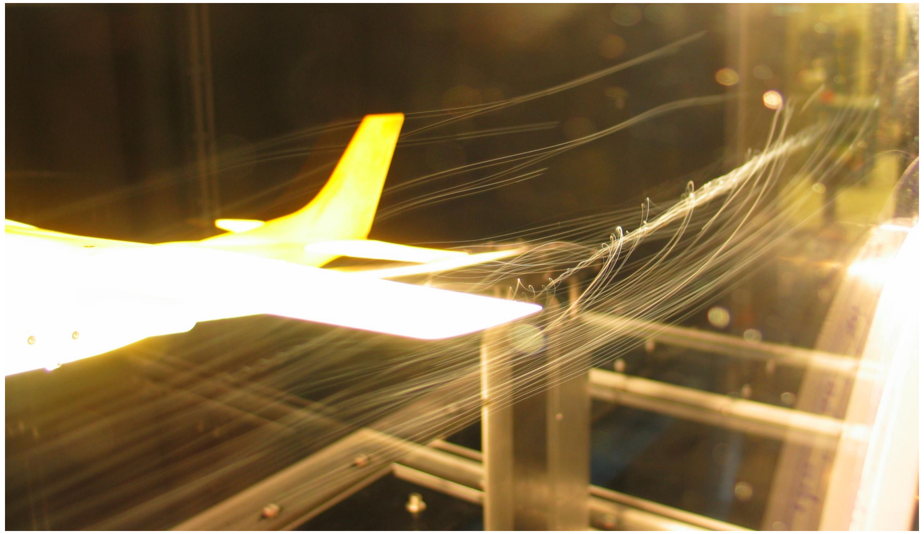
Lecture 9: Streamlines, Vorticity, and the Stream Function

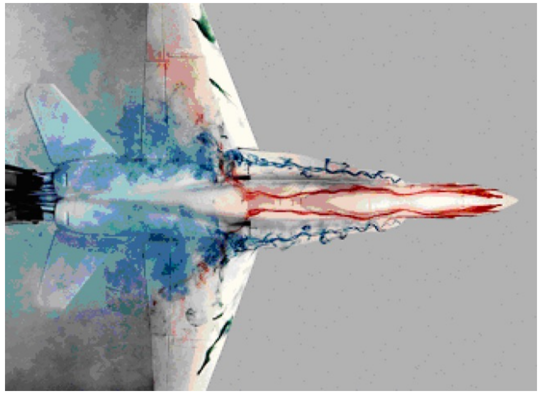
ENAE311H Aerodynamics I

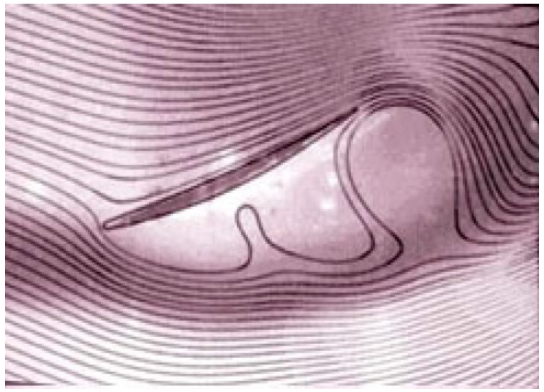
Christoph Brehm

Pathlines, streaklines, and streamlines

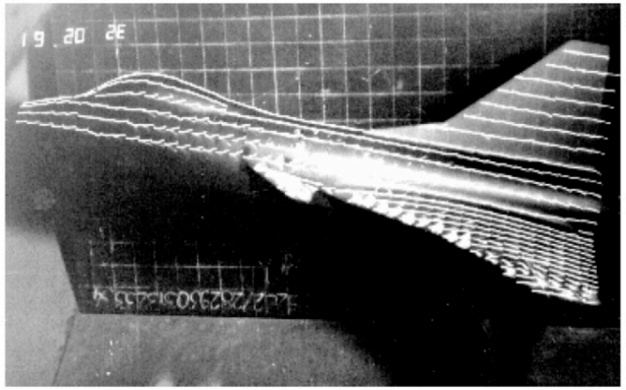


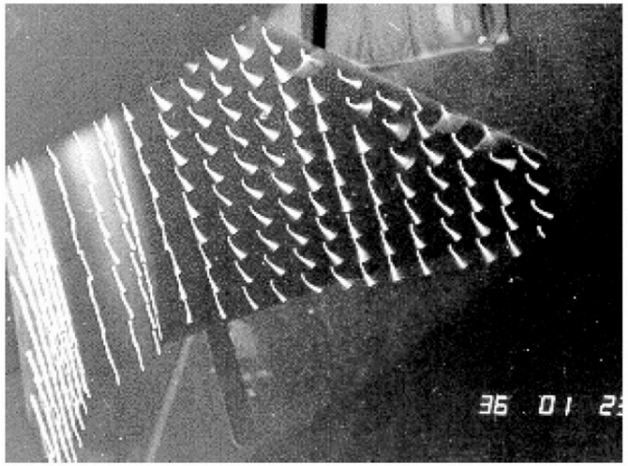
Long-time exposure of illuminated helium-filled bubbles (flow tracers)





Flow visualization by fluorescent dye



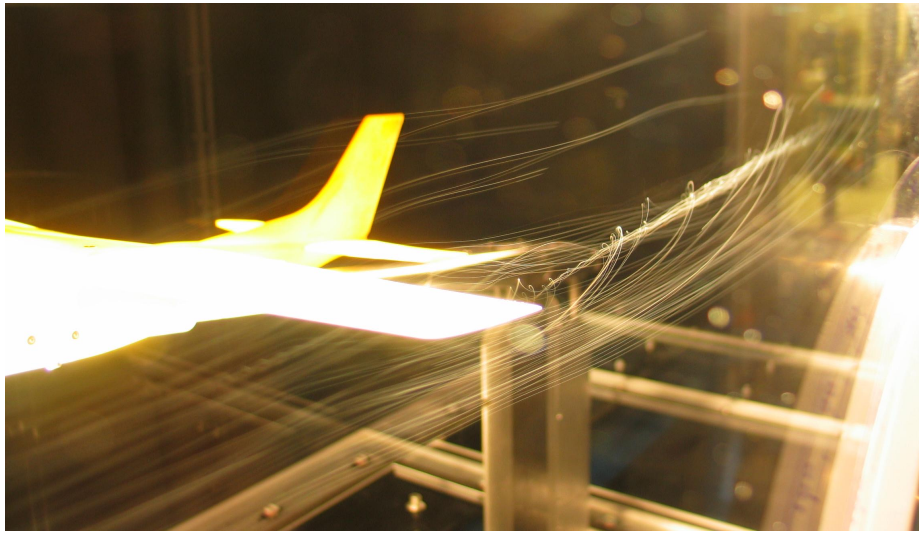


Flow visualization by fluorescent tufts

Pathlines, streaklines, and streamlines

Three ways of spatially visualizing the flowfield (especially for lower-speed flows) are pathlines, streaklines, and streamlines.

- **Pathline**: the trajectory in three-dimensional space followed by an element of the flow. If the flow is steady, all pathlines through a given point will be the same; for unsteady flows, these will be generally different. If, for example, you imagine somehow tagging a fluid element and following its path this would form a pathline.



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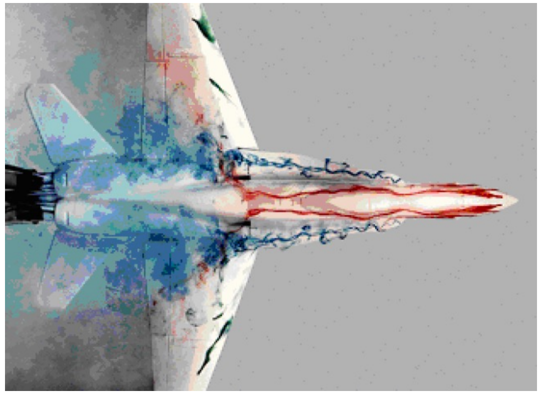
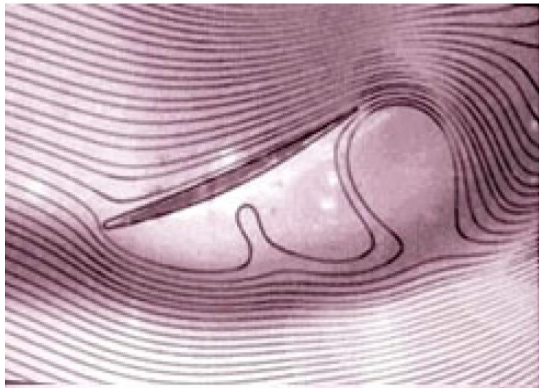
- **Streakline**: the locus of points, at a particular moment in time, corresponding to fluid elements that once passed through a given point in space in the flowfield. If dye were released at a particular point in the flowfield, for example, it would form a streakline.





Flow visualization by fluorescent dye

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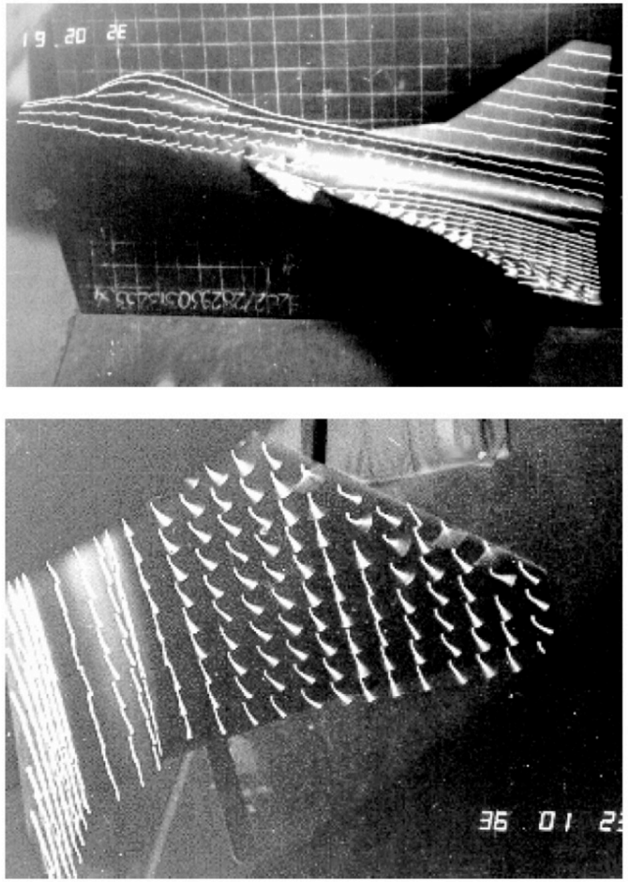
- Streakline: the locus of points, at a particular moment in time, corresponding to fluid elements that once passed through a given point in space in the flowfield. If dye were released at a particular point in the flowfield, for example, it would form a streakline.
- **Streamline**: a curve that is everywhere tangential to the flow at a given moment (i.e., instantaneous snapshot of the flow). A streamtube is a surface formed by collection of streamlines that pass through a closed curve in space.

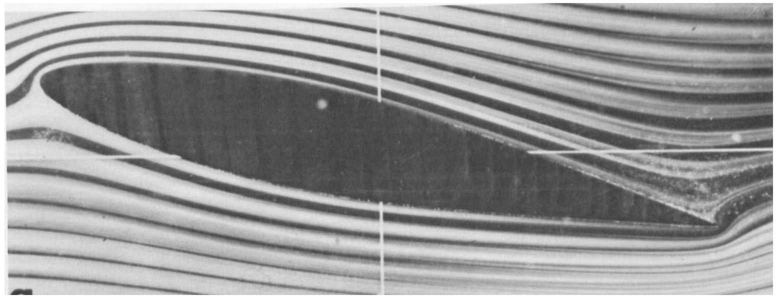
Pathlines, streaklines and streamlines are all the same in *steady* flow.

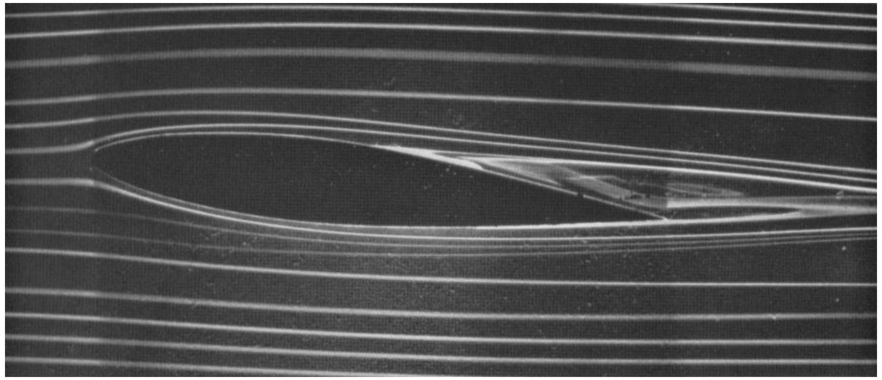


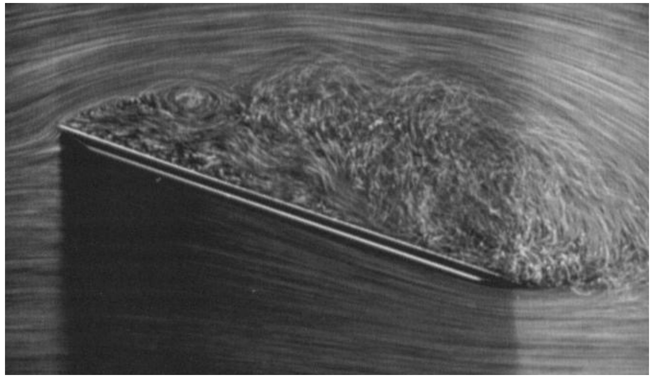
Flow visualization by fluorescent tufts

Inviscid Flow (3)

- Changes in overall velocity or geometry of a problem can change the importance of viscous forces
- Some regions of a flow may be inviscid while others show strong viscous effects



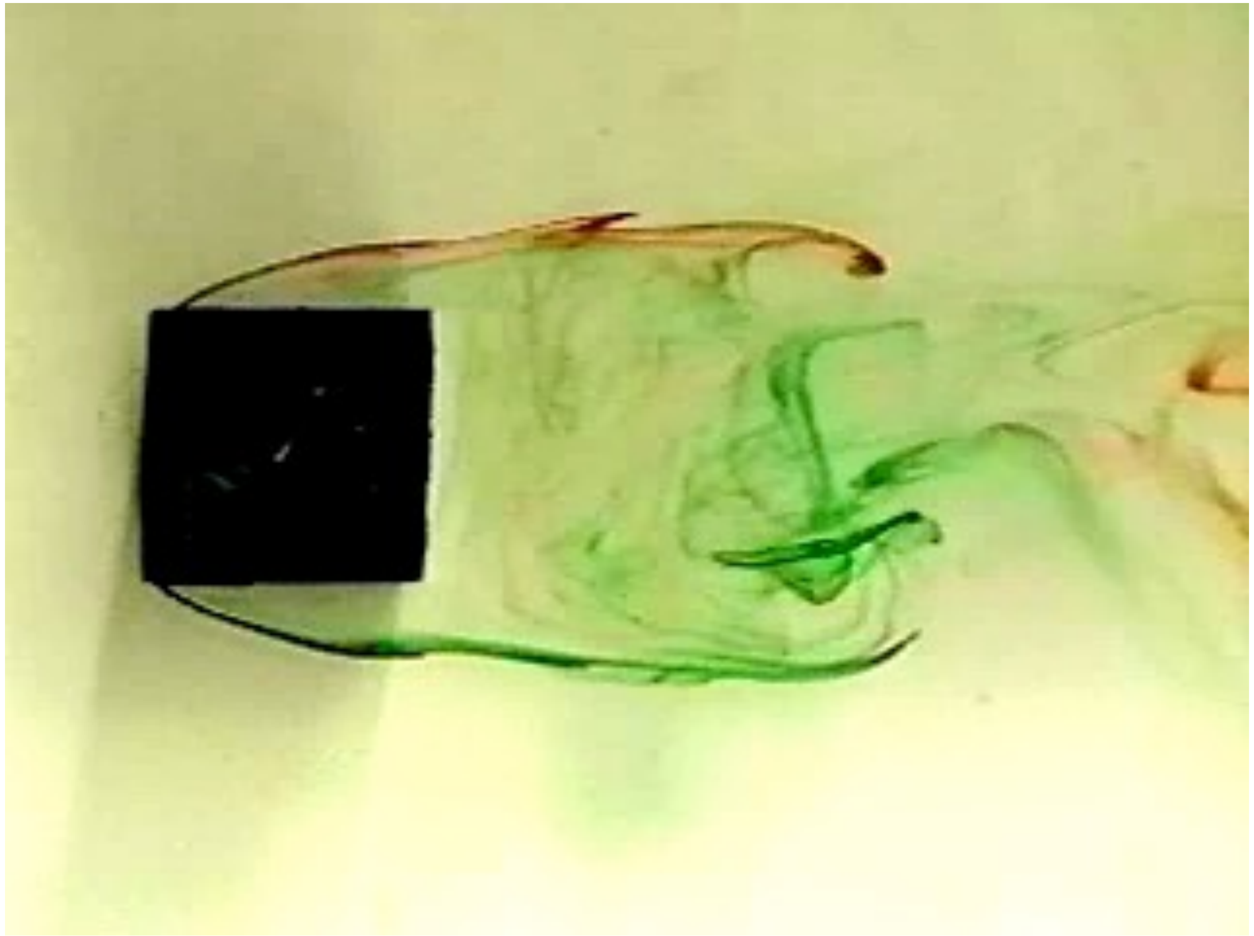




Streamlines (3)

- For steady flows – pathlines, streaklines, and streamlines are identical

NOT FOR UNSTEADY!!!



The stream function

For a two-dimensional flow, we have the following equation for a streamline:

$$\frac{dy}{dx} = \frac{v}{u}.$$

This can be integrated to give some function of x and y:

Stream function $\boxed{\overline{\Psi}(x, y)} = c,$

where c is a constant of integration. Different values of c will give different streamlines.

Now, since the flow velocity is everywhere tangential to streamlines, no fluid can cross a streamline, and thus the mass flux (per unit depth) between two streamlines is the difference in the value of the stream function between the two streamlines, i.e., if streamline 1 is given by $\overline{\Psi}(x, y) = c_1$ and streamline 2 by $\overline{\Psi}(x, y) = c_2$, we have for depth, d

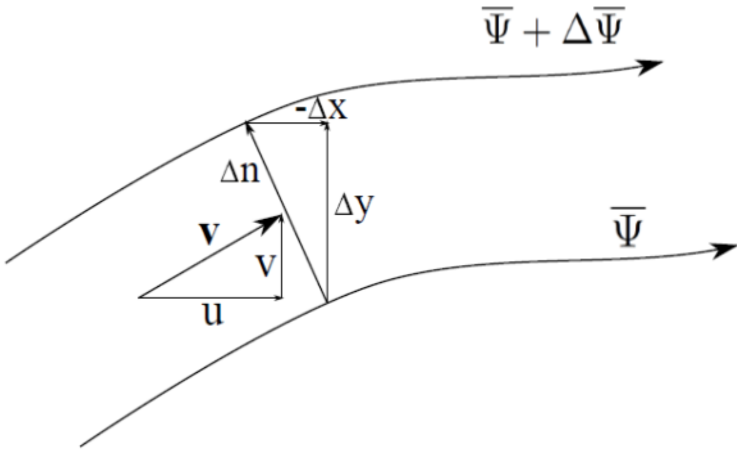
$$\frac{\dot{m}}{d} = c_2 - c_1 = \Delta\overline{\Psi}$$

For incompressible flows (ρ constant), we define $\Psi = \overline{\Psi}/\rho$ and have a relation for the volumetric flow rate, $\dot{Q}$:

$$\frac{\dot{Q}}{d} = \Delta\Psi.$$

The stream function

The stream function has an important relationship to the flow velocity components. To see this, consider two streamlines separated by a small normal distance, Δn, as shown to the right.



Since the difference in stream function values is equal to the mass flux (per unit depth) between them, we have

$$\Delta\overline{\Psi} = \rho V \Delta n,$$

which in the limit of $\Delta n \to 0$ becomes

$$\frac{\partial\overline{\Psi}}{\partial n} = \rho V.$$

Note from the geometry shown that we can also write

$$\Delta\overline{\Psi} = \rho u \Delta y + \rho v(-\Delta x),$$

or, as $\Delta n \to 0$

$$d\overline{\Psi} = \rho u dy - \rho v dx.$$

From the chain rule, however, we also have

$$d\overline{\Psi} = \frac{\partial\overline{\Psi}}{\partial x}dx + \frac{\partial\overline{\Psi}}{\partial y}dy.$$

And thus $$\boxed{\frac{\partial\overline{\Psi}}{\partial x} = -\rho v, \quad \frac{\partial\overline{\Psi}}{\partial y} = \rho u.}$$

The stream function

In a steady flow, it is possible to use the stream function to effectively replace the continuity equation.

To see this, note that the differential form of the conservation of mass for steady flows is $\nabla \cdot (\rho \boldsymbol{v}) = 0$, which in two dimensions is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0.$$

However, we can also write

$$\begin{aligned}
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) &= \frac{\partial}{\partial x}\left(\frac{\partial \overline{\Psi}}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \overline{\Psi}}{\partial x}\right) \\
&= \frac{\partial^2 \overline{\Psi}}{\partial x \partial y} - \frac{\partial^2 \overline{\Psi}}{\partial y \partial x} \\
&= 0,
\end{aligned}$$

by equality of mixed derivatives.

In cylindrical coordinates, our relationship for the stream function with the velocity components is

$$\rho v_r = \frac{1}{r}\frac{\partial \overline{\Psi}}{\partial \theta} \quad \text{and} \quad \rho v_\theta = -\frac{\partial \overline{\Psi}}{\partial r}.$$

6.15 The velocity components for an incompressible, plane flow are

$$v_r = Ar^{-1} + Br^{-2}\cos\theta$$

$$v_\theta = Br^{-2}\sin\theta$$

where A and B are constants. Determine thc corresponding stream function.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

$$\rho v_r = \frac{1}{r}\frac{\partial \overline{\Psi}}{\partial \theta} \qquad \rho v_\theta = -\frac{\partial \overline{\Psi}}{\partial r}.$$

From the definition of the stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{(Eq. 6.42)}$$

so that for the velocity distribution given,

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = A r^{-1} + B r^{-2} \cos\theta \qquad (1)$$

$$\frac{\partial \psi}{\partial r} = -B r^{-2} \sin\theta \qquad (2)$$

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Integrate Eq. (1) with respect to θ to obtain

$$\int d\psi = \int (A + B r^{-1}\cos\theta)\, d\theta + f_1(r)$$

or

$$\psi = A\theta + B r^{-1}\sin\theta + f_1(r) \qquad (3)$$

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Similarly, integrate Eq. (2) with respect to r to obtain

$$\int d\psi = -\int B r^{-2}\sin\theta\, dr + f_2(\theta)$$

or

$$\psi = B r^{-1}\sin\theta + f_2(\theta) \qquad (4)$$

From the definition of the stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad (\text{Eq. } 6.42)$$

so that for the velocity distribution given,

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = A r^{-1} + B r^{-2}\cos\theta \qquad (1)$$

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$$\psi = B r^{-1}\sin\theta + f_2(\theta) \qquad (4)$$

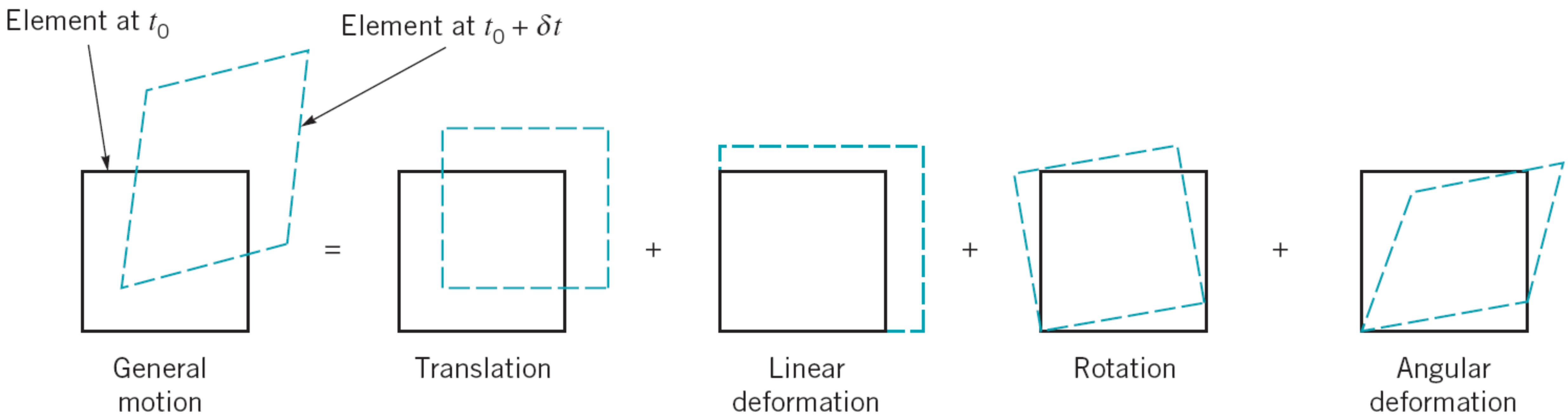
Thus, to satisfy both Eqs. (3) and (4)

$$\psi = \underline{\underline{A\theta + B r^{-1}\sin\theta + C}}$$

where C is an arbitrary constant.

Fluid element motion

- Translation (V)
- Deformation
- Rotation (ζ – vorticity)
- Angular Deformation (γ – shear strain rate)
- Differential mass conservation



Translation

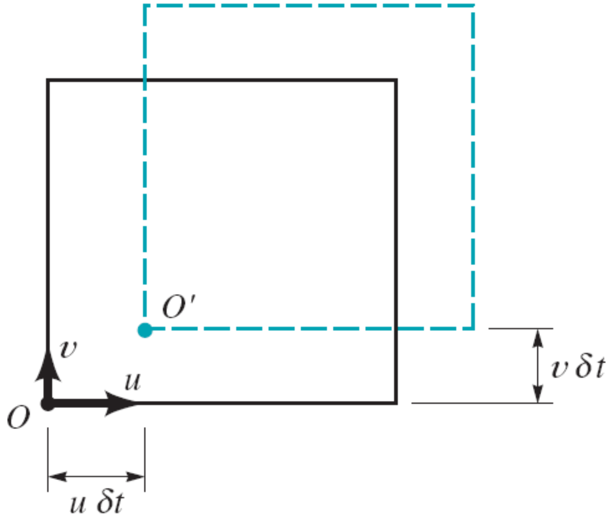
- Translation – fluid elements translate at local fluid velocity

$$\vec{V} = u\hat{x} + v\hat{y} + w\hat{z}$$

$$\Delta x = u\Delta t$$

$$\Delta y = v\Delta t$$

$$\Delta z = w\Delta t$$

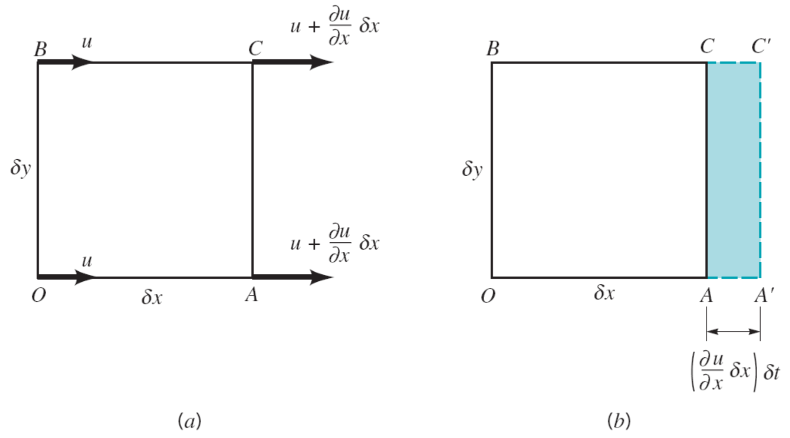


Linear Deformation

- Volume of differential fluid element: $(\delta\forall) = (\delta x\,\delta y\,\delta z)$
- Change in volume of fluid element in x direction

$$d(\delta\forall) = \left(\frac{\partial u}{\partial x}\delta x\right)\delta y\,\delta z\,dt$$

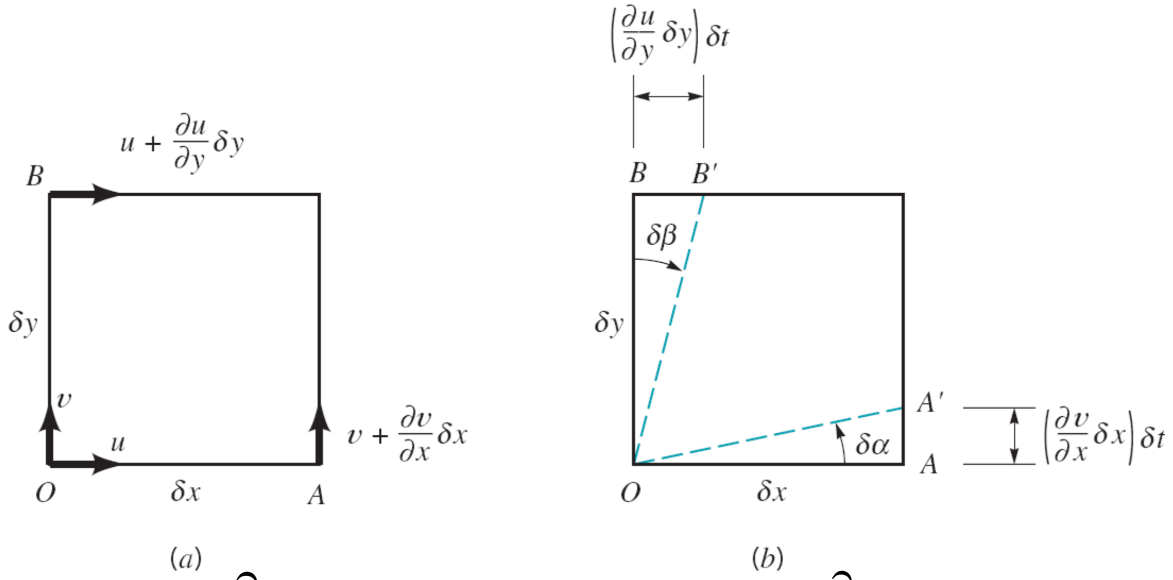
$$\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \left(\frac{\partial u}{\partial x}\right)$$



- In general for 3-D: $\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \vec{\nabla}\cdot\vec{V}$

Rotation/Angular Deformation (1)

- Define angles δα and δβ as rotation of x and y axis



$$\tan \delta\alpha = \frac{\partial v}{\partial x}\delta t \approx \delta\alpha$$

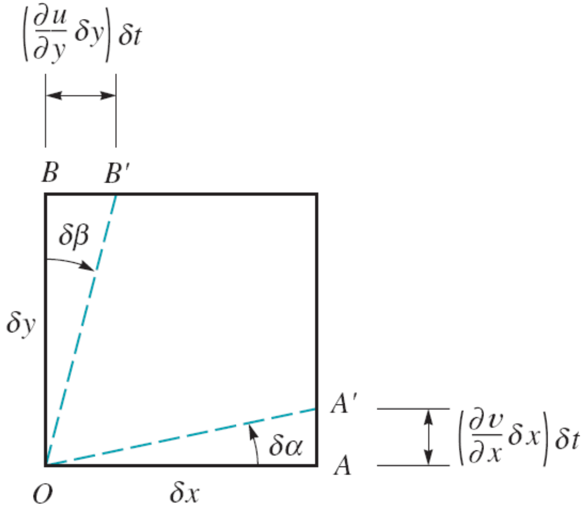
$$\tan \delta\beta = \frac{\partial u}{\partial y}\delta t \approx \delta\beta$$

Rotation/Angular Deformation (2)

- Rate of rotation of x and y axis

$$\varpi_{OA} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t} = \frac{\partial v}{\partial x}$$

$$\varpi_{OB} = \lim_{\delta t \to 0} \frac{\delta \beta}{\delta t} = \frac{\partial u}{\partial y}$$

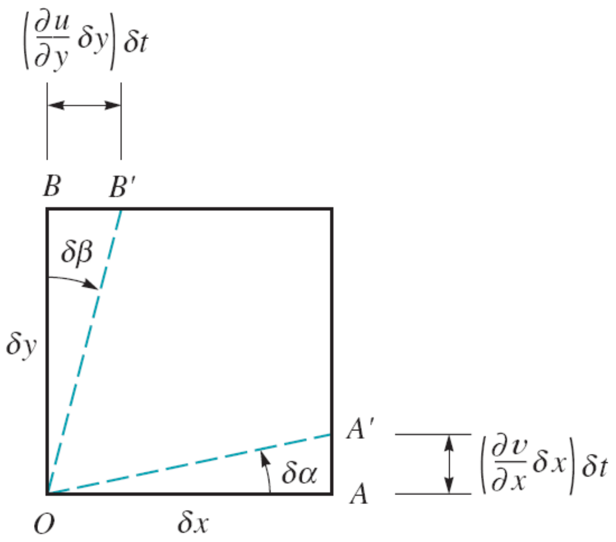


- Note different sign convention for α and β
- If $\omega_{OA}=-\omega_{OB}$ then the fluid element will only rotate and not deform
- If $\omega_{OA}=+\omega_{OB}$ then the fluid element will only deform and not rotate

Rotation/Angular Deformation (3)

- Rate of rotation of fluid element defined as average of ω_{OA} and $-\omega_{OB}$

$$\varpi_z = \frac{\varpi_{OA} - \varpi_{OB}}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$



- Likewise

$$\varpi_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \varpi_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

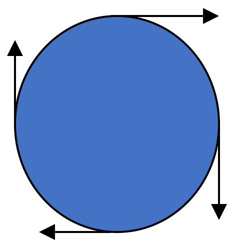
Rotation and Vorticity

$$\vec{\varpi} = \varpi_x \hat{x} + \varpi_y \hat{y} + \varpi_z \hat{z}$$

- Rotation rate is a vector:

$$\varpi_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \varpi_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \varpi_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\vec{\varpi} = \frac{1}{2}\vec{\nabla} \times \vec{V} = \frac{1}{2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$



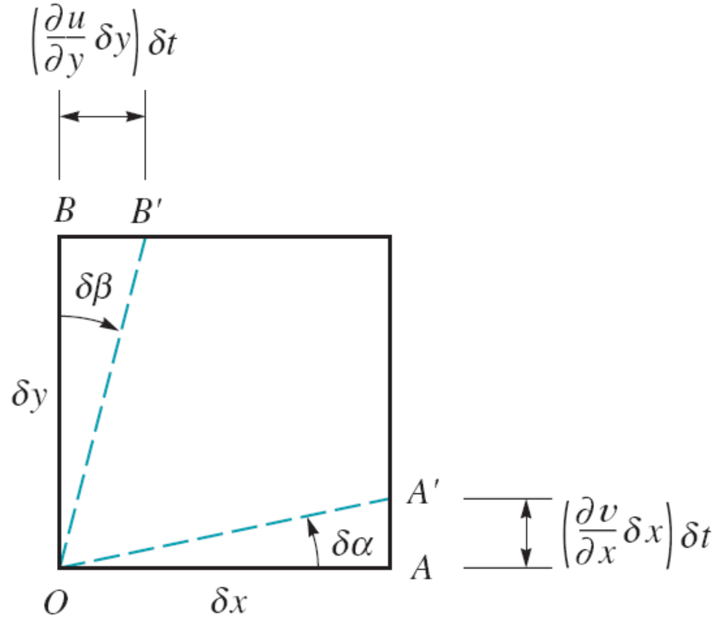
- Vorticity is defined as twice the rotation rate

$$\vec{\zeta} = 2\vec{\varpi} = \vec{\nabla} \times \vec{V}$$

Angular Deformation

- Rate of angular deformation (rate of shearing strain) of fluid element defined as twice the average of ω_{OA} and $+\omega_{OB}$

$$\gamma_z = 2\left(\frac{\varpi_{OA} + \varpi_{OB}}{2}\right) = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$



- Likewise

$$\gamma_x = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \gamma_y = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

6.11 The velocity components of an incompressible, two-dimensional velocity field are given by the equations

$$u = y^2 - x(1 + x)$$
$$v = y(2x + 1)$$

Show that the flow is irrotational and satisfies conservation of mass.

If the two-dimensional flow is irrotational,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

For the velocity distribution given,

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = 2y$$

Thus,

$$\omega_z = \frac{1}{2}\left(2y - 2y\right) = 0$$

and the <u>flow is irrotational.</u>

To satisfy conservation of mass,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since,

$$\frac{\partial u}{\partial x} = -1 - 2x \qquad \frac{\partial v}{\partial y} = 2x + 1$$

then

$$-1 - 2x + 2x + 1 = 0$$

and

<u>conservation of mass is satisfied</u>.

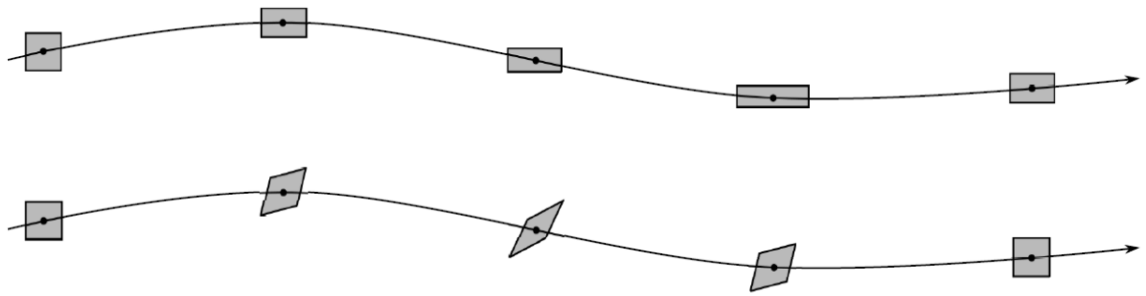
Vorticity

As a fluid element moves along a streamline or pathline, it may rotate and become deformed. The angular velocity, $\boldsymbol{\omega}$, is an important property in fluid mechanics, but a slightly more useful quantity is the vorticity $\boldsymbol{\xi} = 2\boldsymbol{\omega}$. This may be written:

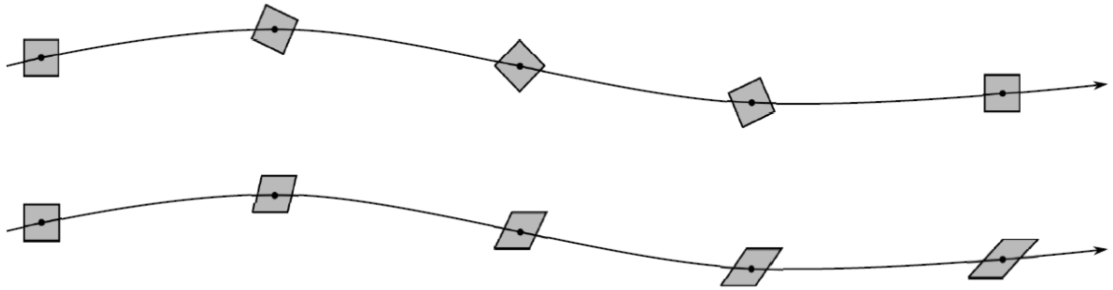
$$\begin{aligned} \boldsymbol{\xi} &= \nabla \times \mathbf{v} \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}. \end{aligned}$$

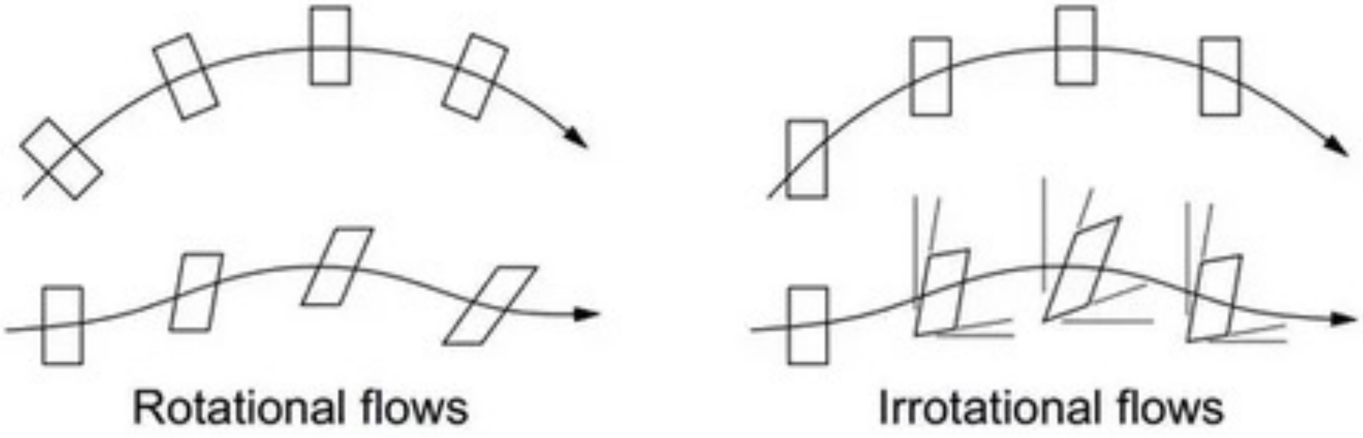
Or, in two dimensions, $\xi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}.$

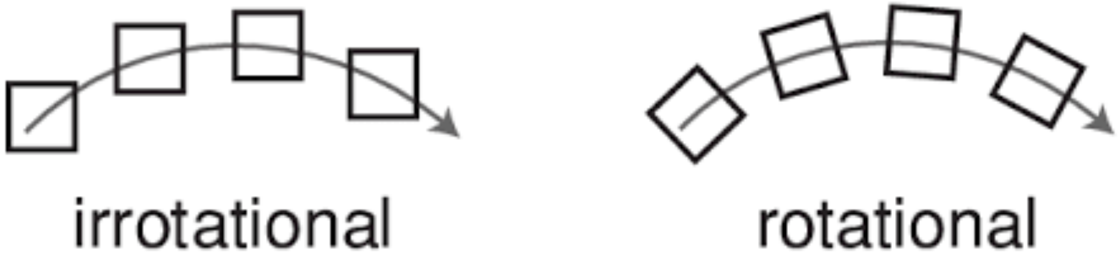
For *irrotational* flows, the vorticity is zero (inviscid flows are typically irrotational).



Rotational flows have nonzero vorticity.




Rotational flows
Irrotational flows


irrotational
rotational

Circulation

A related concept to vorticity is the circulation, Γ, which is defined as

$$\Gamma = -\oint_c \mathbf{v} \cdot \mathbf{ds}$$

i.e., the integral around a closed curve of the dot product of the velocity with the curve element.

Using Stokes' theorem, we can write

$$\begin{aligned} \Gamma &= -\iint_s (\nabla \times \mathbf{v}) \cdot \mathbf{dA} \\ &= -\iint_s \boldsymbol{\xi} \cdot \hat{\mathbf{n}} dA. \end{aligned}$$

The Kutta-Joukowski theorem tells us that the lift produced by an airfoil is directly proportional to the circulation about it.

Velocity Potential

- For an irrotational flow the velocity components can be expressed in terms of a scalar function $\phi(x,y,z)$:

$$\mathbf{V} = \nabla\phi \qquad u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y} \qquad w = \frac{\partial\phi}{\partial z}$$

- $\phi(x,y,z)$ is called velocity potential

- Irrotational: $\nabla \times \mathbf{V} = 0$ (note: $\nabla \times \nabla\phi = 0$)

- The inviscid, incompressible, and irrotational flow fields is governed by the Laplace equation:

  with $\nabla \cdot \mathbf{V} = 0$ and $\nabla^2(\ ) = \nabla \cdot \nabla(\ )$

  we get $\nabla^2\phi = 0$ or $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0$