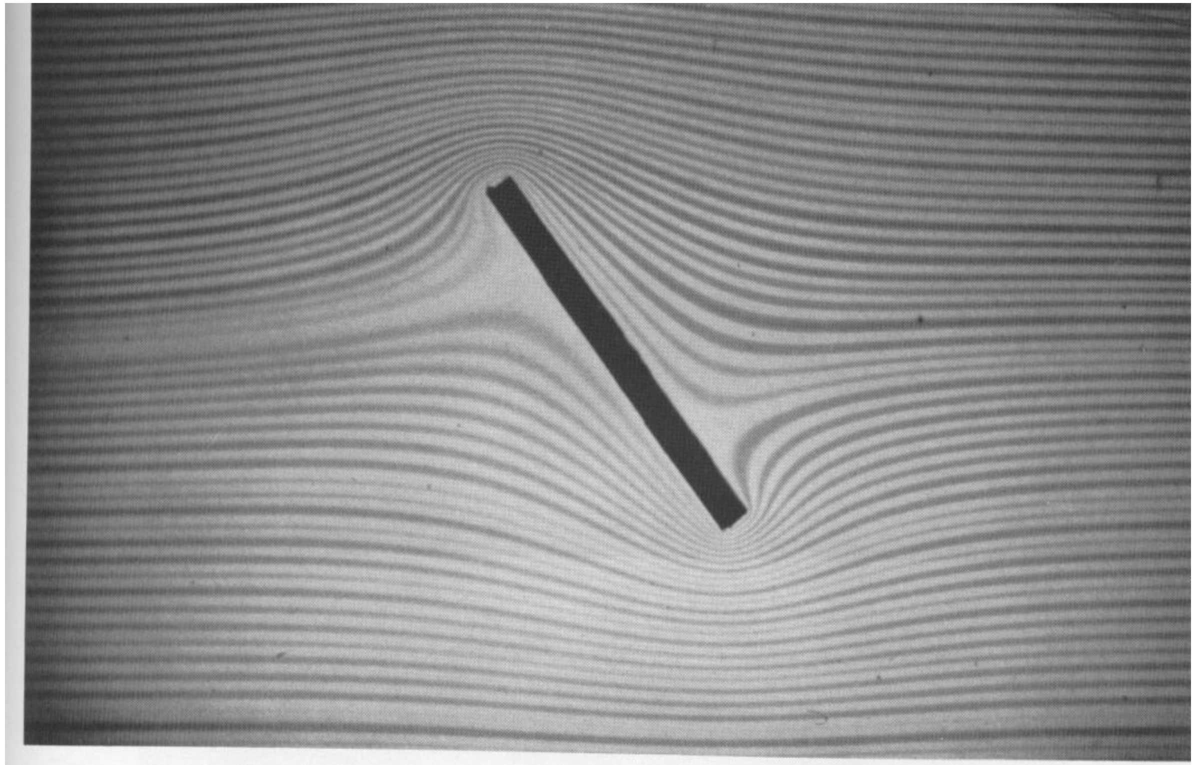
- Streamlines determined by measuring instantaneous velocity and integrating to find tangent lines
- Harder to measure than streaklines
- Most useful to mathematically describe flow





Streamlines (3)

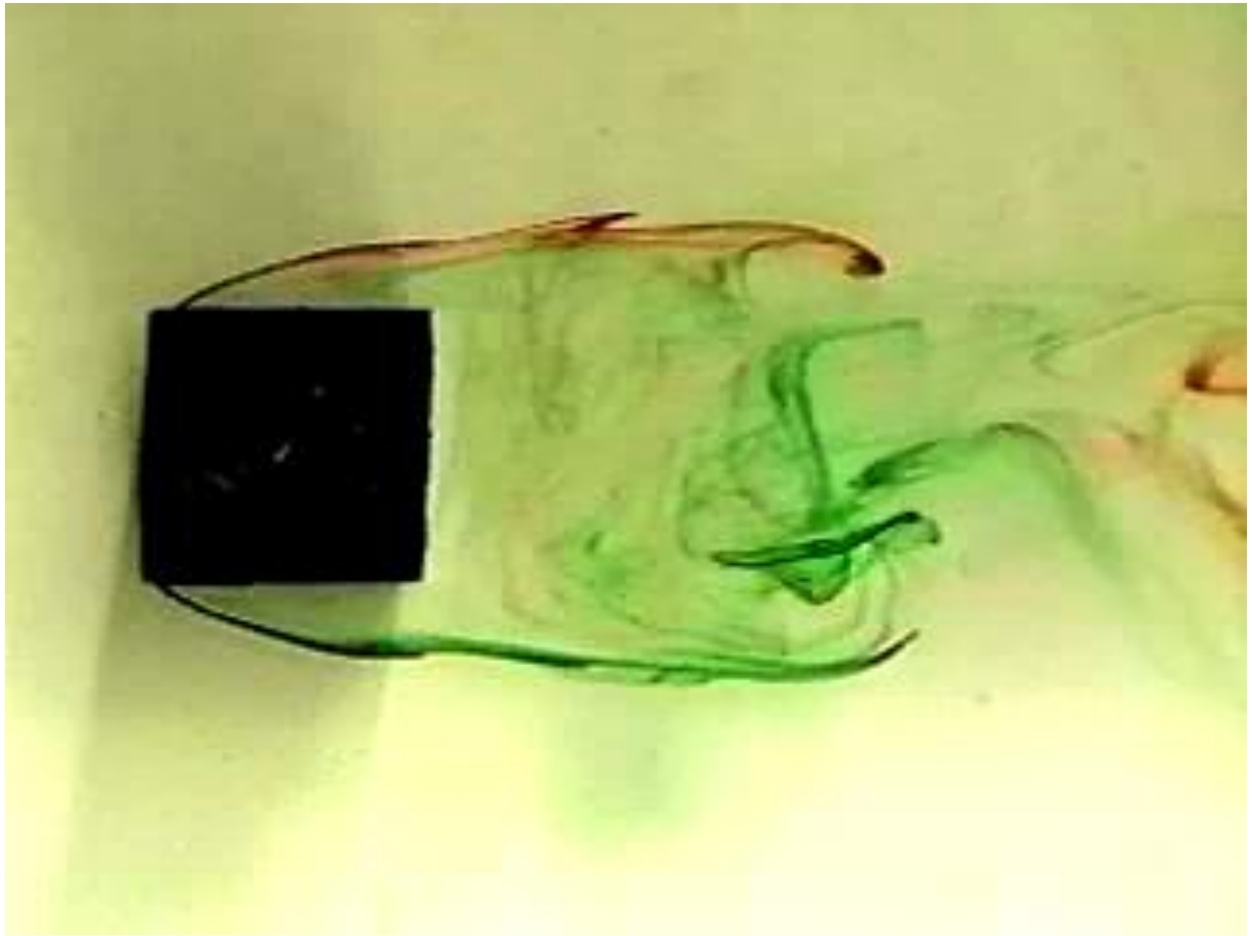
- For steady flows – pathlines, streaklines, and streamlines are identical



3. Hele-Shaw flow past an inclined plate. The Hele-Shaw analogy cannot represent a flow with circulation. It therefore shows the streamlines of potential flow past an inclined plate with zero lift. Dye flows in water between glass plates spaced 1 mm apart. *Photograph by D. H. Peregrine*

Streamlines (3)

- For steady flows – pathlines, streaklines, and streamlines are identical



NOT FOR UNSTEADY!!!

EXAMPLE 4.2 Streamlines for a Given Velocity Field

GIVEN Consider the two-dimensional steady flow discussed in Example 4.1, $\mathbf{V} = (V_0/\ell)(-x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$.

FIND Determine the streamlines for this flow.

SOLUTION

Since

$$u = (-V_0/\ell)x \text{ and } v = (V_0/\ell)y \qquad \textbf{(1)}$$

it follows that streamlines are given by solution of the equation

$$\frac{dy}{dx} = \frac{v}{u} = \frac{(V_0/\ell)y}{-(V_0/\ell)x} = -\frac{y}{x}$$

in which variables can be separated and the equation integrated to give

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

or

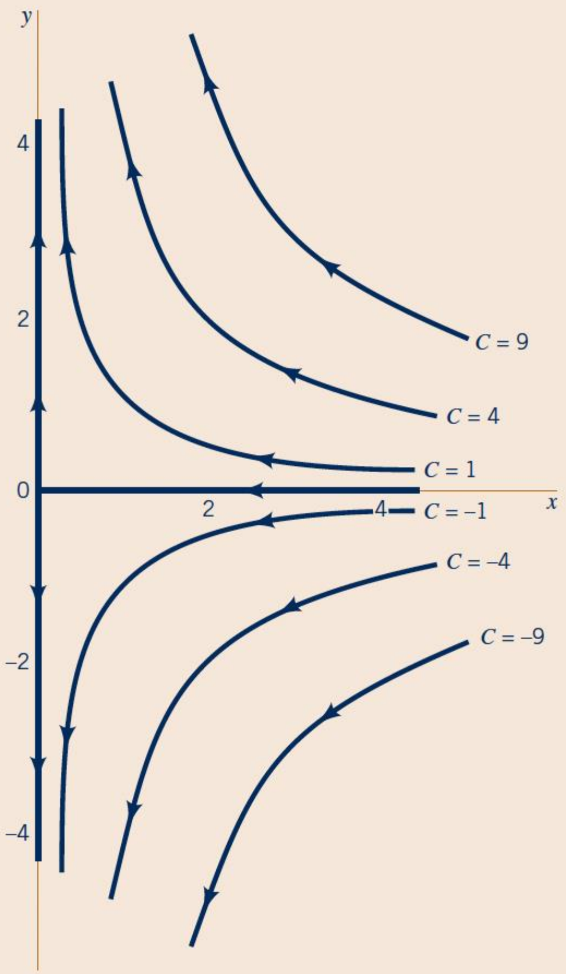
$$\ln y = -\ln x + \text{constant}$$

Thus, along the streamline

$$xy = C, \qquad \text{where } C \text{ is a constant} \qquad \textbf{(Ans)}$$

By using different values of the constant C, we can plot various lines in the x–y plane—the streamlines. The streamlines for $x \geq 0$ are plotted in Fig. E4.2. A comparison of this figure with Fig. E4.1*a* illustrates the fact that streamlines are lines tangent to the velocity field.

COMMENT Note that a flow is not completely specified by the shape of the streamlines alone. For example, the streamlines for the flow with $V_0/\ell = 10$ have the same shape as those for the flow with $V_0/\ell = -10$. However, the direction of the flow is opposite for these two cases. The arrows in Fig. E4.2 representing the flow direction are correct for $V_0/\ell = 10$ since, from Eq. 1, $u = -10x$ and $v = 10y$. That is, the flow is from right to left. For $V_0/\ell = -10$ the arrows are reversed. The flow is from left to right.



■ **Figure E4.2**

EXAMPLE 4.3 Comparison of Streamlines, Pathlines, and Streaklines

GIVEN Water flowing from the oscillating slit shown in Fig. E4.3*a* produces a velocity field given by $\mathbf{V} = u_0 \sin[\omega(t - y/v_0)]\hat{\mathbf{i}} + v_0\hat{\mathbf{j}}$, where u_0, v_0, and ω are constants. Thus, the y component of velocity remains constant ($v = v_0$), and the x component of velocity at $y = 0$ coincides with the velocity of the oscillating sprinkler head $[u = u_0 \sin(\omega t)$ at $y = 0]$.

FIND **(a)** Determine the streamline that passes through the origin at $t = 0$; at $t = \pi/2\omega$.
(b) Determine the pathline of the particle that was at the origin at $t = 0$; at $t = \pi/2$.
(c) Discuss the shape of the streakline that passes through the origin.

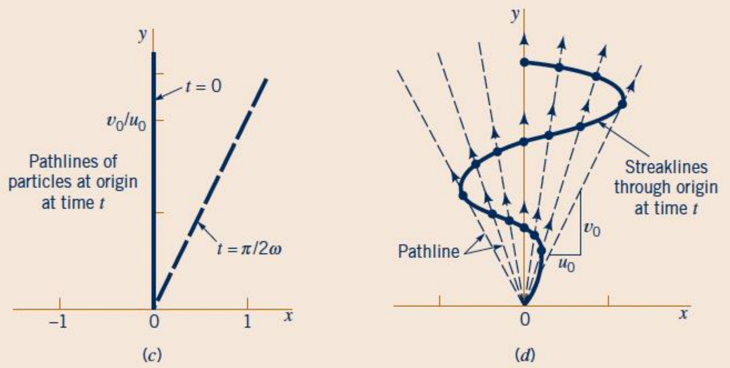
SOLUTION

(a) Since $u = u_0 \sin[\omega(t - y/v_0)]$ and $v = v_0$, it follows from Eq. 4.1 that streamlines are given by the solution of

$$\frac{dy}{dx} = \frac{v}{u} = \frac{v_0}{u_0 \sin[\omega(t - y/v_0)]}$$

in which the variables can be separated and the equation integrated (for any given time t) to give

$$u_0 \int \sin\left[\omega\left(t - \frac{y}{v_0}\right)\right] dy = v_0 \int dx$$



■ Figure E4.3(*c*), (*d*)

or

$$u_0(v_0/\omega)\cos\left[\omega\left(t - \frac{y}{v_0}\right)\right] = v_0 x + C \quad (1)$$

where C is a constant. For the streamline at $t = 0$ that passes through the origin ($x = y = 0$), the value of C is obtained from Eq. 1 as $C = u_0 v_0/\omega$. Hence, the equation for this streamline is

$$x = \frac{u_0}{\omega}\left[\cos\left(\frac{\omega y}{v_0}\right) - 1\right] \quad (2) \quad \text{(Ans)}$$

Similarly, for the streamline at $t = \pi/2\omega$ that passes through the origin, Eq. 1 gives $C = 0$. Thus, the equation for this streamline is

$$x = \frac{u_0}{\omega}\cos\left[\omega\left(\frac{\pi}{2\omega} - \frac{y}{v_0}\right)\right] = \frac{u_0}{\omega}\cos\left(\frac{\pi}{2} - \frac{\omega y}{v_0}\right)$$

or

$$x = \frac{u_0}{\omega}\sin\left(\frac{\omega y}{v_0}\right) \quad (3) \quad \text{(Ans)}$$

COMMENT These two streamlines, plotted in Fig. E4.3*b*, are not the same because the flow is unsteady. For example, at the origin ($x = y = 0$) the velocity is $\mathbf{V} = v_0\hat{\mathbf{j}}$ at $t = 0$ and $\mathbf{V} = u_0\hat{\mathbf{i}} + v_0\hat{\mathbf{j}}$ at $t = \pi/2\omega$. Thus, the angle of the streamline passing through the origin changes with time. Similarly, the shape of the entire streamline is a function of time.

(b) The pathline of a particle (the location of the particle as a function of time) can be obtained from the velocity field and the definition of the velocity. Since $u = dx/dt$ and $v = dy/dt$ we obtain

$$\frac{dx}{dt} = u_0 \sin\left[\omega\left(t - \frac{y}{v_0}\right)\right] \quad \text{and} \quad \frac{dy}{dt} = v_0$$

The y equation can be integrated (since v_0 = constant) to give the y coordinate of the pathline as

$$y = v_0 t + C_1 \quad (4)$$

where C_1 is a constant. With this known $y = y(t)$ dependence, the x equation for the pathline becomes

$$\frac{dx}{dt} = u_0 \sin\left[\omega\left(t - \frac{v_0 t + C_1}{v_0}\right)\right] = -u_0 \sin\left(\frac{C_1\omega}{v_0}\right)$$

This can be integrated to give the x component of the pathline as

$$x = -\left[u_0 \sin\left(\frac{C_1\omega}{v_0}\right)\right]t + C_2 \quad (5)$$

where C_2 is a constant. For the particle that was at the origin ($x = y = 0$) at time $t = 0$, Eqs. 4 and 5 give $C_1 = C_2 = 0$. Thus, the pathline is

$$x = 0 \quad \text{and} \quad y = v_0 t \quad (6) \quad \text{(Ans)}$$

Similarly, for the particle that was at the origin at $t = \pi/2\omega$, Eqs. 4 and 5 give $C_1 = -\pi v_0/2\omega$ and $C_2 = -\pi u_0/2\omega$. Thus, the pathline for this particle is

$$x = u_0\left(t - \frac{\pi}{2\omega}\right) \quad \text{and} \quad y = v_0\left(t - \frac{\pi}{2\omega}\right) \quad (7)$$

The pathline can be drawn by plotting the locus of $x(t)$, $y(t)$ values for $t \geq 0$ or by eliminating the parameter t from Eq. 7 to give

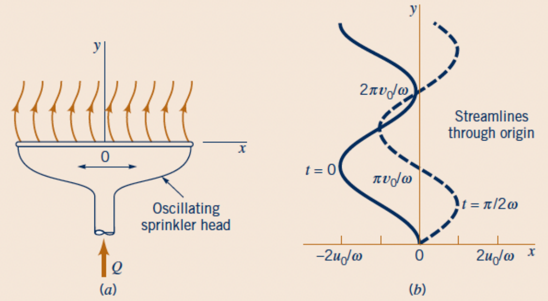
$$y = \frac{v_0}{u_0}x \quad (8) \quad \text{(Ans)}$$

COMMENT The pathlines given by Eqs. 6 and 8, shown in Fig. E4.3*c*, are straight lines from the origin (rays). The pathlines and streamlines do not coincide because the flow is unsteady.

(c) The streakline through the origin at time $t = 0$ is the locus of particles at $t = 0$ that previously ($t < 0$) passed through the origin. The general shape of the streaklines can be seen as follows. Each particle that flows through the origin travels in a straight line (pathlines are rays from the origin), the slope of which lies between $\pm v_0/u_0$ as shown in Fig. E4.3*d*. Particles passing through the origin at different times are located on different rays from the origin and at different distances from the origin. The net result is that a stream of dye continually injected at the origin (a streakline) would have the shape shown in Fig. E4.3*d*. Because of the unsteadiness, the streakline will vary with time, although it will always have the oscillating, sinuous character shown.

COMMENT Similar streaklines are given by the stream of water from a garden hose nozzle that oscillates back and forth in a direction normal to the axis of the nozzle.

In this example neither the streamlines, pathlines, nor streaklines coincide. If the flow were steady, all of these lines would be the same.



■ Figure E4.3(*a*), (*b*)

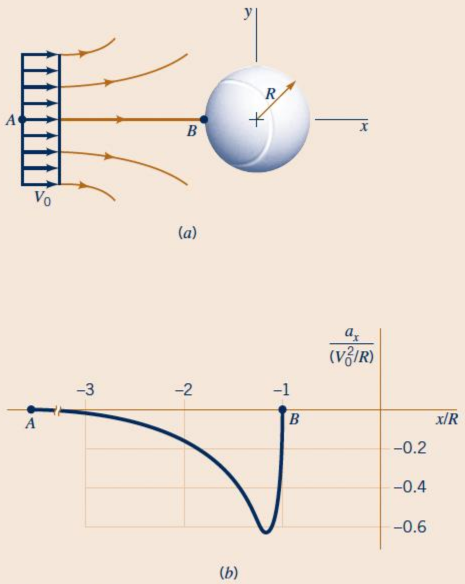
EXAMPLE 4.4 Acceleration along a Streamline

GIVEN An incompressible, inviscid fluid flows steadily past a tennis ball of radius R, as shown in Fig. E4.4*a*. According to a more advanced analysis of the flow, the fluid velocity along streamline A–B is given by

$$\mathbf{V} = u(x)\hat{\mathbf{i}} = V_0\left(1 + \frac{R^3}{x^3}\right)\hat{\mathbf{i}}$$

where V_0 is the upstream velocity far ahead of the sphere.

FIND Determine the acceleration experienced by fluid particles as they flow along this streamline.



■ **Figure E4.4**

SOLUTION

Along streamline A–B there is only one component of velocity ($v = w = 0$) so that from Eq. 4.3

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\hat{\mathbf{i}}$$

or

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \qquad a_y = 0, \qquad a_z = 0$$

Since the flow is steady, the velocity at a given point in space does not change with time. Thus, $\partial u/\partial t = 0$. With the given velocity distribution along the streamline, the acceleration becomes

$$a_x = u\frac{\partial u}{\partial x} = V_0\left(1 + \frac{R^3}{x^3}\right)V_0[R^3(-3x^{-4})]$$

or

$$a_x = -3(V_0^2/R)\frac{1 + (R/x)^3}{(x/R)^4} \qquad \textbf{(Ans)}$$

COMMENTS Along streamline A–B ($-\infty \leq x \leq -R$ and $y = 0$) the acceleration has only an x component, and it is negative (a deceleration). Thus, the fluid slows down from its upstream velocity of $\mathbf{V} = V_0\hat{\mathbf{i}}$ at $x = -\infty$ to its stagnation point velocity of $\mathbf{V} = 0$ at $x = -R$, the "nose" of the ball. The variation of a_x along streamline A–B is shown in Fig. E4.4*b*. It is the same result as is obtained in Example 3.1 by using the streamwise component of the acceleration, $a_x = V\,\partial V/\partial s$. The maximum deceleration occurs at $x = -1.205R$ and has a value of $a_{x,max} = -0.610\ V_0^2/R$. Note that this maximum deceleration increases with increasing velocity and decreasing size. As indicated in the following table, typical values of this deceleration can be quite large. For example, the $a_{x,max} = -4.08 \times 10^4\ \text{ft/s}^2$ value for a pitched baseball is a deceleration approximately 1500 times that of gravity.

Object	V_0 (ft/s)	R (ft)	$a_{x,max}$ (ft/s^2)
Rising weather balloon	1	4.0	-0.153
Soccer ball	20	0.80	-305
Baseball	90	0.121	-4.08×10^4
Tennis ball	100	0.104	-5.87×10^4
Golf ball	200	0.070	-3.49×10^5

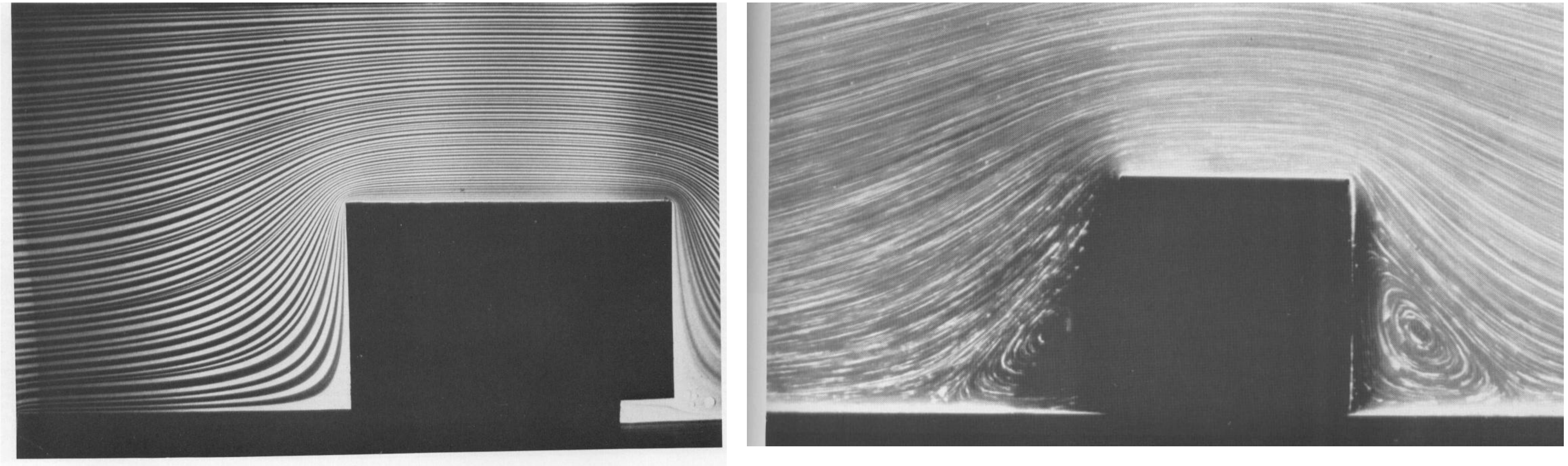
In general, for fluid particles on streamlines other than A–B, all three components of the acceleration (a_x, a_y, and a_z) will be nonzero.

Inviscid Flow (1)

- In real fluids, if there is fluid motion with non-uniform velocity then there will be strain and shear forces
- However, it is often true that these shear forces are much smaller than forces due to pressure gradients or gravity
- In these cases the fluid is assumed to be inviscid (μ=0)

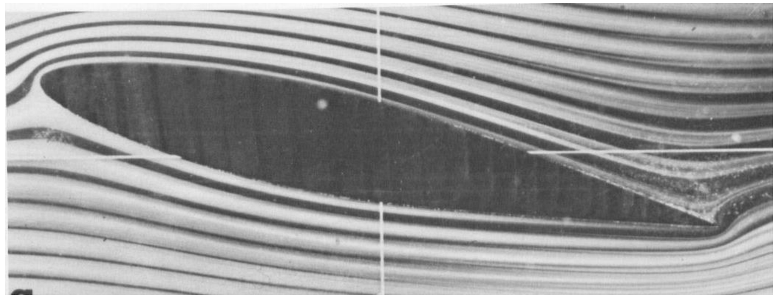
Inviscid Flow (2)

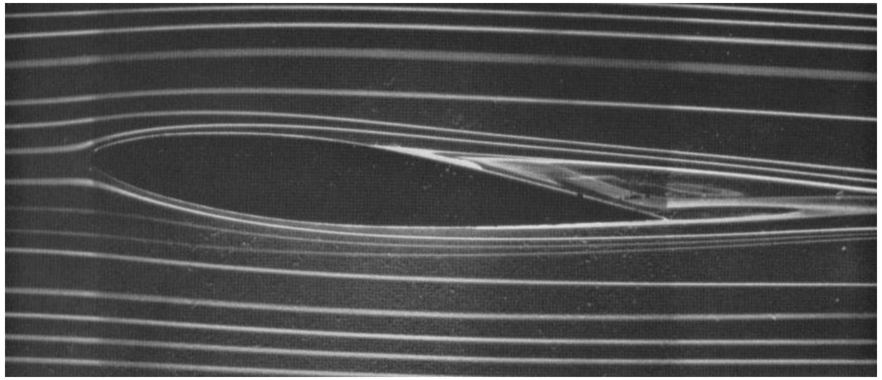
- Inviscid flows are not strongly affected by drag at surfaces and can flow around sharp corners
- Viscid flows are slowed by drag at the surface much more strongly

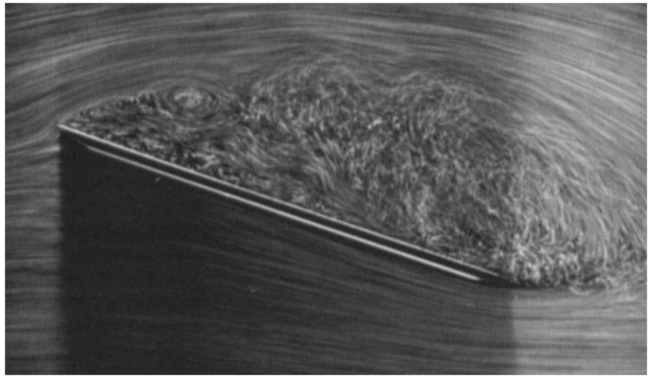


Inviscid Flow (3)

- Changes in overall velocity or geometry of a problem can change the importance of viscous forces
- Some regions of a flow may be inviscid while others show strong viscous effects







Lecture 6: Reynolds' Transport Theorem

ENAE311H Aerodynamics I

Christoph Brehm

Introduction

Definition: A fluid system is an arbitrary region/quantity of fluid of fixed identity (c.f. a control volume, which identifies the region in space rather than the fluid itself).

The Reynolds' Transport Theorem relates changes of properties in a fluid system (to which basic physical laws are more easily applied) to changes in properties of the corresponding control volume (which are easier to deal with in practical contexts).

To begin, note that any property of a fluid system may fall into one of two classes:

1. *Extensive properties* depend on the amount of material present, e.g., mass, momentum, energy – we denote these by N.
2. *Intensive properties* are independent of the amount of material present, e.g., pressure, temperature, density, velocity – these we denote by η.

Note that for every extensive property there is a corresponding intensive property, defined by

$$N = \iiint_M \eta dm = \iiint_V \eta \rho dV,$$

where η is the amount of N per unit mass.

Extensive property, N	Mass (M)	Momentum ($\boldsymbol{P}$)	Energy (E)
Intensive property, η	1	Velocity ($\boldsymbol{v}$)	Specific energy (e)

Derivation of RTT

Define a stationary CV that exactly bounds our chosen system at time t. At time $t+\Delta t$, the system will have moved while the CV remains fixed.

For the change in N of the system we have

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \frac{N_s(t+\Delta t) - N_s(t)}{\Delta t}$$

Now, at time t we have

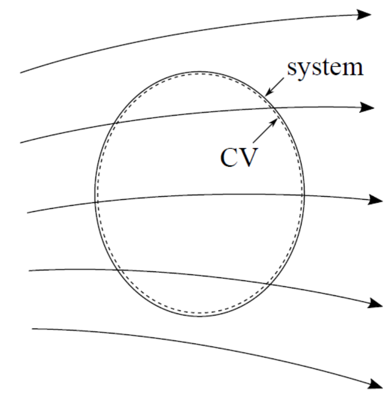
$$N_s = N_{CV}$$

while at time $t+\Delta t$ we have

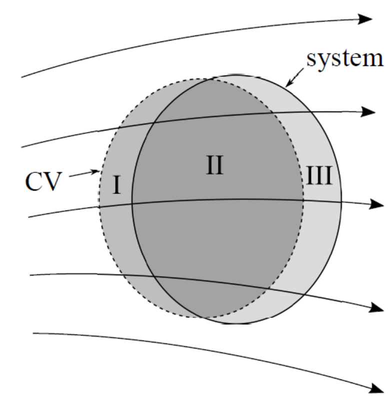
$$N_s = N_{II} + N_{III} = N_{CV} - N_I + N_{III}.$$

We can thus write the above derivative as

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left[\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} - \frac{N_I(t+\Delta t)}{\Delta t} + \frac{N_{III}(t+\Delta t)}{\Delta t} \right]$$



(a) time t



(b) time $t+\Delta t$

Derivation of RTT

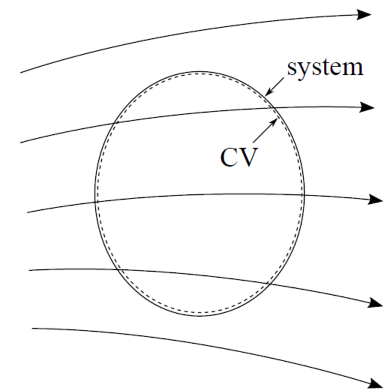
Let us look at the terms on the RHS individually….

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0}\left[\boxed{\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t}} - \frac{N_I(t+\Delta t)}{\Delta t} + \frac{N_{III}(t+\Delta t)}{\Delta t}\right]$$

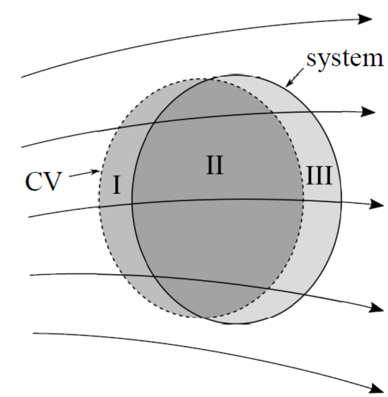
Since the CV is stationary, we have

$$\lim_{\Delta t \to 0}\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} = \frac{\partial N_{CV}}{\partial t}$$

$$= \frac{\partial}{\partial t}\iiint_{CV} \eta \rho dV.$$



(a) time t



(b) time $t+\Delta t$

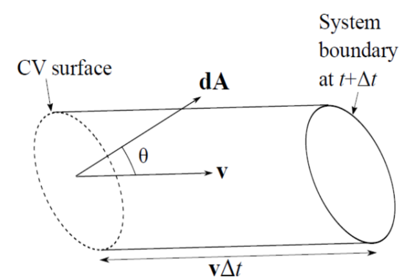
Derivation of RTT

Let us look at the terms on the RHS individually….

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left[\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} - \frac{N_I(t+\Delta t)}{\Delta t} + \boxed{\frac{N_{III}(t+\Delta t)}{\Delta t}} \right]$$

For region III, consider a small subregion dN_{III} at time $t + \Delta t$, as shown to the lower right. The side length is $||\boldsymbol{v}||\Delta t$, and thus the volume of the element is

$$dV = dA||\mathbf{v}||\Delta t \cos\theta = \mathbf{v} \cdot \mathbf{dA}\Delta t$$

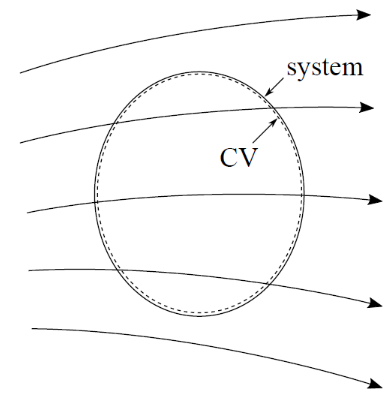


Note that the amount of N in dN_{III} is

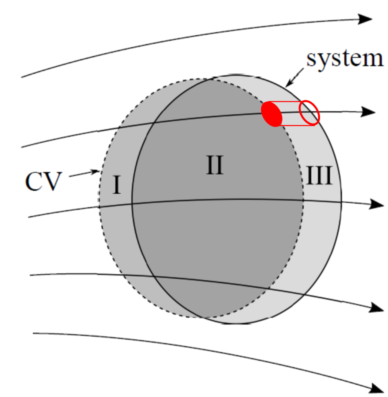
$$dN_{III}(t+\Delta t) = \eta\rho dV.$$

We can thus integrate over the entire region III to obtain

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{N_{III}(t+\Delta t)}{\Delta t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint_{CS_{III}} dN_{III}(t+\Delta t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint_{CS_{III}} \eta\rho \mathbf{v} \cdot \mathbf{dA}\Delta t \\ &= \iint_{CS_{III}} \eta\rho \mathbf{v} \cdot \mathbf{dA}. \end{aligned}$$



(a) time t



(b) time $t+\Delta t$

Derivation of RTT

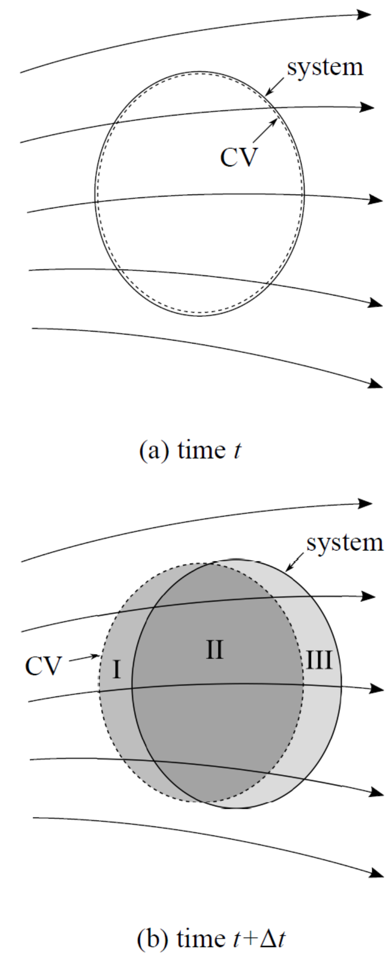
Let us look at the terms on the RHS individually….

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left[\frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} - \boxed{\frac{N_I(t+\Delta t)}{\Delta t}} + \frac{N_{III}(t+\Delta t)}{\Delta t} \right]$$

Similarly, for region I we obtain

$$\lim_{\Delta t \to 0} \frac{N_I(t+\Delta t)}{\Delta t} = -\iint_{CS_I} \eta \rho \mathbf{v} \cdot \mathbf{dA}.$$

Here the negative sign is necessary because the velocity vector $\boldsymbol{v}$ and the outward normal vector $\hat{\boldsymbol{n}}$ point in opposite directions (hence the dot product is negative), but we require N_I to be positive.



Derivation of RTT

Combining these results, we have

$$\begin{aligned}\boxed{\frac{dN_s}{dt}} &= \frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV + \iint_{CS_I} \eta\rho\mathbf{v}\cdot\mathbf{dA} + \iint_{CS_{III}} \eta\rho\mathbf{v}\cdot\mathbf{dA} \\ &= \boxed{\frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV} + \boxed{\iint_{CS} \eta\rho\mathbf{v}\cdot\mathbf{dA},}\end{aligned}$$

since CS_I and CS_{III} constitute the entire control surface.

This is the Reynolds Transport Theorem, which relates the rate of change of an extensive property of a fluid system with variations of the associated intensive property within (and through the boundaries of) the corresponding control volume.

Physical interpretations:

- Rate of change of system extensive property, e.g., mass, momentum
- Rate of change of N inside the control volume
- Rate of flux of N across the control-volume boundaries

Conservation of mass (continuity)

For the case of mass, $N = M$ and $\eta = 1$. Since the mass of a fluid system doesn't change, we have

$$\frac{dM_s}{dt} = 0$$

In this case then, the RTT becomes

$$\underbrace{\frac{\partial}{\partial t}\iiint_{CV} \rho dV}_{\text{rate of change of mass within CV}} + \underbrace{\iint_{CS} \rho \mathbf{v} \cdot \mathbf{dA}}_{\text{Net mass flux through CV boundaries}} = 0.$$

This is the integral form of the conservation of mass equation (sometimes known as the continuity equation). It is often useful for solving the flow through macroscopic configurations.

Conservation of mass – differential form

We can also derive a corresponding differential form of the continuity equation.

Starting from the integral form, for a stationary CV we have

$$\frac{\partial}{\partial t}\iiint_{CV} \rho dV = \iiint_{CV} \frac{\partial \rho}{\partial t} dV.$$

From the divergence theorem, we can write the second term on the RHS of the integral equation as

$$\iint_{CS} (\rho \mathbf{v}) \cdot \mathbf{dA} = \iiint_{CV} \nabla \cdot (\rho \mathbf{v}) dV.$$

Combining these results, we have

$$\iiint_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right] dV = 0.$$

Since the CV is arbitrary, this statement can only hold in general if the integrand in [] is identically zero, i.e.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.}$$

Differential form of the continuity equation

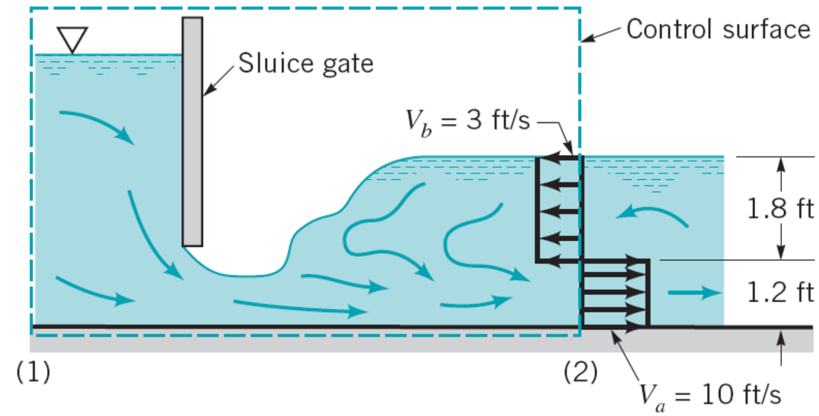
Conservation of mass – differential form

Note that if the flow is steady ($\frac{\partial}{\partial t} = 0$) and incompressible ($\rho =$ const.), the differential form of the continuity equation reduces to simply

$$\nabla \cdot v = 0$$

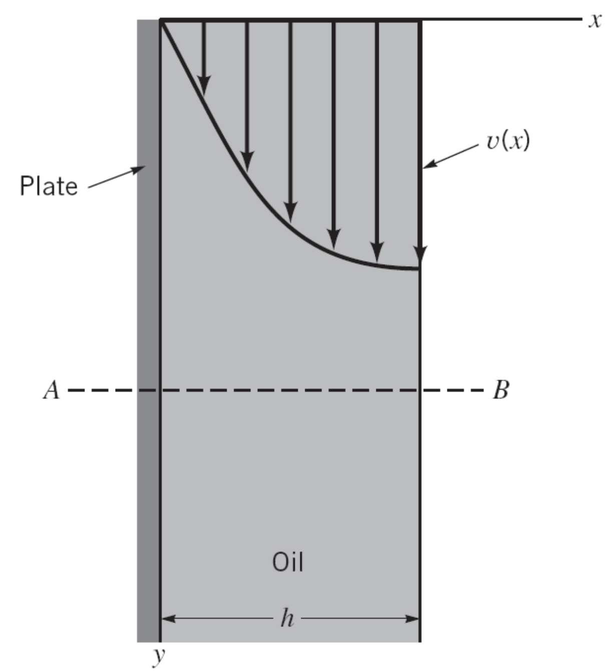
We thus see that, for a steady, incompressible flow, the velocity field is divergence-free.

4.51 In the region just downstream of a sluice gate, the water may develop a reverse flow region as is indicated in Fig. P4.51 and Video V10.5. The velocity profile is assumed to consist of two uniform regions, one with velocity $V_a = 10$ fps and the other with $V_b = 3$ fps. Determine the net flowrate of water across the portion of the control surface at section (2) if the channel is 20 ft wide.



$$Q = V_a A_a - V_b A_b = \left(10\ \frac{\text{ft}}{\text{s}}\right)(1.2\ \text{ft})(20\ \text{ft}) - \left(3\ \frac{\text{ft}}{\text{s}}\right)(1.8\ \text{ft})(20\ \text{ft}) = \underline{\underline{132\ \frac{\text{ft}^3}{\text{s}}}}$$

4.55 A layer of oil flows down a vertical plate as shown in Fig. P4.55 with a velocity of $\mathbf{V} = (V_0/h^2)(2hx - x^2)\,\hat{\mathbf{j}}$ where V_0 and h are constants. (a) Show that the fluid sticks to the plate and that the shear stress at the edge of the layer ($x = h$) is zero. (b) Determine the flowrate across surface AB. Assume the width of the plate is b. (Note: The velocity profile for laminar flow in a pipe has a similar shape. See Video V6.6.)



a) $v = \left(\frac{V_0}{h^2}\right)(2hx - x^2)$

Thus,

$$v\big|_{x=0} = \left(\frac{V_0}{h^2}\right)(0-0) = 0 \text{ and}$$

$$\tau\big|_{x=h} = \mu \left.\frac{dv}{dx}\right|_{x=h} = \mu \frac{V_0}{h^2}[2h - 2x]\big|_{x=h} = 0$$

Hence, the fluid sticks to the plate and there is no shear stress at the free surface.

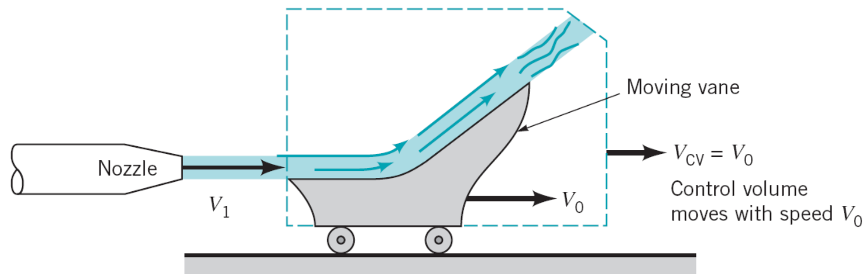
b) $$Q_{AB} = \int v\,dA = \int_{x=0}^{x=h} v\,b\,dx = \int_0^h \frac{V_0}{h^2}(2hx - x^2)\,b\,dx$$

or

$$Q_{AB} = \frac{V_0 b}{h^2}\left[hx^2 - \frac{1}{3}x^3\right]_0^h = \underline{\underline{\frac{2}{3}V_0 h b}}$$

Deforming Control Volumes

- Reynolds Transport Theorem can be applied to any CV even if it moves



- V is relative velocity
- = V_1-V_0

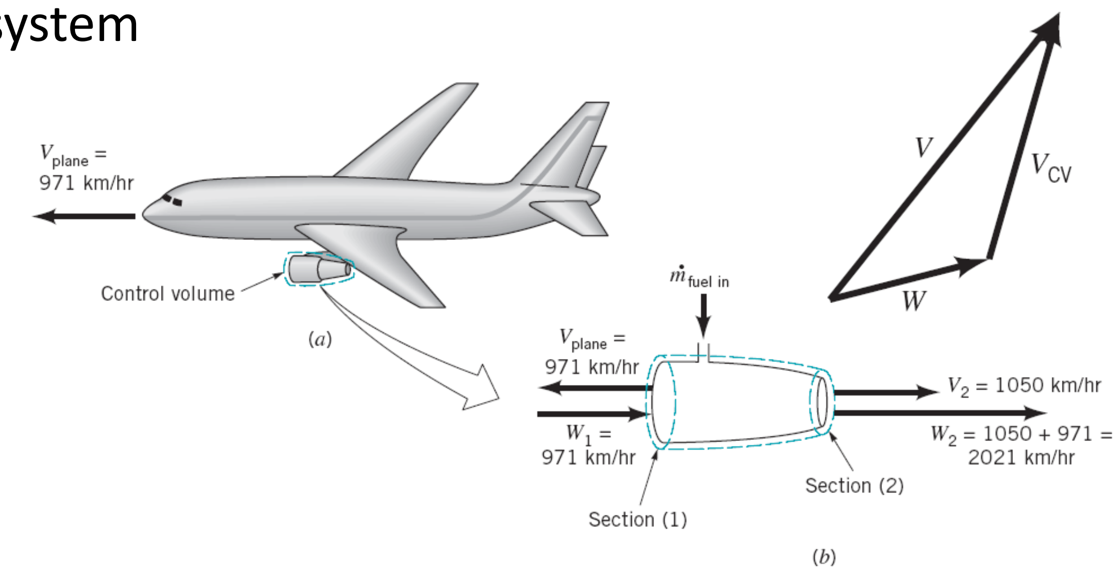
$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b d\forall + \int_{cs} \rho b \vec{V} \cdot \hat{n} dA$$

Moving Control Volumes

- If the control volume is moving, the coordinate system is fixed to the control volume
- The velocity of flow across the control surface is evaluated relative to this coordinate system

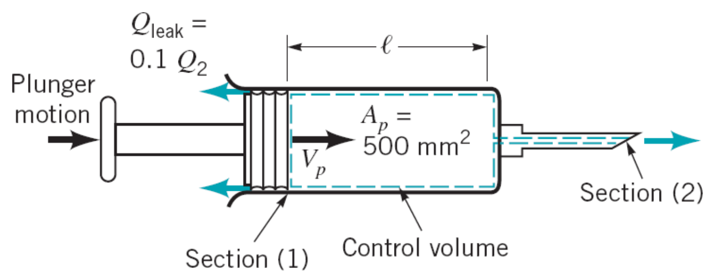
$$\vec{W} = \vec{V} - \vec{V}_{cv}$$

$$\frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{W} \cdot \hat{n} dA = 0$$



Deforming Control Volumes

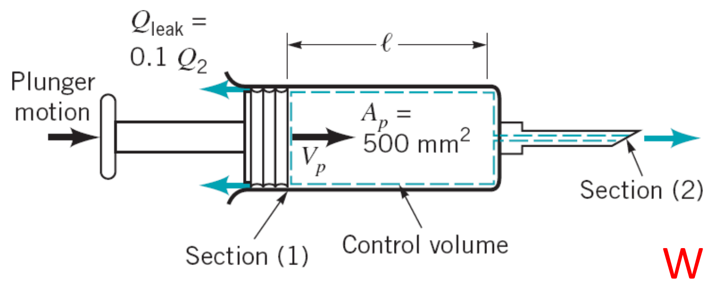
- Deforming control volumes are both moving and unsteady
- Local relative velocities used at all surfaces



$$\frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{W} \cdot \hat{n} dA = 0$$

Deforming Control Volumes

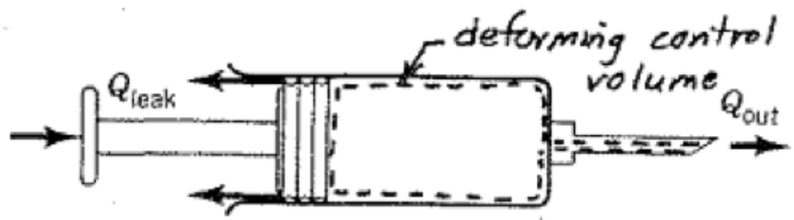
- Deforming control volumes are both moving and unsteady
- Local relative velocities used at all surfaces



$$\frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{W} \cdot \hat{n} dA = 0$$

What is the velocity V_2?

- D_1 and D_2 given



Using a deforming control volume and the conservation of mass principle

$$\frac{DM_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \mathbf{W} \cdot \mathbf{n} dA = 0$$

we obtain $\quad -\rho A_1 V_p + \rho Q_2 + \rho Q_{leak} = 0 \quad (1)$

Since $\rho = \text{constant}$, $Q_{leak} = 0.1 Q_2$ and $Q_2 = A_2 V_2$

we obtain from Eq. 1

$$1.1 A_2 V_2 = A_1 V_p$$

or

$$V_2 = \left(\frac{A_1}{A_2}\right)\frac{V_p}{1.1} = \left(\frac{d_1^2}{d_2^2}\right)\frac{V_p}{1.1} = \frac{(20\text{ mm})^2\left(20\dfrac{\text{mm}}{\text{s}}\right)}{(0.7\text{ mm})^2(1.1)\left(1000\dfrac{\text{mm}}{\text{m}}\right)}$$

and

$$\underline{\underline{V_2 = 14.8\frac{\text{m}}{\text{s}}}}$$

Lecture 7: The Substantial Derivative and Conservation of Momentum

ENAE311H Aerodynamics I

Christoph Brehm

QUIZ

5.11 At cruise conditions, air flows into a jet engine at a steady rate of 65 lbm/s. Fuel enters the engine at a steady rate of 0.60 lbm/s. The average velocity of the exhaust gases is 1500 ft/s relative to the engine. If the engine exhaust effective cross section area is 3.5 ft^2, estimate the density of the exhaust gases in lbm/ft^3.

The substantial derivative

Consider again the differential form of the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

We can expand out the second term to give:

$$\nabla \cdot (\rho \mathbf{v}) = \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}$$

Our original equation thus becomes

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v}$$

We can expand out the LHS to give

$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \equiv \frac{D\rho}{Dt}.$$

The derivative D/Dt is known as the total, substantial, or material derivative. <u>It describes the rate of change of a property of a fluid element moving with the flow.</u>

To see this, imagine we have a fluid element of density ρ, at position (x, y, z) at time t, i.e., $\rho(x, y, z, t)$.

At time $t + \Delta t$, it will have moved to $(x + \Delta x, y + \Delta y, z + \Delta z)$ and its density will be:

$$\begin{aligned} \rho|_{t+\Delta t} &= \rho(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) \\ &= \rho(x, y, z, t) + \frac{\partial \rho}{\partial x} dx + \frac{\partial \rho}{\partial y} dy + \frac{\partial \rho}{\partial z} dz + \frac{\partial \rho}{\partial t} dt \\ &\quad + H.O.T. \end{aligned}$$

The change in density is then

$$d\rho = \frac{\partial \rho}{\partial x} dx + \frac{\partial \rho}{\partial y} dy + \frac{\partial \rho}{\partial z} dz + \frac{\partial \rho}{\partial t} dt$$

and thus

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt} + \frac{\partial \rho}{\partial t} \\ &= v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t} \\ &= \frac{D\rho}{Dt}. \end{aligned}$$

Applies to any intrinsic property of the flow!

Conservation of momentum

Let us return now to the Reynolds Transport Theorem:

$$\frac{dN_s}{dt} = \frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV + \iint_{CS} \eta\rho\mathbf{v}\cdot\mathbf{dA},$$

Let us now consider the case of $N_s = \boldsymbol{P}$, i.e., the momentum of the fluid system. We have seen already that the corresponding intensive variable is the fluid velocity, i.e, $\eta = \boldsymbol{v}$.

From Newton's second law, the rate of change of momentum is equal to the sum of applied forces, i.e.,

$$\frac{d\mathbf{P_s}}{dt} = \sum \mathbf{F}$$

Applying the RTT, we thus have

$$\underbrace{\frac{\partial}{\partial t}\iiint_{CV} \rho\mathbf{v}dV}_{\text{Rate of change of momentum within CV}} + \underbrace{\iint_{CS} \mathbf{v}(\rho\mathbf{v}\cdot\mathbf{dA})}_{\text{Net momentum flux through CV boundaries}} = \underbrace{\sum \mathbf{F}}_{\text{Applied forces}}$$

Conservation of momentum

Let us return now to the Reynolds Transport Theorem:

$$\frac{dN_s}{dt} = \frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV + \iint_{CS} \eta\rho\mathbf{v}\cdot\mathbf{dA},$$

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From Newton's second law, the rate of change of momentum is equal to the sum of applied forces, i.e.,

$$\frac{d\mathbf{P_s}}{dt} = \sum \mathbf{F}$$

Applying the RTT, we thus have

$$\frac{\partial}{\partial t}\iiint_{CV} \rho\mathbf{v}dV + \iint_{CS} \mathbf{v}(\rho\mathbf{v}\cdot\mathbf{dA}) = \sum \mathbf{F}$$

Note that this is a vector equation with three components. The x-component, for example, is

$$\frac{\partial}{\partial t}\iiint_{CV} \rho v_x dV + \iint_{CS} v_x\rho\mathbf{v}\cdot\mathbf{dA} = \sum F_x$$

Conservation of momentum (integral form)

The forces relevant here are of two types:

1. Surface forces (pressure and shear stress), for which we can write

$$\sum \mathbf{F_s} = -\iint_{CS} p\mathbf{dA} + \iint_{CS} \bar{\bar{\tau}} \cdot \mathbf{dA}$$

2. The gravity body force:

$$\sum \mathbf{F_b} = \iiint_{CV} \rho \mathbf{f} dV,$$

where $\boldsymbol{f} = -g\hat{\boldsymbol{j}}$ is the gravitational acceleration (assumed in the y direction).

Substituting into the momentum equation, we have

inviscid gravity negligible

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \mathbf{v} dV + \iint_{CS} \mathbf{v}(\rho \mathbf{v} \cdot \mathbf{dA}) = -\iint_{CS} p\mathbf{dA} + \cancel{\iint_{CS} \bar{\bar{\tau}} \cdot \mathbf{dA}} + \cancel{\iiint_{CV} \rho \mathbf{f} dV}$$

Or, in x-direction (inviscid, no body force):

$$\frac{\partial}{\partial t}\iiint_{CV} \rho v_x dV + \iint_{CS} \rho v_x(\mathbf{v} \cdot \mathbf{dA}) = -\iint_{CS} p dA_x$$

Conservation of momentum (differential form)

We can derive a corresponding differential form by using similar arguments as in the continuity case.

Since the CV is spatially fixed, we have

$$\frac{\partial}{\partial t}\iiint_{CV}\rho\mathbf{v}dV = \iiint_{CV}\frac{\partial}{\partial t}(\rho\mathbf{v})dV.$$

From the divergence theorem:

$$\begin{aligned}\iint_{CS}\mathbf{v}(\rho\mathbf{v}\cdot\mathbf{dA}) &= \iiint_{CV}\nabla\cdot(\rho\mathbf{v}\mathbf{v})dV\\ \iint_{CS}\bar{\bar{\tau}}\cdot\mathbf{dA} &= \iiint_{CV}\nabla\cdot\bar{\bar{\tau}}dV,\end{aligned}$$

And from the gradient theorem:

$$\iint_{CS}p\mathbf{dA} = \iiint_{CV}\nabla p dV.$$

Substituting into the integral momentum equation:

$$\iiint_{CV}\left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \nabla p - \nabla\cdot\bar{\bar{\tau}} - \rho\mathbf{f}\right]dV = 0.$$

We argue, as before, that since the CV is arbitrary, the term in [] must be identically zero, i.e.,

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \nabla p - \nabla\cdot\bar{\bar{\tau}} - \rho\mathbf{f} = 0.$$

A more useful form of this results if we expand the first two terms and use the continuity equation:

$$\boxed{\rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p - \nabla\cdot\bar{\bar{\tau}} - \rho\mathbf{f} = 0.}$$

Or alternatively

$$\boxed{\rho\frac{D\mathbf{v}}{Dt} + \nabla p - \nabla\cdot\bar{\bar{\tau}} - \rho\mathbf{f} = 0.}$$

Conservation of momentum (differential form)

We can derive a corresponding differential form by using similar arguments as in the continuity case.

Often we will have the case that the flow is (approximately) inviscid and the body forces are negligible, in which case these equations simplify to:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = 0$$

and

$$\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p = 0$$

The x-component of this equation is

$$\frac{\partial v_x}{\partial t} + (\mathbf{v}\cdot\nabla)v_x + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

or

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0.$$

Substituting into the integral momentum equation:

$$\iiint_{CV}\left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \nabla p - \nabla\cdot\bar{\bar{\tau}} - \rho\mathbf{f}\right]dV = 0.$$

We argue, as before, that since the CV is arbitrary, the term in [] must be identically zero, i.e.,

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Or alternatively

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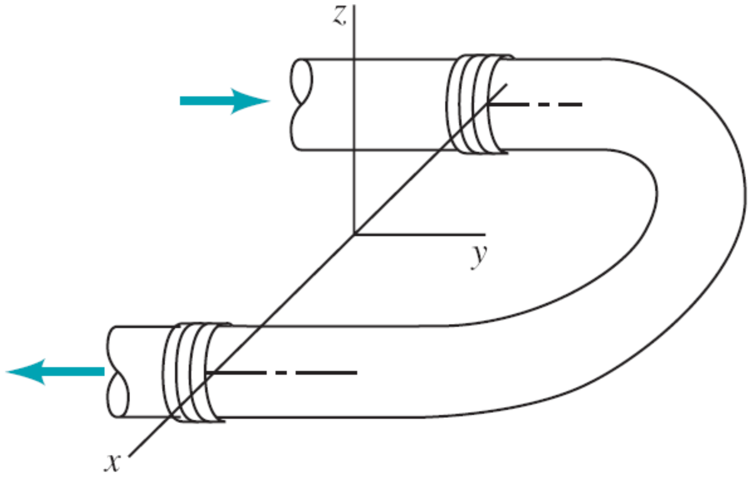
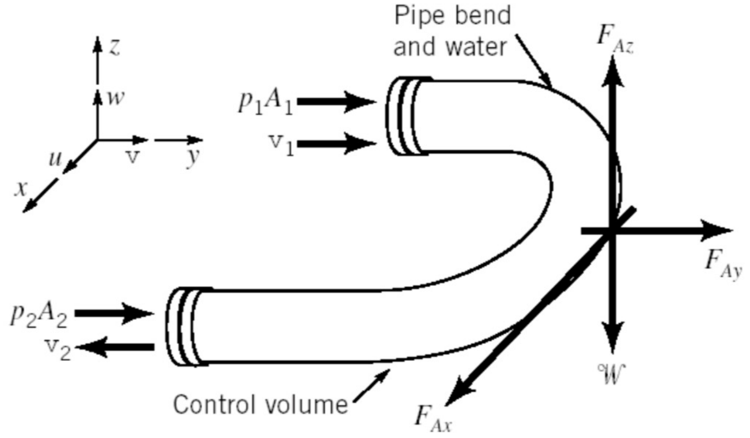
5.28 Water flows through a horizontal, 180° pipe bend as is illustrated in Fig. P5.28. The flow cross section area is constant at a value of 9000 mm^2. The flow velocity everywhere in the bend is 15 m/s. The pressures at the entrance and exit of the bend are 210 and 165 kPa, respectively. Calculate the horizontal (x and y) components of the anchoring force needed to hold the bend in place.



5.28 Water flows through a horizontal, 180° pipe bend as is illustrated in Fig. P5.28. The flow cross section area is constant at a value of 9000 mm^2. The flow velocity everywhere in the bend is 15 m/s. The pressures at the entrance and exit of the bend are 210 and 165 kPa, respectively. Calculate the horizontal (x and y) components of the anchoring force needed to hold the bend in place.



Step. 1 Select a proper CV: Inside of the valve

Step. 2 Find all forces acting on the CV (Free-body diagram)

Then, $\frac{\partial}{\partial t}\int_{CV} \cancel{\vec{V}\rho dV} + \int_{CS}\vec{V}\rho\vec{V}\cdot\hat{n}dA = \sum \vec{F}_{CV}$ (Steady flow)

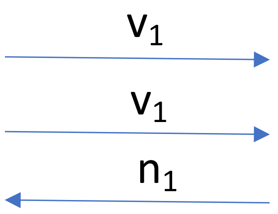
$\vec{V} \perp \hat{n}$
at the side wall

1. $x -$ comp. : $\int_{CS} u\rho\vec{V}\cdot\hat{n}dA = \int_{(1)} u_1\rho\vec{V}\cdot\hat{n}dA + \cancel{\int_{\text{Side}} u\rho\vec{V}\cdot\hat{n}dA} + \int_{(2)} u_2\rho\vec{V}\cdot\hat{n}dA = F_{Ax}$

No x component of fluid velocity at sections (1) and (2), ($u_1 = u_2 = 0$)

$$\therefore \int_{CS} u\rho\vec{V}\cdot\hat{n}dA = F_{Ax} = 0$$

1 $n_1 \cdot v_1 < 0$

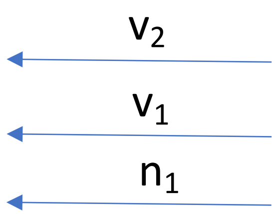


2. $y -$ comp. : $\int_{CS} v\rho\vec{V}\cdot\hat{n}dA = v_1\int_{(1)} \rho\vec{V}\cdot\hat{n}dA + v_2\int_{(2)}\rho\vec{V}\cdot\hat{n}dA = F_{Ay} + p_1A_1 + p_2A_2$

or $$(v_1)(-\dot{m}_1) + (-v_2)(\dot{m}_2) = F_{Ay} + p_1A_1 + p_2A_2$$

$$\therefore F_{Ay} = -\dot{m}(v_1 + v_2) - p_1A_1 - p_2A_2$$

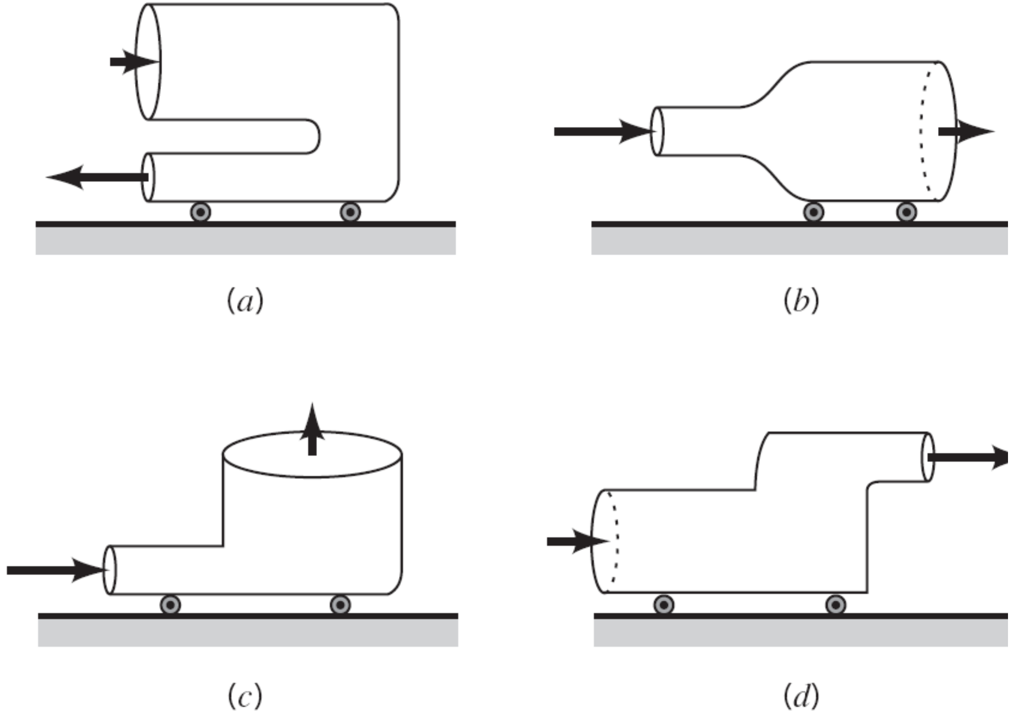
2 $n_2 \cdot v_2 > 0$

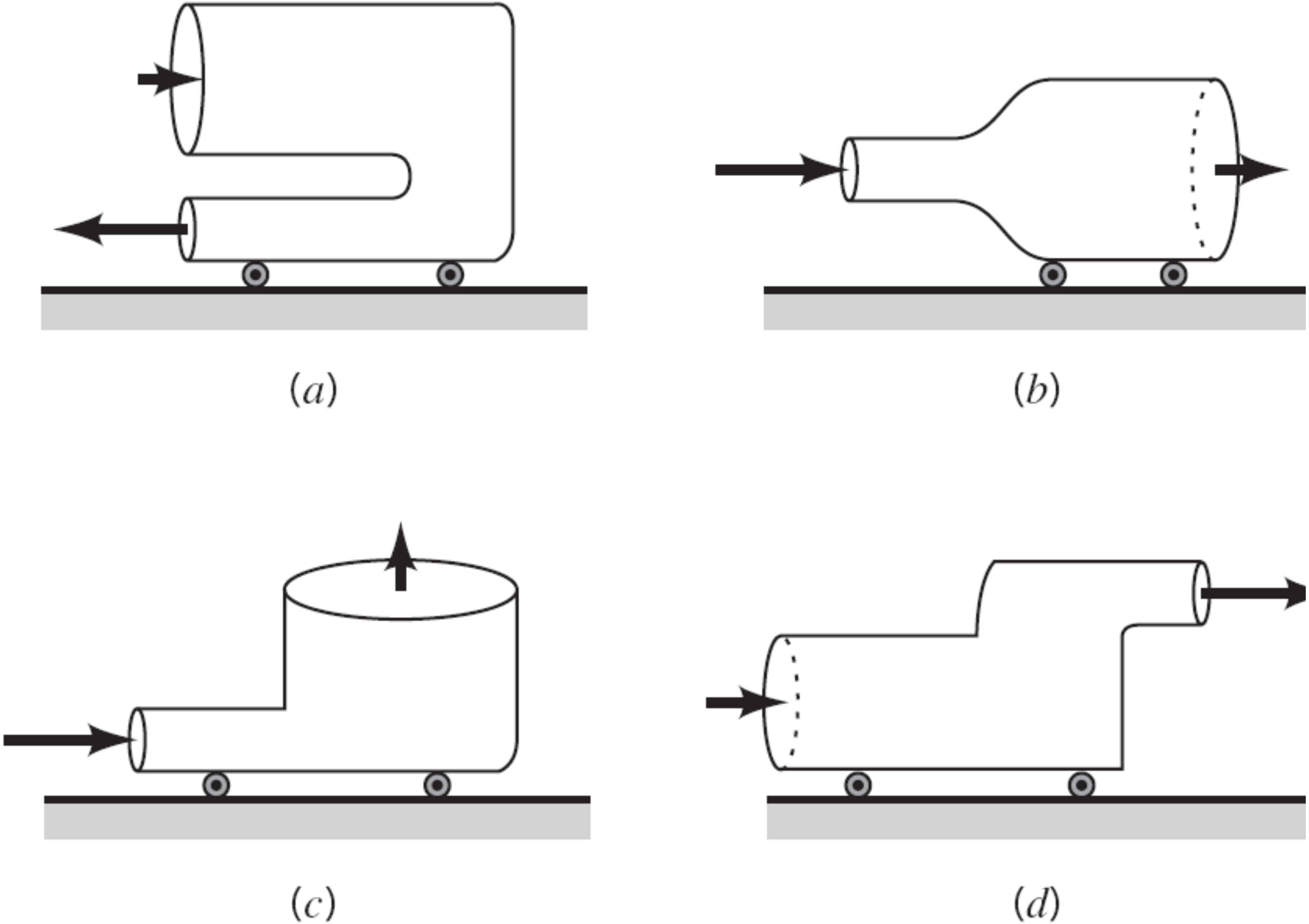


where $\dot{m} = \rho A_1 v_1 = (1.94)(0.1)(50) = 9.70$ slug/s

$p_1 = 30$ psia , $p_2 = 24$ psia , and $A_1 = A_2 = 0.1\ \text{ft}^2 (144\ \text{in}^2/\text{ft}^2) = 14.4\ \text{in}^2$

5.58 The four devices shown in Fig. P5.58 rest on frictionless wheels, are restricted to move in the x direction only and are initially held stationary. The pressure at the inlets and outlets of each is atmospheric, and the flow is incompressible. The contents of each device is not known. When released, which devices will move to the right and which to the left? Explain.




(a)
(b)
(c)
(d)

5.58

5.58 The four devices shown in Fig. P5.58 rest on frictionless wheels, are restricted to move in the x direction only and are initially held stationary. The pressure at the inlets and outlets of each is atmospheric, and the flow is incompressible. The contents of each device is not known. When released, which devices will move to the right and which to the left? Explain.

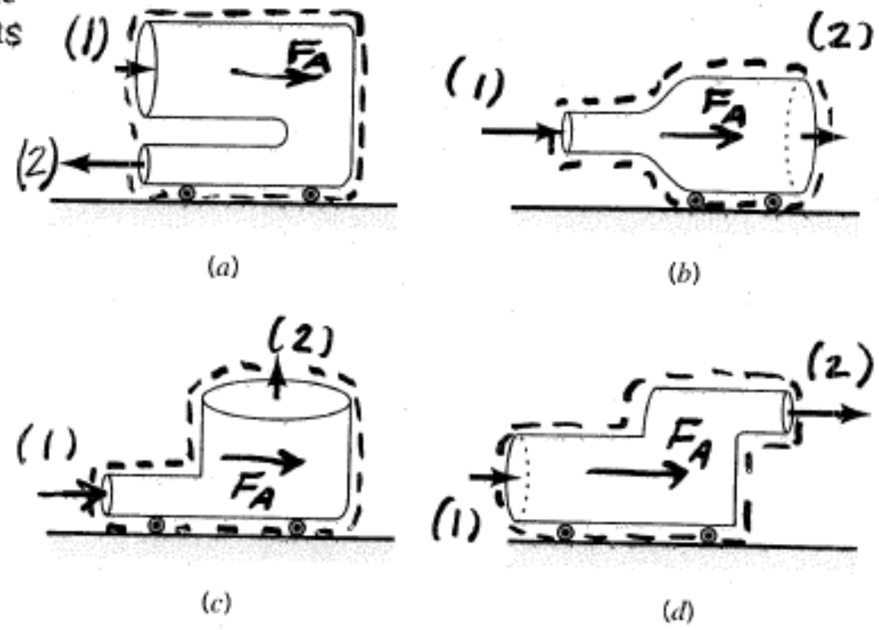


FIGURE P5.58

We apply the horizontal component of the linear momentum equation to the contents of the control volume (broken lines) and determine the sense of the anchoring force F_A. If F_A is in the direction shown in the sketches, motion will be to the left. If F_A is in a direction opposite to that shown, the motion is to the right. If $F_A = 0$, there is no horizontal motion.

For sketch (a)

$$-V_1 \rho V_1 A_1 - V_2 \rho V_2 A_2 = F_A$$

Since F_A is to the left, motion is to the right.

For sketch (b)

$$-V_1 \rho V_1 A_1 + V_2 \rho V_2 A_2 = F$$

and from conservation of mass

$$\rho V_1 A_1 = \rho V_2 A_2$$

and since $V_1 > V_2$, then F_A is to the left and motion is to the right.

For sketch (c) (note: flow is into cv at (1))

$$-V_1 \rho V_1 A_1 = F_A$$

and F_A is to the left so motion is to the right.

For sketch (d)

$$-V_1 \rho V_1 A + V_2 \rho V_2 A_2 = F_A$$

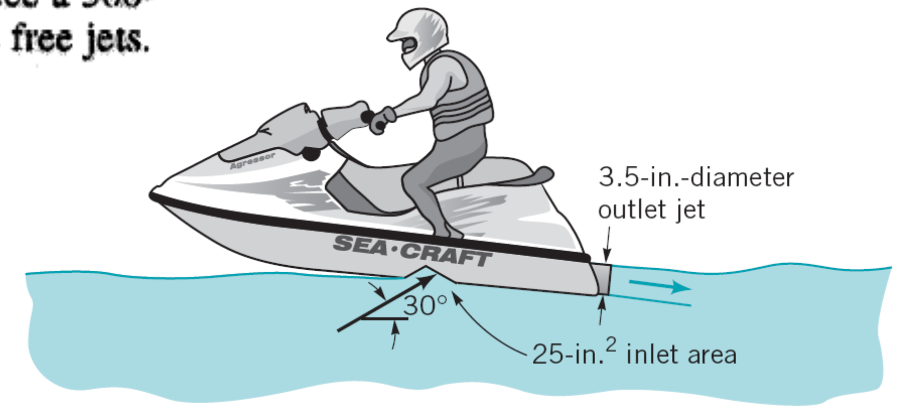
and from conservation of mass

$$\rho V_1 A_1 = \rho V_2 A_2$$

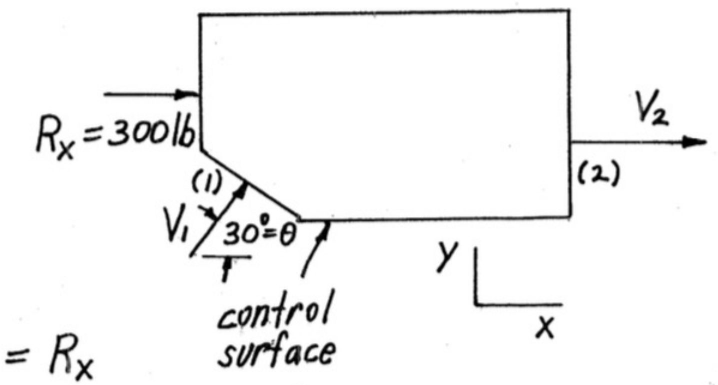
and $V_1 < V_2$

so F_A is to the right and motion is to the left.

5.36 The thrust developed to propel the jet ski shown in **Video V9.7** and Fig. P5.36 is a result of water pumped through the vehicle and exiting as a high-speed water jet. For the conditions shown in the figure, what flowrate is needed to produce a 300-lb thrust? Assume the inlet and outlet jets of water are free jets.



For the control volume indicated the x-component of the momentum equation



$$\int_{cs} u \rho \vec{V} \cdot \hat{n} \, dA = \sum F_x \quad \text{becomes}$$

(1) $$(V_1 \cos 30°)\rho(-V_1)A_1 + V_2 \rho(+V_2)A_2 = R_x$$

where we have assumed that $p = 0$ on the entire control surface and that the exiting water jet is horizontal.

With $\dot{m} = \rho A_1 V_1 = \rho A_2 V_2$ Eq. (1) becomes

$$R_x = \dot{m}(V_2 - V_1 \cos\theta) = \rho V_1 A_1 (V_2 - V_1 \cos 30°) \qquad (1)$$

Also, $A_1 V_1 = A_2 V_2$ so that

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{25\,in.^2}{\frac{\pi}{4}(3.5\,in.)^2} V_1 = 2.60\,V_1 \qquad (2)$$

By combining Eqs. (1) and (2):

$$R_x = \rho V_1^2 A_1 (2.60 - \cos 30°)$$

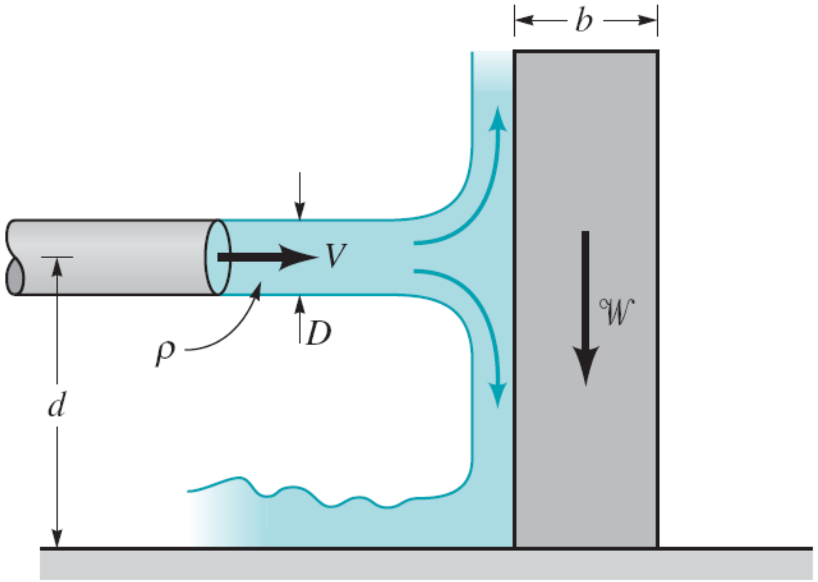
or

$$V_1 = \left[\frac{300\,lb}{\left(1.94 \frac{slugs}{ft^3}\right)\left(\frac{25}{144} ft^2\right)(2.60 - \cos 30°)} \right]^{1/2} = 22.7 \frac{ft}{s}$$

Thus,

$$Q = A_1 V_1 = \left(\frac{25}{144} ft^2\right)\left(22.7 \frac{ft}{s}\right) = \underline{\underline{3.94 \frac{ft^3}{s}}}$$

7.14 As shown in Fig. P7.14 and Video V5.4, a jet of liquid directed against a block can tip over the block. Assume that the velocity, V, needed to tip over the block is a function of the fluid density, ρ, the diameter of the jet, D, the weight of the block, $\mathcal{W}$, the width of the block, b, and the distance, d, between the jet and the bottom of the block. **(a)** Determine a set of dimensionless parameters for this problem. Form the dimensionless parameters by inspection. **(b)** Use the momentum equation to determine an equation for V in terms of the other variables. **(c)** Compare the results of parts **(a)** and **(b)**.



(a) $V = f(\rho, D, \mathcal{W}, b, d)$

(a) $V = f(\rho, D, \mathcal{W}, b, d)$

$V \doteq LT^{-1} \quad \rho \doteq FL^{-4}T^{2} \quad D \doteq L \quad \mathcal{W} \doteq F \quad b \doteq L \quad d \doteq L$

(a) $$V = f(\rho, D, \mathcal{W}, b, d)$$

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From the pi theorem, $6-3 = 3$ pi terms required.

By inspection for Π_1 (containing V)

$$\Pi_1 = VD\sqrt{\frac{\rho}{\mathcal{W}}} \doteq (LT^{-1})(L)\left(\sqrt{\frac{FL^{-4}T^{2}}{F}}\right) \doteq F^0L^0T^0$$

(a) $$V = f(\rho, D, \mathcal{W}, b, d)$$

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Check using MLT:

$$VD\sqrt{\frac{\rho}{\mathcal{W}}} = (LT^{-1})(L)\left(\sqrt{\frac{ML^{-3}}{MLT^{-2}}}\right) \doteq M^0L^0T^0 \quad \therefore \text{ok}$$

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For Π_2 let

$$\Pi_2 = \frac{b}{d}$$

(a) $$V = f(\rho, D, \mathcal{W}, b, d)$$

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For Π_2 let

$$\Pi_2 = \frac{b}{d}$$

and for Π_3

$$\Pi_3 = \frac{d}{D}$$

(a) $$V = f(\rho, D, \mathcal{W}, b, d)$$

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For Π_2 let

$$\Pi_2 = \frac{b}{d}$$

and for Π_3

$$\Pi_3 = \frac{d}{D}$$

and both Π_2 and Π_3 are obviously dimensionless.
Thus,

$$\underline{\underline{VD\sqrt{\frac{\rho}{\mathcal{W}}} = \phi\left(\frac{b}{d}, \frac{d}{D}\right)}}$$

(a) $$V = f(\rho, D, \mathcal{W}, b, d)$$

$$V \doteq LT^{-1} \quad \rho \doteq FL^{-4}T^{2} \quad D \doteq L \quad \mathcal{W} \doteq F \quad b \doteq L \quad d \doteq L$$

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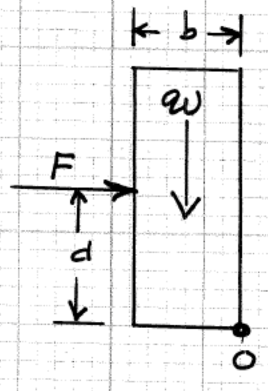


(b) For impending tipping around O

$$\Sigma M_o = 0$$

so that

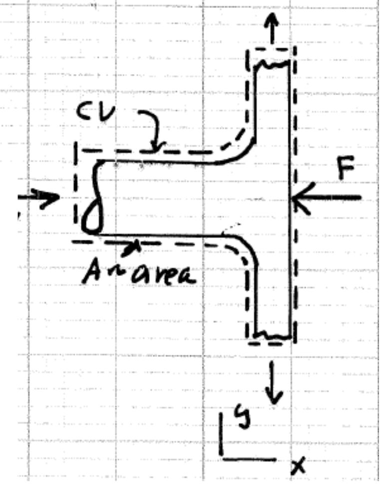
$$Fd = \mathcal{W}\left(\frac{b}{2}\right) \qquad (1)$$

(con't)

From momentum considerations using the CV shown

$\overset{+}{\rightarrow}$ $\int \rho u \vec{V} \cdot \hat{n} \, dA = \sum F_x$

$\rho V^2 A = F$



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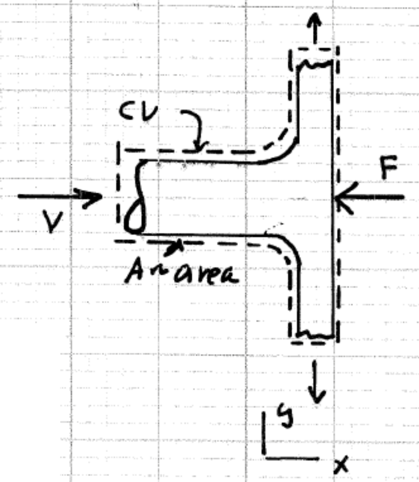
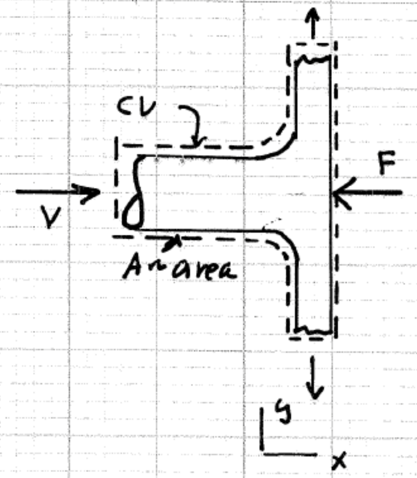


Thus, from Eq. (1)

$$(\rho V^2 A)(d) = \mathcal{W}\left(\frac{b}{2}\right)$$

From momentum considerations using the CV shown



$(\xrightarrow{+})$ $$\int \rho u \vec{V} \cdot \hat{n} \, dA = \sum F_x$$

$$\rho V^2 A = F$$

Thus, from Eq. (1)

$$(\rho V^2 A)(d) = \mathcal{W}\left(\frac{b}{2}\right)$$

so that

$$V = \sqrt{\frac{\mathcal{W}(b)}{2\rho A d}}$$

and with $A = \pi/4 \, D^2$

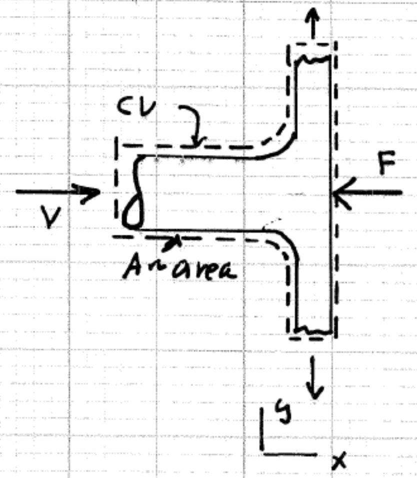
$$V = \sqrt{\frac{2\mathcal{W} b}{\pi \rho d D^2}}$$

(2)

From momentum considerations using the CV shown

$$\overset{+}{\rightarrow} \quad \int \rho u \vec{V} \cdot \hat{n} \, dA = \sum F_x$$

$$\rho V^2 A = F$$



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$$\underline{\underline{V = \sqrt{\frac{2\mathcal{W} \, b}{\pi \rho \, d \, D^2}}}} \qquad (2)$$

(c) From part (a)

$$V = \sqrt{\frac{\mathcal{W}}{\rho D^2}} \, \phi\left(\frac{b}{d}, \frac{d}{D}\right)$$

Eq. (2) can be written as

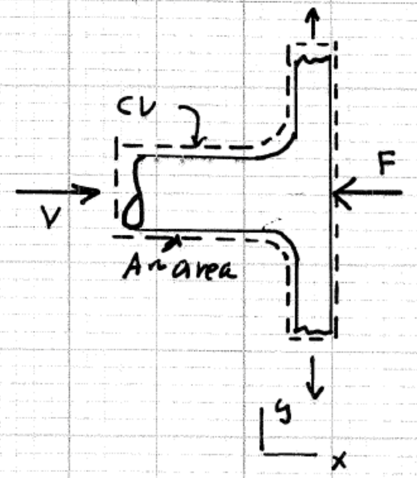
$$V = \sqrt{\frac{\mathcal{W}}{\rho D^2}} \left(\sqrt{\left(\frac{2}{\pi}\right)\left(\frac{b}{d}\right)}\right) \qquad (3)$$

It follows by comparing Eqs. (2) and (3) that

$$\phi\left(\frac{b}{d}, \frac{d}{D}\right) = \sqrt{\left(\frac{2}{\pi}\right)\left(\frac{b}{d}\right)}$$

so that $\phi\left(\frac{b}{d}, \frac{d}{D}\right)$ is actually independent of $\frac{d}{D}$.

From momentum considerations using the CV shown



$$\xrightarrow{+} \quad \int \rho u \vec{V} \cdot \hat{n} \, dA = \sum F_x$$

$$\rho V^2 A = F$$

Thus, from Eq. (1)

$$(\rho V^2 A)(d) = \mathcal{W}\left(\frac{b}{2}\right)$$

so that

$$V = \sqrt{\frac{\mathcal{W}(b)}{2\rho A \, d}}$$

and with $A = \pi/4 \, D^2$

$$\underline{\underline{V = \sqrt{\frac{2\mathcal{W}\, b}{\pi \rho \, d \, D^2}}}} \qquad (2)$$

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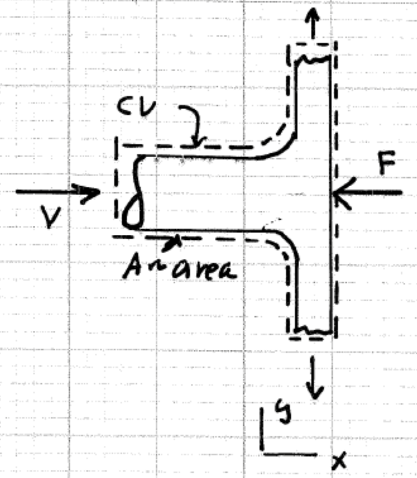

$\xrightarrow{+}$ $\int \rho u \vec{V}\cdot\hat{n}\, dA = \sum F_x$

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Thus, from Eq. (1)

$$(\rho V^2 A)(d) = \mathcal{W}\left(\frac{b}{2}\right)$$

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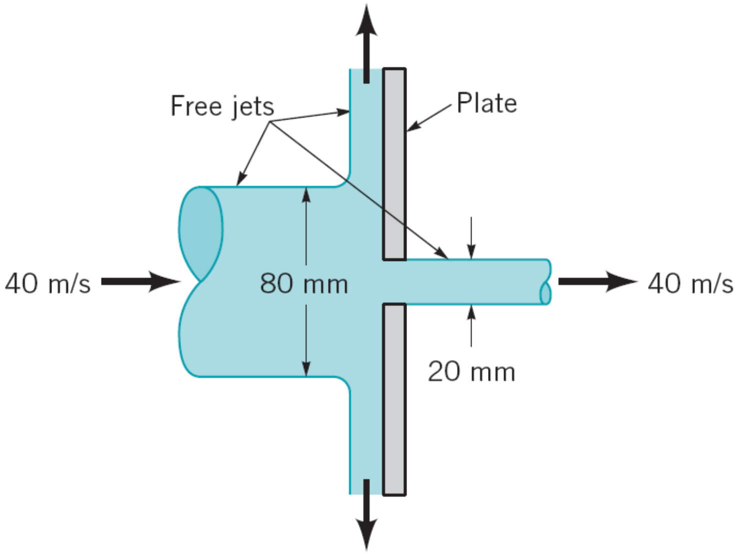
$$\phi\left(\frac{b}{d}, \frac{d}{D}\right) = \sqrt{\left(\frac{2}{\pi}\right)\left(\frac{b}{d}\right)}$$

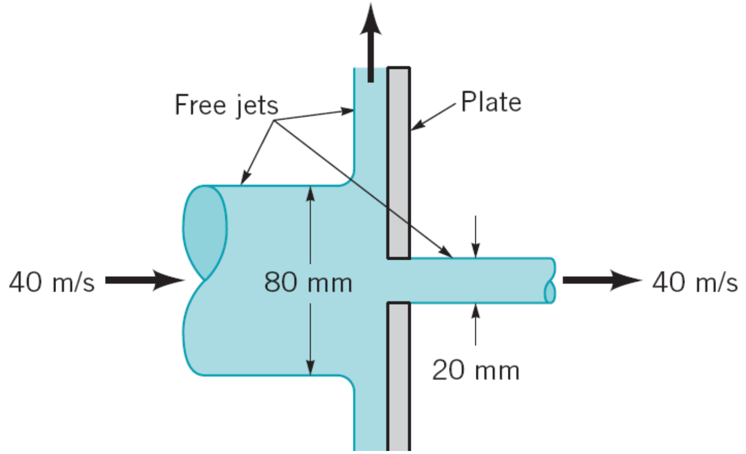
so that $\phi\left(\frac{b}{d}, \frac{d}{D}\right)$ is actually independent of $\frac{d}{D}$.

5.38 A circular plate having a diameter of 300 mm is held perpendicular to an axisymmetric horizontal jet of air having a velocity of 40 m/s and a diameter of 80 mm as shown in Fig. P5.38. A hole at the center of the plate results in a discharge jet of air having a velocity of 40 m/s and a diameter of 20 mm. Determine the horizontal component of force required to hold the plate stationary.



$$\int_{cs} -p$$

5.38 A circular plate having a diameter of 300 mm is held perpendicular to an axisymmetric horizontal jet of air having a velocity of 40 m/s and a diameter of 80 mm as shown in Fig. P5.38. A hole at the center of the plate results in a discharge jet of air having a velocity of 40 m/s and a diameter of 20 mm. Determine the horizontal component of force required to hold the plate stationary.



The control volume contains the plate and flowing air as indicated in the sketch above. Application of the horizontal or x direction component of the linear momentum equation yields

$$-u_1 \rho u_1 A_1 + u_2 \rho u_2 A_2 = -F_{A,x}$$

or

$$F_{A,x} = u_1^2 \rho \frac{\pi D_1^2}{4} - u_2^2 \rho \frac{\pi D_2^2}{4} = u_1^2 \rho \frac{\pi}{4}\left(D_1^2 - D_2^2\right)$$

Thus

$$F_{A,x} = \left(40 \frac{m}{s}\right)^2 \left(1.23 \frac{kg}{m^3}\right) \frac{\pi}{4} \left[\frac{(80\ mm)^2 - (20\ mm)^2}{\left(1000 \frac{mm}{m}\right)^2}\right] \left(\frac{1\ N}{kg \cdot \frac{m}{s^2}}\right)$$

and

$$F_{A,x} = \underline{\underline{9.27\ N}}$$

Lecture 8: Forces on an Airfoil and Conservation of Energy

ENAE311H Aerodynamics I

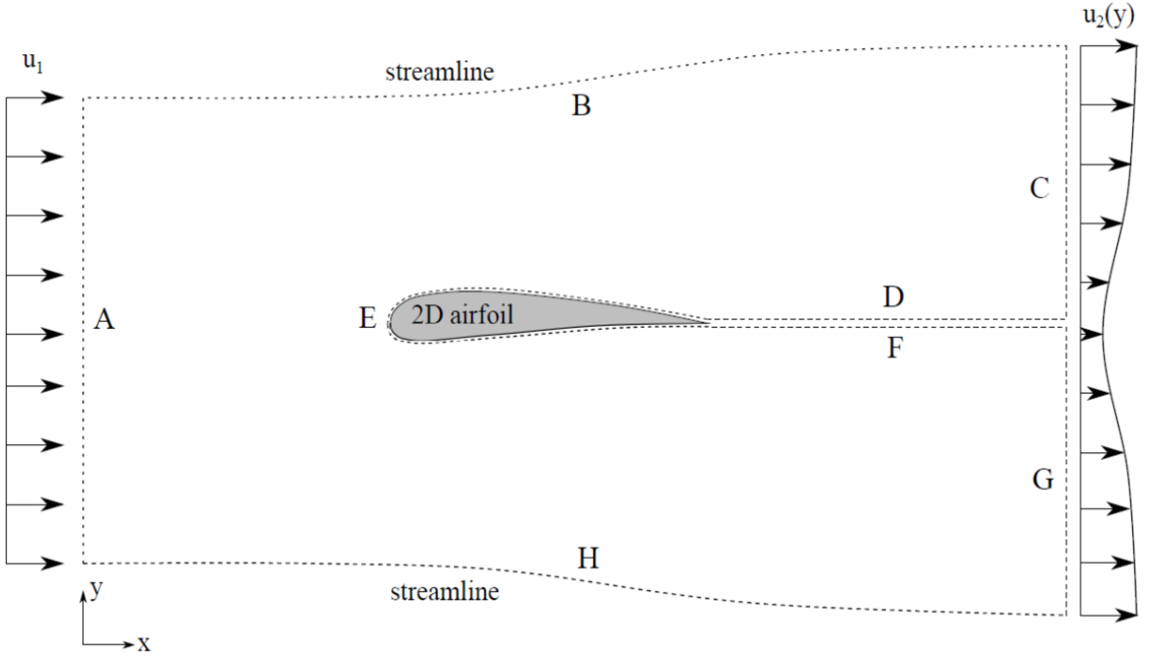
Christoph Brehm

Forces on a 2D airfoil

Early efforts to determine forces acting on airfoils did not actually involve any direct force measurements, but rather relied on measurements of fluid properties downstream of the airfoil and on the wind-tunnel walls.

To see how this worked for the drag force, consider the CV shown to the right. Note the following:

1. Surfaces B, C, G, and H are sufficiently far from the airfoil that the pressure has reverted to ambient (i.e., $p = p_\infty$)
2. No matter how far downstream we go, however, a velocity deficit will remain in the wake (because of momentum transferred from the fluid to the airfoil)
3. The force exerted on the fluid along E will be equal and opposite to the drag on the airfoil, D' (Newton's 3rd law)
4. No fluid crosses B and H since they are streamlines
5. The momentum of the fluid crossing D is exactly balanced by that crossing F; also pressure forces on these two faces exactly balance.



Forces on a 2D airfoil

We can ignore the effects of friction, except along E where these combine with the pressure forces to contribute to D'. We also assume that the effects of gravity on the fluid are negligible.

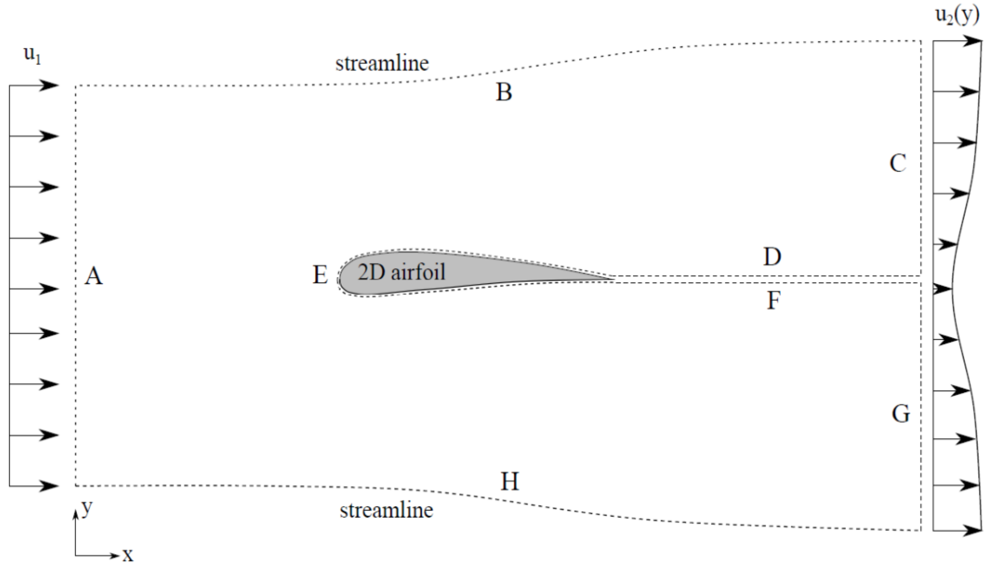
The x-component of the momentum equation then becomes

$$\overbrace{\frac{\partial}{\partial t}\iiint_{CV} v_x \rho dV}^{0\text{ (steady)}} + \iint_{CS'} v_x(\rho \mathbf{v} \cdot \mathbf{dA}) = -\iint_{CS'} p dA_x - D',$$

where CS' is the control surface comprising A, B, C, G and H, and $-D'$ is the force exerted on the fluid by the airfoil along E.

Note that we have chosen B, C, G, and H so that the pressure there has equalized to ambient, so pressure is constant over CS'. Since CS' forms a closed surface, we thus have

$$\iint_{CS'} p dA_x = 0.$$



Forces on a 2D airfoil

Also, since no fluid crosses B and H, we have that $\boldsymbol{v} \cdot \boldsymbol{dA} = \mathbf{0}$ along each (so only need to consider A, C, and G for surface integral). Thus:

$$\iint_{CS'} v_x(\rho \mathbf{v} \cdot \mathbf{dA}) = -\iint_{\mathrm{A}} \rho_1 u_1^2 dA + \iint_{\mathrm{C \cup G}} \rho_2 u_2^2 dA.$$

Since the airfoil is 2D, we consider the integrals per unit depth (i.e., $dA \to dy$). Our momentum conservation equation then reduces to:

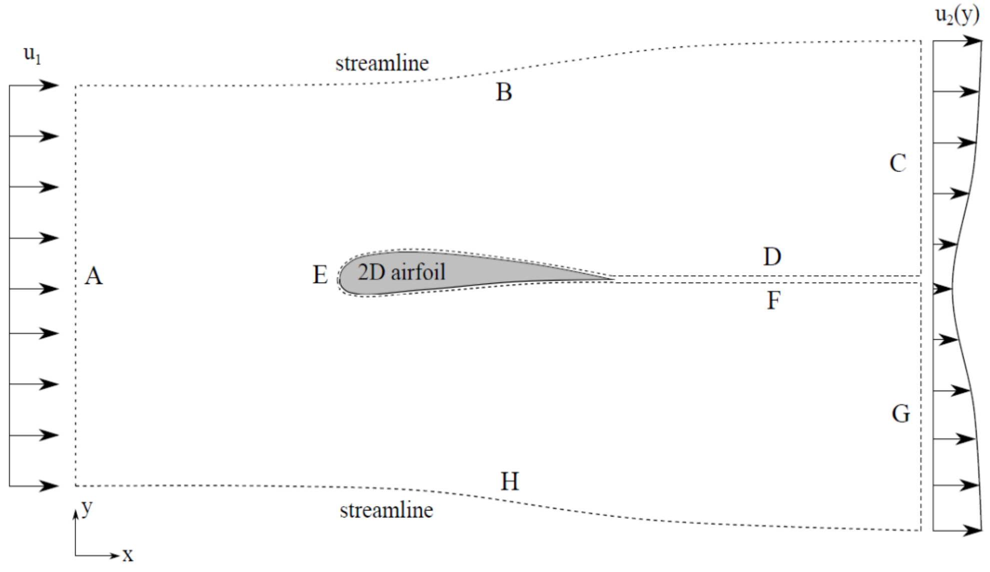
$$D' = \int_{\mathrm{A}} \rho_1 u_1^2 dy - \int_{\mathrm{C \cup G}} \rho_2 u_2^2 dy.$$

We can simplify this expression further by applying mass conservation to the same CV, which yields (per unit depth):

$$-\int_{\mathrm{A}} \rho_1 u_1 dy + \int_{\mathrm{C \cup G}} \rho_2 u_2 dy = 0.$$

Note that, since u_1 is constant, we can multiply through:

$$\int_{\mathrm{A}} \rho_1 u_1^2 dy = \int_{\mathrm{C \cup G}} \rho_2 u_1 u_2 dy.$$



Forces on a 2D airfoil

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Note that, since u_1 is constant, we can multiply through:

$$\int_{\mathrm{A}} \rho_1 u_1^2 dy = \int_{\mathrm{C \cup G}} \rho_2 u_1 u_2 dy.$$

$$\begin{aligned} D' &= \int_{\mathrm{C \cup G}} \rho_2 u_1 u_2 dy - \int_{\mathrm{C \cup G}} \rho_2 u_2^2 dy \\ &= \int_{\mathrm{C \cup G}} \underbrace{\rho_2 u_2}_{\text{mass flux per unit area}} \underbrace{(u_1 - u_2)}_{\text{velocity decrement}} dy. \end{aligned}$$

or, for incompressible flow

$$D' = \rho \int_{\mathrm{C \cup G}} u_2 (u_1 - u_2) dy.$$

Require only velocity measurement in wake!

Conservation of energy

We return once again to the Reynolds Transport Theorem:

$$\frac{dN_s}{dt} = \frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV + \iint_{CS} \eta\rho \mathbf{v}\cdot\mathbf{dA},$$

If the extensive system property is the total system energy, E_0, the corresponding intensive property is

$$\eta = e + \frac{V^2}{2} + gy$$

i.e., the sum of the fluid internal, kinetic, and potential (specific) energies.

The first law of thermodynamics tells us that the rate of change of energy of a system is the sum of the heat addition to and work done on the system:

$$\frac{dE_0}{dt} = \dot{Q} + \dot{W}.$$

(Note the convention that $\dot{W}$ is positive if done on – not by – the system.)

The RTT then becomes:

$$\frac{\partial}{\partial t}\iiint_{CV} \rho\left(e + \frac{V^2}{2} + gy\right) dV + \iint_{CS} \rho\left(e + \frac{V^2}{2} + gy\right) \mathbf{v}\cdot\mathbf{dA} = \dot{Q} + \dot{W}.$$

Work done and heat addition

In a single dimension, work done is force times distance, so rate of work done, $\dot{W}$, is force times velocity, or more generally (multiple dimensions), $\dot{W} = \boldsymbol{F} \cdot \boldsymbol{v}$.

We can thus break down $\dot{W}$ as follows:

$$\dot{W} = \underbrace{-\iint_{CS} \mathbf{v} \cdot (p\mathbf{dA})}_{\text{work done by pressure forces}} + \underbrace{\iint_{CS} \mathbf{v} \cdot (\bar{\bar{\tau}} \cdot \mathbf{dA})}_{\text{work done by shear stresses}} + \underbrace{\dot{W}_s}_{\text{shaft work}}.$$

Note that the CV can often be chosen such that the shear stress term is zero. Also, we have neglected gravitational work, since it has already been included in the potential term.

The heating term can be divided into volumetric heating, $\dot{q}$ (e.g., from radiation) and viscous heating at the CV surface:

$$\dot{Q} = \iiint_{CV} \rho \dot{q} dV + \dot{Q}_{viscous}.$$

Integral and differential forms

The integral form of our energy equation thus becomes

$$\overbrace{\frac{\partial}{\partial t}\iiint_{CV}\rho\left(e+\frac{V^2}{2}+gy\right)dV}^{\text{rate of change of energy inside CV}}+\overbrace{\iint_{CS}\rho\left(e+\frac{V^2}{2}+gy\right)\mathbf{v}\cdot\mathbf{dA}}^{\text{flux of energy through CV boundaries}}=$$

$$\underbrace{-\iint_{CS}\mathbf{v}\cdot(p\mathbf{dA})}_{\substack{\text{work done by}\\ \text{pressure forces}}}+\underbrace{\iint_{CS}\mathbf{v}\cdot(\bar{\bar{\tau}}\cdot\mathbf{dA})}_{\substack{\text{work done by}\\ \text{shear stresses}}}+\underbrace{\dot{W}_s}_{\text{shaft work}}+\underbrace{\iiint_{CV}\rho\dot{q}dV}_{\substack{\text{volumetric}\\ \text{heating}}}+\underbrace{\dot{Q}_{viscous}}_{\text{viscous heating}}.$$

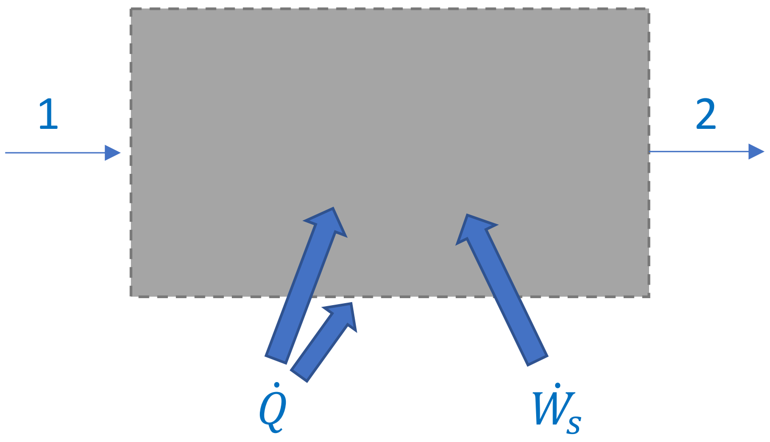
The corresponding differential form (neglecting shaft work, gravity, and viscous heating) is

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\mathbf{v}\right]=\rho\dot{q}-\nabla\cdot(\rho\mathbf{v})+\nabla\cdot(\bar{\bar{\tau}}\cdot\mathbf{v}).$$

Simplified energy equation

In a number of useful flow configurations, the flow is steady ($\frac{\partial}{\partial t} = 0$) and we have uniform conditions across inlet and outlet. We start from the steady energy equation:

$$\iint_{CS} \rho \left(e + \frac{V^2}{2} + gy \right) \mathbf{v} \cdot \mathbf{dA} = \dot{Q} + \dot{W}$$



and use our assumption of uniform inlet/outlet conditions

$$\begin{aligned} \iint_{CS} \rho \left(e + \frac{V^2}{2} + gy \right) \mathbf{v} \cdot \mathbf{dA} &= -(e_1 + \frac{u_1^2}{2} + gy_1)\rho_1 u_1 A_1 + (e_2 + \frac{u_2^2}{2} + gy_2)\rho_2 u_2 A_2 \\ &= \dot{m} \left[e_2 - e_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1) \right], \end{aligned}$$

since (from mass conservation), $\rho_1 u_1 A_1 = \rho_2 u_2 A_2 = \dot{m}$.

If we choose our control volume so that shear stresses don't contribute to the work, we can write

$$\begin{aligned} \dot{W} &= -\iint_{CS} \mathbf{v} \cdot (p\mathbf{dA}) + \dot{W}_s \\ &= u_1 p_1 A_1 - u_2 p_2 A_2 + \dot{W}_s \\ &= \dot{m}\frac{p_1}{\rho_1} - \dot{m}\frac{p_2}{\rho_2} + \dot{W}_s. \end{aligned}$$

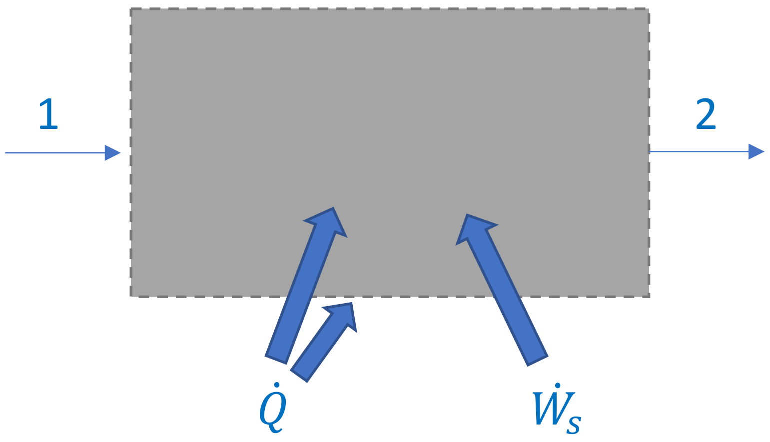
Simplified energy equation

Combining these results, we have:

$$\dot{m}\left[e_2 - e_1 + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.$$

An alternative form is possible if we use the flow enthalpy, $h = e + \frac{p}{\rho}$:

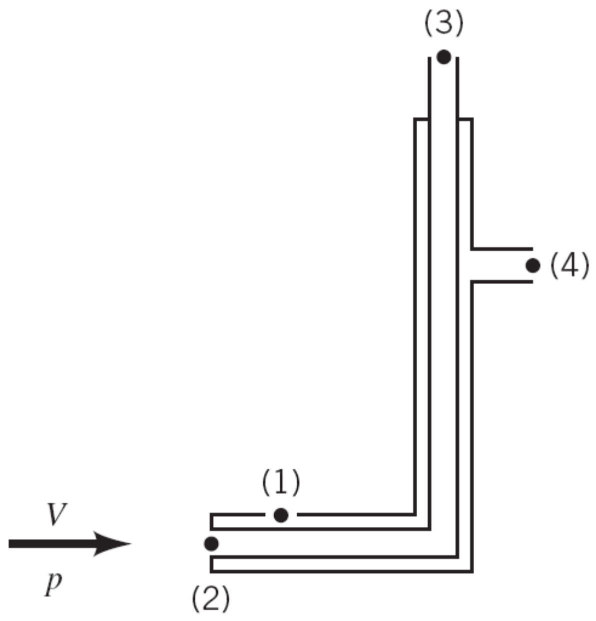
$$\boxed{\dot{m}\left[h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.}$$



This equation will be useful in several important situations later in the course.

Pitot-static Tube

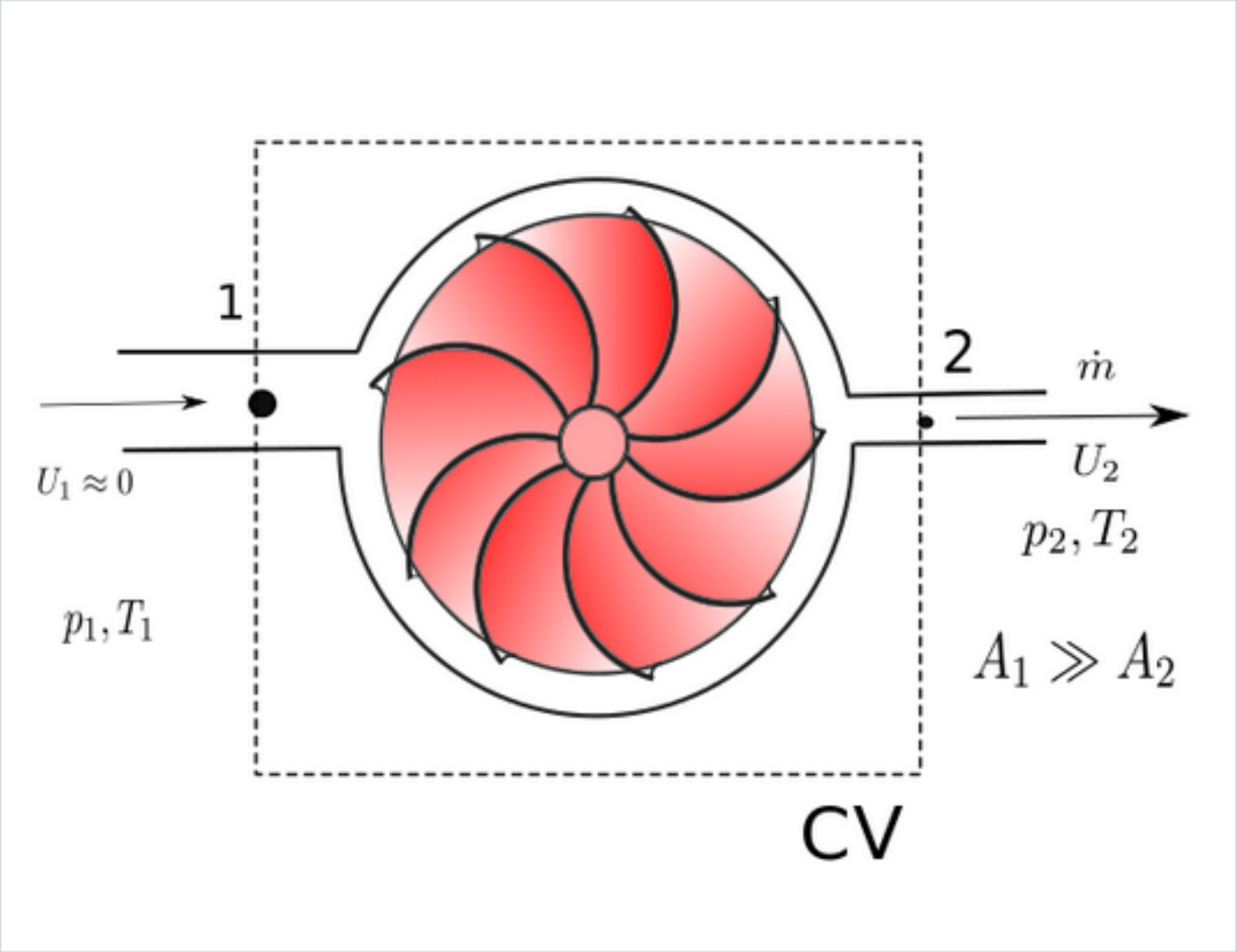
- Two concentric tubes – one with a forward facing tap and the other with a side tap



$$p_3 = p + \frac{1}{2}\rho V^2$$

$$p_4 = p$$

$$V = \sqrt{\frac{2}{\rho}\left(p_3 - p_4\right)}$$



Energy balance for a compressor (example 1)

Example 8 [edit | edit source]

Air enters compressor at inlet 1 with negligible velocity and leaves at outlet 2. The power input to the machine is P_{input} and the volume flow rate is $\dot{V}$. Find a relation for the rate of heat transfer in terms of the power, temperature, pressure, etc.

1:

$$\dot{Q} + \dot{W}_{shaft} + \underbrace{\dot{W}_{shear}}_{=0 = \tau \cdot \vec{U}} + \underbrace{\dot{W}_{other}}_{=0} = \underbrace{\frac{\partial}{\partial t}\int e\rho dV}_{=0 \ steady \ state} + \int_{cs}\left(u + \frac{p}{\rho} + \frac{U^2}{2} + gx_2\right)\rho\vec{U}\cdot\vec{n}dA$$

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{cv}\rho dV}_{=0 \ steady \ state} + \int_{cs}\rho\vec{U}\cdot\vec{n}dA \rightarrow |\rho_1 U_1 A_1| = |\rho_2 U_2 A_2| = \dot{m}$$

2:

$$\dot{Q} = -\dot{W}_{shaft} + \int_{cs}\left(u + \frac{p}{\rho} + \frac{U^2}{2} + gz\right)\rho\vec{U}\cdot\vec{n}dA$$

For uniform properties at 1 and 2 and inserting the inserting the relation for the enthalpy $h = u + \frac{p}{\rho}$.

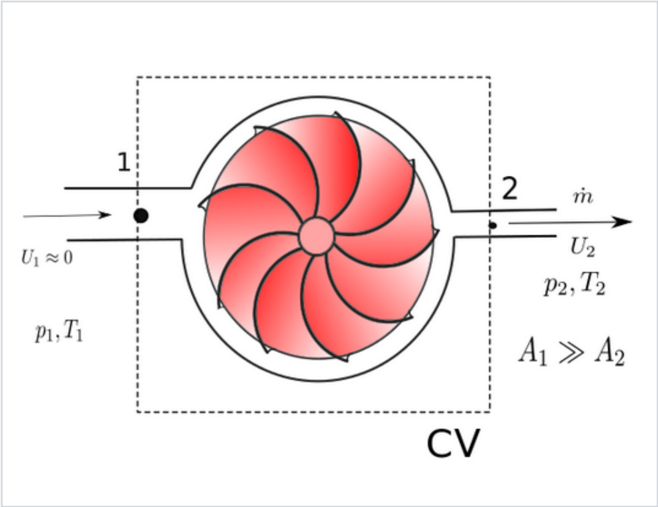
$$\dot{Q} = -\dot{W}_{shaft} - \left(h_1 + \underbrace{\frac{U_1^2}{2}}_{=0} + gz_1\right)|\rho_1 U_1 A_1| + \left(h_2 + \frac{U_2^2}{2} + gz_2\right)|\rho_2 A_2 U_2|$$

$$\dot{Q} = -\dot{W}_{shaft} + \dot{m}\left[h_2 + \frac{U_2^2}{2} - h_1 + \underbrace{g(z_2 - z_1)}_{=0}\right]$$

Assuming that air behaves like an ideal gas with a constant c_p.

$$h_2 - h_1 = c_p(T_2 - T_1)$$

$$\dot{Q} = -\dot{W}_{shaft} + \dot{m}\left[c_p(T_2 - T_1) + \frac{U_2^2}{2}\right]$$



Energy balance for a compressor (example 1)

EXAMPLE 5.22 Energy—Temperature Change

GIVEN The 420-ft waterfall shown in Fig. E5.22*a* involves steady flow from one large body of water to another.

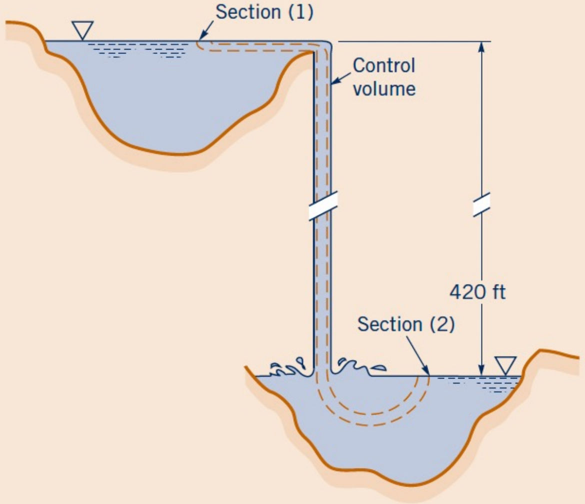
FIND Determine the temperature change associated with this flow.

■ **Figure E5.22*a***
[Photograph of Akaka Falls (Hawaii) courtesy of Scott and Margaret Jones.]

SOLUTION

To solve this problem, we consider a control volume consisting of a small cross-sectional streamtube from the nearly motionless surface of the upper body of water to the nearly motionless surface of the lower body of water as is sketched in Fig. E5.22*b*. We need to determine $T_2 - T_1$. This temperature change is related to the change of internal energy of the water, $\check{u}_2 - \check{u}_1$, by the relationship

$$T_2 - T_1 = \frac{\check{u}_2 - \check{u}_1}{\check{c}} \qquad (1)$$



■ **Figure E5.22*b***

where $\check{c} = 1$ Btu/(lbm · °R) is the specific heat of water. The application of Eq. 5.70 to the contents of this control volume leads to

$$\dot{m}\left[\check{u}_2 - \check{u}_1 + \left(\frac{p}{\rho}\right)_2 - \left(\frac{p}{\rho}\right)_1 + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1)\right] = \dot{Q}_{\substack{\text{net}\\ \text{in}}} \qquad (2)$$

We assume that the flow is adiabatic. Thus $\dot{Q}_{\text{net in}} = 0$. Also,

$$\left(\frac{p}{\rho}\right)_1 = \left(\frac{p}{\rho}\right)_2 \qquad (3)$$

because the flow is incompressible and atmospheric pressure prevails at sections (1) and (2). Furthermore,

$$V_1 = V_2 = 0 \qquad (4)$$

because the surface of each large body of water is considered motionless. Thus, Eqs. 1 through 4 combine to yield

$$T_2 - T_1 = \frac{g(z_1 - z_2)}{\check{c}}$$

so that with

$$\check{c} = [1\ \text{Btu/(lbm} \cdot \text{°R)}]\,(778\ \text{ft} \cdot \text{lb/Btu}) = [778\ \text{ft} \cdot \text{lb/(lbm} \cdot \text{°R)}]$$

$$T_2 - T_1 = \frac{(32.2\ \text{ft/s}^2)(420\ \text{ft})}{[778\ \text{ft} \cdot \text{lb/(lbm} \cdot \text{°R)}][32.2\ \text{(lbm} \cdot \text{ft)/(lb} \cdot \text{s}^2)]} = 0.540\ \text{°R} \qquad \textbf{(Ans)}$$

COMMENT Note that it takes a considerable change of potential energy to produce even a small increase in temperature.

EXAMPLE 5.28 Energy—Fan Performance

GIVEN Consider the fan of Example 5.19.

FIND Show that only some of the shaft power into the air is converted into useful effects. Develop a meaningful efficiency equation and a practical means for estimating lost shaft energy.

SOLUTION

We use the same control volume used in Example 5.19. Application of Eq. 5.82 to the contents of this control volume yields

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 + w_{\substack{\text{shaft}\\ \text{net in}}} - \text{loss} \qquad \textbf{(1)}$$

As in Example 5.26, we can see with Eq. 1 that a "useful effect" in this fan can be defined as

$$\text{useful effect} = w_{\substack{\text{shaft}\\ \text{net in}}} - \text{loss}$$

$$= \left(\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1\right) \qquad \textbf{(2)} \quad \textbf{(Ans)}$$

In other words, only a portion of the shaft work delivered to the air by the fan blades is used to increase the available energy of the air; the rest is lost because of fluid friction.

A meaningful efficiency equation involves the ratio of shaft work converted into a useful effect (Eq. 2) to shaft work into the air, $w_{\text{shaft net in}}$. Thus, we can express efficiency, η, as

$$\eta = \frac{w_{\substack{\text{shaft}\\ \text{net in}}} - \text{loss}}{w_{\substack{\text{shaft}\\ \text{net in}}}} \qquad \textbf{(3)}$$

However, when Eq. 5.54, which was developed from the moment-of-momentum equation (Eq. 5.42), is applied to the contents of the control volume of Fig. E5.19, we obtain

$$w_{\substack{\text{shaft}\\ \text{net in}}} = +U_2 V_{\theta 2} \qquad \textbf{(4)}$$

Combining Eqs. 2, 3, and 4, we obtain

$$\eta = \{[(p_2/\rho) + (V_2^2/2) + gz_2] - [(p_1/\rho) + (V_1^2/2) + gz_1]\}/U_2 V_{\theta 2} \qquad \textbf{(5)} \quad \textbf{(Ans)}$$

Equation 5 provides us with a practical means to evaluate the efficiency of the fan of Example 5.19.

Combining Eqs. 2 and 4, we obtain

$$\text{loss} = U_2 V_{\theta 2} - \left[\left(\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1\right)\right] \qquad \textbf{(6)} \quad \textbf{(Ans)}$$

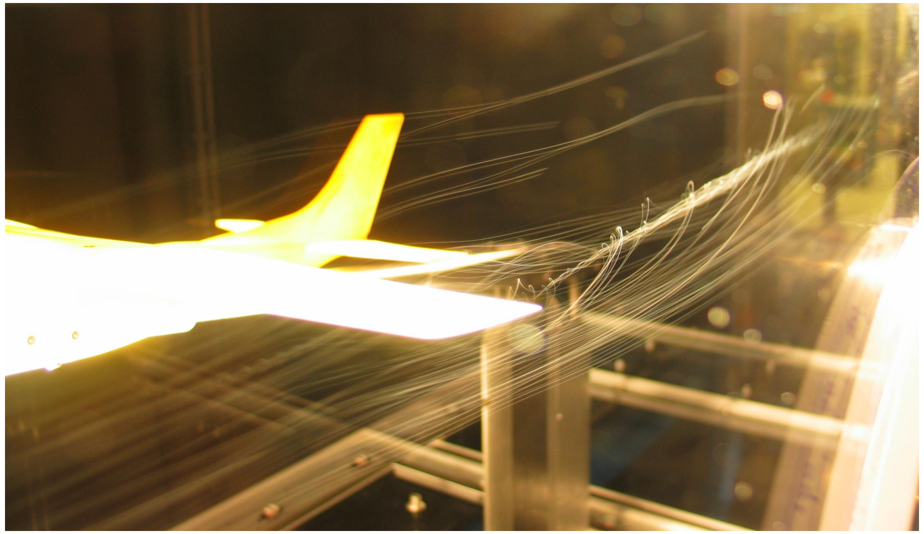
COMMENT Equation 6 provides us with a useful method of evaluating the loss due to fluid friction in the fan of Example 5.19 in terms of fluid mechanical variables that can be measured.

Lecture 9: Streamlines, Vorticity, and the Stream Function

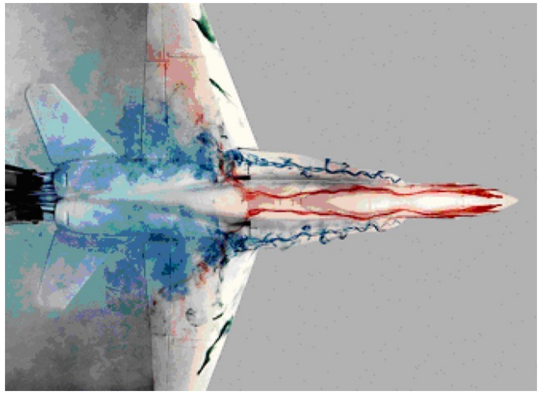
ENAE311H Aerodynamics I

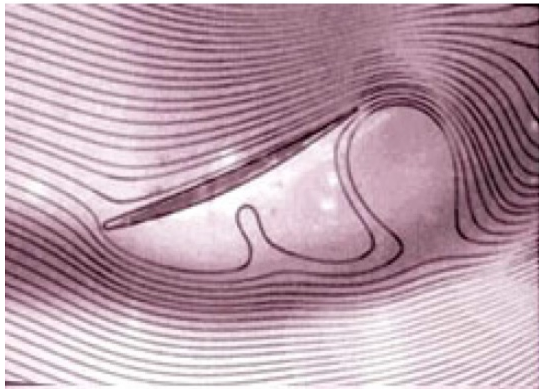
Christoph Brehm

Pathlines, streaklines, and streamlines

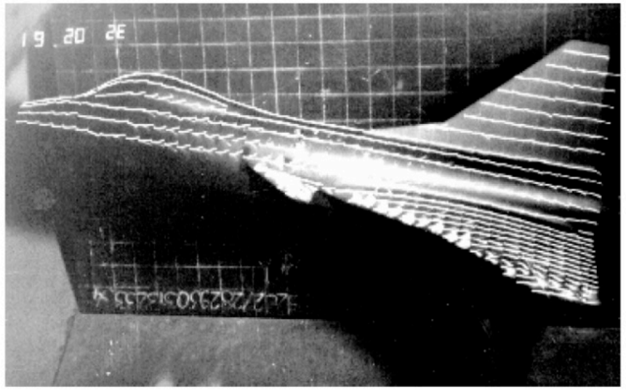


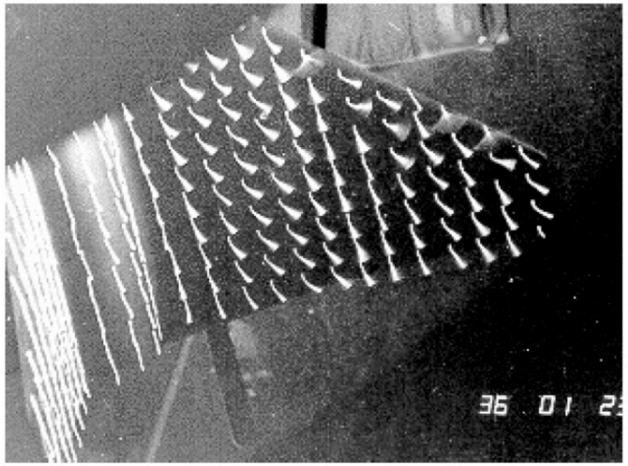
Long-time exposure of illuminated helium-filled bubbles (flow tracers)





Flow visualization by fluorescent dye



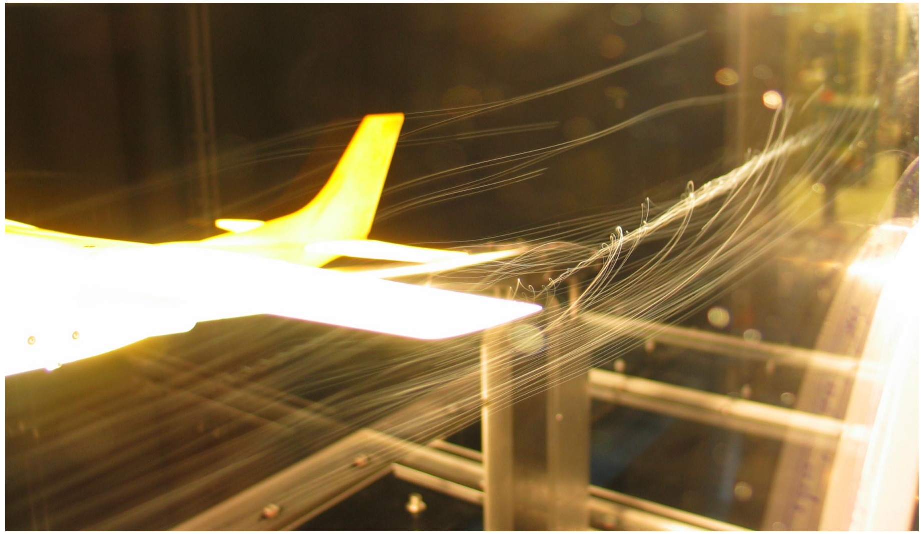


Flow visualization by fluorescent tufts

Pathlines, streaklines, and streamlines

Three ways of spatially visualizing the flowfield (especially for lower-speed flows) are pathlines, streaklines, and streamlines.

- **Pathline**: the trajectory in three-dimensional space followed by an element of the flow. If the flow is steady, all pathlines through a given point will be the same; for unsteady flows, these will be generally different. If, for example, you imagine somehow tagging a fluid element and following its path this would form a pathline.



Long-time exposure of illuminated helium-filled bubbles (flow tracers)

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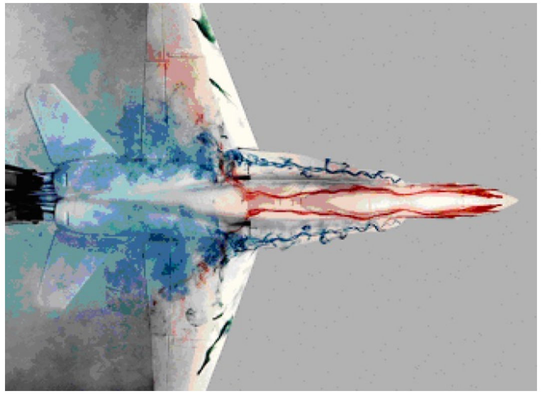
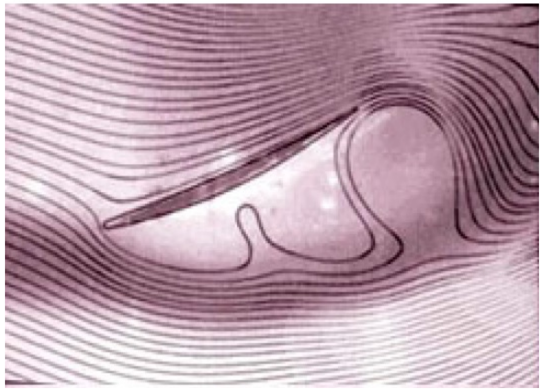
- **Streakline**: the locus of points, at a particular moment in time, corresponding to fluid elements that once passed through a given point in space in the flowfield. If dye were released at a particular point in the flowfield, for example, it would form a streakline.





Flow visualization by fluorescent dye

Pathlines, streaklines, and streamlines

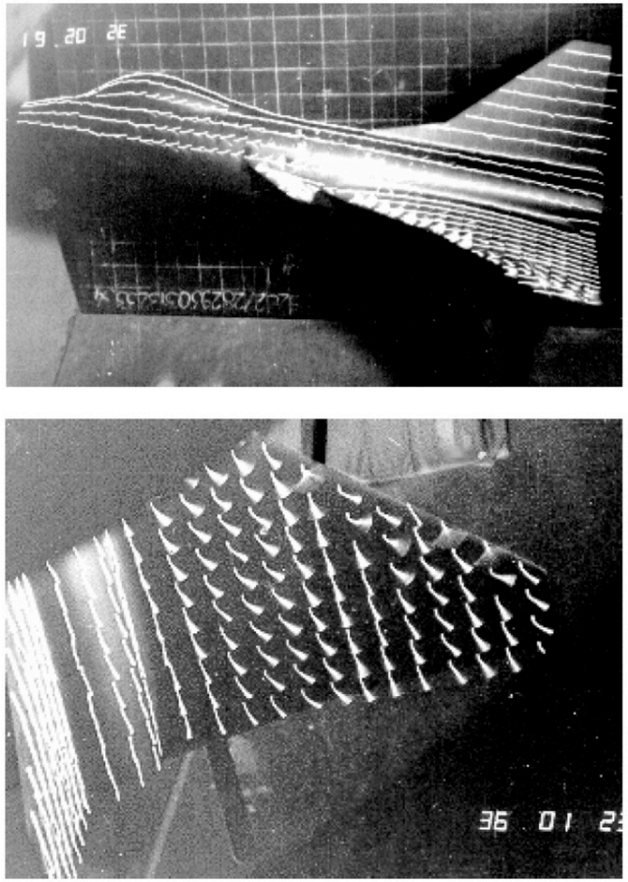
Three ways of spatially visualizing the flowfield (especially for lower-speed flows) are pathlines, streaklines, and streamlines.

- Pathline: the trajectory in three-dimensional space followed by an element of the flow. If the flow is steady, all pathlines through a given point will be the same; for unsteady flows, these will be generally different. If, for example, you imagine somehow tagging a fluid element and following its path this would form a pathline.

- Streakline: the locus of points, at a particular moment in time, corresponding to fluid elements that once passed through a given point in space in the flowfield. If dye were released at a particular point in the flowfield, for example, it would form a streakline.

- **Streamline**: a curve that is everywhere tangential to the flow at a given moment (i.e., instantaneous snapshot of the flow). A streamtube is a surface formed by collection of streamlines that pass through a closed curve in space.

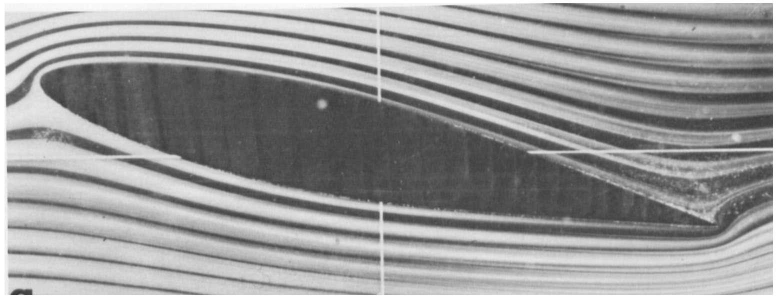
Pathlines, streaklines and streamlines are all the same in *steady* flow.

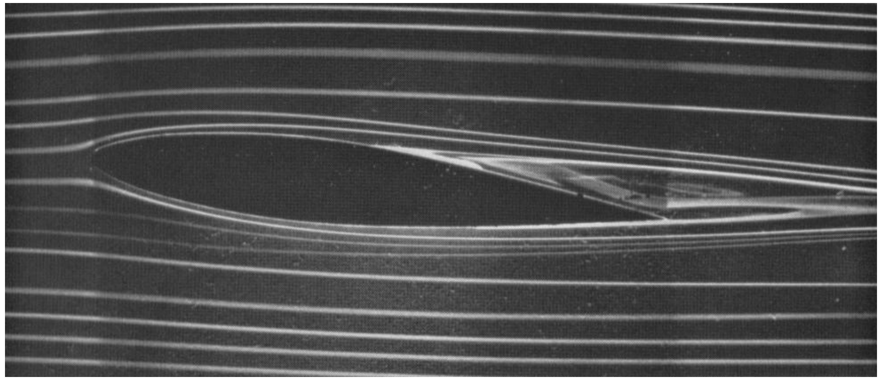


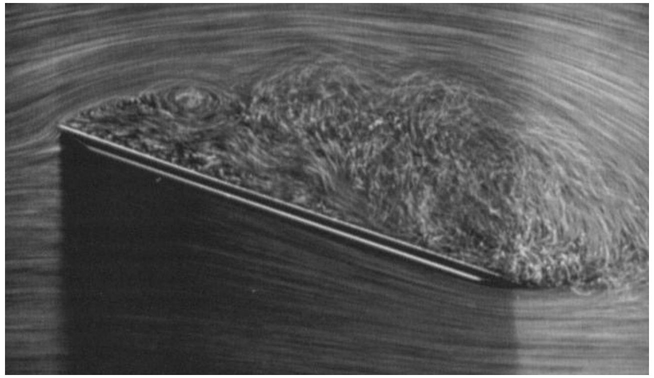
Flow visualization by fluorescent tufts

Inviscid Flow (3)

- Changes in overall velocity or geometry of a problem can change the importance of viscous forces
- Some regions of a flow may be inviscid while others show strong viscous effects



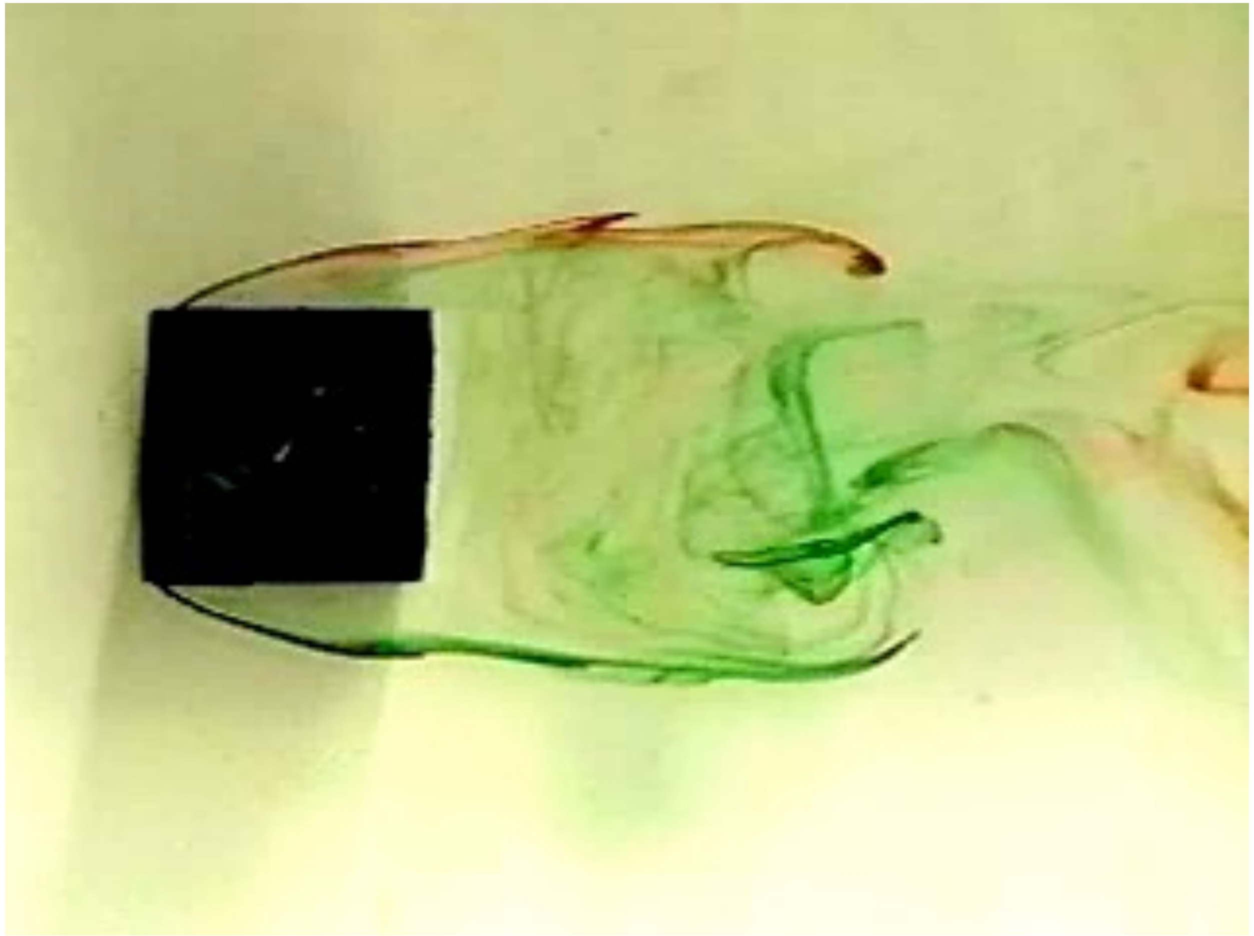




Streamlines (3)

- For steady flows – pathlines, streaklines, and streamlines are identical

NOT FOR UNSTEADY!!!



The stream function

For a two-dimensional flow, we have the following equation for a streamline:

$$\frac{dy}{dx} = \frac{v}{u}.$$

This can be integrated to give some function of x and y:

$$\text{Stream function } \boxed{\overline{\Psi}(x, y)} = c,$$

where c is a constant of integration. Different values of c will give different streamlines.

Now, since the flow velocity is everywhere tangential to streamlines, no fluid can cross a streamline, and thus the mass flux (per unit depth) between two streamlines is the difference in the value of the stream function between the two streamlines, i.e., if streamline 1 is given by $\overline{\Psi}(x, y) = c_1$ and streamline 2 by $\overline{\Psi}(x, y) = c_2$, we have for depth, d

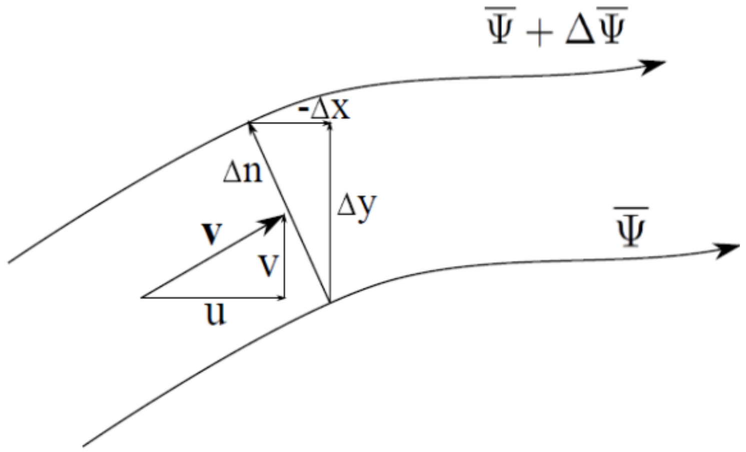
$$\frac{\dot{m}}{d} = c_2 - c_1 = \Delta\overline{\Psi}$$

For incompressible flows (ρ constant), we define $\Psi = \overline{\Psi}/\rho$ and have a relation for the volumetric flow rate, $\dot{Q}$:

$$\frac{\dot{Q}}{d} = \Delta\Psi.$$

The stream function

The stream function has an important relationship to the flow velocity components. To see this, consider two streamlines separated by a small normal distance, Δn, as shown to the right.



Since the difference in stream function values is equal to the mass flux (per unit depth) between them, we have

$$\Delta\overline{\Psi} = \rho V \Delta n,$$

which in the limit of $\Delta n \to 0$ becomes

$$\frac{\partial\overline{\Psi}}{\partial n} = \rho V.$$

Note from the geometry shown that we can also write

$$\Delta\overline{\Psi} = \rho u \Delta y + \rho v(-\Delta x),$$

or, as $\Delta n \to 0$

$$d\overline{\Psi} = \rho u dy - \rho v dx.$$

From the chain rule, however, we also have

$$d\overline{\Psi} = \frac{\partial\overline{\Psi}}{\partial x}dx + \frac{\partial\overline{\Psi}}{\partial y}dy.$$

And thus $\boxed{\dfrac{\partial\overline{\Psi}}{\partial x} = -\rho v, \quad \dfrac{\partial\overline{\Psi}}{\partial y} = \rho u.}$

The stream function

In a steady flow, it is possible to use the stream function to effectively replace the continuity equation.

To see this, note that the differential form of the conservation of mass for steady flows is $\nabla \cdot (\rho \boldsymbol{v}) = 0$, which in two dimensions is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0.$$

However, we can also write

$$\begin{aligned}
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) &= \frac{\partial}{\partial x}\left(\frac{\partial \overline{\Psi}}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \overline{\Psi}}{\partial x}\right) \\
&= \frac{\partial^2 \overline{\Psi}}{\partial x \partial y} - \frac{\partial^2 \overline{\Psi}}{\partial y \partial x} \\
&= 0,
\end{aligned}$$

by equality of mixed derivatives.

In cylindrical coordinates, our relationship for the stream function with the velocity components is

$$\rho v_r = \frac{1}{r}\frac{\partial \overline{\Psi}}{\partial \theta} \quad \text{and} \quad \rho v_\theta = -\frac{\partial \overline{\Psi}}{\partial r}.$$

6.15 The velocity components for an incompressible, plane flow are

$$v_r = Ar^{-1} + Br^{-2}\cos\theta$$

$$v_\theta = Br^{-2}\sin\theta$$

where A and B are constants. Determine thc corresponding stream function.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

$$\rho v_r = \frac{1}{r}\frac{\partial \overline{\Psi}}{\partial \theta} \qquad \rho v_\theta = -\frac{\partial \overline{\Psi}}{\partial r}.$$

From the definition of the stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{(Eq. 6.42)}$$

so that for the velocity distribution given,

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = A r^{-1} + B r^{-2} \cos\theta \qquad (1)$$

$$\frac{\partial \psi}{\partial r} = -B r^{-2} \sin\theta \qquad (2)$$

From the definition of the stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{(Eq. 6.42)}$$

so that for the velocity distribution given,

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = Ar^{-1} + Br^{-2}\cos\theta \qquad (1)$$

$$\frac{\partial \psi}{\partial r} = -Br^{-2}\sin\theta \qquad (2)$$

Integrate Eq. (1) with respect to θ to obtain

$$\int d\psi = \int (A + Br^{-1}\cos\theta)\,d\theta + f_1(r)$$

or

$$\psi = A\theta + Br^{-1}\sin\theta + f_1(r) \qquad (3)$$

From the definition of the stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{(Eq. 6.42)}$$

so that for the velocity distribution given,

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = A r^{-1} + B r^{-2}\cos\theta \qquad (1)$$

$$\frac{\partial \psi}{\partial r} = -B r^{-2}\sin\theta \qquad (2)$$

Integrate Eq. (1) with respect to θ to obtain

$$\int d\psi = \int (A + B r^{-1}\cos\theta)\, d\theta + f_1(r)$$

or

$$\psi = A\theta + B r^{-1}\sin\theta + f_1(r) \qquad (3)$$

Similarly, integrate Eq. (2) with respect to r to obtain

$$\int d\psi = -\int B r^{-2}\sin\theta\, dr + f_2(\theta)$$

or

$$\psi = B r^{-1}\sin\theta + f_2(\theta) \qquad (4)$$

From the definition of the stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{(Eq. 6.42)}$$

so that for the velocity distribution given,

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = A r^{-1} + B r^{-2}\cos\theta \qquad (1)$$

$$\frac{\partial \psi}{\partial r} = -B r^{-2}\sin\theta \qquad (2)$$

Integrate Eq. (1) with respect to θ to obtain

$$\int d\psi = \int (A + B r^{-1}\cos\theta)\, d\theta + f_1(r)$$

or

$$\psi = A\theta + B r^{-1}\sin\theta + f_1(r) \qquad (3)$$

Similarly, integrate Eq. (2) with respect to r to obtain

$$\int d\psi = -\int B r^{-2}\sin\theta\, dr + f_2(\theta)$$

or

$$\psi = B r^{-1}\sin\theta + f_2(\theta) \qquad (4)$$

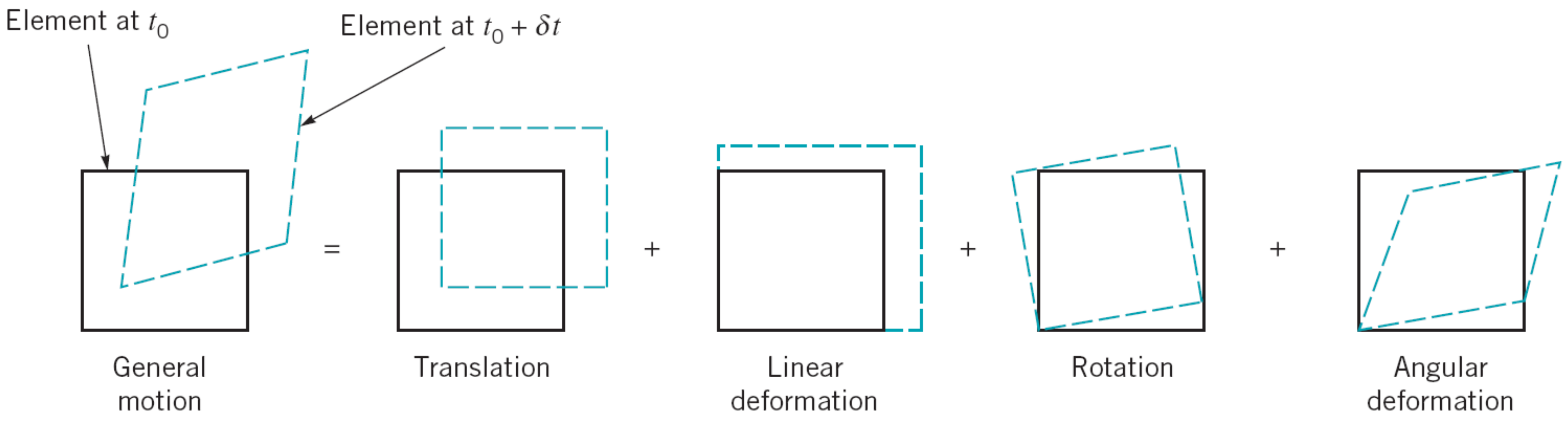
Thus, to satisfy both Eqs. (3) and (4)

$$\psi = \underline{\underline{A\theta + B r^{-1}\sin\theta + C}}$$

where C is an arbitrary constant.

Fluid element motion

- Translation (V)
- Deformation
- Rotation (ζ – vorticity)
- Angular Deformation (γ – shear strain rate)
- Differential mass conservation



Translation

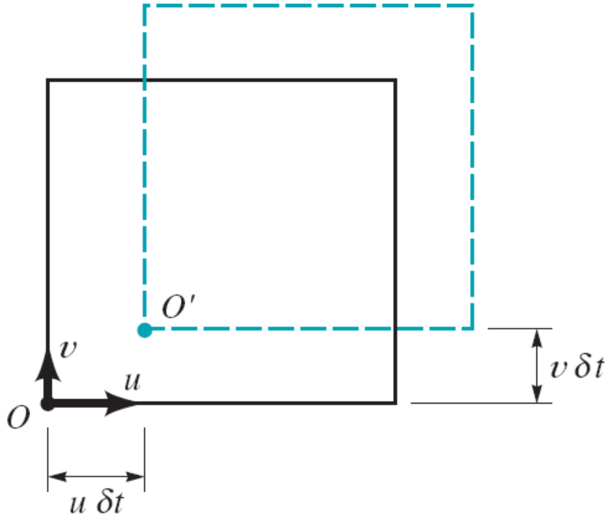
- Translation – fluid elements translate at local fluid velocity

$$\vec{V} = u\hat{x} + v\hat{y} + w\hat{z}$$

$$\Delta x = u\Delta t$$

$$\Delta y = v\Delta t$$

$$\Delta z = w\Delta t$$

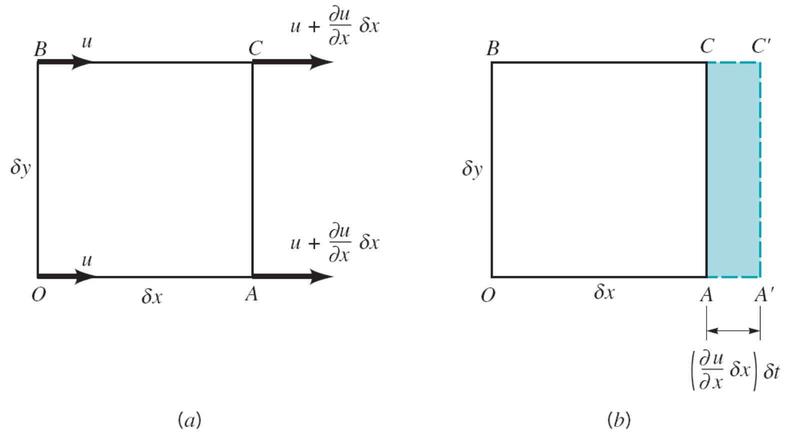


Linear Deformation

- Volume of differential fluid element: $(\delta\forall) = (\delta x \delta y \delta z)$
- Change in volume of fluid element in x direction

$$d(\delta\forall) = \left(\frac{\partial u}{\partial x}\delta x\right)\delta y \delta z dt$$

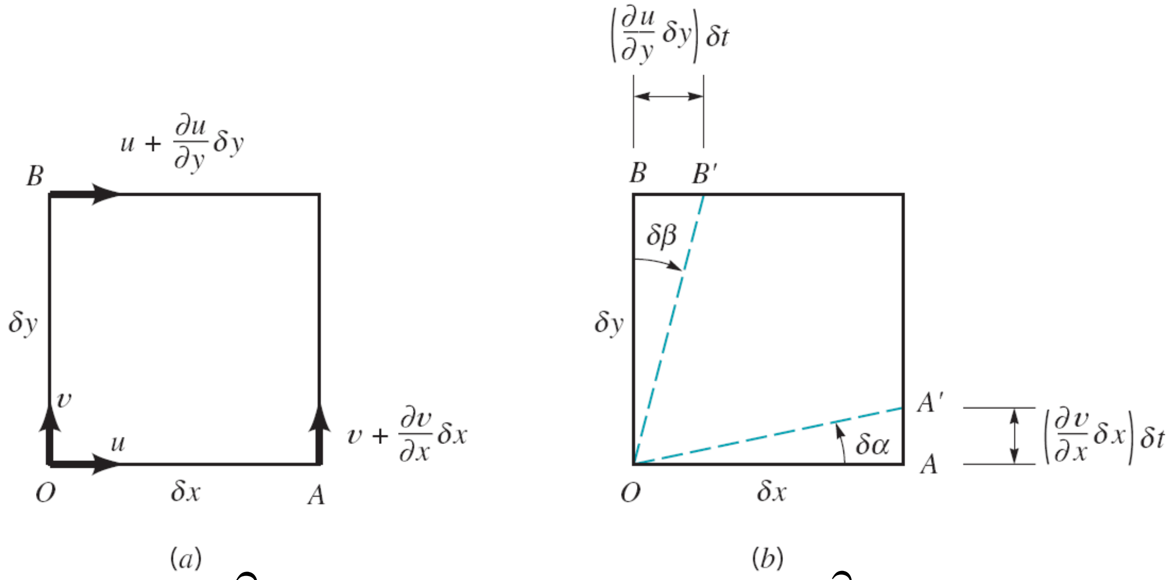
$$\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \left(\frac{\partial u}{\partial x}\right)$$



- In general for 3-D: $\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \vec{\nabla}\cdot\vec{V}$

Rotation/Angular Deformation (1)

- Define angles δα and δβ as rotation of x and y axis



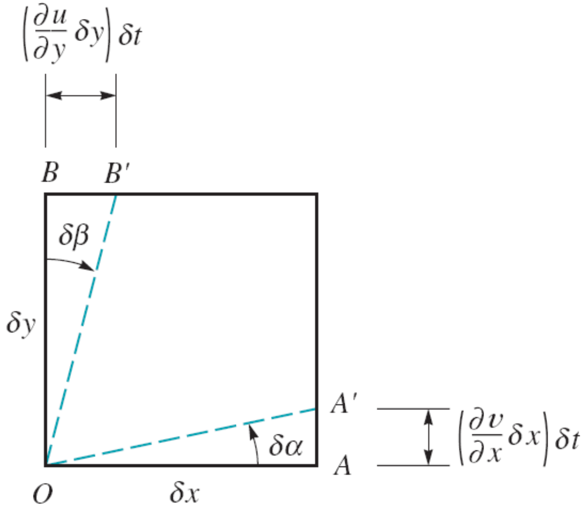
$$\tan \delta\alpha = \frac{\partial v}{\partial x}\delta t \approx \delta\alpha \qquad \tan \delta\beta = \frac{\partial u}{\partial y}\delta t \approx \delta\beta$$

Rotation/Angular Deformation (2)

- Rate of rotation of x and y axis

$$\varpi_{OA} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t} = \frac{\partial v}{\partial x}$$

$$\varpi_{OB} = \lim_{\delta t \to 0} \frac{\delta \beta}{\delta t} = \frac{\partial u}{\partial y}$$

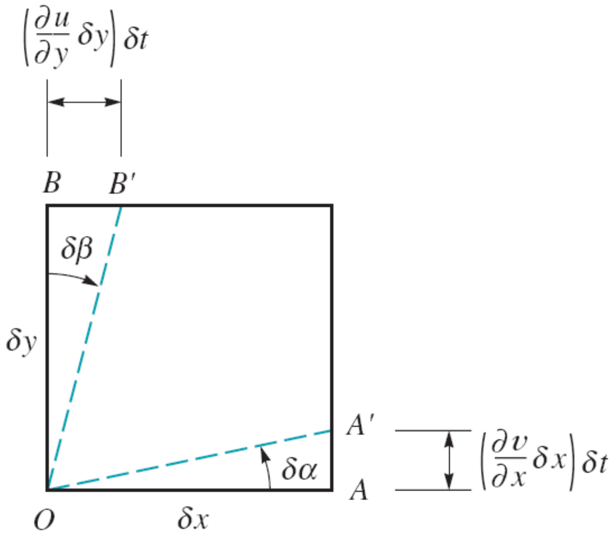


- Note different sign convention for α and β
- If $\omega_{OA}=-\omega_{OB}$ then the fluid element will only rotate and not deform
- If $\omega_{OA}=+\omega_{OB}$ then the fluid element will only deform and not rotate

Rotation/Angular Deformation (3)

- Rate of rotation of fluid element defined as average of ω_{OA} and $-\omega_{OB}$

$$\varpi_z = \frac{\varpi_{OA} - \varpi_{OB}}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$



- Likewise

$$\varpi_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \varpi_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

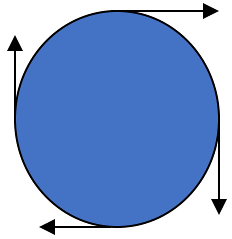
Rotation and Vorticity

$$\vec{\varpi} = \varpi_x \hat{x} + \varpi_y \hat{y} + \varpi_z \hat{z}$$

- Rotation rate is a vector:

$$\varpi_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \varpi_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \varpi_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\vec{\varpi} = \frac{1}{2}\vec{\nabla}\times\vec{V} = \frac{1}{2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$



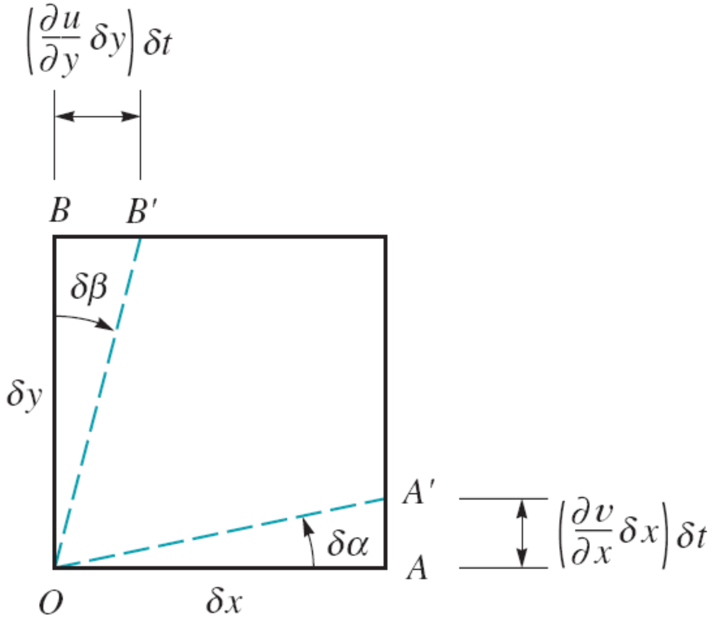
- Vorticity is defined as twice the rotation rate

$$\vec{\zeta} = 2\vec{\varpi} = \vec{\nabla}\times\vec{V}$$

Angular Deformation

- Rate of angular deformation (rate of shearing strain) of fluid element defined as twice the average of ω_{OA} and $+\omega_{OB}$

$$\gamma_z = 2\left(\frac{\varpi_{OA} + \varpi_{OB}}{2}\right) = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$



- Likewise

$$\gamma_x = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \gamma_y = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

6.11 The velocity components of an incompressible, two-dimensional velocity field are given by the equations

$$u = y^2 - x(1 + x)$$
$$v = y(2x + 1)$$

Show that the flow is irrotational and satisfies conservation of mass.

If the two-dimensional flow is irrotational,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

For the velocity distribution given,

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = 2y$$

Thus,

$$\omega_z = \frac{1}{2}\left(2y - 2y\right) = 0$$

and the <u>flow is irrotational.</u>

To satisfy conservation of mass,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since,

$$\frac{\partial u}{\partial x} = -1 - 2x \qquad \frac{\partial v}{\partial y} = 2x + 1$$

then

$$-1 - 2x + 2x + 1 = 0$$

and

<u>conservation of mass is satisfied</u>.

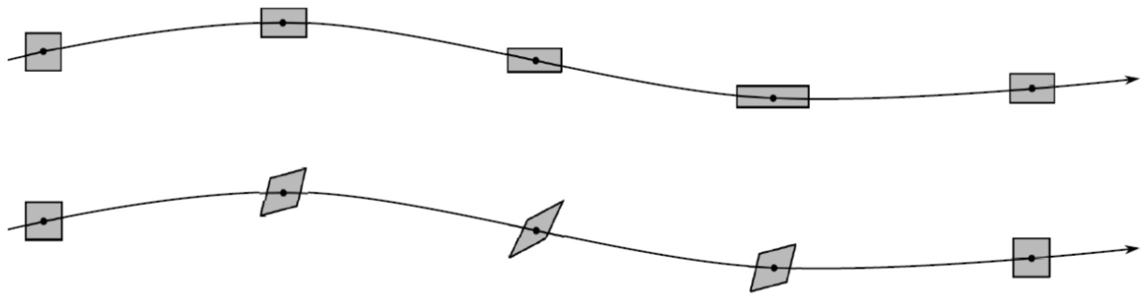
Vorticity

As a fluid element moves along a streamline or pathline, it may rotate and become deformed. The angular velocity, $\boldsymbol{\omega}$, is an important property in fluid mechanics, but a slightly more useful quantity is the vorticity $\boldsymbol{\xi} = 2\boldsymbol{\omega}$. This may be written:

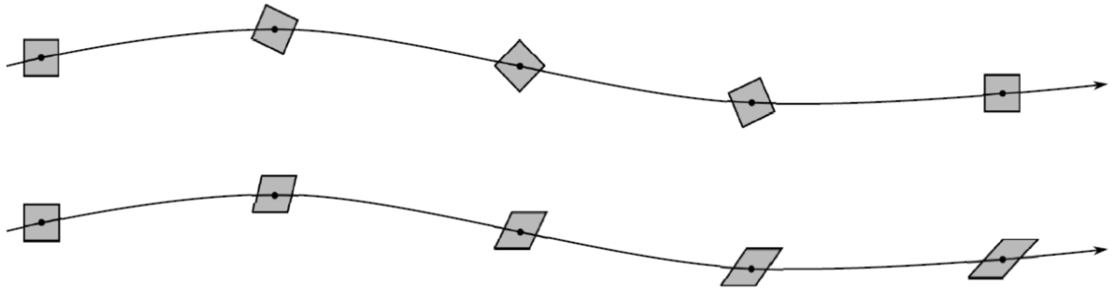
$$\begin{aligned} \boldsymbol{\xi} &= \nabla \times \mathbf{v} \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}. \end{aligned}$$

Or, in two dimensions, $\xi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}.$

For *irrotational* flows, the vorticity is zero (inviscid flows are typically irrotational).



Rotational flows have nonzero vorticity.



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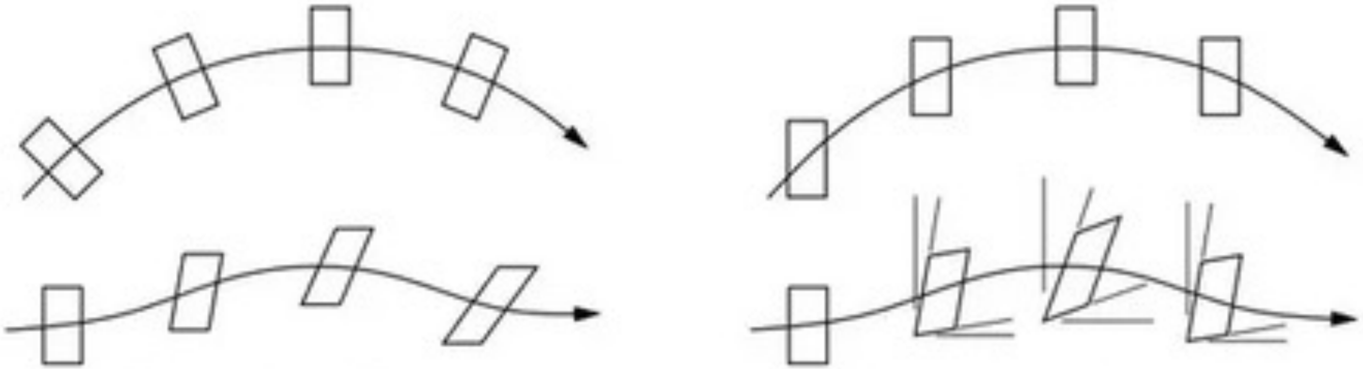

Rotational flows

Irrotational flows

irrotational

rotational

Circulation

A related concept to vorticity is the circulation, Γ, which is defined as

$$\Gamma = -\oint_c \mathbf{v} \cdot \mathbf{ds}$$

i.e., the integral around a closed curve of the dot product of the velocity with the curve element.

Using Stokes' theorem, we can write

$$\begin{aligned} \Gamma &= -\iint_s (\nabla \times \mathbf{v}) \cdot \mathbf{dA} \\ &= -\iint_s \boldsymbol{\xi} \cdot \hat{\mathbf{n}} dA. \end{aligned}$$

The Kutta-Joukowski theorem tells us that the lift produced by an airfoil is directly proportional to the circulation about it.

Velocity Potential

- For an irrotational flow the velocity components can be expressed in terms of a scalar function $\phi(x,y,z)$:

$$\mathbf{V} = \nabla\phi \qquad u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

- $\phi(x,y,z)$ is called velocity potential

- Irrotational: $\nabla \times \mathbf{V} = 0$ (note: $\nabla \times \nabla\phi = 0$)

- The inviscid, incompressible, and irrotational flow fields is governed by the Laplace equation:

with $\nabla \cdot \mathbf{V} = 0$ and $\nabla^2(\ ) = \nabla \cdot \nabla(\ )$

we get $\nabla^2\phi = 0$ or $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0$

Lecture 10: The Velocity Potential and Bernoulli’s Equation

ENAE311H Aerodynamics I

Christoph Brehm

Circulation

A related concept to vorticity is the circulation, Γ, which is defined as

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The Kutta-Joukowski theorem tells us that the lift produced by an airfoil is directly proportional to the circulation about it.

The velocity potential

For an irrotational flow, we have that

$$\boldsymbol{\xi} = \nabla \times \mathbf{v} = 0.$$

From vector calculus, we know that, if ϕ is a scalar function

$$\nabla \times (\nabla \phi) = 0.$$

Thus, the irrotational flow condition will be satisfied if

$$\mathbf{v} = \nabla \phi.$$

The function ϕ is called the *velocity potential* and any flow that can be described in this way is known as a *potential flow*.

In Cartesian and cylindrical coordinates, we have

$$u = \frac{\partial \phi}{\partial x}, \; v = \frac{\partial \phi}{\partial y}, \; w = \frac{\partial \phi}{\partial z},$$

and

$$v_r = \frac{\partial \phi}{\partial r}, \; v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \; v_z = \frac{\partial \phi}{\partial z}.$$

Note that, for steady flow, the differential continuity equation is

$$\nabla \cdot (\rho \mathbf{v}) = \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \mathbf{v} = 0.$$

Substituting $\boldsymbol{v} = \nabla \phi$,

$$\rho \nabla^2 \phi = -\nabla \rho \cdot \nabla \phi.$$

Thus, for an incompressible flow (ρ constant),

$$\nabla^2 \phi = 0$$

i.e., the velocity potential satisfies Laplace’s equation.

The velocity potential and the stream function

Lines of constant ϕ are called *equipotential lines*.

Lines tangential to $\nabla\phi$ are called *gradient lines* and are streamlines of the flow (i.e., if the stream function exists, are lines of constant Ψ).

Equipotential lines and streamlines are perpendicular to one another.

	Dimensions	Flow type
Stream function	Two	Rotational or irrotational
Velocity potential	Two or three	Irrotational

Bernoulli's equation

Bernoulli's equation is probably the most famous equation in fluid mechanics, but is probably also the most widely misused, so we must be very aware of the assumptions used in deriving it.

We by assuming a steady, inviscid flow with negligible gravity. In this case, the x component of the differential momentum conservation equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

We can multiply through by dx to obtain

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx.$$

Let us now assume that we are moving along a streamline. We then have

$$\begin{aligned} vdx &= udy \\ wdx &= udz. \end{aligned}$$

The above equation can thus be written

$$u\underbrace{\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right)}_{=du \text{ for steady flow}} = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx. \quad \Longrightarrow \quad udu = d\left(\frac{u^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx.$$

Bernoulli's equation

We can repeat exactly the same procedure for the y and z components of the momentum conservation equation (multiplying by dy and dz, and using the relevant streamline equations in each case). This results in two further equations:

$$d\left(\frac{v^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy$$

$$d\left(\frac{w^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz.$$

Combining these, we have

$$\frac{1}{2}d(u^2+v^2+w^2) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

Now, since $u^2 + v^2 + w^2 = V^2$ and, for a steady flow,

$$\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = dp$$

we can re-write this as

$$\frac{1}{2}d(V^2) = -\frac{1}{\rho}dp,$$

Or equivalently

$$\boxed{dp = -\rho V\,dV.}$$

Euler's equation

If density is constant, we can integrate immediately:

$$\frac{1}{2}\int_{V_1^2}^{V_2^2} d(V^2) = -\frac{1}{\rho}\int_{p_1}^{p_2} dp,$$

to obtain

$$\boxed{p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2.}$$

Bernoulli's equation

We thus see that, for an <u>inviscid</u>, <u>incompressible</u> flow, $p + \frac{1}{2}\rho V^2$ is constant <u>along a streamline</u>. Different streamlines will, in general, have different values of that constant (but all the same for the special case of <u>irrotational</u> flow).

If gravity is important:

$$p + \frac{1}{2}\rho V^2 + \rho g y = \text{const.}$$

F=ma Normal to Streamline (1)

$$\sum \delta F_n = \delta m a_n = \frac{V^2}{\Re} \rho \delta \forall$$

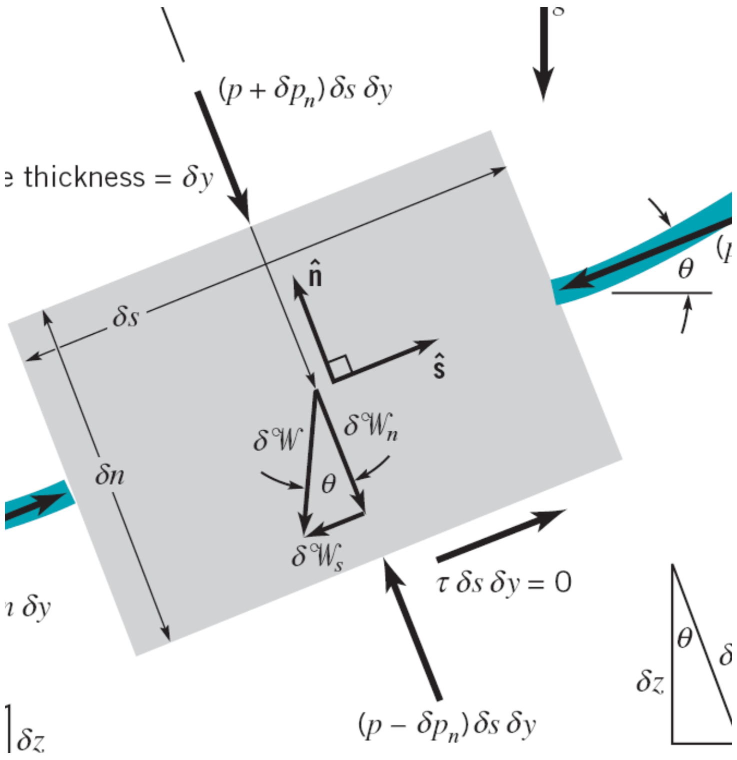
- Force due to gravity across streamline is

$$\delta F_{n,g} = -\delta m g \cos\theta = -\rho g \cos\theta \delta \forall$$

- Force due to pressure

$$\delta F_{n,p} = (p - \delta p_n)\delta s \delta y - (p + \delta p_n)\delta s \delta y$$

$$= -2\delta p_n \delta s \delta y = -\frac{\partial p}{\partial n}\delta \forall$$



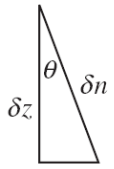
$$\delta p_n = \frac{\partial p}{\partial n}\frac{\delta n}{2}$$

F=ma Normal to Streamline (2)

- Across streamline

$$-\rho g\cos\theta - \frac{\partial p}{\partial n} = \rho\frac{V^2}{\Re}$$

$$\cos\theta = \frac{\partial z}{\partial n}$$



Normal to streamline

$$\frac{\partial p}{\partial n} + \rho\frac{V^2}{\Re} + \rho g\frac{\partial z}{\partial n} = 0 \qquad dp = \frac{\partial p}{\partial s}ds + \frac{\partial p}{\partial n}dn = \frac{\partial p}{\partial n}dn$$

- Since normal to streamline ds=0, for any derivative partial and ordinary derivatives in n are the same (Note: analysis is limited to normal to a streamline)

$$dp + \rho\frac{V^2}{\Re}dn + \rho g dz = 0$$

F=ma Normal to Streamline (3)

- Integrating normal to streamline

$$dp + \rho \frac{V^2}{\Re} dn + \rho g dz = 0$$

$$\int \frac{dp}{\rho} + \int \frac{V^2}{\Re} dn + \int g dz = Const.$$

- If we assume fluid is incompressible

$$p + \rho \int \frac{V^2}{\Re} dn + \rho g z = Const.$$

F=ma Normal to Streamline (3)

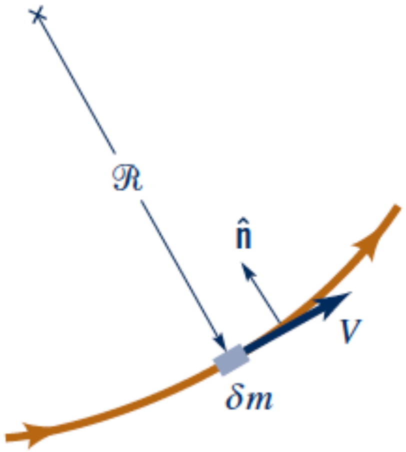
- Integrating normal to streamline

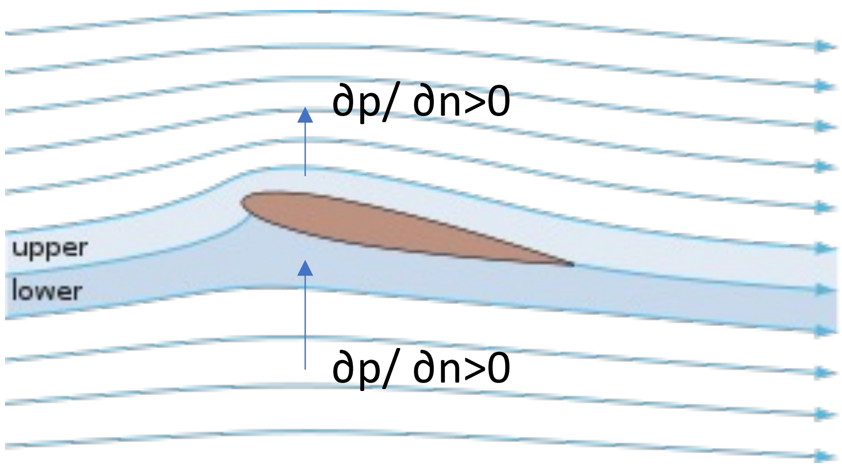
$$dp + \rho \frac{V^2}{\Re} dn + \rho g dz = 0$$

$$\int \frac{dp}{\rho} + \int \frac{V^2}{\Re} dn + \int g dz = Const.$$

- If we assume fluid is incompressible

$$\frac{\partial p}{\partial n} = -\frac{\rho V^2}{\mathscr{R}}$$





F=ma Normal to Streamline (3)

- Integrating normal to streamline

$$dp + \rho \frac{V^2}{\Re} dn + \rho g dz = 0$$

$$\int \frac{dp}{\rho} + \int \frac{V^2}{\Re} dn + \int g dz = Const.$$

- If we assume fluid is incompressible

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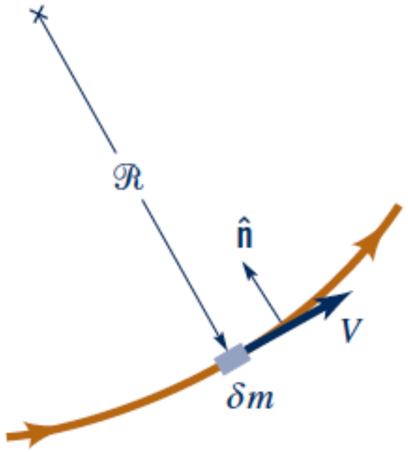
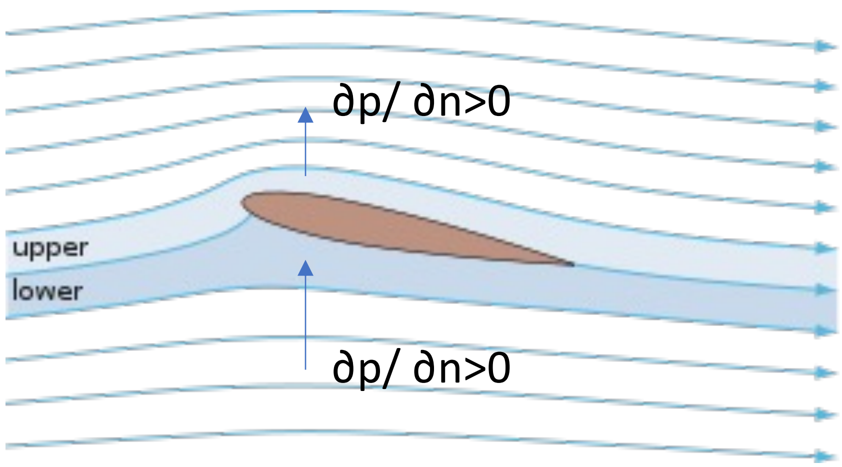




F=ma Normal to Streamline (3)

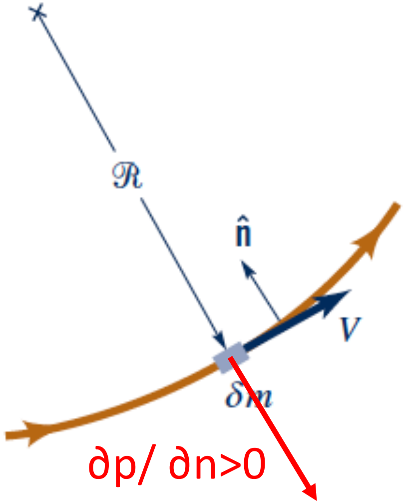
- Integrating normal to streamline

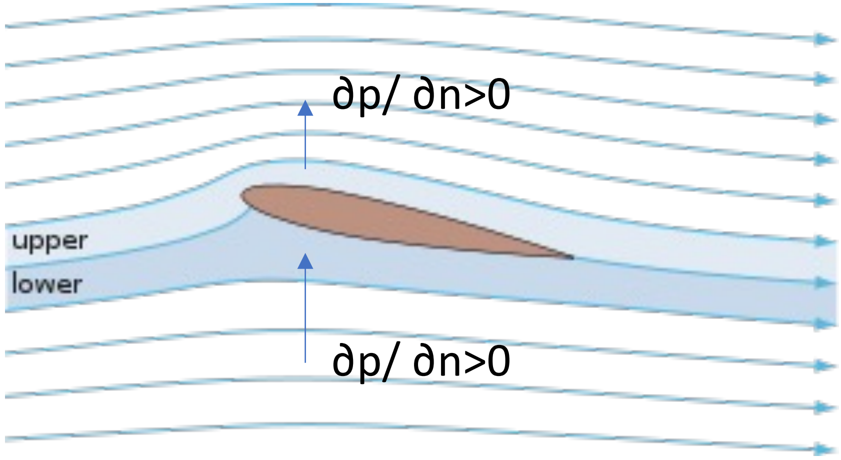
$$dp + \rho \frac{V^2}{\Re} dn + \rho g dz = 0$$

$$\int \frac{dp}{\rho} + \int \frac{V^2}{\Re} dn + \int g dz = Const.$$

- If we assume fluid is incompressible

$$\frac{\partial p}{\partial n} = -\frac{\rho V^2}{\mathcal{R}}$$





Steady, Inviscid, Incompressible Flow

- Along streamline: $$p + \frac{1}{2}\rho V^2 + \rho g z = Const.$$

- Across streamline: $$p + \rho \int \frac{V^2}{\Re} dn + \rho g z = Const.$$

- Pressure changes along streamline accelerates fluid particles
- Pressure changes normal to streamline turns fluid particles (changes streamline direction)

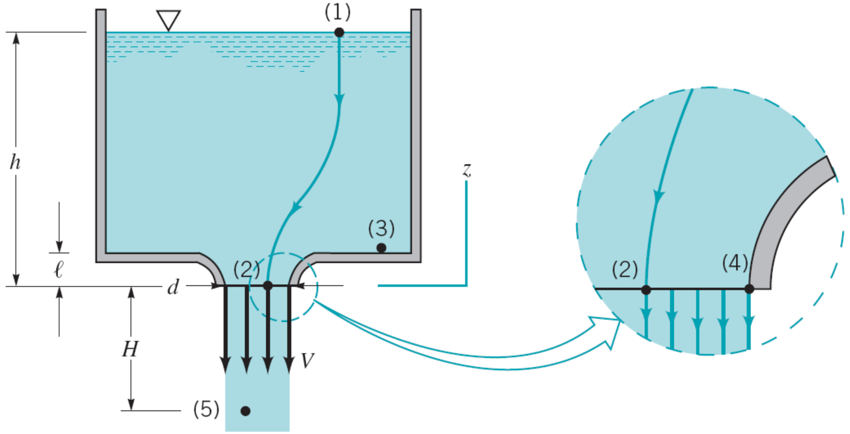
Free Jets

- If streamlines are straight at jet exit (free jet, R=∞) then no pressure gradient across jet, p2=p1
- V1=0

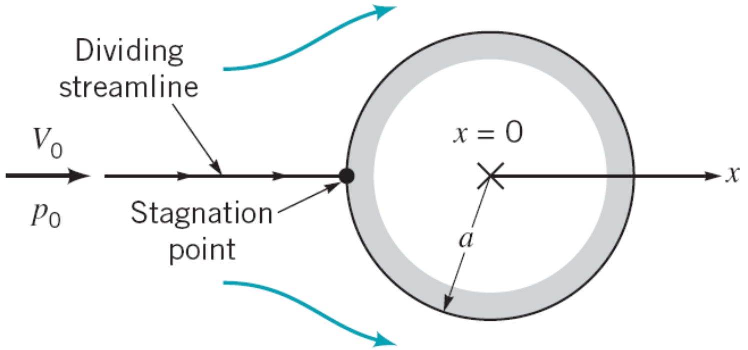
$$p_1 + \frac{1}{2}\rho V_1^{\,2} + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^{\,2} + \rho g z_2$$

$$V_2 = \sqrt{2g(z_1 - z_2)}$$

$$V_2 = \sqrt{2gh}$$



*3.8 A wind of velocity V_0 blows past a smokestack of radius $a = 2.5$ ft as shown in Fig. P3.8. The fluid velocity along the dividing streamline $(-\infty \leq x \leq -a)$ is found to be $V = V_0(1 - a^2/x^2)$. Plot the pressure distribution from a distance 30 ft ahead of the smokestack to the stagnation point on the smokestack for wind speeds of $V_0 = 0, 10, 20, 30, 40$, and 50 mph.



From the Bernulli eqn. with z = constant:

$$p_0 + \frac{1}{2}\rho V_0^2 = p + \frac{1}{2}\rho V^2, \text{ or with } p_0 = 0:$$

$$p = \frac{1}{2}\rho\left[V_0^2 - V^2\right] = \frac{1}{2}\rho\left[V_0^2 - V_0^2(1 - a^2/x^2)^2\right]$$

$$= \frac{1}{2}\rho V_0^2\left[1 - 1 + 2(a/x)^2 - (a/x)^4\right]$$

or

$$p = \frac{1}{2}\rho V_0^2\left[2(a/x)^2 - (a/x)^4\right]$$

Hence, with the given data:

$$p = \frac{1}{2}(0.00238\ slugs/ft^3)\left[2(2.5\,ft/x)^2 - (2.5\,ft/x)^4\right]V_0^2(mph)^2(88\,ft/s/60mph)^2$$

or

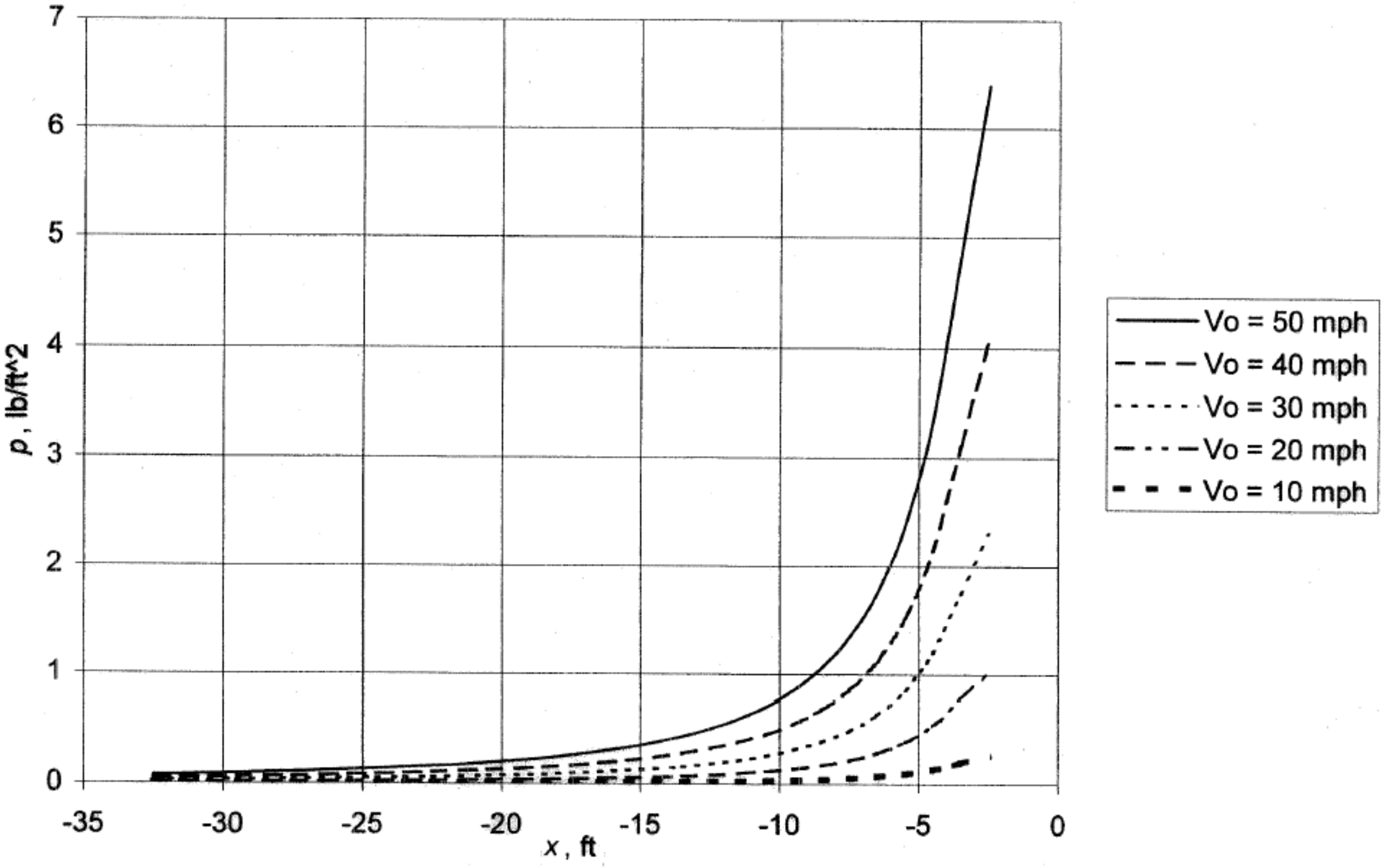
$$p = 0.00256\left[2(2.5/x)^2 - (2.5/x)^4\right]V_0^2\ lb/ft^2, \text{ where } V_0 \sim mph \text{ and } x \sim ft$$

For example, with $x = -3.5$ ft and $V_0 = 50$ mph,

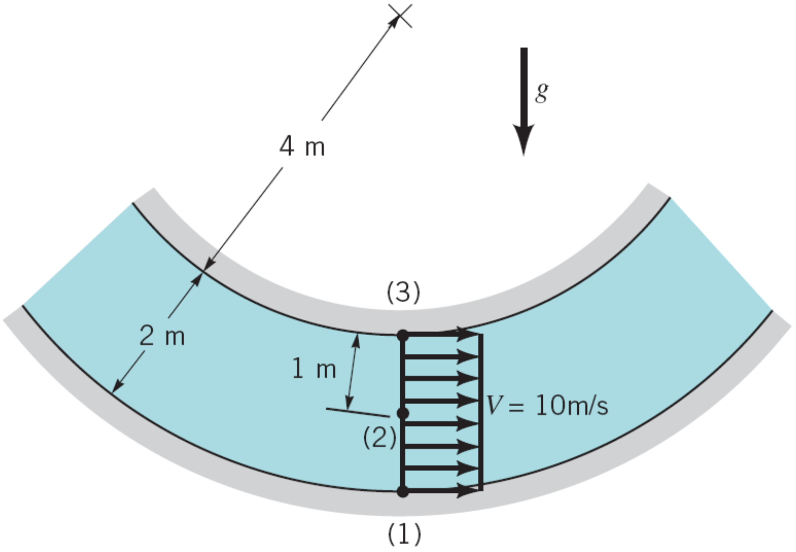
$$p = 0.00256\left[2(2.5/3.5)^2 - (2.5/3.5)^4\right](50)^2 = 4.86\ lb/ft^2$$

The results for various x and V_0 are plotted below.

Pressure Along Dividing Streamline



3.10 Water flows around the vertical two-dimensional bend with circular streamlines and constant velocity as shown in Fig. P3.10. If the pressure is 40 kPa at point (1), determine the pressures at points (2) and (3). Assume that the velocity profile is uniform as indicated.



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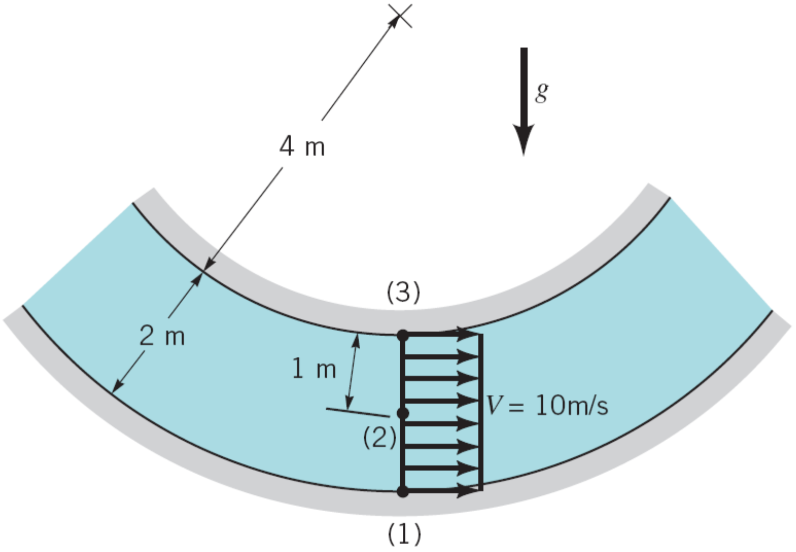
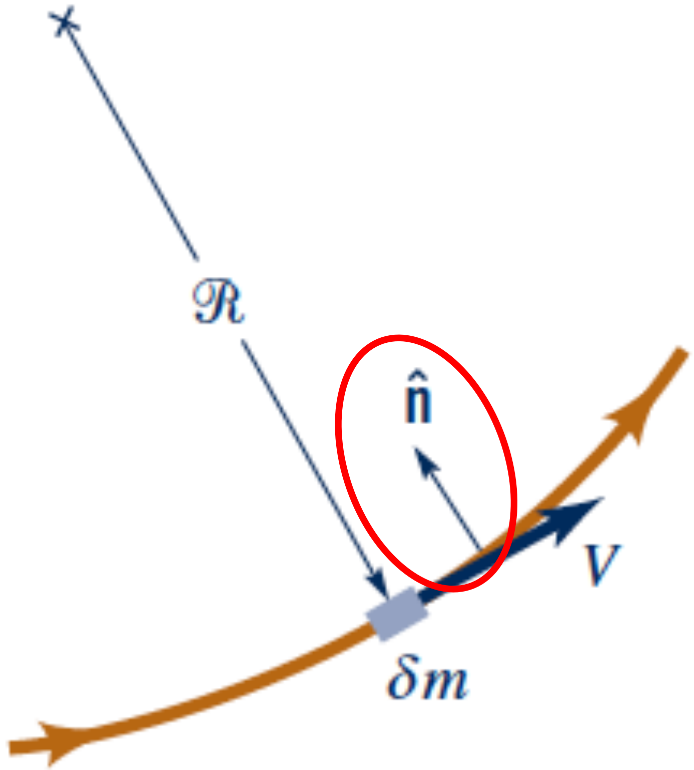




$$-\gamma \frac{dz}{dn} - \frac{\partial p}{\partial n} = \frac{\rho V^2}{\mathcal{R}} \quad \text{with } \frac{dz}{dn} = 1 \text{ and } V = 10\,m/s$$

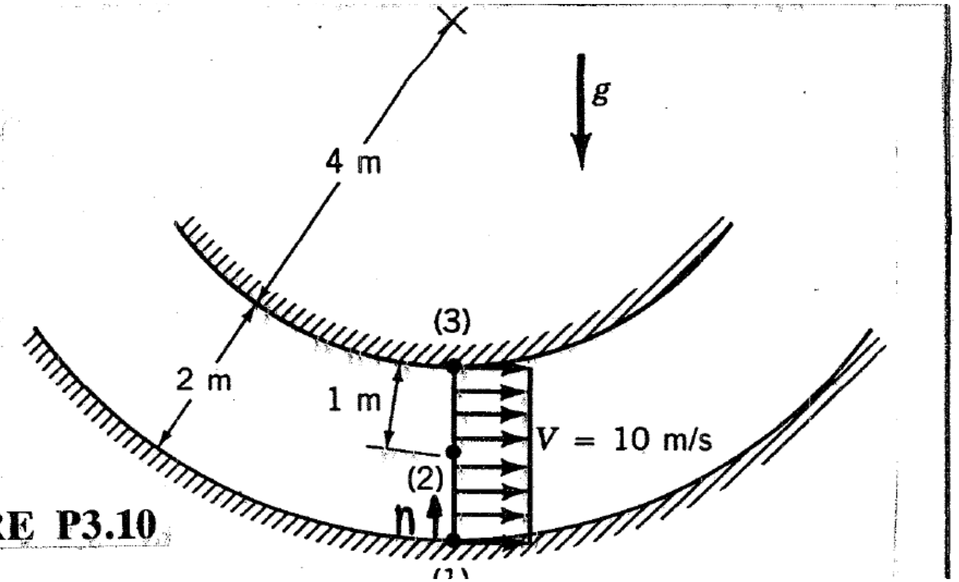
Thus, with $\mathcal{R} = 6 - n$

$$\frac{dp}{dn} = -\gamma - \frac{\rho V^2}{6-n} \quad \text{or}$$

$$\int_{n=0}^{n} \frac{dp}{dn}\,dn = -\int_{n=0}^{n} \gamma\,dn - \int_{n=0}^{n} \frac{\rho V^2\,dn}{6-n}$$

so that since γ and V are constants

$$p - p_1 = -\gamma n - \rho V^2 \int_{n=0}^{n} \frac{dn}{6-n}$$



RE P3.10

Thus,

$$p = p_1 - \gamma n - \rho V^2 \ln\left(\frac{6}{6-n}\right)$$

With $p_1 = 40\,kPa$ and $n_2 = 1\,m$: $p_2 = 40\,kPa - 9.8 \times 10^3 \frac{N}{m^3}(1\,m)$

$$- 999 \frac{kg}{m^3}\left(10\frac{m}{s}\right)^2 \ln\left(\frac{6}{5}\right)$$

or

$$p_2 = \underline{\underline{12.0\,kPa}}$$

and

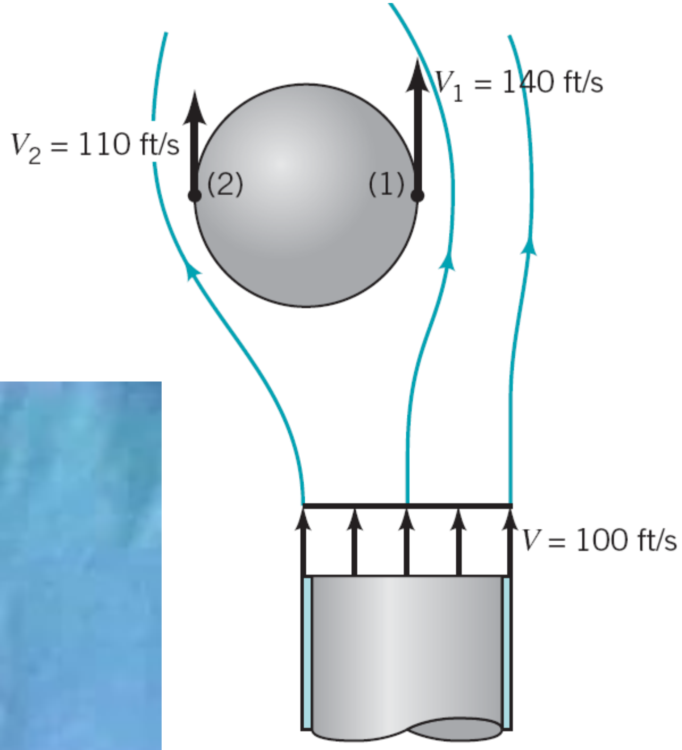
with $p_1 = 40\,kPa$ and $n_3 = 2\,m$: $p_3 = 40\,kPa - 9.80 \times 10^3 \frac{N}{m^3}(2\,m)$

$$- 999 \frac{kg}{m^3}\left(10\frac{m}{s}\right)^2 \ln\left(\frac{6}{4}\right)$$

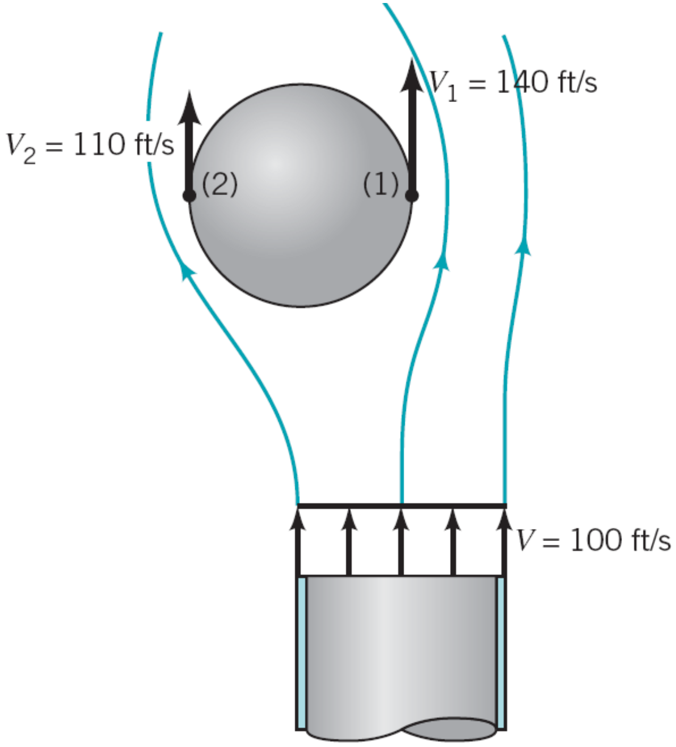
or

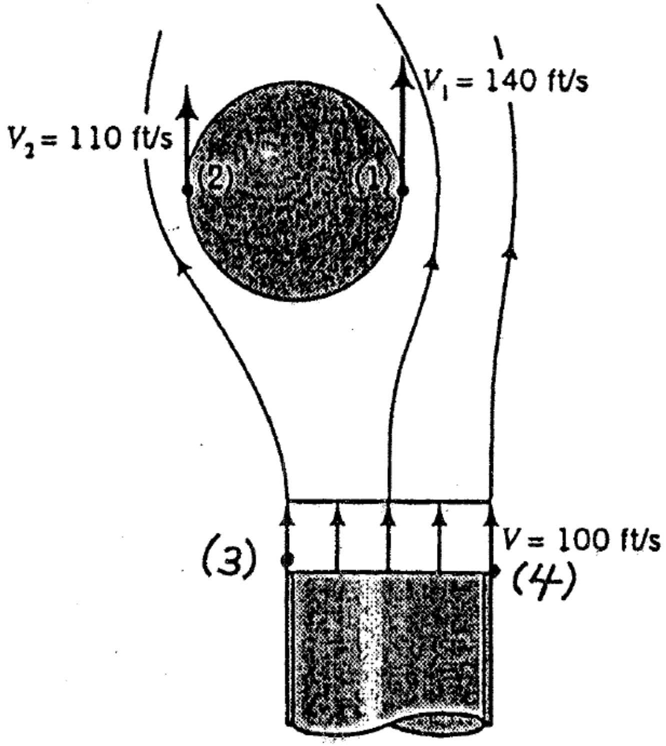
$$p_3 = \underline{\underline{-20.1\,kPa}}$$

3.16 A 100 ft/s jet of air flows past a ball as shown in Video V3.1 and Fig. P3.16. When the ball is not centered in the jet, the air velocity is greater on the side of the ball near the jet center [point (1)] than it is on the other side of the ball [point (2)]. Determine the pressure difference, $p_2 - p_1$, across the ball if $V_1 = 140$ ft/s and $V_2 = 110$ ft/s. Neglect gravity and viscous effects.



3.16 A 100 ft/s jet of air flows past a ball as shown in Video V3.1 and Fig. P3.16. When the ball is not centered in the jet, the air velocity is greater on the side of the ball near the jet center [point (1)] than it is on the other side of the ball [point (2)]. Determine the pressure difference, $p_2 - p_1$, across the ball if $V_1 = 140$ ft/s and $V_2 = 110$ ft/s. Neglect gravity and viscous effects.





The Bernoulli equation from point (3) to (2) and (4) to (1) with gravity neglected gives

$$p_3 + \frac{1}{2}\rho V_3^2 = p_2 + \frac{1}{2}\rho V_2^2 \quad \text{and} \quad p_4 + \frac{1}{2}\rho V_4^2 = p_1 + \frac{1}{2}\rho V_1^2$$

But $p_3 = p_4 = 0$ and $V_3 = V_4$

Thus, even though points (1) and (2) are not on the same streamline,

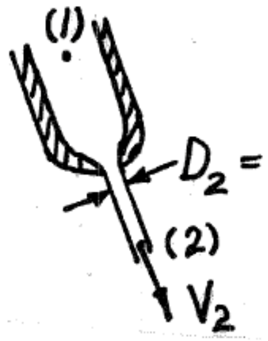
$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

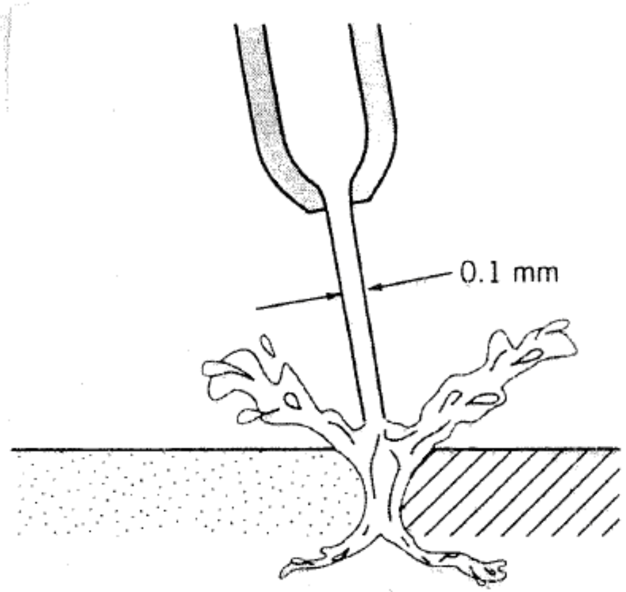
or

$$p_1 - p_2 = \frac{1}{2}\rho\left(V_1^2 - V_2^2\right) = \frac{1}{2}\left(0.00238\,\frac{\text{slugs}}{\text{ft}^3}\right)\left[\left(140\,\frac{\text{ft}}{\text{s}}\right)^2 - \left(110\,\frac{\text{ft}}{\text{s}}\right)^2\right]$$

$$= 8.93\,\frac{\text{slugs}}{\text{ft}\cdot\text{s}^2} = \underline{\underline{8.93\,\frac{\text{lb}}{\text{ft}^2}}}$$

3.26 Small-diameter, high-pressure liquid jets can be used to cut various materials as shown in Fig. P3.26. If viscous effects are negligible, estimate the pressure needed to produce a 0.10-mm-diameter water jet with a speed of 700 m/s. Determine the flowrate.





■ FIGURE P3.26

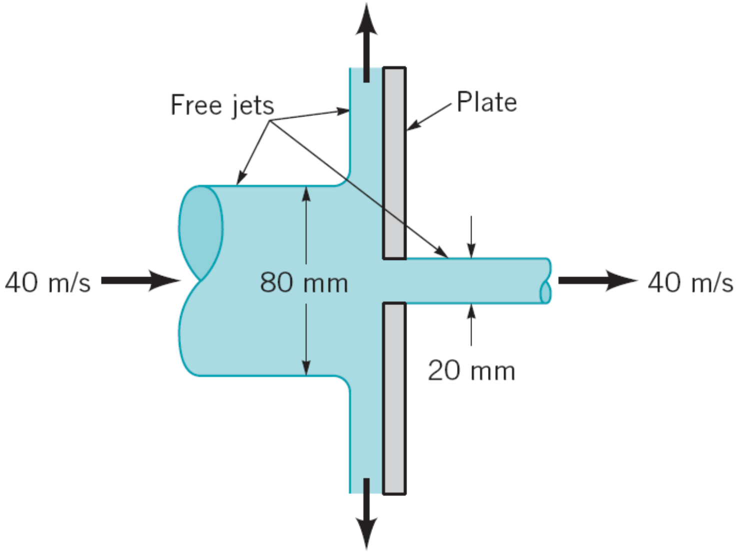
$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \quad \text{where } V_1 \approx 0,\ z_1 \approx z_2, \text{ and } p_2 = 0$$

Thus $$p_1 = \frac{1}{2}\frac{\gamma}{g}V_2^2 = \frac{1}{2}\rho V_2^2 = \frac{1}{2}\left(999\,\frac{kg}{m^3}\right)\left(700\,\frac{m}{s}\right)^2 = \underline{\underline{2.45 \times 10^5\,\frac{kN}{m^2}}}$$

Also,

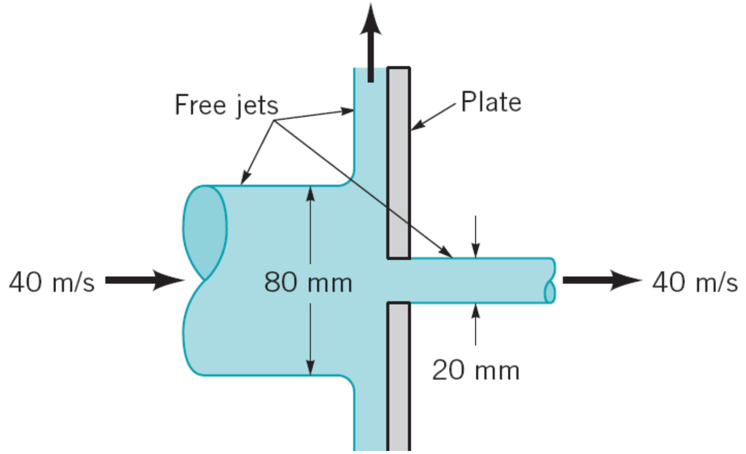
$$Q = V_2 A_2 = 700\,\frac{m}{s}\left[\frac{\pi}{4}(10^{-4}\,m)^2\right] = \underline{\underline{5.50 \times 10^{-6}\,\frac{m^3}{s}}}$$

5.38 A circular plate having a diameter of 300 mm is held perpendicular to an axisymmetric horizontal jet of air having a velocity of 40 m/s and a diameter of 80 mm as shown in Fig. P5.38. A hole at the center of the plate results in a discharge jet of air having a velocity of 40 m/s and a diameter of 20 mm. Determine the horizontal component of force required to hold the plate stationary.



$$\int_{cs} -p$$

5.38 A circular plate having a diameter of 300 mm is held perpendicular to an axisymmetric horizontal jet of air having a velocity of 40 m/s and a diameter of 80 mm as shown in Fig. P5.38. A hole at the center of the plate results in a discharge jet of air having a velocity of 40 m/s and a diameter of 20 mm. Determine the horizontal component of force required to hold the plate stationary.



The control volume contains the plate and flowing air as indicated in the sketch above. Application of the horizontal or x direction component of the linear momentum equation yields

$$-u_1 \rho u_1 A_1 + u_2 \rho u_2 A_2 = -F_{A,x}$$

or

$$F_{A,x} = u_1^2 \rho \frac{\pi D_1^2}{4} - u_2^2 \rho \frac{\pi D_2^2}{4} = u_1^2 \rho \frac{\pi}{4}\left(D_1^2 - D_2^2\right)$$

Thus

$$F_{A,x} = \left(40 \frac{m}{s}\right)^2 \left(1.23 \frac{kg}{m^3}\right) \frac{\pi}{4}\left[\frac{(80\ mm)^2 - (20\ mm)^2}{\left(1000 \frac{mm}{m}\right)^2}\right]\left(\frac{1\ N}{kg \cdot \frac{m}{s^2}}\right)$$

and

$$F_{A,x} = \underline{\underline{9.27\ N}}$$

Lecture 11: Review of Thermodynamics

ENAE311H Aerodynamics I

Christoph Brehm

Thermodynamic systems and state variables

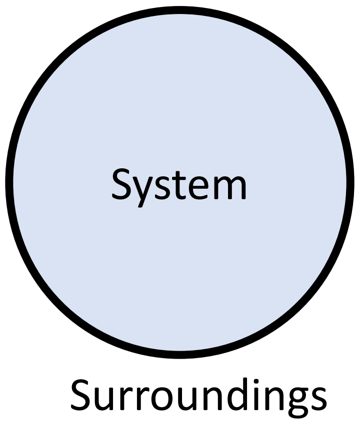
- A thermodynamic *system* is a quantity of matter separated from its surroundings by an enclosure (real or imagined).
- Classical thermodynamics deals with systems in equilibrium, i.e., exhibiting no spatial or temporal gradients (leave a system for long enough with surroundings at constant conditions, and it will reach such a state).
- Variables that depend only on the current state of the system (and not the process by which it got there) are called *variables of state*, e.g., $p, v = 1/\rho\,, T$.
- For a *simple* system, any two state variables are sufficient to fully specify the state of the system (and thus the value of any other state variable), so can write, for example:

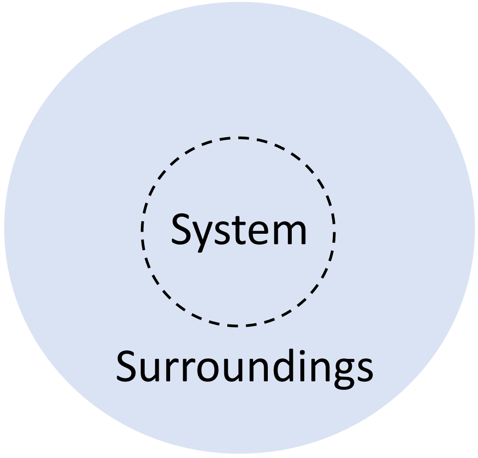
$$p = p(v, T).$$

This is the “thermal equation of state”.

- For a *thermally perfect* gas the thermal equation of state takes the simple form

$$pv = RT, \quad \text{or} \quad p = \rho RT. \qquad \text{(For air, } R\text{=287 J/kg K)}$$





Internal energy and specific heats

- The specific internal energy we denote by e – this in general includes translational, rotational, vibrational and electronic-excitation components.
- As e is a variable of state, we can write

$$e = e(T, v)$$

which for a thermally perfect gas becomes simply

$$e = e(T).$$

- We define the specific heat, $c = \delta q / dT$, i.e., the amount of heat required to raise the temperature by one unit. We will see shortly that the specific heat at constant volume relates differentials of T and e:

$$de = c_v dT.$$

- If the specific heats are constant (reasonable up to ~800 K), the gas is *calorically perfect* and

$$e = c_v T.$$

Enthalpy and specific heats

- We define an additional state variable, the enthalpy h, as

$$h = e + pv.$$

- It is convenient to write

$$h = h(T, p),$$

which for a thermally perfect gas becomes

$$h = h(T).$$

- We then have

$$dh = c_p dT,$$

and for a calorically perfect gas

$$h = c_p T.$$

In this course we will assume a calorically perfect gas, and so we can write

$$h - e = (c_p - c_v)T.$$

and thus

$$c_p - c_v = \frac{pv}{T} = R$$

We now define $\gamma = c_p/c_v$, and dividing above equation by either c_p or c_v, we find

$$\begin{aligned} c_p &= \frac{\gamma R}{\gamma - 1} \\ c_v &= \frac{R}{\gamma - 1}. \end{aligned}$$

For a calorically perfect (or just "perfect") gas, γ is constant (for air, γ=1.4).

The first law of thermodynamics

Let δq be the heat flowing into a system from the surroundings and δw be the work done *by* the system (say, in expanding against a pressure force). Then the first law states

$$de = \delta q - \delta w.$$

Note that neither δq nor δw is a state variable as they depend on the process (not just the end state). We identify three particularly important types of processes:

1. **Adiabatic**: no heat is added to or taken away from the system ($\delta q = 0$)
2. **Reversible**: no dissipative phenomena (e.g., effects of viscosity, heat conduction, diffusion) take place in system
3. **Isentropic**: both adiabatic and reversible

The first law of thermodynamics

For a reversible process, $\delta w = p\, dv$, and the first law becomes

$$de = \delta q - pdv.$$

Since $e = e(T, v)$, however, we can write

$$de = \left.\frac{\partial e}{\partial T}\right|_v dT + \left.\frac{\partial e}{\partial v}\right|_T dv.$$

If the system undergoes a process at constant v, from the first law above

$$\left.\frac{\partial e}{\partial T}\right|_v = \left.\frac{\delta q}{dT}\right|_v = c_v.$$

For a thermally perfect gas, $e = e(T)$, and this becomes $de = c_v dT$, as before.

Similarly, using $dh = de + v\, dp + p\, dv = \delta q + v\, dp$ and $h = h(T, p)$, we recover $dh = c_p dT$.

The second law of thermodynamics

We define another variable of state, the entropy s, such that

$$ds = \left(\frac{\delta q}{T}\right)_{rev},$$

where rev indicates that the heat is being added reversibly.

For an irreversible (i.e., real) process

$$ds = \frac{\delta q}{T} + ds_{irrev},$$

where $ds_{irrev} \geq 0$ is the internal entropy production due to dissipative phenomena.

Thus, we have the second law of thermodynamics, which may be expressed in either of two ways:

$$\begin{aligned} ds &\geq \frac{\delta q}{T} \quad \text{(general process)} \\ ds &\geq 0 \quad \text{(adiabatic process).} \end{aligned}$$

Lecture 12: Isentropic Processes and the Speed of Sound

ENAE311H Aerodynamics I

Christoph Brehm

Entropy change for a perfect gas

To calculate the entropy change (in terms of other variables) for a perfect gas, assume we are undergoing a reversible process. In this case, the first law can be written:

$$de = Tds - pdv, \quad \Longrightarrow \quad Tds = de + pdv.$$

Dividing through by T, and using the ideal gas law and $de = c_v dT$, we can write this as

$$ds = c_v \frac{dT}{T} + R\frac{dv}{v}.$$

Since R and (for a perfect gas) c_v are constant, we can integrate from state 1 to state 2:

$$\int_{s_1}^{s_2} ds = c_v \int_{T_1}^{T_2} \frac{dT}{T} + R \int_{v_1}^{v_2} \frac{dv}{v},$$

to immediately obtain

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}. \quad \Longrightarrow \quad s = s(T, v)$$

Alternatively, starting from $dh = Tds + vdp$,

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}. \quad \Longrightarrow \quad s = s(T, p)$$

All variables of state, so true for a general process!

Isentropic processes

In gas dynamics, isentropic processes (both adiabatic and reversible) are particularly important. For such processes, $s_2 - s_1 = 0$, and from our equation for $s = s(T, v)$ on the last slide we have

$$\begin{aligned} R \ln \frac{v_2}{v_1} &= -c_v \ln \frac{T_2}{T_1} \\ \Longrightarrow \frac{v_2}{v_1} &= \left(\frac{T_2}{T_1}\right)^{-c_v/R}, \end{aligned}$$

or equivalently

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{c_v/R}.$$

Identifying $c_v/R = 1/(\gamma - 1)$, we thus have

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}.$$

Similarly, setting $s_2 - s_1 = 0$, and from our equation for $s = s(T, p)$ and using $c_p/R = \gamma/(\gamma - 1)$, we obtain

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}.$$

So, for isentropic flow

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}.$$

General compressible flow theory

- Earlier we derived the three conservation laws for fluid flow. These give us five equations in six variables: ρ, u, v, w, p and e (or h).

- To close the set, we require additional constitutive relations relating the thermodynamic variables
  - For a perfect gas, these are the ideal gas equation (thermal equation of state), $p = \rho R T$, and the caloric equation of state $e = c_v T$ or $h = c_p T$.

- With this complete set, it is possible to solve for a number of canonical problems in compressible flow.

The speed of sound

The propagation of sound in a gas is brought about by collisions between molecules as they undergo their random thermal motion. Since the speed of the molecules increases with T, we might expect the sound speed to do so, too.

Consider a quiescent gas at conditions $\rho = \rho_0, p = p_0,\ u = 0$, and suppose a sound wave (planar, propagating in the x direction) of infinitesimal strength passes through it, causing small fluctuations in these properties of $\rho'(x,t)$, $p'(x,t)$, and $u'(x,t)$. Note the following:

1. No heat is added or taken away (adiabatic)
2. Infinitesimal wave strength means the process is reversible

isentropic! ($s' = 0$)

From the differential form of conservation of mass, we can write

$$\frac{\partial}{\partial t}(\rho_0 + \rho') + \frac{\partial}{\partial x}[(\rho_0 + \rho')u'] = 0.$$

Expanding and discarding the second-order fluctuation term (i.e., $\frac{\partial}{\partial x}(\rho' u')$), we obtain

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0.$$

The speed of sound

We can similarly write the differential form of the one-dimensional momentum equation as

$$(\rho_0 + \rho')\left(\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x}\right) = -\frac{\partial}{\partial x}(p_0 + p').$$

Again dropping second-order terms, we obtain simply

$$\rho_0 \frac{\partial u'}{\partial t} = \frac{\partial p'}{\partial x}.$$

Differentiating w.r.t. x, we have

$$\rho_0 \frac{\partial^2 u'}{\partial x \partial t} = -\frac{\partial^2 p'}{\partial x^2}.$$

Note, however, that if we differentiate our linearized continuity equation w.r.t. t, we obtain

$$\frac{\partial^2 \rho'}{\partial t^2} + \rho_0 \frac{\partial^2 u'}{\partial t \partial x} = 0$$

Then from equality of mixed derivatives, we can write

$$\frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0.$$

The speed of sound

Now, since p, ρ, and s are variables of state, we can write $p = p(\rho, s)$, and thus

$$p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s'$$

As we have noted already though, $s' = 0$, and so

$$p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho'$$

We write $c = \sqrt{(\partial p/\partial \rho)_s}$ and note that the derivative can be evaluated at reference conditions (p_0, ρ_0) and thus be treated as constant. Our differential equation then becomes:

$$\frac{\partial^2 p'}{\partial t^2} - c^2 \frac{\partial^2 p'}{\partial x^2} = 0.$$

This is the one-dimensional wave equation, which has the general solution

$$p'(x, t) = f(x - ct) + g(x + ct)$$

i.e., a travelling wave with speed c travelling in either the $+x$ or $-x$ direction.

The speed of sound

The speed of sound, i.e., the speed of propagation of sound waves, is typically denoted by a. We then see that

$$\boxed{a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}.}$$

To see what form this takes for a perfect gas, recall that for an isentropic process involving a perfect gas we have

$$p = k\rho^\gamma,$$

where k is constant $(= p_0/\rho_0^\gamma)$.

Therefore

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma k \rho^{\gamma-1} = \gamma \frac{p}{\rho},$$

and thus

$$a = \sqrt{\gamma \frac{p}{\rho}}.$$

From the ideal gas law, we then have

$$\boxed{a = \sqrt{\gamma R T}}$$ (= 341 m/s for air at room temperature)

The Mach number

Earlier we defined the Mach number as $M = V/a$. We now see that, for a perfect gas,

$$M^2 = \frac{V^2}{a^2} = \frac{V^2/2}{\gamma RT/2} = \frac{V^2/2}{\gamma(\gamma-1)c_vT/2}$$

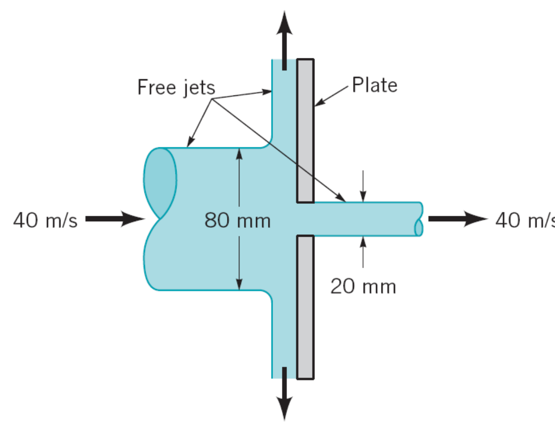
$$= \frac{2}{\gamma(\gamma-1)} \frac{\boxed{V^2/2}}{\boxed{c_vT}}$$

specific kinetic energy

specific internal energy

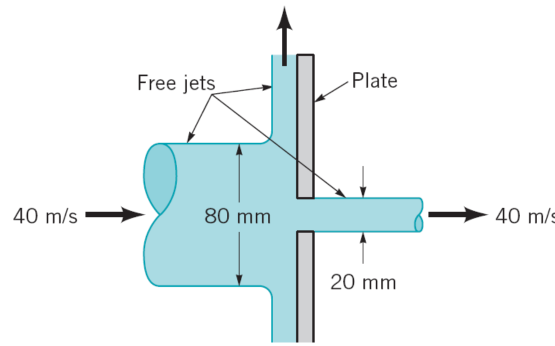
So the square of the Mach number is a measure of the ratio of the directed kinetic energy to the thermal internal energy of the gas.

5.38 A circular plate having a diameter of 300 mm is held perpendicular to an axisymmetric horizontal jet of air having a velocity of 40 m/s and a diameter of 80 mm as shown in Fig. P5.38. A hole at the center of the plate results in a discharge jet of air having a velocity of 40 m/s and a diameter of 20 mm. Determine the horizontal component of force required to hold the plate stationary.



$$\int_{cs} -p$$

5.38 A circular plate having a diameter of 300 mm is held perpendicular to an axisymmetric horizontal jet of air having a velocity of 40 m/s and a diameter of 80 mm as shown in Fig. P5.38. A hole at the center of the plate results in a discharge jet of air having a velocity of 40 m/s and a diameter of 20 mm. Determine the horizontal component of force required to hold the plate stationary.



The control volume contains the plate and flowing air as indicated in the sketch above. Application of the horizontal or x direction component of the linear momentum equation yields

$$-u_1 \rho u_1 A_1 + u_2 \rho u_2 A_2 = -F_{A,x}$$

or

$$F_{A,x} = u_1^2 \rho \frac{\pi D_1^2}{4} - u_2^2 \rho \frac{\pi D_2^2}{4} = u_1^2 \rho \frac{\pi}{4}\left(D_1^2 - D_2^2\right)$$

Thus

$$F_{A,x} = \left(40\,\frac{m}{s}\right)^2 \left(1.23\,\frac{kg}{m^3}\right) \frac{\pi}{4} \left[\frac{(80\text{ mm})^2 - (20\text{ mm})^2}{\left(1000\,\frac{mm}{m}\right)^2}\right] \left(\frac{1\text{ N}}{kg\cdot\frac{m}{s^2}}\right)$$

and

$$F_{A,x} = \underline{\underline{9.27\text{ N}}}$$

Lecture 103: The Energy Equation in Compressible Flow

ENAE311H Aerodynamics I

Christoph Brehm

The adiabatic energy equation

Recall our conservation-of-energy equation for flow through a simple CV:

$$\dot{m}\left[h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.$$

Away from solid boundaries, there is typically no heat transfer (or shaft work), and so we can assume that the flow is adiabatic ($\dot{Q} = 0$). For air flows, we can also typically neglect the influence of gravity. With these assumptions, this equation becomes

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2.$$

If we consider condition 2 to be a stagnation point, so that $u_2 = 0$, and condition 1 to be arbitrary, we have

$$h + \frac{1}{2}u^2 = h_0 = \text{const.},$$

where h_0 is the *stagnation enthalpy*.

The stagnation temperature

Now, assuming a perfect gas, so that $h = c_p T$, this can be written as

$$c_p T + \frac{1}{2}u^2 = c_p T_0, \qquad \text{or} \qquad \frac{T_0}{T} = 1 + \frac{u^2}{2c_p T}.$$

T_0 is the *stagnation* or *total temperature*, i.e., the temperature the flow reaches if brought to rest adiabatically.

Now we use $T = a^2/(\gamma R)$ to write the above equation as

$$\frac{c_p}{\gamma R}a^2 + \frac{1}{2}u^2 = \frac{c_p}{\gamma R}a_0^2.$$

Since $c_p/(\gamma R) = 1/(\gamma - 1)$, this becomes

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma - 1}.$$

Dividing through by $a^2/(\gamma - 1)$, we then have

$$1 + \frac{\gamma - 1}{2}\frac{u^2}{a^2} = \frac{a_0^2}{a^2}, \qquad \text{or} \qquad \boxed{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2.}$$

We thus see that, for a perfect gas, T_0/T, depends only on M and γ.

The stagnation pressure and density

Let us now assume that the flow is undergoing an isentropic process. Then, from last lecture

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)}$$

From our previous equation for T_0/T, we can thus write

$$\begin{aligned} \frac{p_0}{p} &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}, \\ \frac{\rho_0}{\rho} &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}. \end{aligned}$$

p_0 and ρ_0 are the *stagnation* (or *total*) *pressure* and *density*, i.e., the values the flow achieves if brought to rest <u>isentropically</u>.

Earlier we stated that a flow can be considered incompressible (constant density) at low speeds. To see at which Mach number this no longer becomes reasonable, let us assume we have an isentropic process and are willing to accept a 5% variation in density. Then the maximum Mach number allowable is given by

$$M = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{1}{0.95}\right)^{\gamma-1} - 1\right]} = 0.32.$$

Variables in adiabatic and isentropic flow

Assume we have an adiabatic flow and the Mach numbers at two points (1 and 2) are known. Then, since T_0 is constant in the flow, we can write

$$\begin{aligned}\frac{T_2}{T_1} &= \frac{T_2}{T_0}\frac{T_0}{T_1} \\ &= \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\end{aligned}$$

Thus, given T_1, we can immediately determine T_2 .

Similarly, if the flow is isentropic (p_0 constant), we can relate the pressures at points 1 and 2:

$$\begin{aligned}\frac{p_2}{p_1} &= \frac{p_2}{p_0}\frac{p_0}{p_1} \\ &= \left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{\gamma/(\gamma-1)},\end{aligned}$$

and similarly for the density.

Sonic-referenced conditions

Thus far we have used stagnation ($M = 0$) conditions to reference our flow variables, but this is by no means a unique choice. An alternative that is useful in some situations is to use the conditions the flow variables would have if accelerated/decelerated adiabatically or isentropically to sonic ($M = 1$) conditions, which we denote by *. Compared to stagnation conditions, we have:

$$
\begin{aligned}
\text{Adiabatic:} \quad \frac{T^*}{T_0} &= \frac{2}{\gamma+1} \quad (=0.833 \text{ for } \gamma=1.4) \\
\text{Isentropic:} \quad \frac{p^*}{p_0} &= \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \quad (=0.528 \text{ for } \gamma=1.4) \\
\frac{\rho^*}{\rho_0} &= \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \quad (=0.634 \text{ for } \gamma=1.4)
\end{aligned}
$$

The characteristic Mach number

We can also define $a^* = \sqrt{\gamma R T^*}$ and the *characteristic Mach number*, M^*, as

$$M^* = \frac{u}{a^*}.$$

Now, we can write

$$\frac{a^{*2}}{a_0^2} = \frac{T^*}{T_0} = \frac{2}{\gamma + 1}.$$

Substituting into our energy equation from earlier, we obtain

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2}.$$

Dividing through by u^2 and rearranging, we arrive at a relation between M^* and M:

$$M^{*2} = \frac{(\gamma + 1) M^2}{2 + (\gamma - 1) M^2}.$$

Note that, as $M \to \infty$, $M^* \to \sqrt{(\gamma + 1)/(\gamma - 1)}$.

An airplane is flying at a standard altitude of 10,000 ft. A Pitot tube mounted at the nose measures a pressure of 2220 lb/ft^2. The airplane is flying at a high subsonic speed, faster than 300 mph. From our comments in Section 3.1, the flow should be considered *compressible*. Calculate the velocity of the airplane.

Solution

From our discussion in Section 3.4, the pressure measured by a Pitot tube immersed in an incompressible flow is the total pressure. For the same *physical reasons* discussed in Section 3.4, a Pitot tube also measures the total pressure in a high-speed subsonic compressible flow. (This is further discussed in Section 8.7.1 on the measurement of velocity in a subsonic compressible flow.) *Caution*: Because we are dealing with a compressible flow in this example, we *cannot* use Bernoulli's equation to calculate the velocity.

The flow in front of the Pitot tube is compressed isentropically to zero velocity at the mouth of the tube, hence the pressure at the mouth is the total pressure, p_0. From Equation (7.32), we can write:

$$\frac{p_0}{p_\infty} = \left(\frac{T_0}{T_\infty}\right)^{\gamma/(\gamma-1)} \tag{E.7.3}$$

where p_0 and T_0 are the total pressure and temperature, respectively, at the mouth of the Pitot tube, and p_∞ and T_∞ are the freestream static pressure and static temperature, respectively. Solving Equation (E7.3) above for T_0, we get

$$T_0 = T_\infty \left(\frac{p_0}{p_\infty}\right)^{(\gamma-1)/\gamma} \tag{E.7.4}$$

From Appendix E, the pressure and temperature at a standard altitude of 10,000 ft are 1455.6 lb/ft^2 and 483.04 °R, respectively. These are the values of p_∞ and T_∞ in Equation (E7.4). Thus, from Equation (E7.4),

$$T_0 = (483.04)\left(\frac{2220}{1455.6}\right)^{0.4/1.4} = 544.9\ ^\circ\mathrm{R}$$

From the energy equation, Equation (7.54), written in terms of temperature, we have

$$c_p\, T + \frac{V^2}{2} = c_p\, T_0 \tag{E.7.5}$$

In Equation (E7.5), both T and V are the freestream values, hence we have

$$c_p\, T_\infty + \frac{V_\infty^2}{2} = c_p\, T_0 \tag{E.7.6}$$

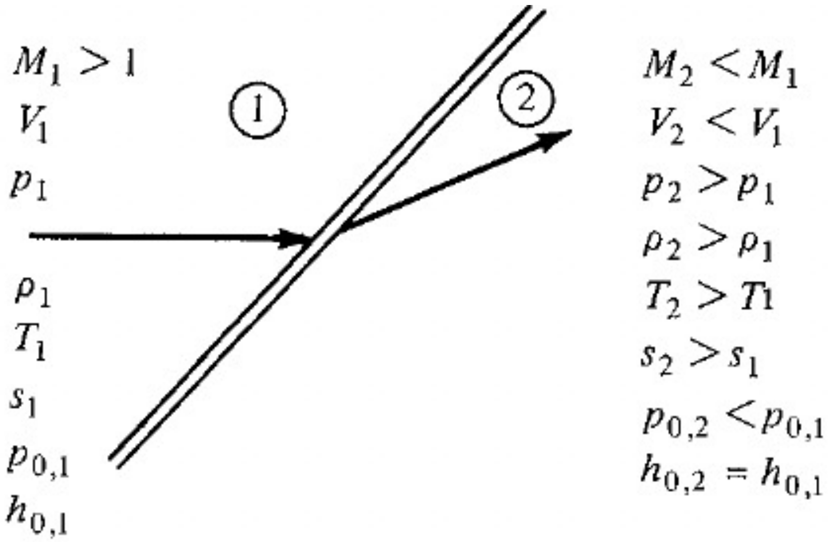
Also,

$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.4)(1716)}{0.4} = 6006\ \frac{\mathrm{ft \cdot lb}}{\mathrm{slug \cdot {}^\circ R}}$$

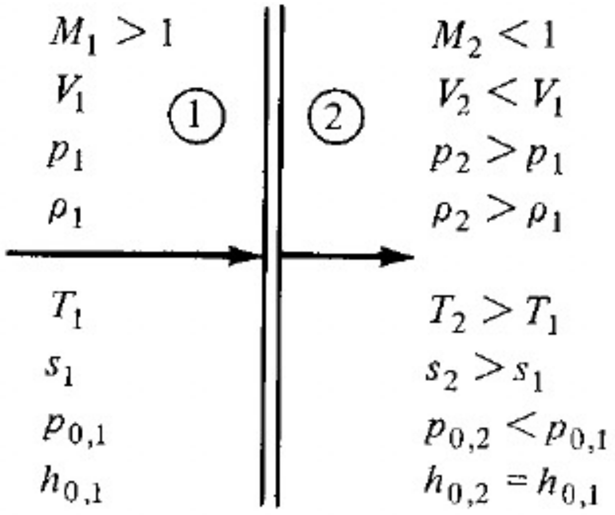
Solving Equation (E7.6) for V_∞, we have

$$\begin{aligned} V_\infty &= [2\, c_p\, (T_0 - T_\infty)]^{1/2} \\ &= [2\,(6006)(544.9 - 483.04]^{1/2} \\ &= \boxed{862\ \mathrm{ft/s}} \end{aligned}$$

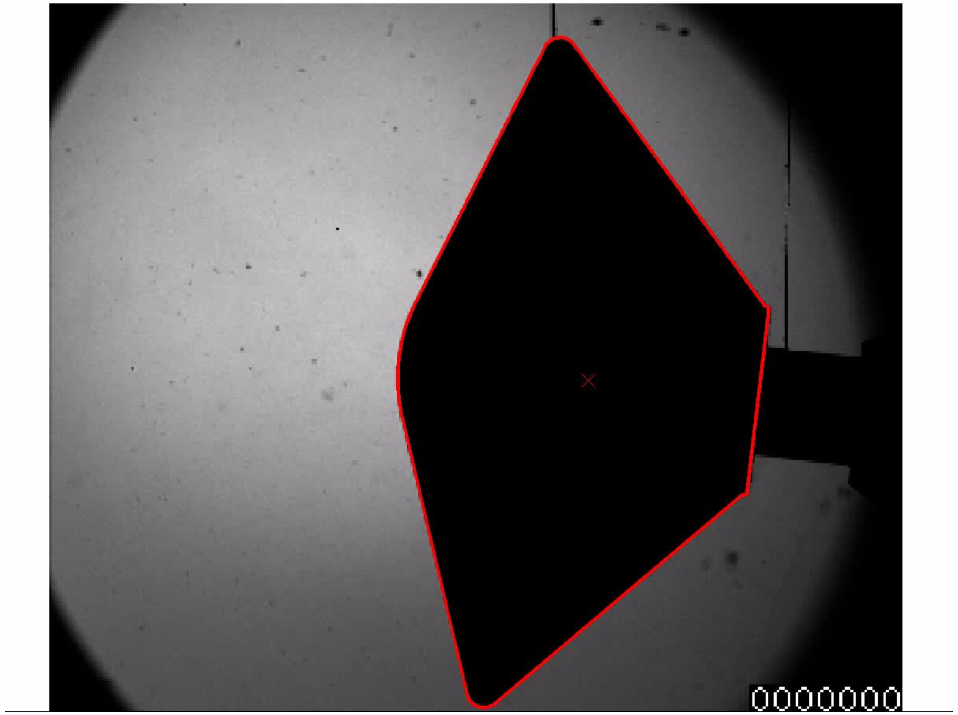
Note: From this example, we see that the total pressure measured by a Pitot tube in a subsonic compressible flow is a measure of the flow velocity, but we need also the value of the flow static temperature in order to calculate the velocity. In Section 8.7, we show more fundamentally that the *ratio* of Pitot pressure to flow static pressure in a compressible flow, subsonic or supersonic, is a *direct measure* of the *Mach number*, not the velocity. But more on this later.

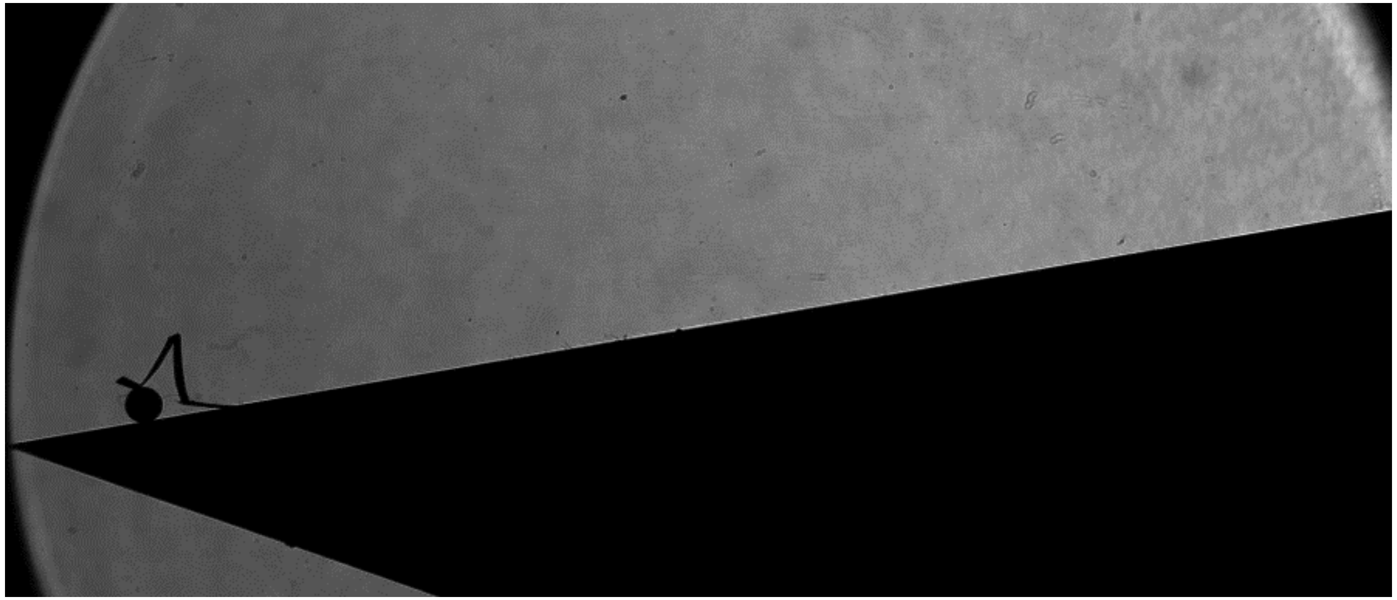


(a) Oblique shock wave



(b) Normal shock wave





Lecture 14: Normal Shock Waves

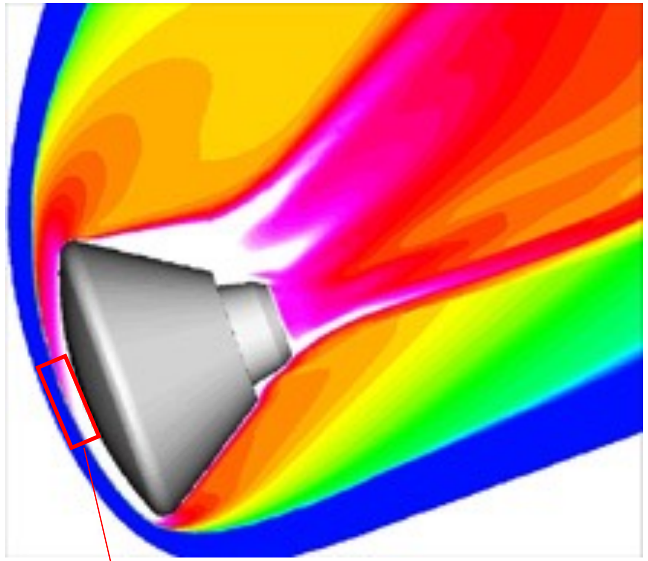
ENAE311H Aerodynamics I

Christoph Brehm

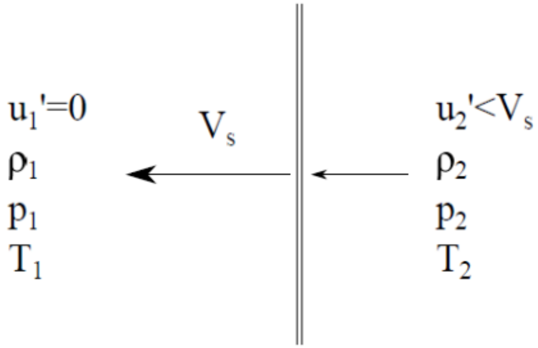
Preliminaries

Imagine we have a normal shock propagating through the atmosphere at speed V_s:

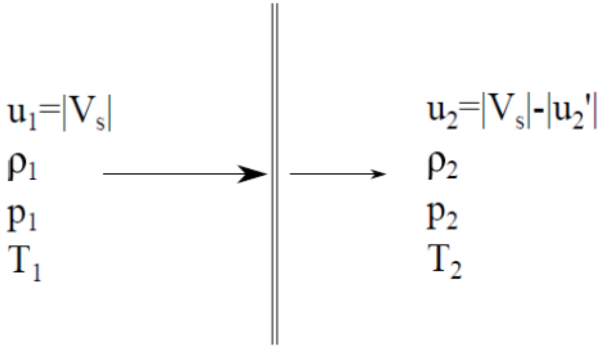
- The shock will see ambient conditions ahead of it, which will change to give the post-shock state
- These changes occur on very small length scales (a few mean free paths) so can treat the shock as a mathematical discontinuity



Now imagine we shift our reference frame from one fixed with the atmosphere to one fixed to the shock. We wish to determine the conditions downstream of the shock, given the upstream conditions and the shock Mach number, $M_1 = V_s/a_1$.



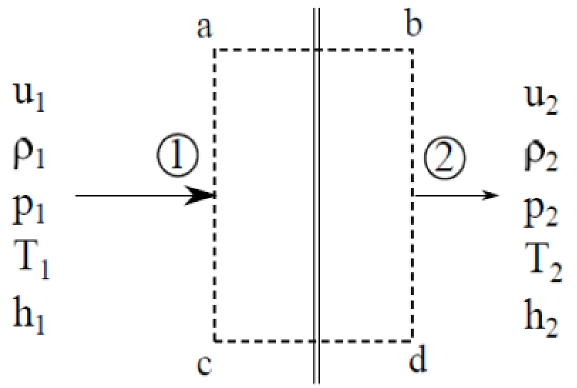
Atmosphere-fixed



Shock-fixed

Conservation laws for normal shocks

Consider a control volume around the shock, as shown to the right. We note or assume the following:



- The surfaces ab and cd are streamlines (no flow crosses them)
- The flow is steady ($\partial/\partial t = 0$)
- There are no viscous effects on surfaces ab and cd (shock infinitely thin)
- Body forces are negligible
- There is no heat addition, so flow is adiabatic

Under these conditions, we can derive particularly simple versions of the conservation equations.

Continuity:

$$\iint_{CS} \rho \mathbf{v} \cdot \mathbf{dA} = 0$$

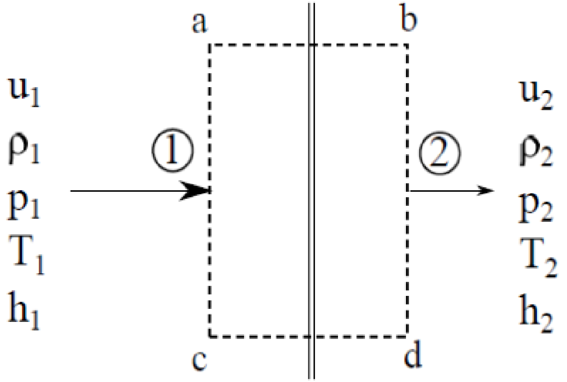
This becomes

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

or, since $A_1 = A_2$

$$\boxed{\rho_1 u_1 = \rho_2 u_2.}$$

Conservation laws for normal shocks



Momentum:

$$\iint_{CS} \mathbf{v}(\rho \mathbf{v} \cdot \mathbf{dA}) = -\iint_{CS} p\mathbf{dA}$$

This becomes

$$-\rho_1 u_1^2 A_1 + \rho_2 u_2^2 A_2 = p_1 A_1 - p_2 A_2,$$

or alternatively

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + +\rho_2 u_2^2.}$$

Energy:

Consider our simplified energy equation from earlier:

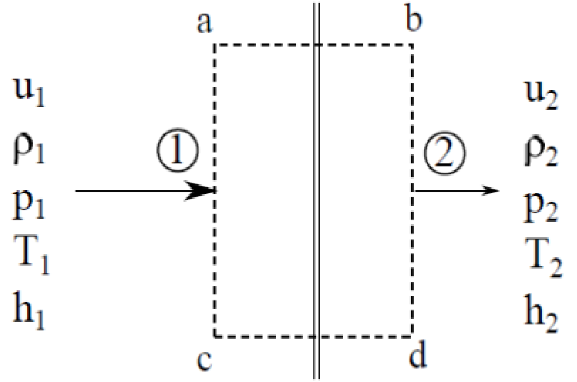
$$\dot{m}\left[h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.$$

In this case, it becomes again

$$\boxed{h_1 + \frac{1}{2}u_1^2 = h_2 + +\frac{1}{2}u_2^2.}$$

Conservation laws for normal shocks

Our conservation laws are thus:



$$\rho_1 u_1 = \rho_2 u_2.$$

$$p_1 + \rho_1 u_1^2 = p_2 + +\rho_2 u_2^2.$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + +\frac{1}{2}u_2^2.$$

To close this set, we additionally require the thermal and caloric equations of state:

$$\begin{aligned} p_2 &= \rho_2 R T_2, \\ h_2 &= c_p T_2. \end{aligned}$$

The Prandtl relation

Dividing the momentum equation by the continuity equation, we have

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1.$$

Noting that $\gamma\, p/\rho = a^2$, this can be expressed as

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1.$$

Since flow is adiabatic, a^* is constant, and we can write a_1 and a_2 in terms of a^* & u_1 and a^* & u_2, respectively (see equation 7.39), to obtain

$$\frac{\gamma+1}{2\gamma u_1 u_2}(u_2 - u_1)a^{*2} + \frac{\gamma-1}{2\gamma}(u_2 - u_1) = u_2 - u_1.$$

Dividing through by $u_2 - u_1$ and simplifying, we arrive at

$$a^{*2} = u_1 u_2.$$

This simple expression is known as the *Prandtl relation.*

The post-shock Mach number

Note that we can rewrite the Prandtl relation

$$a^{*2} = u_1 u_2.$$

as

$$M_2^* = \frac{1}{M_1^*}.$$

Squaring and using our expression relating the characteristic and regular Mach numbers from the previous lecture, we have

$$\frac{(\gamma+1)M_2^2}{2+(\gamma-1)M_2^2} = \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2},$$

which can be rearranged to yield

$$\boxed{M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)}.}$$

This gives the post-shock Mach number (in the shock-fixed frame) in terms of the pre-shock Mach number (in the lab frame, the post-shock Mach number is V_s/a_2 minus this value).

For $M_1 = 1$, $M_2 = 1$; for $M_1 > 1$, $M_2 < 1$. As $M_1 \to \infty$, $M_2 \to \sqrt{(\gamma-1)/2\gamma} = 0.378$ for $\gamma = 1.4$.

Post-shock thermodynamic variables

We now wish to derive ratios of post-shock and pre-shock quantities. For the density, using the continuity equation, we can write

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^{*2}} \\ &= M_1^{*2},\end{aligned}$$

where we have used the Prandtl relation. This can be written as

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}.}$$

Post-shock thermodynamic variables

For the pressure, rearranging the momentum conservation equation, we have

$$\begin{aligned} p_2 - p_1 &= \rho_1 u_1^2 - \rho_2 u_2^2 \\ &= \rho_1 u_1 (u_1 - u_2) \\ &= \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right) \end{aligned}$$

Dividing through by p_1 and using $\gamma\, p/\rho = a^2$, we can write this as

$$\frac{p_2 - p_1}{p_1} = \gamma \frac{u_1^2}{a_1^2}\left(1 - \frac{u_2}{u_1}\right) = \gamma M_1^2 \left(1 - \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2}\right)$$

which can be simplified to

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1).}$$

Meanwhile, for the temperature

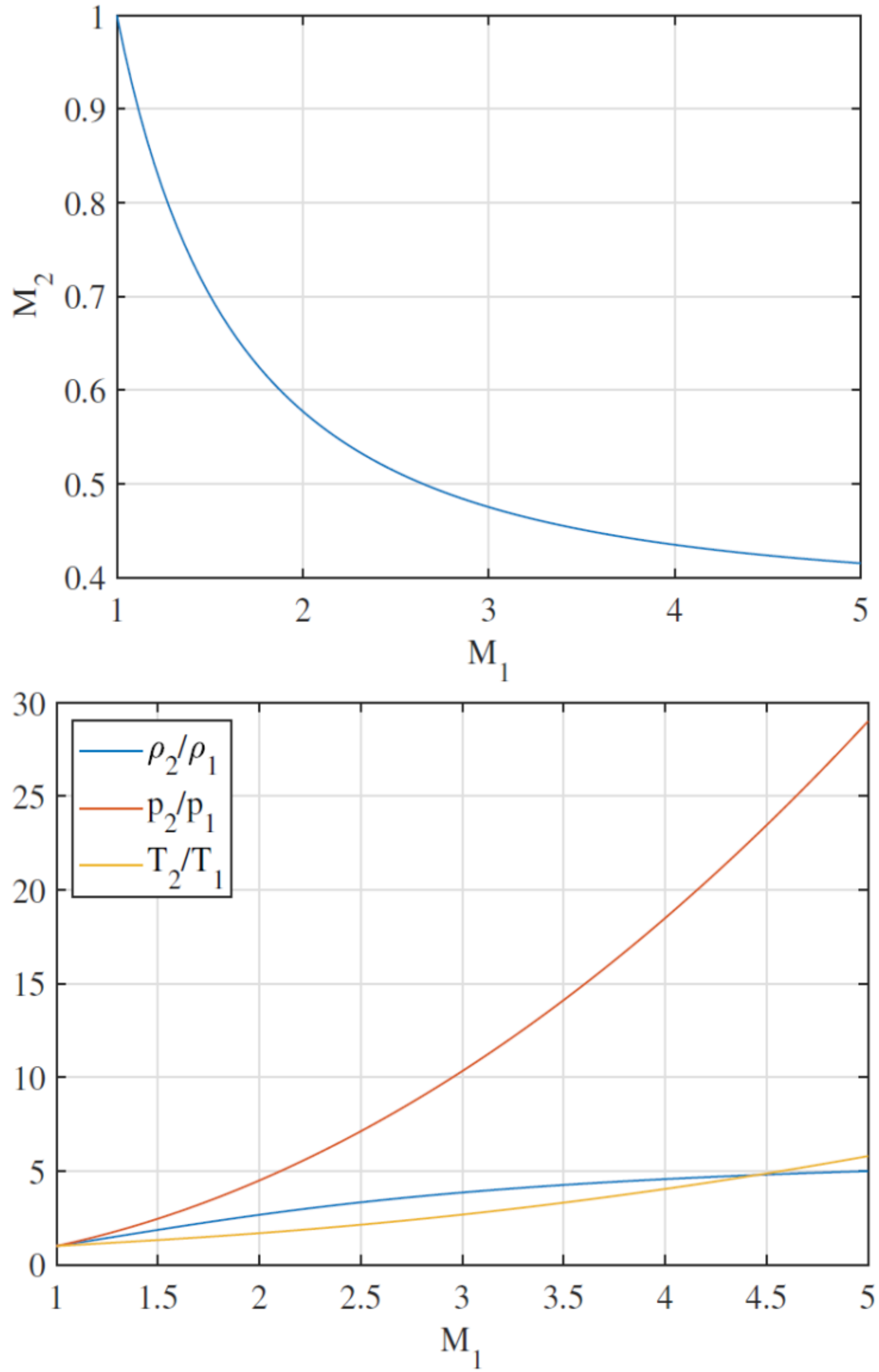
$$\boxed{\begin{aligned} \frac{T_2}{T_1} &= \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \\ &= \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right] \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2}. \end{aligned}}$$

(also holds for h_2/h_1 and a_2^2/a_1^2)

Limiting values

- For $M_1 = 1,\ \rho_2/\rho_1 = p_2/p_1 = T_2/T_1 = 1$.
- For $M_1 > 1$, the ratios are all above 1.
- As $M_1 \to \infty$, we have the following behavior:

$$
\begin{aligned}
\frac{\rho_2}{\rho_1} &\to \frac{\gamma+1}{\gamma-1} = 6 \quad \text{for } \gamma{=}1.4 \\
\frac{p_2}{p_1} &\to \frac{2\gamma}{\gamma+1} M_1^2 \to \infty \\
\frac{T_2}{T_1} &\to \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2 \to \infty.
\end{aligned}
$$



Change in entropy across shock

We can apply our equation for entropy change from earlier to obtain:

$$\begin{aligned}\frac{s_2 - s_1}{R} &= \frac{c_p}{R}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1} \\ &= \frac{\gamma}{\gamma-1}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1} \\ &= \ln\left[\left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}\frac{p_2}{p_1}^{-1}\right].\end{aligned}$$

Using our equations for T_2/T_1 and p_2/p_1, this can be written

$$\frac{s_2 - s_1}{R} = \ln\left[\left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right)^{1/(\gamma-1)}\left(\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right)^{\gamma/(\gamma-1)}\right].$$

For M close to unity, this can be approximated as

$$\frac{s_2 - s_1}{R} \approx \frac{2\gamma}{3(\gamma+1)^2}(M_1^2 - 1)^3.$$

Now, the alternative form of the Prandtl relation tells us that

$$M_2^* = \frac{1}{M_1^*}.$$

Note also that if $M^* < 1$, then $M < 1$, and similarly if $M^* > 1, M > 1$. Thus we must have either supersonic flow upstream and subsonic flow downstream, or subsonic flow upstream and supersonic flow downstream.

However, from the entropy relation to the left, we see that the entropy change would be negative in the latter case.

Therefore, shocks are only possible with $M_1 > 1$ and $M_2 < 1$.

Lecture 15: Pitot Probes in Compressible Flows and Quasi-One-Dimensional Flows

ENAE311H Aerodynamics I

Christoph Brehm

Stagnation quantities through a normal shock

Our energy equation across the normal shock is

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2.$$

We thus see immediately that

$$h_{0,1} = h_{0,2},$$

and so

$$T_{0,1} = T_{0,2}.$$

This is as we would expect for an adiabatic flow.

Stagnation quantities through a normal shock

For the stagnation pressure, imagine we were to bring an element of fluid on either side of the shock to rest isentropically. Then the entropy in each case won't change, and we will have

$$s_2 - s_1 = s_{0,2} - s_{0,1}$$

Using our equation for entropy change, we then have

$$\begin{aligned} s_2 - s_1 &= c_p \ln \frac{T_{0,2}}{T_{0,1}} - R \ln \frac{p_{0,2}}{p_{0,1}} \\ &= -R \ln \frac{p_{0,2}}{p_{0,1}}, \end{aligned}$$

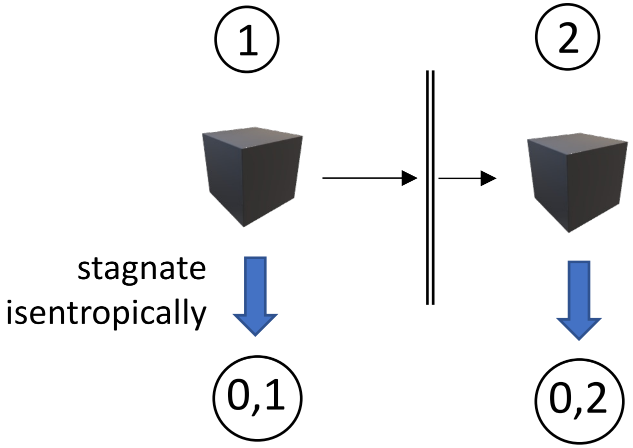
since T_0 is constant across the shock.

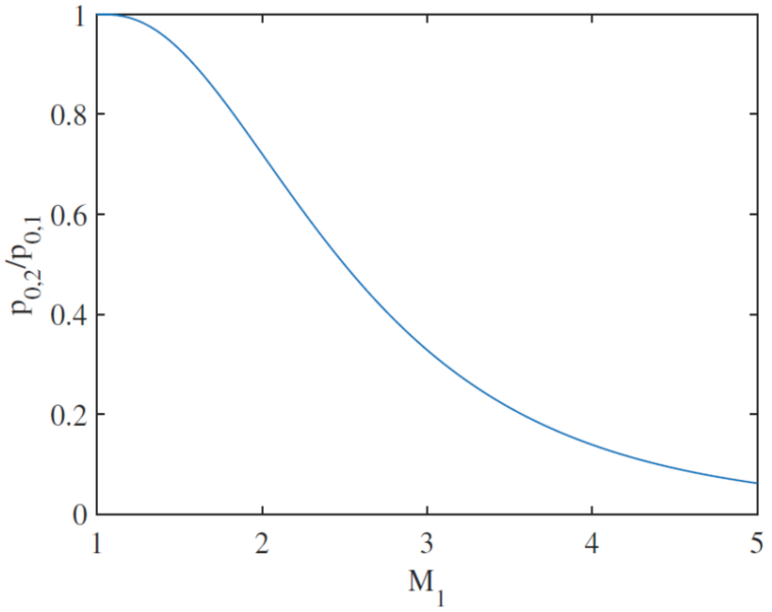
But remember:

$$\frac{s_2 - s_1}{R} = \ln\left[\left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right)^{1/(\gamma-1)} \left(\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right)^{\gamma/(\gamma-1)}\right].$$

Comparing, we see

$$\frac{p_{0,2}}{p_{0,1}} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right]^{-1/(\gamma-1)} \left[\frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2}\right]^{\gamma/(\gamma-1)}$$

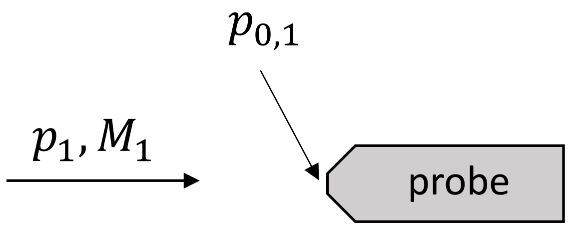




Pitot probes in compressible flow

Imagine we have a Pitot probe in a compressible flow. If the flow is subsonic, we can assume the deceleration from the freestream conditions is isentropic, in which case

$$\frac{p_{0,1}}{p_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\gamma/(\gamma-1)}$$



i.e., the measured Pitot pressure is the freestream total pressure.

This reaches a maximum value of 1.89 (for $\gamma = 1.4$) when $M_1 = 1$.

The Mach number can then be solved according to

$$M_1 = \left\{ \frac{2}{\gamma - 1} \left[\left(\frac{p_{0,1}}{p_1} \right)^{(\gamma-1)/\gamma} - 1 \right] \right\}^{1/2}.$$

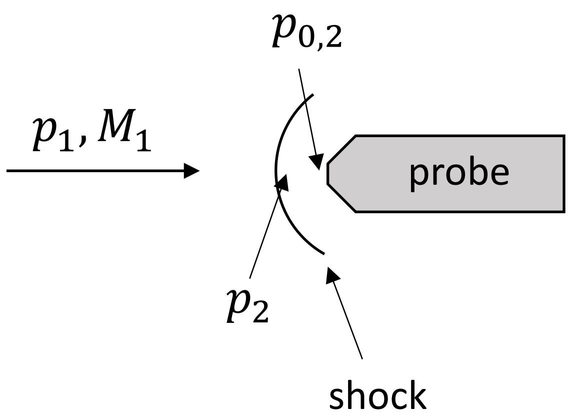
Assuming the freestream temperature (and thus sound speed) is known, the freestream velocity is given by

$$V_1 = M_1 a_1.$$

Pitot probes in compressible flow

If we have a supersonic flow, however, the situation is a little different. We assume that the stagnation streamline passes through a normal shock before reaching the probe tip. Then

$$\frac{p_{0,2}}{p_1} = \frac{p_{0,2}}{p_2}\frac{p_2}{p_1}.$$



The flow behind the shock to the probe tip can be assumed isentropic, so

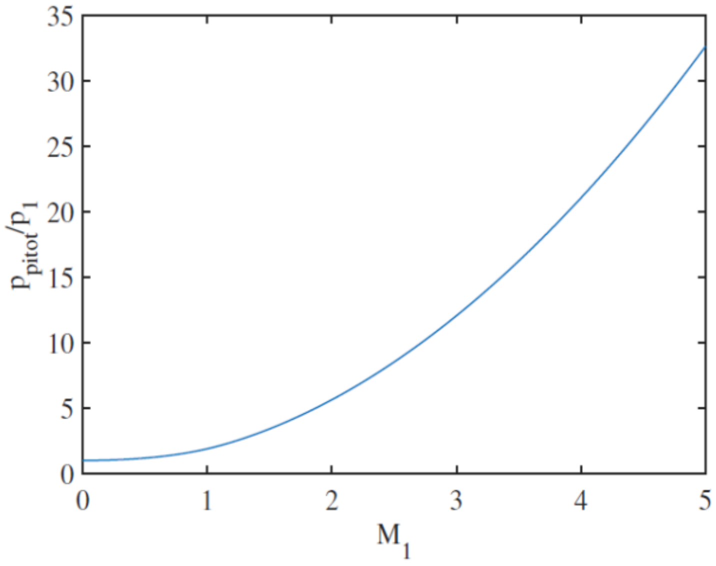
$$\frac{p_{0,2}}{p_2} = \left(1 + \frac{\gamma - 1}{2}M_2^2\right)^{\gamma/(\gamma-1)};$$

where M_2 comes from the normal shock relations, which also give us

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1).$$

Combining, we have

$$\frac{p_{0,2}}{p_1} = \left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right]^{\gamma/(\gamma-1)} \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right].$$

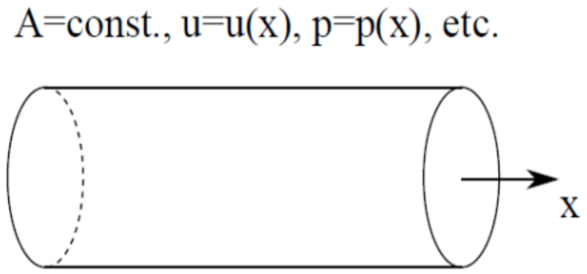


This is the Rayleigh-Pitot formula, which can be solved implicitly for M_1.

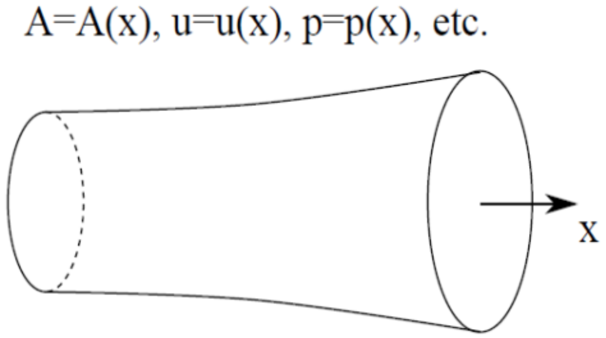
Quasi-one-dimensional flows

A normal shock wave is an example of a true one-dimensional flow: the velocity had only one component (u) and any given flow property changes only in the x direction.

To be strictly one-dimensional, a flow must be constant area; however, there is a class of flows, known as quasi-one-dimensional flows, for which area changes are gradual enough that we can approximate any changes as only taking place in single direction (and so properties are constant across a cross section normal to this direction).

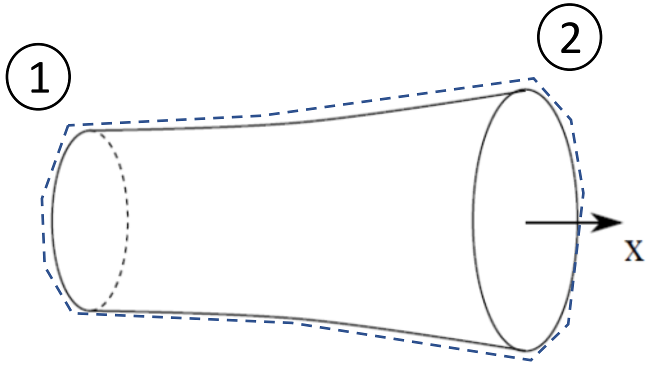


True 1-D flow



Quasi-1-D flow

Conservation equations in quasi-1-D flows



We assume the flow is steady. The continuity equation is then

$$\iint_{CS} \rho \mathbf{v} \cdot \mathbf{dA} = 0,$$

i.e.,

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2.$$

We can also write this as

$$\rho u A = \text{const.},$$

or

$$d(\rho u A) = 0.$$

Any derivatives here are to be understood as being w.r.t. x. Expanding out and dividing by $\rho u A$, we have

$$\boxed{\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0.}$$

Conservation equations in quasi-1-D flows

Instead of a formal momentum analysis, we note that Euler's equation holds for this flow (it is inviscid and adiabatic, and hence isentropic). We can thus write

$$dp + \rho u du = 0.$$

The energy equation is again

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2,$$

or

$$h + \frac{1}{2}u^2 = h_0 = \text{const.}$$

Since the flow is isentropic, we can also write

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2, \quad \Longrightarrow \quad dp = a^2 d\rho.$$

Substituting into Euler's equation:

$$a^2 \frac{d\rho}{\rho} = -u du,$$

or

$$\frac{d\rho}{\rho} = -M^2 \frac{du}{u}.$$

Area-velocity relation

Combining this latest equation with our continuity equation, i.e.,

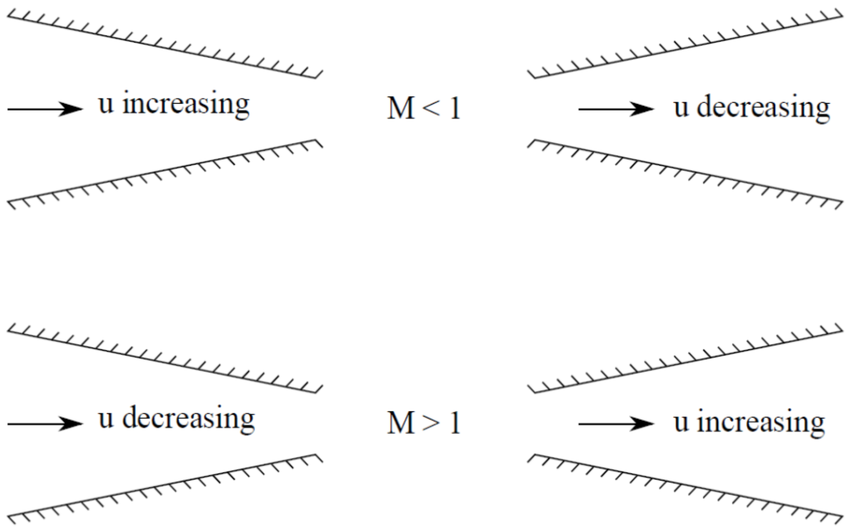
$$\frac{d\rho}{\rho} = -M^2\frac{du}{u} \qquad \& \qquad \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0.$$

we can write

$$(1 - M^2)\frac{du}{u} + \frac{dA}{A} = 0,$$

or

$$\boxed{\frac{du}{u} = -\frac{dA/A}{1 - M^2}.}$$



Note the following:

1. For $M = 0$, an increase in area leads to proportional decrease in velocity, and vice versa
2. For $M < 1$, increase in area leads to decrease in velocity (same qualitative behavior)
3. For $M > 1$, behavior is opposite: increase in area leads to increase in velocity
4. What about $M = 1$?

Throats in quasi-1D flows

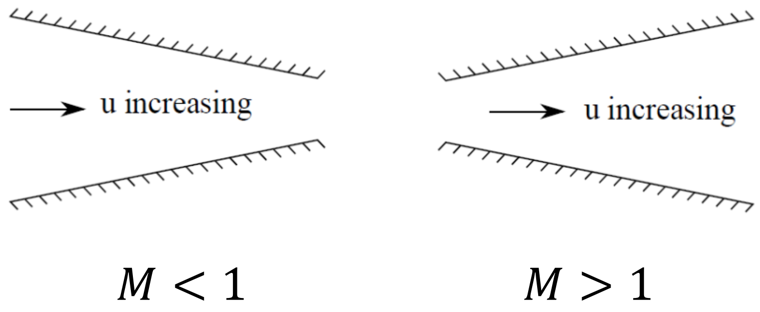
Imagine we have a geometry where the flow velocity is continuously increasing from subsonic to supersonic conditions. Then we note:

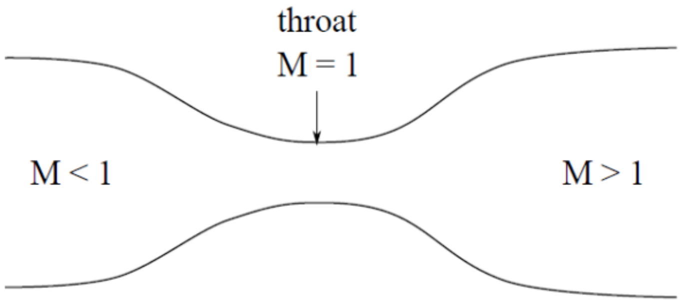
1. The area must be decreasing when $M < 1$
2. The area must be increasing when $M > 1$
3. We must have a throat ($dA/dx = 0$) when $M = 1$

It also flows from the area-velocity relation:

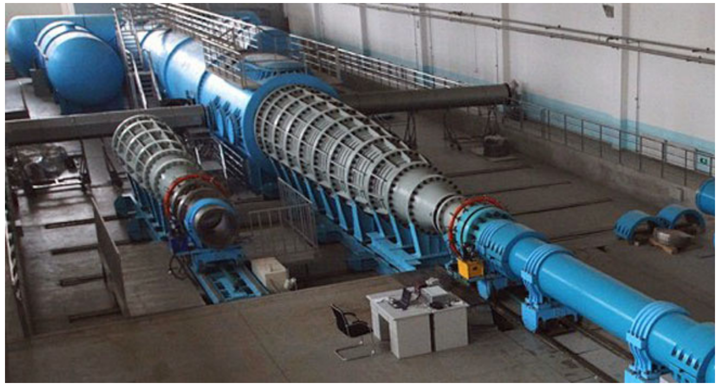
$$\frac{du}{u} = -\frac{dA/A}{1 - M^2}.$$

that dA/dx must be zero when $M = 1$ for du to remain finite.





Supersonic "de Laval" nozzle



Throats in quasi-1D flows

Imagine we have a geometry where the flow velocity is continuously increasing from subsonic to supersonic conditions. Then we note:

1. The area must be decreasing when $M < 1$
2. The area must be increasing when $M > 1$
3. We must have a throat ($dA/dx = 0$) when $M = 1$

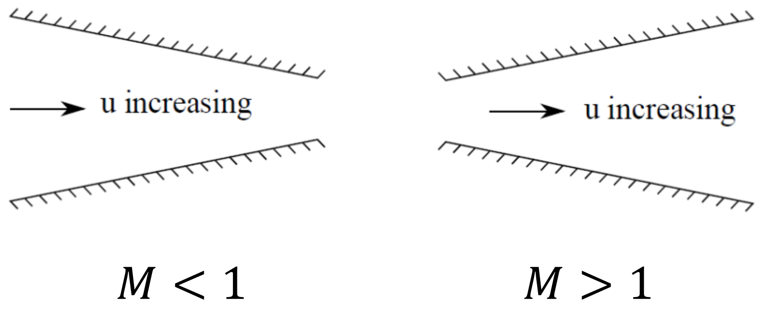
It also flows from the area-velocity relation:

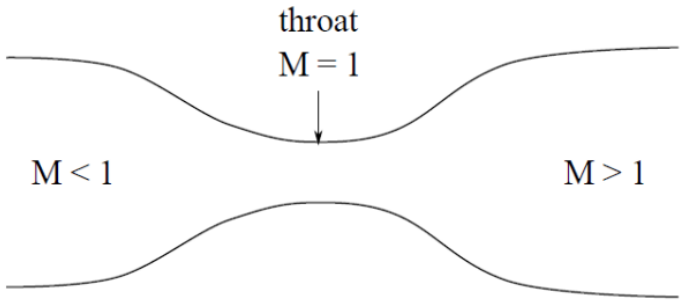
$$\frac{du}{u} = -\frac{dA/A}{1 - M^2}.$$

that dA/dx must be zero when $M = 1$ for du to remain finite.

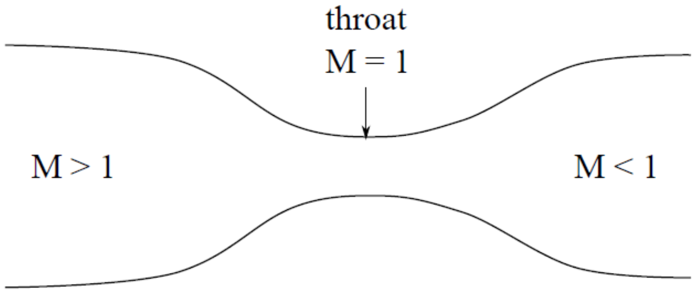
Similarly if we are transitioning isentropically from $M > 1$ to $M < 1$, can only have $M = 1$ at a throat.

Note, of course, that a throat is not a sufficient condition for sonic flow.





Supersonic "de Laval" nozzle



Supersonic diffuser

The area/Mach-number relationship

Imagine we have a converging-diverging supersonic nozzle with $A = A^*$ at the throat. Since at that point $M = 1$, we thus also have that $u = a = a^*$. Thus, we also have $M^* = 1$.

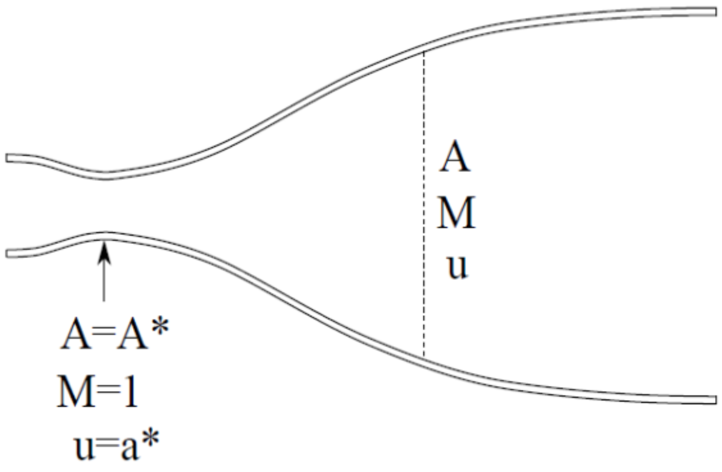
Now consider an arbitrary point downstream of the nozzle. From mass conservation we can write

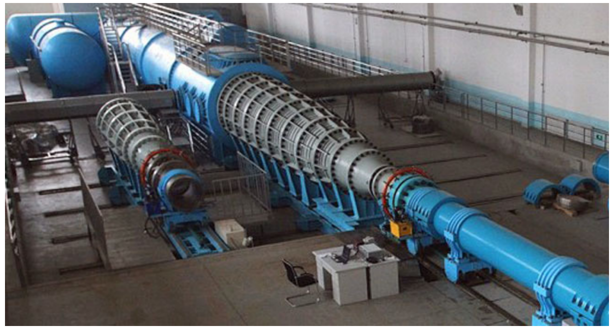
$$\rho^* a^* A^* = \rho u A$$

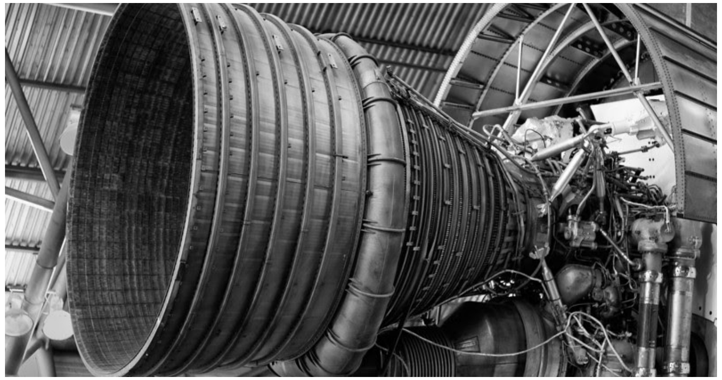
Rearranging, we have

$$\begin{aligned} \frac{A}{A^*} &= \frac{\rho^*}{\rho}\frac{a^*}{u} \\ &= \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{u}, \end{aligned}$$

since the total density is constant (flow is isentropic).







The area/Mach-number relationship

Starting from

$$\frac{A}{A^*} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{u}$$

We note

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)},$$

$$\frac{\rho_0}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)},$$

and

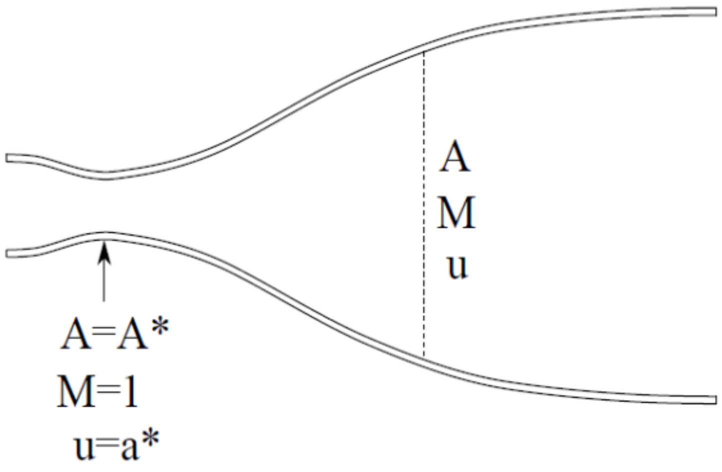
$$M^{*2} = \left(\frac{u}{a^*}\right)^2 = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}.$$

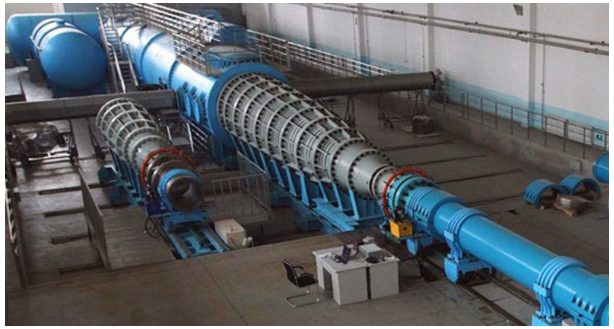
Substituting into our above expression, we have

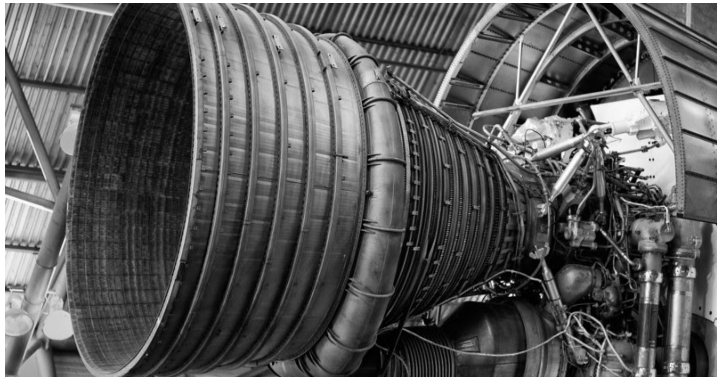
$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\left(1+\frac{\gamma-1}{2}M^2\right)^{2/(\gamma-1)}\frac{2+(\gamma-1)M^2}{(\gamma+1)M^2}.$$

Simplifying:

$$\boxed{\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}.}$$







The area/Mach-number relationship

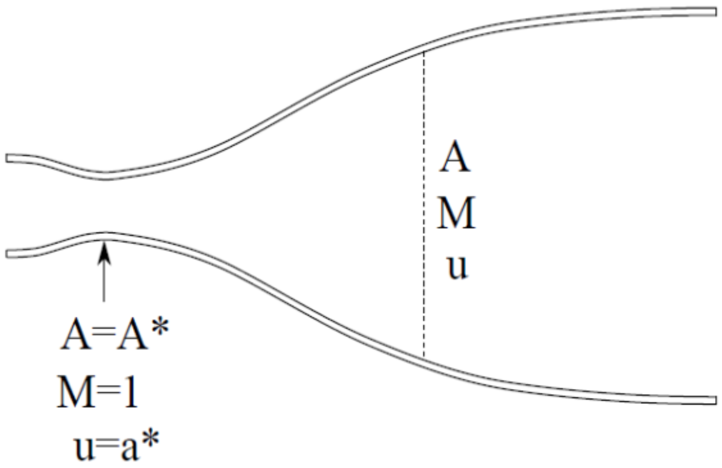
The expression

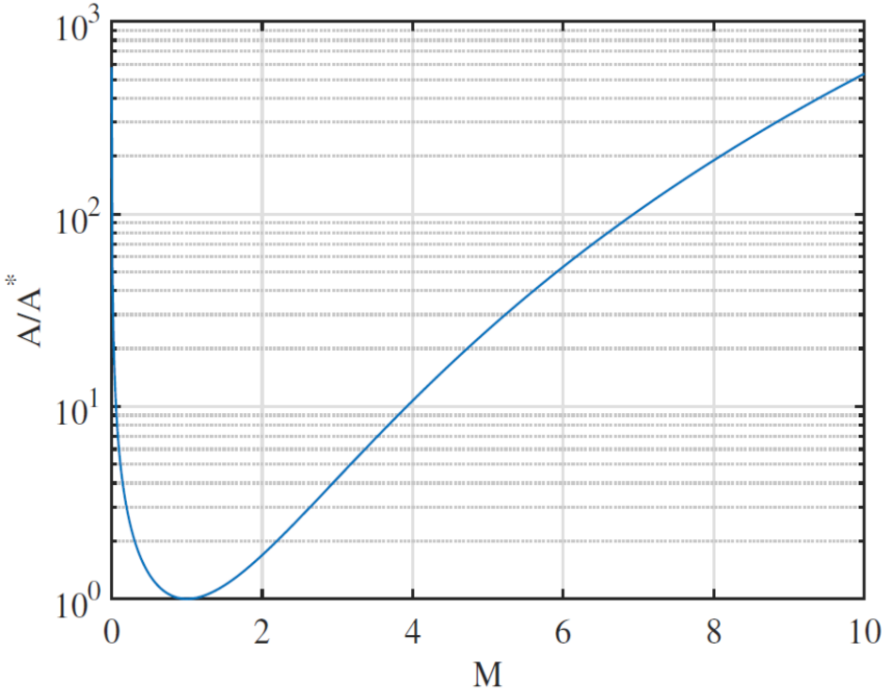
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}.$$

is known as the area/Mach-number relation. It gives the Mach number (implicitly) as a function of the area ratio.

For $A < A^*$, there are no solutions; for $A = A^*$, there is one solution ($M = 1$); for $A > A^*$ there are two solutions (one subsonic, one supersonic). Either way, increasing A pushes M away from one.

For solving the flow inside a wind tunnel, often we will be given an area ratio and reservoir condition. Usually, the reservoir velocity will be so small that we can consider the conditions there to be the stagnation conditions. At a given point downstream, the Mach number can be obtained using the above equation. The other flow conditions can then be derived using our adiabatic/isentropic expressions for T/T_0, p/p_0, etc., since total conditions are constant throughout the nozzle.





Oblique shocks of finite strength

Note, however, that the laws of physics are invariant under a change in inertial reference frame. Thus, the conservation laws we derived for a normal shock will be valid if we use u in place of V, and the shock jump relations that we derived from them will be valid if we replace M_1 by the normal shock Mach number (remember purely thermodynamic variables aren't modified by a change in inertial reference frame), i.e.,

$$
\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2} \\
\frac{p_2}{p_1} &= 1+\frac{2\gamma}{\gamma+1}(M_{n1}^2-1) \\
\frac{T_2}{T_1} &= \left[1+\frac{2\gamma}{\gamma+1}(M_{n1}^2-1)\right]\frac{2+(\gamma-1)M_{n1}^2}{(\gamma+1)M_{n1}^2}.
\end{aligned}
$$

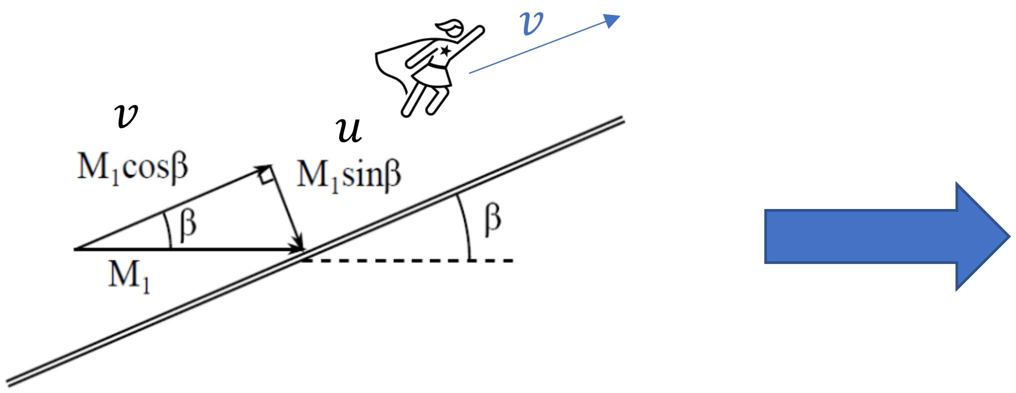
Note that the normal shock and Mach wave are special cases of these equations (for $\beta = \text{asin}(1/M_1)$, we have simply $\rho_2/\rho_1 = p_2/p_1 = T_2/T_1 = 1$). For $\beta < \text{asin}(1/M_1)$, $M_{n1} < 1$, i.e., the normal component of the velocity is subsonic, and so we can't have a shock.

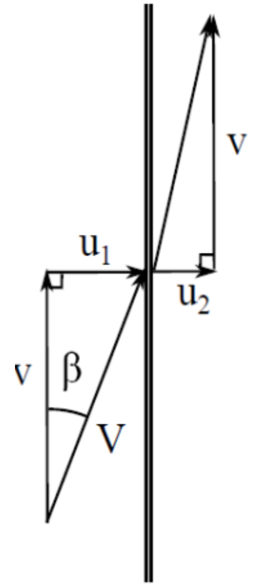
Lecture 17: Oblique and Curved Shocks

ENAE311H Aerodynamics I

Christoph Brehm

Oblique shocks of finite strength





$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2}$$

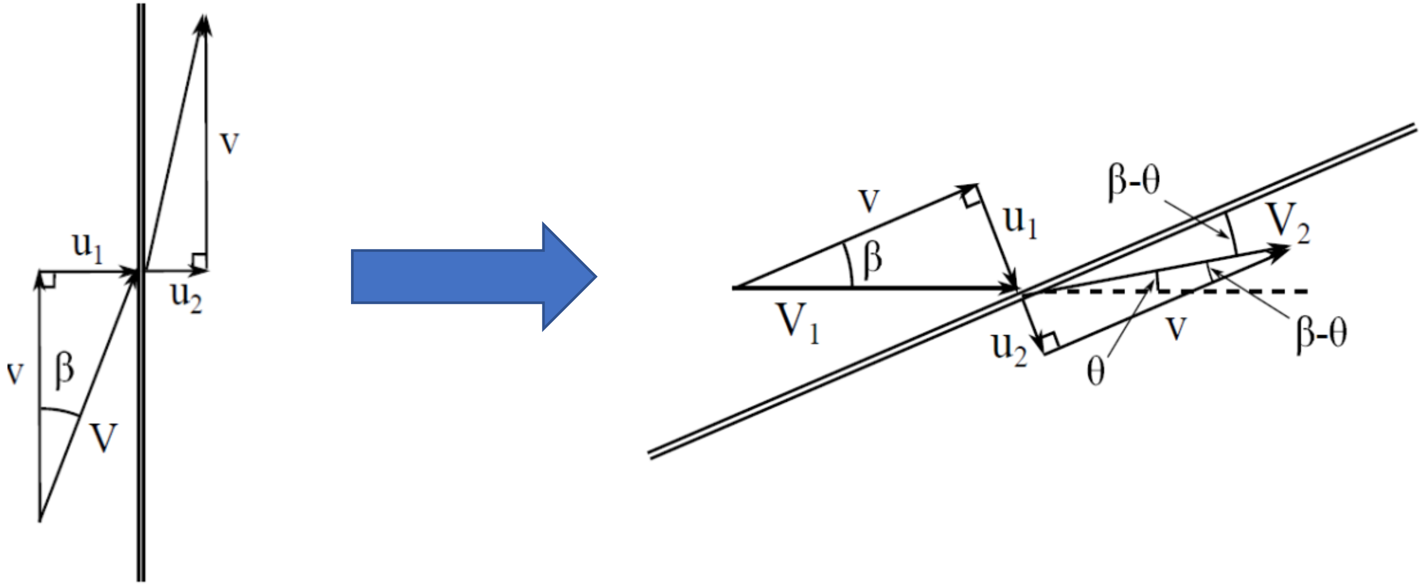
$$\frac{p_2}{p_1} = 1+\frac{2\gamma}{\gamma+1}(M_{n1}^2-1)$$

$$\frac{T_2}{T_1} = \left[1+\frac{2\gamma}{\gamma+1}(M_{n1}^2-1)\right]\frac{2+(\gamma-1)M_{n1}^2}{(\gamma+1)M_{n1}^2}.$$

Flow deflection through an oblique shock

We have seen how the thermodynamic properties vary through a shock – let's now see what happens to the flow velocity.

Consider again our transformed picture of the oblique shock (moving in a reference frame along the shock). Note again that the normal velocity component decreases, whereas the tangential component (which is zero in the transformed frame) is unaffected. If we now transform back to our original reference frame (by re-superimposing the tangential velocity component), the normal component will still decrease and the tangential component (now finite) is unchanged.



The flow is thus deflected towards the shock (though the flow angle remains less than the shock angle).

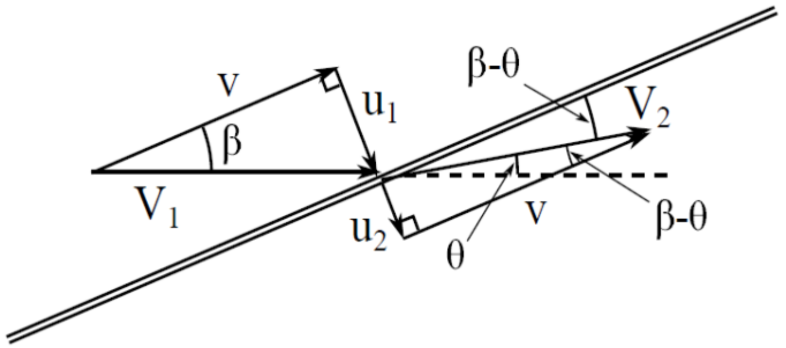
Flow deflection through an oblique shock

Consider the geometry shown to the right. We note the following:

$$\tan\beta = \frac{u_1}{v},$$

and

$$\tan(\beta-\theta) = \frac{u_2}{v}.$$



Using the continuity equation (on the normal velocity components), we can then write

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}.$$

Now, we know

$$\tan(\beta-\theta) = \frac{\tan\beta-\tan\theta}{1+\tan\beta\tan\theta},$$

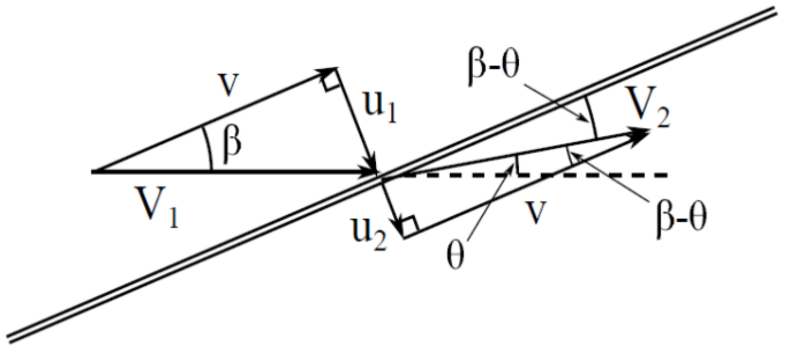
And thus can write our above expression as

$$\frac{1}{\tan\beta}\frac{\tan\beta-\tan\theta}{1+\tan\beta\tan\theta} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}.$$

Flow deflection through an oblique shock

Simplifying, we arrive at

$$\boxed{\tan\theta = 2\cot\beta \frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}.}$$



This expression is known as the $\theta - \beta - M$ relation, and gives the deflection angle in terms of the Mach number and shock angle (unfortunately, there is no simple relation that gives the shock angle in terms of the deflection angle).

To determine the Mach number behind the shock, we first use our normal shock relation to determine the normal component of the post-shock Mach number

$$M_{n2}^2 = \frac{2 + (\gamma - 1)M_{n1}^2}{2\gamma M_{n1}^2 - (\gamma - 1)}.$$

From the geometry above, we see

$$M_{n2} = M_2 \sin(\beta - \theta),$$

And thus

$$\boxed{M_2^2 \sin^2(\beta - \theta) = \frac{2 + (\gamma - 1)M_1^2 \sin^2\beta}{2\gamma M_1^2 \sin^2\beta - (\gamma - 1)}.}$$

Flow deflection through an oblique shock

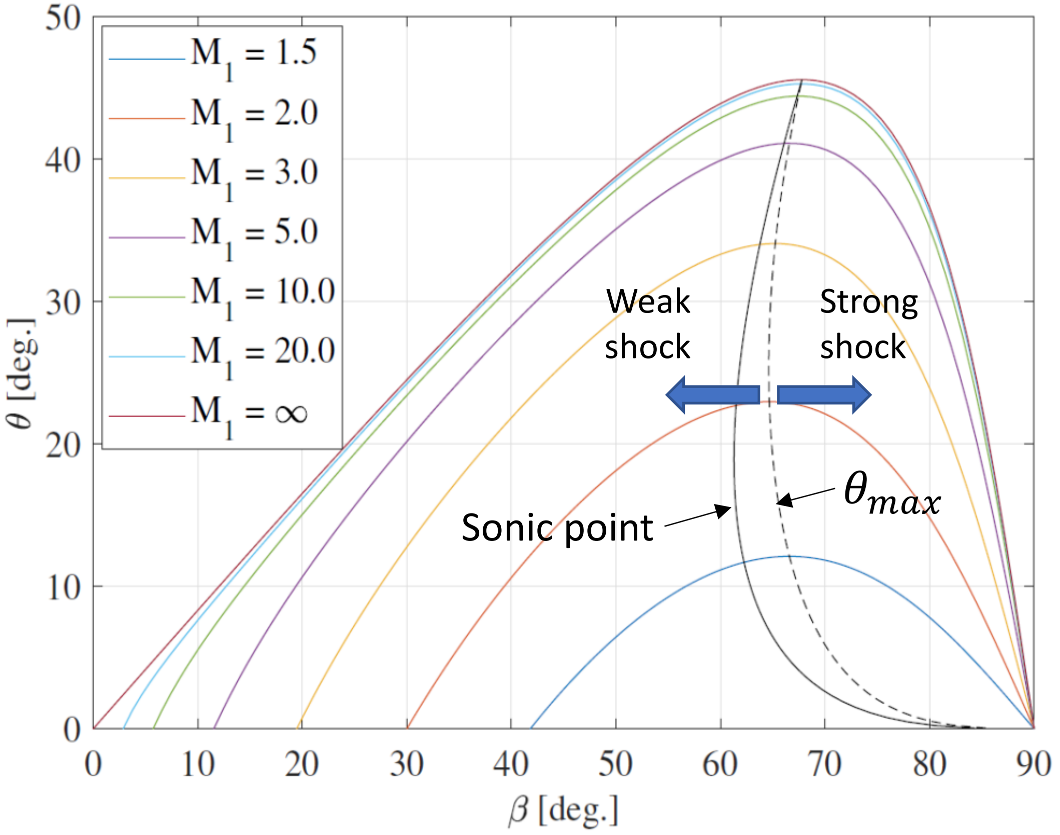
For a given M, the $\theta - \beta - M$ relation has two zeros – one at $\beta = \pi/2$ and one at $\beta = \text{asin}(1/M_1)$. In between, θ is everywhere positive, and thus must reach a maximum, θ_{max}.

The shock angle at this point, $\beta_{\theta max}$, is given by

$$\sin^2 \beta_{\theta max} = \frac{1}{4\gamma M_1^2}\left\{(\gamma+1)M_1^2 - 4 + \sqrt{(\gamma+1)[(\gamma+1)M_1^4 + 8(\gamma-1)M_1^2 + 16]}\right\}.$$

This maximum divides the curve into two: a strong-shock branch ($\beta > \beta_{\theta max}$) and a weak-shock branch ($\beta < \beta_{\theta max}$).

The post-shock flow is always subsonic on the strong-shock branch, and primarily supersonic on the weak-shock branch, but with a small portion on the latter where $M_2 < 1$. The sonic shock angle, $\beta *$, at which $M_2 = 1$, is given by



$$\sin^2 \beta^* = \frac{1}{4\gamma M_1^2}\left\{(\gamma+1)M_1^2 - (3-\gamma) + \sqrt{(\gamma+1)[(\gamma+1)M_1^4 - 2(3-\gamma)M_1^2 + \gamma + 9]}\right\}.$$

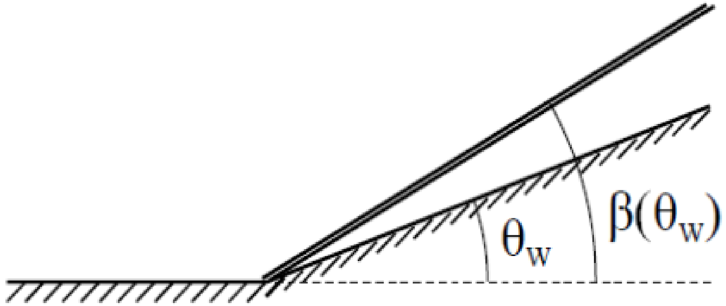
Flow over compression corners and wedges

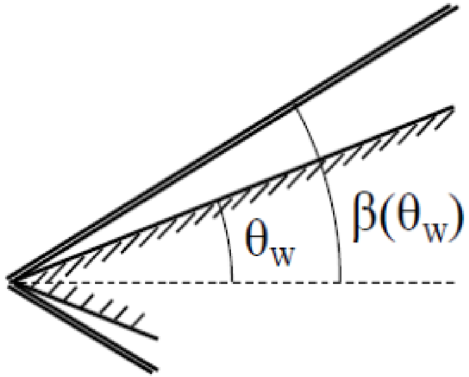
If we place a sharp compression corner or a wedge in a supersonic flow, provided the compression angle θ_w is less than θ_{max} for the given Mach number M_1, an attached oblique shock will form at the corner/vertex.

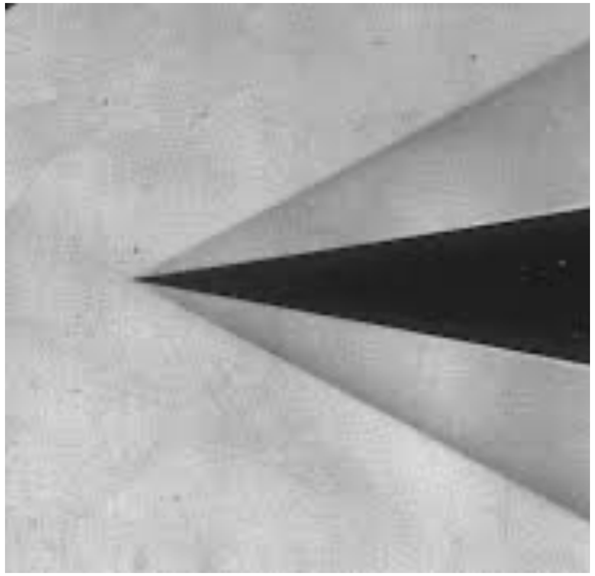
Since the flow angle downstream of the shock has to match the corner/wedge angle, the shock angle will be that corresponding to θ_w and M_1 in the $\theta - \beta - M$ relation. Note that the flow will typically choose the weak solution rather than the strong one.

Streamlines downstream of the shock will be straight and the flow conditions uniform.

Note that if the shock is attached in the case of the wedge, the top and bottom surfaces cannot communicate with one another, so for an asymmetric wedge the top and bottom shock angles will be different (to match the local flow angle).



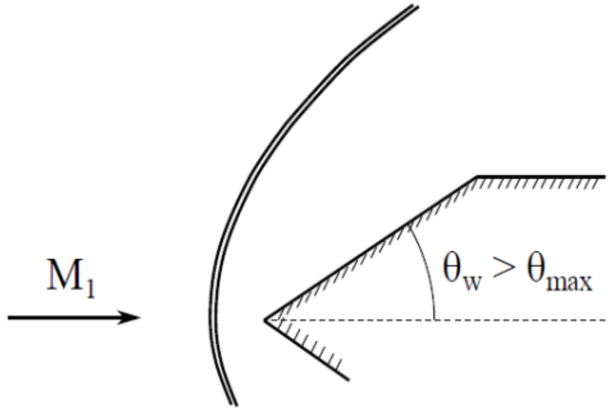


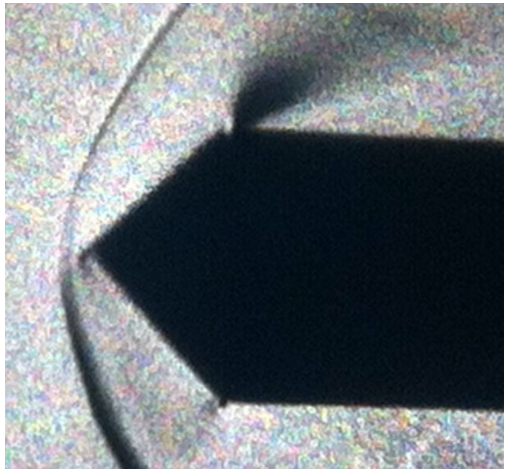


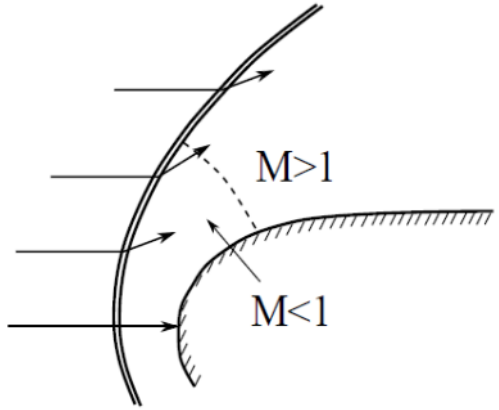
Flow over blunt bodies

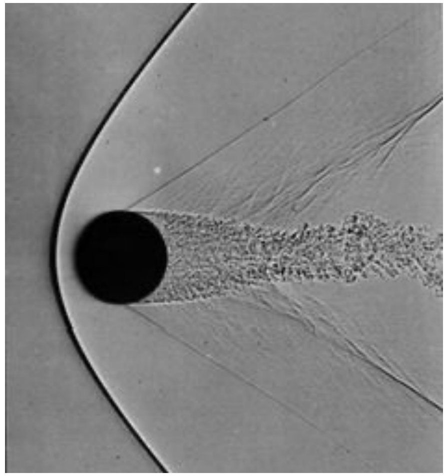
If the wedge angle is increased to above θ_{max}, and attached solution is no longer possible. Instead, a detached shock forms in front of the wedge with a finite shock stand-off distance.

The flowfield in this case will be qualitatively the same as that over a blunt body.









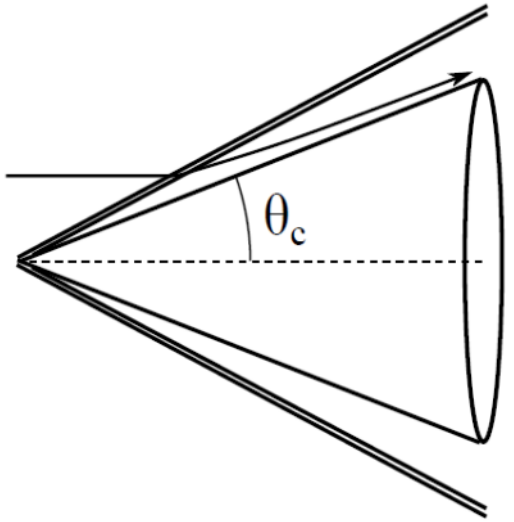
The shock will traverse all points from a normal shock to a Mach wave in the farfield. A finite subsonic region will form near the nose, with the flow accelerating to supersonic conditions downstream. The maximum-deflection will occur inside the subsonic region.

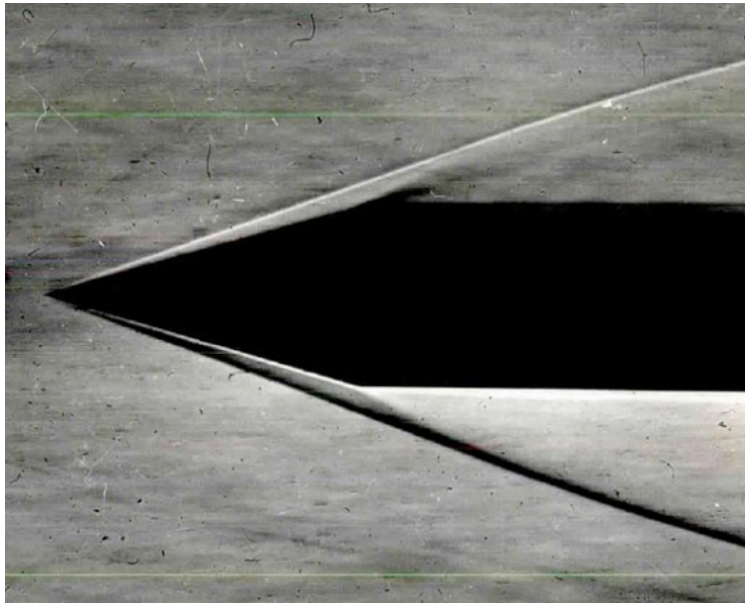
Flow over cones

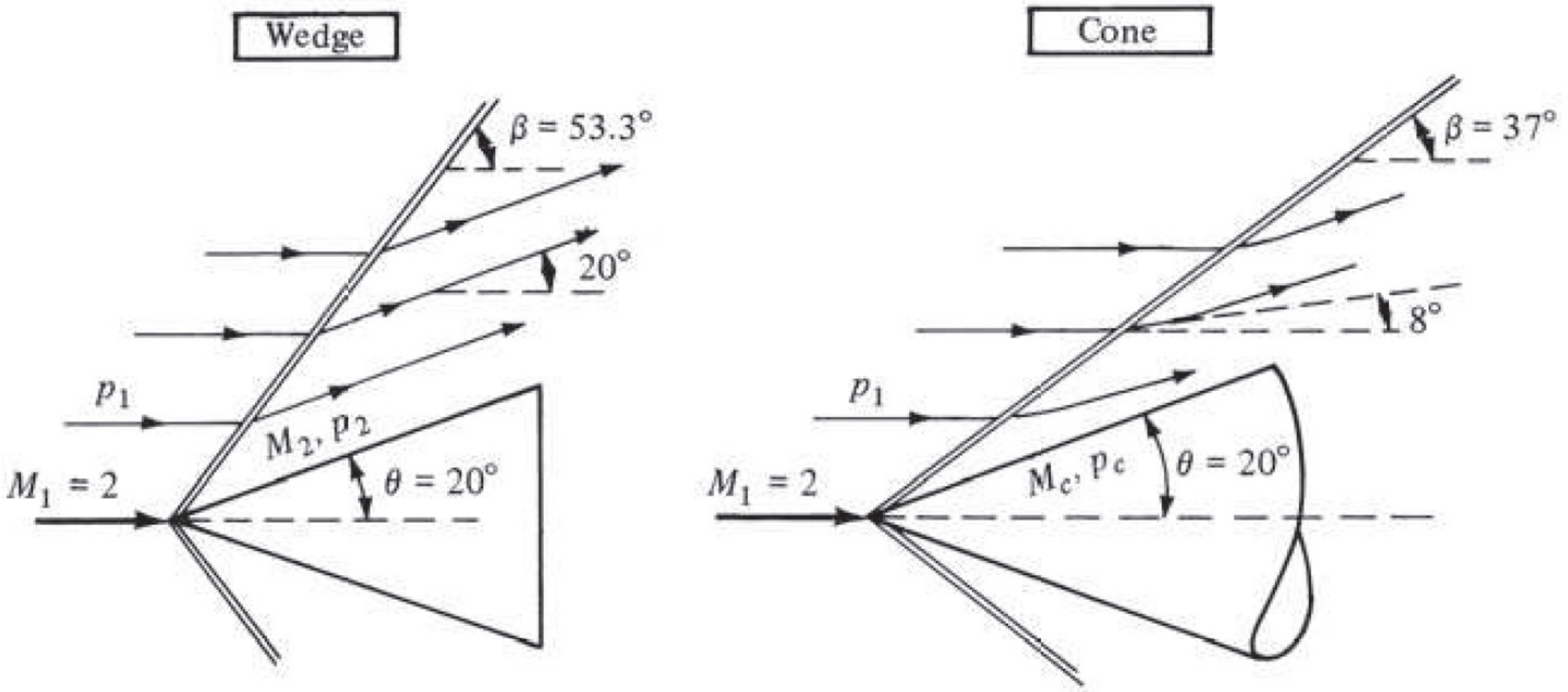
For a slender, infinite cone in supersonic flow, the shock will again be straight (since there is no length scale to scale any curvature).

However, the flow conditions cannot be uniform (this would violate continuity), so the streamlines curve towards the cone surface, with an additional post-shock compression taking place. Conditions are uniform along rays drawn from the cone vertex.

Because of 3-D relieving effects, the disturbance strength (shock angle and pressure jump) produced by a cone is much less than that of a wedge of the same half angle. Also, although the flow will eventually detach with increasing θ_c, this will happen at a much larger angle than for the equivalent symmetrical wedge.



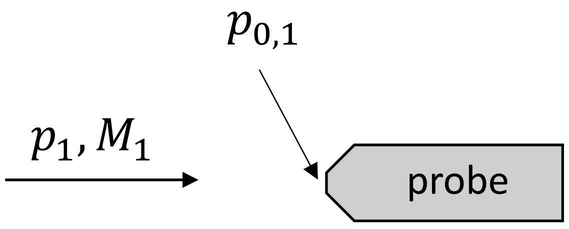




Wedge
$\beta = 53.3°$
20°
p_1
M_2, p_2
$M_1 = 2$
$\theta = 20°$
Cone
$\beta = 37°$
8°
p_1
M_c, p_c
$\theta = 20°$
$M_1 = 2$

Pitot probes in compressible flow

Subsonic:



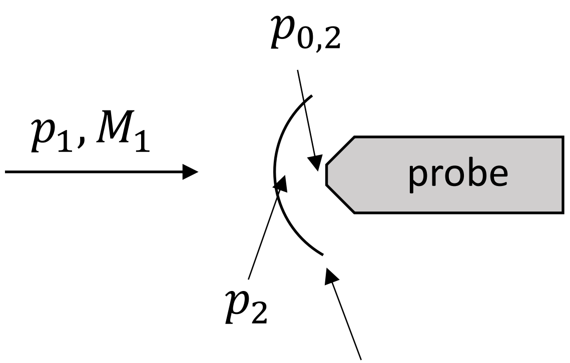
isentropic

$$\frac{p_{0,1}}{p_1} = \left(1 + \frac{\gamma-1}{2} M_1^2\right)^{\gamma/(\gamma-1)}$$

$$M_1 = \left\{\frac{2}{\gamma-1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right]\right\}^{1/2}.$$

$$V_1 = M_1 a_1.$$

Supersonic:



across shock

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1).$$

behind shock

$$\frac{p_{0,2}}{p_2} = \left(1 + \frac{\gamma-1}{2} M_2^2\right)^{\gamma/(\gamma-1)}$$

$$\frac{p_{0,2}}{p_1} = \left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right]^{\gamma/(\gamma-1)} \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right].$$

Lecture 18: Prandtl-Meyer Expansions

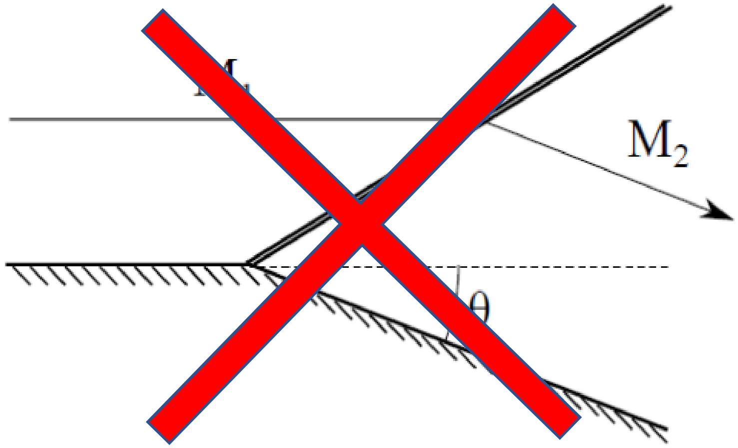
ENAE311H Aerodynamics I

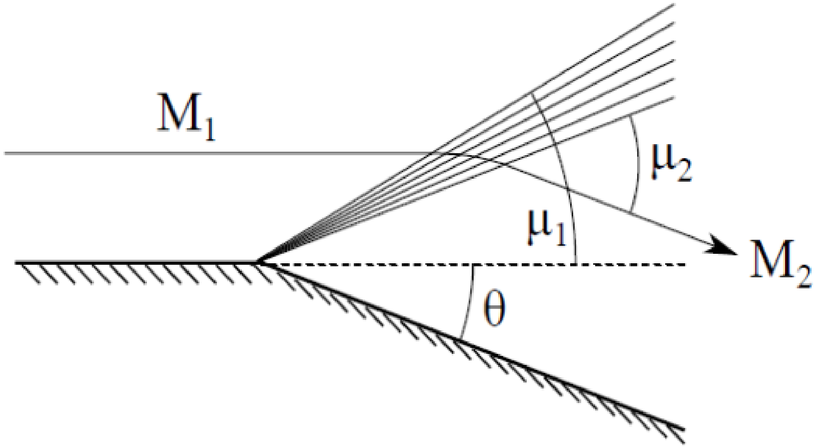
Christoph Brehm

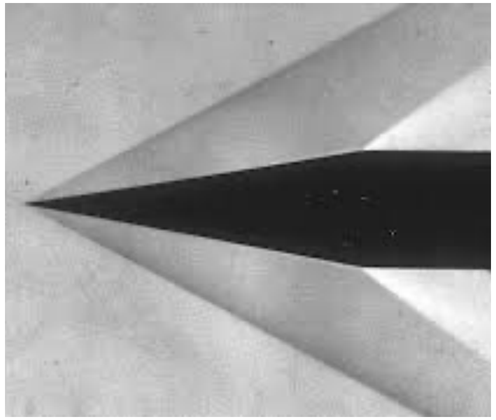
Centered expansion fans

Consider now the supersonic flow over a convex corner. Since the area is increasing, we know that this should lead to an increase in Mach number and velocity.

Unlike for a compression corner, however, this expansion cannot be achieved through a single shock, as an expansion shock would lead to a decrease in entropy.



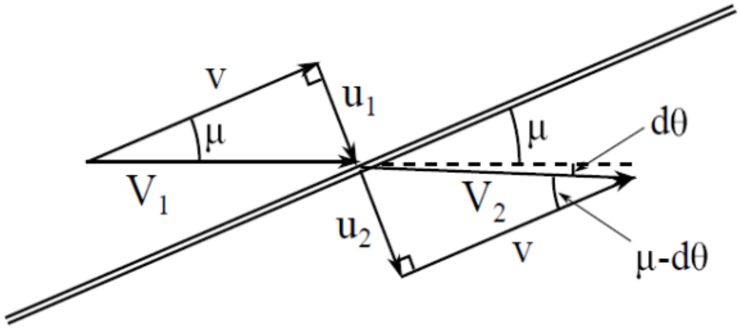




The expansion is thus achieved via a centered expansion fan, with the flow properties changing gradually (and thus isentropically). Each ray emanating from the corner is a Mach line/wave, along which conditions are constant. The angle of the leading wave is $\mathrm{asin}(1/M_1)$, while that of the trailing wave is $\mathrm{asin}(1/M_2)$ relative to the local flow direction. Since M is increasing (and the flow is turning away), the Mach lines are diverging.

The Prandtl-Meyer function

To see how the changes in flow properties (particularly Mach number) are related to the change in flow angle, consider an infinitesimal change over one of the Mach lines.



Note that $d\theta < 0$, by convention

Using the same argument as in the oblique-shock case, the tangential component of the velocity will be unchanged, while the normal component will (in this case) increase.

We can write:

$$\begin{aligned}\frac{V_2^2}{V_1^2} &= \frac{u_2^2+v^2}{u_1^2+v^2} = \frac{(u_2/v)^2+1}{(u_1/v)^2+1} = \frac{\tan^2(\mu-d\theta)+1}{\tan^2\mu+1} \\ &= \frac{\cos^2\mu}{\cos^2(\mu-d\theta)},\end{aligned}$$

where we have used

$$\tan^2\phi = \frac{\sin^2\phi}{\cos^2\phi} = \frac{1-\cos^2\phi}{\cos^2\phi} = \frac{1}{\cos^2\phi} - 1.$$

The Prandtl-Meyer function

We have

$$\frac{V_2^2}{V_1^2} = \frac{\cos^2\mu}{\cos^2(\mu - d\theta)}$$

Now, since $d\theta$ is small, we can approximate

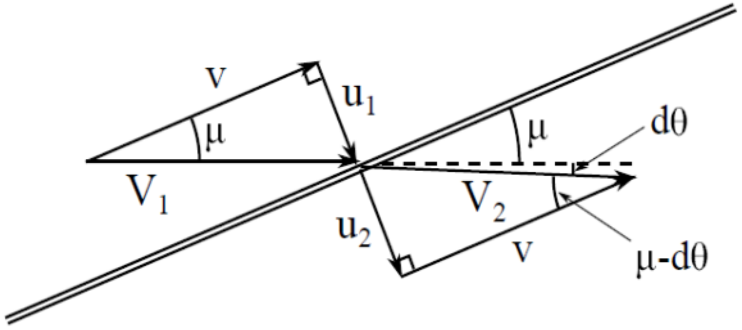
$$\begin{aligned}\cos(\mu - d\theta) &= \cos\mu\cos d\theta + \sin\mu\sin d\theta \\ &\approx \cos\mu + \sin\mu\, d\theta,\end{aligned}$$

and thus

$$\begin{aligned}\cos^2(\mu - d\theta) &\approx (\cos\mu + \sin\mu\, d\theta)^2 \\ &\approx \cos^2\mu + 2\sin\mu\cos\mu\, d\theta.\end{aligned}$$

Substituting into the above equation, we obtain

$$\begin{aligned}\frac{V_2^2}{V_1^2} &\approx \frac{\cos^2\mu}{\cos^2\mu + 2\sin\mu\cos\mu\, d\theta} \\ &= \frac{1}{1 + 2\sin\mu\, d\theta/\cos\mu}\end{aligned}$$



Note that $d\theta < 0$, by convention

The Prandtl-Meyer function

And thus

$$\frac{V_2}{V_1} \approx \left(1 + 2\frac{\sin\mu}{\cos\mu}d\theta\right)^{-1/2}.$$

Since $d\theta$ is small, and using $(1+\varepsilon)^{-1/2} \approx 1 - \frac{1}{2}\varepsilon$, we can approximate this as

$$\frac{V_2}{V_1} \approx 1 - \frac{\sin\mu}{\cos\mu}d\theta.$$

Now, since $\sin\mu = 1/M$, we can write

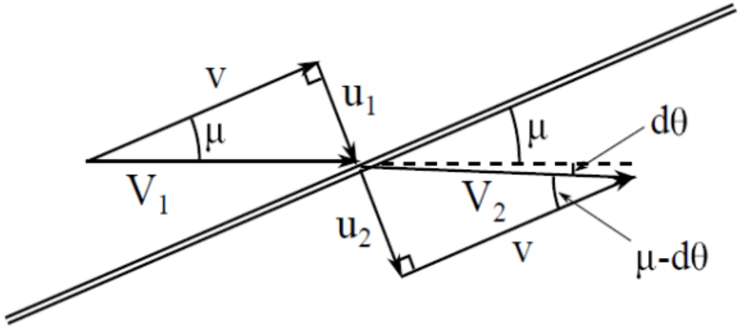
$$\cos\mu = \sqrt{1-\sin^2\mu} = \sqrt{1-\frac{1}{M_1^2}} = \frac{1}{M_1}\sqrt{M_1^2-1}.$$

And our above expression can be written

$$\frac{V_2}{V_1} \approx 1 - \frac{d\theta}{\sqrt{M_1^2-1}},$$

or

$$\frac{\Delta V}{V_1} = \frac{V_2 - V_1}{V_1} \approx -\frac{d\theta}{\sqrt{M_1^2-1}}.$$



Note that $d\theta < 0$, by convention

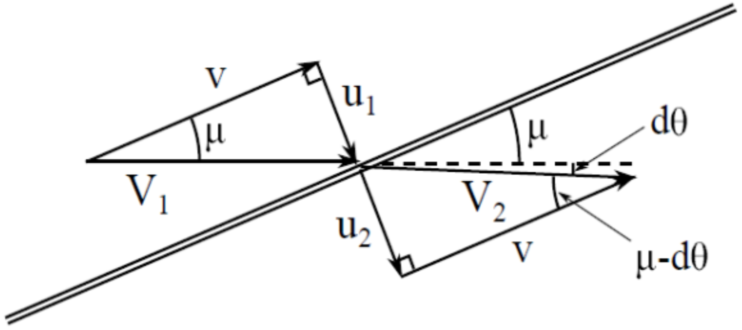
The Prandtl-Meyer function

In the limit of $d\theta \to 0$, our approximate expression

$$\frac{\Delta V}{V_1} = \frac{V_2 - V_1}{V_1} \approx -\frac{d\theta}{\sqrt{M_1^2 - 1}}.$$

becomes exact, i.e.,

$$d\theta = -\sqrt{M^2 - 1}\frac{dV}{V}.$$



Note that $d\theta < 0$, by convention

For a finite turning angle, we can thus integrate:

$$\theta = \int_0^\theta d\theta' = -\int_{V_1}^{V_2} \sqrt{M^2 - 1}\frac{dV}{V}.$$

To evaluate the integral on the RHS, we need to write dV/V in terms of the Mach number. To do so, note first that from $V = Ma$ we can write

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}.$$

Also, since the flow is adiabatic, we have

$$\left(\frac{a_0}{a}\right)^2 = \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2$$

Differentiating, we have

$$\frac{da}{a} = -\frac{(\gamma - 1)M^2}{2 + (\gamma - 1)M^2}\frac{dM}{M}$$

and so

$$\frac{dV}{V} = \frac{2}{2 + (\gamma - 1)M^2}\frac{dM}{M}.$$

The Prandtl-Meyer function

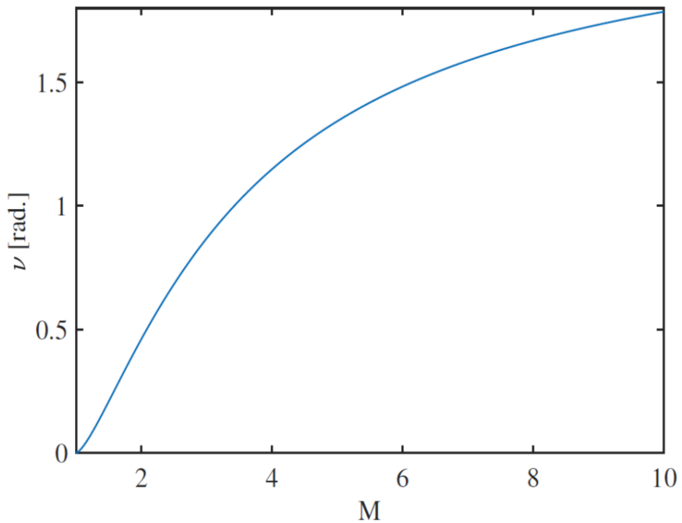
Our integral therefore becomes

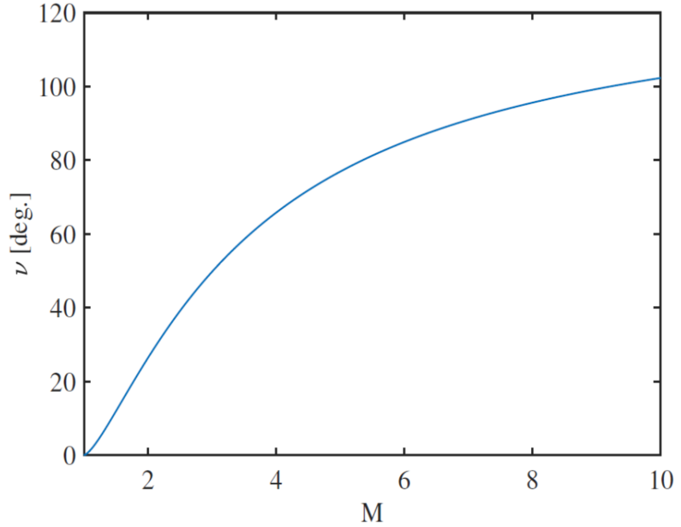
$$\theta = -\int_{M_1}^{M_2} \frac{2\sqrt{M^2-1}}{2+(\gamma-1)M^2}\frac{dM}{M}.$$

The function

$$\begin{aligned}\nu(M) &= \int \frac{2\sqrt{M^2-1}}{2+(\gamma-1)M^2}\frac{dM}{M} \\ &= \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \arctan\sqrt{M^2-1}\end{aligned}$$

is known as the *Prandtl-Meyer function*. The integration constant has been set such that $\nu(1) = 0$.





Expansions and compressions

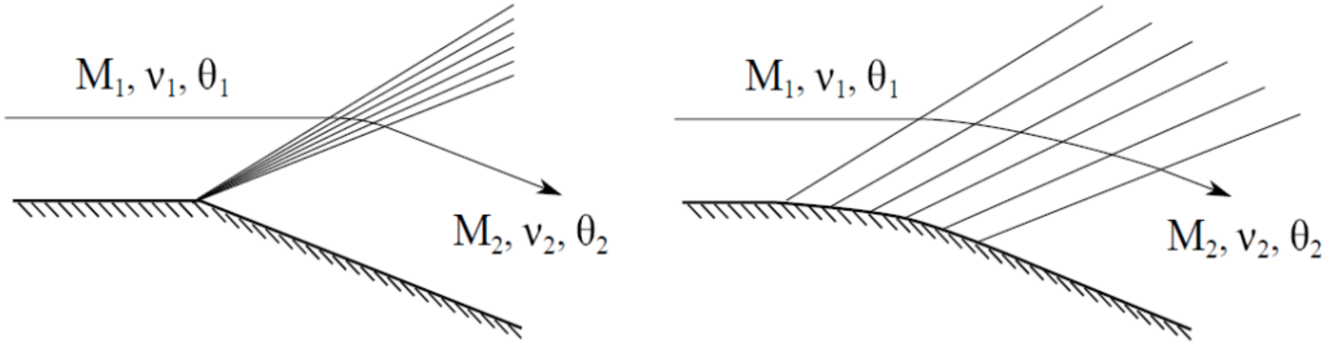
We thus have

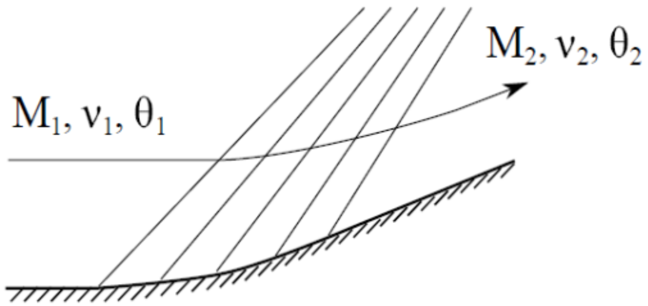
$$\theta = -[\nu(M_2) - \nu(M_1)]$$

where

$$\nu(M) \;=\; \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \arctan\sqrt{M^2-1}$$

Since the expansion is isentropic, however, it doesn’t matter whether it is achieved at a single corner or gradually over a continuous bend. Also, if a compression is gradual enough that no shocks form, it will also be isentropic and the foregoing analysis applies (with $d\theta > 0$). To avoid ambiguity in sign then, we distinguish as:





Expansion: $\nu \;=\; \nu_1 + |\theta - \theta_1|$

Compression: $\nu \;=\; \nu_1 - |\theta - \theta_1|$

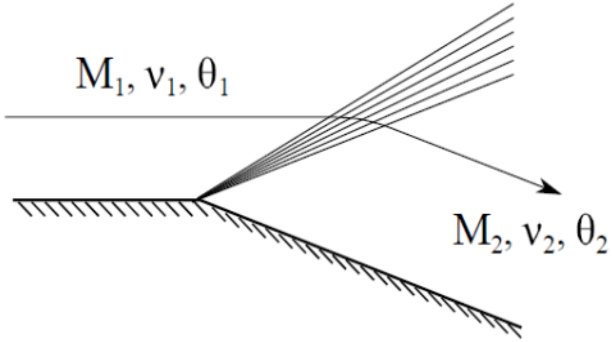
Expansions and compressions

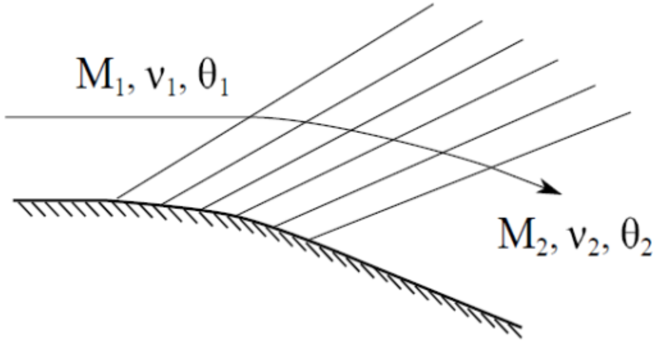
To solve then for the flow through an expansion or isentropic compression with a given M_1 and $\Delta\theta = \theta - \theta_1$, first calculate $\nu(M_1)$ (using the Prandtl-Meyer formula directly or tables); $\nu_2 = \nu(M_2)$, can then be derived using the appropriate formula below.

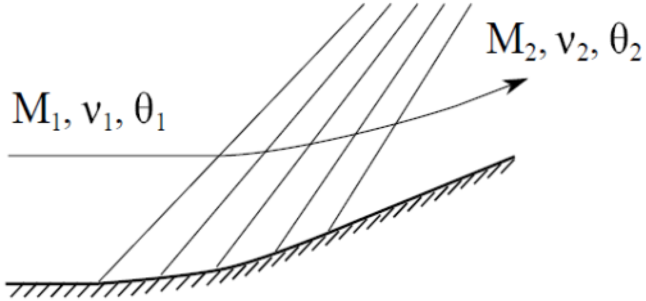
Knowing $\nu(M_2)$, M_2 can then be obtained either numerically, graphically, or through tables.

Assuming the other upstream flow properties are available, we can use the fact that T_0, p_0, and ρ_0 are all constant (flow is adiabatic and isentropic) to derive the downstream flow conditions, e.g.,

$$\frac{T_2}{T_1} = \frac{T_2}{T_0}\frac{T_0}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2} \qquad \text{and} \qquad \frac{p_2}{p_1} = \frac{p_2}{p_0}\frac{p_0}{p_1} = \left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{\gamma/(\gamma-1)}$$







$$\text{Expansion:} \quad \nu = \nu_1 + |\theta - \theta_1|$$

$$\text{Compression:} \quad \nu = \nu_1 - |\theta - \theta_1|$$

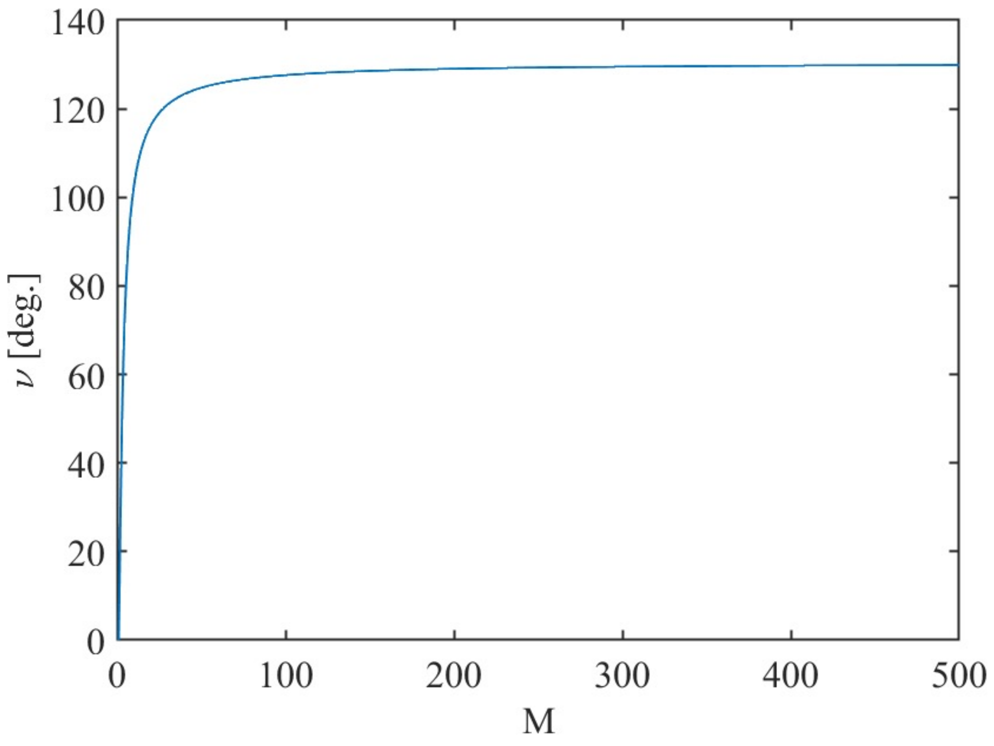
Maximum expansion angle

Finally, note that although $\nu(M)$ increases monotonically with M, it asymptotes to a maximum value given by

$$\nu_{max} = \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right).$$

For $\gamma = 1.4$, this takes the value of 130.45°.

Thus, for a flow initially at a Mach number of 1, if it encounters an expansion with a larger angle than this, it will not be able to turn all the way and will separate once it reaches ν_{max}. For flows initially at higher Mach numbers, the maximum turning angle is reduced accordingly (for Mach 5, for instance, it is 53.45°).



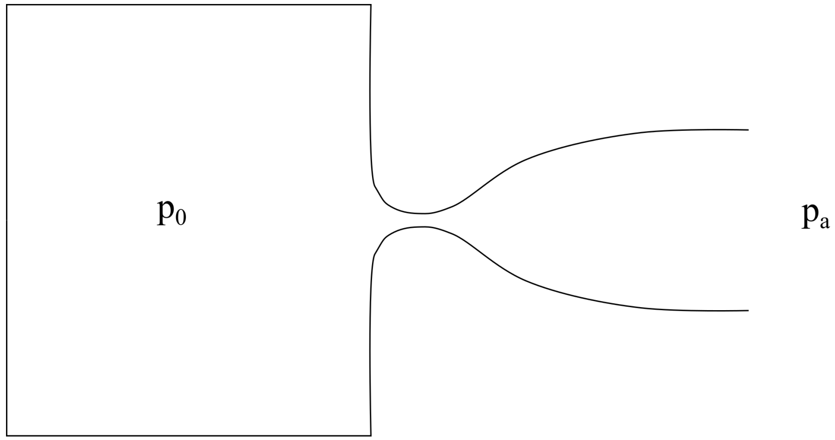
Lecture 19: Flow through Converging-Diverging Nozzles and Supersonic Wind Tunnels

ENAE311H Aerodynamics I

Christoph Brehm

Subsonic solutions

Imagine we have a converging-diverging nozzle, supplied by a large reservoir (at pressure p_0) and exhausting into an ambient atmosphere (of pressure p_a).



If p_a is matched to p_0, there won't be any flow through the nozzle. If we begin to lower p_a, however, a subsonic flow will establish itself through the nozzle (including at the throat). Note the following:

- The nozzle exit pressure must match p_a, which fixes the Mach number throughout the rest of the nozzle via the area/Mach-number relationship (flow is isentropic throughout).
- However, A^* is now an imagined reference area ($A^* < A_t$) and is not achieved within the nozzle.

This situation will continue until the flow becomes exactly sonic at the throat, at which point

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} = 0.528 \quad \text{for } \gamma = 1.4.$$

Subsonic solutions

Imagine we have a converging-diverging nozzle, supplied by a large reservoir (at pressure p_0) and exhausting into an ambient atmosphere (of pressure p_a).

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This situation will continue until the flow becomes exactly sonic at the throat, at which point

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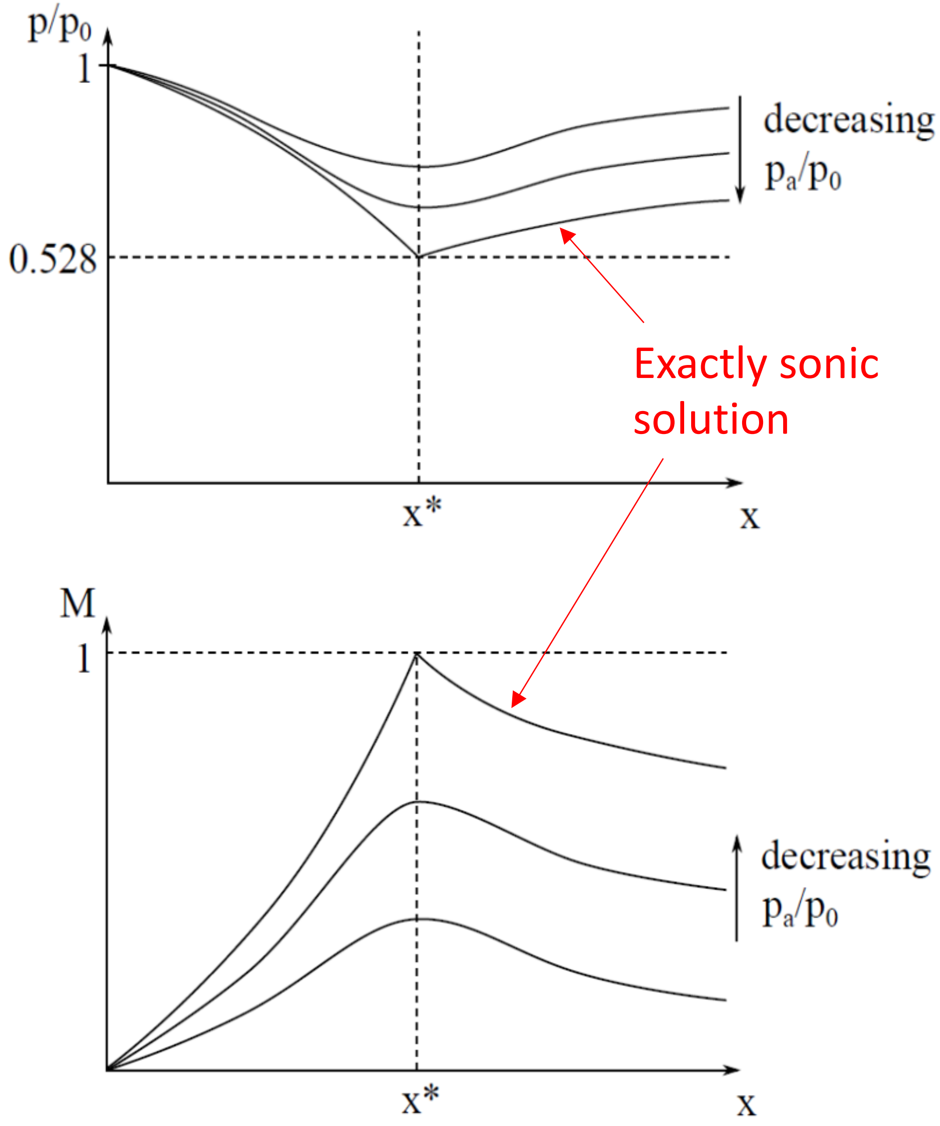


Choked solutions

Now imagine we decrease p_a further from that exactly choked point. There is no longer a solution for which the flow remains fully subsonic throughout the nozzle, so we must have supersonic flow downstream of the throat.

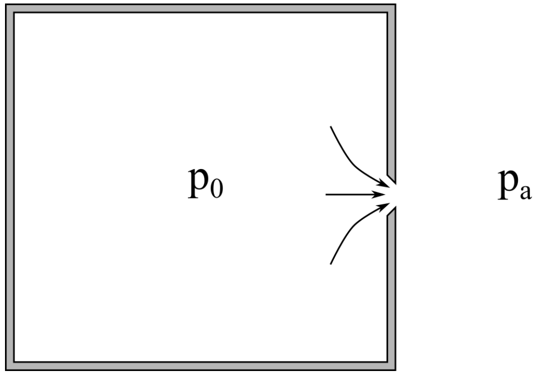
Note, however, that now there is no way for information to travel upstream from the exit to the throat, so any further reduction in p_a will not affect the flow between the reservoir and the throat

→ we say that the throat is "choked".

For a choked throat, the (maximum) mass flux through the nozzle (independent of p_a) is

$$\dot{m} = \rho^* u^* A^* = \rho^* a^* A_t$$

Note that we can also infer from these results the behavior of a pressurized reservoir discharging to an ambient atmosphere through an orifice (with minimum area at the exit). Provided $p_a < 0.528 p_0$, the flow at the exit will be sonic/choked, and the mass flux will be given by the relation above.



In particular, we will have

$$\dot{m}_{max} \propto \frac{p_0 A_t}{\sqrt{T_0}}.$$

Supersonic solutions

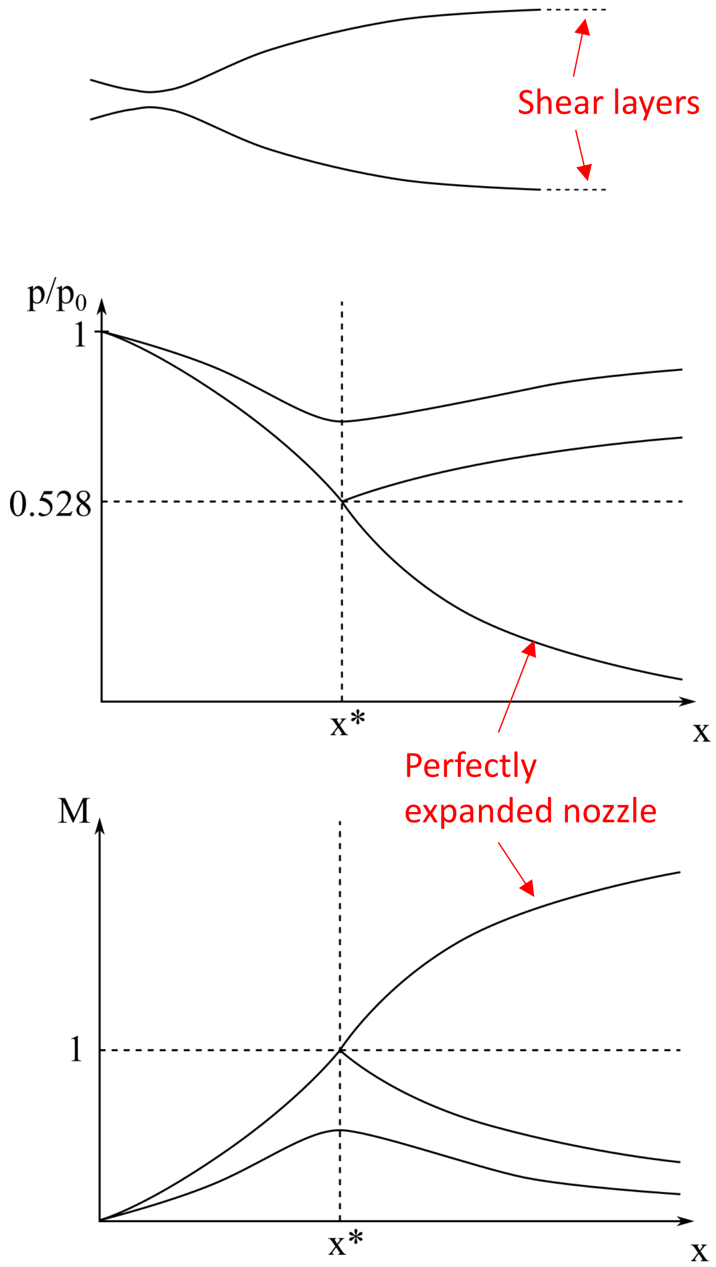
Returning to the converging/diverging nozzle, let us explore the range of solutions available with a sonic throat.

- If the ratio of ambient to reservoir pressure is exactly the value corresponding to the isentropic solution

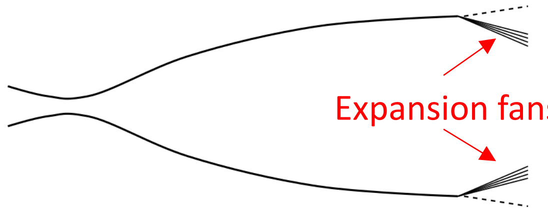
$$\frac{p_a}{p_0} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\gamma/(\gamma-1)}$$

with M matching the value for A_e/A^* in the area/Mach-number relationship, then the flow will be smoothly varying and isentropic throughout the nozzle:

→ nozzle is "perfectly expanded"



Supersonic solutions



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- If the ratio of ambient to reservoir pressure is exactly the value corresponding to the isentropic solution

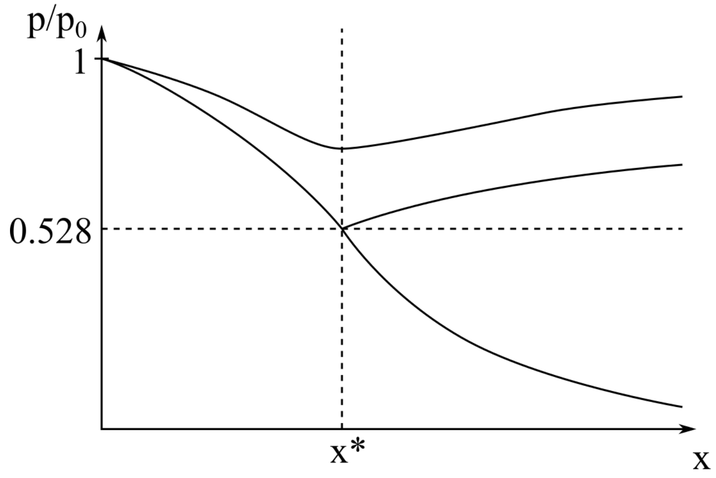
$$\frac{p_a}{p_0} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\gamma/(\gamma-1)}$$

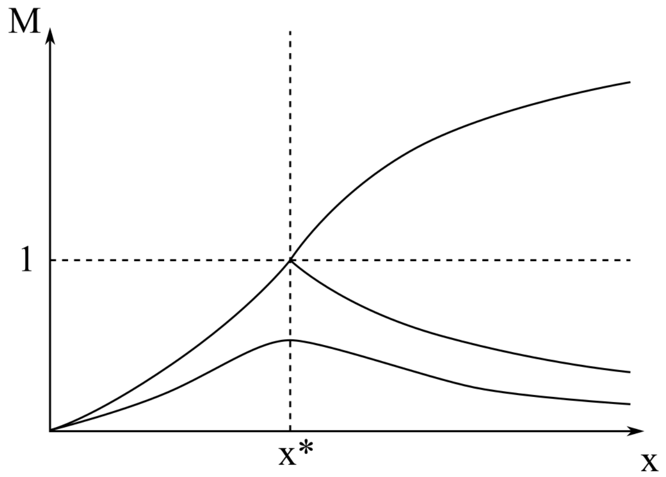
with M matching the value for A_e/A^* in the area/Mach-number relationship, then the flow will be smoothly varying and isentropic throughout the nozzle:

→ nozzle is "perfectly expanded"

- If p_a is below this perfectly expanded value, expansion waves will form at the nozzle exit, with shear layers deflected outwards

→ nozzle is "under-expanded"





Supersonic solutions

Returning to the converging/diverging nozzle, let us explore the range of solutions available with a sonic throat.

- If the ratio of ambient to reservoir pressure is exactly the value corresponding to the isentropic solution

$$\frac{p_a}{p_0} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\gamma/(\gamma-1)}$$

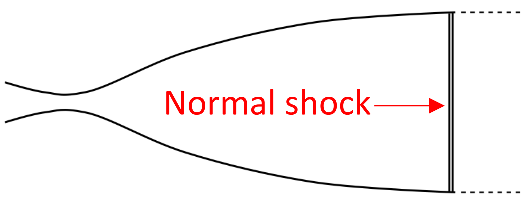
  with M matching the value for A_e/A^* in the area/Mach-number relationship, then the flow will be smoothly varying and isentropic throughout the nozzle:

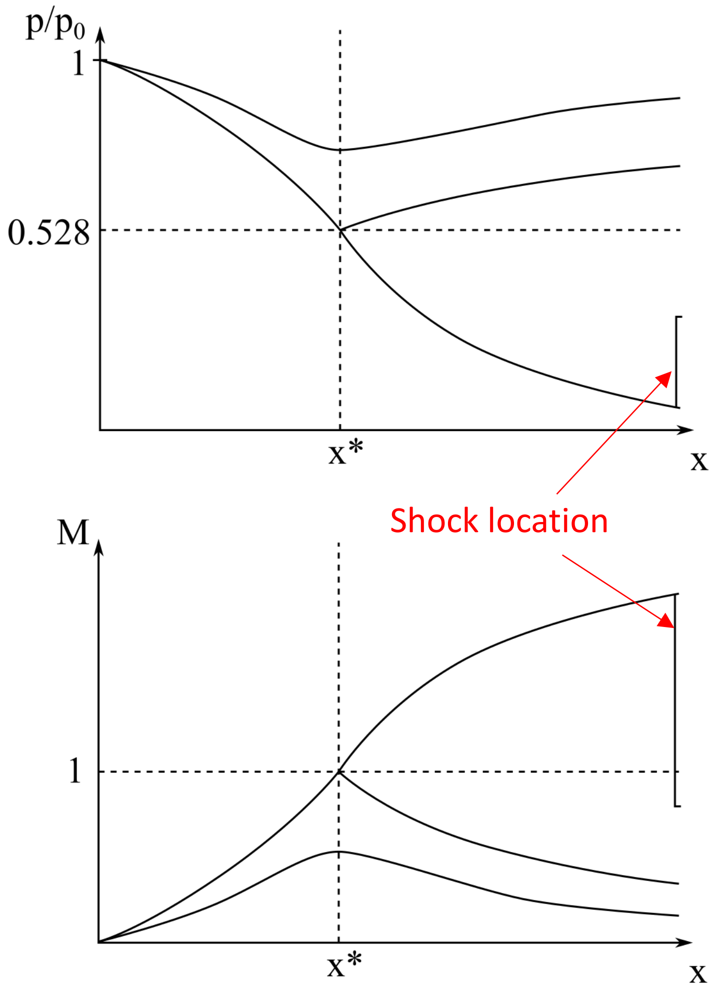
  → nozzle is "perfectly expanded"

- If p_a is between the perfectly expanded and exactly sonic values, then a shock(s) will need to be present to bring the isentropic solution back up to the ambient pressure

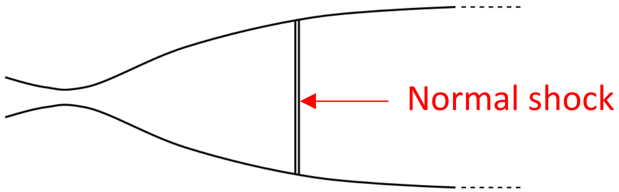
  → nozzle is "over-expanded"

- The shock can form exactly at the nozzle exit (if the ambient pressure is exactly that post-normal-shock value), inside, or outside the nozzle.





Supersonic solutions



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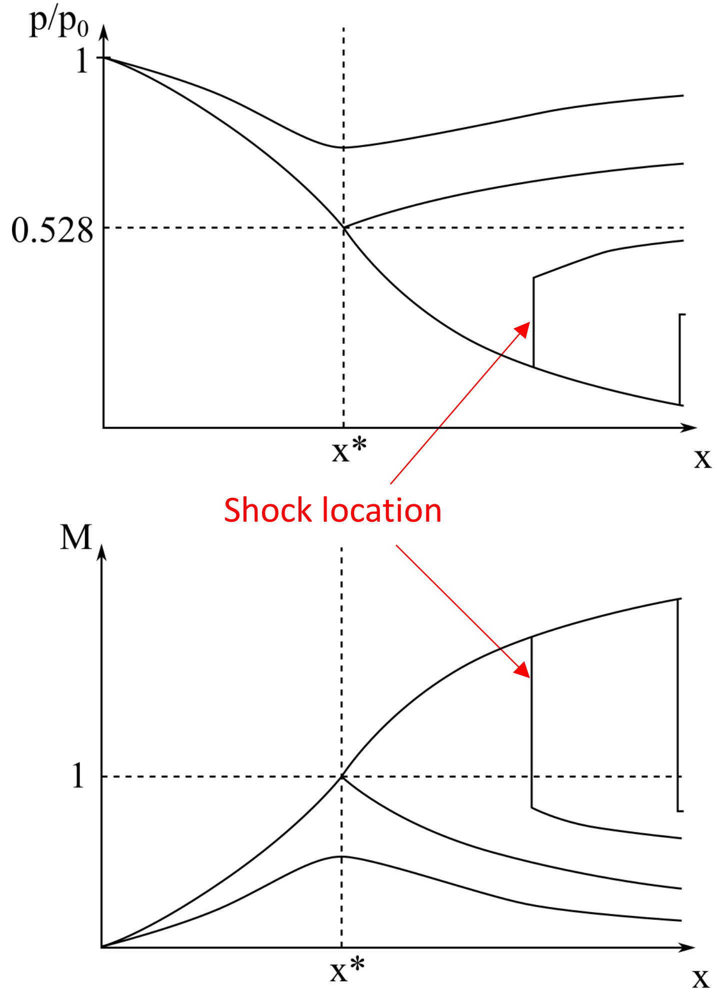

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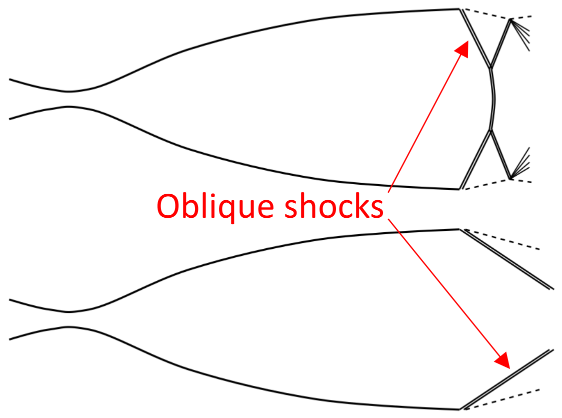
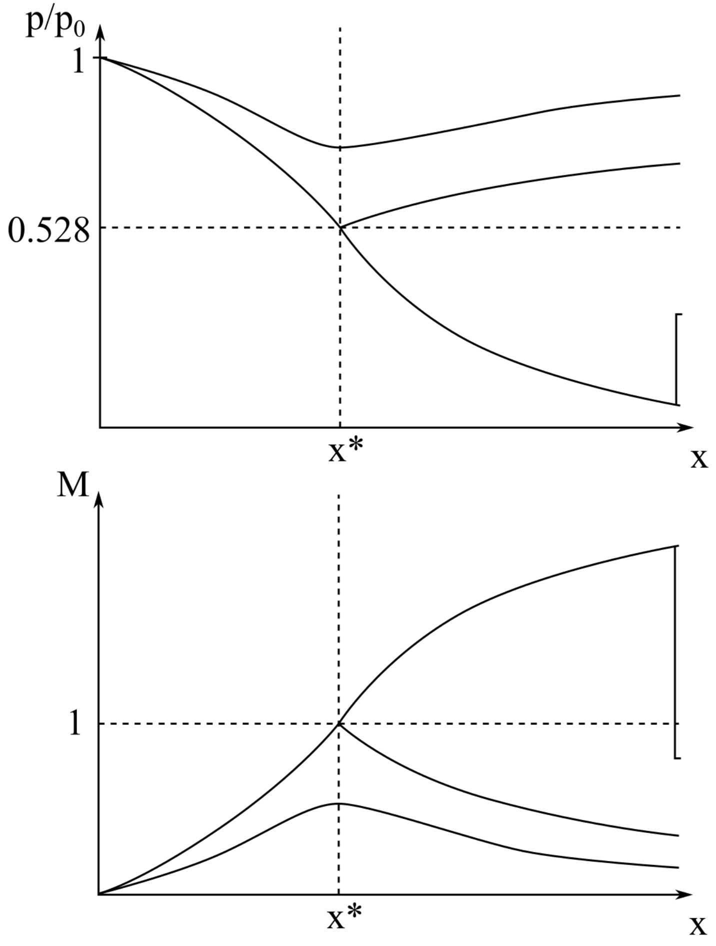
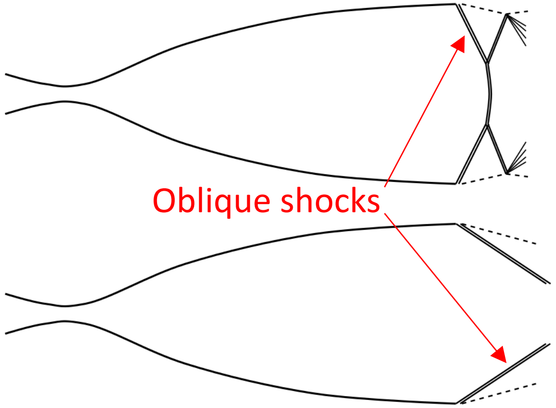




Supersonic solutions



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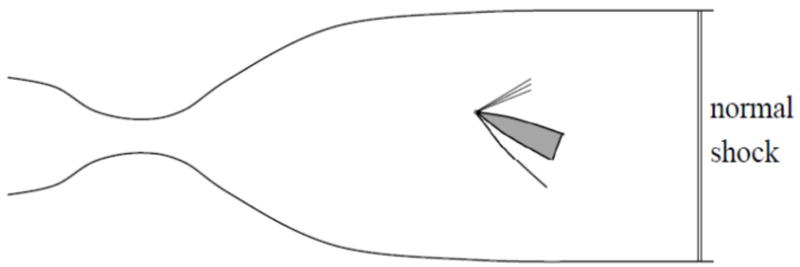
- The shock can form exactly at the nozzle exit (if the ambient pressure is exactly that post-normal-shock value), inside, or outside the nozzle.

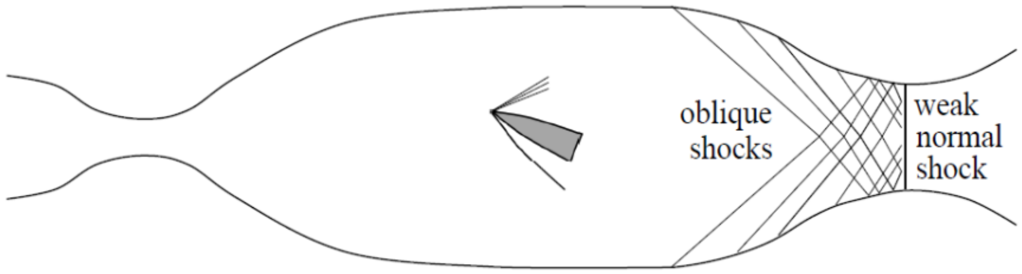
Flow through supersonic wind tunnels

- In a closed-circuit, continuous-flow supersonic wind tunnel, dissipative effects lead to stagnation-pressure losses, requiring a continual power input to maintain the pressure difference across the nozzle.
- It is highly undesirable to maintain supersonic flow for any longer than necessary, so we typically desire to decelerate the flow to subsonic conditions downstream of the test section.
- One option to achieve this is through a normal shock:



However, the shock is unstable (and sensitive to model introduction), and pressure losses are large.

- A better option is a supersonic diffuser, which is less sensitive and (generally) has lower pressure losses:



Supersonic diffusers

Consider the flow then through a supersonic diffuser.

If there were no losses in the test section, the flow would be isentropic and the diffuser throat area would be the same as that of the supersonic nozzle.

In a real tunnel, however, the stagnation pressure at the second throat will be lower than at the first. For a steady flow, however, the mass flux will be the same and we thus have

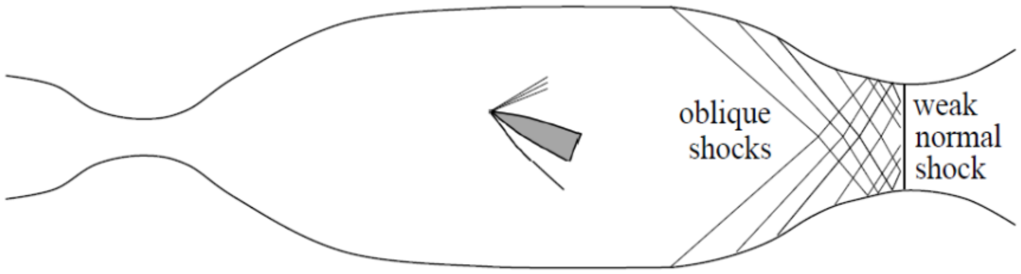
$$\rho_1^* a_1^* A_{t1} = \rho_2^* a_2^* A_{t2}.$$

We can to good approximation assume the flow to be adiabatic, so $T_2^* = T_1^*$, $a_2^* = a_1^*$, and also

$$\frac{\rho_1^*}{\rho_2^*} = \frac{p_1^*}{T_1^*}\frac{T_2^*}{\rho_2^*} = \frac{p_1^*}{p_2^*}.$$

Thus, from above

$$\frac{A_{t2}}{A_{t1}} = \frac{p_1^*}{p_2^*}.$$



Note, however, that

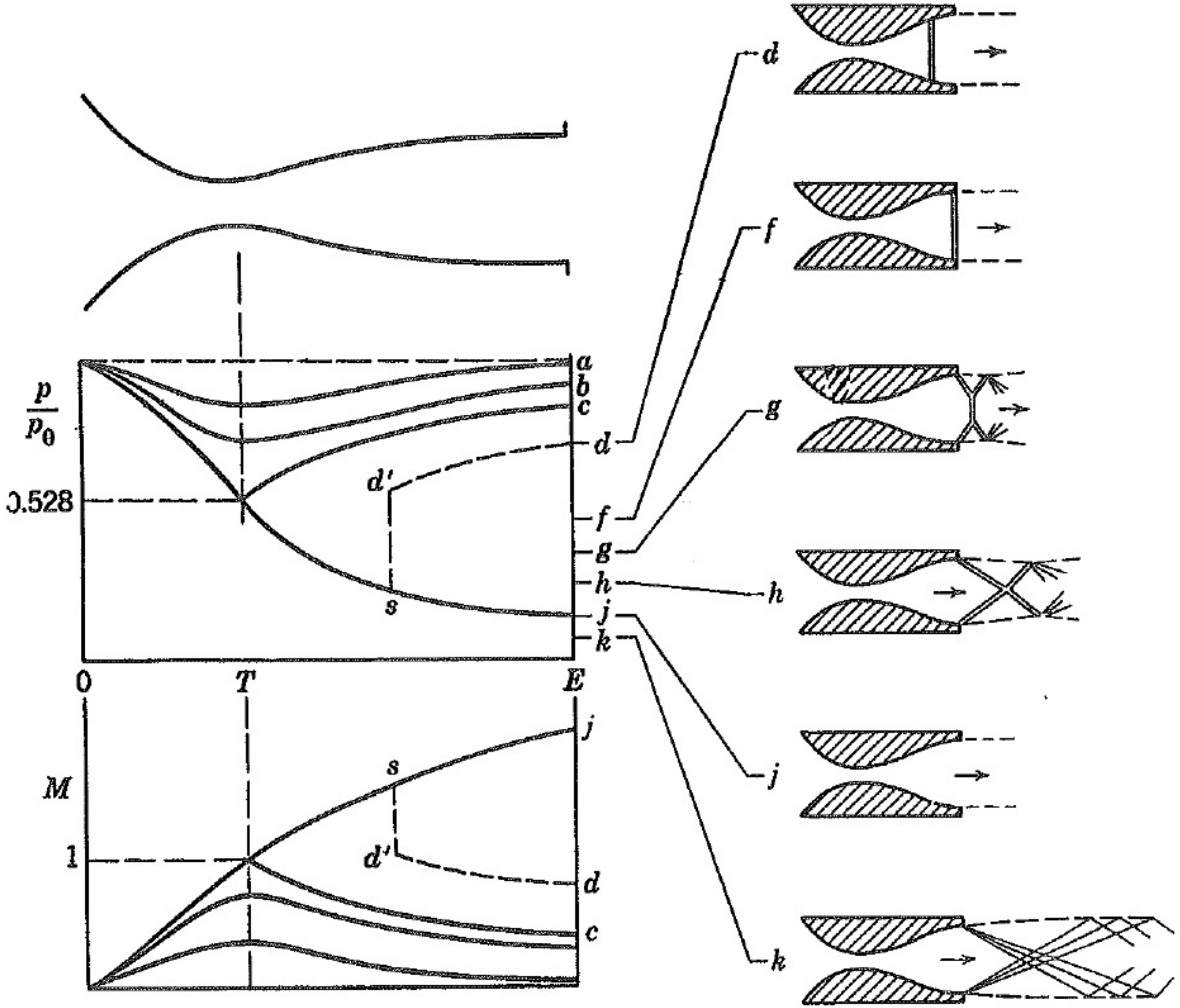
$$p^* = p_0 \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma+1)}$$

We can thus conclude

$$\frac{A_{t2}}{A_{t1}} = \frac{p_{01}}{p_{02}}.$$

Since $p_{02} < p_{01}$, we must therefore have that $A_{t2} > A_{t1}$.

Review Lecture 19

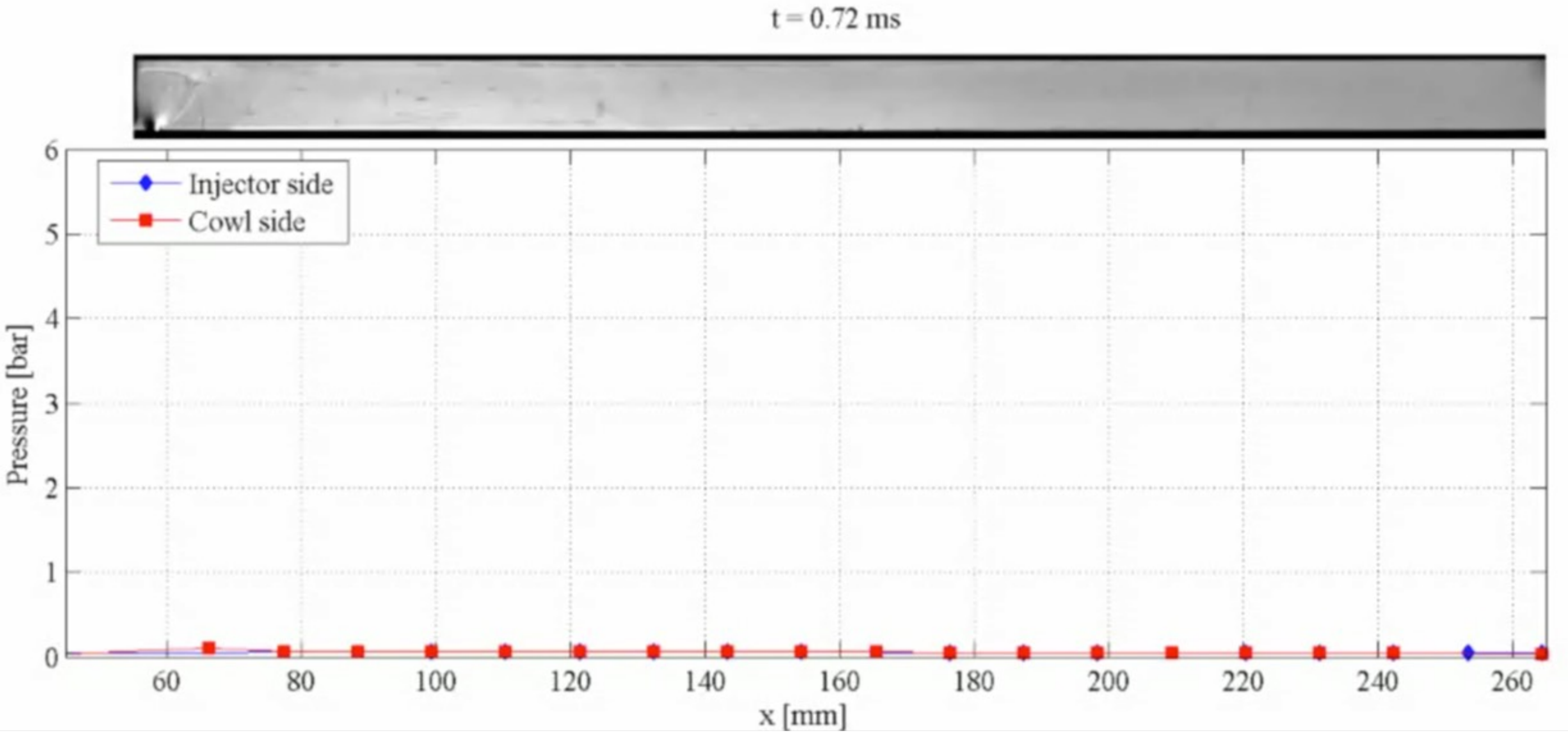


Lecture 20: Rayleigh Flow

ENAE311H Aerodynamics I

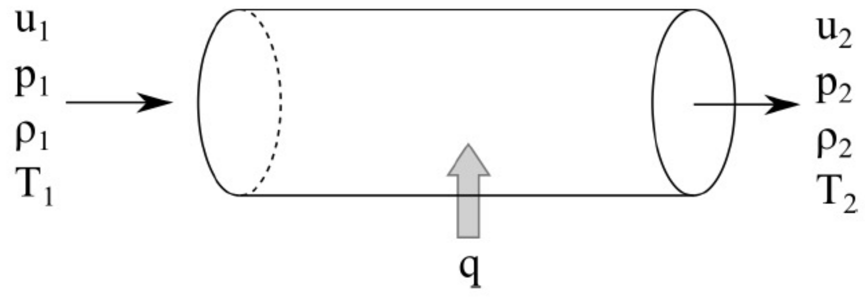
Stuart Laurence

Unstart inside a scramjet combustor



Governing equations

Consider the steady, one-dimensional flow inside a constant-area duct with no frictional effects at the walls. We assume that heat is added or subtracted in an idealized fashion that otherwise leaves the nature of the fluid (e.g., composition) unchanged. We additionally assume a perfect gas.



Using a simple control volume around the pipe, the continuity and momentum equations are:

$$\text{Mass:} \qquad \rho_1 u_1 = \rho_2 u_2$$
$$\text{Momentum:} \qquad p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

(note that these are the same as for a normal shock).

For the energy, we can use our simple CV equation from earlier, i.e.,

$$\dot{m}\left[h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) + g(y_2 - y_1)\right] = \dot{Q} + \dot{W}_s.$$

Here, this becomes

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2},$$

with $q = \dot{Q}/\dot{m}$.

This can be written as

$$h_{01} + q = h_{02},$$

which, for a perfect gas is

$$c_p T_{01} + q = c_p T_{02},$$

or

$$\boxed{T_{02} = T_{01} + \frac{q}{c_p}.}$$

Heat addition produces an increase in stagnation temperature

Flow variables

We wish to relate flow variables at the downstream station to those upstream, given the heat addition or subtraction. The Mach number is again a convenient means to do this.

We first note

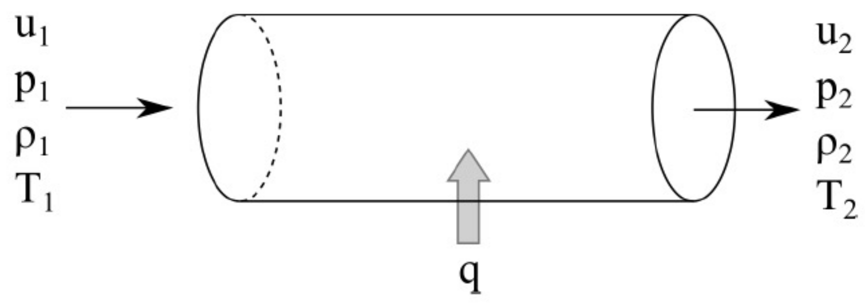
$$\frac{M_2}{M_1} = \frac{u_2}{u_1}\frac{a_1}{a_2} = \frac{u_2}{u_1}\sqrt{\frac{T_1}{T_2}}.$$

Now, since $\rho u^2 = \gamma p M^2$ for a perfect gas, we can write the momentum equation as

$$p_1 + \gamma p_1 M_1^2 = p_2 + \gamma p_2 M_2^2$$

which can be rearranged as

$$\boxed{\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}}$$



For the temperature ratio, we have

$$\begin{aligned}\frac{T_2}{T_1} &= \frac{\rho_1}{\rho_2}\frac{1+\gamma M_1^2}{1+\gamma M_2^2} \\ &= \frac{u_2}{u_1}\frac{1+\gamma M_1^2}{1+\gamma M_2^2}.\end{aligned}$$

Substituting our expression for u_2/u_1 from left:

$$\frac{T_2}{T_1} = \sqrt{\frac{T_2}{T_1}}\frac{M_2}{M_1}\frac{1+\gamma M_1^2}{1+\gamma M_2^2},$$

And so

$$\boxed{\frac{T_2}{T_1} = \left(\frac{M_2}{M_1}\right)^2\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2.}$$

Flow variables

For the density, we can use the ideal gas equation, $\rho = p/(RT)$, together with our equations for pressure and temperature, to obtain

$$\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{M_1}{M_2}\right)^2 \frac{1+\gamma M_2^2}{1+\gamma M_1^2} = \frac{u_1}{u_2}.}$$

Now, for the ratios of stagnation properties, we can use, for example,

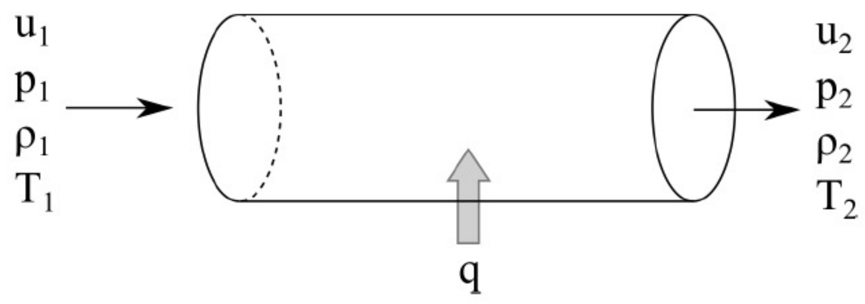
$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

to write

$$\boxed{\frac{p_{02}}{p_{01}} = \frac{p_2}{p_1}\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right)^{\gamma/(\gamma-1)}},$$

and

$$\boxed{\frac{T_{02}}{T_{01}} = \frac{T_2}{T_1}\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}.}$$



But remember that we also have another expression relating T_{01} and T_{02}, i.e.,

$$T_{02} = T_{01} + \frac{q}{c_p}.$$

Thus, given conditions at 1 and q, we can solve this equation for T_{02}, which allows us to solve (implicitly) for M_2 using the equation to the left.

The remaining flow conditions at 2 then follow immediately from the relations we have just derived.

Entropy changes in Rayleigh flow

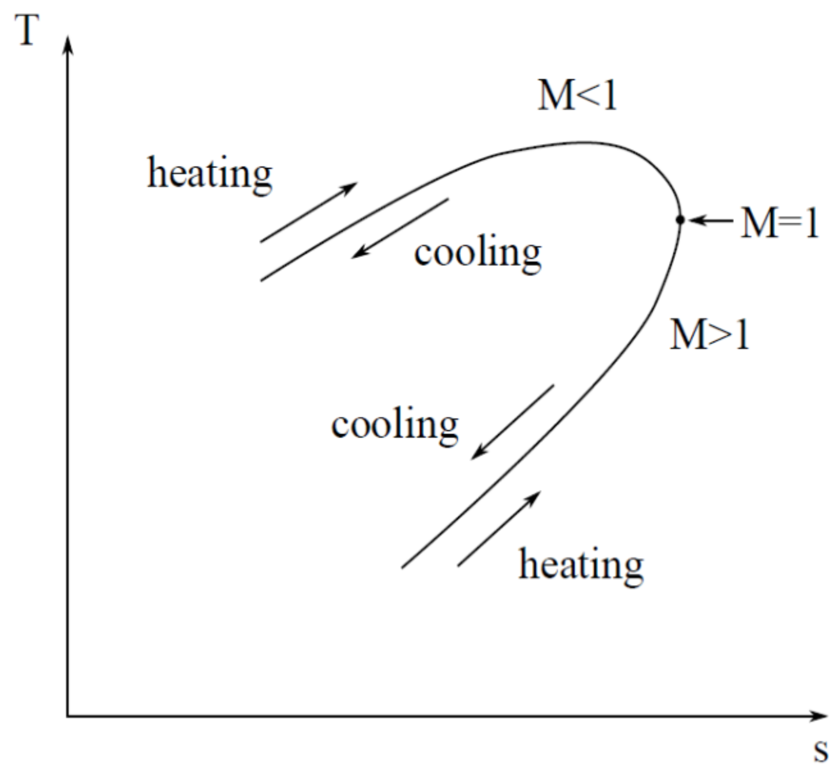
The change in entropy can be written as

$$\frac{s_2 - s_1}{c_p} = \ln \frac{T_2}{T_1} - \ln \left[\left(\frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma} \right].$$

If we plot temperature versus entropy on a $T - s$ diagram, we see that the maximum entropy point occurs when the flow is sonic, i.e., $M = 1$.

Whether the flow is initially subsonic or supersonic, heating will drive the Mach number towards unity (increasing the entropy), while cooling will shift it away from sonic conditions.

Behavior of other flow variables depends on whether the flow is subsonic or supersonic.



Sonic-referenced conditions

Conditions at the sonic point provide convenient reference values for normalization of the flow variables. Setting $M = 1$ in our earlier expressions, we have

$$
\begin{aligned}
\frac{p}{p^*} &= \frac{\gamma+1}{1+\gamma M^2} \\
\frac{T}{T^*} &= \left(\frac{(\gamma+1)M}{1+\gamma M^2}\right)^2 \\
\frac{\rho}{\rho^*} &= \frac{1+\gamma M^2}{(\gamma+1)M^2} = \frac{u^*}{u} \\
\frac{p_0}{p_0^*} &= \frac{\gamma+1}{1+\gamma M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\gamma/(\gamma-1)} \\
\frac{T_0}{T_0^*} &= \frac{2(\gamma+1)M^2}{(1+\gamma M^2)^2}\left(1+\frac{\gamma-1}{2}M^2\right) \\
\frac{s-s^*}{c_p} &= \ln\left[M^2\left(\frac{\gamma+1}{1+\gamma M^2}\right)^{(\gamma+1)/\gamma}\right].
\end{aligned}
$$

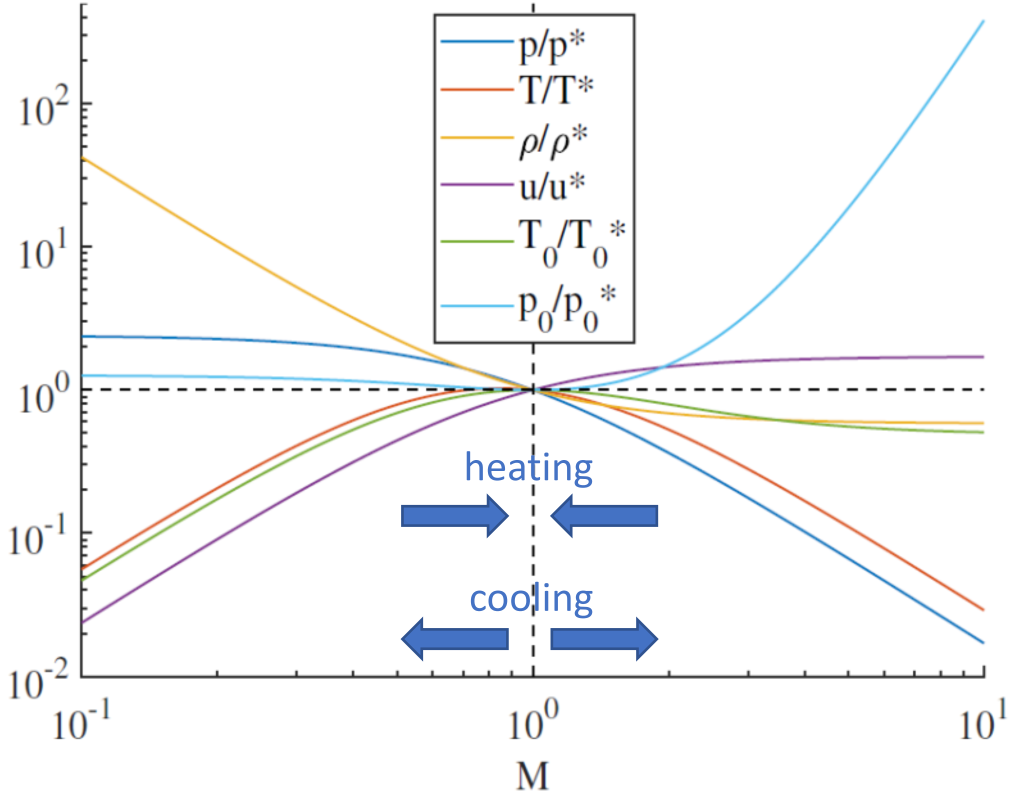
Since the sonic point is constant in Rayleigh flow, conditions at two other points can then be related as, e.g.,

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1}.$$

Behavior of flow variables

The qualitative behavior for positive heat addition ($q > 0$) in Rayleigh flow can be summarized as follows:

	$M < 1$	$M > 1$
T_0	increases	increases
M	increases	decreases
T	increases for $M < 1/\sqrt{\gamma}$ decreases for $M > 1/\sqrt{\gamma}$	increases
p	decreases	increases
ρ	decreases	increases
p_0	decreases	decreases
u	increases	decreases



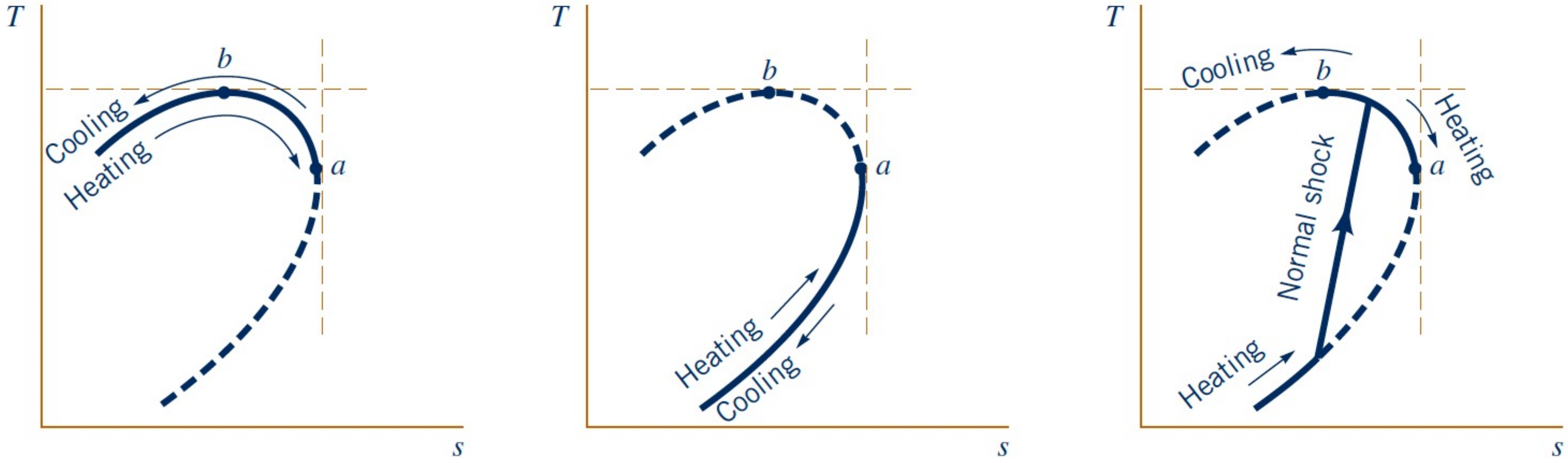
Trends for cooling ($q < 0$) are reversed.

Choking in Rayleigh flow

Note that if the flow reaches $M = 1$ from either above or below, no more heat can be added to the flow while maintaining a steady flow

→ we say the flow is *thermally choked*.

If we attempt to add more heat beyond this point, the flow responds by sending disturbances upstream to modify the inlet flow.



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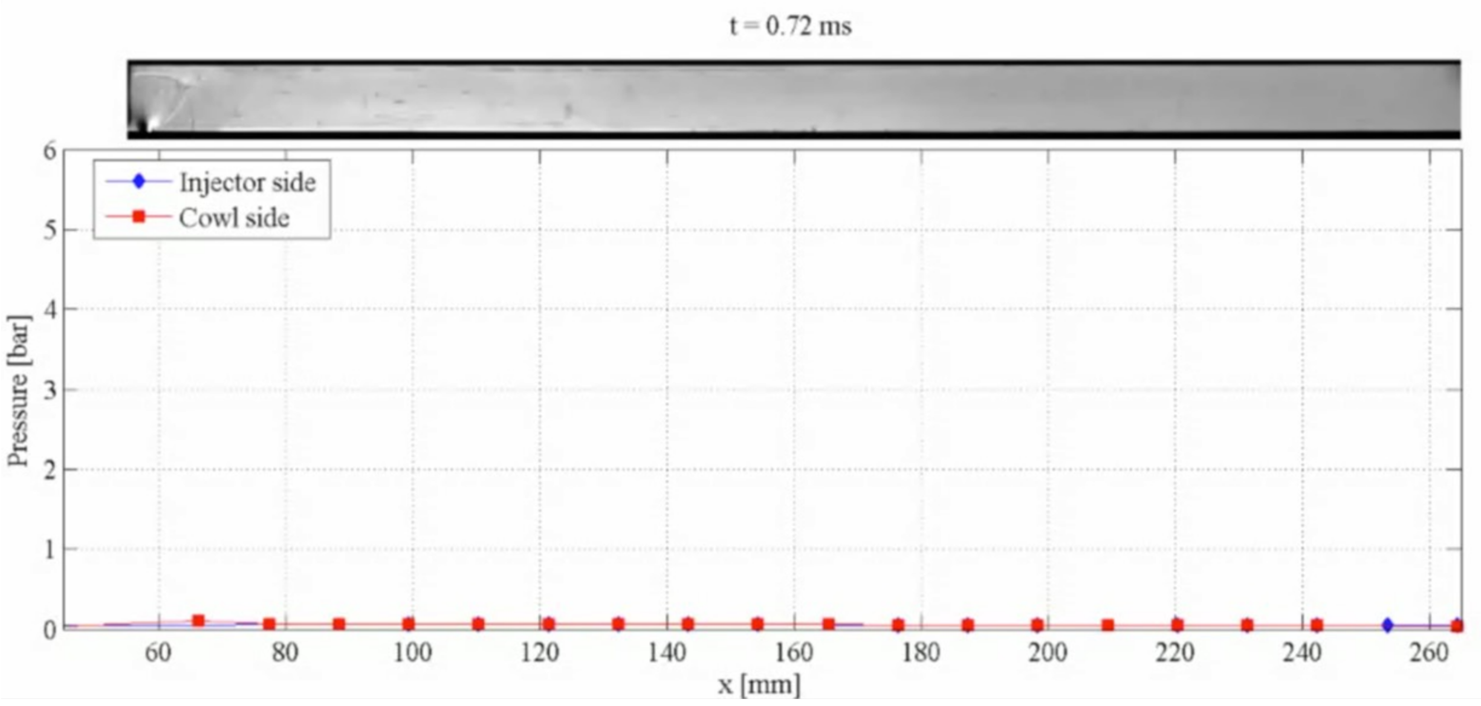
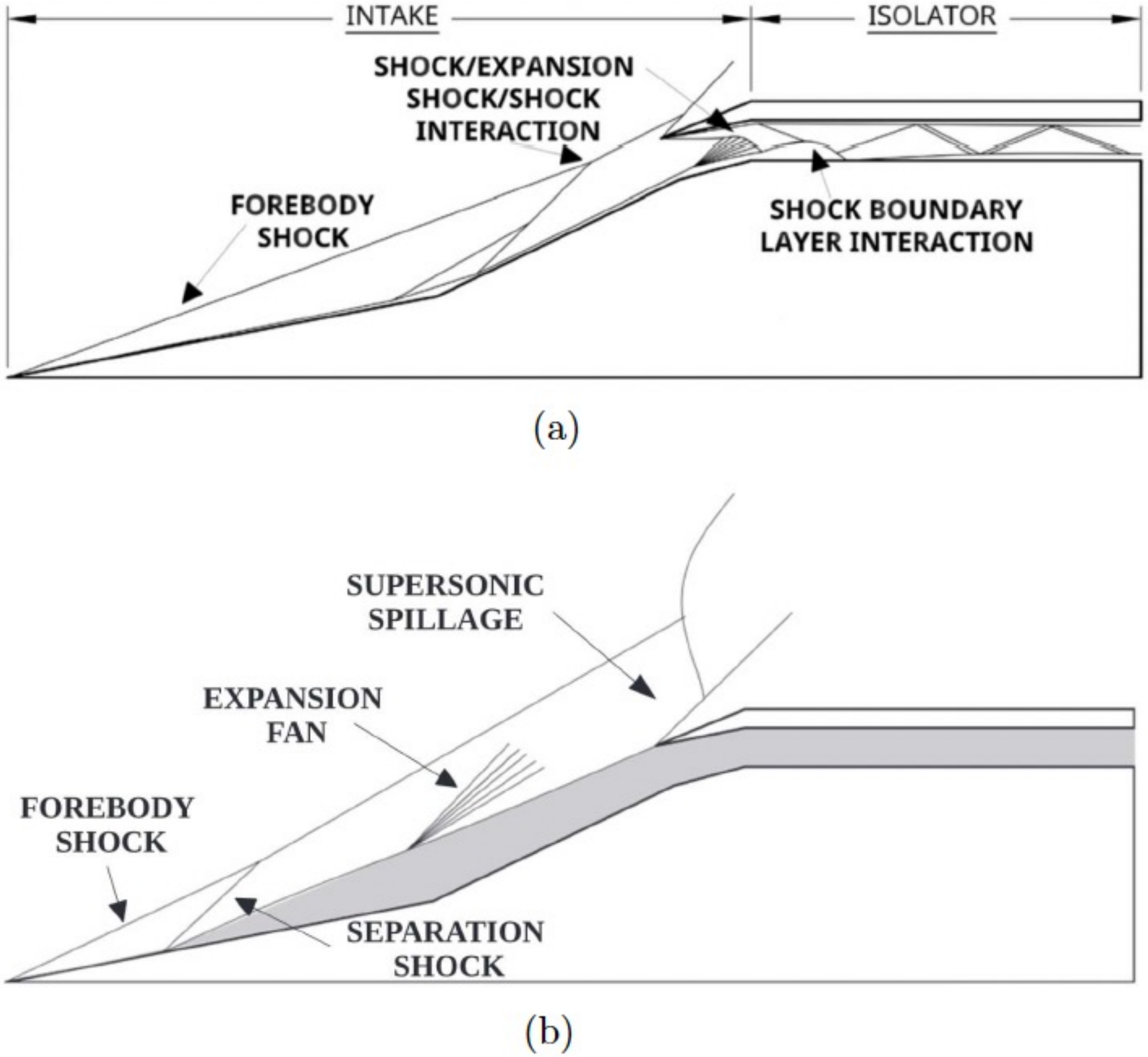
If the flow is supersonic, these disturbances must be shock waves (or a "shock train") to move upstream.





(a) Diagram of a scramjet at nominal conditions and (b) scramjet at unstarted conditions.

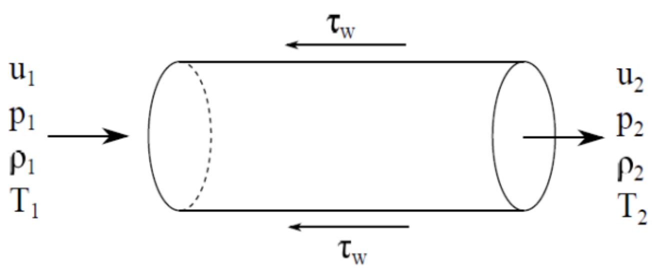
Lecture 21: Fanno Flow

ENAE311H Aerodynamics I

Christoph Brehm

Governing equations – mass and energy

Now consider the one-dimensional flow inside a constant-area duct, but instead of heat addition or subtraction, we have a nonzero shear stress acting at the walls. We again assume this process to be somewhat idealized in that the frictional effects are experienced immediately across the duct area at any location downstream.



The conservation of mass and energy equations can be written

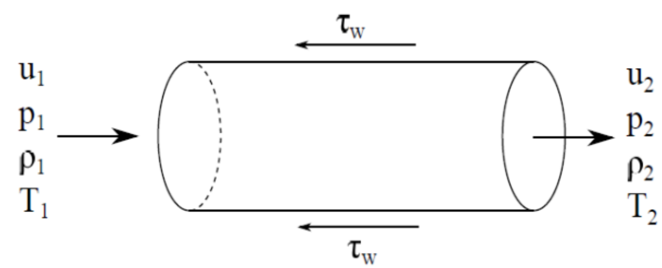
$$\begin{aligned} \text{Mass:} \quad & \rho_1 u_1 = \rho_2 u_2 \\ \text{Energy:} \quad & h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}, \end{aligned}$$

or simply, $\rho u = \text{const.}$, and $T_0 = \text{const.}$

Note that these are again the same equations as for a normal shock.

Conservation of momentum

If we draw a control volume around the geometry to the right and apply conservation of momentum, we have



$$\rho_2 u_2^2 A - \rho_1 u_1^2 A = p_1 A - p_2 A - \iint_{wall} \tau_w dA$$

Using continuity, we can write this as

$$\rho_1 u_1 (u_2 - u_1) = p_1 - p_2 - \frac{1}{A} \iint_{wall} \tau_w dA$$

If we now let the distance between 1 and 2 become small, we can write this in differential form:

$$\rho u \, du = -dp - \frac{1}{A} \tau_w dA_w,$$

where dA_w is the wetted area of the duct between 1 and 2.

Now we introduce the following definitions

friction coefficient

$$D \equiv \frac{4A}{dA_w/dx} = \frac{4A}{dAw} dx \qquad \text{and} \qquad f \equiv \frac{2\tau_w}{\rho u^2}.$$

hydraulic diameter

Conservation of momentum

Our momentum equation can then be written

$$\rho u^2 \frac{du}{u} + dp = -4f \frac{\rho u^2}{2} \frac{dx}{D}.$$

Dividing through by p and using $\rho u^2 = \gamma p M^2$, we can write this as

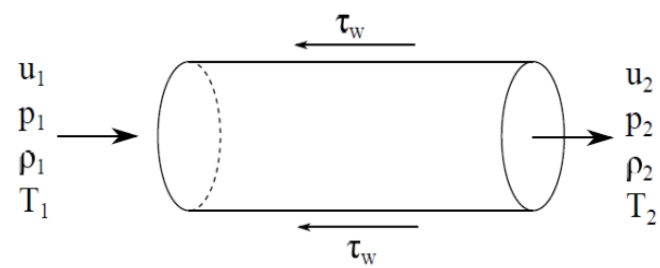
$$\gamma M^2 \frac{du}{u} + \frac{dp}{p} = -4f \frac{\gamma M^2}{2} \frac{dx}{D}.$$

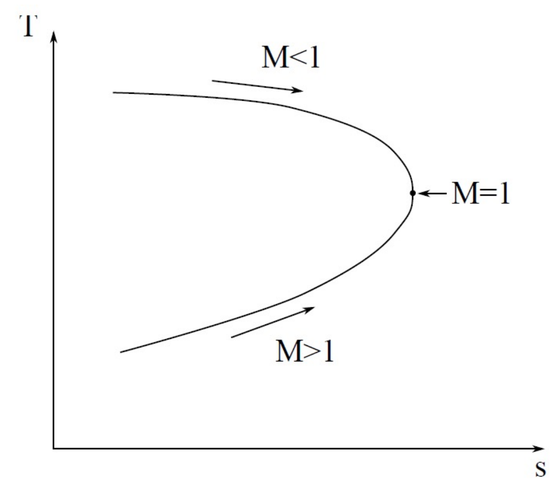
After much manipulation and use of the other conservation equations, we arrive at

$$4f \frac{dx}{D} = \frac{1}{\gamma M^2} \frac{1 - M^2}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM^2}{M^2}.$$

From this equation we see that friction always acts to push the flow towards sonic conditions (whether it is initially subsonic or supersonic).

This we also see in the $T - s$ diagram for Fanno flow, which also shows that the sonic point is the point of maximum entropy.





Frictional choking

Since the sonic point is the maximum-entropy point, no solution can exist beyond it. If it is reached, we say the flow is *frictionally choked*.

Therefore, given initial conditions and a friction coefficient, f, there is a maximum length the duct can have before the sonic point is reached. This can be determined by integrating

$$\int_0^{L_{max}} 4f\frac{dx}{D} = \int_{M^2}^{1} \frac{1}{\gamma M^4}\frac{1-M^2}{1+\frac{\gamma-1}{2}M^2}dM^2$$

to give

$$\boxed{4\overline{f}\frac{L_{max}}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)},} \qquad \text{with } \overline{f} = \frac{1}{L_{max}}\int_0^{L_{max}} f\,dx.$$

If no better estimate of f is available, e.g., from a Moody chart, a reasonable value for a smooth pipe is $f = 0.0025$. With this value and $\gamma = 1.4$, we calculate:

M	0	0.25	0.5	0.75	1	1.5	2	3	∞
L_{max}/D	∞	850	110	12	0	14	31	52	82

So frictional effects are particularly severe for supersonic flows.

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We can also use this equation to solve for the Mach number at a second point down the duct with our frictional flow. In particular,

$$\boxed{\frac{4\overline{f}}{D}\Delta x = -\frac{4\overline{f}}{D}(L_{max,2} - L_{max,1}) = g(M_1) - g(M_2),}$$

with

$$g(M) = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)}$$

Flow conditions in Fanno flow

Assuming we now know the Mach number at some downstream station (together with upstream conditions) we can determine the remaining downstream flow properties as follows.

We know that the total temperature is constant, so can write

$$\boxed{\frac{T_2}{T_1} = \frac{T_2}{T_0}\frac{T_0}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}.}$$

Now, using continuity, we have

$$\frac{p_1}{RT_1}u_1 = \frac{p_2}{RT_2}u_2.$$

which we can write as

$$\sqrt{\frac{\gamma}{RT_1}}p_1\frac{u_1}{\sqrt{\gamma RT_1}} = \sqrt{\frac{\gamma}{RT_2}}p_2\frac{u_2}{\sqrt{\gamma RT_2}},$$

and rearrange to give

$$\frac{p_2}{p_1} = \sqrt{\frac{T_2}{T_1}}\frac{M_1}{M_2}.$$

Substituting in our temperature ratio we have

$$\boxed{\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{1/2}.}$$

Again using the ideal gas equation, we can derive the density ratio from the pressure and temperature:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right)^{1/2}.}$$

Finally, for the stagnation-pressure ratio:

$$\boxed{\frac{p_{02}}{p_{01}} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}.}$$

Sonic-referenced conditions

As with Rayleigh flow, it is customary to take the sonic conditions as reference flow values for normalization, in which case we have:

$$
\begin{aligned}
\frac{p}{p^*} &= \frac{1}{M}\left(\frac{\gamma+1}{2+(\gamma-1)M^2}\right)^{1/2} \\
\frac{\rho}{\rho^*} &= \frac{1}{M}\left(\frac{2+(\gamma-1)M^2}{\gamma+1}\right)^{1/2} = \frac{u^*}{u} \\
\frac{T}{T^*} &= \frac{\gamma+1}{2+(\gamma-1)M^2} = \frac{a^2}{a^{*2}} \\
\frac{p_0}{p_0^*} &= \frac{1}{M}\left(\frac{2+(\gamma-1)M^2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}.
\end{aligned}
$$

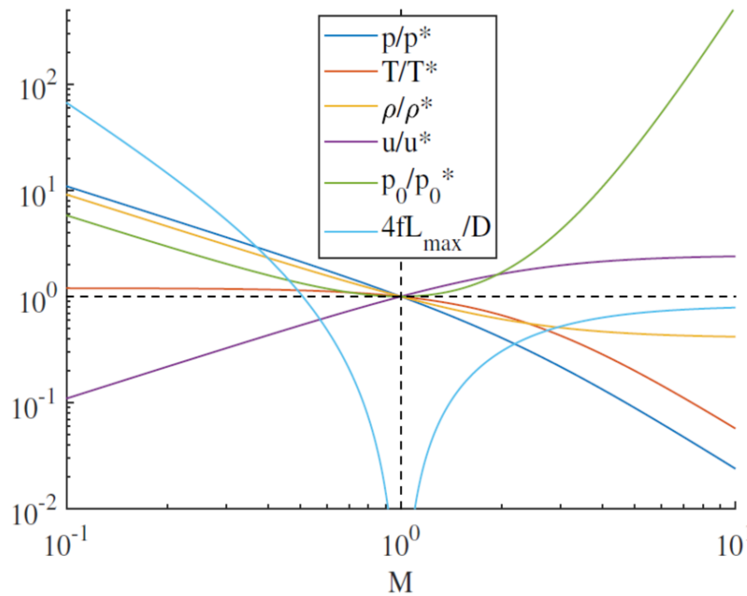
We can then again relate conditions at any two points in the flow using, for example,

$$
\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1}.
$$

Behavior or flow variables in Fanno flow

The trends in the flow properties for increasing distance downstream can be summarized as follows:

	$M < 1$	$M > 1$
M	increases	decreases
T	decreases	increases
p	decreases	increases
ρ	decreases	increases
p_0	decreases	decreases
u	increases !	decreases



	Flow	
Parameter	**Subsonic Flow**	**Supersonic Flow**
Stagnation temperature	Constant	Constant
Ma	Increases (maximum is 1)	Decreases (minimum is 1)
Friction	Accelerates flow	Decelerates flow
Pressure	Decreases	Increases
Temperature	Decreases	Increases

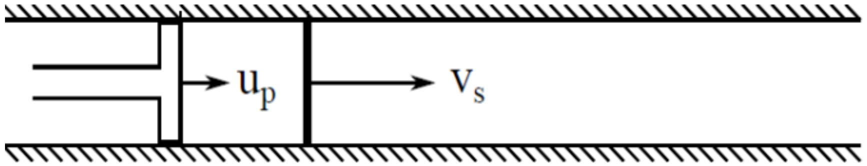
Lecture 22: Introduction to Unsteady Gas Dynamics

ENAE311H Aerodynamics I

Christoph Brehm

Propagating shock waves

Consider a constant area duct, with a piston that is impulsively started to u_p at $t = 0$ (with zero velocity before that).



We note the following:

- Since the flow won't have time to respond in a smooth fashion, a shock wave must propagate ahead of the piston at a speed $v_s > u_p$.
- The flow conditions behind the shock are uniform, so the fluid velocity must be equal to the fluid velocity, u_p.
- The shock speed will be precisely that required to accelerate the flow to u_p, according to the normal-shock relations.

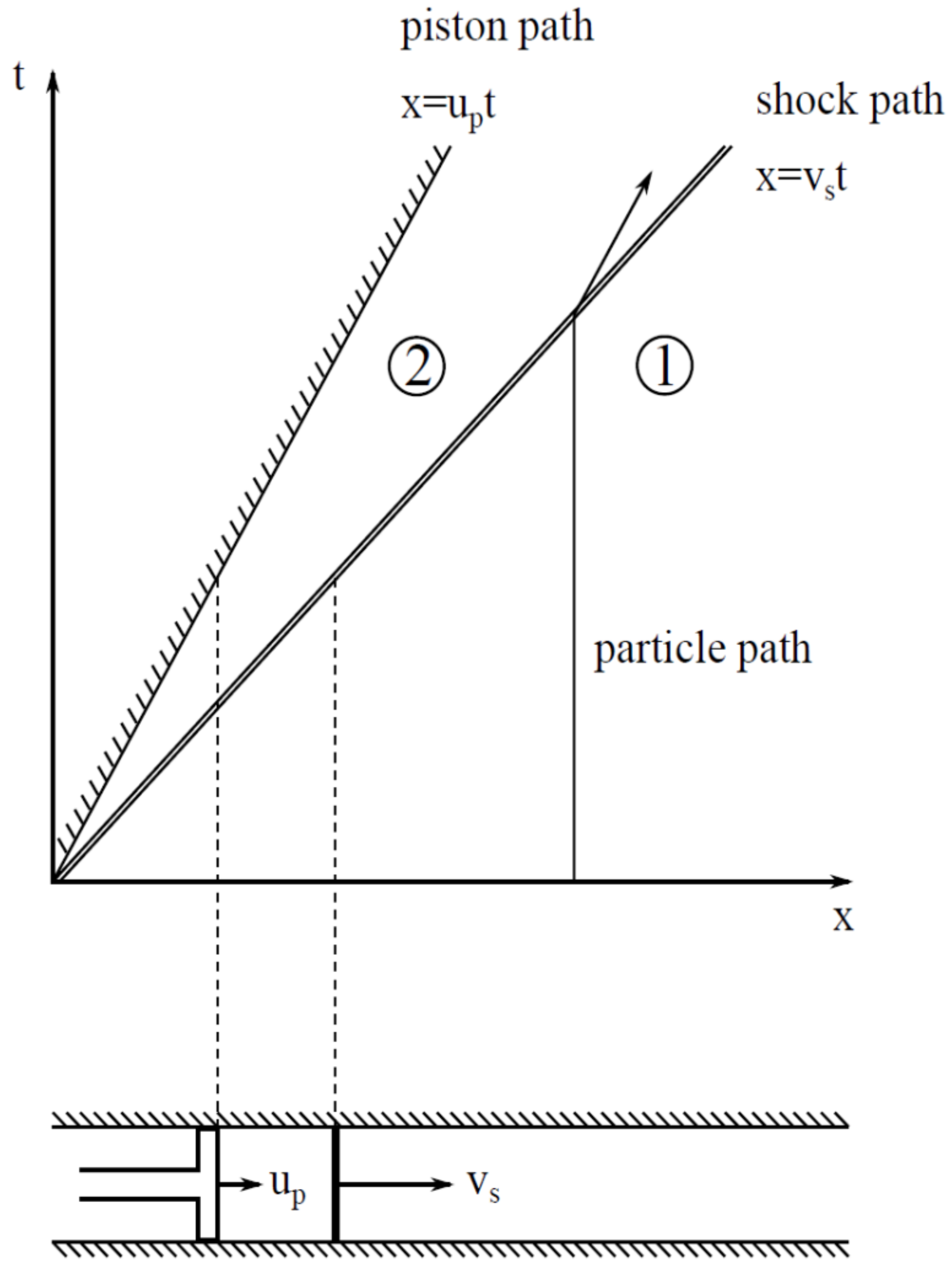
Propagating shock waves

Consider a constant area duct, with a piston that is impulsively started to u_p at $t = 0$ (with zero velocity before that).

We note the following:

- A useful way to visualize such unsteady, one-dimensional flow phenomena is through an x-t diagram (note the similarity to the 2-D flow over a compression corner).
- Knowing the conditions in region 1 and the shock speed, v_s, all conditions in region 2 can be determined using the normal – shock relations. One further expression that will be useful relates the pressure jump to the piston speed:

$$u_p = a_1 \left(\frac{p_2}{p_1} - 1\right) \left(\frac{2/\gamma}{(\gamma+1)\frac{p_2}{p_1} + \gamma - 1}\right)^{1/2}.$$



Propagating shock waves

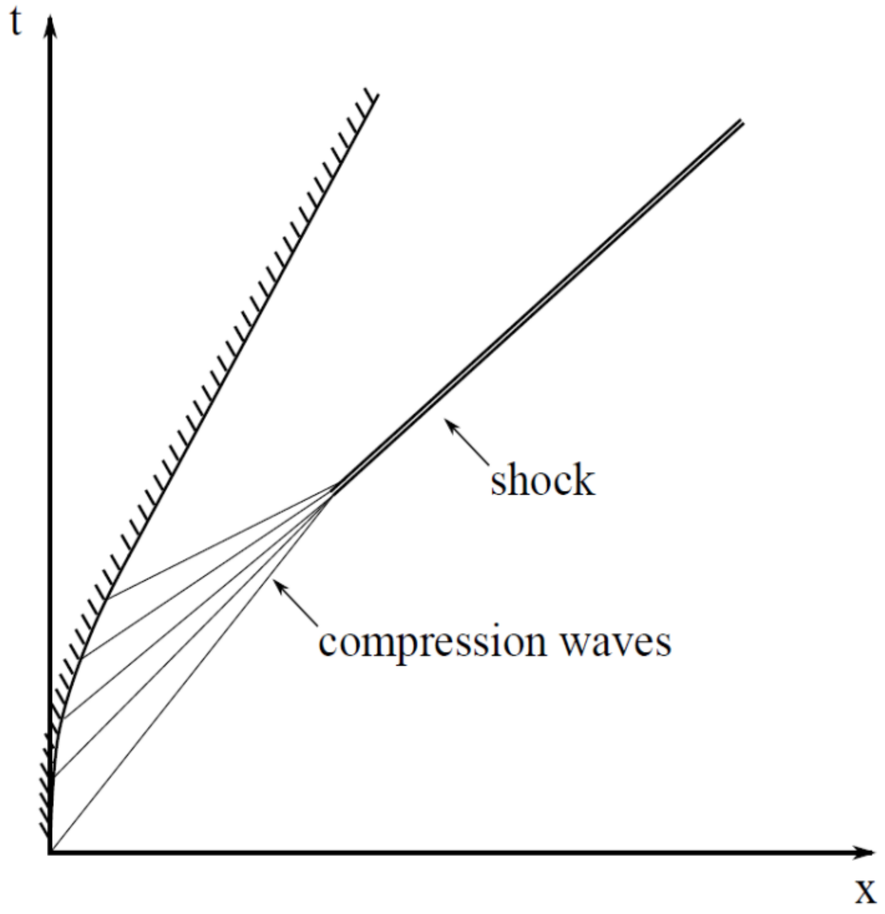
Consider a constant area duct, with a piston that is impulsively started to u_p at $t = 0$ (with zero velocity before that).

We note the following:

- A useful way to visualize such unsteady, one-dimensional flow phenomena is through an x-t diagram (note the similarity to the 2-D flow over a compression corner).
- Knowing the conditions in region 1 and the shock speed, v_s, all conditions in region 2 can be determined using the normal – shock relations. One further expression that will be useful relates the pressure jump to the piston speed:

$$u_p = a_1 \left(\frac{p_2}{p_1} - 1\right) \left(\frac{2/\gamma}{(\gamma+1)\frac{p_2}{p_1} + \gamma - 1}\right)^{1/2}.$$

- Even if the piston acceleration is gradual, the compression waves will eventually coalesce to form a shock wave.



Propagating expansion waves

Now imagine that the piston is impulsively withdrawn away from the fluid at speed $|u_p|$, rather than accelerated into it.

We now note the following:

- An expansion wave must propagate into the fluid.

Propagating expansion waves

Now imagine that the piston is impulsively withdrawn away from the fluid at speed $|u_p|$, rather than accelerated into it.

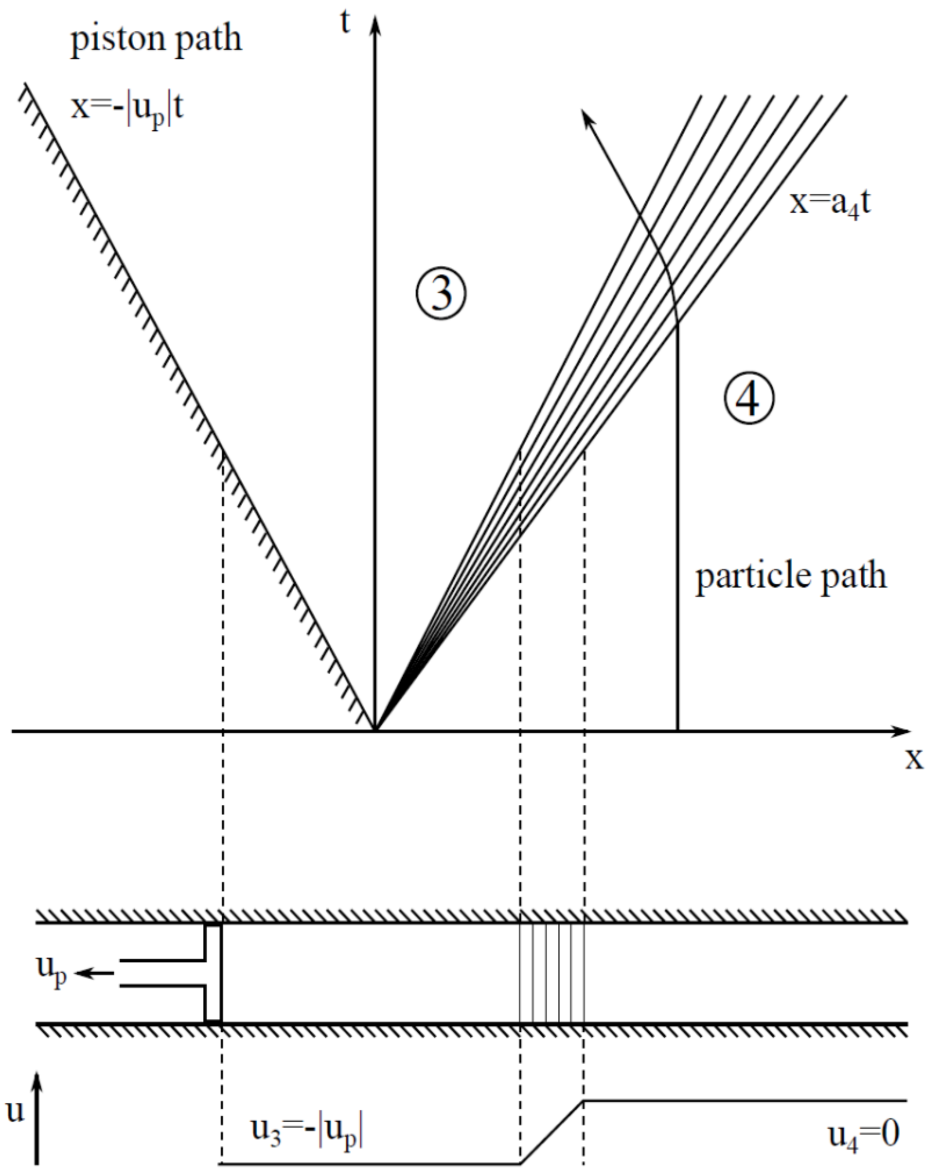
We now note the following:

- An expansion wave must propagate into the fluid.
- Since a finite expansion wave is forbidden by the second law of thermodynamics, we must have a gradual (isentropic) expansion through a centered expansion fan. This can again be represented in an x-t diagram.
- The leading wave will have a propagation speed of a_4. For all other waves, the propagation speed (in the lab frame) is

$$c = a_4 + \frac{\gamma + 1}{2} u,$$

where u is the local fluid velocity. Since the final velocity matches the piston speed, the terminal wave propagates at

$$c_{terminal} = a_4 - \frac{\gamma + 1}{2} |u_p|.$$



Propagating expansion waves

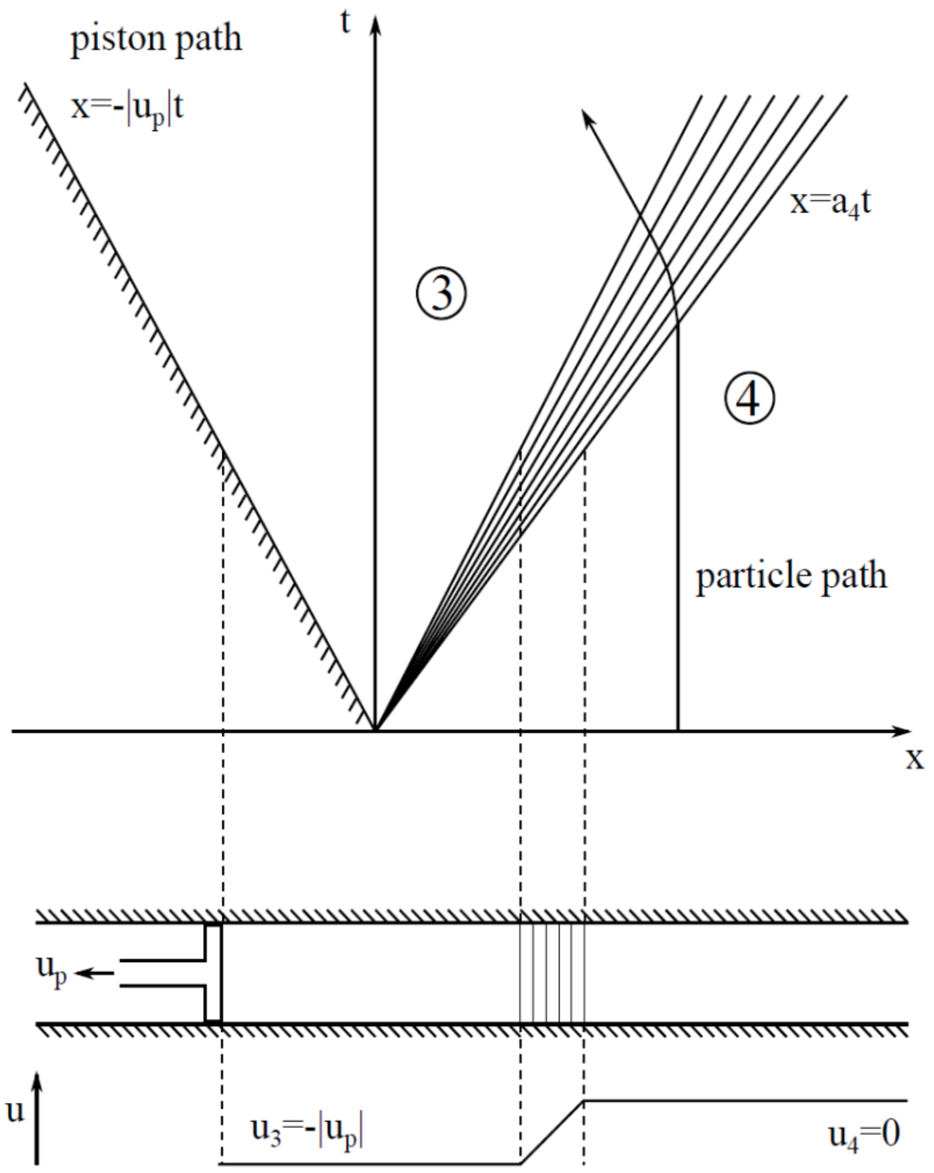
Now imagine that the piston is impulsively withdrawn away from the fluid at speed $|u_p|$, rather than accelerated into it.

We now note the following:

- The flow velocity decreases linearly through the expansion fan (and of course matches the piston speed at the trailing end).
- The strength of the expansion can be characterized in terms of the pre- and post-expansion pressures:

$$\frac{p_3}{p_4} = \left(1 - \frac{\gamma-1}{2}\frac{|u_p|}{a_4}\right)^{2\gamma/(\gamma-1)}.$$

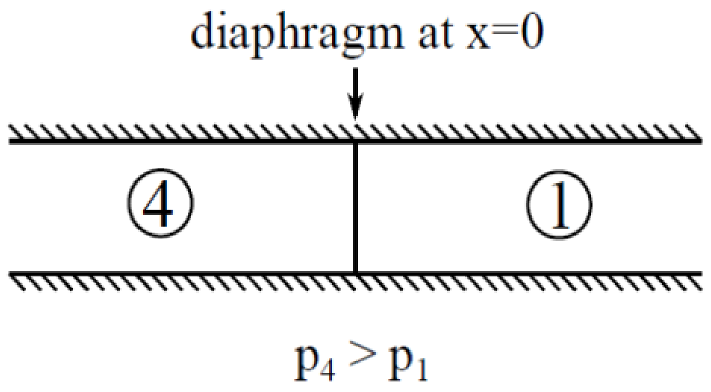
- Note that this entails a maximum velocity that the flow can achieve through an unsteady expansion.



Shock tubes

Now consider a constant area duct with high- and low-pressure regions (possibly of different compositions) separated by a diaphragm.

If the diaphragm is suddenly burst, the unsupported pressure difference will cause the fluid interface (also referred to as the *contact surface*) to start propagating towards the low-pressure region.



Shock tubes

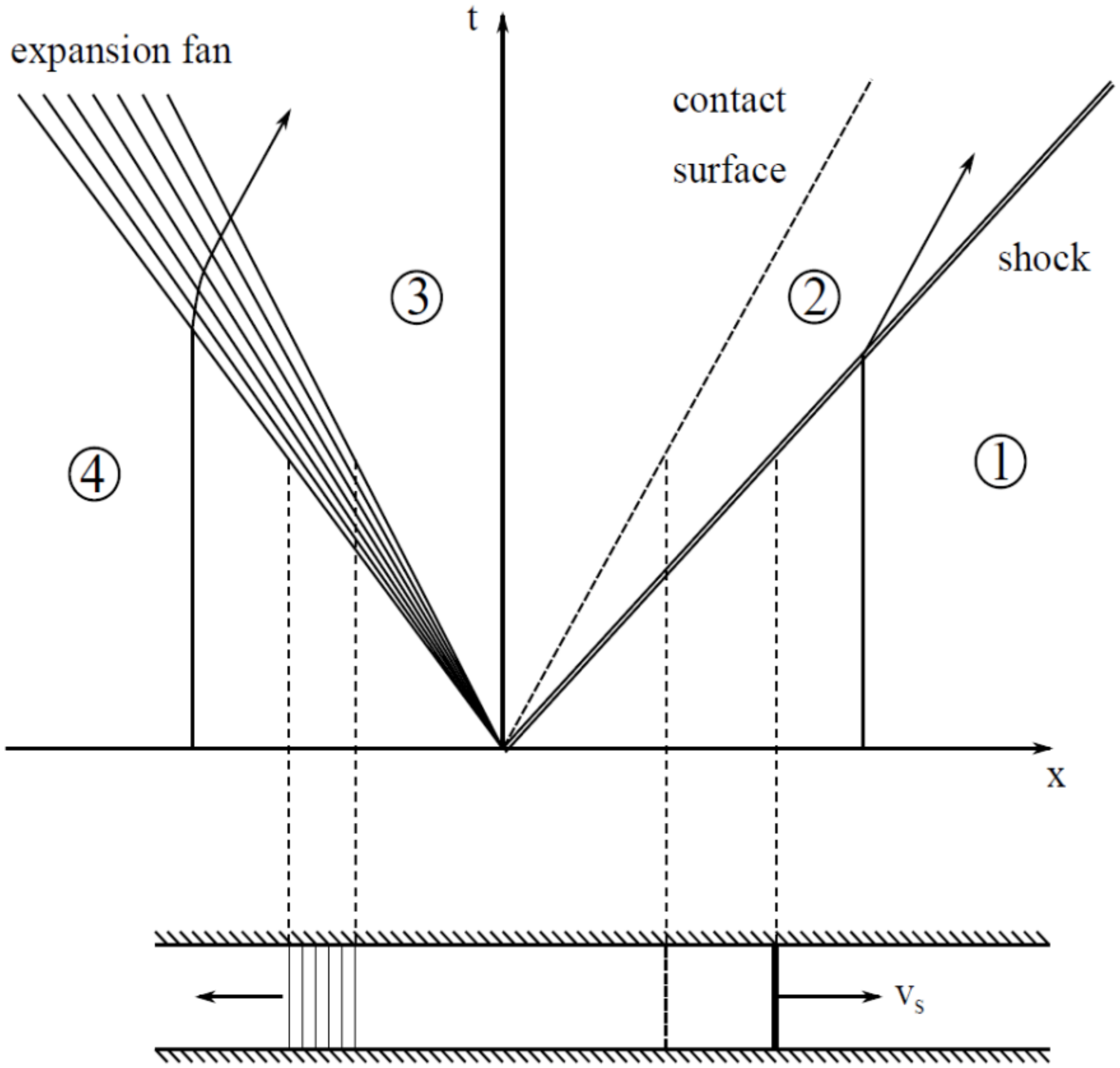
Now consider a constant area duct with high- and low-pressure regions (possibly of different compositions) separated by a diaphragm.

If the diaphragm is suddenly burst, the unsupported pressure difference will cause the fluid interface (also referred to as the *contact surface*) to start propagating towards the low-pressure region.

The contact surface will act as a fluid piston, causing a shock wave to propagate into the low-pressure region and an expansion wave into the high-pressure region. This can again be represented on an x-t diagram.

A contact surface is the 1-D, unsteady equivalent of a shear layer; the conditions across it are

$$\begin{aligned} p_2 &= p_3 \\ u_2 &= u_3 (= u_{cs}). \end{aligned}$$



Shock tubes

If we treat the contact surface as an equivalent fluid piston, we can rewrite our earlier equations for the propagating shock and expansion waves as:

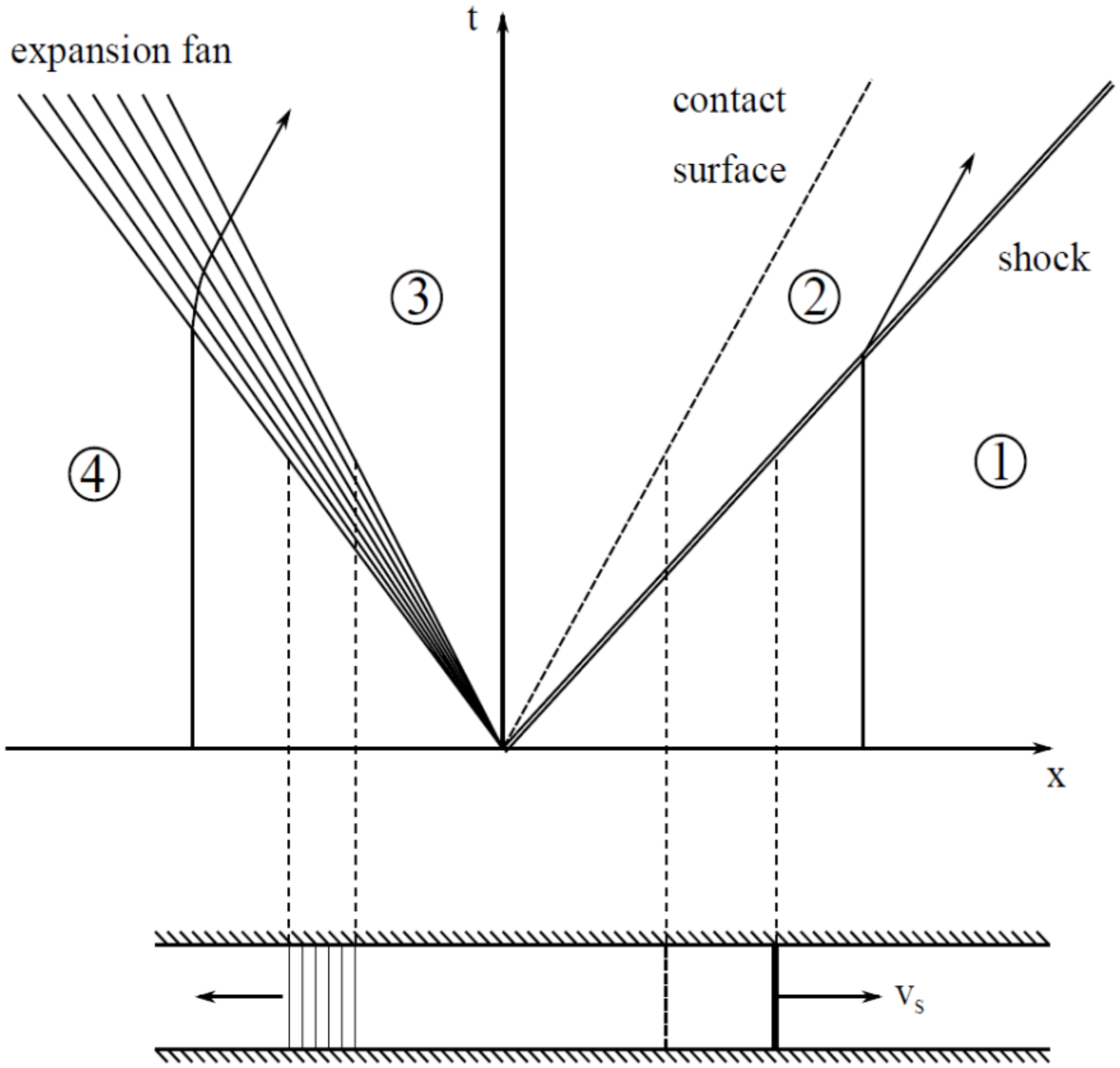
$$u_2 = a_1\left(\frac{p_2}{p_1}-1\right)\left(\frac{2/\gamma_1}{(\gamma_1+1)\frac{p_2}{p_1}+\gamma_1-1}\right)^{1/2}$$

$$u_3 = \frac{2a_4}{\gamma_4+1}\left[1-\left(\frac{p_3}{p_4}\right)^{(\gamma_4-1)/2\gamma_4}\right].$$

Matching pressures and velocities, we can then derive the shock-tube equation:

$$\boxed{\frac{p_4}{p_1} = \frac{p_2}{p_1}\left[1-\frac{(\gamma_4-1)\frac{a_1}{a_4}\left(\frac{p_2}{p_1}-1\right)}{\left\{2\gamma_1\left[2\gamma_1+(\gamma_1+1)\left(\frac{p_2}{p_1}-1\right)\right]\right\}^{1/2}}\right]^{-2\gamma_4/(\gamma_4-1)}.}$$

This gives the shock strength (p_2/p_1) implicitly as a function of the initial pressure ratio and sound speeds.



Shock tubes

If we treat the contact surface as an equivalent fluid piston, we can rewrite our earlier equations for the propagating shock and expansion waves as:

$$\begin{aligned} u_2 &= a_1 \left(\frac{p_2}{p_1} - 1\right) \left(\frac{2/\gamma_1}{(\gamma_1+1)\frac{p_2}{p_1} + \gamma_1 - 1}\right)^{1/2} \\ u_3 &= \frac{2a_4}{\gamma_4+1}\left[1 - \left(\frac{p_3}{p_4}\right)^{(\gamma_4-1)/2\gamma_4}\right]. \end{aligned}$$

Matching pressures and velocities, we can then derive the shock-tube equation:

$$\boxed{\frac{p_4}{p_1} = \frac{p_2}{p_1}\left[1 - \frac{(\gamma_4-1)\frac{a_1}{a_4}\left(\frac{p_2}{p_1}-1\right)}{\left\{2\gamma_1\left[2\gamma_1 + (\gamma_1+1)\left(\frac{p_2}{p_1}-1\right)\right]\right\}^{1/2}}\right]^{-2\gamma_4/(\gamma_4-1)}.}$$

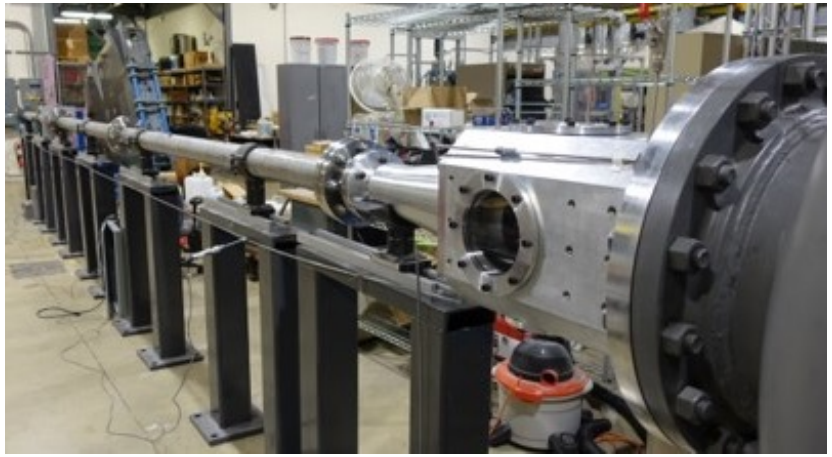
This gives the shock strength (p_2/p_1) implicitly as a function of the initial pressure ratio and sound speeds.

Once we know p_2/p_1, the shock Mach number can be derived using the shock-jump relation

$$M_s = \left[1 + \frac{\gamma_1+1}{2\gamma_1}\left(\frac{p_2}{p_1} - 1\right)\right]^{1/2}.$$

All other properties in region 2 can then be easily derived.

Shock tubes are used for studying, e.g., unsteady flow phenomena and combustion ignition. They are also used to generate high-enthalpy conditions for shock tunnels.



Lecture 23: The Compressible Velocity Potential Equation and Its Linearized Form

ENAE311H Aerodynamics I

Christoph Brehm

The velocity potential equation

Recall our introduction of the velocity potential, ϕ, for irrotational flows, i.e., those for which

$$\boldsymbol{\xi} = \nabla \times \mathbf{v} = 0.$$

This irrotationality assumption generally holds in inviscid external flows, as long as no strongly curved shocks are present (curved shocks are sources of vorticity).

The velocity field is then given by

$$\mathbf{v} = \nabla\phi.$$

Using this relation, we can write the differential form of the steady continuity equation, i.e.,

$$\rho\nabla \cdot \mathbf{v} + \nabla\rho \cdot \mathbf{v} = 0,$$

as

$$\rho\nabla^2\phi + \nabla\rho \cdot \nabla\phi = 0.$$

In two dimensions, the expanded form of this equation is

$$\boxed{\rho}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \boxed{\frac{\partial\rho}{\partial x}}\frac{\partial\phi}{\partial x} + \boxed{\frac{\partial\rho}{\partial y}}\frac{\partial\phi}{\partial y} = 0.$$

The velocity potential equation

To eliminate ρ and its derivatives, we make use of Euler's equation, which was introduced during our derivation of the Bernoulli equation, i.e.,

$$dp = -\rho V\,dV = -\frac{1}{2}\rho d(V^2).$$

If the flow is irrotational (as is the case here), this equation holds along any direction in the flowfield. Using the velocity potential, we can write this as

$$dp = -\frac{1}{2}\rho d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right].$$

Now, if we additionally assume that the flow is isentropic, we can write

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial\rho}\right)_s = a^2,$$

and the above equation becomes

$$d\rho = -\frac{1}{2}\frac{\rho}{a^2}d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

$$\frac{\partial\rho}{\partial x} = -\frac{1}{2}\frac{\rho}{a^2}\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] \Rightarrow \frac{\partial\rho}{\partial x} = -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$\frac{\partial\rho}{\partial y} = -\frac{1}{2}\frac{\rho}{a^2}\frac{\partial}{\partial y}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] \Rightarrow \frac{\partial\rho}{\partial y} = -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial y\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right)$$

The velocity potential equation

Our velocity potential equation then becomes

$$a^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}-\left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2}=0.$$

This equation holds for an inviscid, irrotational, isentropic flow.

To eliminate the factor of a^2, we note that the energy equation for an adiabatic, two-dimensional flow is

$$\frac{a^2}{\gamma-1}+\frac{1}{2}(u^2+v^2)=\frac{a_0^2}{\gamma-1}.$$

This can be written as

$$a^2=a_0^2-\frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right],$$

where the stagnation sound speed, a_0, is constant and can be assumed known.

We thus have an equation purely in terms of ϕ and its derivatives. Note, however, that it is highly nonlinear, and no general solutions are available.

The linearized equation

Linearization is a technique used to transform complex, nonlinear equations into simpler linear ones.

HOWEVER, the price we pay is that any solution we obtain is no longer an exact solution to the original equation, and must not differ too much from a known, exact solution to be even approximately valid.

Take a uniform freestream flow with velocity components

$$\begin{aligned} u &= V_1, \\ v &= 0. \end{aligned}$$

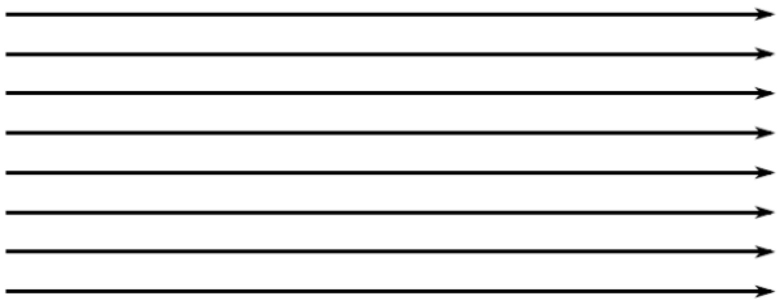
This is an exact solution to the full velocity potential equation.

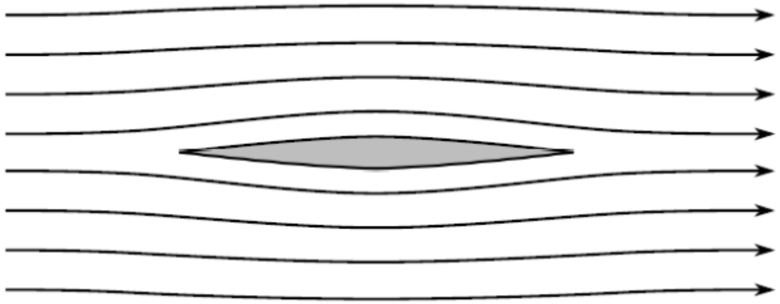
Now consider the flow over a thin 2D airfoil in such a freestream.

The presence of the airfoil will introduce changes in the two velocity components, i.e.,

$$\begin{aligned} u &= V_1 + \hat{u}, \\ v &= \hat{v}, \end{aligned}$$

but because the airfoil is thin, these "perturbation velocities", $\hat{u}$ and $\hat{v}$, will be much smaller than V_1 in magnitude, i.e., $\hat{u}, \hat{v} \ll V_1$.


Uniform flow


Perturbed flow

The linearized equation

We can also define a "perturbation velocity potential", $\hat{\phi}$, which in this case takes the form

$$\phi = V_1 x + \hat{\phi}.$$

We then see that

$$\hat{u} \;=\; \frac{\partial \hat{\phi}}{\partial x}, \quad \hat{v} \;=\; \frac{\partial \hat{\phi}}{\partial y}.$$

Substituting into our full potential equation, we obtain

$$a^2 \left(\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} \right) - \left(V_1 + \frac{\partial \hat{\phi}}{\partial x} \right)^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - 2 \left(V_1 + \frac{\partial \hat{\phi}}{\partial x} \right) \frac{\partial \hat{\phi}}{\partial y} \frac{\partial^2 \hat{\phi}}{\partial x \partial y} - \left(\frac{\partial \hat{\phi}}{\partial y} \right)^2 \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0.$$

We can also write this equation in terms of perturbation velocities:

$$a^2 \left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} \right) - (V_1 + \hat{u})^2 \frac{\partial \hat{u}}{\partial x} - 2 (V_1 + \hat{u}) \hat{v} \frac{\partial \hat{u}}{\partial y} - \hat{v}^2 \frac{\partial \hat{v}}{\partial y} = 0.$$

The sound speed can be written as

$$a^2 = a_1^2 - \frac{\gamma - 1}{2} (2 \hat{u} V_1 + \hat{u}^2 + \hat{v}^2) \qquad \text{(since } a_0^2 = a_1^2 + \frac{\gamma - 1}{2} V_1^2 \text{)}$$

The linearized equation

Substituting

$$a^2 = a_1^2 - \frac{\gamma - 1}{2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2)$$

into

$$a^2\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y}\right) - (V_1 + \hat{u})^2 \frac{\partial \hat{u}}{\partial x} - 2(V_1 + \hat{u})\,\hat{v}\frac{\partial \hat{u}}{\partial y} - \hat{v}^2\frac{\partial \hat{v}}{\partial y} = 0$$

dividing through by a_1^2, and rearranging, we arrive at a complicated equation containing a large number of terms that involve either $\hat{u}/V_1$ or $\hat{v}/V_1$ (see Anderson, equation 11.16).

Provided the freestream Mach number is neither too large (i.e., $M_1 \lesssim 5$) nor too close to unity (i.e., in the transonic range between approximately 0.8 and 1.2), we can neglect all terms containing $\hat{u}/V_1$ or $\hat{v}/V_1$ to arrive at

$$(1 - M_1^2)\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0.$$

or

$$\boxed{(1 - M_1^2)\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0.}$$

This is the *linearized* velocity potential equation, which is much simpler than the original equation.

Lecture 24: Linearized Supersonic Flow

ENAE311H Aerodynamics I

Christoph Brehm

Boundary conditions for the linearized equation

Recall our linearized potential equation:

$$(1 - M_1^2)\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0.$$

To solve this for a particular geometry, we also require boundary conditions for $\hat{\phi}$.

- Far from any solid boundaries, we require $\hat{\phi}$ to tend to a constant (so that the perturbation velocities are zero).
- At any solid boundaries we have a no-throughflow condition (can't enforce no-slip for an inviscid wall), so if θ is angle of wall relative to freestream direction, we have

$$\frac{v}{u} = \frac{\hat{v}}{V_1 + \hat{u}} = \tan\theta.$$

Since $\hat{u} \ll V_1$, however, an appropriate form of this equation (under the present assumptions) is

$$\boxed{\frac{\partial \hat{\phi}}{\partial y} = V_1 \tan\theta.}$$

Pressure coefficient in linearized flow

We can also derive a simple relation for the pressure coefficient under our linearized assumptions. Recall the pressure coefficient is defined as

$$C_p \equiv \frac{p - p_1}{q_1}.$$

For a perfect gas, we can write

$$q_1 = \frac{1}{2}\rho_1 V_1^2 = \frac{\gamma}{2} p_1 M_1^2,$$

and thus

$$C_p = \frac{2}{\gamma M_1^2}\left(\frac{p}{p_1} - 1\right).$$

Now, according to our isentropic assumption, we can write

$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{\gamma/(\gamma-1)}.$$

Now recall the form of the energy equation we used in deriving the linearized potential equation:

$$a^2 = a_1^2 - \frac{\gamma - 1}{2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2)$$

Since $a \propto \sqrt{T}$, this can be written as

$$\frac{T}{T_1} = 1 - \frac{\gamma - 1}{2a_1^2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2).$$

Substituting into our isentropic relation, we have

$$\frac{p}{p_1} = \left[1 - \frac{\gamma - 1}{2a_1^2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2)\right]^{\gamma/(\gamma-1)},$$

or

$$\frac{p}{p_1} = \left[1 - \frac{\gamma - 1}{2}M_1^2\left(\frac{2\hat{u}}{V_1} + \frac{\hat{u}^2 + \hat{v}^2}{V_1^2}\right)\right]^{\gamma/(\gamma-1)}.$$

Pressure coefficient in linearized flow

Now, since $\hat{u}, \hat{v} \ll V_1$, we can discard second-order (i.e., squared) terms in favor of first-order ones. Then, provided M_1 is not too large, we can write

$$\frac{p}{p_1} = (1-\epsilon)^{\gamma/(\gamma-1)},$$

with

$$\epsilon = \frac{\gamma-1}{2} M_1^2 \frac{2\hat{u}}{V_1} \ll 1.$$

Also,

$$(1-\epsilon)^a \approx 1 - a\epsilon$$

so we can write

$$\frac{p}{p_1} = 1 - \frac{\gamma}{2} M_1^2 \frac{2\hat{u}}{V_1}.$$

Our expression for the linearized pressure coefficient is then

$$\boxed{C_p = -\frac{2\hat{u}}{V_1}.}$$

Now recall the form of the energy equation we used in deriving the linearized potential equation:

$$a^2 = a_1^2 - \frac{\gamma-1}{2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2)$$

Since $a \propto \sqrt{T}$, this can be written as

$$\frac{T}{T_1} = 1 - \frac{\gamma-1}{2a_1^2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2).$$

Substituting into our isentropic relation, we have

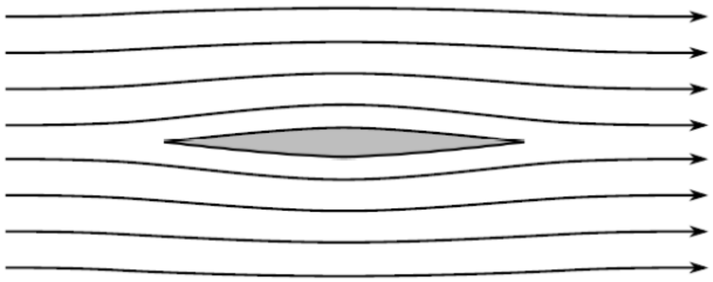
$$\frac{p}{p_1} = \left[1 - \frac{\gamma-1}{2a_1^2}(2\hat{u}V_1 + \hat{u}^2 + \hat{v}^2)\right]^{\gamma/(\gamma-1)},$$

or

$$\frac{p}{p_1} = \left[1 - \frac{\gamma-1}{2} M_1^2 \left(\frac{2\hat{u}}{V_1} + \frac{\hat{u}^2 + \hat{v}^2}{V_1^2}\right)\right]^{\gamma/(\gamma-1)}.$$

Supersonic linearized flow

Now we concentrate on the case of supersonic flow, e.g., over a thin airfoil.



The linearized velocity potential equation can then be written as

$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0$$

with

$$\lambda = \sqrt{M_1^2 - 1}.$$

Now, recall our discussion of the sound speed. The equation we derived governing the propagation of sound was

$$\frac{\partial^2 p'}{\partial t^2} - c^2 \frac{\partial^2 p'}{\partial x^2} = 0.$$

which had the general solution

$$p'(x,t) = f(x - ct) + g(x + ct)$$

Comparing to our linearized potential equation above, we might expect

$$\hat{\phi} = f(x - \lambda y) + g(x + \lambda y)$$

However, we can neglect the g solution (disturbances can only propagate downstream in supersonic flow), so

$$\boxed{\hat{\phi} = f(x - \lambda y).}$$

Supersonic linearized flow

Consider our general solution:

$$\hat{\phi} = f(x - \lambda y).$$

It does not tell us anything about the precise form of f (this will depend on the boundary conditions), but we do see that $\hat{\phi}$, and thus $\hat{u}, \hat{v}$, and all other flow properties will be constant along lines for which $x - \lambda y$ is constant.

The slope of these lines will be

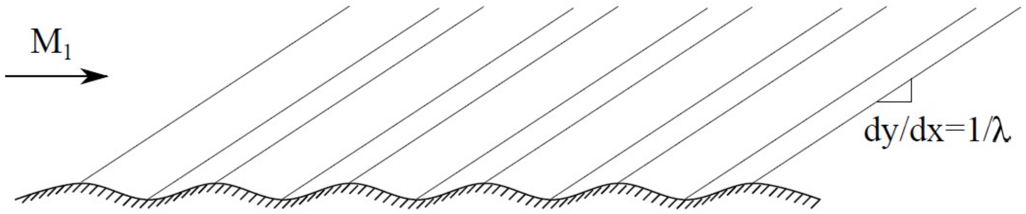
$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_1^2 - 1}}.$$

Recall that Mach lines in supersonic flow have an angle given by

$$\sin \mu = 1/M_1$$

and thus (from trig. identities)

$$\tan \mu = 1/\sqrt{M_1^2 - 1}.$$



We thus see that the flow properties are constant along Mach lines in linearized supersonic flow (note that the Mach number is, to first order, constant throughout the flow field).

Pressure coefficient in supersonic linearized flow

Recall that the pressure coefficient in linearized flow is

$$C_p = -\frac{2\hat{u}}{V_1}.$$

Also, we have at a solid boundary with angle θ to the freestream flow

$$\hat{v} = \frac{\partial \hat{\phi}}{\partial y} = V_1 \tan\theta \approx V_1\theta,$$

where we have assumed that θ is small.

Now from our general solution to the linearized equation,

$$\hat{u} = \frac{\partial \hat{\phi}}{\partial x} = f', \quad \text{and} \quad \hat{v} = \frac{\partial \hat{\phi}}{\partial y} = -\lambda f'$$

Equating f' in these two equations:

$$\hat{u} = -\frac{\hat{v}}{\lambda} = \frac{V_1\theta}{\lambda}$$

Substituting into our expression for C_p above, we arrive at

$$\boxed{C_p = \frac{2\theta}{\sqrt{M_1^2 - 1}}.}$$