Lecture 7: Cartesian to/from Orbital Elements

Longitude of the Ascending Node: Ω : measured from the Greenwich meridian

Right Ascension of the Ascending Node (RAAN): measured from the vernal equinox

Conversion from Cartesian to OE's:

Cartesian State: $[\vec{r}, \vec{v}] = [x, y, z, v_x, v_y, v_z]$

Also know μ.

a]

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

e] $\vec{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{r} - (\vec{r}\cdot\vec{v})\vec{v}\right]$

take the magnitude to get e.

i] Angle between $\hat{z}$ & $\vec{h}$

$$\frac{\vec{h}\cdot\hat{z}}{h} = \cos(i) \qquad \vec{h} = \vec{r}\times\vec{v}$$

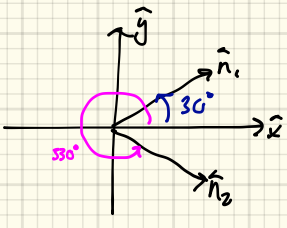
Ω] Angle from $\hat{x}$ to ascending node

$\vec{n} = \hat{z} \times \vec{h}$ (points at the ascending node)

$\cos(\Omega) = \hat{n}\cdot\hat{x}$ $\qquad 0 \le \Omega \le 360°$

$\cos(30) = \cos(330)$

Need to check calculator output



If $\hat{n}\cdot\hat{y} > 0$, $0 < \Omega < 180°$

If $\hat{n}\cdot\hat{y} < 0$, $180 < \Omega < 360°$

$\underline{\omega}$ | angle from ascending node to periapsis
$\vec{n}$ $\vec{e}$

$$\cos(\omega) = \frac{\vec{n}\cdot\vec{e}}{ne}$$

Do the quadrant check
If $\vec{e}\cdot\hat{z} > 0$, $0 < \omega < 180°$.
else $180 < \omega < 360°$

$\underline{\nu}$ | From periapsis to s/c location:

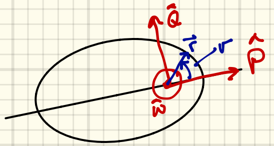
$$\cos(\nu) = \frac{\vec{e}\,\vec{r}}{er}$$

Do the quadrant check:
if $\vec{r}\cdot\vec{v} > 0$, then $0 < \nu < 180°$ (b/c $|\vec{r}|$ is growing here)

else, $180 < \nu < 360°$.

<u>OE to Cartesian:</u>

We know: $a, e, i, \Omega, \omega, \nu$ & μ



Use the perifocal frame: $\hat{P}, \hat{Q}, \hat{W}$

$\hat{P}$: points towards periapsis

$\hat{W}$: points along $\vec{h}$

$\hat{Q}$: Completes the RH'd system

1. Express $\vec{r}, \vec{v}$ in the PQW frame

2. Rotate into the XYZ frame.

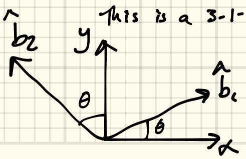
1. $\vec{r} = r\cos\nu\,\hat{P} + r\sin\nu\,\hat{Q}$ $\qquad r = \dfrac{p}{1+e\cos\nu}$, $p = a(1-e^2)$

$$\vec{v} = \sqrt{\frac{\mu}{p}}\left[-\sin\nu\,\hat{P} + (e+\cos\nu)\hat{Q}\right]$$

2. Rotate from PQW to XYZ.

We will write the rotation matrix from XYZ → PQW & then take the inverse.

This is a 3-1-3 rotation



$$\hat{b}_1 = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} + 0\,\hat{z}$$
$$\hat{b}_2 = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} + 0\,\hat{z}$$
$$\hat{b}_3 = \hat{z}$$

$$[R_3] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

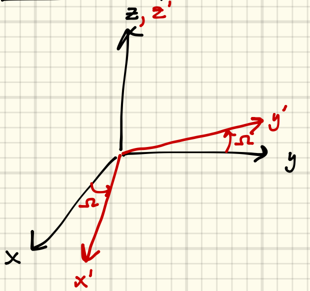
$$[R_2] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$^{C}\vec{a} = [BC][AB]\,^{A}\vec{a}$$

Another property of rotation matrices: $[R]^{-1} = [R]^{T}$ (b/c orthonormal matrix)

From $XYZ \rightarrow PQW$: 3-1-3 (z-x-z)

Rotation #1: Rotate about $\hat{z}$ by Ω.

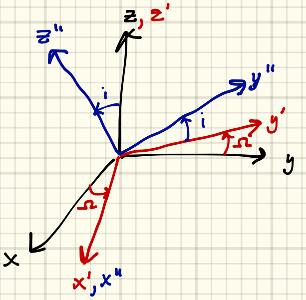


x' points at the ascending node

$$[R_\Omega] = \begin{bmatrix} c\Omega & s\Omega & 0 \\ -s\Omega & c\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

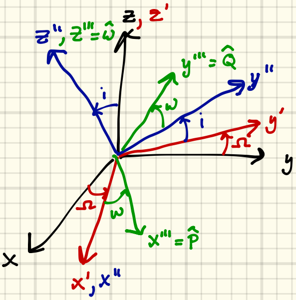
Rotation #2: Rotation about x' by i

After this rotation, $\hat{z}'' \parallel \vec{h} \parallel \hat{w}$



$$[R_i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & ci & si \\ 0 & -si & ci \end{bmatrix}$$

Rotation #3: Rotate about $\hat{z}''$ by ω



$$[R_\omega] = \begin{bmatrix} c\omega & s\omega & 0 \\ -s\omega & c\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⇒ To Rotate from XYZ → PQW

$$^{P}\vec{b} = [R_\omega][R_i][R_\Omega]\,^{I}\vec{b}$$

We want to rotate $\vec{r}, \vec{v}$ from PQW → XYZ

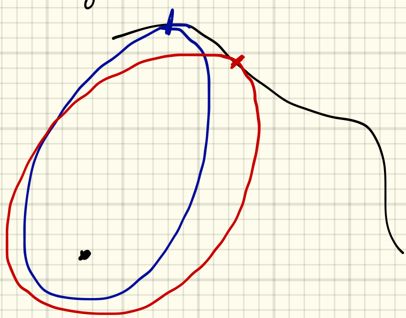
$$^{I}\vec{r} = \left[[R_\omega][R_i][R_\Omega]\right]^{T}\,^{P}\vec{r}$$

Keys: 1. Transpose

2. Rotation matrices in the correct order

$$^{I}\vec{v} = \left[[R_\omega][R_i][R_\Omega]\right]^{T}\,^{P}\vec{v}$$

Osculating Orbital elements: orbital elements calculated at each time step.



Plot the OE's as a f(t) → Constant for 2BP, but varying if other forces are included.