Converting from $G(s)$ to state space

- Sometimes it is useful to reverse process described above, i.e.

  Given $G(s)$, find $[A,B,C,D]$

- In fact, for a given $G(s)$ there are infinitely many equivalent $[A,B,C,D]$

  $\Rightarrow$ many more DoF in $A(n^2)$, $B(n)$, $C(n)$ than in polynomials $r(s)$ $(n+1)$ and $q(s)$ $(m \leq n)$

- One "canonical" conversion is easy to obtain where the coefs. of polys $r(s)$ and $q(s)$ appear as rows and/or cols of $[A,B,C]$

- Known as "companion forms"

Companion form (one possibility)

Given $G(s) = \frac{q(s)}{r(s)} = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$

Take

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix} \qquad D = 0$$

(Note: $A_1 = A^T$, $B_1 = C^T$, $C_1 = B^T$ works too!)

Example

$$G(s) = \frac{3s^2 - 4s + 5}{s^3 + 2s^2 - s + 7}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & -4 & 3 \end{bmatrix} \qquad D = 0$$

One possible state space model for this TF

Philosophical Question: What is $t=0$?

$\Rightarrow$ The instant we start acting on the system with external input.

$\Rightarrow$ In control Theory, we assume these inputs are completely "off" for $t<0$.

$\Rightarrow$ $u(t), \dot{u}(t), \ddot{u}(t)$, etc all zero for $t<0$

$\Rightarrow$ Discontinuities exist when $u(0) \neq 0$

$$u(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0 & t<0 \end{cases}$$

$$u(t) = \begin{cases} e^{pt}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \boxed{u(t) = e^{pt}\,\mathbb{1}(t)}$$

where

$$\boxed{\mathbb{1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}}$$

"Unit step function" (Very important!)

Now, Laplace is concerned about behavior of functions only for $t \geq 0$.

For all intents and purposes, functions in Laplace are considered 0 for $t < 0$

Implication

Formally:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = e^{pt}\mathbb{1}(t)$$

$$= \begin{cases} e^{pt}, & t \geq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

Now generally, our diff'l eq'ns will involve derivatives of these discontinuous functions

$\Rightarrow$ creates singularities in analysis at $t=0$

$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \emptyset & t \neq \emptyset \\ & \end{cases}$$

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Now generally, our diff'l eq'ns will involve <u>derivatives</u> of these discontinuous functions

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$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \emptyset & t \neq \emptyset \\ \infty & t = \emptyset \;\; (???) \end{cases}$$

Theoretical problems in integrals when discontinuities or singularities at one of the end points.

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

$\Rightarrow$ possible problem here

Resolve these by taking lower limit at $t=\emptyset^-$ (the instant <u>before</u> $t=\emptyset$).

$\Rightarrow$ integral "sees" effect of singularities at $t=\emptyset$.

Starting the integral at 0^- instead of 0

- Avoids singularities at endpoints
- Causes transform to "see" singularities and discontinuities at $t=0$, so their effects will be reflected in the solutions for $y(t)$.

Hence:

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st}\, dt$$

<u>Implications</u>:

Assumed ICs for $y(t)$: just <u>before</u> $t=0$

$$\mathcal{L}\{\dot{y}(t)\} = sY(s) - \boxed{y(0^-)}$$

$$\mathcal{L}\{\ddot{y}(t)\} = s^2Y(s) - \boxed{\dot{y}(0^-) - sy(0^-)}$$

etc.

$$\mathcal{L}\{\dot{u}(t)\} = sU(s) - \boxed{u(0^-)}$$

$$\mathcal{L}\{\ddot{u}(t)\} = s^2U(s) - \boxed{\dot{u}(0^-) - su(0^-)}$$

etc.

Always $=0$ in our analysis!

Thus:

$$Y(s) = \left[\frac{q(s)}{r(s)}\right]U(s) + \left[\frac{c(s) - \cancel{b(s)}^{\to \emptyset}}{r(s)}\right]$$

$$= \left[\frac{q(s)}{r(s)}\right]U(s) + \left[\frac{c(s)}{r(s)}\right]$$

$\Rightarrow$ IC polynomial $b(s)$ for input vanishes

$\Rightarrow$ specific to controls convention for $u(t)$

$\Rightarrow$ Not a common assumption in regular math classes.

$\Rightarrow$ In controls, want to know effect of discontinuities

Common, discontinuous "test functions"

$u(t) = \mathbb{1}(t)$ (unit step) $\quad \mathcal{L}\{\mathbb{1}(t)\} = \frac{1}{s}$

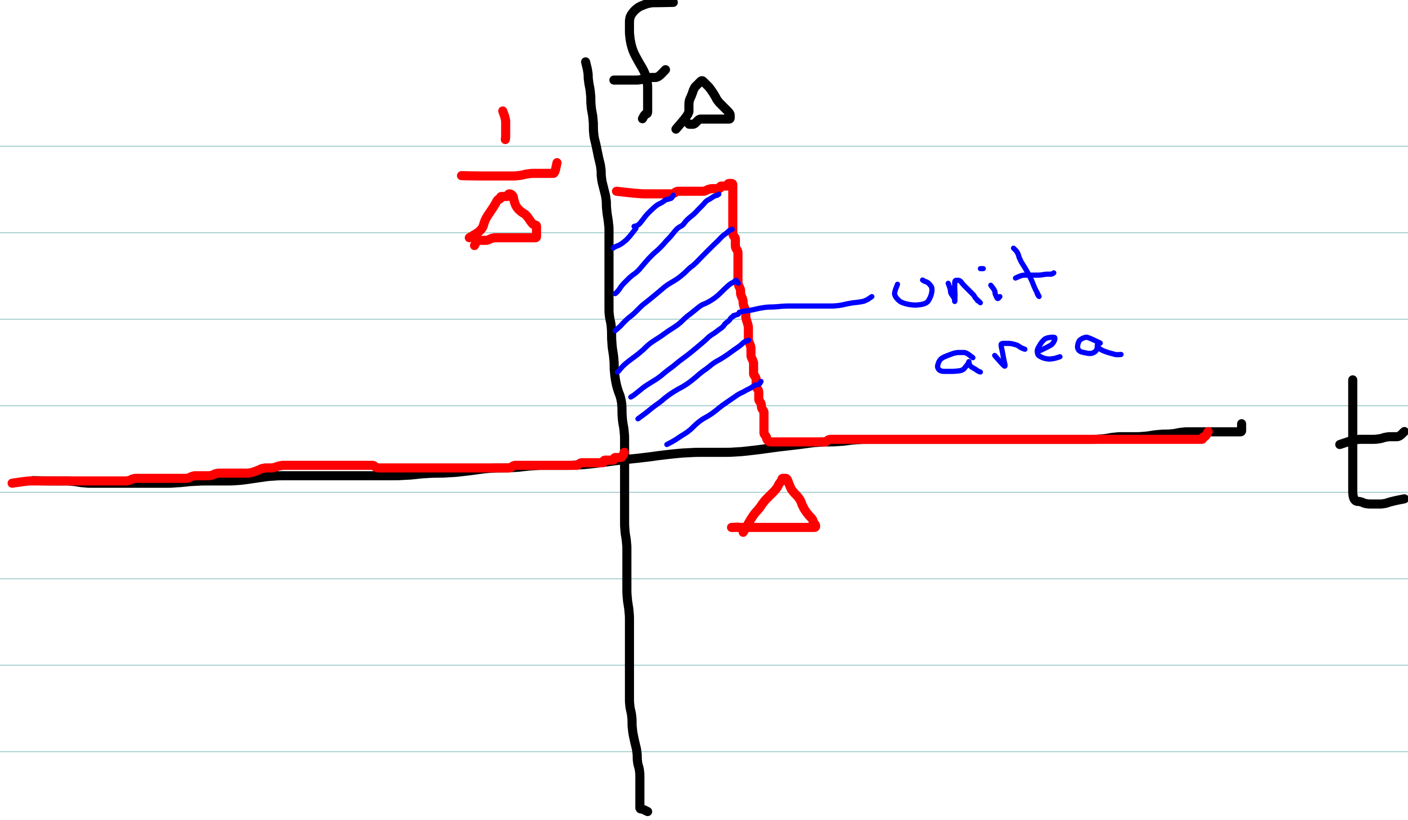
$$u(t) = \cos(\omega t)\,\mathbb{1}(t)$$

$$= \begin{cases} \cos(\omega t) & t \geq \emptyset \\ \emptyset & t < \emptyset \end{cases}$$

$\Longrightarrow$ $u(t) = f_\Delta(t)$

$$= \begin{cases} \frac{1}{\Delta} & \emptyset \leq t \leq \Delta \\ \emptyset & \text{otherwise} \end{cases}$$

"Unit pulse function"



Note: for <u>any</u> $\Delta > 0$

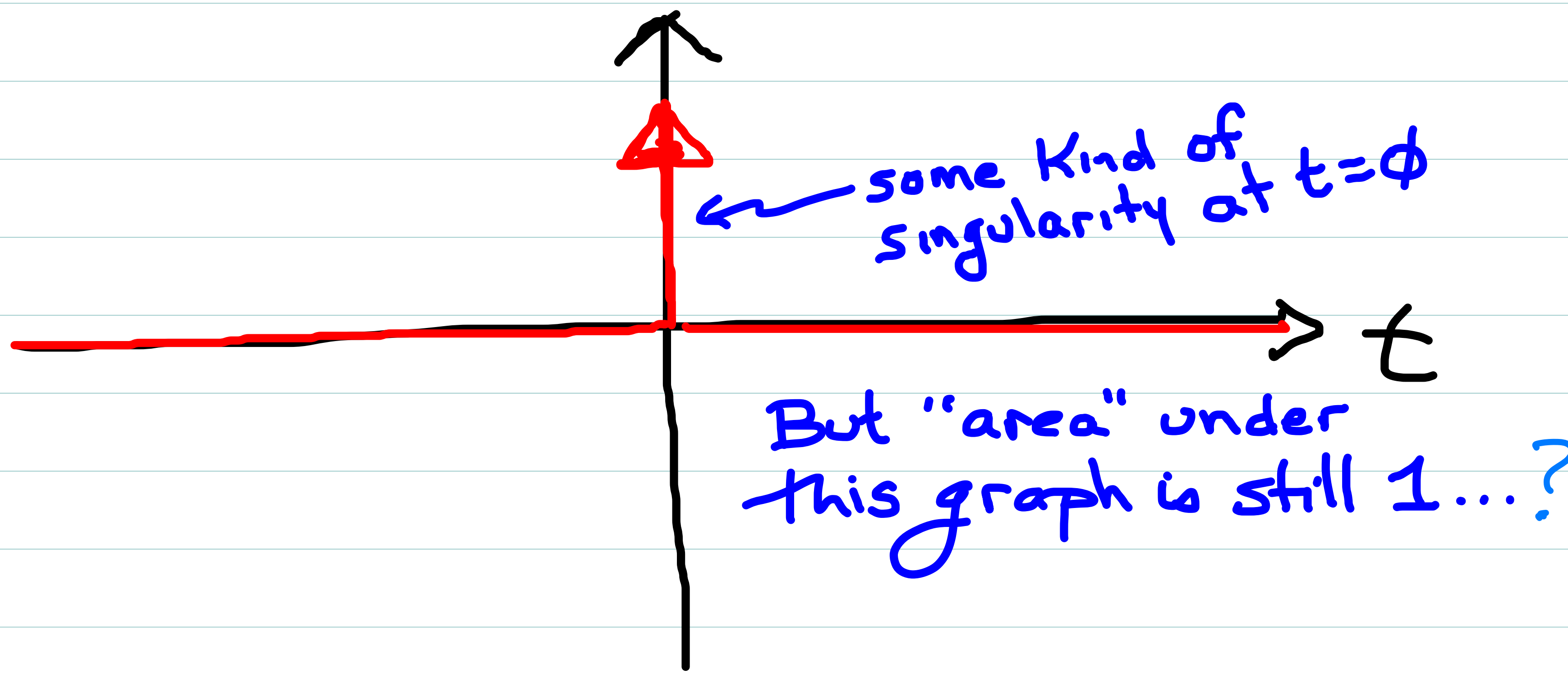
$$\int_{0^-}^{\infty} f_\Delta(t)\,dt = \int_{0^-}^{\Delta} \left(\frac{1}{\Delta}\right) dt$$

$$= 1$$

What is $\lim_{\Delta \to 0} f_\Delta(t)$?

$$= \lim_{\Delta \to 0} \begin{cases} \frac{1}{\Delta} & \phi \le t \le \Delta \\ \phi & \text{otherwise} \end{cases}$$

$$= \begin{cases} \infty & t = \phi \\ \phi & \text{otherwise} \end{cases}$$



Define:

$$\boxed{\delta(t) = \lim_{\Delta \to 0} f_\Delta(t)}$$

"ideal impulse": models delivering a unit of input energy over negligibly small time.
(sharp "kick")

Alternate names:

"delta function"
"impulse function"
"Dirac delta"

Note: Not really a meaningful function at all!

More formally, belongs to a class of mathematical objects called

"distributions" or "generalized functions"

Suppose $\delta(t)$ appears in an integral

$$\int_{-\infty}^{\infty} \delta(t) h(t)\, dt \quad , \quad h(t) \text{ arbitrary f'n}$$

$$= \int_{-\infty}^{\infty} \left[\lim_{\Delta \to 0} f_\Delta(t) \right] h(t)\, dt$$

$$= \lim_{\Delta \to 0} \int_{-\infty}^{\infty} f_\Delta(t) h(t)\, dt$$

$$= \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \int_{0^-}^{\Delta} h(t)\, dt \right\}$$

$$\cong \left(\frac{1}{\Delta}\right) \left(\Delta\, h(\phi) \right)$$

$$= h(\phi)$$

Note: with $h(t) = 1$ for all t, we get

$$\int_{0^-}^{\infty} \delta(t)\, dt = 1$$

Defining Property of $\delta(t)$

$$\int_a^b \delta(t)h(t)\,dt = \begin{cases} h(\emptyset) & \text{if } \phi \in (a,b) \\ \emptyset & \text{otherwise} \end{cases}$$

"sifting property"

$\delta(t)$ is defined by what it does in an integral

Not as an ordinary function

Now we can compute:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt$$

$$= \left[e^{-st}\right]_{t=\emptyset} = 1$$

Thus:

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall $\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t=0 \\ 0 & \text{otherwise} \end{cases}$

which looks like $\frac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\{\tfrac{d}{dt}\mathbb{1}(t)\} = s\,\mathcal{L}\{\mathbb{1}(t)\} - \mathbb{1}(0^-)$$

Thus:

$$\mathcal{L}\{\delta(t)\} = 1$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall $\displaystyle \frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t=\varnothing \\ \varnothing & \text{otherwise} \end{cases}$

which looks like $\displaystyle \frac{d}{dt}\mathbb{1}(t) = \delta(t).$

Is this formally true?

$$\mathcal{L}\left\{\frac{d}{dt}\mathbb{1}(t)\right\} = s\,\overbrace{\mathcal{L}\{\mathbb{1}(t)\}}^{1/s} - \cancel{\mathbb{1}(\varnothing^-)} \rightarrow \varnothing$$

$$= 1 = \mathcal{L}\{\delta(t)\}$$

YES

Recap: Unit Impulse

$$\delta(t) = \lim_{\Delta \to 0} f_\Delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

"sifting property":

$$\int_a^b \delta(t) h(t)\, dt = \begin{cases} h(0) & \text{if } 0 \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Laplace Transform:

$$\mathcal{L}\{\delta(t)\} = 1$$

Useful property:

$$\frac{d}{dt} \mathbb{1}(t) = \delta(t)$$

Impulse Response

The *impulse response* of a system is the output $y(t)$ when $u(t) = \delta(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[c(s) - b(s)]}{r(s)}$$

$\Rightarrow u(t) = \delta(t) \Rightarrow b(s) = \emptyset$ and $U(s) = 1$

$\Rightarrow$ all ICs on $y(t)$ zero $\Rightarrow c(s) = \emptyset$

So:

$$Y(s) = G(s)$$

and thus

$$y(t) = \mathcal{L}^{-1}\{G(s)\} \triangleq g(t)$$

$\boxed{\text{The } \underline{\text{impulse response}}\ g(t) \text{ is the inverse transform of the transfer function } G(s)}$

Conversely, knowledge (or <u>measurement</u>) of $g(t)$ tells us what the transfer function is, and hence the governing diff'l eq'ns.

$\Rightarrow$ Foundation of "system identification" theory.