Chapter 3

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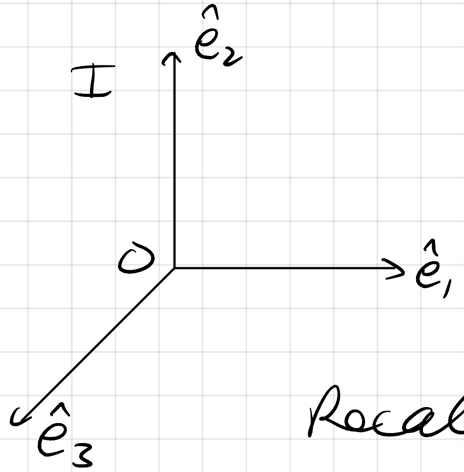


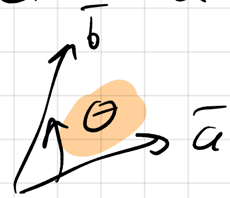
$I = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ inertial frame

$\hat{e}_1 \cdot \hat{e}_2 = 0$ orthogonal

$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$ RHR right-hand rule

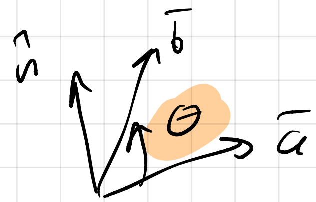
Recall $\bar{a} \cdot \bar{b} = \|\bar{a}\| \, \|\bar{b}\| \cos\theta$

angle between the vectors



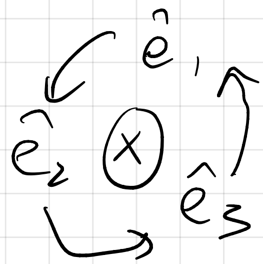
Recall $\bar{a} \times \bar{b} = \|\bar{a}\| \, \|\bar{b}\| \sin\theta \, \hat{n}$

unit vector perp. to both $\bar{a}$ & $\bar{b}$

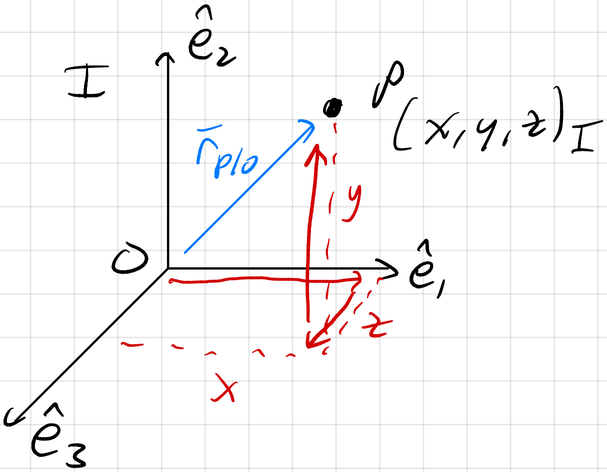


$$\hat{e}_j \cdot \hat{e}_k = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\hat{e}_1 \cdot \hat{e}_1 = \|\hat{e}_1\| \, \|\hat{e}_1\| \cos\theta = 1 \cdot 1 \cdot 1 = 1$$



$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$$
$$\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$$
$$\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$$



$$\bar{r}_{P/O} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$$

vector components

$$[\bar{r}_{P/O}]_I = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_I$$

matrix notation

Cartesian Coordinates $(x, y, z)_I$

$$\bar{r}_{P/O} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$$

$${}^I\bar{v}_{P/O} = \frac{{}^I d}{dt}\left(\bar{r}_{P/O}\right)$$

$$= \frac{{}^I d}{dt}\left(x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3\right)$$

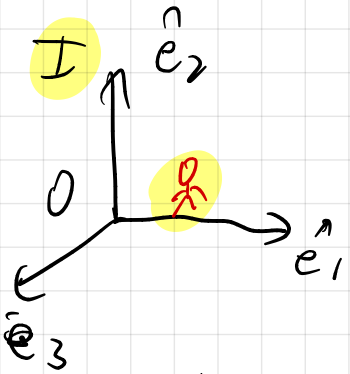
distributive property

$$= \frac{{}^I d}{dt}(x\hat{e}_1) + \frac{{}^I d}{dt}(y\hat{e}_2) + \frac{{}^I d}{dt}(z\hat{e}_3)$$

product rule

$$\frac{d}{dt}(ab) = \frac{d}{dt}(a)b + a\frac{d}{dt}(b)$$

$$= \dot{x}\hat{e}_1 + x\underbrace{\frac{{}^I d}{dt}(\hat{e}_1)}_{=0} + \dot{y}\hat{e}_2 + \cancel{y\frac{{}^I d}{dt}(\hat{e}_2)}^{\,0} + \dot{z}\hat{e}_3 + \cancel{z\frac{{}^I d}{dt}(\hat{e}_3)}^{\,0}$$

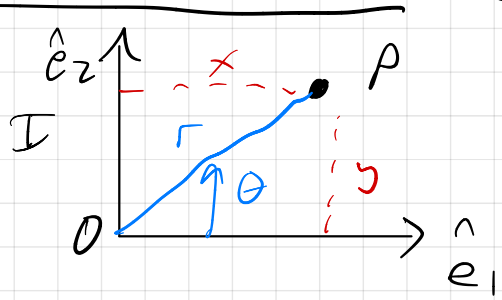


$$= \dot{x}\hat{e}_1 + \dot{y}\hat{e}_2 + \dot{z}\hat{e}_3$$

$${}^I\bar{a}_{P/O} = \ddot{x}\hat{e}_1 + \ddot{y}\hat{e}_2 + \ddot{z}\hat{e}_3$$

inertial kinematics in Cartesian Coords

Polar Coordinates $(r, \theta)_I$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$^I\bar{r}_{P/O} = x\,\hat{e}_1 + y\,\hat{e}_2$$

$$= r\cos\theta\,\hat{e}_1 + r\sin\theta\,\hat{e}_2$$

$$^I\bar{v}_{P/O} = \frac{^I d}{dt}\left(r\cos\theta\,\hat{e}_1 + r\sin\theta\,\hat{e}_2\right)$$

$$= \dot{r}\cos\theta\,\hat{e}_1 - r\sin\theta\,\dot{\theta}\,\hat{e}_1 + \dot{r}\sin\theta\,\hat{e}_2 + r\cos\theta\,\dot{\theta}\,\hat{e}_2$$

$$= \left(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta}\right)\hat{e}_1 + \left(\dot{r}\sin\theta + r\cos\theta\,\dot{\theta}\right)\hat{e}_2$$

$$^I\bar{a}_{P/O} = \frac{^I d}{dt}\left({}^I\bar{v}_{P/O}\right)$$

$$= \dots$$

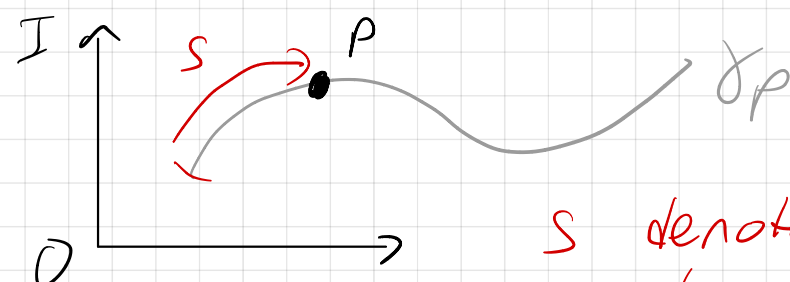
chainrule

$$\frac{d}{dt}(\sin\theta) = \frac{d}{d\theta}\sin\theta\,\frac{d\theta}{dt}$$

$$\cos\theta\,\dot{\theta}$$

inertial kinematics in polar coords expressed as components in the inertial frame

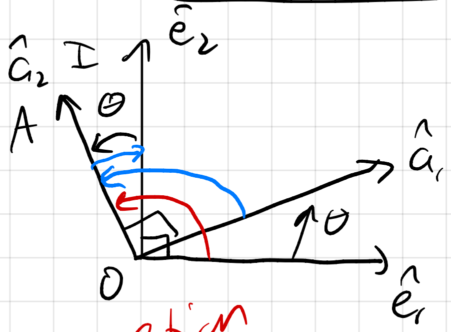
path coords



s denotes the displacement along the path

$\dot{s}$ denotes the rate of change of displacement along the path.

Relative Orientations



$$I = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$$

$$A = (O, \hat{a}_1, \hat{a}_2, \hat{a}_3)$$

$$\hat{a}_3 = \hat{e}_3$$

$$\hat{a}_1 \cdot \hat{e}_1 = \cos\theta$$

$$\hat{a}_2 \cdot \hat{e}_2 = \cos\theta$$

$$\hat{a}_2 \cdot \hat{e}_1 = \cos\left(\frac{\pi}{2} + \theta\right)$$

$$= \cancel{\cos\frac{\pi}{2}}^{\,0} \cos\theta - \sin\frac{\pi}{2}\sin\theta$$

$$= -\sin\theta$$

$$\hat{a}_1 \cdot \hat{e}_2 = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$= \sin\theta$$

Transformation Table

	$\hat{a}_1$	$\hat{a}_2$
$\hat{e}_1$	$\hat{a}_1 \cdot \hat{e}_1$	$\hat{a}_2 \cdot \hat{e}_1$
$\hat{e}_2$	$\hat{a}_1 \cdot \hat{e}_2$	$\hat{a}_2 \cdot \hat{e}_2$

$C\theta = \cos\theta$

$S\theta = \sin\theta$

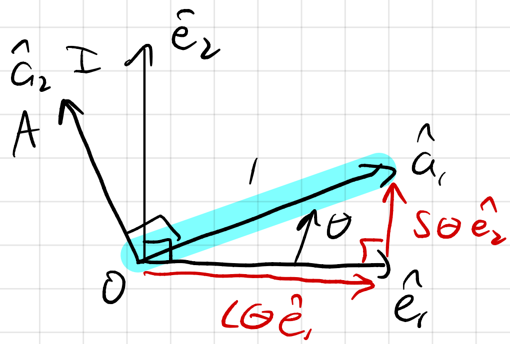
	$\hat{a}_1$	$\hat{a}_2$
$\hat{e}_1$	$C\theta$	$-S\theta$
$\hat{e}_2$	$S\theta$	$C\theta$

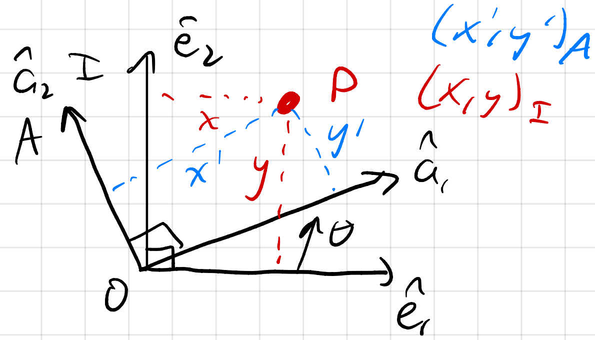
$$\hat{e}_1 = C\theta\,\hat{a}_1 - S\theta\,\hat{a}_2$$

$$\hat{e}_2 = S\theta\,\hat{a}_1 + C\theta\,\hat{a}_2$$

$$\hat{a}_1 = C\theta\,\hat{e}_1 + S\theta\,\hat{e}_2$$

$$\hat{a}_2 = -S\theta\,\hat{a}_1 + C\theta\,\hat{a}_2$$





$(x', y')_A$

$(x, y)_I$

	$\hat{a}_1$	$\hat{a}_2$
$\hat{e}_1$	$C\theta$	$-S\theta$
$\hat{e}_2$	$S\theta$	$C\theta$

$$\bar{r}_{P/O} = x\,\hat{e}_1 + y\,\hat{e}_2 \quad *$$

$$\bar{r}_{P/O} = x'\hat{a}_1 + y'\hat{a}_2$$

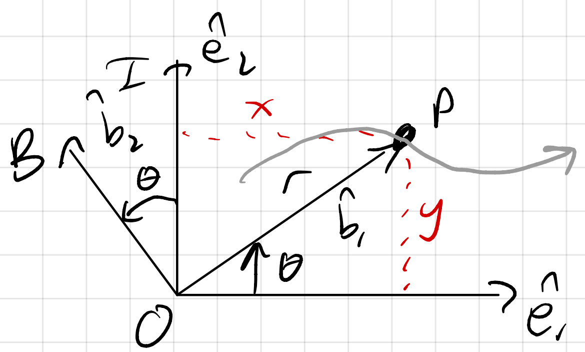
$$r_{P/O} = x(C\theta\,\hat{a}_1 - S\theta\,\hat{a}_2) +$$
$$y(S\theta\,\hat{a}_1 + C\theta\,\hat{a}_2)$$

$$\bar{r}_{P/O} = (x c\theta + y s\theta)\hat{a}_1 + (-x s\theta + y c\theta)\hat{a}_2$$

$$\Rightarrow \quad x' = x c\theta + y s\theta$$
$$y' = -x s\theta + y c\theta$$

$$\bar{r}_{P/O} = x'\hat{a}_1 + y'\hat{a}_2$$
$$= x'(c\theta\,\hat{e}_1 + s\theta\,\hat{e}_2) + y'(-s\theta\,\hat{e}_1 + c\theta\,\hat{e}_2)$$
$$= \underbrace{(x' c\theta - y' s\theta)}_{=x}\hat{e}_1 + \underbrace{(x' s\theta + y' c\theta)}_{=y}\hat{e}_2 \quad *$$

Polar Frame aka Polar Reference Frame



$I = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ inertial

$B = (O, \hat{b}_1, \hat{b}_2, \hat{b}_3)$ polar frame

$$\hat{b}_3 = \hat{e}_3$$

$$\hat{b}_1 = \hat{r}_{P/O} = \frac{\bar{r}_{P/O}}{\|\bar{r}_{P/O}\|}$$

* key idea: $\hat{b}_1$ always points at P

Goal: Find inertial kinematics in polar coords as components in the polar frame

Step #1: write down the position

Step #2: differentiate wrt the inertial frame

$$\bar{r}_{P/O} = r\hat{b}_1$$

$$^I\bar{v}_{P/O} = \frac{^I d}{dt}(\bar{r}_{P/O}) = \frac{^I d}{dt}(r\hat{b}_1)$$

product rule

$$= \dot{r}\,\hat{b}_1 + r\,\boxed{{}^{I}\frac{d}{dt}(\hat{b}_1)}$$

Recall ${}^{I}\frac{d}{dt}(\hat{e}_1) = 0$

${}^{I}\frac{d}{dt}(\hat{e}_2) = 0$

	$\hat{b}_1$	$\hat{b}_2$
$\hat{e}_1$	$C\theta$	$-S\theta$
$\hat{e}_2$	$S\theta$	$C\theta$

$$\hat{b}_1 = C\theta\,\hat{e}_1 + S\theta\,\hat{e}_2$$

$$
\begin{aligned}
{}^{I}\frac{d}{dt}(\hat{b}_1) &= {}^{I}\frac{d}{dt}\left(C\theta\,\hat{e}_1 + S\theta\,\hat{e}_2\right) \\
&= -S\theta\,\dot{\theta}\,\hat{e}_1 + C\theta\,\dot{\theta}\,\hat{e}_2 \\
&= \dot{\theta}\left(-S\theta\,\hat{e}_1 + C\theta\,\hat{e}_2\right) \\
&\overset{\checkmark}{=} \dot{\theta}\,\hat{b}_2
\end{aligned}
$$

$$\dot{\theta}\,\hat{b}_3 \times \hat{b}_1 = \dot{\theta}(\hat{b}_3 \times \hat{b}_1)$$

$${}^{I}\bar{v}_{P/O} = \dot{r}\,\hat{b}_1 + r\dot{\theta}\,\hat{b}_2$$

Dfn <u>angular velocity</u> $\boxed{{}^{I}\bar{\omega}^{B} = \dot{\theta}\,\hat{b}_3}$

$$\boxed{{}^{I}\frac{d}{dt}(\hat{b}_1) = {}^{I}\bar{\omega}^{B} \times \hat{b}_1}$$

$$= \dot{\theta}\,\hat{b}_3 \times \hat{b}_1 \overset{\checkmark}{=} \dot{\theta}\,\hat{b}_2$$

* key idea: ang. vel. can take derivatives!!

$${}^{I}\frac{d}{dt}(\hat{b}_2) = \dot{\theta}\,\hat{b}_3 \times \hat{b}_2 = -\dot{\theta}\,\hat{b}_1$$

$${}^{I}\bar{a}_{P/O} = {}^{I}\frac{d}{dt}\left({}^{I}\bar{v}_{P/O}\right) = \frac{d}{dt}\left(\dot{r}\,\hat{b}_1 + r\dot{\theta}\,\hat{b}_2\right)$$

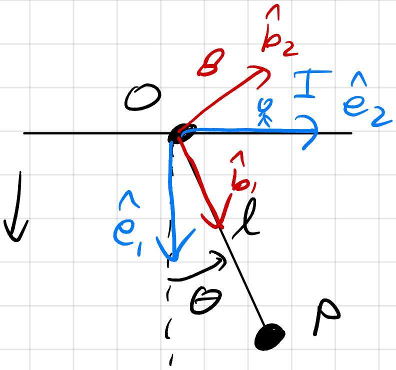
$$= \ddot{r}\,\hat{b}_1 + \dot{r}\dot{\theta}\,\hat{b}_2 + \dot{r}\dot{\theta}\,\hat{b}_2 + r\ddot{\theta}\,\hat{b}_2 - r\dot{\theta}^2\,\hat{b}_1$$

$$\boxed{{}^{I}\bar{a}_{P/O} = (\ddot{r} - r\dot{\theta}^2)\,\hat{b}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{b}_2}$$

inertial kinematics in polar coords in the polar frame

$$\bar{F}_P = m_P\,{}^{I}\bar{a}_{P/O}$$

Ex Simple pendulum



$g \downarrow$

$I = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ inertial

$B = (O, \hat{b}_1, \hat{b}_2, \hat{b}_3)$ polar

${}^I\bar{\omega}^B = \dot{\theta}\hat{b}_3$ angular velocity

$$M = 3N - K = 3 \cdot 1 - 2 = 1$$

$$\bar{F}_P = m_P \, {}^I\bar{a}_{P/O}$$

Free-body diagram

$-T\hat{b}_1 = T(-\hat{b}_1)$

P

$mg\hat{e}_1$

$$\bar{F}_P = -T\hat{b}_1 + mg\hat{e}_1$$

inertial kinematics

$$\bar{r}_{P/O} = \ell \hat{b}_1 \qquad (r = \ell)$$

$${}^I\bar{v}_{P/O} = \frac{{}^I d}{dt}(\ell \hat{b}_1)$$

$$= \ell \frac{{}^I d}{dt}(\hat{b}_1) = \ell \, {}^I\bar{\omega}^B \times \hat{b}_1$$

$$= \ell \dot{\theta} \hat{b}_3 \times \hat{b}_1 = \ell \dot{\theta} \hat{b}_2$$

$${}^I\bar{a}_{P/O} = \ell \ddot{\theta} \hat{b}_2 - \ell \dot{\theta}^2 \hat{b}_1$$

$$-T\hat{b}_1 + mg\hat{e}_1 = m(\ell\ddot{\theta}\hat{b}_2 - \ell\dot{\theta}^2\hat{b}_1)$$

$\hat{b}_1$: $\quad -T + mg C\theta = -m\ell\dot{\theta}^2 \Rightarrow T = mg C\theta + m\ell\dot{\theta}^2$

$\hat{b}_2$: $\quad -mg S\theta = m\ell\ddot{\theta} \Rightarrow \boxed{\ddot{\theta} = -\frac{g}{\ell}\sin\theta}$