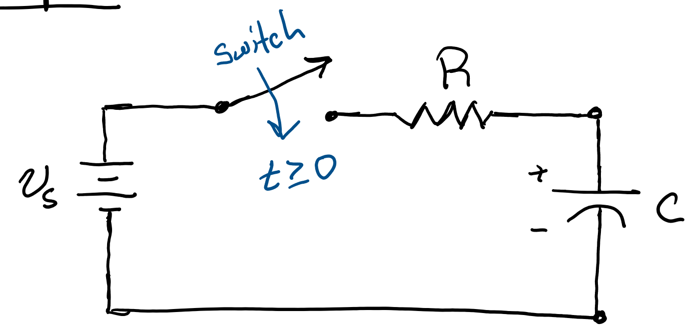
First Order Circuits

A circuit consisting of a single energy storage element (e.g. capacitor or inductor) will exhibit a 1st order transient response when there is a change to one of the circuit inputs (e.g. toggling a switch).

Example: Consider



Circuit begins from an equilibrium state.

Find

a) Formulate governing equation

b) Determine system response

Solution

a) Guess a clockwise current flow and apply KVL

(1) $v_s - iR - v_c = 0$

Recall,

(2) $$i_c = C\frac{dv_c}{dt}$$

Take $\frac{d}{dt}$ of eq (1) and substitute (2),

$$\cancel{\frac{dv_s}{dt}} - \frac{di}{dt}R - \frac{dv_c}{dt} = 0$$

$$\underbrace{\frac{dv_s}{dt}}_{0} - \frac{di}{dt}R - \frac{dv_c}{dt} = 0$$

$$\Rightarrow \quad \frac{di}{dt}R + \frac{dv_c}{dt} = 0$$

$$\Rightarrow \quad \frac{di}{dt}R + \frac{1}{C}i = 0$$

$$\Rightarrow \quad \boxed{\frac{di}{dt} + \frac{1}{RC}i = 0} \qquad (3)$$

b) Rewrite (3) as

$$\frac{di}{i} = -\frac{1}{RC}dt$$

Integrate

$$\int \frac{1}{i}\,di = -\frac{1}{RC}\int dt$$

$$\Rightarrow \quad \ln i = -\frac{1}{RC}t + K_1$$

$$\Rightarrow \quad i(t) = K_2 e^{-t/RC}$$

Apply Initial Conditions (IC's),

$$i(0) = i_0 = K_2$$

$\therefore$

$$i(t) = i_0 e^{-t/RC}$$

Also find $v_c(t)$

$$i = C\frac{dv_c}{dt}$$

$$\Rightarrow \quad \int dv_c = \frac{1}{C}\int i\,dt$$

$$\Rightarrow \quad v_c(t) = \frac{1}{C}\int i_0 e^{-t/RC}\,dt$$

$$\Rightarrow \quad v_c(t) = \frac{1}{C}\int i_0 e^{-t/RC}\,dt$$

$$\Rightarrow \quad v_c(t) = -\frac{i_0}{\cancel{C}} R\cancel{C}\, e^{-t/RC} + K$$

$$\Rightarrow \quad v_c(t) = -i_0 R e^{-t/RC} + K$$

Apply IC's,

$$v_c(0) = v_{c_0} = -i_0 R + K$$

$$\Rightarrow \quad K = v_{c_0} + i_0 R$$

Apply KVL at $t=0$

$$v_s - i_0 R - v_{c_0} = 0$$

$$\Rightarrow \quad v_s = v_{c_0} + i_0 R$$

$\therefore$

$$K = v_s$$

$\therefore$

$$v_c(t) = v_s - i_0 R e^{-t/RC}$$

What is i_0? By KVL,

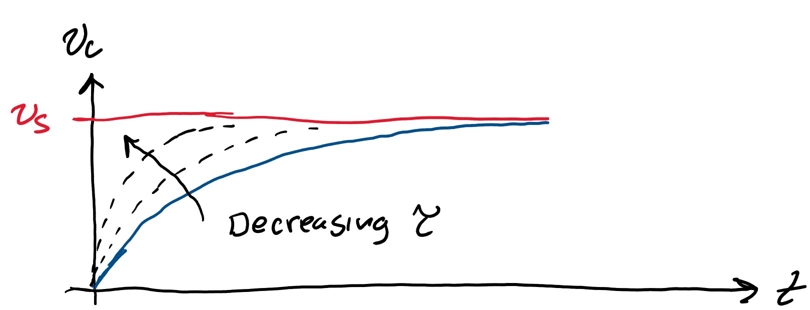
$$i_0 R = v_s - v_{c_0}$$

$\therefore$

$$v_c(t) = v_s - (v_s - v_{c_0}) e^{-t/RC}$$

If $v_{c_0} = 0$ then

$$v_c(t) = v_s\left[1 - e^{-t/RC}\right] = v_s\left[1 - e^{-\frac{t}{\tau}}\right] \; ; \; \tau = RC$$



We call the parameter τ the "time constant"

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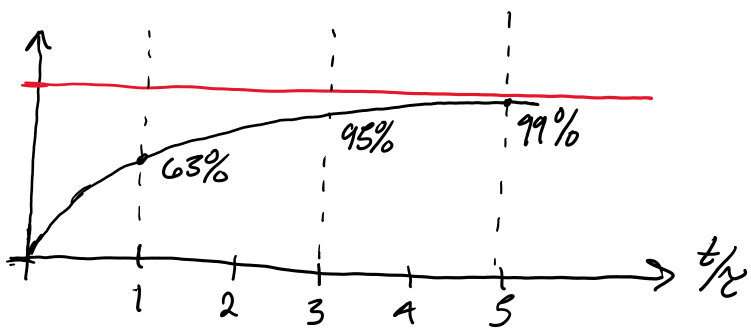
Large $\tau \Rightarrow$ slow response, takes longer to reach equilibrium

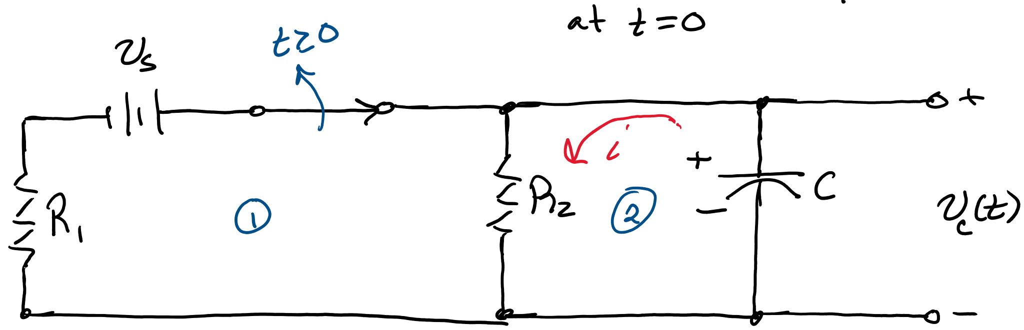
Large $C \Rightarrow$ more charge is needed to fill capacitor
Large $R \Rightarrow$ reduces current flow, takes longer for charge to fill capacitor



In practice, the circuit reaches equilibrium in $\sim 5\tau$.

Example

Circuit is in equilibrium at $t=0$



Find $v_c(t)$

Solution

Since circuit starts from equilibrium with a closed switch, the initial voltage across the capacitor $v_c(0)$ must equal voltage across R_2.

To get governing equation, apply KVL to mesh 2

To get governing equation, apply KVL to mesh 2 (no current flow in mesh 1 when switch is open).

$$v_c - i R_2 = 0$$

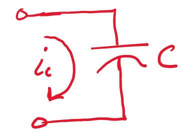
$$\Rightarrow \frac{dv_c}{dt} - \frac{di}{dt} R_2 = 0$$

$$\Rightarrow -\frac{1}{C} i - \frac{di}{dt} R_2 = 0$$

$$\Rightarrow \frac{di}{dt} + \frac{1}{CR_2} i = 0$$

Same equation as before

Note: $i_c = C \dfrac{dv_c}{dt} \Rightarrow$



Since we chose ccw current flow in our analysis, we must flip sign

$$i = -i_c$$