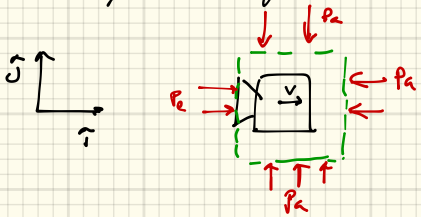
Lecture 17 - Rocket Eqn

Rocket eqn: This is why we calculate ΔV.



rocket w/ velocity V & mass m.

it expells exhaust ΔM at speed V_e.

All forces & velocities are in the $\hat{i}$ direction.

Momentum of the system @ $t+\Delta t$ − initial momentum = external impulse

$$\left[(m-\Delta m)(V+\Delta V)\hat{i} + \Delta m(-V_e)\hat{i}\right] - mV\hat{i} = (p_e - p_a)A_e\,\Delta t\,\hat{i}$$

Pressure = F/A

A_e = area of the exit nozzle

Define $\dot{m}_e$ = exhaust mass flow rate > 0

Assume $\dot{m}$ = constant

Note $\dot{m}$ is the time rate of change of the s/c mass $= -\dot{m}_e$

$$\left[(m-\dot{m}_e\Delta t)(V+\Delta V)\hat{i} + \dot{m}_e\Delta t(-V_e)\hat{i}\right] - mV\hat{i} = (p_e - p_a)A_e\,\Delta t\,\hat{i}$$

$$m\Delta V - \dot{m}_e\Delta t(V+V_e) - \underbrace{\dot{m}_e\Delta t\Delta V}_{} = (p_e - p_a)A_e\,\Delta t$$

= 0 b/c it's the product of 2 small terms

Divide by Δt:

$$\underbrace{m\frac{dv}{dt}}_{=T} - \dot{m}_e(v+v_e) = (p_e - p_a)A_e$$

$$T = m\frac{dv}{dt} = \dot{m}_e(v+v_e) + (p_e - p_a)A_e$$

Define an effective exhaust velocity:

$$T = \dot{m}_e V_{eff}$$

$$V_{eff} = V + V_e + \frac{(p_e - p_a)A_e}{\dot{m}_e}$$

Note: $g_0 = 9.81\ m/s^2$

Specific Impulse:

$$\boxed{I_{sp} = \frac{T}{\dot{m}_e g_0}} = \frac{\text{Thrust}}{\text{rate of propellant weight consumption}}$$

Units: seconds

Large I_{sp} if we have a large thrust or if $\dot{m}_e$ is small

Typical Chemical propulsion:

Solid: $I_{sp} \sim 200-300\,s$

liquid: $I_{sp} \sim 250-450\,s$

Electric Propulsion: $I_{sp} \sim 1000-5000$ sec, but low thrust

$$T = I_{sp}\,\dot{m}_e\,g_0 = -I_{sp}\,g_0\,\frac{dm}{dt}$$

$$\Rightarrow \frac{dm}{dt} = \frac{-T}{I_{sp}\,g_0}$$

$$\frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g\sin\gamma$$

gravity on rocket launching from Earth's surface

$$\frac{dv}{dt} = -\frac{I_{sp}\,g_0}{m}\frac{dm}{dt} - \frac{D}{m} - g\sin\gamma$$

Integrate wrt time:

$$\Delta V = I_{sp}\,g_0 \ln\left(\frac{m_i}{m_f}\right) - \Delta V_D - \Delta V_G$$

ΔV due to gravity

ΔV due to drag

If we are in free space, we can neglect drag & gravity.

$$\boxed{\Delta V = I_{sp}\,g_0 \ln\left(\frac{m_i}{m_f}\right)}$$

← Approximation! B/c dropped H.O.T.

& T = const.

We can rearrange to solve for the mass ratio ($n = m_i/m_f$)

$$\frac{m_i}{m_f} = \exp\left(\frac{\Delta V}{I_{sp}\,g_0}\right)$$

Solve for Δm required to produce ΔV: $\Delta m = m_i - m_f$

$$\boxed{\frac{\Delta m}{m_i} = 1 - \exp\left(\frac{-\Delta V}{I_{sp}\,g_0}\right)}$$

As $\Delta V \uparrow$, exponential term $\downarrow$ & $\frac{\Delta m}{m_i} \to 1 \Rightarrow$ which means that the whole s/c mass is fuel mass

Example of benefits of Electric Propulsion: Dawn Mission: W/Chemical Propulsion: 26,000 kg

W/ electric propulsion $\Rightarrow$ 1,240 kg

$$m_i = m_E + m_p + m_{PL}$$

empty mass (structures)

propellant mass

Payload mass

Define payload ratio: $\lambda = \dfrac{m_{PL}}{m_i - m_{PL}}$

Structural ratio: $\epsilon = \dfrac{m_E}{m_i - m_{PL}}$

Mass fraction: $n = \dfrac{m_i}{m_f} = \dfrac{m_E + m_p + m_{PL}}{m_E + m_{PL}}$

$$\Rightarrow n = \frac{1+\lambda}{\epsilon + \lambda}$$

For a chemical rocket with $I_{sp} = 300\,s$, $\epsilon = 0.1$, $\lambda = 0.05$ $\Rightarrow$ what's the max ΔV provided?

$\Delta V = 5.7$ km/s

Low Thrust Trajectories: the maneuver can take months → years

more complicated to solve b/c the s/c $\mathcal{E}$ ($\vec{h}$) is constantly changing

↳ typically need to numerically integrate to design these trajectories.

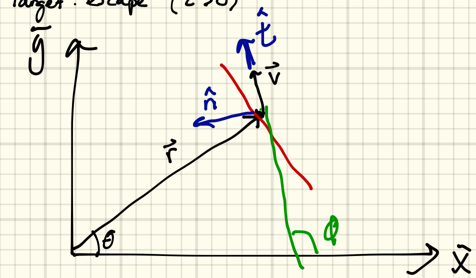
Can modify 2BP numerical integration code to include an additional thrust term.

The complicated part of low-thrust trajectory design is figuring out the control law that gets the s/c to the desired orbit in the min time with min fuel.

We will consider a special case where the s/c is thrusting tangentially to the orbit at all times.

Assume the s/c is initially in a circular orbit with radius r_0.

Target: escape ($\mathcal{E} > 0$)



Define the radius of curvature: $\rho = \dfrac{ds}{d\phi}$

Note: $\vec{v} = v\hat{t}$

$\dot{\vec{v}} = \dot{v}\hat{t} + v\dot{\phi}\hat{n}$

$v = \dfrac{ds}{dt}$ = arc-length per time

$\dfrac{d\phi}{dt} = \dfrac{d\phi}{ds}\dfrac{ds}{dt} = \dfrac{v}{\rho}$

$\dot{\vec{v}} = \dot{v}\hat{t} + \dfrac{v^2}{\rho}\hat{n}$

$\hat{t} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\dot{\hat{t}} = -\dot{\phi}\sin\phi\,\hat{x} + \dot{\phi}\cos\phi\,\hat{y}$

$\hat{n} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\dot{\hat{t}} = \dot{\phi}\hat{n}$