Lecture 20: More Rigid Body Dynamics

Take the angular momentum about the origin of the coordinate system O:

$$\vec{H}_O = \int_B \vec{R} \times \dot{\vec{R}}\, dm$$

$$\vec{R} = \vec{R}_c \times \vec{r}$$

$$\vec{H}_O = \int_B (\vec{R}_c + \vec{r}) \times (\dot{\vec{R}}_c + \dot{\vec{r}})\, dm$$

$$= \int \vec{R}_c \times \dot{\vec{R}}_c\, dm + \int \vec{r}\, dm \times \dot{\vec{R}}_c + \vec{R}_c \times \int \dot{\vec{r}}\, dm + \int \vec{r} \times \dot{\vec{r}}\, dm$$

Remember that $\vec{r}$ points from the CM to some point on the body.

$$\Rightarrow \int \vec{r}\, dm = 0$$

Also, the body is rigid: $\int \dot{\vec{r}}\, dm = 0$

$$\boxed{\vec{H}_O = \underbrace{M \vec{R}_c \times \dot{\vec{R}}_c}_{\text{Ang. Mom. of the CM about the origin.}} + \underbrace{\int_B \vec{r} \times \dot{\vec{r}}\, dm}_{\text{Ang. Mom. about the CM. (ie the rotation or spin of the body)}}}$$

Angular momentum with respect to the CM:

$$\vec{H}_c = \int_B \vec{r} \times \dot{\vec{r}}\, dm$$

Using the transport theorem:

$$\dot{\vec{r}} = {}^{I}\frac{d\vec{r}}{dt} = {}^{B}\frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r}$$

$\vec{\omega}$ is the rotation of the body frame wrt the inertial frame

B/c rigid body: ${}^{B}\frac{d\vec{r}}{dt} = 0$

$$\dot{\vec{r}} = {}^{I}\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\vec{H}_c = \int_B \vec{r} \times (\vec{\omega} \times \vec{r})\, dm$$

Use the tilde matrix = skew-symmetric matrix notation

$\vec{a}$ is a vector: $[\tilde{a}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

$$\vec{a} \times \vec{b} = [\tilde{a}]\vec{b}$$

$$\vec{H}_c = \underbrace{\left(\int_B -[\tilde{r}][\tilde{r}]\, dm \right)}\vec{\omega}$$

The integral term is the inertia matrix:

$$\boxed{\vec{H}_c = [I_c]\vec{\omega}}$$

$$[I_c] = \int_B \begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} dm$$

We would like to diagonalize $[I_c]$.

Diagonalize $[I_c]$ by rotating the body fixed coordinate system to a principal axis system.

1. Calculate the eigenvalues & eigenvectors for the inertia matrix.
2. Make sure the vectors are unit vectors & form a RH'd system.
3. The rotation matrix is:

$$[C] = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix} \quad (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \text{eigenvectors}$$

4. The inertia matrix is:

$$[I] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \lambda_1, \lambda_2, \lambda_3 = \text{eigenvalues}$$

Principal inertia matrix.

Euler's Rotational EOM:

$$\dot{\vec{H}}_c = \frac{{}^I d\vec{H}_c}{dt} = \frac{{}^B d\vec{H}_c}{dt} + \vec{\omega} \times \vec{H}_c = \vec{L}_c$$

$$\vec{H}_c = [I_c]\vec{\omega}$$

$$\frac{{}^B d\vec{H}_c}{dt} = \cancelto{0}{\frac{{}^B d}{dt}[I]}\vec{\omega} + [I]\frac{{}^B d\vec{\omega}}{dt} = [I]\dot{\vec{\omega}}$$

$$\frac{{}^I d\vec{\omega}}{dt} = \frac{{}^B d\vec{\omega}}{dt} + \cancelto{0}{\vec{\omega} \times \vec{\omega}}$$

$$[I]\dot{\vec{\omega}} + \vec{\omega} \times [I]\vec{\omega} = \vec{L}_c$$

$$\boxed{[I]\dot{\vec{\omega}} = -[\tilde{\omega}][I]\vec{\omega} + \vec{L}_c}$$ Valid if calculated about CM -or- an arbitrary inertial point.

If $[I]$ is diagonal (principal body-fixed coordinates):

$$\boxed{\begin{aligned} I_{11}\dot{\omega}_1 &= -(I_{33}-I_{22})\,\omega_2\omega_3 + L_1 \\ I_{22}\dot{\omega}_2 &= -(I_{11}-I_{33})\,\omega_3\omega_1 + L_2 \\ I_{33}\dot{\omega}_3 &= -(I_{22}-I_{11})\,\omega_1\omega_2 + L_3 \end{aligned}}$$

← Tell us how rotation rate change given a torque.

– Even if $\vec{L}=0$, ${}^{B}\vec{\omega}$ is changing. if the body is rotating about more than 1 axis.

In other words, $\dot{\vec{\omega}}=0$ if $\vec{L}=0$ & rotating about just one axis (eg. $\omega_2=\omega_3=0$)

★ The rotation rate (or spin) in the body fixed frame is constant only if torque is zero AND the body is rotating about only 1 of the <u>principal</u> axes.

<u>Kinetic Energy:</u>

$$T=\frac{1}{2}\int_B \dot{\vec{R}}\cdot\dot{\vec{R}}\,dm \qquad \vec{R}=\vec{R}_c+\vec{r}$$

$$T=\frac{1}{2}\int_B dm\,\dot{\vec{R}}_c\cdot\dot{\vec{R}}_c + \dot{\vec{R}}_c\cdot\cancelto{0}{\int_B \dot{\vec{r}}\,dm} + \frac{1}{2}\int_B \dot{\vec{r}}\cdot\dot{\vec{r}}\,dm$$

$$T=\underbrace{\frac{1}{2}M\dot{\vec{R}}_c\cdot\dot{\vec{R}}_c}_{\text{Translational}} + \underbrace{\frac{1}{2}\int_B \dot{\vec{r}}\cdot\dot{\vec{r}}\,dm}_{\text{Rotational}}$$

$$T_{rot}=\frac{1}{2}\int_B \dot{\vec{r}}\cdot\dot{\vec{r}}\,dm \qquad \frac{{}^{I}d\vec{r}}{dt}=\cancelto{0,\text{ Rigid body}}{\frac{{}^{B}d\vec{r}}{dt}}+\vec{\omega}\times\vec{r}$$

$$=\frac{1}{2}\int_B (\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})\,dm$$

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})$$

$$T_{rot}=\frac{1}{2}\vec{\omega}\cdot\underbrace{\int_B \vec{r}\times(\vec{\omega}\times\vec{r})\,dm}_{=\vec{H}_c}$$

$$=\frac{1}{2}\vec{\omega}\cdot\vec{H}_c=\frac{1}{2}\vec{\omega}^T[I]\vec{\omega}$$

$$\boxed{T=\frac{1}{2}M\dot{\vec{R}}_c\cdot\dot{\vec{R}}_c+\frac{1}{2}\vec{\omega}^T[I]\vec{\omega}}$$

Torque-free Motion:

$$\vec{L}_c=0 \Rightarrow \dot{\vec{H}}_c=0=\frac{{}^{I}d\vec{H}_c}{dt}.$$

★ [However, in the body-fixed frame, $\vec{H}_c$ may be rotating. The magnitude of $\vec{H}_c$ is constant in all frames.

Assume principal axes for the body-fixed frame $\Rightarrow [I]$ is diagonal

$$^{B}\vec{H}_c = I_{11}\omega_1\hat{b}_1 + I_{22}\omega_2\hat{b}_2 + I_{33}\omega_3\hat{b}_3$$

B/C $|\vec{H}_c| = \text{constant}$

$$H^2 = I_{11}^2\omega_1^2 + I_{22}^2\omega_2^2 + I_{33}^2\omega_3^2$$

↳ Eqn for the surface of an ellipsoid.

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

Kinetic energy is also constant:

$$T_{rot} = T = \frac{1}{2}I_{11}\omega_1^2 + \frac{1}{2}I_{22}\omega_2^2 + \frac{1}{2}I_{33}\omega_3^2$$

$^{B}\vec{\omega}(t)$ must satisfy both $H = \text{constant}$ & $T = \text{constant}$

We will graphically investigate how the body-fixed angular velocity (due to the rotation of the spacecraft) will change ($\vec{L}_c = 0$)