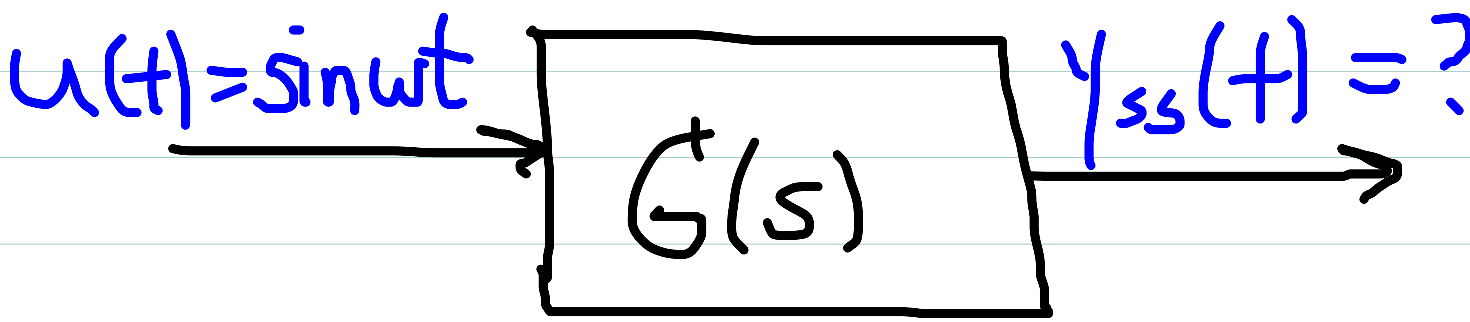
Sinusoidal Response

Here we wish to understand the properties of the steady-state response of a $\boxed{\text{stable}}$ system when $\boxed{u(t) = \sin\omega t.}$

Note: our focus is shifting (temporarily) away from the transient response



Of course, we've already solved this problem:

$$u(t) = \sin\omega t = \operatorname{Im}\{e^{j\omega t}\}$$

$$\Rightarrow y_f(t) = \operatorname{Im}\{G(j\omega)e^{j\omega t}\} = |G(j\omega)|\sin(\omega t + \measuredangle G(j\omega))$$

Then $y(t) = y_f(t) + y_h(t)$

But if system is stable, $y_h(t) \to \phi$ as $t \to \infty$ for any set of initial cond'ns.

Hence $y_{tr}(t) = y_h(t)$ leaving us with

$$y_{ss}(t) = |G(j\omega)|\sin(\omega t + \measuredangle G(j\omega))$$

So:

$$u(t) = \sin \omega t \implies y_{ss}(t) = |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

<u>Note:</u>

$y_{ss}(t)$ is sinusoidal at same frequency as $u(t)$

<u>But:</u>

Amplitude and phase of $y_{ss}(t)$ different.

Now, more generally suppose:

$$u(t) = B \sin(\omega t + \psi) = \operatorname{Im}\{U e^{j\omega t}\}, \quad U = B e^{j\psi}$$

then

$$y_{ss}(t) = \operatorname{Im}\{G(j\omega) U e^{j\omega t}\}$$

$$= |G(j\omega)| \cdot |U| \sin(\omega t + \angle G(j\omega) + \angle U)$$

or

$$y_{ss}(t) = |G(j\omega)| B \sin(\omega t + \angle G(j\omega) + \psi)$$

Thus generally:

$$u(t) = B\sin(\omega t + \psi) \implies y_{ss}(t) = A\sin(\omega t + \varphi)$$

where: $A = |G(j\omega)|B$

$$\varphi = \measuredangle G(j\omega) + \psi$$

Define:

Amplitude ratio: A/B (ratio of output ampl. to input ampl.

Phase shift: $\varphi - \psi$ (Diff. between output and input phase)

Then note:

$$\boxed{\begin{aligned} A/B &= |G(j\omega)| \\ \varphi - \psi &= \measuredangle G(j\omega) \end{aligned}}$$

So generally

$|G(j\omega)|$ quantifies the ratio between output and input amplitude

$\angle G(j\omega)$ quantifies the change in phase of output compared to input

Note: these are frequency dependent

i.e. the amplitude ratio and phase shift depend on frequency of input.

Very useful to quantify this dependence!

Example

$$G(s) = \frac{3}{s+2}$$

$$|G(j\omega)| = \frac{3}{\sqrt{\omega^2+4}} \qquad \measuredangle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

$$\omega = 1/2 \Rightarrow |G(j/2)| = 3/\sqrt{4.25} \approx 1.46$$

$$\measuredangle G(j/2) = -\tan^{-1}(1/4) = -.245 \text{ rad or } -14.04^\circ$$

$$\omega = 2 \Rightarrow |G(2j)| = 3/\sqrt{8} \approx 1.06$$

$$\measuredangle G(2j) = -\tan^{-1}(1) = -\pi/4 = -45^\circ$$

$$\omega = 20 \Rightarrow |G(20j)| = 3/\sqrt{404} = 0.15$$

$$\measuredangle G(20j) = -\tan^{-1}(10) = -1.47 \approx -84.3^\circ$$

$\Rightarrow$ Want to learn to <u>predict</u> these changes based on ZPK structure of $G(s)$

$\Rightarrow$ Useful also to visualize <u>graphically</u>

$\Rightarrow$ Three methods

1. Plot $|G(j\omega)|$ and $\measuredangle G(j\omega)$ vs. $\omega \geq 0$ (2 plots)

2. Plot $G(j\omega)$ as ω varies from 0 to ∞ as points in complex plane.

3. Plot $|G(j\omega)|$ vs. $\measuredangle G(j\omega)$ for $0 \leq \omega < \infty$

$\Rightarrow$ Want to learn to predict these changes based on ZPK structure of $G(s)$

$\Rightarrow$ Useful also to visualize graphically

$\Rightarrow$ Three methods

1. Plot $|G(j\omega)|$ and $\measuredangle G(j\omega)$ vs. $\omega \geq 0$ (2 plots) "Bode diagrams"

2. Plot $G(j\omega)$ as ω varies from 0 to ∞ as points in complex plane. "Polar diagram"

3. Plot $|G(j\omega)|$ vs. $\measuredangle G(j\omega)$ for $0 \leq \omega < \infty$ "Nichols chart"

Bode is most fundamental, start there

⇒ want to see behavior for large range of $\omega \geq \emptyset$

⇒ $|G(j\omega)|$ will vary enormously in size

⇒ Use logarithmic scales for plots.

⇒ Horizontal Axis on Bode diagram is freq on a log scale

⇒ equally spaced divisions on this scale are factors of 10 apart.

⇒ We call one of these divisions a "decade"

$\left.\begin{matrix} 1/10 \to 1 \\ 2 \to 20 \end{matrix}\right\}$ one decade

$\left.\begin{matrix} 1/10 \to 10 \\ 2 \to 200 \end{matrix}\right\}$ two decades

Decibels

$|G(j\omega)|$ is shown on Bode diagrams in special units called decibels.

Def'n: For any real number $X \geq \emptyset$

$$X_{db} = 20 \log X$$

Conversely

$$X = 10^{(X_{db}/20)}$$

Example (from above): $X = 1.46 \Rightarrow X_{db} = 3.25$

$X = 1.06 \Rightarrow X_{dB} = 0.51$

$X = 0.15 \Rightarrow X_{dB} = -16.5$

Common Shorthand

$$X = 0.15 = -16.5\,dB$$

Note common conversions

X	X (dB)
.01	-40
.1	-20
1	0
10	20
100	40

Important ⟹

Zero on dB axis means magnitude of 1!!

Bode diagrams show

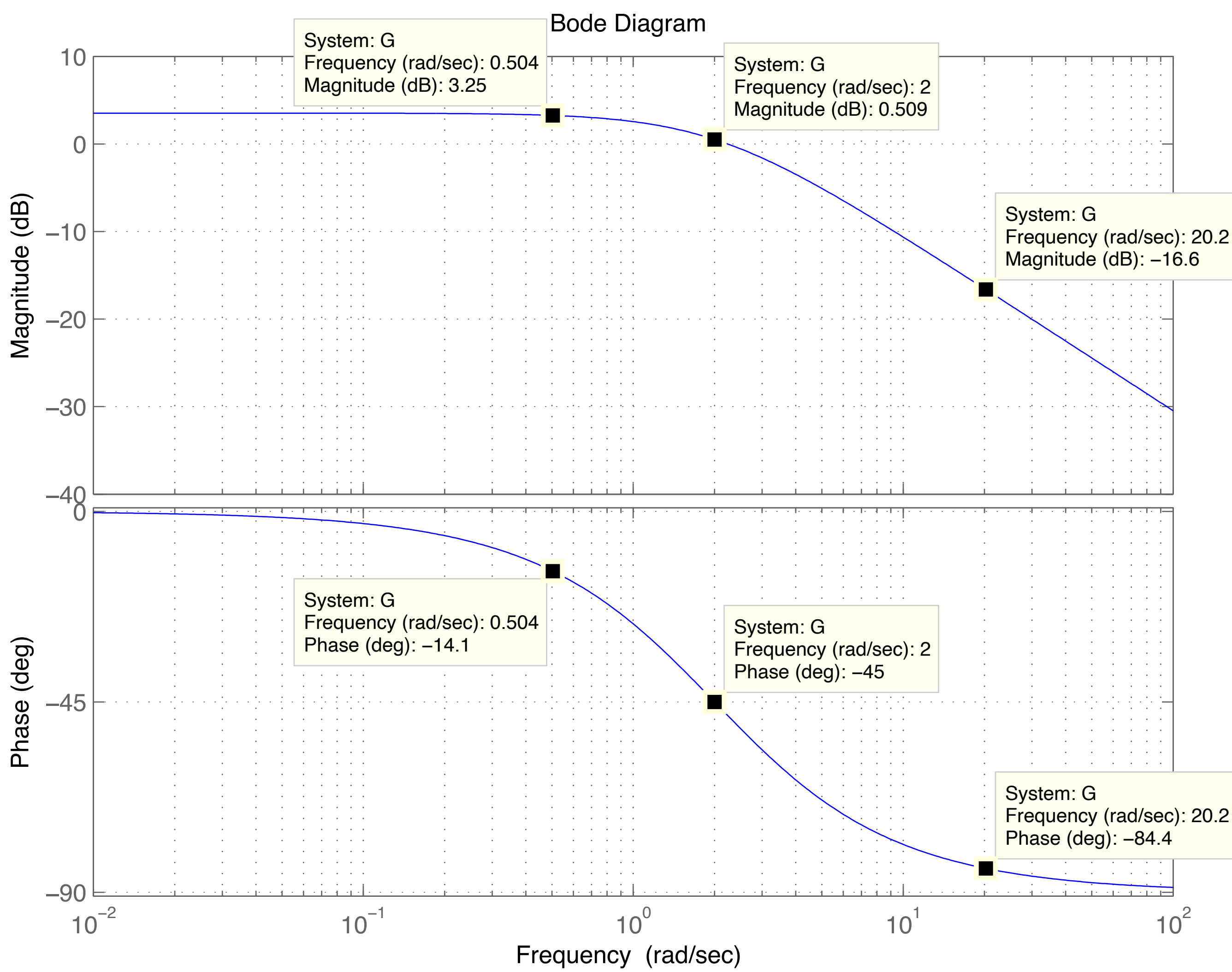
(1.) $|G(j\omega)|$ in dB vs ω on a log scale

(2.) $\measuredangle G(j\omega)$ in deg " " "

See example

Note: there are no negative frequencies on a Bode diagram!

The left limit of the horizontal scale corresponds to $\omega \to \emptyset$!


Bode Diagram
System: G
Frequency (rad/sec): 0.504
Magnitude (dB): 3.25
System: G
Frequency (rad/sec): 2
Magnitude (dB): 0.509
System: G
Frequency (rad/sec): 20.2
Magnitude (dB): −16.6
Magnitude (dB)
10
0
−10
−20
−30
−40
0
−45
−90
Phase (deg)
System: G
Frequency (rad/sec): 0.504
Phase (deg): −14.1
System: G
Frequency (rad/sec): 2
Phase (deg): −45
System: G
Frequency (rad/sec): 20.2
Phase (deg): −84.4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Recap: Frequency Response Analysis

$$u(t) = B\sin(\omega t + \psi) \Rightarrow y_{ss}(t) = A\sin(\omega t + \varphi)$$

$$A = B|G(j\omega)|, \quad \varphi = \measuredangle G(j\omega) + \psi$$

Bode diagrams: show

$|G(j\omega)|$ (dB) vs. ω (log scale) "Magnitude diagram"

$\measuredangle G(j\omega)$ (deg) vs. ω (log scale) "Phase diagram"

$\boxed{\text{Want to learn to rapidly predict the shapes of these diagrams from the ZPK structure of transfer function } G(s)}$

How?

Will Show:

1. Effect of each pole p_k and zero z_i is concentrated in a narrow band of frequencies near $\omega = |p_k|$ (or $|z_i|$, as appropriate)

   $\Rightarrow$ remember: $\omega \geq 0$ on Bode diagrams. There are no negative frequencies shown!

2. Effect of individual poles/zeros on *total* Bode diagrams are <u>additive</u>

"Bode form" of transfer function

ZPK form:

$$G(s) = K\left[\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{k=1}^{n}(s-p_k)}\right]$$

Bode form:

$$G(s) = K_B \frac{\prod_{i=1}^{m}(1-s/z_i)}{s^N \prod_{k=N+1}^{n}(1-s/p_k)}$$

N = # of poles at origin "Type" of system

K_B = "Bode gain"; note $N=\emptyset \Rightarrow K_B = G(\emptyset)$

Bode and ZPK forms are two different ways of writing the same transfer function

Example:

$$G(s) = \frac{5(s+2)}{s(s+3)(s+4)} \qquad \text{(ZPK)}$$

(Bode) $$= \left(\frac{5}{6}\right)\left[\frac{(1+s/2)}{s(1+s/3)(1+s/4)}\right]$$

Here $N = 1$ and $K_B = 5/6$

Algebraically equivalent to ZPK form.

i.e. both are the same TF

So:

$$G(j\omega) = K_B \left[\frac{\prod_{i=1}^{m} (1 - j\omega/z_i)}{(j\omega)^N \prod_{k=N+1} (1 - j\omega/p_k)} \right]$$

for any real $\omega \geq 0$, $G(j\omega)$ is complex and so are each individual factor (except K_B, which is real)

recall for any $s_1, s_2 \in \mathbb{C}$

$$\boxed{\begin{aligned} \measuredangle(s_1 s_2) &= \measuredangle s_1 + \measuredangle s_2 \\ \measuredangle\left(\frac{s_1}{s_2}\right) &= \measuredangle s_1 - \measuredangle s_2 \\ \measuredangle s_1^N &= N \measuredangle s_1 \end{aligned}}$$

Thus:

$$\measuredangle G(j\omega) = \measuredangle K_B + \underline{\sum_{i=1}^{m} \measuredangle(1 - {j\omega}/{z_i})} - N\measuredangle(j\omega) - \underline{\sum_{k=N+1}^{n} \measuredangle(1 - {j\omega}/{p_k})}$$

Note: (1.) Each factor contributes additively

(2.) Zeros add to angle, poles subtract

(3) $\measuredangle K_B$ same for any ω:

$$\measuredangle K_B = \emptyset \ (K_B > \emptyset), \quad \measuredangle K_B = \pm 180° \ (K_B < \emptyset)$$

(3.) $\measuredangle(j\omega)$ is same for any $\omega \geq \emptyset$

$$\measuredangle(j\omega) = 90°$$

(4.) Changes to $\measuredangle G(j\omega)$ as ω varies depends on specific z_i and nonzero p_k.

What about Magnitudes?

Recall: for $s_1, s_2 \in \mathbb{C}$

$$|s_1 s_2| = |s_1|\,|s_2|$$

$$\left|\frac{s_1}{s_2}\right| = \frac{|s_1|}{|s_2|}$$

$$|s_1^N| = |s_1|^N$$

So:

$$|G(j\omega)| = |K_B| \frac{\prod_{i=1}^{m} \left|1 - j\omega/z_i\right|}{|j\omega|^N \prod_{k=N+1}^{n} \left|1 - j\omega/p_k\right|}$$

UGLY...

But Bode shows $|G(j\omega)|$ in dB

i.e. $20\log|G(j\omega)|$

Now recall:

$$\log(xy) = \log x + \log y$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\log(x^N) = N\log x$$

Hence in dB:

$$|G(j\omega)|_{dB} = |K_B|_{dB} + \boxed{\sum_{i=1}^{m}\left|1-{}^{j\omega}/z_i\right|_{dB}} - N|j\omega|_{dB} - \boxed{\sum_{K=N+1}^{n}\left|1-\frac{j\omega}{P_K}\right|_{dB}}$$

Notes:

1. Magnitudes in dB are additive for each factor
2. zeros add to magnitude, poles subtract
3. $|K_B|$ is constant for all ω, like with phase
4. $|j\omega|$ is Not constant, unlike phase.

So, we see effect of individual parts of $G(s)$ contribute additively to

$$\angle G(j\omega) \quad \text{and} \quad |G(j\omega)|_{dB}$$

Look at effect of individual factors

Look at how each $(1 - j\omega/z_i)$ or $(1 - j\omega/p_k)$

Changes with ω.

To simplify notation, we'll look at $(1 + j\omega\tau)$, where

$\tau = -1/z_i$ or $\tau = -1/p_k$ as appropriate

Then:

$$|1 + j\omega\tau| = \sqrt{1 + \omega^2\tau^2}$$

and

$$\measuredangle(1 + j\omega\tau) = \tan^{-1}\omega\tau$$

Study how these vary with ω

Consider first magnitude

$$|1+j\omega\tau| = \sqrt{1+(\omega\tau)^2} = \begin{cases} 1 & \text{if } \omega \ll 1/|\tau| \\ \sqrt{2} & \text{if } \omega = 1/|\tau| \\ \omega|\tau| & \text{if } \omega \gg 1/|\tau| \end{cases}$$

and thus:

$$|1+j\omega\tau|_{dB} = \begin{cases} \emptyset & \omega \ll 1/|\tau| \quad \text{"Low freq. Limit"} \\ 3 & \omega = 1/|\tau| \\ 20\log\omega|\tau| & \omega \gg 1/|\tau| \quad \text{"high freq Limit"} \end{cases}$$

Look at 3rd case:

$$20\log\omega|\tau| = 20\left[\log\omega + \log|\tau|\right]$$

Note when $\omega = 1/|\tau|$, $\log\omega = -\log|\tau|$ + 3rd case evaluates to $\emptyset$.

Also:

in high freq limit $\omega \gg 1/|\tau|$

$$|1+j\omega\tau|_{dB} = 20\left[\log\omega + \log|\tau|\right]$$

Suppose we have two freqs, ω_1, ω_2 both $\gg 1/|\tau|$

with $\omega_2 = 10\omega_1$, then:

$$|1+j\omega_2\tau|_{dB} = |1+j(10\omega_1)\tau|_{dB}$$

$$= 20\left[\log(10\omega_1) + \log|\tau|\right]$$

$$= 20\left[\log\omega_1 + \log 10 + \log|\tau|\right]$$

$$= 20\left[\log\omega_1 + \log|\tau|\right] + 20$$

So

$$|1+j\omega_2\tau|_{dB} = |1+j\omega_1\tau|_{dB} + 20 \Leftarrow \text{+20 dB increase}$$

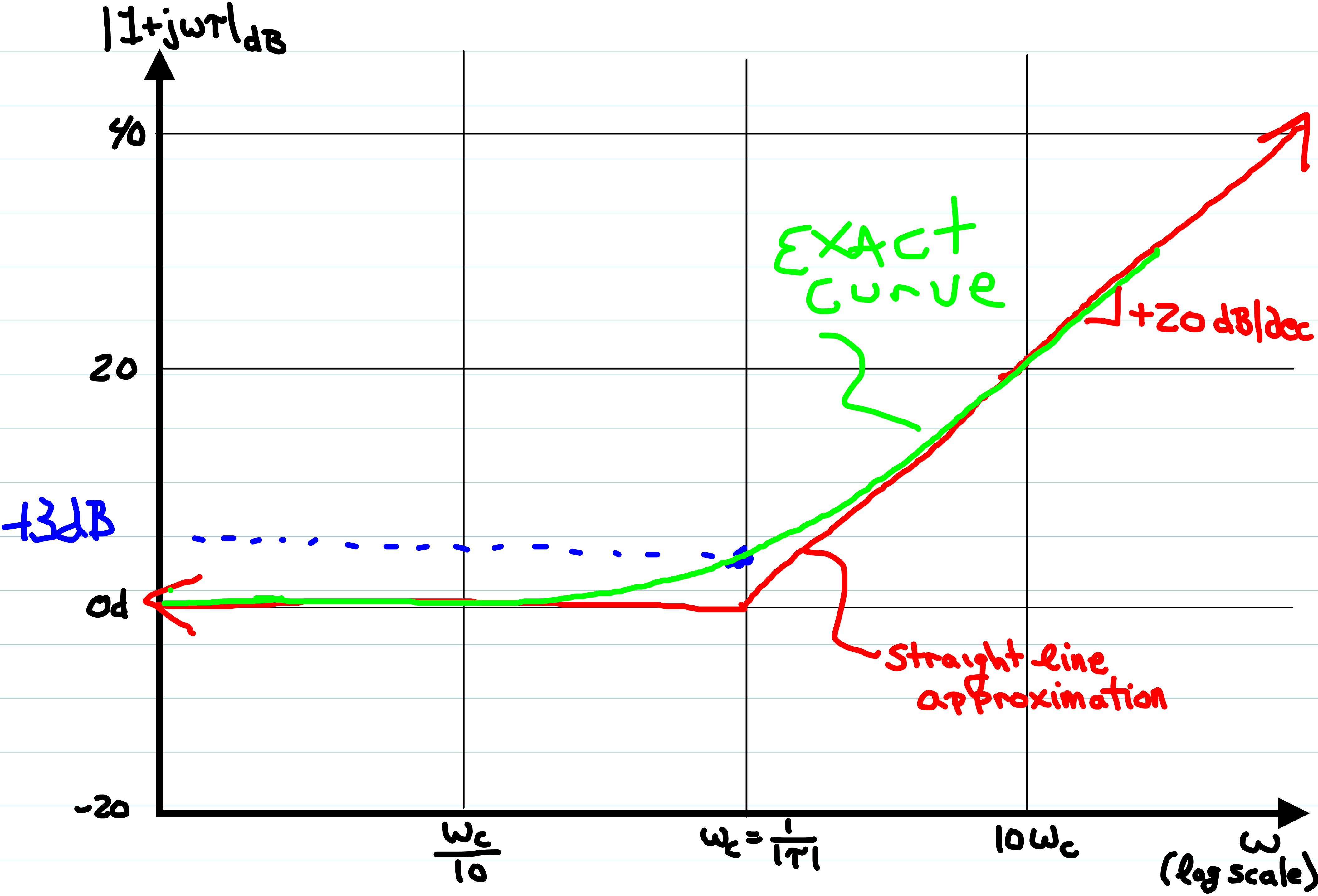
Hence:

in high frequency region $|1+j\omega\tau|_{dB}$ increases by 20dB for every factor of 10 increase in frequency (decade)

$\Rightarrow$ graph has a slope of 20dB/decade in high freq. region

$\Rightarrow$ Recall graph is constant at 0dB in low freq. region

$\Rightarrow$ The two limiting cases come together at the "corner frequency", $\omega_c = \dfrac{1}{|\tau|}$.



$|1+j\omega\tau|_{dB}$
40
20
0d
-20
+3dB
EXACT curve
+20 dB/dec
straight line approximation
$\frac{\omega_c}{10}$
$\omega_c = \frac{1}{|\tau|}$
$10\,\omega_c$
ω (log scale)

Things to note:

- Graph changes slope by +20 dB/dec

- Think in terms of this slope change, not the total shape

- Recall $(1+j\omega\tau)$ is a generic representation of a factor of $G(s)$, either

$$\left(1-{}^{j\omega}/_{z_i}\right) \quad \text{or} \quad \left(1-{}^{j\omega}/_{p_K}\right)$$

i.e. $\tau = {}^{-1}/_{z_i}$ or $\tau = {}^{-1}/_{p_K}$

Thus the corner freq. $\omega_c = {}^{1}/_{|\tau|} = |z_i|$ or $|p_K|$

$\boxed{\text{Corner freq is the absolute VALUE of a pole or zero of } G(s)}$

$\Rightarrow$ Because $|G(j\omega)|_{dB}$ is the sum of the effects of the individual terms $|1 - j\omega/z_i|_{dB}$ $|1 - j\omega/p_k|_{dB}$

Each pole or zero will create a "corner" on the complete graph

$\Rightarrow$ The total graph will have corners at every freq. corresponding to $|z_i|$ and $|p_k|$.

$\Rightarrow$ zeros add to overall $|G(j\omega)|_{dB}$ $\Rightarrow$ slope changes of $+20$ dB/dec at $\omega = |z_i|$, $i = 1 \dots m$

$\Rightarrow$ Poles subtract from overall $|G(j\omega)|_{dB}$ $\Rightarrow$ Slope changes of -20 dB/dec at $\omega = |p_k|$.