1. **Homogeneous Transforms** (20 points)
   Consider the robot arm in Figure 1.
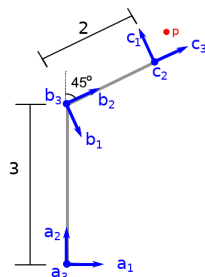


Figure 1: Robot arm diagram. Note that $a_3$, $b_3$, and $c_2$ are pointed out of the page.

There are three coordinate frames $A$, $B$, $C$ with their corresponding basis vectors. Let $^Ap$ denote the coordinates of the point $p$ in frame $A$, $^Bp$ be the same point in frame $B$, and $^Cp$ the same point in frame $C$. Define the homogeneous transforms $T_1$, $T_2$ , $T_3$ as follows:

$$\begin{bmatrix} ^Ap \\ 1 \end{bmatrix} = T_1 \begin{bmatrix} ^Bp \\ 1 \end{bmatrix} \qquad \begin{bmatrix} ^Bp \\ 1 \end{bmatrix} = T_2 \begin{bmatrix} ^Cp \\ 1 \end{bmatrix} \qquad \begin{bmatrix} ^Ap \\ 1 \end{bmatrix} = T_3 \begin{bmatrix} ^Cp \\ 1 \end{bmatrix}$$

   (a) Find the homogeneous transform $T_1$. (6 points)

   (b) Find the homogeneous transform $T_2$. (6 points)

   (c) Find the homogeneous transform $T_3$. (Hint: Use $T_1$ and $T_2$). (8 points)

2. **Rotation Matrix Sudoku** (20 points)
   Only three elements of the rotation matrix below are known (entries with $*$ are unknown). Use the properties of rotation matrices to determine the missing elements. Either prove your solution is unique, or provide an alternative solution. (*Note: Show your work to receive partial credit. You may show your math or include a code segment.) [1]

$$\begin{bmatrix} * & 0.892 & 0.423 \\ * & * & * \\ -0.186 & * & * \end{bmatrix}$$

3. **Rodrigues' Formula** (20 points)

   (a) Are the axis and angle always uniquely defined for a rotation? If not, explain the conditions under which the axis and angle are not uniquely defined. (7 points)

   (b) Write the axis-angle representation and the quaternion corresponding to the rotation matrix. (13 points)

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos(\pi/6) & \sin(\pi/6) \\ 0 & \sin(\pi/6) & \cos(\pi/6) \end{pmatrix}$$

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[1]Given the computational nature of this problem, you should use this opportunity to get comfortable doing calculations in Python 3 using PyCharm's interactive console. You'll find Numpy arrays invaluable, including the associated Numpy functions for 'inv', 'cross,' 'dot,' and matrix multiplication using the '@' symbol.

4. **Transforms and and Python Plotting** (30 points)
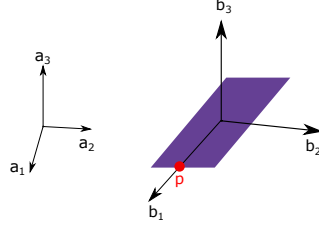   Figure 2 shows a robot coordinate system (B) and a world coordinate system (A).



Figure 2: Robot frame $B$ (with basis vectors $b_1$, $b_2$, $b_3$) in the world frame $A$ (with basis vectors $a_1$, $a_2$, $a_3$).

Assume that the position of a robot in meters $d(t)$ and the orientation $R(t)$ are given (both with respect to the world frame) as

$$d(t) = \begin{bmatrix} \cos(0.1 * t) \\ \sin(0.12 * t) \\ \sin(0.08 * t) \end{bmatrix} \qquad R(t) = \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Note: All angles in this problem are in radians!

(a) Compute, as a function of time, the linear velocity $v(t)$ of the robot in the world frame. (5 points)

(b) Suppose there is a point on the robot we are interested in, e.g. a camera that is offset on the $x$ axis. Say the position of this point is given by

$$^B p(t) =^B p = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$$

for some constant $c$. Note that the above is represented in the robot frame. What is the position of this point in the world frame as a function of time? In other words, find $^A p(t)$. Your expression should include $c$. (8 points)

(c) Using Python, graph both robot position and $^A p(t)$ over time for $c = 0.25$ on the same plot. Include a title, axes labels, and legend. You may find the online documentation for matplotlib helpful (e.g. https://matplotlib.org/mpl_toolkits/mplot3d/tutorial.html#line-plots). An example is shown in Figure 3. (8 points)
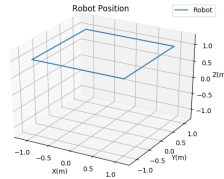


Figure 3: An example plot of robot position.

(d) Suppose at time $t = 1$ the gravity vector measured in body coordinates is $[1.1, 0.9, -9.7]^T$. What is the expected gravity measured in body coordinates at time $t = 5$? (9 points)