Thus, generally the control calculations required by $H(s)$ can be implemented using:

$$\boxed{u(t) = \mathcal{L}^{-1}\{d(s)E(s)\} + \sum_{k=1}^{M} c_k x_k(t)}$$

where

$$\boxed{\dot{x}_k(t) = \ell_k x_k(t) + e(t)}$$

M different 1^{st} order DEs for $x_k(t)$

ℓ_k are poles of $H(s)$, and c_k are the residues:

$$c_k = \left\{(s-\ell_k)\left[\frac{a(s)}{b(s)}\right]\right\}_{s=\ell_k}$$

What about $\mathcal{L}^{-1}\{d(s)E(s)\}$? Recall $d(s)$ is a polynomial with degree $\deg\{a(s)\} - \deg\{b(s)\}$

If $\deg\{d(s)\} > 1$ $\quad (\deg\{a(s)\} > \deg\{b(s)\})$

i.e.

$$d(s) = d_0 + d_1 s + d_2 s^2 + \cdots$$

This term is OK!

then $\mathcal{L}^{-1}\{d(s)E(s)\} = d_0 e(t) + \underbrace{d_1 \dot{e}(t) + d_2 \ddot{e}(t) + \cdots}$

Cannot be implemented with assumed measurements.

Thus, these add'l terms can only be implemented if

$$\deg\{d(s)\} = \emptyset$$ (i.e. $d(s)$ is just a <u>constant</u>)

Or equivalently $\deg\{\underline{a(s)}\} \le \deg\{\underline{b(s)}\}$

numerator of $H(s)$

Denom $H(s)$

Eq'ns are naturally in state-space format:

$$\Rightarrow \begin{cases} \dot{x}_K(t) = \ell_K x_K(t) + e(t) \\ u(t) = \sum c_K x_K(t) + d_o e(t) \end{cases}$$

$K = 1, \ldots, M$ = # Poles $H(s)$

ℓ_K = pole of $H(s)$

Note this is a state-space model for $H(s)$!

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \ell_1 & 0 & \cdots & 0 \\ 0 & \ell_2 & 0 \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & \ell_m \end{bmatrix}}_{A_c}\begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{B_c} e(t)$$

$$u(t) = \underbrace{\begin{bmatrix} c_1 & c_2 & \cdots & c_m \end{bmatrix}}_{C_c}\begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} + \underbrace{d_o}_{D_c} e(t)$$

"0" if $\rho(H) > 0$

$[A_c, B_c, C_c, D_c]$ state-space "realization" of Compensator TF $H(s)$

(One possible form $\Rightarrow$ others are possible)

Relative Degree

The relative degree of a transfer function $G(s)$ is:

$$\rho(G) = \text{Degree of Denom poly} - \text{Degree of num poly}$$
$$= \#\text{poles of } G - \#\text{zeros of } G$$

From the above, the constraint for real-time implementation of compensator $H(s)$ is:

$$\rho(H) \geq \emptyset$$

i.e. $H(s)$ must have No more zeros than it has poles.

$\Rightarrow$ Will be a significant constraint on our designs!

Examples, cont

1.) $H(s) = 6(s+1)^2 = 6s^2 + 12s + 6$ $\quad (\rho = -2)$ ✗

$\Rightarrow u(t) = \underline{6\ddot{e}(t) + 12\dot{e}(t)} + 6e(t)$ $\quad$ **Not implementable**

2.) $H(s) = \dfrac{6(s+1)^2}{(s+3)} = 6s - 6 + \dfrac{24}{s+3}$ $\quad (\rho = -1)$ ✗

$$\Rightarrow \begin{cases} u(t) = \underline{6\dot{e}(t)} - 6e(t) + 24x_1(t) \\ \dot{x}_1(t) = -3x_1(t) + e(t) \end{cases}$$

Not implementable

3.) $H(s) = \dfrac{6(s+1)^2}{(s+3)(s+5)} = 6 + \dfrac{12}{s+3} - \dfrac{48}{s+5}$ $\quad (\rho = 0)$ ✓

$$\Rightarrow \begin{cases} u(t) = 6e(t) + 12x_1(t) - 48x_2(t) \\ \dot{x}_1(t) = -3x_1(t) + e(t) \\ \dot{x}_2(t) = -5x_2(t) + e(t) \end{cases}$$

Implementable!

- Now that we have a practical constraint on $H(s)$ we can start to consider designing it to meet these and other constraints. Let's consider some simple examples.
- Typical freq domain constraints are to have a mag xover (ω_γ) sufficiently large to ensure a fast settling time, and phase marg. (γ) sufficiently large to ensure small overshoot.
- Most physical systems have phase that approaches or falls below $-180°$ at higher freqs, which means typically $H(s)$ will need to increase the phase in at least a band of higher freqs.

Design Study I:

Suppose $G(s) = \frac{3}{s(s+2)}$, and we <u>want</u> a stable CL system with $\omega_\gamma = 6$, $\gamma = 45°$.

With $H(s) = K$, these constraints are not achievable since $\angle G(6j) < -135°$.

Using above example, we know specs are met if:

$$L(s) = \frac{6^2\sqrt{2}}{s(s+6)} \quad (\alpha = 6 \text{ in prev. Example})$$

$\Rightarrow$ Choose

$$H(s) = \left(\frac{6^2\sqrt{2}}{3}\right)\frac{(s+2)}{(s+6)}$$

So $L(s) = G(s)H(s)$ has desired properties

Note: Design here uses <u>stable</u> pole-zero cancellation.

Design Study, II

Suppose instead want $\omega_\gamma = 6$, $\gamma = 60°$. Specs can't be met so easily as above.

<u>Need</u>: $\measuredangle L(j\omega_{Des}) = -120° = \gamma_{Des} - 180°$ ($\gamma_{des} = 60°$ here, and $\omega_{Des} = 6$ here)

But $\measuredangle L(j\omega_{Des}) = \measuredangle G(j\omega_{Des}) + \measuredangle H(j\omega_{Des})$

Hence: $\gamma_{Des} - 180° = \measuredangle G(j\omega_{Des}) + \measuredangle H(j\omega_{Des})$

Or: $\measuredangle H(j\omega_{Des}) = \gamma_{Des} - 180° - \measuredangle G(j\omega_{Des})$

$= \phi_{req}$ "phase Deficit"

ϕ_{req} is required phase (typically positive) that compensator must provide at ω_{Des} to meet specs.

For $G(s) = \frac{3}{s(s+2)}$, $\gamma_{Des} = 60°$, $\omega_{Des} = 6$

$\angle G(6j) = -161.6°$

$$\phi_{req} = 60° - 180° - 161.56°$$

$\Rightarrow \phi_{req} = 41.56°$

Now suppose we could ideally implement only a LHP zero in $H(s)$

$$\Rightarrow H(s) = K(s - z_c) \quad \text{in ZPK form}$$

(Note we can't do this generally, but it is a convenient hypothetical starting point to illustrate the thought process).

$\Phi_{req} = 41.56°$, $H(s) = K(s - z_c)$ $\quad K, z_c > \emptyset$

Choose K, z_c so that

$$\measuredangle(j\omega_{Des} - z_c) = \Phi_{req}$$

and

$$|L(j\omega_{Des})| = 1$$

<u>Decoupled!</u>

$$\measuredangle(j\omega_{Des} - z_c) = \tan^{-1}\left(\frac{\omega_{Des}}{-z_c}\right) = \tan^{-1}\left(\frac{\omega_{Des}}{|z_c|}\right) \text{ since } z_c < \emptyset$$

$$\Rightarrow \frac{\omega_{Des}}{|z_c|} = \tan\Phi_{req} \quad \text{or} \quad z_c = -\left[\frac{\omega_{Des}}{\tan\Phi_{req}}\right]$$

Here $z_c = -\left[\dfrac{6}{\tan 41.56°}\right] = -6.77$

So now $H(s) = K(s + 6.77)$. Find K

Finding K

Let $L_o(s) = L(s)\Big]_{K=1} \Rightarrow L(s) = K L_o(s)$

Then choose

$$K = \frac{1}{|L_o(j\omega_{des})|}$$

Since then

$$|L(j\omega_{Des})| = |K L_o(j\omega_{Des})| = |K|\,|L_o(j\omega_{Des})|$$

$$= 1$$

i.e. ω_{Des} is mag xover freq. for $L(j\omega)$, as desired

Here: $$L_o(s) = \frac{3(s+6.77)}{s(s+2)}$$

$$|L_o(6j)| = 0.715 \Rightarrow K \cong 1.4$$

$$H(s) = 1.4(s+6.77)$$

$$U(s) = H(s)E(s)$$

$$= [1.4s + 1.4 \times 6.77]E(s)$$

$$\Rightarrow u(t) = \underbrace{1.4}_{K_D}\dot{e}(t) + \underbrace{1.4 \times 6.77}_{K_P}e(t)$$

"PD" controller

Above is not generally implementable $\quad p(H) < \emptyset$

$\Rightarrow$ $H(s)$ must contain at least 1 (LHP) pole to balance the zero (make $p(H) \geq \emptyset$)

$\Rightarrow$ LHP poles contribute negative phase, hence work against our objectives

One strategy (not necessarily the best, but easy to do): put pole of $H(s)$ so that its impact on phase of $L(j\omega)$ is negligible at least near desired crossover ω_{Des}

i.e. $$H(s) = K\left[\frac{(s - z_c)}{(s - p_c)}\right]$$

with $|p_c| \geq 10\omega_{Des}$

(Recall p_c will change phase starting for $\omega \gtrsim \frac{|p_c|}{10}$; want this above ω_{Des}).

If $P_c = -10\omega_{Des}$, then

$$\measuredangle(j\omega_{Des} - P_c) = \measuredangle(j\omega_{Des} + 10\omega_{Des})$$

$$= \tan^{-1}\left(\frac{1}{10}\right) \approx 5.7^\circ$$

and $\measuredangle H(j\omega_{Des}) = \measuredangle(j\omega_{Des} - Z_c) - \measuredangle(j\omega_{Des} - P_c)$

$$= \measuredangle(j\omega_{Des} - Z_c) - 5.7^\circ$$

Still need: $\measuredangle H(j\omega_{Des}) = \Phi_{req}$

So choose:

$$\measuredangle(j\omega_{Des} - Z_c) = \boxed{\Phi_{req} + 5.7^\circ}$$

Then choose K as before.

For our example we need

$\measuredangle(6j - z_c) = 41.56° + 5.7° = 47.26°$

$\Rightarrow \quad z_c = -5.54$

$$\Rightarrow \; L_0(s) = \frac{3(s+5.54)}{s(s+2)(s+60)}$$

$\Rightarrow \; K = 93.37$

$$\boxed{H(s) = \frac{93.37(s+5.54)}{(s+60)}}$$

$\dot{x}_1 = -60x_1 + e$

$u = \underline{K_1}\, e + \underline{K_2}\, x_1$

Note big increase in K! Generally associated with bigger u(t). Must check for saturation!

Ideal (pure zero) result obtained As $p_c \to -\infty$ (pole very far into LHP), but is associated with very large control inputs. Can do some simple z_c, p_c optimization to moderate control magnitude

Simple optimization of required location for p_c

We have seen a simple strategy for choosing required pole in $H(s)$ is to make $p_c < -10\,\omega_{Des}$

$\Rightarrow$ Ensures p_c subtracts no more than $5.7°$ from $\measuredangle H(j\omega_{Des})$, easy to adjust location of z_c to "make up" this phase loss to maintain $\measuredangle H(j\omega_{Des}) = \phi_{req}$.

However, such a strategy often results in undesireably large $u(t)$.

Try to balance the competing requirements by finding minimum possible ratio p_c/z_c which still provides $\measuredangle H(j\omega_{Des}) = \phi_{req}$.