FIXED INCOME MATHEMATICS

Analytical and Statistical Techniques

FIFTH EDITION

FRANK J. FABOZZI & FRANCESCO A. FABOZZI

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McGraw Hill

New York Chicago San Francisco Athens London
Madrid Mexico City Milan New Delhi
Singapore Sydney Toronto

ISBN: 978-1-26-425828-4
MHID: 1-26-425828-3

The material in this eBook also appears in the print version of this title: ISBN: 978-1-26-425827-7, MHID: 1-26-425827-5.

eBook conversion by codeMantra
Version 1.0

To my wife, Donna

—Frank J. Fabozzi

To my Princeton wrestling teammates

—Francesco A. Fabozzi

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PREFACE

In the past four decades, participants in the fixed-income markets have been introduced to new analytical frameworks for analyzing fixed-income securities and formulating fixed-income portfolio strategies. In discussing fixed-income securities and strategies, we often hear terms such as *model duration, empirical duration, effective duration, spread duration, positive and negative convexity, option-adjusted spread, duration times spread, prepayment rates, spot rates, forward rates, yield volatility, lattice model, value-at-risk, factor models, optimization, simulation, machine learning, fat tails, and default correlation,* and the list goes on. What do these concepts mean? Why are these concepts useful in the analysis of fixed-income securities and the formulation of fixed-income strategies? Moreover, what are the dangers of using these concepts without a complete understanding of what they mean and their limitations?

Fixed Income Mathematics: Analytical and Statistical Techniques not only explains these and many other important concepts that players in the bond market need to know, but also sets forth the foundation needed to understand them, their computation, their limitations, and their application to fixed-income analysis and portfolio management. It begins with the basic concepts of the mathematics of finance (the time value of money) and systematically builds on these, taking you through the state-of-the-art methodologies for evaluating fixed-income securities with embedded options: mortgage-backed securities (mortgage pass-through securities, collateralized mortgage obligations, and stripped mortgage-backed securities). The concepts are illustrated with numerical examples and graphs. The material is self-contained and requires only a basic knowledge of elementary algebra to understand.

Frank J. Fabozzi
Francesco A. Fabozzi

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FIFTH VERSUS FOURTH EDITION

The fourth edition of the book contained eight parts and 31 chapters as shown in the table following:

Chapter 1 Introduction
Chapter 2 Overview of Fixed-Income Securities and Derivatives
PART ONE TIME VALUE OF MONEY
Chapter 3 Future Value
Chapter 4 Present Value
Chapter 5 Yield (Internal Rate of Return)
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Chapter 6 The Price of a Bond
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PART THREE RETURN ANALYSIS
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PART FOUR PRICE VOLATILITY FOR OPTION-FREE BONDS
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Chapter 23 Prepayment Models for Mortgage-Backed Securities
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Chapter 25 Analysis of Agency Mortgage-Backed Securities
PART EIGHT STATISTICAL AND OPTIMIZATION TECHNIQUES
Chapter 26 Basics of Probability Theory and Statistics
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Chapter 29 Tracking Error and Multifactor Risk Models
Chapter 30 Simulation
Chapter 31 Optimization Models

The fifth edition has 10 parts and 35 chapters. The 15 chapters that are almost identical to the chapters in the fourth edition (shown in parentheses is the chapter number in the fourth edition) are:

2 Future Value (3)
3 Present Value (4)
4 Yield (Internal Rate of Return) (5)
5 The Price of a Bond (6)
6 Bond Yield Measures (7)
7 The Yield Curve, Spot-Rate Curve, and Forward Rates (8)
8 Potential Sources of Dollar Return (9)
9 Total Return (10)
12 Price Volatility of Properties of Option-Free Bonds (12)
14 Combining Duration and Convexity to Measure Price Volatility (14)
15 Duration and the Yield Curve (15)
20 Call Options: Investment and Price Characteristics (17)
21 Valuation and Price Volatility of Bonds with Embedded Options (18)
25 Cash-Flow Characteristics of Fixed-Rate Amortizing Mortgage Loans (21)
26 Cash-Flow Characteristics of Mortgage-Backed Securities (22)

The 11 new chapters in the fifth edition are

10 Historical Return Measures
11 Risk-Adjusted Returns/ Reward-Risk Ratios
16 Empirical Duration
17 Measuring Historical Return Volatility
18 Measuring and Forecasting Yield Volatility
22 Analysis of Floating-Rate Securities
24 Measuring Bond Liquidity

28	Holdings-Based Performance Attribution Analysis
29	Returns-Based Style Attribution Analysis
32	Multifactor Risk Models and Their Application to Portfolio Construction
35	Machine Learning

The 9 substantially revised chapters are (shown in parentheses is the chapter number in the fourth edition):

Ch.	Chapter title	Change
1	Introduction (1)	Completely revised to explain the importance of the topics in the book and the new organization of the book.
13	Duration as a Measure of Price Volatility (13)	Differentiating between model and empirical duration, and a discussion of the interpretation of duration.
19	Interest-Rate Modeling (16)	Completely revised to describe the different arbitrage-free interest rate models and their implementation using the lattice method.
23	Credit Risk Concepts and Measures (23 and 25)	Completely revised to introduce new analytical concepts.
27	Analysis of Agency Mortgage-Backed Securities (25)	Revised to include prepayment modeling in this chapter rather than having a separate chapter on prepayment modeling.
30	Probability Distributions and Statistics (26)	Previous chapter title: Basics of Probability Theory and Statistics. Expanded to include hypothesis testing and types of errors in hypothesis testing.
31	Regression and Principal Component Analysis (27)	Significantly revised to include the steps in applying regression analysis and to include principal component analysis.
33	Monte Carlo Simulation (30)	Revised to include the application to backtesting investment strategies.
34	Optimization Models (31)	Completely revised to discuss convex optimization problems and optimization under uncertainty.

The following four chapters, which provide background material that appeared in the fourth edition, have been removed and are available online:

Overview of Fixed-Income Securities and Derivatives (Chapter 2, now online supplement A)
Basics of Agency CMO Structuring (from Chapter 24, now online supplement B)
Measures Used for Securitized Products (Chapter 20, now online supplement E)
Basic Concepts in Probability Theory (from Chapter 26, now online supplement F)

Two new online supplements are also available:

Online Supplement C. Illustration of Holdings-Based Performance Attribtion Analysis
Online Supplement D. Econometric Issues Associated with the Regression Model Used for Returns-Based Style Attribution Analysis

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ACKNOWLEDGMENTS

The first edition of this book was published in 1993. Several individuals assisted in various ways in subsequent editions of the book, and we acknowledge them below:

- Jan Mayle and Dragomir Krgin provided information on the day count conventions discussed in Chapter 5.
- Chapter 19 on interest rate modeling is coauthored with Gerald W. Buetow, Jr. (BFRC Services), Bernd Hanke (BFRC Services), and Brian J. Henderson (School of Business, The George Washington University).
- Parts of Chapter 21 on valuing bonds with embedded options draws from the work of Frank Fabozzi with Andrew Kalotay (Andrew Kalotay Associates) and George Williams.
- Chapter 24 benefitted from discussions and feedback about liquidity measures provided by Stefano Pasquale (BlackRock) and Harshdeep Singh Ahluwalia (Vanguard).
- The illustration of the Campisi Attribution Model in Chapter 28 was provided by Bruce J. Feibel (State Street).
- The illustration of attribution analysis of a hypothetical UK corporate bond/credit portfolio in the online supplement C was prepared by René Alby (Insight Investment).
- Chapter 29 was coauthored with Gueorgui S. Konstantinov (LBBW Asset Management).
- In Chapter 32, the illustration of the Axioma Factor-Based Fixed Income Risk Model developed by Qontigo's Analytics Research group was provided by Bill Morokoff (Qontigo).
- Amundi Asset Management fixed-income multifactor risk model described and illustrated in Chapter 32, was provided by Amina Cherief and Mohamed Ben Slimane (Amundi Asset Management).
- In Chapters 17 and 34 we used the software by Portfolio Visualizer (https://www.portfoliovisualizer.com/).

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CHAPTER
1

INTRODUCTION

Before the 1980s, the analysis of fixed-income securities was relatively simple. In an economic environment that exhibited relatively stable interest rates, investors purchased fixed-income securities intending to hold them to maturity. Yield to maturity was used as a proxy measure of their relative value. Risk was gauged in terms of default risk based on the credit rating assigned by the major credit-rating agencies. When a fixed-income security was callable, a second measure—yield to call—was used to assess its relative value. For a callable bond, the long-standing rule of thumb for a conservative investor at the time was to select the lower of the yield to maturity and the yield to call as the potential return. Moreover, prior to the 1980s, there was little trading of fixed-income securities. The strategy was simply a buy-and-hold strategy.

Those days of reliance on simple analytics to manage a fixed-income portfolio are gone. This is because fixed-income portfolios are actively traded, requiring the use of analytics that draw from the fields of statistics, data science, mathematics, and operations research. Moreover, prior to the 1970s, corporate bond issuers were primarily those with an investment-grade rating. Non-investment-grade corporate bonds that were traded were those of one-time investment-grade bond issues that were subsequently downgraded, referred to as *fallen angels*. In 1977, Bear Stearns underwrote non-investment-grade corporate bonds, popularly referred to as *junk bonds* and *high-yield bonds*, with bond market participants seeing opportunities to enhance returns by constructing a diversified portfolios of such issues despite higher default risk. Other market participants developed credit analytics to identify junk bond issuers that were candidates for an upgraded rating, thereby enhancing returns. In fact, 6 years after the Bear Stearns underwriting of the first junk bond, roughly a third of all corporate bonds were non-investment grade.[1] Investing in non-investment-grade corporate bonds made investors recognize the need to forecast default rates for a diversified portfolio of such bonds and recovery rates.

The need for more rigorous analytics became clear with the development of the mortgage-backed securities (MBS) market. When the first MBS were issued in 1968, these securities were acquired primarily because of their greater offered yield than Treasury securities, and they were not traded actively. Purchasing MBS based purely on a potential higher yield than Treasury securities was clearly naive.

1. Jared Cummans, "A Brief History of Bond Investing," BondFunds.com, October 1, 2014. Available at http://bondfunds.com/education/a-brief-history-of-bond-investing/.

Once the concept of prepayment risk was understood and that different MBS issues backed by different pools of mortgages paid at different prepayment speeds, the importance of prepayment modeling in the selection of the specific MBS to include in a portfolio made investors realize the need to bring in statistical modeling and ushered in the individuals trained in mathematics, statistics, and physics. The modeling of prepayments became even more important with the development of mortgage-derivative products (collateralized mortgage obligations and mortgage strips) where the pricing of some of these products is highly sensitive to changes in prepayment rates and interest rates. Although the initial MBS issues were viewed as having the same credit risk as U.S. Treasuries, in the late 1980s, MBS issued by private entities, referred to a *nonagency MBS*, began to appear. The pricing of nonagency MBS made investors realize the need for not only modeling prepayments but also forecasting default rates and recovery rates.

A look at the history of interest rates and the properties of a bond's price volatility provides insights into the need for analytics beyond those used in traditional fixed-income analytics. Let's look at 30-year Treasury yields in the following 3 years:

1977: 7.8%
1981: 15.21% (historical high)
2021: 1.90%

Let's suppose that a 30-year Treasury bond was purchased in each year with a coupon rate equal to the yield. As explained in Chapter 5, each bond would trade at par value or 100. Suppose that after the bond is purchased, interest rates increase by 50 basis points.[2] The new market price, the change in the price, and the percentage price change are shown below:

Year	Yield/Coupon Rate (%)	Initial Price ($)	New Yield (%)	New Price ($)	Price Change ($)	Percentage Price Change (%)
1977	7.80	100	8.30	94.50	–5.50	–5.50
1981	15.21	100	15.71	96.85	–3.15	–3.15
2021	1.90	100	2.40	89.35	–10.65	–10.65

Look at the price sensitivity of the three bonds. In the historical high interest-rate environment of 1981, a 50 basis point change would have resulted in a percentage price change of only about half of 1977, about a third of that of 2021, and almost triple that of the lowest of the three yields in this illustration. This follows from a property of the price sensitivity of a bond described in Chapter 13: for a given maturity, the lower the market yield, the greater is the price volatility. As interest rates decline, bond portfolio managers focused increasingly on measures of interest-rate sensitivity, the two most popular measures being duration and convexity (the subject of Chapters 13 and 14).

2. One basis point is equal to 0.01%.

With the complexity of fixed-income products—debt instruments with embedded options, callable/putable bonds and MBS, and certain types of asset-backed securities—traditional analytical techniques for valuing these products had severe limitations. The new army of market researchers that dealer firms recruited from academia brought tools not previously used in the bond market. These included models and techniques for valuing embedded options. Market researchers drew from the fields of option theory and operations research (e.g., Monte Carlo simulation) not only to value these complex products but also to quantify how sensitive the products were to changes in interest rates (concepts such as effective duration and effective convexity were introduced). But it became apparent that even measures such as duration had their limitations using analytical models, leading to the use of regression analysis to estimate the price sensitivity of securities to changes in interest rates for some complex fixed-income products. The resulting interest-rate sensitivity measure is empirical duration, the subject of Chapter 16.

Because a greater understanding of interest-rate sensitivity of a portfolio (duration and convexity) was used in structuring a portfolio compared with the same exposure for a bond market benchmark, the limitations of these measures became apparent. Although duration and convexity offered portfolio managers a means to quantify a portfolio's exposure to changes in interest rates, they were for only one type of an interest-rate change: a parallel shift in the yield curve. Thus market researchers developed frameworks for quantifying exposure to changes in the shape of the yield curve, more specifically changes in the slope of the yield curve (key rate duration and yield-curve duration), the subject of Chapter 15.

Returning to the valuation of complex fixed-income products, models for doing so required the modeling of how interest rates might change over a security's life. Along with valuation modeling came the need to model how interest rates can vary randomly over a security's life. To do so, researchers introduced the notion of stochastic interest-rate models, drawing on an advanced statistical technique called *stochastic differential equations*.[3]

In equity portfolio management, at one time the key and sole driver of returns was assumed to be the market as proxied by the concept of beta. As more studies demonstrated that there are multiple factors that drive equity returns, quantitative (systematic) approaches to equity portfolio management began to identify those other systematic factors These studies have led to the quantitative strategy of factor investing and the construction of equity portfolios based on those factors. Similarly in the fixed-income market, empirical studies have investigated the drivers of bond returns. Early studies empirically demonstrated that changes in the level and shape of the yield curve were major drivers of bond returns. Further studies identified other factors that should be considered in constructing a bond portfolio. Since the

3. As one well-known bond markets analytics guru once remarked: "At one time we hired salespeople based on their ability to drink with clients. Now we hire them based on their ability to solve differential equations."

turn of the century, increasingly research has investigated whether the same factors that drive equity returns also drive corporate bond returns.

Clearly, to be successful as a fixed-income portfolio manager, one needs an understanding of the fundamentals of bond analytics, probability theory, statistics (including financial econometrics and machine learning), and operations research (optimization and Monte Carlo simulation).

ORGANIZATION OF THIS BOOK

There are 34 chapters divided into 10 parts. Part One covers the time value of money with three chapters covering future value (Chapter 2), present value (Chapter 3), and yield measures (Chapter 4). Part Two covers how option-free bonds are priced and conventional yield measures. Option-free bonds are bonds that do not have any embedded options. That is, option-free bonds are not callable or putable. Chapter 5 explains and illustrates how option-free bonds are priced. (The pricing of bonds with embedded options such as callable bonds, MBS, and floating-rate securities is covered in Part Six.) Chapter 6 covers conventional yield measures such as yield to maturity, several yield-to-call measures, yield to put, and yield to worst. The limitations of these measures, as well as the limitation of portfolio yield measures, are described in this chapter. The yield curve, spot-rate curve, and forward rates are covered in Chapter 7.

The four chapters in Part Three cover return analysis and return measures. The performance of a bond portfolio is measured by its total return. To understand total return, which is the subject of Chapter 9, the potential sources of a bond's return are covered in Chapter 8. It may sound simple to calculate a bond portfolio's historical return. However, as explained in Chapter 10, that is not the case. Typically, returns are computed for subperiods and then averaged. There are three measures of averaging subperiod returns: arithmetic average rate of return, time-weighted rate of return, and dollar-weighted rate of return. These measures are described in Chapter 10, along with the advantages and disadvantages of each measure. Also explained in that chapter is how to calculate trailing returns, rolling returns, contribution of a bond to a portfolio's return, and relative return measures (e.g., excess return, active return, abnormal return, and residual return). The final chapter in Part Three, Chapter 11, covers how to adjust returns for risk. Risk-adjusted returns are basically reward/risk measures where the numerator is a measure of return and the denominator is some measure of risk. The three risk-adjusted return measures covered in this chapter are the Sharpe ratio, the Sortino ratio, and the information ratio.

The measurement of price volatility for option-free bonds is the subject of Part Four. The first of the five chapters in this part of the book, Chapter 12, describes the price volatility properties of option-free bonds when interest rates change. Chapter 13 then explains the most popular measure used to estimate a bond's price and a portfolio's value sensitivity to a parallel shift in interest rates (i.e., the yield for all maturities changes by the same number of basis points), duration. We begin by distinguishing between model duration and empirical duration. Model duration for a bond is based on prices generated from a bond pricing model when interest rates change, whereas empirical duration is based on observed market prices when interest rates change. Chapter 13 covers model duration, which includes Macaulay duration, modified duration, and effective duration. Duration, model or empirical, is the first approximation as to how a bond's price or a portfolio's value will change when interest rates change in a parallel fashion. A second approximation is measured by a bond's convexity, the subject of Chapter 14. Because duration is a measure that assumes a parallel shift in the yield curve, a measure or measures are needed to assess the impact on the value of a portfolio for a nonparallel shift in the yield curve. Such measures are the subject of Chapter 15, which covers key rate durations, level-slope-curvature durations, and yield-curve-reshaping durations. Empirical duration is the subject of Chapter 16. As noted earlier, empirical duration is based on observed market prices and prices estimated using regression analysis.

The two chapters in Part Five explain how to measure historical return volatility and yield volatility. In Chapter 17, where we explain how to measure historical return volatility, we begin with describing the measures of dispersion commonly used in bond analytics (i.e., range, interquartile range, mean absolute deviation, variance/standard deviation, semivariance/semi–standard deviation, and lower partial moment). We then describe skewness and kurtosis measures. In this chapter, we introduce the concept of tracking error. More specifically, we discuss backward-looking tracking error. Chapter 18 covers how to measure historical yield volatility and the difference between historical and implied yield volatility. Implied volatility is a by-product of option pricing models. We explain the observed volatility smile and implied volatility surface. After describing these basic concepts, we discuss the methods used to forecast yield volatility (i.e., moving averages and autoregressive conditional heteroscedasticity).

While we explained the valuation of option-free bonds in Part Two, the four chapters in Part Six describe how to value and analyze bonds with embedded options. Because the value of a bond depends on the expected cash flows, and in the case of a bond with an embedded option, the cash flow depends on the level of interest rates, modeling the path of future interest rates is critical in valuing a bond with an embedded option. Therefore, we start Part Six with a discussion of interest-rate modeling (Chapter 19). More specifically, we explain no-arbitrage interest-rate modeling and commonly used models (e.g., Ho–Lee model, Kalotay–Williams–Fabozzi model, Black–Derman–Toy model, Hull–White model, and Black–Karasinski model) and their implementation using the lattice method. We also explain the impact of these models on effective duration, effective convexity, and option-adjusted spread. Because a commonly embedded option is a call option, Chapter 20 is devoted to describing the investment and price characteristics of call options. The valuation of bonds with options and the price volatility of such bonds are the subjects of Chapter 21. We demonstrate how to value bonds with embedded options using the lattice model and introduce the concept of an option-adjusted spread. We postpone discussion of the valuation of MBS (which have an embedded call option because of the borrower's right to prepay a mortgage) until Chapter 27 in Part Eight. We conclude Part Six with an explanation of how to value floating-rate securities (Chapter 22). These securities often have call and put provisions, as well as caps and floors. We begin with a description of spread measures for floating-rate securities (e.g., spread for life, adjusted simple margin, adjusted total margin, and discount margin) and then explain their price-volatility characteristics. The lattice method described in Chapter 21 is then applied to value three complex floating-rate securities (a range note, a step-up callable note, and a callable capped floating-rate bond). We also explain how to analyze an inverse-floating-rate security.

Part Seven covers measures of credit risk and liquidity risk. Credit-risk concepts and measures are covered in Chapter 23. The measures include credit default risk and credit spread risk. Statistical models for predicting corporate bankruptcy (e.g., multiple discriminant analysis, linear probability model, probit regression model, and logit regression model) are explained. Other important concepts covered in Chapter 23 include the statistical concepts of default correlation and copula and the analytical concepts of credit spread duration and duration times spread. The deteriorating liquidity in the bond market due to dealers withdrawing capital commitments to this market has heightened the concern about liquidity risk. In Chapter 24 we first define (or at least provide the general properties) of liquidity and then explain the relationship between liquidity and transaction costs (i.e., investment delay cost, opportunity cost, and market [price] impact cost). We then describe the challenges to measuring liquidity and metrics proposed by market professionals and academics for measuring the liquidity of individual financial instruments and portfolios.

The analysis of securitized products is the subject of the three chapters in Part Eight. Chapter 25 explains the cash-flow characteristics of fixed-rate level-payment mortgage loans. In Chapter 26 we show how to estimate the cash flow for the largest sector in the securitized market, residential mortgage-backed securities (RMBS). The pool of residential mortgage loans is used as collateral for the creation of these securities. How to analyze agency MBS, explaining the traditional analysis of this product versus the Monte Carlo method for doing so, is the subject of Chapter 27.

Although the risk-adjusted return measures described in Chapter 11 provide a starting point for assessing the performance of a bond portfolio manager, these measures fail to identify the reasons why a portfolio manager may have matched, outperformed, or underperformed a benchmark. The decomposition of performance results to explain why the results were achieved is called *performance attribution analysis*. The two most common approaches to performance attribution models are the holdings-based attribution approach and the returns-based style attribution approach. The two chapters in Part Nine describe these two approaches, Chapter 28 covering holdings-based attribution and Chapter 29 returns-based attribution.

The last part of the book, Part Ten, explains statistical and optimizations techniques. This part begins by describing probability distributions (Chapter 30). Regression analysis, the major statistical tool employed in bond analysis, as well as the steps in applying regressions analysis (i.e., model selection, model estimation, and model testing), is the subject of Chapter 31. Also covered in this chapter is principal component analysis and its application to explaining yield-curve dynamics and identifying bond risk factors. A description of multifactor risk models and an illustration of how these models are used in portfolio construction are covered in Chapter 32. Two tools from the field of operations research, Monte Carlo simulation and optimization models, are the subjects of Chapters 33 and 34, respectively. In Chapter 33 we describe the various methodologies for backtesting bond portfolio strategies and explain why Monte Carlo simulation has several advantages over other methodologies. The final chapter in the book, Chapter 35, describes machine learning and how machine-learning algorithms provide bond portfolio managers with the opportunity to use modern nonlinear and highly dimensional techniques needed to build predictive models regarding information that contains complex patterns that when properly analyzed can be used in formulating bond investment strategies so as to enhance portfolio returns. To accomplish this, members of the portfolio management team must be capable of analyzing both structured and unstructured data. Machine learning, which is a subfield of artificial intelligence, provides the analytical tools for extracting insights from a wide range of data sets.

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PART

ONE

TIME VALUE OF MONEY

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CHAPTER

2

FUTURE VALUE

The notion that money has a time value is one of the most basic concepts in financial analysis. Money has a time value because of the opportunities for investing money at some interest rate. In the three chapters of Part One of this book, we review the three fundamental concepts involved in understanding the time value of money. First, we explain how to determine the future value of an investment. In the next chapter, we explain the procedure for determining how much money must be invested today (called the *present value*) in order to realize a specific amount in the future. In the last section of this chapter we explain the special case where continuous compounding is assumed. In Chapter 4 we show how to compute the yield on any investment.

FUTURE VALUE OF AN INVESTMENT

Suppose that an investor places \$1,000 in a bank account and the bank agrees to pay interest of 7% a year. At the end of 1 year, the account will contain \$1,070, that is, \$1,000 of the original principal plus \$70 of interest. Suppose that the investor decides to let the \$1,070 remain in the bank account for another year and that the bank continues paying interest of 7% a year. The amount in the bank account at the end of the second year will equal \$1,144.90, determined as follows:

Principal at beginning of year 2	\$1,070.00
Interest for year 2 (\$1,070 × 0.07)	74.90
Total in bank account	\$1,144.90

In terms of our original \$1,000 investment, the \$1,144.90 represents the following:

Original investment at beginning of year 1	\$1,000.00
Interest for year 1 (\$1,000 × 0.07)	70.00
Interest for year 2 based on the original investment	70.00
Interest for year 2 earned on the interest from year 1 (\$70 × 0.07)	4.90
Total	\$1,144.90

The interest of \$4.90 in year 2 above the \$70 interest earned on the original principal of \$1,000 is interest earned on interest.

After 8 years, the $1,000 investment—if allowed to accumulate tax free at an annual interest rate of 7%—will be $1,718.19, as shown below:

Original investment at beginning of year 1	$1,000.00
At the end of year 1 ($1,000.00 × 1.07)	$1,070.00
At the end of year 2 ($1,070.00 × 1.07)	$1,144.90
At the end of year 3 ($1,144.90 × 1.07)	$1,225.04
At the end of year 4 ($1,225.04 × 1.07)	$1,310.79
At the end of year 5 ($1,310.79 × 1.07)	$1,402.55
At the end of year 6 ($1,402.55 × 1.07)	$1,500.73
At the end of year 7 ($1,500.73 × 1.07)	$1,605.78
At the end of year 8 ($1,605.78 × 1.07)	$1,718.19

After 8 years, $1,000 will grow to $1,718.19 if allowed to accumulate tax free at an annual interest rate of 7%. We refer to the amount at the end of 8 years as the *future value*.

Notice that the total interest at the end of 8 years is $718.19. The total interest represents $560 of interest earned on the original principal ($70 × 8) plus $158.19 ($718.19 – $560) earned by the reinvestment of the interest.

Computing the Future Value of an Investment

To compute the amount that $1,000 will grow to at the end of 8 years if interest is earned at an annual rate of 7%, $1,000 can be multiplied by 1.07 eight times, as shown below:

$$\$1{,}000(1.07)(1.07)(1.07)(1.07)(1.07)(1.07)(1.07)(1.07) = \$1{,}718.19.$$

A shorthand notation for this calculation is

$$\$1{,}000(1.07)^8 = \$1{,}718.19.$$

To generalize the formula, suppose that $1,000 is invested for N periods at an annual interest rate of i (expressed as a decimal). Then, the future value N years from now can be expressed as follows:

$$\$1{,}000(1 + i)^N.$$

For example, if $1,000 is invested for 4 years at an annual interest rate of 10% ($i = 0.10$), then the future value will be $1,464.10:

$$\$1{,}000(1.10)^4 = \$1{,}000(1.4641) = \$1{,}464.10.$$

The expression $(1 + i)^N$ is the amount to which $1 will grow at the end of N years if an annual interest rate of i is earned. This expression is called the *future value of $1*. The future value of $1 multiplied by the original principal yields the future value of the original principal.

For example, we just demonstrated that the future value of $1,000 invested for 4 years at an annual interest rate of 10% would be $1,464.10. The future value of \$1 is \$1.4641. Therefore, if instead of \$1,000, \$50,000 is invested, the future value would be

$$\$50{,}000(1.4641) = \$73{,}205.$$

We can generalize the formula for the future value as follows:

$$FV = P(1 + i)^N,$$

where

FV = future value (\$);
P = original principal (\$);
i = interest rate (in decimal form);
N = number of years.

The following five illustrations show how to apply the future value formula.

Illustration 2–1. A pension fund manager invests \$10 million in a financial instrument that promises to pay 8.7% per year for 5 years. The future value of the \$10 million investment is \$15,175,665, as shown below:

$P = \$10{,}000{,}000$; $i = 0.087$; $N = 5$.

$$\begin{aligned} FV &= \$10{,}000{,}000(1.087)^5 \\ &= \$10{,}000{,}000(1.5175665) = \$15{,}175{,}665. \end{aligned}$$

Illustration 2–2. A life insurance company receives a premium of \$10 million, which it invests for 5 years. The investment promises to pay an annual interest rate of 9.25%. The future value of \$10 million at the end of 5 years is \$15,563,500, as shown below:

$P = \$10{,}000{,}000$; $i = 0.0925$; $N = 5$.

$$\begin{aligned} FV &= \$10{,}000{,}000(1.0925)^5 \\ &= \$10{,}000{,}000(1.5563500) = \$15{,}563{,}500. \end{aligned}$$

Illustration 2–3. Suppose that a life insurance company has guaranteed a payment of \$14 million to a pension fund 4 years from now. If the life insurance company receives a premium of \$11 million and can invest the entire premium for 4 years at an annual interest rate of 6.5%, will it have sufficient funds from this investment to meet the \$14 million obligation?

The future value of the \$11 million investment at the end of 4 years is \$14,151,130, as shown below:

$P = \$11{,}000{,}000$; $i = 0.065$; $N = 4$.

$$\begin{aligned} FV &= \$11{,}000{,}000(1.065)^4 \\ &= \$11{,}000{,}000(1.2864664) = \$14{,}151{,}130. \end{aligned}$$

Because the future value is expected to be \$14,151,130, the life insurance company will have sufficient funds from this investment to satisfy the \$14 million obligation to the pension fund.

Illustration 2–4. The portfolio manager of a tax-exempt fund is considering investing \$400,000 in an instrument that pays an annual interest rate of 5.7% for 4 years. At the end of 4 years, the portfolio manager plans to reinvest the proceeds for 3 more years and expects that, for the 3-year period, an annual interest rate of 7.2% can be earned. The future value of this investment is \$615,098, as shown below.

Future value of the \$400,000 investment for 4 years at 5.7%:

$P = \$400{,}000$; $i = 0.057$; $N = 4$.

$$\begin{aligned} FV &= \$400{,}000(1.057)^4 \\ &= \$400{,}000(1.248245) = \$499{,}298. \end{aligned}$$

Future value of \$499,298 reinvested for 3 years at 7.2%:

$P = \$499{,}298$; $i = 0.072$; $N = 3$.

$$\begin{aligned} FV &= \$499{,}298(1.072)^3 \\ &= \$499{,}298(1.231925) = \$615{,}098. \end{aligned}$$

Illustration 2–5. Suppose that the portfolio manager in the previous illustration has the opportunity to invest the \$400,000 for 7 years in an instrument that promises to pay an annual interest rate of 6.15%. Is this alternative a more attractive investment than the one analyzed in the previous illustration?

The future value of \$400,000 invested for 7 years at 6.15% is \$607,435:

$P = \$400{,}000$; $i = 0.0615$; $N = 7$.

$$\begin{aligned} FV &= \$400{,}000(1.0615)^7 \\ &= \$400{,}000(1.518588) = \$607{,}435. \end{aligned}$$

Assuming that both investments have the same default risk, the investment in the previous illustration will provide a greater future value at the end of 7 years *if* the expectation of the portfolio manager—concerning the annual interest rate at which the rolled-over funds can be reinvested—is realized.

Fractional Periods

In our illustrations, we have computed the future value for whole years. The future value formula, however, is the same if an investment is made for part of a year.

For example, suppose that \$100,000 is invested for 7 years and 3 months. Because 3 months is 0.25 of 1 year, N in the future value formula is 7.25. Assuming an annual interest rate of 5%, the future value of \$100,000 invested for 7 years and 3 months is \$142,437, as shown below:

P = \$100,000; i = 0.05; N = 7.25.

$$FV = \$100{,}000(1.05)^{7.25}$$
$$= \$100{,}000(1.424369) = \$142{,}437.$$

Compounding More than One Time per Year

An investment may pay interest more than one time per year. For example, interest may be paid semiannually, quarterly, monthly, weekly, or daily. The future value formula handles interest payments that are made more than once per year by adjusting the annual interest rate and the exponent. The annual interest rate is adjusted by dividing by the number of times that interest is paid per year. The exponent, which represents the number of years, is adjusted by multiplying the number of years by the number of times that interest is paid per year.

Mathematically, we can express the future value when interest is paid m times per year as follows:

$$FV = P(1 + i)^{n},$$

where

i = *annual* interest rate divided by m;
n = number of interest payments (= $N \times m$).

Illustration 2–6. Suppose that a portfolio manager invests \$1 million in an investment that promises to pay an annual interest rate of 6.4% for 6 years. Interest on this investment is paid semiannually. The future value is \$1,459,340, as shown below:

P = \$1,000,000; m = 2; i = 0.032 (= 0.064/2); N = 6; n = 12 (6 × 2).

$$FV = \$1{,}000{,}000(1.032)^{12}$$
$$= \$1{,}000{,}000(1.459340) = \$1{,}459{,}340.$$

If interest were paid only once per year, the future value would be \$1,450,941 instead of \$1,459,340. The higher future value when interest is paid semiannually reflects the more frequent opportunity for reinvesting the interest paid.

Illustration 2–7. Suppose that in the previous illustration, interest is paid quarterly rather than semiannually. The future value is \$1,463,690, as shown below:

P = \$1,000,000; m = 4; i = 0.016 (= 0.064/4); N = 6; n = 24 (6 × 4).

$$FV = \$1{,}000{,}000(1.016)^{24}$$
$$= \$1{,}000{,}000(1.463690) = \$1{,}463{,}690.$$

The future value is greater than if interest were paid semiannually.

FUTURE VALUE OF AN ORDINARY ANNUITY

Suppose that an investor expects to receive \$10,000 a year from some investment for each of the next 5 years starting 1 year from now. Each time the investor receives the \$10,000, he plans to invest it. Let's assume that the investor can earn an annual interest rate of 6% each time \$10,000 is invested. How much money will the investor have at the end of 5 years?

Our future value formula makes it simple to determine how much each \$10,000 investment will grow to. This calculation is illustrated graphically in Exhibit 2–1.

When the same amount of money is received (or paid) periodically, it is referred to as an *annuity*. When the first investment (or payment) occurs one period from now, it is referred to as an *ordinary annuity*.

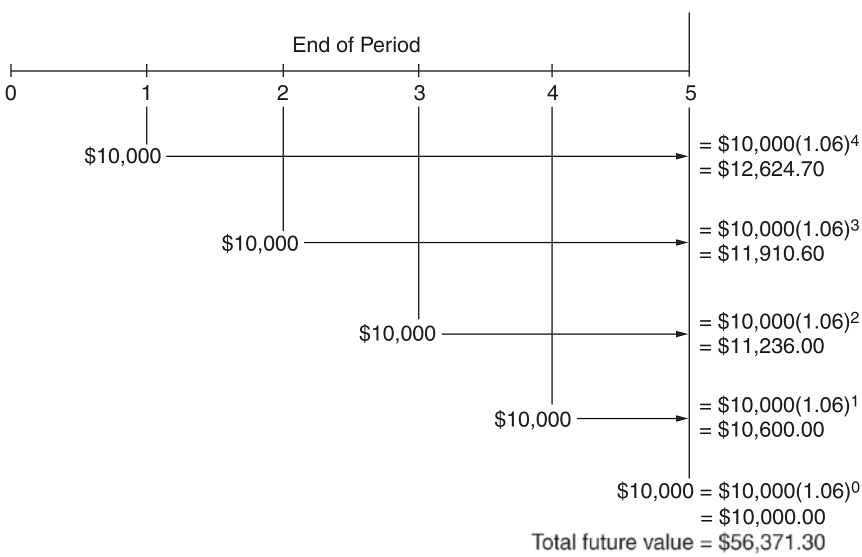
Computing the Future Value of an Ordinary Annuity

In Chapter 10, you will need to know how to compute the future value of an ordinary annuity. Of course, the procedure we just followed—whereby we computed the future value of each investment—can be used. Fortunately, there is a formula that can be used to speed up this computation. The formula is

$$FV = A\left[\frac{(1+i)^N - 1}{i}\right],$$

EXHIBIT 2–1

Future Value of an Ordinary Annuity of \$10,000 per Year for 5 Years



where

A = amount of the annuity ($);
i = annual interest rate (in decimal form).

The term in the brackets is the *future value of an ordinary annuity of $1 per year*. Multiplying the future value of an ordinary annuity of $1 by the amount of the annuity produces the future value of an ordinary annuity.

Using the previous example in which $10,000 is invested each year for the next 5 years, starting 1 year from now, we would have

$A = \$10{,}000$; $i = 0.06$; $N = 5$; therefore,

$$FV = \$10{,}000\left[\frac{(1.06)^5 - 1}{0.06}\right]$$
$$= \$10{,}000\left[\frac{1.3382256 - 1}{0.06}\right]$$
$$= \$10{,}000(5.63710) = \$56{,}371.$$

This value agrees with our earlier calculation in Exhibit 2-1.

Illustration 2–8. Suppose that a portfolio manager purchases $5 million of par value of a 10-year bond that promises to pay 8% interest per year.[1] The price of the bond is $5 million. The interest payment is made once per year by the issuer; the first annual interest payment will be made 1 year from now. How much will the portfolio manager have if she (1) holds the bond until it matures 10 years from now and (2) can reinvest the annual interest payments at an annual interest rate of 6.7%?

The amount that the portfolio manager will have at the end of 10 years will be equal to

1. The $5 million when the bond matures;
2. 10 annual interest payments of $400,000 (0.08 × $5 million);
3. The interest earned by investing the annual interest payments.

We can determine the sum of (2) and (3) by applying the formula for the future value of an ordinary annuity. In this illustration, the annuity is $400,000 per year (0.08 × $5,000,000). Therefore,

$A = \$400{,}000$; $i = 0.067$; $N = 10$.

$$FV = \$400{,}000\left[\frac{(1.067)^{10} - 1}{0.067}\right]$$
$$= \$400{,}000\left[\frac{1.912688 - 1}{0.067}\right]$$
$$= \$400{,}000(13.62221) = \$5{,}448{,}884.$$

1. Eurodollar bonds, for example, pay coupon interest once per year.

The future value of the ordinary annuity of \$400,000 per year for 10 years invested at 6.7% is \$5,448,884. Because \$4,000,000 of this future value represents the total annual interest payments made by the issuer and invested by the portfolio manager, \$1,448,884 (\$5,448,884 − \$4,000,000) must be the interest earned by reinvesting the annual interest payments. Therefore, the total amount that the portfolio manager will have at the end of 10 years by making the investment will be

Par (maturity) value	\$5,000,000
Interest payments	4,000,000
Interest on reinvestment of interest payments	1,448,884
Total	\$10,448,884

As you will see in Chapter 10, it will be necessary to determine the total future amount at the end of some investment horizon to assess the relative performance of a bond.

Future Value of an Ordinary Annuity When Payments Occur More than Once per Year

We can easily generalize the future value of an ordinary annuity formula to handle situations in which payments are made more than one time per year. For example, instead of assuming that an investor receives and then invests \$10,000 per year for 5 years, starting 1 year from now, suppose that the investor receives \$5,000 every 6 months for 5 years, starting 6 months from now.

The general formula for the future value of an ordinary annuity when payments occur m times per year is

$$FV = A\left[\frac{(1+i)^n - 1}{i}\right],$$

where

A = amount of the annuity (\$);
i = annual interest rate divided by m (in decimal form);
$n = N \times m$.

The value in the brackets is the *future value of an ordinary annuity of \$1 per period*.

Illustration 2–9. Let's rework the analysis for the bond in Illustration 2–8 assuming that the interest is paid every 6 months and that the first payment is to be received and invested 6 months from now. The interest payment every 6 months is \$200,000. The future value of the 20 semiannual interest payments of \$200,000 to be received plus the interest earned by investing the interest payments is found as follows:

A = \$200,000; $m = 2$; $i = 0.0335$ (0.067/2); $n = 20$ (10 × 2).

$$FV = \$200,000\left[\frac{(1.0335)^{20} - 1}{0.0335}\right]$$
$$= \$200,000\left[\frac{1.932901 - 1}{0.0335}\right]$$
$$= \$200,000[27.84779] = \$5,569,558.$$

Because the interest payments are equal to \$4,000,000, the interest earned on the interest payments invested is \$1,569,558. Because of the more frequent reinvestment of the interest payments received, the interest of \$1,569,558 earned by investing the interest payments exceeds the interest earned if interest is paid only one time per year, that is, \$1,448,884 (see Illustration 2–8).

The total amount that the portfolio manager will have at the end of 10 years by making the investment will be

Par (maturity) value	\$5,000,000
Interest payments	4,000,000
Interest on reinvestment of interest payments	1,569,558
Total	\$10,569,558

CONTINUOUS COMPOUNDING

Thus far we have explained how to deal with compounding more than once per year. There is a special-case formula for dealing with continuous compounding. Mathematically, it can be shown that the future value of P invested for 1 year compounded continuously is

$$FV = Pe^{i},$$

where e is the base of the natural logarithm (2.71828 . . .).

For example, suppose that P is \$10,000 and i is 8%. Then

$$FV = \$10,000e^{0.08} = \$10,832.87.$$

This formula is for continuous compounding for 1 year. If there is more than 1 year, say N years, then the future value over the N years is computed as follows:

$$FV = Pe^{N \times i}\ c.$$

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CHAPTER

3

PRESENT VALUE

In the previous chapter we illustrated how to compute the future value of an investment. In this chapter we show how to work the process in reverse; that is, given the future value of an investment, we show how to determine the amount of money that must be invested today in order to realize the future value. The amount of money that must be invested today is called the *present value*. Because, as we shall explain later in this chapter, the price of *any* financial instrument is the present value of its expected cash flows, it is necessary to understand present value in order to be able to price a fixed-income instrument. In the last section of this chapter we explain how to compute the present value when continuous compounding is assumed.

PRESENT VALUE OF A SINGLE AMOUNT TO BE RECEIVED IN THE FUTURE

Recall from the previous chapter that the future value of a sum invested for N years can be expressed as

$$FV = P(1 + i)^N,$$

where

FV = future value ($);
P = original principal ($);
i = interest rate (in decimal form);
N = number of years;
$(1 + i)^N$ = future value of $1 invested at i for N years.

How do we determine the amount of money that must be invested today, earning an interest rate of i for N years, in order to produce a specific future value? This is done by solving the future value formula for P, the original principal:

$$P = FV\left[\frac{1}{(1+i)^N}\right].$$

Instead of using P in the formula, we denote the present value as PV. Therefore, the present value formula can be rewritten as

$$PV = FV\left[\frac{1}{(1+i)^N}\right].$$

The term in brackets is equal to the present value of \$1; that is, it indicates how much must be set aside today, earning an interest rate of i, in order to have \$1 N years from now.

The process of computing the present value is referred to as *discounting*. Therefore, the present value is sometimes referred to as the *discounted value*, and the interest rate is referred to as the *discount rate*.

The following four illustrations demonstrate how to compute the present value.

Illustration 3–1. A pension fund manager must satisfy a liability of \$9 million 6 years from now. Assuming that an annual interest rate of 7.5% can be earned on any sum invested today, the pension fund manager must invest \$5,831,654 today in order to have \$9 million 6 years from now, as shown below:

$FV = \$9{,}000{,}000$; $i = 0.075$; $N = 6$.

$$\begin{aligned} PV &= \$9{,}000{,}000\left[\frac{1}{(1.075)^6}\right] \\ &= \$9{,}000{,}000\left[\frac{1}{1.543302}\right] \\ &= \$9{,}000{,}000(0.647961) = \$5{,}831{,}654. \end{aligned}$$

Illustration 3–2. Suppose now that the pension fund manager could earn 8.3% instead of 7.5%; then the present value of the \$9 million to be paid 6 years from now would be \$5,577,912, as shown below:

$FV = \$9{,}000{,}000$; $i = 0.083$; $N = 6$.

$$\begin{aligned} PV &= \$9{,}000{,}000\left[\frac{1}{(1.083)^6}\right] \\ &= \$9{,}000{,}000\left[\frac{1}{1.613507}\right] \\ &= \$9{,}000{,}000(0.619768) = \$5{,}577{,}912. \end{aligned}$$

Illustration 3–3. Suppose that a money manager has the opportunity to purchase a financial instrument that promises to pay \$800,000 in 4 years. The price of the financial instrument is \$572,000. Should the money manager invest in this financial instrument if she wants a 7.8% annual interest rate?

To answer this, the money manager must determine the present value of the \$800,000 to be received 4 years from now. The present value is \$592,400, as shown below:

$FV = \$800{,}000$; $i = 0.078$; $N = 4$.

$$PV = \$800{,}000\left[\frac{1}{(1.078)^4}\right]$$
$$= \$800{,}000\left[\frac{1}{1.350439}\right]$$
$$= \$800{,}000(0.740500) = \$592{,}400 \text{ (the present value).}$$

Because the price of the financial instrument is \$572,000, the money manager will realize more than a 7.8% annual interest rate if the financial instrument is purchased and the issuer pays \$800,000 in 4 years. In the next chapter we'll show how to compute the annual interest rate that the money manager would realize.

Another way of looking at the problem faced by this money manager is to ask how much the \$572,000 would grow to in 4 years if invested at 7.8%. Using the formula for the future value of an investment, we find that the future value is \$772,451, as shown below:

$P = \$572{,}000$; $i = 0.078$; $N = 4$.

$$FV = \$572{,}000(1.078)^4$$
$$= \$572{,}000(1.350439) = \$772{,}451.$$

A \$572,000 investment at 7.8% would grow to only \$772,451. Yet an investment of \$572,000 in the particular financial instrument produces \$800,000 in 4 years. Consequently, the financial instrument offers more than a 7.8% annual interest rate. The present value of \$592,400 tells the money manager that as long as she pays no more than \$592,400 (the present value), an annual interest rate of at least 7.8% will be earned from this investment.

Illustration 3–4. Instead of promising \$800,000 in 4 years, suppose that the financial instrument in the previous illustration promises to pay \$800,000 in 5 years. Assume that the money manager still wants an annual interest rate of 7.8%. Is the investment still attractive if it is selling for \$572,000?

As shown below, the present value of the \$800,000 in 5 years is \$549,536:

$FV = \$800{,}000$; $i = 0.078$; $N = 5$.

$$PV = \$800{,}000\left[\frac{1}{(1.078)^5}\right]$$
$$= \$800{,}000\left[\frac{1}{1.455733}\right]$$
$$= \$800{,}000(0.686920) = \$549{,}536.$$

Here the present value is less than the price of \$572,000, so the financial instrument offers an annual interest rate that is less than 7.8%.

PRESENT VALUE FOR A FRACTIONAL PERIOD

If a future value is to be received or paid over a fractional part of a year, the number of years is adjusted accordingly. For example, if \$1,000 is to be received 9 years and 3 months from now and the interest rate is 7%, the present value is determined as follows:

$FV = \$1{,}000$; $i = 0.07$; $N = 9.25$ years (3 months is 0.25 year).

$$
\begin{aligned}
PV &= \$1{,}000\left[\frac{1}{(1.07)^{9.25}}\right] \\
&= \$1{,}000\left[\frac{1}{1.86982}\right] \\
&= \$1{,}000(0.53481) = \$534.81.
\end{aligned}
$$

PROPERTIES OF PRESENT VALUE

There are two properties of present value that you should recognize. First, for a given future value at a specified time in the future, the higher the interest rate (or discount rate), the lower is the present value. To see this, compare the present value in Illustration 3–1 to that in Illustration 3–2. When the annual interest rate increased from 7.5% to 8.3%, the present value of the \$9 million needed 6 years from now decreased from \$5,831,654 to \$5,577,912. The reason that the present value decreases as the interest rate increases should be easy to understand. The higher the interest rate that can be earned on any sum invested today, the less has to be invested today to realize a specified future value.

The second property of the present value is that for a given interest rate (discount rate), the farther into the future the future value will be received, the lower is the present value. This is demonstrated in Illustrations 3–3 and 3–4. When the amount of \$800,000 is to be received 4 years from now, the present value is \$592,400; if the \$800,000 is to be received 5 years from now, the present value declines to \$549,536. The reason is that the farther into the future a given future value is to be received, the more time there is for interest to accumulate. The result is that fewer dollars have to be invested.

PRESENT VALUE OF A SERIES OF FUTURE VALUES

In most applications in investment management and asset/liability management, a financial instrument will offer a series of future values, or a financial institution will have multiple liabilities in the future. To determine the present value of a series of future values, the present value of each future value must first be computed. Then the present values are added to obtain the present value of the series of future values. This procedure is demonstrated in the following three illustrations.

Illustration 3–5. A pension fund manager knows that the following liabilities must be satisfied:

Years from Now	Liability ($)
1	200,000
2	340,000
3	500,000
4	580,000

Suppose that the pension fund manager wants to invest a sum of money that will satisfy this liability stream. Let's assume that any amount that can be invested today can earn an annual interest rate of 8.5%. How much must be invested to satisfy this liability stream?

The answer is the present value of the liability stream. Consequently, the present value of each liability must be calculated, and the results must be totaled, as shown below.

Years from Now	Future Value of Liability ($)	Present Value of $1 at 8.5%	Present Value of Liability ($)
1	200,000	0.921659	184,332
2	340,000	0.849455	288,815
3	500,000	0.782908	391,454
4	580,000	0.721574	418,513
		Total present value =	$1,283,114

The present value of $1,283,114 means that if this sum is invested today at an annual interest rate of 8.5%, it will provide sufficient funds to satisfy the liability stream.

For those who must be convinced that this is true, let's look at what would happen if $1,283,114 were invested at 8.5% in a bank account and at the end of each year enough money is withdrawn from the bank account to satisfy the annual liability.

(1) Year	(2) Amount at Beginning of Year ($)	(3) Interest at 8.5% ($) [0.085 × (2)]	(4) Amount Withdrawn to Pay Liability ($)	(5) Amount at End of Year ($) [(2) + (3) – (4)]
1	1,283,114	109,065	200,000	1,192,179
2	1,192,179	101,335	340,000	953,514
3	953,514	81,049	500,000	534,563
4	534,563	45,437	580,000	0

As these computations show, the $1,283,114 investment will provide enough money to pay the liability stream. At the end of the fourth year (after the last liability is paid), there is no money left in the account.

Illustration 3–6. An investor is considering the purchase of a financial instrument that promises to make the following payments:

Years from Now	Promised Payment by Issuer ($)
1	100
2	100
3	100
4	100
5	1,100

This financial instrument is selling for $1,243.83. Assume that the investor wants a 6.25% annual interest rate on this investment. Should the investor purchase this investment?

To answer this question, the investor first must compute the present value of the future amounts that are expected to be received, as follows:

Years from Now	Future Value of Payment ($)	Present Value of $1 at 6.25%	Present Value of Payment ($)
1	100	0.9412	94.12
2	100	0.8858	88.58
3	100	0.8337	83.37
4	100	0.7847	78.47
5	1,100	0.7385	812.35
		Total present value =	$1,156.89

The present value of the series of future values promised by the issuer of this financial instrument is less than the price of $1,243.83, so the investor would earn an annual interest rate of less than 6.25%. Thus the financial instrument is unattractive.

PRESENT VALUE OF AN ORDINARY ANNUITY

When the same amount of money is received or paid each year, the series is referred to as an *annuity*. When the first payment is received or paid 1 year from now, the annuity is called an *ordinary annuity*. When the first payment or receipt is immediate, the annuity is called an *annuity due*. In all the applications discussed in this book, we deal with ordinary annuities.

Of course, one way to compute the present value of an ordinary annuity is to follow the procedure explained in the previous section: compute the present value of each future value and then total the present values. Fortunately, there is a formula that can be employed to compute—in one step—the present value of an ordinary annuity:

$$PV = A\left[\frac{1-\left[\frac{1}{(1+i)^N}\right]}{i}\right],$$

where

A = amount of the annuity ($).

The term in the large brackets is the *present value of an ordinary annuity of $1 for N years*.

Two illustrations will show how to apply this formula.

Illustration 3–7. An investor has the opportunity to purchase a financial instrument that promises to pay $500 a year for the next 20 years beginning 1 year from now. The financial instrument is being offered for a price of $5,300. The investor seeks an annual interest rate of 5.5% on this investment. Should the investor purchase this financial instrument?

Because the first payment is to be received 1 year from now, the financial instrument is offering a 20-year annuity of $500 per year. The present value of this ordinary annuity is calculated as follows:

$A = \$500$; $i = 0.055$; $N = 20$.

$$\begin{aligned} PV &= \$500\left[\frac{1-\left[\frac{1}{(1.055)^{20}}\right]}{0.055}\right] \\ &= \$500\left[\frac{1-\left[\frac{1}{2.917757}\right]}{0.055}\right] \\ &= \$500\left[\frac{1-0.342729}{0.055}\right] \\ &= \$500(11.950382) = \$5{,}975.19. \end{aligned}$$

Because the present value of an ordinary annuity of $500 per year when discounted at 5.5% exceeds the price of the financial instrument ($5,300), this financial instrument offers an annual interest rate in excess of 5.5%. Therefore, it is an attractive investment for this investor.

Illustration 3–8. In Illustration 3–6 we computed the present value of a financial instrument that offers $100 a year for 4 years and $1,100 at the end of the fifth year. This payment series is equivalent to an ordinary annuity of $100 a year for 5 years *plus* a future value payment of $1,000 in 5 years. Viewing the payments of the financial instrument in this way, let's compute the present value.

The present value of an ordinary annuity of \$100 per year for 5 years at an annual interest rate of 6.25% is

$A = \$100;\ i = 0.0625;\ N = 5.$

$$
\begin{aligned}
PV &= \$100\left[\frac{1-\left[\dfrac{1}{(1.0625)^5}\right]}{0.0625}\right] \\
&= \$100\left[\frac{1-\left[\dfrac{1}{1.354081}\right]}{0.0625}\right] \\
&= \$100\left[\frac{1-0.738508}{0.0625}\right] \\
&= \$100(4.1838) = \$418.38.
\end{aligned}
$$

The present value of the \$1,000 to be received 5 years from now is \$738.51, as shown below:

$FV = \$1{,}000;\ i = 0.0625;\ N = 5.$

$$
\begin{aligned}
PV &= \$1{,}000\left[\frac{1}{(1.0625)^5}\right] \\
&= \$1{,}000\left[\frac{1}{1.354081}\right] \\
&= \$1{,}000(0.738508) = \$738.51.
\end{aligned}
$$

The present value of the series offered by this financial instrument is then

Present value of ordinary annuity of \$100 for 5 years at 6.25%	\$418.38
Present value of \$1,000 in 5 years at 6.25%	738.51
Total present value	\$1,156.89

This agrees with the computation in Illustration 3–6.

Perpetual Annuity: Special Case

So far we have shown how to compute the present value of an ordinary annuity over a specific time period. Suppose, instead, that the annuity will last forever. This is called a *perpetual annuity*. The formula for a perpetual annuity is[1]

$$PV = \frac{A}{i}.$$

1. The formula is derived from the formula for the present value of an ordinary annuity. As the number of years gets very large, the value of $1/(1 + i)^N$ approaches zero. The numerator in the large brackets is then equal to 1, producing the formula for the present value of a perpetual annuity.

Illustration 3–9. An investor can purchase for \$1,000 a financial instrument that promises to pay \$80 per year forever. The investor wants an annual interest rate of 10% from this investment. Is this investment attractive to the investor?

The present value of the \$80 perpetual annuity is equal to \$800, as shown below:

$A = \$80$; $i = 0.10$

$$PV = \frac{\$80}{0.10} = \$800.$$

Because the \$1,000 price for the financial instrument is greater than the present value of the perpetual annuity (\$800), the investment offers an annual interest rate that is less than 10%; therefore, it is not an attractive investment given the minimum annual interest rate required by the investor.

PRESENT VALUE WHEN THE FREQUENCY IS MORE THAN ONCE PER YEAR

In the computations of present value we have assumed that the future value is to be received or paid once each year. In practice, the future value may be received or paid more than once per year. In this situation, the formulas for the present value given earlier in this chapter must be modified in two ways. First, the annual interest rate is divided by the frequency per year.[2] For example, if the future values are received or paid semiannually, the annual interest rate is divided by 2; if quarterly, the annual interest rate is divided by 4. Second, the periods when the future value will be received or paid must be adjusted by multiplying the number of years by the frequency per year.

The general formula for the present value of a future sum is

$$PV = FV\left[\frac{1}{(1+i)^n}\right],$$

where

i = periodic interest rate [annual interest rate (in decimal form) divided by m];
n = number of periods [number of years (N) times m];
m = frequency of receipt or payment of the future value.

2. Technically, this is not the proper way for adjusting the annual interest rate. For example, an 8% annual interest rate is not equal to a quarterly interest rate of 2%. However, in the computation of the yield on bonds, where we will use this calculation, the market has adopted a convention that embodies this approach. This will be made clearer in the next two chapters.

Illustration 3–10. An investor is considering the purchase of a financial instrument that promises to make the following payments every 3 months (quarterly):

Period (3 months)	Promised Payments ($)
1	1,000
2	1,200
3	1,500
4	1,700
5	1,800
6	2,000

If the investor seeks an annual interest rate of 12% from this investment, what is the most that the investor should pay for it?

The most that the investor should pay in order to earn an annual interest rate of at least 12% is the present value of the future payments promised. As shown below, the present value is $8,212.79.

Periods from Now	Future Value of Payment ($)	Present Value of $1 at 3.0%*	Present Value of Payment ($)
1	1,000	0.97087	970.87
2	1,200	0.94260	1,131.12
3	1,500	0.91514	1,372.71
4	1,700	0.88849	1,510.43
5	1,800	0.86261	1,552.70
6	2,000	0.83748	1,674.96
		Total present value =	$8,212.79

*12% annual interest rate divided by 4.

When the present value of an ordinary annuity is sought, the general formula is

$$PV = A\left[\frac{1-\left[\frac{1}{(1+i)^n}\right]}{i}\right],$$

where

A = amount of the annuity ($ *per period*).

Illustration 3–11. In Illustration 3–6, we computed the present value of the following series of future amounts, assuming an annual interest rate of 6.25%:

Years from Now	Promised Payment by Issuer ($)
1	100
2	100
3	100
4	100
5	1,100

Instead of annual payments, let's assume that the payments are made by the issuer every 6 months, in the following way:

6-Month Periods from Now	Promised Payment by Issuer ($)
1	50
2	50
3	50
4	50
5	50
6	50
7	50
8	50
9	50
10	1,050

This is equivalent to an ordinary annuity of $50 per 6-month period for ten 6-month periods and $1,000 to be paid ten 6-month periods from now. Notice that the $1,000 is treated on the same time-period basis as the annuity.

The present value of the ordinary annuity for

$A = \$50$; $m = 2$ (i.e., payments every 6 months); $i = 0.03125$ (0.0625 annual interest rate divided by 2); $n = 10$ (5 years times 2)

is

$$PV = \$50\left[\frac{1-\left[\dfrac{1}{(1.03125)^{10}}\right]}{0.03125}\right]$$

$$= \$50\left[\frac{1-\left[\dfrac{1}{1.360315}\right]}{0.03125}\right]$$

$$= \$50\left[\frac{1-.735124}{0.03125}\right]$$

$$= \$50(8.4760) = \$423.80.$$

The present value of the $1,000 to be received after ten 6-month periods is

$FV = \$1{,}000$; $m = 2$ (i.e., payments every 6 months);

$i = 0.03125$ (0.0625 annual interest rate divided by 2);

$n = 10$ (5 years times 2)

is

$$PV = \$1{,}000\left[\frac{1}{(1.03125)^{10}}\right]$$

$$= \$1{,}000\left[\frac{1}{1.360315}\right]$$

$$= \$1{,}000(0.735124) = \$735.12.$$

The present value of the future value series offered by the financial instrument is then

Present value of ordinary annuity of $50 for ten 6-month periods at 3.125%	$423.80
Present value of $1,000 ten 6-month periods at 3.125%	735.12
Total present value	$1,158.92

Notice that because the payments are made more often, the present value of the future payments has increased from $1,156.89 to $1,158.92.

Illustration 3–12. Suppose that a banker agrees to make a $100,000 thirty-year loan to an individual to purchase a home. Under the terms of the loan, the monthly payments to be made by the individual will all be the same. The annual interest rate that the banker charges for the loan is 12%. How much must the fixed monthly payment be in order for the banker to realize an annual interest rate of 12%?

We can employ the formula for the present value of an ordinary annuity to determine the fixed monthly payment. The banker wants to receive an annuity of some fixed monthly amount such that the present value of that ordinary annuity, at an annual interest rate of 12%, is $100,000. In the formula for the present value of an ordinary annuity, the number of monthly payments is 360 (30 years times 12), and the interest rate is 1% (12% divided by 12). Therefore, we know the following:

$$\$100{,}000 = A\left[\frac{1-\left[\dfrac{1}{(1.01)^{360}}\right]}{0.01}\right].$$

The unknown is A, the monthly annuity or monthly loan payment. We can solve for A as follows:

$$\$100{,}000 = A\left[\frac{1-\left[\dfrac{1}{35.949641}\right]}{0.01}\right]$$

$$\$100{,}000 = A\left[\frac{1-0.0278167}{0.01}\right]$$

$$\$100{,}000 = A(97.21833).$$

Solving for A,

$$A = \left[\frac{\$100{,}000}{97.21833}\right] = \$1{,}028.61.$$

Therefore, the fixed monthly payment must be $1,028.61.

Unequal Discount Rates

Throughout this chapter we have assumed that the same interest rate or discount rate should be used to calculate the present value of each payment to be received. This assumption is unwarranted. As we explain in Chapter 7, there is a reason why each payment to be received should be discounted at a unique rate. The present value of a series of payments is then the sum of the present value of each payment where each payment is discounted at a unique rate.

Illustration 3–13. Let's consider Illustration 3–6, where the financial instrument makes the following five payments: $100 for 4 years and $1,100 in the fifth year. When each payment is discounted at a 6.25% annual interest rate, the total present value is $1,156.89. Suppose, instead, that the interest rate for each year is as shown below:

Years from Now	Promised Payment by Issuer ($)	Interest Rate (%)
1	100	4.80
2	100	5.25
3	100	5.50
4	100	6.00
5	1,100	6.25

The present value of each promised payment is shown below. Each payment is discounted at the required yield.

Years from Now	Promised Payment by Issuer ($)	Required Yield (%)	Present Value of $1	Present Value of Payment ($)
1	100	4.80	0.954198	95.4198
2	100	5.25	0.902726	90.2725
3	100	5.50	0.851614	85.1613
4	100	6.00	0.792094	79.2093
5	1,100	6.25	0.738508	812.3590
			Total present value =	$1,162.4219

The present value of the cash flows is $1,162.42.

PRICING ANY FINANCIAL INSTRUMENT

The price of any financial instrument is equal to the present value of the *expected* cash flows from investing in the financial instrument. Determining the price, therefore, requires the following inputs:

1. Estimation of the expected cash flows;
2. Determination of the appropriate interest rate or discount rate so that the present value of the cash flows can be computed.

The cash flow in any period is simply the difference between the cash inflow and the cash outflow from investing in the financial instrument. The expected cash flows of some financial instruments are simple to compute; for others, the task is not as simple. The determination of the interest rate or discount rate reflects the required yield for financial instruments with *comparable* risk.

CONTINUOUS COMPOUNDING

In Chapter 2 we explained that the future value based on continuous compounding is

$$FV = P\,e^{i},$$

where e is the base of the natural logarithm (2.71828 . . .). The present value when there is continuous compounding is a special-case formula. Solving this equation for P and replacing P with PV (for present value), we obtain

$$PV = FV\,e^{-i}$$

For example, suppose that $1,000 is to be received 1 year from now. Assuming that interest can be earned at 5% continuously compounded, the present value is

$$PV = \$1{,}000\,e^{-0.05} = \$951.23.$$

The formula above is for the present value when there is continuous compounding for 1 year. If it is more than 1 year into the future, say N years, then the present value after N years is computed as follows:

$$PV = FV\,e^{-N \times i}.$$

For example, the present value of $1,000 to be received 5 years from now assuming an interest rate of 5% is

$$PV = \$1{,}000\,e^{-(5 \times 0.05)} = \$778.80.$$

CHAPTER

4

YIELD (INTERNAL RATE OF RETURN)

In the previous chapter we showed how to use present value to determine whether a financial instrument provides a minimum annual interest rate specified by an investor. For example, if the present value of the promised future value payments of some financial instrument selling for \$944.14 is \$1,039.57 when discounted at 9%, then the investment offers an annual interest rate greater than 9%. But how much greater? What yield will the investor earn by buying the financial instrument for \$944.14? The purpose of this chapter is to explain how to compute the yield on any investment.

COMPUTING THE YIELD ON ANY INVESTMENT

The yield on any investment is computed by determining the interest rate that will make the present value of the cash flow from the investment equal to its price. Mathematically, the yield y on any investment is the interest rate that will make the following relationship hold:

$$p = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_N}{(1+y)^N},$$

where

p = price;
C_t = cash flow in year t;
N = number of years.

The individual terms summed to produce the price are the present values of the cash flow. The yield calculated from the above relationship is also called the *internal rate of return*.

Alternatively, using the capital Greek letter sigma to denote summation, the above expression can be rewritten as

$$p = \sum_{t=1}^{N} \frac{C_t}{(1+y)^t}.$$

Solving for the yield y requires an iterative procedure. The objective is to find the interest rate that will make the present value of the cash flows equal to the

EXHIBIT 4–1

Step-by-Step Summary of Yield Computation for Any Investment

Objective	Find the interest rate that will make the present value of the cash flow equal to the price of the investment.
Step 1	Select an interest rate.
Step 2	Compute the present value of each cash flow by using the interest rate selected in Step 1.
Step 3	Total the present value of the cash flows found in Step 2.
Step 4	Compare the total present value found in Step 3 with the price of the investment. Then, if the total present value of the cash flows found in Step 3 is equal to the price of the investment, the interest rate selected in Step 1 is the yield. if the total present value of the cash flows found in Step 3 is more than the price of the investment, the interest rate used is not the yield. Go back to Step 1 and use a higher interest rate. if the total present value of the cash flows found in Step 3 is less than the price of the investment, the interest rate used is not the yield. Go back to Step 1 and use a lower interest rate.

price. Exhibit 4–1 explains the iterative procedure. The following two illustrations demonstrate how it is carried out.

Illustration 4–1. A financial instrument offers the following annual payments:

Years from Now	Promised Annual Payments (Cash Flow to Investor; $)
1	2,000
2	2,000
3	2,500
4	4,000

Suppose that the price of this financial instrument is $7,704. What is the yield or internal rate of return offered by this financial instrument?

To compute the yield, we must try different interest rates until we find one that makes the present value of the cash flows equal to $7,704 (its price). Trying an annual interest rate of 10% gives the following present value:

Years from Now	Promised Annual Payments (Cash Flow to Investor; $)	Present Value of Cash Flow at 10% ($)
1	2,000	1,818
2	2,000	1,652
3	2,500	1,878
4	4,000	2,732
		Total present value = $8,080

The present value computed using a 10% interest rate exceeds the price of \$7,704, so a higher interest rate must be tried. If a 14% interest rate is tried, the present value is \$7,348, as shown below:

Years from Now	**Promised Annual Payments (Cash Flow to Investor; \$)**	**Present Value of Cash Flow at 14% (\$)**
1	2,000	1,754
2	2,000	1,538
3	2,500	1,688
4	4,000	2,368
		Total present value = \$7,348

At 14%, the present value of the cash flows is less than the \$7,704 price of the financial instrument. Therefore, a lower interest rate must be tried. The present value at a 12% interest rate is shown below:

Years from Now	**Promised Annual Payments (Cash Flow to Investor, \$)**	**Present Value of Cash Flow at 12% (\$)**
1	2,000	1,786
2	2,000	1,594
3	2,500	1,780
4	4,000	2,544
		Total present value = \$7,704

The present value of the cash flows is now equal to the price of the financial instrument when a 12% interest rate is used. Therefore, the yield is 12%.

Although the formula for the yield is based on annual cash flows, the formula can be generalized to any number of periodic payments in a year. The generalized formula for determining the yield is

$$p = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n},$$

where

C_t = cash flow in period t;
n = number of periods.

In shorthand notation, this can be expressed as

$$p = \sum_{t=1}^{n} \frac{C_t}{(1+y)^t}.$$

Keep in mind that the yield computed is now the yield for the period. That is, if the cash flows are semiannual, the yield is a semiannual yield. If the cash flows are monthly, the yield is a monthly yield. The annual interest rate must be computed by multiplying the yield for the period by the appropriate factor (*m*).

Illustration 4–2. In Illustration 3–11 of the previous chapter, an investor considered purchasing a financial instrument that promised the following *semiannual* cash flows:

10 payments of $50 every 6 months;

$1,000 ten 6-month periods from now.

Suppose that the price of this financial instrument is $1,243.88. At the 6.5% annual interest rate sought by the investor, the present value of the cash flows is equal to $1,158.92; thus the financial instrument would not be an attractive investment for this investor. What yield is this financial instrument offering?

The yield can be computed as summarized in the table below:

Annual Interest Rate (%)	Semiannual Interest Rate (%)	Present Value of Ten 6-Month Payments of $50 ($)*	Present Value of $1,000 Ten 6-Month Periods from Now ($)**	Total Present Value ($)
6.000	3.000	426.51	744.09	1,160.60
5.500	2.750	432.00	762.40	1,194.40
5.000	2.500	437.60	781.20	1,218.80
4.500	2.250	443.31	800.51	1,243.83

*$50 × present value of an ordinary annuity of $1 for 10 periods.

**$1,000 × present value of $1 ten periods from now.

As can be seen from the calculation, when a semiannual interest rate of 2.250% is used to find the present value of the cash flows, the present value is equal to the price of $1,243.83. Hence, 2.250% is the 6-month yield. Doubling this yield gives the annual interest rate of 4.5%. This agrees with our earlier conclusion: this financial instrument is unattractive because it offers a yield that is less than the 6.5% annual interest rate required by the investor.

Illustration 4–3. Suppose that the financial instrument analyzed in the previous illustration is selling for $944.14 instead of $1,243.83. What is the yield offered on this financial instrument at this lower price?

The table below shows the calculation of the yield:

Annual Interest Rate (%)	Semiannual Interest Rate (%)	Present Value of Ten 6-month Payments of \$50 (\$)*	Present Value of \$1,000 Ten 6-month Periods from Now (\$)**	Total Present Value (\$)
9.000	4.500	395.64	643.93	1,039.57
9.500	4.750	390.82	628.72	1,019.54
10.000	5.000	386.09	613.91	1,000.00
10.500	5.250	381.44	599.49	980.93
11.000	5.500	376.88	585.43	962.31
11.500	5.750	372.40	571.74	944.14

* \$50 × present value of an ordinary annuity of \$1 for 10 periods.

** \$1,000 × present value of \$1 ten periods from now.

An interest rate of 5.75% equates the present value of the cash flows to the price of the financial instrument; hence, 5.75% is the 6-month yield, and 11.50% is the annual interest rate.

YIELD CALCULATION WHEN THERE IS ONLY ONE CASH FLOW

There is a special case where it is unnecessary to go through the iterative procedure to determine the yield. This occurs when there is only one cash flow provided by the investment. We'll introduce the formula by means of an illustration.

Illustration 4–4. A financial instrument that can be purchased for \$6,805.82 promises to pay \$10,000 in 5 years. The yield is the interest rate that will make \$6,805.82 grow to \$10,000 in 5 years. That is, we are looking for the value of y that will satisfy the following relationship:

$$\$10{,}000 = \$6{,}805.82\,(1 + y)^5.$$

We can solve this equation as follows. Divide both sides by \$6,805.82:

$$\frac{\$10{,}000}{\$6{,}805.82} = (1 + y)^5$$

$$1.46933 = (1 + y)^5.$$

Take the fifth root of both sides:

$$1.0800 = (1 + y)$$

Subtracting 1 from both sides gives the yield on this investment of 8%.

$$1.0800 - 1 = y$$

$$0.08 = y.$$

It is not necessary to go through all the steps in Illustration 4–4 to compute the yield. The following formula is consistent with those steps:

$$y = (\text{future value per dollar invested})^{1/n} - 1,$$

where

n = number of periods until the cash flow will be received;

$$\text{Future value per dollar invested} = \frac{\text{cash flow from investment}}{\text{amount invested (or price)}}.$$

Illustration 4–5 An investment offers a payment 20 years from now of $84,957. The price of the investment is $20,000. The yield for this investment is 7.50%, as shown below:

$n = 20$;

$$\text{Future value per dollar invested} = \frac{\$84{,}957}{\$20{,}000} = 4.24785.$$

$$\begin{aligned} y &= (4.24785)^{1/20} - 1 = 1.07499 - 1 \\ &= 0.074999 \text{ or } 7.5\%. \end{aligned}$$

Illustration 4–6 In Illustration 2–8, we computed how many dollars would be available to a portfolio manager if he invests $5 million (the par value) in a bond that matures in 10 years and promises to pay an annual interest rate of 8%. The interest is assumed to be paid once per year, and these payments are assumed to be reinvested at an annual interest rate of 6.7%. We calculated that at the end of 10 years, the portfolio manager would have $10,448,884, consisting of $5 million in par value, $4 million in annual interest payments, and the balance, $1,448,884, from interest earned on the reinvestment of the annual interest payments.

The yield on this investment based on the portfolio manager's expectations can be computed by finding the yield that will make a $5 million investment grow to $10,448,884 in 10 years. Because we have reduced the problem to that of an investment that provides the portfolio manager with one cash flow, the yield can be found as follows:

$n = 10\ (= N)$;

$$\text{Future value per dollar invested} = \frac{\$10{,}448{,}884}{\$5{,}000{,}000} = 2.08978.$$

$$\begin{aligned} y &= (2.08978)^{1/10} - 1 \\ &= 1.07649 - 1 = 0.07649 \text{ or } 7.65\%. \end{aligned}$$

ANNUALIZING YIELDS

So far in this book we have annualized interest rates by simply multiplying by the frequency of payments per year. We called the resulting rate the *annual interest rate*. For example, if we computed a semiannual yield, we annualized it by multiplying by 2. Alternatively, if we had an annual interest rate and wanted to use a semiannual interest rate, we divided by 2.

This single procedure for computing the annual interest rate given a periodic (weekly, monthly, quarterly, semiannual, etc.) interest rate is not correct. To see why, suppose that \$100 is invested for 1 year at an annual interest rate of 8%. At the end of 1 year, the interest is \$8. Suppose, instead, that \$100 is invested for 1 year at an annual interest rate of 8%, but interest is paid semiannually at 4% (one-half the annual interest rate). The interest at the end of 1 year is found by first calculating the future value of \$100 at the end of 1 year:

$$\$100(1.04)^2 = \$100(1.0816) = \$108.16$$

Interest is therefore \$8.16 on a \$100 investment. The interest rate or yield on the \$100 investment is therefore 8.16% (\$8.16/\$100). The 8.16% is called the *effective annual yield*.

Investors who are familiar with certificates of deposit offered by banks and thrifts should recognize the difference between the annual interest rate and effective annual yield. Typically, both of these interest rates are quoted for a certificate of deposit, the higher interest rate being the effective annual yield.

To obtain the effective annual yield associated with a periodic interest rate, the following formula can be used:

$$\text{Effective annual yield} = (1 + \text{periodic interest rate})^m - 1,$$

where

m = frequency of payments per year.

For example, in the previous example, the periodic yield is 4%, and the frequency of payments is twice per year. Therefore,

$$\begin{aligned}\text{Effective annual yield} &= (1.04)^2 - 1\\ &= 1.0816 - 1 = 0.0816 \text{ or } 8.16\%.\end{aligned}$$

If interest is paid quarterly, then the periodic interest rate is 2% (8%/4), and the effective annual yield is 8.24%, as shown below:

$$\begin{aligned}\text{Effective annual yield} &= (1.02)^4 - 1\\ &= 1.0824 - 1 = 0.0824 \text{ or } 8.24\%.\end{aligned}$$

We can also determine the periodic interest rate that will produce a given annual interest rate. For example, suppose that we want to know what quarterly interest rate would produce an effective annual yield of 12%. The following formula can be used:

$$\text{Periodic interest rate} = (1 + \text{effective annual yield})^{1/m} - 1.$$

Applying this formula to determine the quarterly interest rate to produce an effective annual yield of 12%, we find

$$\begin{aligned}\text{Periodic interest rate} &= (1.12)^{1/4} - 1\\ &= 1.0287 - 1 = 0.0287 \text{ or } 2.87\%.\end{aligned}$$

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PART

TWO

BOND PRICING FOR OPTION-FREE BONDS AND CONVENTIONAL YIELD MEASURES

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CHAPTER

5

THE PRICE OF A BOND

In Chapter 3 we explained that the price of any financial instrument is equal to the present value of the expected cash flows. The interest rate or discount rate used to compute the present value depends on the yield offered on comparable securities in the market. In this chapter we explain how to compute the price of an option-free bond (i.e., a bond that is not callable, putable, or convertible). Options and the valuation of securities with embedded options are discussed in later chapters.

DETERMINING THE CASH FLOWS

The first step in determining the price of a bond is to determine its cash flows. The cash flows of an option-free bond consist of (1) periodic coupon interest payments to the maturity date and (2) the par (or maturity) value at maturity. While the periodic coupon payments can be made over any time period during the year (weekly, monthly, quarterly, semiannually, or annually), most bonds issued in the United States pay coupon interest semiannually.

In our illustrations, we shall assume that the coupon interest is paid semiannually. To simplify the analysis, we also assume that the next coupon payment for the bond will be made exactly 6 months from now. Later in the chapter we generalize the pricing model to allow for a coupon payment that is less than 6 months from now.

Consequently, the cash flows for an option-free bond consist of an annuity (i.e., the fixed coupon interest paid every 6 months) and the par or maturity value. For example, a 20-year bond with a 9% coupon rate (4.5% every 6 months) and a par or maturity value of $1,000 has the following cash flows:

$$\text{Semiannual coupon interest} = \$1{,}000 \times 0.045 = \$45;$$
$$\text{Maturity value} = \$1{,}000.$$

Therefore, there are 40 semiannual cash flows of $45 and a $1,000 cash flow forty 6-month periods from now.

Notice the treatment of the par value. It is *not* treated as if it is received 20 years from now. Instead, it is treated on a consistent basis with the coupon payments, which are semiannual.

DETERMINING THE REQUIRED YIELD

The interest rate or discount rate that an investor wants from investing in a bond is called the *required yield*. The required yield is determined by investigating the yields offered on comparable bonds in the market. By comparable, we mean option-free bonds of the same credit quality and same maturity.[1]

The required yield is typically specified as an annual interest rate. When the cash flows are semiannual, the convention is to use one-half the annual interest rate as the periodic interest rate with which to discount the cash flows. As explained at the end of the previous chapter, a periodic interest rate that is one-half the annual yield will produce an effective annual yield that is greater than the annual interest rate.

PRICING A BOND

Given the cash flows of a bond and the required yield, we have all the necessary data to price the bond. The price of a bond is the present value of the cash flows, which can be determined by adding

1. The present value of the semiannual coupon payments;
2. The present value of the par or maturity value.

In general, the price of a bond can be computed using the formula

$$p = \frac{c}{(1+i)^1} + \frac{c}{(1+i)^2} + \frac{c}{(1+i)^3} + \cdots + \frac{c}{(1+i)^n} + \frac{M}{(1+i)^n},$$

where

p = price (\$);
c = semiannual coupon payment (\$);
i = periodic interest rate (required yield/2) (in decimal form);
n = number of periods (number of years $\times$ 2);
M = maturity value.

Because the semiannual coupon payments are equivalent to an ordinary annuity, the present value of the coupon payments, that is, the present value of

$$p = \frac{c}{(1+i)^1} + \frac{c}{(1+i)^2} + \frac{c}{(1+i)^3} + \cdots + \frac{c}{(1+i)^n},$$

1. In Chapter 14, we introduce a measure of interest-rate risk known as *duration*. Instead of talking in terms of a bond with the same maturity as being comparable, the analysis can properly be recast in terms of the same duration.

can be expressed as

$$c\left[\frac{1-\left[\frac{1}{(1+i)^n}\right]}{i}\right].$$

This formula is the same as the formula for the present value of an ordinary annuity for n periods introduced in Chapter 3 (see Illustration 3–10). Instead of using A to represent the annuity, we have used c, the semiannual coupon payment.

Illustration 5–1. Compute the price of a 9% coupon bond with 20 years to maturity and a par value of $1,000 if the required yield is 12%.

The cash flows for this bond are as follows:

1. 40 semiannual coupon payments of $45;
2. $1,000 forty 6-month periods from now.

The semiannual or periodic interest rate is 6%.

The present value of the 40 semiannual coupon payments of $45 discounted at 6% is $677.08, as shown below:

$c = \$45$; $n = 40$; $i = 0.06$.

$$\begin{aligned}\$45&\left[\frac{1-\left[\frac{1}{(1.06)^{40}}\right]}{0.06}\right]\\ &= \$45\left[\frac{1-\left[\frac{1}{10.28572}\right]}{0.06}\right]\\ &= \$45(15.04630) = \$677.08.\end{aligned}$$

The present value of the par or maturity value forty *6-month periods from now* discounted at 6% is $97.22, as shown below:

$M = \$1{,}000$; $n = 40$; $i = 0.06$.

$$\begin{aligned}\$1{,}000&\left[\frac{1}{(1.06)^{40}}\right]\\ &= \$1{,}000\left[\frac{1}{10.28572}\right]\\ &= \$1{,}000(0.097222) = \$97.22.\end{aligned}$$

The price of the bond is then equal to the sum of the two present values, $744.30 ($677.08 + $97.22).

Illustration 5–2. Compute the price of the bond in Illustration 5–1 assuming that the required yield is 7%.

The cash flows are unchanged, but the periodic interest rate is now 3.5% (7%/2).

The present value of the 40 semiannual coupon payments of \$45 discounted at 3.5% is \$960.98, as shown below:

$c = \$45;\ n = 40;\ i = 0.035.$

$$\$45\left[\frac{1-\left[\frac{1}{(1.035)^{40}}\right]}{0.035}\right]$$

$$= \$45\left[\frac{1-\left[\frac{1}{3.95926}\right]}{0.035}\right]$$

$$= \$45(21.35509) = \$960.98.$$

The present value of the par or maturity value of \$1,000 forty *6-month periods from now* discounted at 3.5% is \$252.57, as shown below:

$M = \$1{,}000;\ n = 40;\ i = 0.035.$

$$\$1{,}000\left[\frac{1}{(1.035)^{40}}\right]$$

$$= \$1{,}000\left[\frac{1}{3.95926}\right]$$

$$= \$1{,}000(0.252572) = \$252.57.$$

The price of the bond is then equal to the sum of the two present values, \$1,213.55 (\$960.98 + \$252.57).

Illustration 5–3. Compute the price of the bond in Illustration 5–1 assuming that there are 16 years to maturity rather than 20 years. (Assume that the required yield is still 12%.)

The cash flows for this bond are as follows:

1. 32 semiannual coupon payments of \$45;

2. \$1,000 thirty-two 6-month periods from now.

The semiannual or periodic interest rate is 6%.

The present value of the 32 semiannual coupon payments of \$45 discounted at 6% is

$c = \$45;\ n = 32;\ i = 0.06.$

$$\$45\left[\frac{1-\left[\frac{1}{(1.06)^{32}}\right]}{0.06}\right]$$

$$= \$45\left[\frac{1-\left[\frac{1}{6.45339}\right]}{0.06}\right]$$

$$= \$45(14.08404) = \$633.78.$$

The present value of the par or maturity value thirty-two *6-month periods from now* discounted at 6% is

$M = \$1{,}000;\ n = 32;\ i = 0.06.$

$$\begin{aligned}\$1{,}000\left[\frac{1}{(1.06)^{32}}\right] \\ = \$1{,}000\left[\frac{1}{6.45339}\right] \\ = \$1{,}000(0.154957) = \$154.96.\end{aligned}$$

The price of the bond is then equal to the sum of the two present values, \$788.84 (\$633.78 + \$154.96).

RELATIONSHIP BETWEEN REQUIRED YIELD AND PRICE AT A GIVEN TIME

The price of a bond changes in the direction opposite from the change in the required yield. The reason is that the price of the bond is the present value of the cash flows. As the required yield increases, the present value of the cash flows decreases; hence, the price decreases. The opposite is true when the required yield decreases: the present value of the cash flows increases and, therefore, the price of the bond increases.

We can see this by comparing the price of the 20-year, 9% coupon bond that we priced in Illustrations 5–1 and 5–2. When the required yield is 12%, the price of the bond is \$774.30. If, instead, the required yield is 7%, the price of the bond is \$1,213.55. Exhibit 5–1 shows the price of the bond for required yields from 5% to 14% for the 20-year, 9% coupon bond.

If we graph the price/yield relationship for any option-free bond, we will find it has the "bowed" shape shown in Exhibit 5–2. This shape is referred to as *convex*. The convexity of the price/yield relationship has important implications for the investment properties of a bond. In Chapter 13 we examine this relationship more closely.

RELATIONSHIPS AMONG COUPON RATE, REQUIRED YIELD, AND PRICE

For a bond issue, the coupon rate and the term to maturity are fixed. Consequently, as yields in the marketplace change, the only variable that can change to compensate for the new yield required in the market is the price of the bond. As we saw in the previous section, as the required yield increases (decreases), the price of the bond decreases (increases).

Generally, a bond's coupon rate at the time of issuance is set at approximately the prevailing yield in the market.[2] The price of the bond will then be approximately

2. The exception is an original-issue deep-discount bond such as a zero-coupon bond. We'll discuss zero-coupon bonds later in this chapter.

EXHIBIT 5–1

Price/Yield Relationship for a 20-Year, 9% Coupon Bond

Required Yield (%)	Present Value of 40 Coupon Payments ($)*	Present Value of Par Value in 40 Periods ($)**	Price of Bond ($)
5	1,129.62	372.43	1,502.05
6	1,040.16	306.56	1,346.72
7	960.98	252.57	1,213.55
8	890.67	208.29	1,098.96
9	828.07	171.93	1,000.00
10	772.16	142.05	914.21
11	722.08	117.46	839.54
12	677.08	97.22	774.30
13	636.55	80.54	717.09
14	599.93	66.78	666.71

*Computed as follows:

$$\$45\left[\frac{1-\left[\frac{1}{(1+i)^{40}}\right]}{i}\right],$$

where *i* is one-half the required yield.

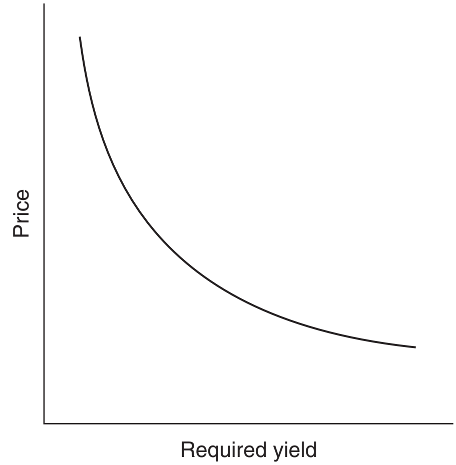
**Computed as follows:

$$\$1{,}000\left[\frac{1}{(1+i)^{40}}\right],$$

where *i* is one-half the required yield.

EXHIBIT 5–2

Price/Yield Relationship for an Option-Free Bond



equal to its par value. For example, in Exhibit 5–1, we see that when the required yield is equal to the coupon rate, the price of the bond is its par value ($1,000). Consequently, we have the following properties:

When the coupon rate equals the required yield, then the price equals the par value.

When the price equals the par value, then the coupon rate equals the required yield.

When yields in the marketplace rise above the coupon rate at a *particular time*, the price of the bond has to adjust so that the investor can realize additional interest income. This adjustment happens when the bond's price falls below the par value. The difference between the par value and the price is a capital gain and represents a form of interest income to the investor to compensate for the coupon rate being lower than the required yield.

When a bond sells below its par value, it is said to be selling at a *discount*. We can see this in Exhibit 5–1. When the required yield is greater than the coupon rate of 9%, the price of the bond is always less than the par value ($1,000). Consequently, we have the following properties:

When the coupon rate is less than the required yield, then the price is less than the par value.

When the price is less than the par value, then the coupon rate is less than the required yield.

Finally, when the required yield in the market is below the coupon rate, the bond must sell above its par value. This occurs because investors who would have the opportunity to purchase the bond at par would be getting a coupon rate in excess of what the market requires. Because its yield is attractive, investors would bid up the price of the bond to a price that offers the required yield in the market.

A bond whose price is above its par value is said to be selling at a *premium*. Exhibit 5–1 shows that for a required yield less than the coupon rate of 9%, the price of the bond is higher than its par value. Consequently, we have the following properties:

When the coupon rate is higher than the required yield, then the price is higher than the par value.

When the price is higher than the par value, then the coupon rate is higher than the required yield.

TIME PATH OF A BOND

If the required yield is unchanged between the time a bond is purchased and the maturity date, what will happen to the price of the bond? For a bond selling at par value, the coupon rate is equal to the required yield. As the bond moves closer to maturity, the bond will continue to sell at par value. Thus the price of a bond selling at par will remain at par as the bond moves toward the maturity date.

The price of a bond will *not* remain constant for a bond selling at a premium or a discount. This can be seen for a discount bond by comparing the price found in Illustration 5–1 to that found in Illustration 5–3. In both illustrations the bond has a 9% coupon rate, and the required yield is 12%. In Illustration 5–1 the maturity of the bond is 20 years, while in Illustration 5–3 the maturity is 16 years. With 20 years to maturity, the price of the bond is $774.30. Four years later, when the bond has 16 years remaining to maturity, the price of the bond increases to $788.74. For all discount bonds, the following is true: as a bond moves toward maturity, its price will increase if *the required yield* does not change.

Exhibit 5–3 shows the price of the 20-year, 9% coupon bond as it moves toward maturity, assuming that the required yield remains at 12%. The price of the bond is decomposed into the present value of the coupon payments and the present value of the par value. Notice that as the bond moves toward maturity, there are fewer coupon payments to be received by the bondholder. The present value of the coupon payments decreases. Since the maturity date is closer, however, the present value of the par value increases. The increase in the present value of the par value is greater than the decline in the present value of the coupon payments, resulting in a price increase.

For a bond selling at a premium, the price of the bond declines as it moves toward maturity. When the bond has 20 years to maturity, its price is $1,213.55. Six years later, when the bond has 14 years remaining to maturity, the price of the bond declines to $1,176.67.

EXHIBIT 5–3

Time Path of the Price of a Discount Bond: 20-Year, 9% Coupon, 12% Required Yield

Years Remaining to Maturity	Present Value of Coupon Payments of $45 at 6% ($)	Present Value of Par Value at 6% ($)	Price of Bond ($)
20	677.08	97.22	774.30
18	657.94	122.74	780.68
16	633.78	154.96	788.74
14	603.28	195.63	798.91
12	564.77	256.98	811.75
10	516.15	311.80	827.95
8	454.77	393.65	848.42
6	377.27	496.97	874.24
4	279.44	627.41	906.85
2	155.93	792.09	948.02
1	82.50	890.00	972.50
0	0.00	1,000.00	1,000.00

EXHIBIT 5–4

Time Path of the Price of a Premium Bond: 20-Year, 9% Coupon, 7% Required Yield

Years Remaining to Maturity	Present Value of Coupon Payments of $45 at 3.5% ($)	Present Value of Par Value at 3.5% ($)	Price of Bond ($)
20	960.98	252.57	1,213.55
18	913.07	289.83	1,202.90
16	858.10	332.59	1,190.69
14	795.02	381.65	1,176.67
12	722.63	437.96	1,160.59
10	639.56	502.57	1,142.13
8	544.24	576.71	1,120.95
6	434.85	661.78	1,096.63
4	309.33	759.41	1,068.74
2	165.29	871.44	1,036.73
1	85.49	933.51	1,019.00
0	0.00	1,000.00	1,000.00

The time path of the 20-year, 9% coupon bond selling to yield 7% is shown in Exhibit 5–4. As the bond moves toward maturity, the present value of the coupon payments decreases, and the present value of the par value increases. Unlike a bond selling at a discount, the increase in the present value of the par value is not sufficient to offset the decline in the present value of the coupon payments. As a result, the price of a bond selling at a premium decreases over time if the required yield does not change.

ANALYSIS OF BOND PRICE CHANGES

A money manager is interested in assessing the expected performance of a bond over an investment horizon given certain assumptions about the future direction of interest rates. We'll demonstrate how this is done in Chapter 9. Doing so requires that we know how to analyze the way a bond's price will change under a specified set of assumptions.

The price of a bond can change for one or more of the following three reasons:

1. A change in the required yield due to changes in the credit quality of the issuer;
2. A change in the maturity of the bond as it moves toward maturity without any change in the required yield (i.e., the time path of the bond);
3. A change in the required yield due to a change in the yield on comparable bonds (i.e., a change in the yield required by the market).

Predicting the change in an issue's credit quality before that change is recognized by the market is one of the challenges of investment management. We will describe some of the tools used for credit risk analysis in later chapters. For purposes of our illustrations below, let's suppose that the issue's credit quality is unchanged so that we can focus on the last two reasons.

It is informative to separate the effect of the change in price due to the time path of a bond from that of the change due to a change in the required yield. The next two illustrations show how this is done.

Illustration 5–4. Suppose that a money manager purchases a 20-year, 9% coupon bond at a price of $774.30 to yield 12%. The money manager expects to hold this bond for 4 years, at which time the money manager believes that the required yield on comparable 16-year bonds will be 8%. On the basis of these expectations, we can investigate what will happen to the price of the bond 4 years from now.

After this bond is held for 4 years, it becomes a 16-year bond. If the required yield for a 16-year bond is 8%, the price of the bond 4 years from now will be $1,089.37. The price of this bond is therefore expected to increase by $315.07 ($1,089.37 – $744.30). Not all of the price change, however, is due to the decline in market yield. If the required yield remains at 12%, the price of the bond in 4 years will have increased to $788.74, an increase of $14.44 ($788.74 – $744.30). Therefore, we can decompose the expected price change after 4 years as follows:

Price change due to the time path of a discount bond	$ 14.44
Price change due to the change in the required yield	300.63
Total price change	$315.07

Illustration 5–5. Suppose that a money manager is considering the purchase of a 20-year, 9% coupon bond selling at $1,213.55 to yield 7%. If the money manager purchases this bond, she expects to hold it for 6 years, at which time she expects that the required yield on 14-year bonds may be 11%. What would be the price performance of this bond based on the money manager's expectations?

The price of a 14-year, 9% coupon bond if an 11% required yield is assumed is $858.79. If the required yield remained at 7%, however, the price of a 14-year bond with a coupon rate of 9% would be $1,176.67 (see Exhibit 5–4). The price change can be broken down as follows:

Price change due to the time path of a discount bond	–$ 36.88
Price change due to the change in the required yield	–317.88
Total price change	–$354.76

THE PRICE OF A ZERO-COUPON BOND

So far we have determined the price of coupon-bearing bonds. There are bonds that do not make any periodic coupon payments. Instead, the investor realizes interest by the amount of the difference between the maturity value and the purchase price.

The pricing of a zero-coupon bond is no different from the pricing of a coupon bond: its price is the present value of the expected cash flows. In the case of a zero-coupon bond, the only cash flow is the maturity value. Therefore, the formula for the price of a zero-coupon bond that matures N years from now is

$$p = M\left[\frac{1}{(1+i)^n}\right],$$

where

p = price;
M = maturity value;
i = periodic interest rate (annual interest rate/2);
$n = 2 \times N$, where
N = number of years.

Pay particular attention to the number of periods used in the pricing of a zero-coupon bond. Although an issue may mature in N years, the number of 6-month periods is used in the exponent, and the periodic interest rate is the required yield divided by 2. The reason is that the pricing of a zero-coupon bond must be made consistent with the pricing of a coupon bond. Recall that with a coupon bond the present value of the maturity value is computed using twice the number of years to maturity. Therefore, we handle the maturity value for the zero-coupon bond the same way.

Illustration 5–6. Compute the price of a zero-coupon bond that matures 10 years from now if the maturity value is \$1,000 and the required yield is 8.6%.

The price is determined as follows:

$M = \$1{,}000$; $i = 0.043\,(0.086/2)$; $N = 10$; $n = 20\,(2 \times 10)$.

$$\begin{aligned} p &= \$1{,}000\left[\frac{1}{(1.043)^{20}}\right] \\ &= \$1{,}000\left[\frac{1}{2.321059}\right] \\ &= \$1{,}000(0.43083) = \$430.83. \end{aligned}$$

PRICE QUOTATIONS

In all of our illustrations, we have assumed that the maturity or par value of the bond is \$1,000. A bond can have a maturity or par value of any amount. Consequently, traders quote bond prices as a percentage of par value.

A bond selling at par is quoted as 100, meaning 100% of its par value. A bond selling at a discount will be selling for less than 100; a bond selling at a premium will be selling for more than 100. The following examples illustrate how a price quote is converted into a dollar price.

(1) Price Quote	(2) Converted to a Decimal [(1)/100]	(3) Par Value $	(4) Dollar Price $ [(2) × (3)]
95	0.9500000	1,000	950.00
$95\frac{1}{2}$	0.9550000	100,000	95,500.00
$98\frac{1}{4}$	0.9825000	5,000	4,912.50
$80^{1}/_{8}$	0.8012500	10,000	8,012.50
$74^{1}/_{32}$	0.7403125	1,000,000	740,312.50
100	1.0000000	10,000	10,000.00
103	1.0300000	1,000	1,030.00
$106\frac{3}{4}$	1.0675000	500,000	533,750.00
$108^{3}/_{8}$	1.0837500	25,000	27,093.75
$111^{11}/_{32}$	1.1134375	100,000	111,343.75

Treasury notes and bonds are quoted in the secondary market on a price basis in points, where one point equals 1% of par.[3] The points are split into units of thirty-seconds, so a price of 98-14, for example, refers to a price of 98 and $^{14}/_{32}$, or 98.4375. The thirty-seconds are themselves split by the addition of a plus sign or a number, with a plus sign indicating that half a thirty-second (or $^{1}/_{64}$) is added to the price and a number indicating how many eighths of thirty-seconds (or 256ths) is added to the price. A price of 98-14+ therefore refers to a price of 98 and $14\frac{1}{2}$ thirty-seconds, or 98.453125, whereas a price of 98-142 refers to a price of 98 and $14^{2}/_{8}$ thirty-seconds, or 98.4453125.

DETERMINING THE PRICE WHEN THE SETTLEMENT DATE FALLS BETWEEN COUPON PERIODS

Dates for Computations

In computations in the bond market, there are several dates that have specific meaning. These include the trade date, settlement date, issue date, dated date, coupon dates, and the next coupon date.

The date on which a transaction is initiated is the *trade date*. It is also referred to as the *transaction date*. The date when the transaction is cleared by the delivery of the securities by the seller to the buyer and the payment of funds from the buyer to the seller is the *settlement date*. The earliest date on which settlement can occur

3. Treasury coupon securities are quoted in yield terms in when-issued trading because coupon rates for new Treasury securities are not set until after these securities are auctioned.

is the *issue date*. A more important date for calculation of interest is the *dated date*, which is the date when interest begins to accrue. This date, also referred to as the *interest accrue date*, can be either before, on, or after the issue date.

With the exception of zero-coupon bonds, there are dates when the issuer agrees to make coupon payments. These are called the *coupon dates*. For a bond purchased in the secondary market, the *next coupon date* is the coupon date that is just after the settlement date.

Our illustrations have assumed that the next coupon payment is 6 months away. This means that settlement occurs the day after a coupon date. Typically, an investor purchases a bond between coupon dates, so the next coupon payment is less than 6 months away. To compute the price, we have to answer three questions:

1. How many days are there until the next coupon payment?
2. How should we determine the present value of cash flows received over fractional periods?
3. How much must the buyer compensate the seller for the coupon interest earned by the seller for the fraction of the period that the bond was held?

The first question is the "day count" question. The second is the "compounding" question. The last question asks how accrued interest is determined.

Day Count Conventions

Market conventions for the number of days in a coupon period and the number of days in a year differ by type of bond issuer (i.e., government, government-related entity, local government, or corporate) and by country. The following notation is typically used to denote a day count convention:

Number of days in a month/number of days in a year

and "NL" is used to denote no leap year and "E" to denote European.

In practice, there are eight day count conventions: actual/actual (in period); actual/365; actual (NL)/365; actual/365 (366 in leap year); actual/360; 30/360; 30/365; 30E/360.[4]

In the calculation of the actual number of days, only one of the two bracketing dates in question is included. For example, the actual number of days between August 20 and August 24 is 4 days. Actual (NL)/365 is the same as actual/365 with the exception that February 29 is not counted in the former method.

In the day count conventions where "30" is used for the number of days (i.e., the last three methods), there are rules for computing the number of days in between two days in assuming a 30-day month.

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4. Dragomir Krgin, *Handbook of Global Fixed Income Calculation* (Hoboken, NJ: John Wiley & Sons, 2002), p. 21. This book is the most comprehensive source for formulas and rules for not only day count conventions but also for accrued interest, which we discuss next.

Day count conventions are used for calculating:

- Accrued interest to be paid;
- Accrued interest for price/yield;
- Next coupon payment;
- The exponent used to compute the present value of the cash flow from the next coupon date back to the settlement date;
- The coupon payment on the maturity date;
- The exponent used to compute the present value of the cash flow at the maturity date back to the last coupon date.

Below we describe the day count conventions for U.S. Treasury, corporate, and municipal bonds.

Application to U.S. Treasury Coupon Securities

In the U.S. Treasury coupon securities market, the day count convention used is to determine the actual number of days between two dates and the actual number of days in a year. This is referred to as the *actual/actual* day count convention. For example, consider a Treasury bond whose previous coupon payment was on March 1 and whose next coupon payment is on September 1. Suppose this bond is purchased with a settlement date of July 17. The actual number of days between July 17 (the settlement date) and September 1 (the date of the next coupon payment) is 46 days, as shown below:

July 17 to July 31	14 days
August	31 days
September 1	1 day
	46 days

The number of days in the coupon period is the actual number of days between March 1 and September 1, which is 184 days.

Application to Federal Agency, Corporate, and Municipal Securities

In contrast to the actual/actual day count convention for coupon-bearing Treasury securities, for corporate and municipal bonds and agency securities, the day count convention is 30/360. That is, each month is assumed to have 30 days and each year 360 days. For example, suppose that the security in our previous example is not a coupon-bearing Treasury security but instead either a coupon-bearing corporate bond, municipal bond, or agency security. The number of days between July 17 and September 1 is as shown below:

July 17 to July 31	13 days
August	30 days
September 1	1 day
	44 days

Basic financial calendars provide the day count between settlement and the next coupon payment. Most money managers, however, use software programs that will furnish this information.

Compounding

Once the number of days between the settlement date and the next coupon date is determined, the present value formula must be modified to take into account that the cash flows will not be received 6 months (one full period) from now. The "street" convention is to compute the price as follows:

1. Determine the number of days in the coupon period.
2. Compute the ratio

$$w = \frac{\text{number of days between settlement and next coupon payment}}{\text{number of days in the coupon period}}.$$

   For a corporate bond, municipal bond, or agency security, the number of days in the coupon period will be 180, because a year is assumed to have 360 days. For a coupon-bearing Treasury security, the number of days is the actual number of days. The number of days in the coupon period is called the *basis*.

3. For a bond with n coupon payments remaining to maturity, the price is

$$p = \frac{c}{(1+i)^{w}} + \frac{c}{(1+i)^{1+w}} + \frac{c}{(1+i)^{2+w}} + \cdots + \frac{c}{(1+i)^{n-1+w}} + \frac{M}{(1+i)^{n-1+w}},$$

   where

   p = price ($);
   c = *semiannual* coupon payment ($);
   i = periodic interest rate (required yield divided by 2) (in decimal form);
   n = number of coupon payments remaining;
   M = maturity value.

The period (exponent) in the formula for determining the present value can be expressed generally as $t - 1 + w$. For example, for the first cash flow, the period is $1 - 1 + w$, or simply w. For the second cash flow it is $2 - 1 + w$, or simply $1 + w$. If the bond has 20 coupon payments remaining, the period is $20 - 1 + w$, or simply $19 + w$.

Illustration 5–7. Suppose that a corporate bond with a coupon rate of 10% maturing March 1, 2028, is purchased with a settlement date of July 17, 2022. What would the price of this bond be if it is priced to yield 6.5%?

The next coupon payment will be made on September 1, 2022. Because the bond is a corporate bond, the 30/360 day count convention is that there are 44 days between the settlement date and the next coupon date. The number of days in the coupon period is 180 days. Therefore,

$$w = \frac{44}{180} = 0.24444.$$

The number of coupon payments remaining n is 12. The semiannual interest rate is 3.25% (6.5%/2).

The price of this corporate bond is \$120.0281. The price calculated in this way is called the *full price* or *dirty price* because it reflects the portion of the coupon interest that the buyer will receive but that the seller has earned.

Accrued Interest

The buyer must compensate the seller for the portion of the next coupon interest payment the seller has earned but will not receive from the issuer because the issuer will send the next coupon payment to the buyer. This amount is called *accrued interest*. Interest accrues on a bond from and including the date of the previous coupon up to but *excluding* a date called the *value date*. The value date is usually, but not always, the same as the settlement date. Unlike the settlement date, the value date is not constrained to fall on a business day.[5]

Calculation of accrued interest assumes that the coupon payment takes place on the scheduled date, even if in practice it will be delayed because the scheduled date is a nonbusiness day. In the formulas below we use the settlement date rather than the value date.

The accrued interest is calculated as follows:

$$AI = c\left[\frac{\text{number of days from last coupon payment to settlement date}}{\text{number of days in coupon period}}\right],$$

where

AI = accrued interest (\$);
c = semiannual coupon payment (\$).

Accrued interest is not computed for all bonds. No accrued interest is computed for bonds in default and income bonds. A bond that trades without accrued interest is said to be traded *flat*.

5. The term *value date* is not used consistently across markets—the definition in the text is that used by the Association of International Bond Dealers (AIBD). In some markets the interest accrues up to and *including* a date that is called the value date. The *dated date* is the date from which accrued interest is calculated on a newly issued security.

Illustration 5–8. Let's continue with the hypothetical corporate bond in Illustration 5–7. Because the number of days between settlement (July 17, 2022) and the next coupon payment (September 1, 2022) is 44 days and the number of days in the coupon period is 180, the number of days from the last coupon payment date (March 1, 2022) to the settlement date is 136 (180 – 44). The accrued interest per $100 of par value is

$$AI - \$5\left(\frac{136}{180}\right) = \$3.777778.$$

Illustration 5–9. If the bond in the previous illustration were a Treasury bond rather than a corporate bond, the accrued interest would be computed as follows. The number of days in the coupon period is based on the actual number of days. Between March 1 and September 1 the actual number of days is 184. The actual number of days between March 1 and July 17 is 138. Accrued interest per $100 of par value is then

$$AI = \$5\left(\frac{138}{184}\right) = \$3.75.$$

Cum-Dividend and Ex-Dividend Trading

When the buyer receives the next coupon, the bond is said to be traded *cum-dividend* (or *cum-coupon*), and the buyer pays the seller accrued interest. If the buyer forgoes the next coupon, the bond is said to be traded *ex-dividend* (or *ex-coupon*), and the seller pays the buyer accrued interest. In some markets (the United States is one) bonds are always traded cum-dividend. In other markets, bonds are traded ex-dividend for a certain period before the coupon date.

PRICE BUYER PAYS AND PRICE QUOTES

The full or dirty price includes the accrued interest that the seller is entitled to receive. The *clean price* or *flat price* is the dirty price of the bond minus the accrued interest; that is,

$$\text{Clean price} = \text{dirty price} - \text{accrued interest}.$$

The price that the buyer pays the seller is the dirty price. It is important to note that in calculation of the dirty price, the next coupon payment is a discounted value, but in calculation of accrued interest it is an undiscounted value. Because of this market practice, if a bond is selling at par and the settlement date is not a coupon date, the yield will be slightly less than the coupon rate. Only when the settlement date and coupon date coincide is the yield equal to the coupon rate for a bond selling at par.

In the U.S. market, the convention is to quote a bond's clean or flat price. The buyer, however, pays the seller the dirty price. In some non-U.S. markets, the dirty price is quoted.

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CHAPTER

6

BOND YIELD MEASURES

In the previous chapter we explained how to compute the price of a bond given the required yield. In this chapter we show how to compute various yield measures for a bond given its price. We focus largely on four conventional yield measures commonly quoted by dealers and traders: (1) current yield, (2) yield to maturity, (3) yield to call, and (4) yield to put. While our focus in Part Two of this book is on option-free bonds, it is convenient to introduce yield measures for callable and putable bonds here. In Chapter 8 we evaluate these yield measures; in Chapter 9 we suggest a yield measure that is more useful for determining the potential return by investing in a bond.

CURRENT YIELD

The current yield relates the *annual* coupon interest to the market price. The formula for the current yield is

$$\text{Current yield} = \frac{\text{annual dollar coupon interest}}{\text{price}}.$$

Illustration 6–1. The current yield for an 18-year, 6% coupon bond selling for \$700.89 is 8.56%, as shown below:

$$\text{Annual dollar coupon interest} = \$1{,}000 \times 0.06 = \$60$$

$$\text{Current yield} = \frac{\$60}{\$700.89} = 0.0856 \text{ or } 8.56\%.$$

Illustration 6–2. The current yield for a 19-year, 11% coupon bond selling for \$1,233.64 is 8.92%, as shown below:

$$\text{Annual dollar coupon interest} = \$1{,}000 \times 0.11 = \$110$$

$$\text{Current yield} = \frac{\$110}{\$1{,}233.64} = 0.0892 \text{ or } 8.92\%.$$

The current yield considers only the coupon interest and no other source of return that will affect an investor's yield. For example, in Illustration 6–1, no consideration is given to the capital gain that the investor will realize when the bond matures; in Illustration 6–2, no recognition is given to the capital loss that the investor will realize when the bond matures.

YIELD TO MATURITY

In Chapter 4 we explained how to compute the yield or internal rate of return on any investment. The yield is the interest rate that will make the present value of the cash flows equal to the price (or initial investment). The yield to maturity is computed in the same way as the yield; the cash flows are those that the investor would realize by holding the bond to maturity. For a semiannual-pay bond whose next coupon payment will be received 6 months from now, the yield to maturity is computed by solving the following relationship for y:

$$p = \frac{c}{(1+y)^1} + \frac{c}{(1+y)^2} + \frac{c}{(1+y)^3} + \cdots + \frac{c}{(1+y)^n} + \frac{M}{(1+y)^n},$$

where

p = price ($);
c = *semiannual* coupon interest ($);
y = one-half the yield to maturity;
n = number of periods (number of years $\times$ 2);
M = maturity value ($).

For a semiannual-pay bond, doubling the interest rate or discount rate (y) gives the yield to maturity.

Using the Greek letter sigma to denote summation, we can express this relationship as follows:

$$p = \sum_{t=1}^{n} \frac{c}{(1+y)^t} + \frac{M}{(1+y)^n}.$$

The yield to maturity considers not only the current coupon income but any capital gain or loss that the investor will realize by *holding the bond to maturity*. The yield to maturity also considers the timing of the cash flows.

Recall from Chapter 4 that computation of the yield requires an iterative procedure. The next two illustrations show how to compute the yield to maturity for a bond.

Illustration 6–3. In Illustration 6–1 we computed the current yield for an 18-year, 6% coupon bond selling for $700.89. The maturity or par value for this bond is $1,000. The yield to maturity for this bond is 9.5%, as shown below:

Cash flows for the bond are

1. 36 coupon payments of $30 every 6 months;

2. $1,000 thirty-six 6-month periods from now.

To get y, different interest rates must be tried until one is found that makes the present value of the cash flows equal to the price of $700.89. Because the coupon rate on the bond is 6% and the bond is selling at a discount, the yield must be greater than 6%. Exhibit 6–1 shows the present value of the cash flows of the bond for periodic interest rates from 3.25% to 4.75% (corresponding to annual interest rates from 6.50% to 9.50%, respectively). As can be seen from this exhibit, at an interest rate of 4.75%, the present value of the cash flows is $700.89. Therefore, y is 4.75%, and the yield to maturity is 9.50%.

Illustration 6–4. In Illustration 6–2 we computed the current yield for a 19-year, 11% coupon bond selling for $1,233.64. The maturity or par value for this bond is $1,000. The yield to maturity for this bond is 8.50%, as shown below:

Cash flows for the bond are

1. 38 coupon payments of $55 every 6 months;
2. $1,000 thirty-eight 6-month periods from now.

We are looking for the interest rate y that will make the present value of the cash flows equal to $1,233.64 (the price of the bond). Because the bond is selling at a premium and the coupon rate is 11%, the yield must be less than 11%. The present value of the cash flows of the bond for periodic interest rates from 3.00% to

EXHIBIT 6–1

Computation of Yield to Maturity for an 18-Year, 6% Coupon Bond Selling at $700.89

Objective: Find the semiannual interest rate that will make the present value of the following cash flows equal to $700.89:

36 coupon payments of $30 every 6 months;
$1,000 thirty-six 6-month periods from now.

Annual Interest Rate (%)	Semiannual Rate (%)	Present Value of 36 Payments of $30 ($)*	Present Value of $1,000 Thirty-Six Periods from Now ($)**	Present Value of Cash Flows ($)
6.50	3.25	631.20	316.20	947.40
7.00	3.50	608.71	289.83	898.54
7.50	3.75	587.42	265.72	583.14
8.00	4.00	567.25	243.67	810.92
8.50	4.25	548.12	223.49	771.61
9.00	4.50	529.98	205.03	735.01
9.50	4.75	512.76	188.13	700.89

$$*\$30\left[\frac{1-\left[\dfrac{1}{(1+\text{semiannual interest rate})^{36}}\right]}{\text{semiannual interest rate}}\right]$$

$$**\$1{,}000\left[\frac{1}{(1+\text{semiannual interest rate})^{36}}\right]$$

4.25% (corresponding to annual interest rates from 6.0% to 8.50%, respectively) can be easily determined. The present value of the cash flows is equal to the price when a 4.25% interest rate is used. The value for y is therefore 4.25%. Doubling 4.25% produces the yield to maturity of 8.50%.

Yield to Maturity for a Zero-Coupon Bond

In Chapter 4 we explained that when there is only one cash flow, it is much easier to compute the yield on an investment. A zero-coupon bond is characterized by a single cash flow resulting from the investment. Consequently, the following formula, presented in Chapter 4 (Illustration 4–4), can be applied to compute the yield to maturity for a zero-coupon bond:

$$y = (\text{future value per dollar invested})^{1/n} - 1,$$

where

y = one-half the yield to maturity;

$$\text{Future value per dollar invested} = \frac{\text{maturity value}}{\text{price}}$$

Once again, doubling y gives the yield to maturity. *Remember that the number of periods used in the formula is double the number of years.*

Illustration 6–5. The yield to maturity for a zero-coupon bond selling for \$274.78 with a maturity value of \$1,000, maturing in 15 years, is 8.8%, as computed below:

$$\text{Future value per dollar invested} = \frac{\$1{,}000.00}{\$274.78} = 3.639275;$$

$n = 30(15 \times 2);$

$$\begin{aligned} y &= (3.639275)^{1/30} - 1 \\ &= (3.639275)^{0.033333} - 1 \\ &= 1.044 - 1 = 0.044 \text{ or } 4.4\%. \end{aligned}$$

Doubling 4.4% gives the yield to maturity of 8.8%.

Computing the Yield to Maturity When the Settlement Date Falls Between Coupon Payments

In the previous chapter we explained how to determine the price that the buyer pays the seller when a bond is purchased between coupon payments. The amount paid is the dirty or full price. The yield to maturity for a bond when settlement falls between coupon dates is the interest rate that will make the present value of the cash flows equal to the dirty price. That is, for a semiannual-pay bond with n coupon payments remaining, we must solve the following relationship for y:

$$tp = \frac{c}{(1+y)^{w}} + \frac{c}{(1+y)^{1+w}} + \frac{c}{(1+y)^{2+w}} + \cdots + \frac{c}{(1+y)^{n-1+w}} + \frac{M}{(1+y)^{n-1+w}},$$

where

tp = dirty price ($);
c = *semiannual* coupon payment ($);
y = one-half the yield to maturity;

$$w = \frac{\text{number of days between settlement and the next coupon payment}}{\text{number of days in the coupon period}};$$

n = number of coupon payments remaining;
M = maturity value.

Doubling y gives the yield to maturity.

Using the Greek letter sigma to denote summation, the formula can be expressed as follows:

$$tp = \sum_{t=1}^{n} \frac{c}{(1+y)^{t-1+w}} + \frac{M}{(1+y)^{n-1+w}}.$$

Illustration 6–6. Suppose that a 10% coupon corporate bond maturing on March 1, 2028, has a dirty price of $118.778. The settlement date is July 17, 2022. The cash flows for this bond per $100 of par value and the corresponding periods they will be received are

Period	Cash Flow
0.24444 through 10.2444	5.00
11.24444	105.00

The semiannual interest rate that will make the present value of the cash flows equal to the dirty price of $118.778 is 3.3735%. Doubling the semiannual interest rate gives a yield to maturity of 6.747%.

Illustration 6–7. If the bond in Illustration 6–6 were a Treasury bond rather than a corporate bond, the cash flows for the bond would be as follows:

Number of days from settlement to next coupon payment = 46;

Number of days in coupon period = 184;

$$w = \frac{46}{184} = 0.25.$$

The cash flows for this bond per $100 of par value and the corresponding periods they will be received are

Period	Cash Flow $
0.25 through 10.25	5.00
11.25	105.00

Suppose that the dirty price is $118.75. The interest rate that will make the present value of the cash flows equal to $118.75 is 3.374%. The yield to maturity is then 6.748%.

Relationships Among Coupon Rate, Current Yield, and Yield to Maturity

These relationships pertain among coupon rate, current yield, and yield to maturity:

Bond Selling at	Relationship
Par	Coupon rate = current yield = yield to maturity
Discount	Coupon rate < current yield < yield to maturity
Premium	Coupon rate > current yield > yield to maturity

The relationships for discount and premium bonds can be verified from the illustrations presented earlier in this chapter.

Problem with the Annualizing Procedure

As we pointed out in Chapter 4, multiplying a semiannual interest rate by 2 will produce an underestimate of the effective annual yield. The proper way to annualize the semiannual yield is by applying the following formula presented in Chapter 4 (see the "Annualizing Yields" section in Chapter 4):

$$\text{Effective annual yield} = (1 + \text{periodic interest rate})^m - 1,$$

where

m = number of payments per year.

For a semiannual-pay bond, the formula can be modified as follows:

$$\text{Effective annual yield} = (1 + \text{semiannual interest rate})^2 - 1$$

or

$$\text{Effective annual yield} = (1 + y)^2 - 1.$$

For example, in Illustration 6–3, the semiannual interest rate is 4.75%, and the effective annual yield is 9.73%, as shown below:

$$\text{Effective annual yield} = (1.0475)^2 - 1 = 0.0973 \text{ or } 9.73\%$$

Although the proper way to annualize a semiannual interest rate is given in the formula above, the *convention* adopted in the bond market is to double the semiannual interest rate. The yield to maturity computed in this manner—doubling the semiannual yield—is called a *bond equivalent yield*. In fact, this convention is carried over to yield calculations for other types of fixed-income securities.[1]

Yield on an Annual-Pay Bond

While the practice in the United States is to pay interest semiannually, in other bond markets interest is paid annually. The calculation of the yield to maturity for an annual-pay bond is the same as for a semiannual-pay bond: it is the interest rate that makes the present value of the cash flow equal to the dirty price.

Illustration 6–8. Suppose that an annual-pay bond with a coupon rate of 9.125% and 10 years to maturity is selling for \$993.33 per \$1,000 par value. The yield to maturity for this bond is 9.23%.

An adjustment is required to make a direct comparison between the yield to maturity on a U.S. bond issue and that on an annual-pay bond issue. Given the yield to maturity on an annual-pay bond issue, its yield to maturity on a bond-equivalent basis is computed as follows:

$$\text{Yield to maturity on a bond-equivalent basis}$$
$$= 2[(1 + \text{yield to maturity})^{1/2} - 1].$$

Illustration 6–9. For the annual-pay bond issue in the previous illustration whose yield to maturity is 9.23%, the yield to maturity on a bond-equivalent basis is

$$2[(1.0923)^{1/2} - 1] = 0.0903 = 9.03\%.$$

Notice that the yield to maturity on a bond-equivalent basis will always be less than the yield to maturity on an annual-pay bond.

Alternatively, to convert the yield to maturity on a bond-equivalent basis of a U.S. bond issue to an annual-pay basis so that it can be compared to the yield to maturity of an annual-pay bond issue, the following formula can be used:

$$\text{Yield to maturity on an annual-pay basis}$$
$$= \left(1 + \frac{\text{yield to maturity on a bond-equivalent basis}}{2}\right)^2 - 1.$$

1. For example, in Chapter 25 we discuss the yield calculation for mortgage pass-through securities. The periodic interest rate is computed on a monthly basis. To compute an annual yield, the practice is first to compute an effective semiannual yield as follows:

$$\text{Effective semiannual yield} = (1 + \text{monthly interest rate})^6 - 1.$$

Then the effective semiannual yield is doubled.

Illustration 6–10. Suppose that an investor wants to compare the yield to maturity of a 10-year, 9.125% coupon Eurobond and the yield to maturity of a 10-year, 9% coupon U.S. bond issue. A Eurobond issue is an annual-pay bond. Suppose that the price of the Eurobond is \$993.33 and the price of the U.S. bond issue is \$990.31. From Illustration 6–8, we know that the yield to maturity is 9.23%, and from Illustration 6–9, we know that the yield to maturity on a bond-equivalent basis is 9.03%. For the U.S. bond issue, the yield to maturity on a bond-equivalent basis is 9.15% if the price is \$990.31. To convert this yield to an annual-pay basis, the previous formula is used:

$$\left(1+\frac{0.0915}{2}\right)^2-1=0.0936=9.36\%.$$

Notice that the yield to maturity on an annual basis is always greater than the yield to maturity on a bond-equivalent basis.

Below we have a comparison for our two hypothetical bonds:

Issue	Coupon Rate (%)	Price (\$)	Yield on a Bond-Equivalent Basis (%)	Yield on an Annualized Basis (%)
Eurobond	9.125	993.33	9.03	9.23
U.S.	9.000	990.31	9.15	9.36

This summary shows that the yield on the U.S. bond issue is higher than the yield on the Eurobond issue whether calculated on a bond-equivalent basis or an annual basis. An improper comparison of the unadjusted yield—9.23% for the Eurobond issue and 9.15% for the U.S. bond issue—would give a different conclusion.

YIELD TO CALL

An issue may have a provision granting the issuer an option to buy back all or part of the issue prior to the stated maturity date. The right of the issuer to retire the issue prior to the stated maturity date is referred to as a *call option*, and the bond is said to be a *callable bond*. If an issuer exercises this right, the issuer is said to *call the bond*. The price which the issuer must pay to retire the issue is referred to as the *call price*. Typically, there is not one call price but a call schedule that sets forth a call price based on when the issuer can exercise the call option.

When a bond is issued, typically the issuer may not call the bond for a number of years. That is, the issue is said to have a *deferred call*. The date at which the bond may first be called is referred to as the *first call date*.

Generally, the call schedule is such that the call price at the first call date is a premium over the par value and scaled down to the par value over time. The date at which the issue is first callable at par value is referred to as the *first par call date*.

When a bond is callable, an investor can calculate a yield to an assumed call date. The *yield to call* is the interest rate that makes the present value of the cash flows to the assumed call date equal to the dirty price of the bond. Mathematically, the yield to an assumed call date for a bond on which the next coupon payment will be due 6 months from now can be expressed as follows:

$$p=\frac{c}{(1+y)^1}+\frac{c}{(1+y)^2}+\frac{c}{(1+y)^3}+\cdots+\frac{c}{(1+y)^{n^*}}+\frac{CP}{(1+y)^{n^*}},$$

where

p = price ($);
c = *semiannual* coupon interest ($);
y = one-half the yield to call (in decimal form);
n^* = number of periods until assumed call date (number of years × 2);
CP = call price at assumed call date from the call schedule.

For a semiannual-pay bond, doubling the interest rate (y) gives the yield to call.

Alternatively, the yield to call can be expressed as follows:

$$p=\sum_{t=1}^{n^*}\frac{c}{(1+y)^t}+\frac{CP}{(1+y)^{n^*}}.$$

Two commonly used call dates are the first call date and the first par call date. To calculate the yield to the first call date, the value for *CP* used in the formula is the call price on the first call date as given in the call schedule. In computing the yield to the first par call date, the par value is used for *CP* in the formula.

The next two illustrations show how to calculate the yield to the first call date.

Illustration 6–11. In Illustrations 6–1 and 6–3 we computed the current yield and yield to maturity for an 18-year, 6% coupon bond selling for $700.89. Suppose that this bond is first callable in 5 years at $1,030. The cash flows for this bond if it is called at the first call date are

1. 10 coupon payments of $30 every 6-months;
2. $1,030 ten 6-month periods from now.

The value for y that we seek is the one that will make the present value of the cash flows equal to $700.89. It can be shown that when y is 7.6%, the present value of the cash flows is $700.11, which is close enough to $700.89 for our purposes. Therefore, the yield to the first call date on a bond-equivalent basis is 15.2% (double the periodic interest rate of 7.6%).

Illustration 6–12. In Illustrations 6–2 and 6–4 we computed the current yield and yield to maturity for a 19-year, 11% coupon bond selling for $1,233.64. Suppose that this bond is first callable in 6 years at $1,055. If the bond is called on the first call date, the cash flows for this bond would be

1. 12 coupon payments of $55 every 6 months;
2. $1,055 twelve 6-month periods from now.

It can be shown that the interest rate that equates the present value of the cash flows to the price of $1,233.64 is approximately 3.55%. Therefore, the yield to the first call date on a bond-equivalent basis is 7.1%.

For the call provision we just described, the call price is fixed. It can be par or a premium over par based on the call date. There is a call provision where the call price is not fixed but instead the call price the bondholder receives is determined by the present value of the remaining payments discounted at a small spread over a maturity-matched Treasury yield. This type of call provision is called a *make-whole call provision* with the specified spread that is fixed over the bond's life called the *make-whole premium*. The calculation of the call price is done as follows. The issuer at the time of exercise of the call option must pay the bondholder the maturity value plus the lost coupon interest that the bondholder loses by having the bond called before maturity. However, the amount of the lost coupon interest is not simply the sum of the lost coupon interest payments but the present value of the lost coupon interest payments. For the maturity value, the present value also must be computed. The indenture specifies the discount rate that must be used to compute the present value of the maturity value which is a spread over a comparable Treasury security at the time the issuer exercises the call option. So, the issuer must pay to the bondholder the present value of the maturity value plus the present value of the lost coupon payments discounted at a spread over a comparable Treasury at the time the issue is called.

Illustration 6–13. Suppose that six years ago, a corporation issued a bond with a maturity of 15 years and a coupon rate of 5.8%. The issue has a make-whole provision where the indenture specifies a make-whole premium of 50 basis points over the yield on a comparable Treasury at the time the call provision is exercised by the issuer. The issuer calls the bond after nine years immediately after the payment of a coupon payment. There are then six years of coupon payments remaining and six years remaining to maturity. At the exercise date, suppose that the yield on a 6-year U.S. Treasury is 2.5%, and therefore the discount rate to calculate the present value of the remaining coupon payments is 3% (2.5% + 0.5%). Per $1,000 of par value, the issuer must pay the bondholder the following price by exercising the call option:

- The present value of $29 (one-half $58) of interest every six months for 12 six-month periods;
- The present value of $1,000 (the maturity value) 12 six-month periods from now.

The discount rate for calculating the present value each six months is 1.5% (one half of the 3% rate). It can be shown that the present value of the coupon payments is equal to $316.32. The present value of the maturity value is $836.39. The call price is therefore $1,152.71 ($316.32 + $836.39).

YIELD TO PUT

A call provision grants the issuer the right to change the maturity of the bond. An issue with a *put provision* grants the bondholder the right to sell the issue back to the issuer at par value on designated dates. Consequently, a bond with a put provision gives the bondholder the right to change the maturity of the bond. A bond with this provision is called a *putable bond*. There may be one put price or a schedule of put prices. Most putable bonds have just one put price.

For a putable bond, a yield to put can be calculated for any possible put date. The formula for the yield to put for any assumed put date is

$$p = \frac{c}{(1+y)^1} + \frac{c}{(1+y)^2} + \frac{c}{(1+y)^3} + \cdots + \frac{c}{(1+y)^{n^*}} + \frac{PP}{(1+y)^{n^*}},$$

where

p = price ($);
c = *semiannual* coupon interest ($);
y = one-half the yield to put (in decimal form);
n^* = number of periods until assumed put date (number of years × 2);
PP = put price at assumed put date.

Doubling y gives the yield to put on a bond-equivalent basis.

Alternatively, the yield to put can be expressed as follows:

$$p = \sum_{t=1}^{n^*} \frac{c}{(1+y)^t} + \frac{PP}{(1+y)^{n^*}}.$$

YIELD TO WORST

A bond issue may be callable and/or putable. Thus, in addition to the yield to maturity, a bond can have a yield to call for all possible call dates and a yield to put for all possible put dates. Some practitioners calculate the *yield to worst* for a bond. This is the smallest yield measure of all the possible yields that can be computed for the bond issue.

PORTFOLIO YIELD

There are two conventions that have been adopted by practitioners to calculate a portfolio yield: (1) weighted-average portfolio yield and (2) internal rate of return.

Weighted-Average Portfolio Yield

Probably the most common—and most flawed—method for calculating a portfolio yield is to calculate the weighted average of the yield of all the securities in the portfolio. The yield is weighted by the proportion of the portfolio that a security makes up. In general, if we let

w_i = market value of security i relative to the total market value of the portfolio;

y_i = yield on security i;

K = number of securities in the portfolio;

then the *weighted-average portfolio yield* is

$$w_1 y_1 + w_2 y_2 + w_3 y_3 + \cdots + w_K y_K.$$

Illustration 6–14. Consider the following three-bond portfolio:

Bond	Coupon Rate (%)	Maturity (years)	Par Value ($)	Market Value ($)	Yield to Maturity (%)
A	7.0	5	10,000,000	9,209,000	9.0
B	10.5	7	20,000,000	20,000,000	10.5
C	6.0	3	30,000,000	28,050,000	8.5

In this illustration, the total market value of the portfolio is $57,259,000, K is equal to 3, and

$$w_1 = 9{,}209{,}000/57{,}259{,}000 = 0.161 \qquad y_1 = 0.090;$$
$$w_2 = 20{,}000{,}000/57{,}259{,}000 = 0.349 \qquad y_2 = 0.105;$$
$$w_3 = 28{,}050{,}000/57{,}259{,}000 = 0.490 \qquad y_3 = 0.085.$$

The weighted-average portfolio yield is then

$$0.161\,(0.090) + 0.349\,(0.105) + 0.490\,(0.085) = 0.0928 = 9.28\%.$$

While it is the most commonly used measure of portfolio yield, the average yield measure provides little insight into the potential yield of a portfolio. To see this, consider a portfolio consisting of only two bonds: a 6-month bond offering a yield to maturity of 11% and a 30-year bond offering a yield to maturity of 8%. Suppose that 99% of the portfolio is invested in the 6-month bond and 1% in the 30-year bond. The weighted-average yield for this portfolio would be 10.97%. But what does this yield mean? How can it be used within any asset/liability framework? The portfolio is basically a 6-month portfolio even though it has a 30-year bond. Would a manager of a depository institution feel confident offering a 2-year CD with a yield of 9%? This would suggest a spread of 197 basis points above the yield on the portfolio based on the weighted-average portfolio yield. This would be an imprudent policy because the yield on this portfolio over the next 2 years will depend on interest rates 6 months from now.

Portfolio Internal Rate of Return

Another measure used to calculate a portfolio yield is the internal rate of the portfolio's cash flow. It is computed by first determining the cash flows for all the securities in the portfolio and then finding the interest rate that will make the present value of the cash flows equal to the market value of the portfolio.[2]

2. In Chapter 13 we discuss the concept of duration. A good approximation to the yield for a portfolio can be obtained by using duration to weight the yield to maturity of the individual bonds in the portfolio.

Illustration 6–15. To illustrate how to calculate a portfolio's internal rate of return, we will use the three bonds in Illustration 6–14. To simplify the illustration, it is assumed that the coupon payment date is the same for each bond.

The portfolio's total market value is $57,259,000. The cash flows for each bond in the portfolio and for the whole portfolio are given in Exhibit 6–2. To determine the yield (internal rate of return) for this three-bond portfolio, the interest rate that makes the present value of the cash flows shown in the last column of Exhibit 6–2 equal to $57,259,000 (the total market value of the portfolio) must be found. If an interest rate of 4.77% is used, the present value of the cash flows will equal $57,259,000. Doubling 4.77% gives 9.54%, which is the yield on the portfolio on a bond-equivalent basis.

The portfolio internal rate of return, while superior to the weighted-average portfolio yield, suffers from the same problems as yield measures in general that will be discussed in subsequent chapters.

EXHIBIT 6–2

Cash Flows for a Three-Bond Portfolio

Period Cash Flow Received	Bond A ($)	Bond B ($)	Bond C ($)	Portfolio ($)
1	350,000	1,050,000	900,000	2,300,000
2	350,000	1,050,000	900,000	2,300,000
3	350,000	1,050,000	900,000	2,300,000
4	350,000	1,050,000	900,000	2,300,000
5	350,000	1,050,000	900,000	2,300,000
6	350,000	1,050,000	30,900,000	32,300,000
7	350,000	1,050,000	—	1,400,000
8	350,000	1,050,000	—	1,400,000
9	350,000	1,050,000	—	1,400,000
10	10,350,000	1,050,000	—	11,400,000
11	—	1,050,000	—	1,050,000
12	—	1,050,000	—	1,050,000
13	—	1,050,000	—	1,050,000
14	—	21,050,000	—	21,050,000

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CHAPTER

7

THE YIELD CURVE, SPOT-RATE CURVE, AND FORWARD RATES

Up to this point, we know the basic principles for pricing a bond and the various yield measures. One of the assumptions made in the illustrations of how to price a bond is that the cash flow for each period should be discounted at the same interest rate or yield. In this chapter we modify this assumption and explain how the appropriate interest rate that should be used to discount the cash flow for each period is determined. The process begins with the Treasury yield curve. From the Treasury yield curve, two important interest rates are derived: (1) spot rates and (2) forward rates. These are the rates that should be used to value bonds.

A BOND IS A PACKAGE OF ZERO-COUPON INSTRUMENTS

Financial theory tells us that a "bond is not a bond." Well, if it is not a bond, what is it? A bond should be viewed as a package of cash flows, with each cash flow viewed as a zero-coupon instrument, with the maturity date the date that the cash flow will be paid and the maturity value equal to the cash flow. This means that a 30-year, 8% coupon Treasury bond with a par value of $100,000 should be viewed as a package of 60 zero-coupon instruments: 59 zero-coupon instruments maturing every 6 months for the next 29.5 years with a maturity value equal to the semiannual coupon of $4,000, and one zero-coupon instrument that matures 30 years from now, with a maturity value equal to $104,000 (the semiannual coupon plus $100,000).

If a bond is viewed as a package of zero-coupon instruments, how then should a bond be valued? The value of a bond is the total value of all the zero-coupon instruments. The value of each zero-coupon instrument is determined in turn by discounting its maturity value at a rate that is unique to that zero-coupon instrument. But what is the yield that should be used to value each zero-coupon instrument? The minimum yield is the yield that the U.S. Treasury would have to pay if it issued a zero-coupon bond with the maturity of the cash flow analyzed.

Of course, the U.S. Treasury does not issue zero-coupon bonds. While there are stripped Treasury securities created by dealers, the yield offered on these securities is not the same as what the U.S. Treasury would have to pay if it issued zero-coupon bonds; stripped Treasuries have less liquidity. Fortunately, we can use arbitrage arguments to calculate from the yield curve the theoretical zero-coupon rates that the Treasury would have to pay.

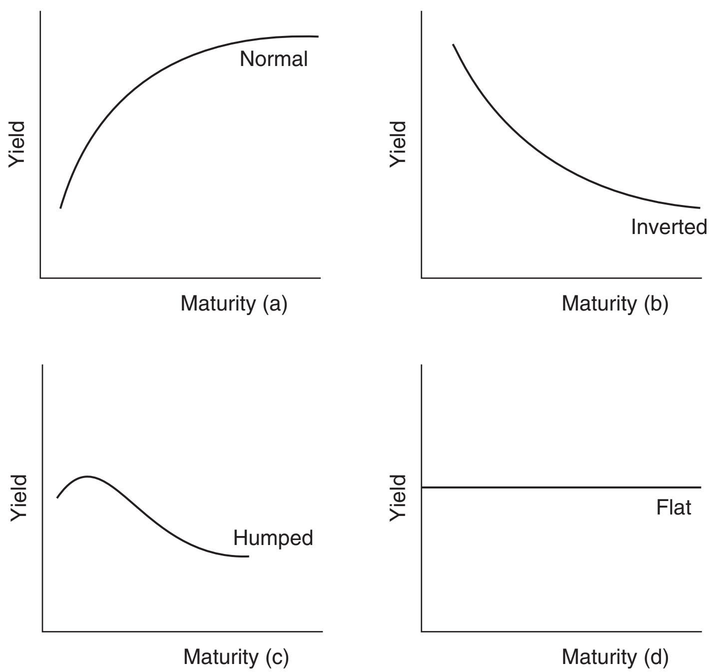
THE YIELD CURVE

The graphical depiction of the relationship between the yield to maturity on securities of the same credit risk and different maturity is called the *yield curve*. The yield curve is constructed from the maturity and observed yield of Treasury securities because Treasuries reflect the pure effect of maturity alone on yield, given that market participants perceive U.S. government securities to have minimal credit risk. When market participants refer to the "yield curve," they usually mean the U.S. Treasury yield curve.

Exhibit 7–1 shows four yield curves that have been observed in the U.S. Treasury market (and occur in other major government bond markets). In the yield curve in panel (a), the yield increases with maturity. This shape is commonly referred to as an *upward-sloping* or *normal yield curve*. The yield curve in panel (b) is a *downward-sloping* or *inverted yield curve* because yield decreases with

EXHIBIT 7–1

Four Hypothetical Yield Curves



maturity. In a *humped yield curve*, depicted in panel (c) of the exhibit, the yield curve initially is upward sloping, but after a certain maturity, it becomes downward sloping. Finally, a *flat yield curve* is one where the yield is the same regardless of the maturity. A flat yield curve is shown in panel (d).

Several theories have been proposed to explain the shape of the yield curve, although discussion of these theories is beyond the scope of this book.[1] Analysis of the yield curve in this chapter is based on what is called the *pure expectations theory of interest rates*. According to the pure expectations theory, the only factor that affects the shape of the yield curve is the market expectation about future interest rates.

THE SPOT-RATE CURVE

The Treasury yield curve shows the relationship between the yield on Treasury securities (Treasury bills and coupon securities) and maturity. According to the pure expectations theory and arbitrage arguments, we can determine the theoretical relationship between the yield on zero-coupon Treasury securities and maturity. This relationship is called the *Treasury spot-rate curve*. The yield on a zero-coupon instrument is called a *spot rate*.

To illustrate how to construct a theoretical spot-rate curve from the yield offered on Treasury securities, we use the 20 hypothetical Treasury securities shown in Exhibit 7–2.

The basic principle is that the value of a Treasury coupon security should be equal to the value of a package of zero-coupon Treasury securities. Consider first the 6-month Treasury bill in Exhibit 7–2. Because a Treasury bill is a zero-coupon instrument, its yield of 8% is equal to the spot rate. Similarly, for the 1-year Treasury, the yield of 8.3% is the 1-year spot rate. Given these two spot rates, we can compute the spot rate for a 1.5-year zero-coupon Treasury. The value or price of a 1.5-year zero-coupon Treasury should equal the present value of the three cash flows from the 1.5-year coupon Treasury, where the yield used for discounting is the spot rate corresponding to the cash flow. Using \$100 as par, the cash flows for the 1.5-year coupon Treasury are as follows:

0.5 year	$0.085 \times \$100 \times 0.5$	$= \$4.25$
1.0 year	$0.085 \times \$100 \times 0.5$	$= \$4.25$
1.5 years	$0.085 \times \$100 \times 0.5 + 100$	$= \$104.25$

The present value of the cash flows is then

$$\frac{4.25}{(1+z_1)^1}+\frac{4.25}{(1+z_2)^2}+\frac{104.25}{(1+z_3)^3},$$

1. The interested reader is referred to Chapter 6 of Frank J. Fabozzi, *Bond Markets, Analysis and Strategies*, 10th ed., for a discussion of these theories.

EXHIBIT 7–2

Maturity and Yield to Maturity for 20 Hypothetical Treasury Securities

Maturity (years)	Coupon	Yield to Maturity	Price ($)
0.50	0.0000	0.080	96.15
1.00	0.0000	0.083	92.19
1.50	0.0850	0.089	99.45
2.00	0.0900	0.092	99.64
2.50	0.1100	0.094	103.49
3.00	0.0950	0.097	99.49
3.50	0.1000	0.100	100.00
4.00	0.1000	0.104	98.72
4.50	0.1150	0.106	103.16
5.00	0.0875	0.108	92.24
5.50	0.1050	0.109	98.38
6.00	0.1100	0.112	99.14
6.50	0.0850	0.114	86.94
7.00	0.0825	0.116	84.24
7.50	0.1100	0.118	96.09
8.00	0.0650	0.119	72.62
8.50	0.0875	0.120	82.97
9.00	0.1300	0.122	104.30
9.50	0.1150	0.124	95.06
10.00	0.1250	0.125	100.00

where

z_1 = one-half the 6-month theoretical spot rate;
z_2 = one-half the 1-year theoretical spot rate;
z_3 = one-half the 1.5-year theoretical spot rate.

Because the 6-month spot rate and 1-year spot rate are 8.0% and 8.3%, respectively, then

$$z_1 = 0.04 \text{ and } z_2 = 0.0415.$$

Therefore, the present value of the 1.5-year coupon Treasury security is

$$\frac{4.25}{(1.0400)^1} + \frac{4.25}{(1.0415)^2} + \frac{104.25}{(1+z_3)^3}.$$

Because the price of the 1.5-year coupon Treasury security is \$99.45, the following relationship must hold:

$$99.45 = \frac{4.25}{(1.0400)^1} + \frac{4.25}{(1.0415)^2} + \frac{104.25}{(1+z_3)^3}.$$

We can now solve for the theoretical 1.5-year spot rate as follows:

$$99.45 = 4.08654 + 3.91805 + \frac{104.25}{(1+z_3)^3};$$

$$91.4451 = \frac{104.25}{(1+z_3)^3};$$

$$(1+z_3)^3 = 1.140024;$$

$$z_3 = 0.04465.$$

Doubling this yield, we obtain the bond-equivalent yield of 0.0893 or 8.93%, which is the theoretical 1.5-year spot rate.

Given the theoretical 1.5-year spot rate, we can obtain the theoretical 2-year spot rate. The cash flows for the 2-year coupon Treasury in Exhibit 7–2 are given below:

0.5 year	$0.090 \times \$100 \times 0.5$	= \$4.50
1.0 year	$0.090 \times \$100 \times 0.5$	= \$4.50
1.5 years	$0.090 \times \$100 \times 0.5$	= \$4.50
2.0 years	$0.090 \times \$100 \times 0.5 + 100$	= \$104.50

The present value of the cash flows is as follows:

$$\frac{4.50}{(1+z_1)^1} + \frac{4.50}{(1+z_2)^2} + \frac{4.50}{(1+z_3)^3} + \frac{4.50}{(1+z_4)^4},$$

where

z_4 = one-half the 2-year theoretical spot rate.

The 6-month spot rate, 1-year spot rate, and 1.5-year spot rate are 8.0%, 8.3%, and 8.93%, respectively, so

$$z_1 = 0.04,\ z_2 = 0.0415, \text{ and } z_3 = 0.04465.$$

Therefore, the present value of the 2-year coupon Treasury security is

$$\frac{4.50}{(1.0400)^1} + \frac{4.50}{(1.0415)^2} + \frac{4.50}{(1.04465)^3} + \frac{104.50}{(1+z_4)^4}.$$

Because the price of the 2-year coupon Treasury security is \$99.64, the following relationship must hold:

$$99.64 = \frac{4.50}{(1.0400)^1} + \frac{4.50}{(1.0415)^2} + \frac{4.50}{(1.04465)^3} + \frac{104.50}{(1+z_4)^4}.$$

We can now solve for the theoretical 2-year spot rate as follows:

$$99.64 = 4.32692 + 4.14853 + 3.94730 + \frac{104.50}{(1+z_4)^4};$$

$$87.21725 = \frac{104.50}{(1+z_4)^4};$$

$$(1+z_4)^4 = 1.198158;$$

$$z_4 = 0.046235.$$

Doubling this yield, we obtain the theoretical 2-year spot-rate bond-equivalent yield of 9.247%.

We can then use the theoretical 2-year spot rate and the 2.5-year coupon Treasury in Exhibit 7–2 to compute the 2.5-year theoretical spot rate. In general, to compute the theoretical spot rate for the nth 6-month period, the following equation must be solved:

$$P_n = \frac{c^*}{(1+z_1)^1} + \frac{c^*}{(1+z_2)^2} + \frac{c^*}{(1+z_3)^3} + \cdots + \frac{c^* + 100}{(1+z_n)^n},$$

where

P_n = price of the coupon Treasury with n periods to maturity (per \$100 of par value);

c^* = *semiannual* coupon interest for the coupon Treasury with n periods to maturity per \$100 of par value;

and z_t for $t = 1, 2, \ldots, n - 1$ are the theoretical spot rates that are known.

This expression can be rewritten as

$$P_n = c^* \sum_{t=1}^{n-1} \frac{1}{(1+z_t)^t} + \frac{c^* + 100}{(1+z_n)^n}.$$

Solving for z_n, we get

$$z_n = \left[\frac{c^* + 100}{P_n - c^* \sum_{t=1}^{n-1} \frac{1}{(1+z_t)^t}} \right]^{1/n} - 1.$$

Doubling z_n gives the theoretical spot rate on a bond-equivalent basis.

The equation above is used to determine the theoretical spot rates for each hypothetical Treasury security shown in Exhibit 7–2. The theoretical spot rates are presented in Exhibit 7–3. It is this yield/maturity structure that would be used to construct the theoretical spot-rate curve that is referred to as the *term structure of interest rates*. The methodology described above for deriving the spot-rate curve is called *bootstrapping*.

In practice, the Treasury securities used to determine the spot rates are the "on-the-run Treasury securities." These are the most recently auctioned Treasury securities. Currently, Treasury bills are auctioned on a regular basis with initial maturities of 4 weeks, 3 months, and 6 months, while Treasury coupon securities are auctioned on a regular basis with initial maturities of 2 years, 5 years, 10 years, and 30 years.

The preference for using the on-the-run Treasury securities is twofold. First, seasoned Treasury coupon securities that are not the most recently auctioned Treasury coupon securities (called "off-the-run Treasuries") may trade at a substantial premium or discount from par. Consequently, their price, and hence yield,

EXHIBIT 7–3

Theoretical Spot Rates

Maturity (years)	Yield to Maturity	Theoretical Spot Rate
0.50	0.0800	0.08000
1.00	0.0830	0.08300
1.50	0.0890	0.08930
2.00	0.0920	0.09247
2.50	0.0940	0.09468
3.00	0.0970	0.09787
3.50	0.1000	0.10129
4.00	0.1040	0.10592
4.50	0.1060	0.10850
5.00	0.1080	0.11021
5.50	0.1090	0.11175
6.00	0.1120	0.11584
6.50	0.1140	0.11744
7.00	0.1160	0.11991
7.50	0.1180	0.12405
8.00	0.1190	0.12278
8.50	0.1200	0.12546
9.00	0.1220	0.13152
9.50	0.1240	0.13377
10.00	0.1250	0.13623

reflects any favorable or unfavorable tax treatment of the discount or premium from par. The on-the-run Treasury coupon securities trade close to par, so their prices are not biased by tax considerations. Second, on-the-run issues trade with greater liquidity than off-the-run issues. Consequently, the latter have a liquidity premium built into their yield.[2]

Thus, unlike in our illustration where all 6-month points on the yield curve are used to begin the analysis, beyond 1 year, only 2-, 5-, 10-, and 30-year maturities are available in practice. To obtain the yield for any maturity between two maturity dates, a simple linear interpolation is used. The resulting yield curve that is then used to generate the spot-rate curve via bootstrapping is called the *par yield curve*.

PRICING A BOND

Given the cash flows of a bond and the Treasury spot-rate curve, we can determine the price of a Treasury security by discounting each cash flow by the corresponding spot rate and adding up the present value of the cash flows. For example, consider an 8.8% coupon Treasury bond with 25 years to maturity. Exhibit 7–4 shows the cash flow for this bond per $100 of par value. Also shown in the exhibit are the assumed Treasury spot rates. The price of this Treasury bond is $96.6133.

Suppose instead that the 25-year bond is an option-free corporate bond rather than a Treasury bond. The corporate bond would trade at a lower price than the Treasury bond. The appropriate spot rate at which each cash flow will be discounted is the Treasury spot rate plus a premium to reflect the credit risk associated with investing in a corporate bond. For example, if an appropriate premium is 100 basis points, discounting the cash flows in Exhibit 7–4 by the assumed Treasury spot-rate plus 100 basis points would give a price of $88.5473. While by this description we assume that the premium is the same for each period, a spot rate default spread curve can be developed and used to determine the spot rate at which to discount each cash flow.[3]

DRAWBACK OF TRADITIONAL YIELD SPREAD ANALYSIS

Traditional analysis of the yield spread for a non-Treasury bond involves calculating the difference between the yield to maturity (or yield to call) of the bond in question and the yield to maturity of a comparable-maturity coupon Treasury. The

2. Some practitioners prefer to use all Treasury issues—on-the-run and off-the-run issues—to estimate the spot rates. Those who do so believe that there is information contained in all Treasury issues and that this information is not used when only the on-the-run issues are used. Complicated statistical techniques are used when all Treasury issues are used to estimate the spot rates.
3. Credit-risk modeling has been used to estimate a spot-rate default spread.

EXHIBIT 7–4

Calculation of Price of a 25-Year, 8.8% Coupon Bond Using Treasury Spot Rates

Period	Cash Flow ($)	Treasury Spot Rate (%)	Present Value ($)
1	4.4	7.00000	4.2512
2	4.4	7.04999	4.1055
3	4.4	7.09998	3.9628
4	4.4	7.12498	3.8251
5	4.4	7.13998	3.6922
6	4.4	7.16665	3.5622
7	4.4	7.19997	3.4351
8	4.4	7.26240	3.3077
9	4.4	7.33315	3.1820
10	4.4	7.38977	3.0611
11	4.4	7.44517	2.9434
12	4.4	7.49135	2.8302
13	4.4	7.53810	2.7200
14	4.4	7.57819	2.6141
15	4.4	7.61959	2.5112
16	4.4	7.66205	2.4111
17	4.4	7.70538	2.3139
18	4.4	7.74391	2.2207
19	4.4	7.78888	2.1291
20	4.4	7.83434	2.0404
21	4.4	8.22300	1.8879
22	4.4	8.33333	1.7923
23	4.4	8.40000	1.7080
24	4.4	8.50000	1.6204
25	4.4	8.54230	1.5465
26	4.4	8.72345	1.4500
27	4.4	8.90000	1.3581
28	4.4	9.00000	1.2829
29	4.4	9.01450	1.2252
30	4.4	9.23000	1.1367
31	4.4	9.39000	1.0611
32	4.4	9.44840	1.0045
33	4.4	9.50000	0.9514
34	4.4	9.50000	0.9083
35	4.4	9.50000	0.8671
36	4.4	9.50000	0.8278
37	4.4	9.55000	0.7833

EXHIBIT 7–4

Calculation of Price of a 25-Year, 8.8% Coupon Bond Using Treasury Spot Rates (*Continued*)

Period	Cash Flow ($)	Treasury Spot Rate (%)	Present Value ($)
38	4.4	9.56000	0.7462
39	4.4	9.58000	0.7095
40	4.4	9.58000	0.6671
41	4.4	9.60000	0.6436
42	4.4	9.70000	0.6020
43	4.4	9.80000	0.5625
44	4.4	9.90000	0.5251
45	4.4	10.00000	0.4897
46	4.4	10.10000	0.4563
47	4.4	10.30000	0.4154
48	4.4	10.50000	0.3774
49	4.4	10.60000	0.3503
50	104.4	10.80000	7.5278
		Theoretical price = $96.6133	

latter is obtained from the Treasury yield curve. For example, consider the following two 8.8% coupon, 25-year bonds:

Issue	Price ($)	Yield to Maturity (%)
Treasury	96.6133	9.15
Corporate	87.0798	10.24

The yield spread for these two bonds as traditionally computed is 109 basis points (10.24% minus 9.15%). The drawbacks of this convention, however, are (1) for both bonds, the yield fails to take into consideration the yield curve or spot-rate curve, and (2) in the case of callable and/or putable bonds, expected interest-rate volatility may alter the expected cash flow of a bond. In this chapter we focus only on the first problem: failure to consider the spot-rate curve. This can be overcome by calculating a measure called *zero-volatility spread* or *static spread*. We will deal with the second problem in Chapter 18.

Zero-Volatility Spread

In traditional yield spread analysis, the investor compares the yield to maturity of a particular bond with the yield to maturity of a similar-maturity on-the-run Treasury security. This means that the yield to maturity of both a 25-year zero-coupon corporate bond and a 25-year, 8.8% coupon corporate bond would be compared to the on-the-run 25-year Treasury security. Such a comparison makes little sense because the cash-flow characteristics of the two corporate bonds will not be the same as that of the benchmark Treasury.

The proper way to compare non-Treasury bonds of the same maturity but with different coupon rates is to compare them to a portfolio of Treasury securities that have the same cash flows. For example, consider the 8.8%, 25-year corporate bond selling for $87.0798. The cash flow per $100 par value for this corporate bond, assuming that interest rates do not change (i.e., assuming interest rates are static) is forty-nine 6-month payments of $4.40 and a payment in 25 years (at the end of fifty 6-month periods) of $104.40. A portfolio that will replicate this cash flow would include 50 zero-coupon Treasury securities with maturity values and maturities coinciding with the amounts and timing of the cash flows of the corporate bond.

The corporate bond's value is equal to the present value all its cash flows. Assuming that the cash flows are riskless, the value will equal the present value of the replicating portfolio of Treasury securities. In turn, these cash flows are valued at the Treasury spot rates. The price of the risk-free 8.8%, 25-year bond assuming the Treasury spot-rate curve shown in Exhibit 7–4 is $96.6133. The corporate bond's price is $87.0798, less than that package of zero-coupon Treasury securities, because investors in fact require a yield spread for the risk associated with holding a corporate bond rather than a riskless package of Treasury securities.

The *zero-volatility spread* or *static spread* is a measure of the spread that the investor would realize over the entire Treasury spot-rate curve if the bond is held to maturity. It is not a spread off one point on the Treasury yield curve, as is the traditional yield spread. The zero-volatility spread, which is found by an iterative procedure, is the spread that makes the present value of the cash flows from the corporate bond when discounted at the Treasury spot rate plus the spread equal to the corporate bond's price.

Let's use the corporate bond in the previous illustration to explain how to compute the zero-volatility spread. Select some spread, say, 100 basis points, and add that to each Treasury spot rate shown in the third column in Exhibit 7–4. For example, the 14-year (period 28) spot rate becomes 10.0% (9% plus 1%). The spot rate plus 100 basis points is then used to calculate the present value, which can be shown to be $88.5473. Because the present value is not equal to the corporate bond's price, which we know to be $87.0798, the zero-volatility spread is not 100 basis points. If a spread of 110 basis points is tried, that results in a present value of $87.8031; again, because this is not equal to the corporate bond's price, 110 basis points is not the zero-volatility spread. It can be shown that if a spread of 120 basis points is used, the present value is $87.0798, which is equal to the corporate bond's price. Therefore, 120 basis points is the static spread, in comparison with the traditional yield spread of 109 basis points.

It can be demonstrated that the shorter the maturity of the bond, the less the zero-volatility spread differs from the traditional yield spread. The magnitude of the difference between the traditional yield spread and the zero-volatility spread also depends on the shape of the yield curve. The steeper the yield curve, the greater is the difference for a given coupon and maturity. The difference between the traditional yield spread and the zero-volatility spread will be considerably greater for sinking-fund bonds and mortgage-backed securities in a steep yield curve environment.

FORWARD RATES

We have seen how the theoretical spot-rate curve can be constructed from the yield curve. But there may be more information contained in the yield curve. Specifically, can we use the yield curve to infer the market's expectations of future interest rates? Let's explore this possibility.

Suppose that an investor with a 1-year investment horizon is considering two alternatives:

Alternative 1: Buy a 1-year Treasury bill.

Alternative 2: Buy a 6-month Treasury bill, and when it matures in 6 months, buy another 6-month Treasury bill.

The investor will be indifferent between the two alternatives if they produce the same yield or the same number of dollars per dollar invested over the 1-year investment horizon. The investor knows the spot rates on the 6-month Treasury bill and the 1-year Treasury bill but not what yield will be available on a 6-month Treasury bill purchased 6 months from now. The yield on a 6-month Treasury bill 6 months from now is what's referred to as a *forward rate*. Given the spot rates for the 6-month Treasury bill and the 1-year Treasury bill rate, it is possible to determine the forward rate on a 6-month Treasury bill that will make investors indifferent to the two alternatives.

By investing in the 1-year Treasury bill, the investor will receive the maturity value at the end of 1 year. Suppose that the maturity value of the 1-year Treasury bill is \$100. The price (cost) of the 1-year Treasury bill would be as follows:

$$\frac{100}{(1+z_2)^2},$$

where z_2 is one-half the bond-equivalent yield of the theoretical 1-year spot rate.

Suppose that an investor purchases a 6-month Treasury bill for P dollars. At the end of 6 months, the value of this investment would be

$$P(1 + z_1),$$

where z_1 is one-half the bond-equivalent yield of the theoretical 6-month spot rate.

Let f be one-half the forward rate on a 6-month Treasury bill available 6 months from now. Then the future dollars available at the end of 1 year from the P dollars invested would be given by

$$P(1 + z_1)\,(1 + f).$$

Suppose that today we want to know how many P dollars an investor must invest in order to get \$100 in 1 year. This can be found as follows:

$$P(1 + z_1)(1 + f) = 100.$$

Solving for P, we get

$$P = \frac{100}{(1+z_1)(1+f)}.$$

The investor will be indifferent between the two alternatives if the same dollar investment is made and \$100 is received from both investments at the end of 1 year. That is, an investor will be indifferent if

$$\frac{100}{(1+z_2)^2} = \frac{100}{(1+z_1)(1+f)}.$$

Solving for f, we get

$$f = \frac{(1+z_2)^2}{(1+z_1)} - 1.$$

Doubling f gives the bond-equivalent yield for the 6-month forward rate.

To illustrate application of this equation, we will use the theoretical spot rates shown in Exhibit 7–3. Because we use the theoretical spot rates to compute the forward rate, the resulting forward rate is also called the *implied forward rate*. We know that

6-month Treasury bill rate = 0.080; therefore, $z_1 = 0.0400$;

1-year Treasury bill rate = 0.083; therefore, $z_2 = 0.0415$.

Substituting into the equation, we have

$$f = \frac{(1.0415)^2}{1.0400} - 1 = 0.043.$$

The forward rate on a 6-month security, quoted on a bond-equivalent basis, is 8.60% (0.043×2).

We can confirm these results. The price of a 1-year Treasury bill with a \$100 maturity value is

$$\frac{100}{(1.0415)^2} = 92.19.$$

If \$92.19 is invested for 6 months at the 6-month spot rate of 4% for 6 months (8% annually), the amount at the end of 6 months would be

$$92.19(1.0400) = 95.8776.$$

If \$95.8776 is reinvested for another 6 months in a 6-month Treasury offering 4.3% for 6 months (8.6% annually), the amount at the end of 1 year is

$$95.8876(1.043) = 100.$$

Each alternative has the same \$100 payoff if the 6-month Treasury bill yield 6 months from now is 4.3% (8.6% on a bond-equivalent basis). This means that if an investor is guaranteed a 4.3% yield (8.6% bond-equivalent basis) on a 6-month Treasury bill 6 months from now, he will be indifferent between the two alternatives.

While we have calculated only the implied 6-month forward rate 6 months from now, we can follow the same methodology to determine the implied forward rate 6 months from now for an investment for a period longer than 6 months. That is, we can use the yield curve or, more specifically, the spot-rate curve generated from the yield curve to construct an implied forward rate 6 months from now for 1-year investments, 1.5-year investments, 2-year investments, 2.5-year investments, and so on.

We can even take this one step further. It is not necessary to limit ourselves to implied forward rates 6 months from now. The yield curve can be used to calculate the implied forward rate for any time into the future for any investment length. As examples, the following can be calculated:

- The 2-year implied forward rate 5 years from now;
- The 6-year implied forward rate 10 years from now;
- The 7-year implied forward rate 3 years from now.

How is this done? To demonstrate how, we must introduce some notation. We will continue to let f represent the forward rate. But now we must identify two aspects of the forward rate. First, we want to denote when the forward rate begins. Second, we want to denote the length of time of the forward rate. To identify these two aspects of the forward rate, we use the following notation:

${}_nf_t$ = forward rate n periods from now for t periods.

Remember that for our bond examples, each period is equal to 6 months. Consider first the earlier example of the 6-month forward rate 6 months from now. In this case, because we are looking at a forward rate 6 months from now, this is equal to one period from now. Thus n is 1. The length of the forward rate is 6 months, so t is equal to 1. Consequently, the 6-month forward rate 6 months from now is denoted by ${}_1f_1$. The 6-month forward rates are then expressed as follows:

${}_2f_1$ = 6-month forward rate 1 year (two periods) from now;

${}_3f_1$ = 6-month forward rate 1.5 years (three periods) from now;

${}_4f_1$ = 6-month forward rate 2 years (four periods) from now, etc.

For forward rates 4 years (eight periods) from now, we would have the following:

${}_8f_1$ = 6-month forward rate 4 years (eight periods) from now;

${}_8f_2$ = 1-year (two period) forward rate 4 years (eight periods) from now;

${}_8f_3$ = 1.5-year forward rate 4 years (eight periods) from now, etc.

Now let's see how the spot rates can be used to calculate the forward rate. We assume in the illustration that there are zero-coupon Treasury securities available.[4] Suppose that an investor with a 5-year investment horizon is considering two alternatives:

Alternative 1: Buy a 5-year (10-period) zero-coupon Treasury security.

Alternative 2: Buy a 3-year (6-period) zero-coupon Treasury security, and when it matures in 3 years, buy a 2-year Treasury security.

An investor will be indifferent between the two alternatives if they produce the same yield or the same number of dollars per dollar invested over the 5-year investment horizon. The spot rates on the 5-year Treasury security and the 3-year Treasury security are known, but the yield available on a 2-year Treasury security purchased 3 years from now is not known. That is, an investor does not know the 2-year forward rate 3 years from now. In terms of our notation, that unknown is ${}_6f_4$.

The price of the 5-year zero-coupon Treasury security with a maturity value of $100 would be

$$\frac{100}{(1+z_{10})^{10}},$$

where z_{10} is one-half the bond-equivalent yield of the theoretical 5-year spot rate.

Suppose that an investor purchases a 3-year zero-coupon Treasury security for P dollars. At the end of 3 years, the value of this investment would be

$$P(1 + z_6)^6,$$

where z_6 is one-half the bond-equivalent yield of the theoretical 3-year spot rate. Let ${}_6f_4$ be the semiannual 2-year forward rate 3 years from now. Then the future dollars available at the end of 5 years from the P dollars invested are

$$P(1 + z_6)^6 (1 + {}_6f_4)^4.$$

Suppose that today we want to know how many P dollars an investor must invest in order to get $100 one year from now. This can be found as follows:

$$P(1 + z_6)^6 (1 + {}_6f_4)^4 = 100.$$

Solving for P,

$$P = \frac{100}{(1+z_6)^6(1+{}_6f_4)^4}.$$

4. The existence of zero-coupon Treasury securities is not necessary for determination of the forward rates. The assumption just simplifies the presentation.

An investor will be indifferent between the two alternatives if he makes the same dollar investment and receives \$100 at the end of 5 years from both alternatives. That is, an investor will be indifferent if

$$\frac{100}{(1+z_{10})^{10}}=\frac{100}{(1+z_6)^6(1+{}_6f_4)^4}.$$

Solving for ${}_6f_4$, we get

$${}_6f_4=\left[\frac{(1+z_{10})^{10}}{(1+z_6)^6}\right]^{1/4}-1.$$

Doubling ${}_6f_4$ gives the bond-equivalent yield for the 2-year forward rate 3 years from now.

To illustrate this, we use the theoretical spot rates shown in Exhibit 7–4. We know that

3-year spot rate = 0.09787; therefore, z_6 = 0.048935;

5-year spot rate = 0.11021; therefore, z_{10} = 0.055105.

Substituting into the equation, we have

$$\begin{aligned}{}_6f_4&=\left[\frac{(1.055105)^{10}}{(1.048935)^6}\right]^{1/4}-1\\&=\left[\frac{(1.709845)^{10}}{(1.331961)^6}\right]^{1/4}-1\\&=0.0644.\end{aligned}$$

The forward rate on a 2-year Treasury security 3 years from now, quoted on a bond-equivalent basis, is calculated as 12.88% (0.0644 × 2). Let's confirm our results. The price of a 5-year zero-coupon Treasury security with a \$100 maturity value is

$$\frac{100}{(1.055105)^{10}}=58.48.$$

If \$58.48 is invested for 3 years at the 3-year spot rate of 9.787%, the amount at the end of 6 periods will be

$$\$58.48\,(1.048935)^6=\$77.8931.$$

If \$77.8931 is reinvested for another 2 years (four periods) at 6.44% (12.88% annually), the amount at the end of the fifth year will be

$$\$77.8931\,(1.0644)^4=\$100.$$

Each alternative does have the same \$100 payoff if the 2-year Treasury rate 3 years from now is 6.44% (12.88% on a bond-equivalent basis).

In general, the formula for the forward rate is

$$ {}_{n}f_{t} = \left[\frac{(1 + z_{n+1})^{n+t}}{(1 + z_{n})^{n}} \right]^{1/t} - 1, $$

where z_n is a semiannual spot rate. Doubling ${}_{n}f_{t}$ gives the forward rate on a bond-equivalent basis.

To illustrate application of the general equation for the forward rate, consider the earlier example where we sought the 6-month forward rate 6 months from now. That is, we sought ${}_{1}f_{1}$. Because n is equal to 1 and t is equal to 1,

$$ {}_{1}f_{1} = \left[\frac{(1 + z_{1+1})^{1+1}}{(1 + z_{1})^{1}} \right]^{1/1} - 1 $$

or

$$ = \frac{(1 + z_{2})^{2}}{(1 + z_{1})^{1}} - 1. $$

This agrees with our earlier equation for a 1-period spot rate 1 period from now.

Relationship Between Long Spot Rates and Short-Term Forward Rates

Suppose that an investor purchases a 5-year zero-coupon Treasury security for \$58.48 with a maturity value of \$100. The investor instead could buy a 6-month Treasury bill and reinvest the proceeds every 6 months for 5 years. The number of dollars that will be realized will depend on the 6-month forward rates. If an investor could actually reinvest the proceeds maturing every 6 months at the 6-month forward rates, let's see how many dollars would accumulate at the end of 5 years. The 6-month forward rates were calculated for the yield curve given in Exhibit 7–2. The semiannual forward rates from Exhibit 7–3 are

${}_{1}f_{1} = 0.043000$; ${}_{2}f_{1} = 0.050980$; ${}_{3}f_{1} = 0.051005$;

${}_{4}f_{1} = 0.051770$; ${}_{5}f_{1} = 0.056945$; ${}_{6}f_{1} = 0.060965$;

${}_{7}f_{1} = 0.069310$; ${}_{8}f_{1} = 0.064625$; ${}_{9}f_{1} = 0.062830$.

By investing the \$58.48 at the 6-month spot rate of 4% (8% on a bond-equivalent basis) and reinvesting at the forward rates above, the number of dollars accumulated at the end of 5 years will be

$$ \$58.48(1.04)(1.043)(1.05098)(1.051005)(1.05177)(1.056945) \\ \times (1.060965)(1.06931)(1.064625)(1.06283) = \$100. $$

Therefore, we see that if the forward rates are realized, the \$58.48 investment will produce the same number of dollars as an investment in a 5-year zero-coupon Treasury security at the 5-year spot rate. From this illustration, we can see that the 5-year spot rate is related to the current 6-month spot rate and the 6-month forward rates.

In general, the relationship among a t-period spot rate, the current 6-month spot rate, and the 6-month forward rates is as follows:

$$z_t = [(1 + z_1)(1 + {}_1f_1)(1 + {}_2f_1)(1 + {}_3f_1) \cdots (1 + {}_pf_1)]^{1/t} - 1.$$

For example, the 5-year spot rate $(2 \times z_{10})$ can be computed from the current 6-month rate (z_1) of 4% and the nine 6-month forward rates. This is demonstrated below:

$$\begin{aligned} z_{10} &= [(1.04)(1.043)(1.05098)(1.051005)(1.05177)(1.056945) \\ &\quad \times (1.060965)(1.06931)(1.064625)(1.06283)]^{1/10} - 1 \\ &= 0.055105. \end{aligned}$$

The 5-year spot rate is then 11.021%, which agrees with the value in Exhibit 7–4.

Valuing Cash Flows and Bonds Using Forward Rates

Because spot rates and forward rates are related, it makes no difference whether spot rates or forward rates are used to determine the present value of a cash flow. In general, the present value of a cash flow c in period t using forward rates is

$$\frac{c}{(1+z_1)(1+{}_1f_1)(1+{}_2f_1)\cdots(1+{}_nf_1)}.$$

The general formula for using forward rates to value an n-period-maturity bond whose semiannual cash flow is denoted by c and maturity value is denoted by M is

$$\begin{aligned} \text{Price} &= \frac{c}{(1+z_t)} + \frac{c}{(1+z_1)(1+f_1)} + \frac{c}{(1+z_1)(1+{}_1f_1)(1+{}_2f_1)} \\ &\quad + \cdots + \frac{c+M}{(1+z_1)(1+{}_1f_1)(1+{}_2f_1)\cdots(1+{}_nf_1)}. \end{aligned}$$

Often vendors of analytical systems will state how they value cash flows. Some state that spot rates are used, while others state that forward rates are used. The results from the discounting process will be the same.

Forward Rate as a Hedgeable Rate

A natural question about forward rates is how well they do at predicting future interest rates. Studies have demonstrated that forward rates do not do a good job in predicting future interest rates.[5] Then why the big deal about understanding

5. Eugene F. Fama, "Forward Rates as Predictors of Future Spot Rates," *Journal of Financial Economics*, Vol. 3, No. 4, 1976, pp. 361–377.

forward rates? As we demonstrated in our illustration of how to select between two alternative investments, the reason is that forward rates indicate how an investor's expectations must differ from the rate built into bond prices in order to make an investment decision.

In our illustration, the 6-month forward rate may not be realized. That is irrelevant. The fact is that the 6-month forward rate indicated to the investor that if her expectation about the 6-month rate 6 months from now is less than 4.3% (the forward rate in our illustration), she would be better off with Alternative 1.

For this reason, some market participants prefer not to talk about forward rates as being market consensus rates. Instead, they refer to forward rates as being *hedgeable rates*. For example, by buying the 1-year security, the investor is able to hedge the 6-month rate 6 months from now.

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PART
THREE

RETURN ANALYSIS AND RETURN MEASURES

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CHAPTER
8

POTENTIAL SOURCES OF DOLLAR RETURN

To make an intelligent decision about the attractiveness of a bond, an investor must be able to measure the potential yield from owning it. This requires an understanding of the potential sources of dollar return from investing in a bond and then converting the total dollar return from all sources into a yield measure that can be used to compare different bonds.

The purpose of this chapter is threefold: (1) to explain the potential sources of dollar return, (2) to examine the characteristics of a bond that affect its dollar return, and (3) to analyze whether the conventional measures of yield discussed in Chapter 6 appropriately account for the potential sources of dollar return. In the next chapter we present a measure of yield that is more useful to an investor in assessing the potential performance of bonds.

SOURCES OF DOLLAR RETURN

The dollar return a bond investor expects to receive takes the form of the periodic interest payments made by the issuer (i.e., the coupon interest payments) and any capital appreciation realized when the bond is sold. For example, suppose that an investor buys a 20-year, 7% coupon bond with a par value of $1,000 for $816. Holding this bond until it matures, the investor will receive 40 coupon interest payments of $35, one every 6 months; then at the end of 20 years he will receive $1,000 for the bond he purchased for $816. This will result in a capital gain of $184. In other words, an investor who buys this bond and holds it to maturity expects a return in the form of 40 semiannual coupon interest payments of $35 and a capital gain of $184.

There is, however, another potential source for a bond's dollar return that may not be recognized by many investors. That source is the interest income that can be realized by reinvesting the coupon interest payments. For example, an investor receiving the first $35 coupon interest payment from the bond issuer must do something with that money. If the money is invested in another instrument for the next 19.5 years, this reinvestment will add to the return from holding the bond. This is also true for the second coupon interest payment of $35, which can be invested for 19 years; and so on for subsequent coupon interest payments. This potential source of return is referred to as the *interest-on-interest* component of a bond's dollar return. In fact, for certain bonds, the interest-on-interest component can represent more than 80% of a bond's potential dollar return.

CONVENTIONAL YIELD MEASURES AND THE THREE POTENTIAL SOURCES OF A BOND'S DOLLAR RETURN

To summarize, an investor who purchases a bond can expect to receive a dollar return from one or more of the following sources:

1. The periodic interest payments made by the issuer (i.e., the coupon interest payments);
2. Any capital gain (or capital loss, which reduces the dollar return) when the bond matures or is sold;
3. Income from reinvestment of the periodic interest payments (the interest-on-interest component).

Any measure of a bond's potential yield should consider all three of these potential sources of return. Do the three conventional bond measures (current yield, yield to maturity, and yield to call) take into account these three potential sources of return?

Current yield considers only the coupon interest payments. No consideration is given to any capital gain (or loss) or interest on interest.

Yield to maturity takes into account coupon interest and any capital gain (or loss). It also considers the interest-on-interest component, although implicit in this computation is an assumption that the coupon payments can be reinvested at the computed yield to maturity. The yield to maturity, therefore, is a *promised* yield. That is, the promised yield to maturity will be realized if (1) the bond is held to maturity and (2) the coupon interest payments are reinvested at the yield to maturity. If either (1) or (2) does not occur, the actual yield realized by an investor will be greater than or less than the yield to maturity.

For example, for our 20-year, 7% coupon bond selling at $816, the yield to maturity is 9%. Thus, if an investor buys this bond, holds it to maturity, and reinvests each semiannual coupon payment at 9% until the maturity date, the promised yield to maturity of 9% at the time of the purchase will be realized. We'll demonstrate this in the next section.

Although the yield to maturity does consider the interest-on-interest component of a bond's potential dollar return, the assumption that the coupon interest payments can be reinvested at the yield to maturity is not very realistic. For example, in 1981 an investor could have purchased a long-term investment-grade coupon bond offering a yield to maturity of 16%. To realize that promised yield, each coupon payment had to be reinvested at a rate of interest equal to 16%. In recent years, however, yields on investment-grade bonds have been nowhere near this rate. Thus it is unlikely that a 16% yield would be realized by the investor holding the bond to maturity.

The yield to call also takes into account all three potential sources of return. In this case, the assumption is that the coupon payments can be reinvested at the yield to call. This yield measure therefore suffers from the same drawback concerning the reinvestment rate of the coupon interest payments.

COMPUTATION OF THE INTEREST-ON-INTEREST COMPONENT OF A BOND'S DOLLAR RETURN

The interest-on-interest component can represent a substantial portion of a bond's potential return. In this section we explain how to determine the contribution of the interest-on-interest component.

The portion of the potential total dollar return from coupon interest and interest on interest can be computed by using the formula for the future value of an annuity (see Chapter 2):

$$FV = A\left[\frac{(1+i)^n - 1}{i}\right],$$

where

A = amount of the annuity ($);
i = annual interest rate divided by m (in decimal form);
m = number of payments per year;
n = number of periods.

For purposes of computing the coupon interest and interest on interest for a bond paying interest semiannually,

A = semiannual coupon interest ($);

i = semiannual reinvestment rate (in decimal form);

m = 2;

n = double the number of years to maturity.

Letting

c = *semiannual* coupon interest ($)

and

r = *semiannual* reinvestment rate (in decimal form),

we can rewrite the formula for the future value of an annuity for computing the coupon interest plus interest on interest as

$$\text{Coupon interest plus interest on interest} = c\left[\frac{(1+r)^n - 1}{r}\right].$$

The total coupon interest is found by multiplying the semiannual coupon interest by the number of periods; that is,

$$\text{Total coupon interest} = n \times c.$$

The interest-on-interest component is then the difference between the coupon interest plus interest on interest and the total coupon interest. Mathematically, this can be expressed as

$$\text{Interest on interest} = c\left[\frac{(1+r)^n - 1}{r}\right] - n \times c.$$

The reinvestment rate assumed for the yield to maturity is the yield to maturity.

The three illustrations that follow demonstrate the application of these formulas, as well as the importance of the interest-on-interest component as a source of potential return from a bond.

Illustration 8–1. Suppose that an investor is considering purchasing a 7-year bond selling at par ($1,000) and having a coupon rate of 9%. Because this bond is selling at par, the yield to maturity is 9%.

Remember that a yield to maturity of 9% as conventionally computed means a 4.5% semiannual yield.[1] If an investor is promised a yield of 4.5% for fourteen 6-month periods (7 years) on a $1,000 investment, the amount at the end of fourteen 6-month periods would be[2]

$$\$1{,}000\,(1.045)^{14} = \$1{,}852.$$

Because the investment is $1,000, the total dollar return that the investor expects is $852.

Let's look at the total dollar return from holding this bond to maturity. The total dollar return comes from two sources:

1. Coupon interest of $45 every 6 months for 7 years;
2. Interest earned from reinvesting the semiannual coupon interest payments at 4.5%.

Because the bond is selling at par, no capital gain will be realized by holding the bond to maturity.

1. See Chapter 6.
2. This is just an application of the future value of $1 formula; see Chapter 2.

For this bond,

$c = \$45$; $m = 2$; $r = 0.045$ (= 0.09/2); $n = 14$ (= 7 × 2).

$$\text{Coupon interest plus interest on interest } = \$45\left[\frac{(1.045)^{14}-1}{0.045}\right]$$

$$= \$45\left[\frac{(1.8519)-1}{0.045}\right]$$

$$= \$45(18.9321) = \$852.$$

Notice that the total dollar return for this bond is the same as the return that we computed for an investment of $1,000 for fourteen 6-month periods at 4.5%.

The total coupon interest is 14 × $45 = $630.

The interest-on-interest component is then $222, as shown below:

$$\text{Interest on interest} = \$45\left[\frac{(1.045)^{14}-1}{0.045}\right] - 14(\$45)$$

$$= \$852 - \$630 = \$222.$$

Interest on interest as a percentage of total dollar return is therefore 26% ($222/$852),

For this 7-year, 9% coupon bond selling at par to offer a yield to maturity of 9%, 26% of this bond's total dollar return must come from reinvesting the coupon payments at a simple annual interest rate of 9%.

Illustration 8–2. Suppose that an investor is considering a 20-year, 7% coupon bond selling for $816. The yield to maturity for this bond is 9%.

First, let's consider how much the total dollar return should be for an investment of $816 for forty 6-month periods if the semiannual yield is 4.5%. The future value would be $4,746 because $\$816(1.045)^{40} = \$4{,}746$. Because the investment is $816, the total dollar return should be $3,930.

Now let's look at the $816 investment in the bond. The total dollar return from holding this bond to maturity comes from all three sources:

1. Coupon interest of $35 every 6 months for 20 years;
2. Interest earned from reinvesting the semiannual coupon interest payments at 4.5%;
3. A capital gain of $184 (= $1,000 – $816).

For this bond,

$c = \$35$; $m = 2$; $r = 0.045$ (= 0.09/2); $n = 40$ (= 20 × 2).

$$\text{Coupon interest plus interest on interest} = \$35\left[\frac{(1.045)^{40}-1}{0.045}\right]$$

$$= \$35\left[\frac{5.8164-1}{0.045}\right]$$

$$= \$35(107.031) = \$3,746.$$

The total coupon interest is 40 × \$35 = \$1,400.

The interest-on-interest component is then \$2,346, as shown by the following calculation:

$$\text{Interest on interest} = \$35\left[\frac{(1.045)^{40}-1}{0.045}\right] - 40(\$35)$$

$$= \$3,746 - \$1,400 = \$2,346.$$

The total dollar return is then

Total coupon interest	\$1,400
Interest on interest	2,346
Capital gain	184
Total	\$3,930

Once again, the total dollar return is the same as the return that would be expected from investing \$816 for forty 6-month periods at 4.5%. The percentage breakdown of the total dollar return is total coupon interest of 35%, interest on interest of 60%, and capital gain of 5%. Hence, for this bond, the interest-on-interest component must represent 60% of total dollar return if the investor is to realize a 9% yield to maturity.

Illustration 8–3. The two previous illustrations have shown the computation of the interest-on-interest component for a bond selling at par (Illustration 8–1) and a bond selling at a discount (Illustration 8–2), both with a yield to maturity of 9%. Now let's consider a bond selling at a premium with a yield to maturity of 9%. Suppose that an investor is considering a 12% coupon bond with 25 years to maturity selling for \$1,296. The yield to maturity is 9%.

The total dollar return from holding this bond to maturity is composed of

1. Coupon interest of \$60 every 6 months for 25 years;
2. Interest earned from reinvesting the semiannual coupon interest payments at 4.5%;
3. A capital loss of \$296 (= \$1,000 – \$1,296).

For this premium bond,

$c = \$60$; $m = 2$; $r = 0.045$ (= 0.09/2); $n = 50$ (= 25 × 2).

$$\text{Coupon interest plus interest on interest} = \$60\left[\frac{(1.045)^{50}-1}{0.045}\right]$$

$$= \$60\left[\frac{9.0326-1}{0.045}\right]$$

$$= \$60(178.503) = \$10,710.$$

The total coupon interest is $50 \times \$60 = \$3,000$.

The interest-on-interest component is then \$7,710, as shown below:

$$\text{Interest on interest} = \$60\left[\frac{(1.045)^{50}-1}{0.045}\right] - 50(\$60)$$

$$= \$10,710 - \$3,000 = \$7,710.$$

The total dollar return is then

Total coupon interest	\$3,000
Interest on interest	7,710
Capital loss	(296)
Total	\$10,414

The percentage breakdown of the total dollar return is total coupon interest of 29%, interest on interest of 74%, and capital loss of 3%. For this long-term bond selling at a premium, the interest on interest represents 74% of the total dollar return necessary to produce a 9% yield to maturity.

To see that the total dollar return from investing in this bond agrees with an investment of \$1,296 for fifty 6-month periods at 4.5%, the future value is \$1,296 $(1.045)^{50}$ = \$11,706. Subtracting from the future value the investment of \$1,296 gives the total dollar return of \$10,410. (The difference between \$10,414 and \$10,410 is due to rounding.)

BOND CHARACTERISTICS THAT AFFECT THE IMPORTANCE OF THE INTEREST-ON-INTEREST COMPONENT

There are two characteristics of a bond that determine the importance of the interest-on-interest component: (1) maturity and (2) coupon rate. The impact of each is as follows:

- For a given yield to maturity and a given coupon rate, the longer the maturity of a bond, the greater is the interest-on-interest component as a percentage of the total dollar return.

- For a given maturity and a given yield to maturity, the higher the coupon rate, the more dependent is the bond's total dollar return necessary to produce some yield to maturity on the reinvestment of coupon interest.

The implication of the impact of the coupon rate is that holding maturity and yield to maturity constant, premium bonds will be more dependent on the interest-on-interest component than bonds selling at par. Discount bonds will be less dependent on the interest-on-interest component than bonds selling at par. For zero-coupon bonds, none of the bond's total dollar return is made up of interest on interest. The reason, of course, is that no coupon interest is paid.

CHAPTER
9

TOTAL RETURN

In the previous chapter we explained the three potential sources of dollar return from investing in a bond. We also demonstrated the importance of the interest-on-interest component as a source of dollar return. In this chapter we present a yield measure that is more meaningful than the commonly used yield to maturity and yield to call for assessing the potential performance of a bond or a portfolio. This yield measure is called the *total return*.[1]

ANOTHER LOOK AT THE DRAWBACKS OF THE YIELD TO MATURITY AND YIELD TO CALL

Yield to Maturity

In the previous chapter we explained that the yield to maturity is a *promised* yield. The promised yield assumes that

1. The bond is held to maturity;

2. All coupon interest payments are reinvested at the yield to maturity.

We focused on the second assumption in the previous chapter, where we showed that the interest-on-interest component for a bond may constitute a substantial portion of the bond's promised total dollar return. Failure to reinvest the coupon interest payments at a rate of interest at least equal to the yield to maturity will produce a yield less than the yield to maturity. This risk is called *reinvestment risk*.

Rather than assume that the coupon interest payments are reinvested at a rate of interest equal to the yield to maturity, nothing prevents an investor from making an explicit assumption about the reinvestment rate on the basis of different expectations. The total return is a measure of yield that incorporates an explicit assumption about the reinvestment rate.

1. Bond market participants also refer to this yield measure as *horizon return* or *realized compound yield*. The term *realized compound yield* was first used by Sidney Homer and Martin Leibowitz in *Inside the Yield Book* (Prentice-Hall/New York Institute of Finance, 1972).

Let's take a careful look at the first assumption. An investor need not hold a bond until maturity. Suppose, for example, that an investor who has a 5-year investment horizon is considering four bonds:

Bond	Coupon (%)	Maturity (years)	Yield to Maturity (%)
A	5	3	9.0
B	6	20	8.6
C	11	15	9.2
D	8	5	8.0

Assuming that all four bonds are of the same credit quality, which one is the most attractive to this investor? An investor who selects bond C because it offers the highest yield to maturity is failing to recognize that the bond must be sold after 5 years at a price that will depend on the yield required in the market for 10-year, 11% coupon bonds at the time. Hence there could be a capital gain or capital loss that will make the return higher or lower than the yield to maturity promised now. Moreover, the higher coupon on bond C relative to the other three bonds means that more of this bond's return will be dependent on reinvestment of coupon interest payments.

Bond A offers the second highest yield to maturity. On the surface, it seems to be particularly attractive because it eliminates the problem in the bond C purchase of realizing a possible capital loss when the bond must be sold prior to the maturity date. Moreover, the reinvestment risk seems to be less than for the other three bonds because the coupon rate is the lowest. The investor would not be eliminating the reinvestment risk, however, because after 3 years the proceeds received at maturity must be reinvested for 2 more years. The yield that the investor will realize then will depend on the interest rates 3 years from now, when the investor must reinvest the proceeds.

Which is the best bond? The yield to maturity doesn't seem to be helping us answer this question. The answer depends on the expectations of the investor. Specifically, it depends on the interest rate at which the coupon interest payments can be reinvested until the end of the investor's investment horizon. Also, for bonds with a maturity greater than the investment horizon, it depends on the investor's expectations about interest rates at the end of the investment horizon. Consequently, any of these bonds could be the best investment vehicle on the basis of some reinvestment rate and some future interest rate at the end of the investment horizon. The total return measure takes these expectations into account.

Yield to Call

The yield to call is subject to the same problems as the yield to maturity. First, it assumes that the bond will be held until the assumed call date. Second, it assumes that the coupon interest payments will be reinvested at the yield to call. If an investor's investment horizon is shorter than the time to the assumed call date, there remains the potential of having to sell the bond below its acquisition cost. If, in contrast, the investment horizon is longer than the time to the assumed call date, there is the problem of reinvesting the proceeds when the bond is called until the end of the investment horizon. Consequently, the yield to call doesn't tell us very much. The total return, however, can accommodate the analysis of callable bonds.

COMPUTING THE TOTAL RETURN FOR A BOND HELD TO MATURITY

The idea underlying the total return is simple. The objective is to compute the total future amount that will result from an investment. The total return is the interest rate that will make the initial investment in the bond grow to the computed total future amount. To illustrate the computation, we first show how the total return is computed assuming that a bond is held to the maturity date.

For an assumed reinvestment rate, the dollar return that will be available at maturity can be computed from the coupon interest payments and the interest-on-interest component. In the previous chapter we explained how to do this. In addition, at maturity the investor will receive the par value. The total return is then the interest rate that will make the amount invested in the bond (i.e., the current market price plus accrued interest) equal to the total future amount available at the maturity date.

More formally, the steps for computing the total return for a bond *held to maturity* are:

Step 1. Compute the total future amount that will be received from the investment. This is the sum of the amount that will be received from the coupon payments, interest on interest based on the coupon payments and the assumed reinvestment rate, and the par value. The coupon payments plus the interest on interest can be computed by using the formula presented in the previous chapter:

$$\text{Coupon interest plus interest on interest} = c\left[\frac{(1+r)^n - 1}{r}\right],$$

where

c = semiannual coupon interest ($);
r = semiannual reinvestment rate (in decimal form);
n = number of periods to maturity.

In the case of the yield to maturity, the reinvestment rate is assumed to be one-half the yield to maturity. In computing the total return, the reinvestment rate is set equal to one-half the simple annual interest rate that the investor assumes can be earned by reinvesting the coupon interest payments.

Notice that the total future amount is different from the total dollar return that we have used in showing the importance of the interest-on-interest component in the previous chapter. The total dollar return includes only the capital gain (or capital loss if there is one), not the entire par value that is used to compute the total future amount. That is,

$$\text{Total dollar return} = \text{total future amount} - \text{price of bond}.$$

Step 2. To obtain the semiannual total return, use the following formula:

$$\left[\frac{\text{Total future amount}}{\text{Price of bond}}\right]^{1/n} - 1.$$

This formula is an application of the yield discussed in Chapter 3 for an investment that has only one cash flow.

Step 3. Because interest is assumed to be paid semiannually, double the interest rate found in Step 2. The resulting interest rate is the total return on a bond-equivalent basis. The total return can also be computed by compounding the semiannual rate as follows:

$$(1 + \text{semiannual total return})^2 - 1.$$

The total return calculated in this manner is said to be on an *effective rate basis*.

The two illustrations that follow show how to compute the total return for a bond held to maturity.

Illustration 9–1. The total return for a 7-year, 9% coupon bond selling at par assuming a reinvestment rate of 5% (simple annual interest rate) is 8.1%, as shown in Exhibit 9–1. Because this bond is selling at par, its yield to maturity is 9%, yet its total return is only 8.1%. The total return is less than the yield to maturity because the coupon interest payments are assumed to be reinvested at 5% rather than 9%.

Illustration 9–2. The computation of the total return for the bond in the previous illustration assuming a reinvestment rate of 9% is shown in Exhibit 9–2. The total return is 9%. In this case, the total return is equal to the yield to maturity because the coupon interest payments are assumed to be reinvested at 9%.

EXHIBIT 9–1

Computation of the Total Return for a 7-Year, 9% Bond Selling at Par and Held to Maturity: Reinvestment Rate = 5%

Step 1 The total future amount from this bond includes

1. Coupon interest of \$45 every 6 months for 7 years;
2. Interest earned from reinvesting the semiannual coupon interest payments at 2.5% (one-half the assumed annual reinvestment rate);
3. The par value of \$1,000.

The coupon interest plus interest on interest can be found as follows:

$c = \$45$; $m = 2$; $r = 0.025$ (= 0.05/2); $n = 14$ (= 7 × 2).

$$
\begin{aligned}
\text{Coupon interest plus interest on interest} &= \$45\left[\frac{(1.025)^{14}-1}{0.025}\right] \\
&= \$45\left[\frac{1.4130-1}{0.025}\right] \\
&= \$45(16.5189) \\
&= \$743.35.
\end{aligned}
$$

Total future amount:

Coupon interest plus interest on interest	\$743.35
Par value	\$1,000.00
Total	\$1,743.35

Step 2 Compute the following:

$$
\left[\frac{\$1{,}743.35}{\$1{,}000.00}\right]^{1/14} - 1
$$

$$
= (1.74335)^{0.07143} - 1 = 0.0405 \text{ or } 4.05\%.
$$

Step 3 Doubling 4.05% gives a total return of 8.1%.

EXHIBIT 9–2

Computation of the Total Return for a 7-Year, 9% Bond Selling at Par and Held to Maturity: Reinvestment Rate = 9%

Step 1 The total future amount for this bond includes

1. Coupon interest of $45 every 6 months for 7 years;
2. Interest earned from reinvesting the semiannual coupon interest payments at 4.5% (one-half the assumed annual reinvestment rate);
3. The par value of $1,000.

The coupon interest plus interest on interest can be found as follows:

c = $45; $m = 2$; $r = 0.045$ (= 0.09/2); $n = 14$ (= 7 × 2).

$$\text{Coupon interest plus interest on interest} = \$45\left[\frac{(1.045)^{14}-1}{0.045}\right]$$

$$= \$45\left[\frac{1.8519-1}{0.045}\right]$$

$$= \$45(18.9321)$$

$$= \$851.94.$$

Total future amount:

Coupon interest plus interest on interest	$851.94
Par value	$1,000.00
Total	$1,851.94

Step 2 Compute the following:

$$\left[\frac{\$1{,}851.94}{\$1{,}000.00}\right]^{1/14}-1$$

$$= (1.85194)^{0.07143}-1 = 0.0450 \text{ or } 4.50\%.$$

Step 3 Doubling 4.50% gives a total return of 9.0%.

COMPUTING THE TOTAL RETURN FOR A BOND TO BE SOLD PRIOR TO MATURITY

The total return as calculated above suffers from the same problem as the yield to maturity in that it assumes holding the bond until maturity. Fortunately, it is quite simple to modify the calculation of the total return to determine the potential yield from holding a bond until the end of a predetermined investment horizon. We need to make only one adjustment to the three steps given above to calculate the total return.

In Step 1, the total future amount is calculated on the basis of (1) the coupon interest payments until the end of the investment horizon, (2) the interest on interest from reinvesting the coupon interest payments until the end of the investment horizon, and (3) the expected price of the bond at the end of the investment horizon.

How does one know the expected price of the bond at the end of the investment horizon? The expected price depends on the investor's expectations about what interest rates will be at the end of the investment horizon. Given the expected yield that the bond will be selling for at the end of the investment horizon and the remaining time to maturity of the bond, the expected price can easily be determined from the formula for the price of a bond given in Chapter 5.

In calculating the price, it is important to remember that n is the number of periods remaining to maturity *at the end of the investment horizon*. For example, if the investment horizon is 5 years and the number of years to maturity at the present time is 20 years, there would remain 15 years to maturity at the end of the investment horizon. In this case, n would be 30.

Illustration 9–3. Suppose that an investor with a 5-year investment horizon is considering purchasing a 7-year, 9% coupon bond selling at par. The investor expects that the coupon interest payments can be reinvested at an annual interest rate of 9.4% and that at the end of the investment horizon 2-year bonds will be selling to offer a yield to maturity of 11.2%. As shown in Exhibit 9–3, the total return for this bond is 8.54%. The yield to maturity for this bond is 9%.

EXHIBIT 9–3

Computation of the Total Return for a 7-Year, 9% Bond Selling at Par and Held for 5 Years: Reinvestment Rate = 9.4%

Step 1 The total future amount from this bond includes

1. Coupon interest of $45 every 6 months for 5 years (the investment horizon);
2. Interest earned from reinvesting the semiannual coupon interest payments at 4.7% (one-half the assumed annual reinvestment rate) until the end of the investment horizon;
3. The expected price of the bond at the end of 5 years.

The coupon interest plus interest on interest can be found as follows:

$c = \$45$; $m = 2$; $r = 0.047$ $(= 0.094/2)$; $n = 10$ $(= 5 \times 2)$.

$$\text{Coupon interest plus interest on interest} = \$45\left[\frac{(1.047)^{10} - 1}{0.047}\right]$$

$$= \$45\left[\frac{1.5830 - 1}{0.047}\right]$$

$$= \$45(12.4032)$$

$$= \$588.14.$$

EXHIBIT 9–3

(Continued)

The expected price of the bond 5 years from now is determined as follows:

(a) Present value of coupon interest payments, assuming that the required yield to maturity at the end of the investment horizon is 11.2%:

c = \$45; i = 0.056 (= 0.112/2); n = 4 (= remaining years to maturity × 2).

$$PV = \$45\left[\frac{1-\left[\frac{1}{(1.056)^4}\right]}{0.056}\right]$$

$$= \$45\left[\frac{1-0.8042}{0.056}\right]$$

$$= \$45(3.4964) = \$157.34.$$

(b) Present value of the maturity value:

$$\frac{\$1{,}000}{(1.056)^4} = \$804.16.$$

$$\text{Expected price} = \$157.34 + \$804.16 = \$961.50.$$

Total future amount:

Coupon interest plus interest on interest	\$558.14
Expected price	\$961.50
Total	\$1,519.64

Step 2 Compute the following:

$$\left[\frac{\$1{,}519.64}{\$1{,}000.00}\right]^{1/10} - 1$$

$$= (1.51964)^{0.10} - 1 = 0.0427 \text{ or } 4.27\%.$$

Step 3 Doubling 4.27% gives a total return of 8.54%.

ANALYZING CALLABLE BONDS WITH THE TOTAL RETURN

The total return can be used to assess the potential return from holding a callable bond. If the first call date falls within the investment horizon, the proceeds from the coupon interest payments plus the interest on interest and the proceeds from calling the issue (the call price) will be available at the first call date. These proceeds can then be reinvested until the end of the investment horizon. Given the total future amount at the end of the investment horizon, the total return can then be computed. The next illustration shows how this is done.

Illustration 9–4. An investor is considering bond C, which we presented earlier in this chapter: 11% coupon, 15 years to maturity, and 9.2% yield to maturity. The price for this bond is $1,144.88. Suppose that this bond is callable in 3 years at $1,055. The yield to call for this bond is 7.48%. Because the yield to call is less than the yield to maturity, this investor might use the 7.48% to assess the relative attractiveness of this bond. However, suppose that this investor's investment horizon is 5 years (a period extending beyond the first call date but shorter than the maturity of the bond). Also assume that this investor believes that any proceeds can be reinvested at a 6% annual interest rate (3% semiannually).

Exhibit 9–4 shows the computation of the total return on the assumption that the bond is called in 3 years. The total return is 6.66%. This investor's reinvestment assumption and the investment horizon make this the appropriate yield to use in assessing the attractiveness of this bond compared with other bonds that are purchase candidates.

EXHIBIT 9–4

Computation of the Total Return for a 15-Year, 11% Bond Selling for $1,144.88, Callable in 3 Years at $1,055 and Held for 3 Years: Reinvestment Rate = 6%

Step 1 The total future amount from this bond includes:

1. Coupon interest of $55 every 6 months for 3 years (to the first call date);
2. Interest earned from reinvesting the first 6 semiannual coupon interest payments at 3% (one-half the assumed annual reinvestment rate) until the first call date;
3. Proceeds from reinvesting the call price plus (1) and (2) above for 2 years at 3% until the end of the investment horizon.

EXHIBIT 9–4

(Continued)

The coupon interest plus interest on interest for the first six coupon payments can be found as follows:

$c = \$55;\ m = 2;\ r = 0.03\ (= 0.06/2);\ n = 6.$

$$\text{Coupon interest plus interest on interest} = \$55\left[\frac{(1.03)^6 - 1}{0.03}\right]$$
$$= \$55\left[\frac{1.1941 - 1}{0.03}\right]$$
$$= \$55(6.4683)$$
$$= \$355.76.$$

This gives the coupon interest plus interest on interest as of the end of the third year (six periods) when the bond is called. Adding the call price of $1,055 gives the total proceeds that must be reinvested at 3% until the end of the investment horizon, for 2 years or 4 periods.

$$\text{Proceeds to be reinvested} = \$355.76 + \$1{,}055$$
$$= \$1{,}410.76.$$

Reinvesting $1,410.76 for 4 periods at 3%:

$$\$1{,}410.76(1.03)4 = \$1{,}587.82.$$

Therefore, the total future amount is $1,587.82.

Step 2 Compute the following:

$$\left[\frac{\$1{,}587.82}{\$1{,}144.88}\right]^{1/10}$$
$$= (1.38689)^{0.10} - 1$$
$$= 1.0304 - 1 = 0.0333 \text{ or } 3.33\%.$$

Step 3 Doubling 3.33% gives a total return of 6.66%.

SCENARIO ANALYSIS

Because the total return depends on the reinvestment rate and the yield at the end of the investment horizon, portfolio managers assess performance over a wide range of scenarios for these two variables. This approach is referred to as *scenario analysis*.

Illustration 9–5. Suppose that a portfolio manager is considering the purchase of bond A, a 20-year, 9% noncallable bond selling at $109.896 per $100 of par value. The yield to maturity for this bond is 8%. Assume also that the portfolio manager's investment horizon is 3 years and that the portfolio manager believes that the reinvestment rate range can vary from 3% to 6.5% and the yield at the end of the investment horizon from 5% to 12%.

The top panel of Exhibit 9–5 shows the total future dollars at the end of 3 years under various scenarios. The bottom panel shows the total return (based on the effective annualizing of the 6-month total return). The portfolio manager knows that the maximum and minimum total return will be 14.64% and –0.2%, respectively, and the scenarios under which each will be realized. If the portfolio manager faces 3-year liabilities guaranteeing, say, 6%, the major consideration is scenarios that will produce a 3-year total return of less than 6%. These scenarios can be determined from Exhibit 9–5.

EXHIBIT 9–5

Scenario Analysis for Bond A

Bond A = 9% coupon, 20-year noncallable bond;
Price = $109.896;
Yield to maturity = 8.00%;
Investment horizon = 3 years.

	Yield at End of Horizon							
	5.00%	**6.00%**	**7.00%**	**8.00%**	**9.00%**	**10.00%**	**11.00%**	**12.00%**
	Horizon Price							
	145.448	**131.698**	**119.701**	**109.206**	**100.000**	**91.9035**	**84.763**	**78.4478**
Reinv. Rate (%)	*Total Future Dollars*							
	5.00%	**6.00%**	**7.00%**	**8.00%**	**9.00%**	**10.00%**	**11.00%**	**12.00%**
3.0	173.481	159.731	147.734	137.239	128.033	119.937	112.796	106.481
3.5	173.657	159.907	147.910	137.415	128.209	120.113	112.972	106.657
4.0	173.834	160.084	148.087	137.592	128.387	120.290	113.150	106.834
4.5	174.013	160.263	148.266	137.771	128.565	120.469	113.328	107.013
5.0	174.192	160.443	148.445	137.950	128.745	120.648	113.508	107.193
5.5	174.373	160.623	148.626	138.131	128.926	120.829	113.689	107.374
6.0	174.555	160.806	148.809	138.313	129.108	121.011	113.871	107.556
6.5	174.739	160.989	148.992	138.497	129.291	121.195	114.054	107.739
Reinv. Rate (%)	*Total Return (Effective Rate)*							
	5.00%	**6.00%**	**7.00%**	**8.00%**	**9.00%**	**10.00%**	**11.00%**	**12.00%**
3.0	16.44	13.28	10.37	7.69	5.22	2.96	0.87	–1.05
3.5	16.48	13.32	10.41	7.73	5.27	3.01	0.92	–0.99
4.0	16.52	13.36	10.45	7.78	5.32	3.06	0.98	–0.94
4.5	16.56	13.40	10.50	7.83	5.37	3.11	1.03	–0.88
5.0	16.60	13.44	10.54	7.87	5.42	3.16	1.08	–0.83
5.5	16.64	13.49	10.59	7.92	5.47	3.21	1.14	–0.77
6.0	16.68	13.53	10.63	7.97	5.52	3.26	1.19	–0.72
6.5	16.72	13.57	10.68	8.02	5.57	3.32	1.25	–0.66

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CHAPTER
10

HISTORICAL RETURN MEASURES

In the previous two chapters we looked at the potential dollar sources of return from investing and the potential total return. Our focus in this chapter is measurement of the historical performance of a bond portfolio. Performance measurement involves the calculation of the total return realized by a portfolio manager over some evaluation period.

PORTFOLIO PERIOD RETURN

Let's begin with a basic concept. The dollar return realized on a portfolio for any evaluation period (i.e., a year, a month, or a week) is equal to the sum of

1. The difference between the market value of the portfolio at the end of the evaluation period and the market value at the beginning of the evaluation period, and
2. Any distributions made from the portfolio.

It is important that any capital or income distributions from the portfolio to a client or beneficiary of the portfolio be included.

The *rate of return*, or simply *return*, expresses the dollar return in terms of the amount of the market value at the beginning of the evaluation period. Thus the return can be viewed as the amount (expressed as a fraction of the initial portfolio value) that can be withdrawn at the end of the evaluation period while maintaining the initial market value of the portfolio intact.

In equation form, the *portfolio period return* realized can be expressed as follows:

$$R_p = \frac{V_1 - V_0 + D}{V_0},$$

where

R_p = portfolio period return;
V_1 = portfolio market value at the end of the evaluation period;
V_0 = portfolio market value at the beginning of the evaluation period;
D = cash distributions from the portfolio to the client during the evaluation period.

Illustration 10–1. Assume the following information for an external manager for a pension plan sponsor: the portfolio's market values at the beginning and end of the evaluation period are \$100 million and \$112 million, respectively, and during the evaluation period, \$5 million is distributed to the plan sponsor from investment income. Thus

$$V_1 = \$112{,}000{,}000;\ V_0 = \$100{,}000{,}000;\ D = \$5{,}000{,}000.$$

$$R_p = \frac{\$112{,}000{,}000 - \$100{,}000{,}000 + \$5{,}000{,}000}{\$100{,}000{,}000}$$

$$= 0.17 = 17\%.$$

Assumptions in Calculating the Return

There are three assumptions in calculating the portfolio period return. First, it assumes that cash inflows into the portfolio from interest income that occur during the evaluation period but are not distributed are reinvested in the portfolio. For example, suppose that during the evaluation period \$7 million is received from interest income. This amount is reflected in the market value of the portfolio at the end of the period.

The second assumption is that if there are distributions from the portfolio, they occur at the end of the evaluation period or are held in the form of cash until the end of the evaluation period. In our example, \$5 million is distributed to the plan sponsor. But when did that distribution actually occur? To understand why the timing of the distribution is important, consider two extreme cases:

Case 1. The distribution is made at the end of the evaluation period, as assumed in the portfolio period return calculation.

Case 2. The distribution is made at the beginning of the evaluation period.

In Case 1, the manager had the use of the \$5 million to invest for the entire evaluation period. By contrast, in the second case, the manager loses the opportunity to invest the funds until the end of the evaluation period. Consequently, the timing of the distribution will affect the return, but this is not considered in the portfolio period return calculation.

The third assumption is that there is no cash paid into the portfolio by the client. For example, suppose that sometime during the evaluation period the plan sponsor gives an additional \$8 million to the manager to invest. Consequently, the market value of the portfolio at the end of the evaluation period, \$112 million in our example, would reflect the contribution of \$8 million. The portfolio period return calculation does not reflect that the ending market value of the portfolio is affected by the cash paid in by the sponsor. Moreover, the timing of this cash inflow will affect the calculated return.

Thus, while the portfolio period return calculation can be determined for an evaluation period of any length of time such as 1 day, 1 month, or 5 years, from a practical point of view, the assumptions discussed above limit its application. The longer the evaluation period, the more likely it is that the assumptions will be violated. For example, it is highly likely that there may be more than one distribution to the client and more than one contribution from the client if the evaluation period is 5 years. Thus a return calculation made over a long period of time, if longer than a few months, would not be very reliable because of the assumption underlying the calculations that all cash payments and inflows are made and received at the end of the period.

Subperiod Returns

Not only does the violation of the assumptions make it difficult to compare the returns of two managers over some evaluation period, but it is also not useful for evaluating performance over different periods. For example, the portfolio period return will not give reliable information to compare the performance of a 1-month evaluation period and a 3-year evaluation period. To make such a comparison, the return must be expressed per unit of time, for example, per year.

The way to handle these practical issues is to calculate the return for a short unit of time such as a month or a quarter. We call the return so calculated the *subperiod return*. The subperiod return is calculated in the same way as the portfolio period return. To get the return for the evaluation period, the subperiod returns are then averaged. Therefore, for example, if the evaluation period is 1 year and 12 monthly returns are calculated, the monthly returns are the subperiod returns, and they are averaged to get the 1-year return. If a 3-year return is sought and 12 quarterly returns can be calculated, quarterly returns are the subperiod returns, and they are averaged to get the 3-year return. The 3-year return then can be converted into an annual return by the straightforward procedure described later.

AVERAGING SUBPERIOD RETURNS

There are three methodologies that have been used in practice to calculate the average of the subperiod returns:

- Arithmetic average rate of return;
- Time-weighted rate of return;
- Dollar-weighted return.

Arithmetic Average Rate of Return

The *arithmetic average rate of return* is an unweighted average of the subperiod returns. The general formula is

$$R_A = \frac{R_{p1} + R_{p2} + \cdots + R_{pN}}{N},$$

where

R_A = arithmetic average rate of return;
R_{Pk} = portfolio return for subperiod k, $k = 1, \ldots, N$;
N = number of subperiods in the evaluation period.

Illustration 10–2. Suppose that the portfolio subperiod returns were 12%, 25%, −15%, and −2% in months 1, 2, 3, and 4, respectively. The arithmetic average monthly return is 5%, as shown below:

$$N = 4;\ R_{P1} = 0.12;\ R_{P2} = 0.25;\ R_{P3} = -0.15;\ R_{P4} = -0.02.$$

$$R_A = \frac{0.12 + 0.25 + (-0.15) + (-0.2)}{4} = 0.05 = 5\%.$$

There is a major problem with using the arithmetic average rate of return. To see this problem, suppose that the initial market value of a portfolio is \$140 million and that the market values at the end of the next 2 months are \$280 million and \$140 million, and assume that there are no distributions or cash inflows from the client for either month. The subperiod return for the first month R_{P1} is 100%, and the subperiod return for the second month R_{P2} is –50%. The arithmetic average rate of return is then 25%. Not a bad return! But think about this number. The portfolio's initial market value was \$140 million. Its market value at the end of 2 months is \$140 million. The return over this 2-month evaluation period is zero. Yet the arithmetic average rate of return says it is a whopping 25%.

Thus it is improper to interpret the arithmetic average rate of return as a measure of the average return over an evaluation period. The proper interpretation is as follows: *it is the average value of the withdrawals (expressed as a fraction of the initial portfolio market value) that can be made at the end of each subperiod while keeping the initial portfolio market value intact*. In Illustration 10–2, in which the average monthly return is 5%, the investor can withdraw 12% of the initial portfolio market value at the end of the first month, can withdraw 25% of the initial portfolio market value at the end of the second month, must add 15% of the initial portfolio market value at the end of the third month, and must add 2% of the initial portfolio market value at the end of the fourth month. In our example of the limitation of the average arithmetic return, the average monthly return of 25% means that 100% of the initial portfolio market value (\$140 million) can be withdrawn at the end of the first month and that 50% must be added at the end of the second month.

Time-Weighted Rate of Return

The *time-weighted rate of return* measures the compounded rate of growth of the initial portfolio market value during the evaluation period. It is also commonly referred to as the *geometric rate of return* because it is computed by taking the geometric average of the portfolio subperiod returns. The general formula is

$$R_T = \left[\left(1+R_{p1}\right)\left(1+R_{p2}\right)\cdots\left(1+R_{pN}\right)\right]^{1/N} - 1;$$

where

R_T = time-weighted rate of return;
R_{Pk} and N are as defined earlier.

Illustration 10–3. Assume that the portfolio returns were 12%, 25%, −15%, and −2% in month 1, month 2, and month 3, as in Illustration 10–2. Then the time-weighted rate of return is

$$\begin{aligned} R_T &= \left[(1+0.12)(1+0.25)\left(1+(-0.15)\right)\left(1+(-0.02)\right)\right]^{1/4} - 1 \\ &= \left[(1.12)(1.25)(0.85)(0.98)\right]^{1/4} - 1 = 0.0392 = 3.92\%. \end{aligned}$$

Because the time-weighted rate of return is 3.92% per month, $1 invested in the portfolio at the beginning of month 1 would have grown at a rate of 3.92% per month during the 4-month evaluation period.

The time-weighted rate of return in the example where the first month's return is 100% followed by a return of 50% in the second month is 0%, as expected, as shown below:

$$\begin{aligned} R_T &= \left[(1+1.00)\left(1+(-0.50)\right)\right]^{1/2} - 1 \\ &= \left[(2.00)(0.50)\right]^{1/2} - 1 = 0\%. \end{aligned}$$

In general, the arithmetic and time-weighted average returns will give different values for the portfolio return over some evaluation period. This is because in computing the arithmetic average rate of return, the amount invested is assumed to be maintained (through additions or withdrawals) at its initial portfolio market value. The time-weighted return, in contrast, is the return on a portfolio that varies in size because of the assumption that all proceeds are reinvested.

In general, the arithmetic average rate of return will exceed the time-weighted average rate of return. The exception is in the special situation where all the subperiod returns are the same, in which case the averages are identical. The magnitude of the difference between the two averages is smaller the less the variation in the subperiod returns over the evaluation period. For example, suppose that the evaluation period is 4 months and that the 4 monthly returns are as follows:

$$R_{P1} = 0.04, \quad R_{P2} = 0.06, \quad R_{P3} = 0.02 \quad \text{and} \quad R_{P4} = -0.02.$$

The average arithmetic rate of return is 2.5%, and the time-weighted average rate of return is 2.46%—not much of a difference. In our earlier example in which we calculated an average rate of return of 25% but a time-weighted average rate of return of 0%, the large discrepancy is due to the substantial variation in the two monthly returns.

Dollar-Weighted Rate of Return

The *dollar-weighted rate of return*, also referred to as the *money-weighted return*, is computed by finding the interest rate that will make the present value of the cash flows from all the subperiods in the evaluation period plus the terminal market value of the portfolio equal to the initial market value of the portfolio. The cash flow for each subperiod reflects the difference between the cash inflows due to investment income (i.e., coupon interest) and the contribution(s) made by the client to the portfolio and the cash outflows reflecting distributions to the client. Notice that it is not necessary to know the market value of the portfolio for each subperiod to determine the dollar-weighted rate of return.

The dollar-weighted rate of return is simply an internal rate of return calculation; hence it is also called the *internal rate of return*. The general formula for the dollar-weighted return is

$$V_0 = \frac{C_1}{(1+R_D)} + \frac{C_2}{(1+R_D)^2} + \cdots + \frac{C_N + V_N}{(1+R_D)^n},$$

where

R_D = dollar-weighted rate of return;
V_0 = initial market value of the portfolio;
V_N = terminal market value of the portfolio;
C_k = cash flow for the portfolio (cash inflows minus cash outflows) for subperiod k, with $k = 1, \ldots, N$.

Illustration 10–4. Consider a portfolio with a market value of $100,000 at the beginning of month 1; capital withdrawals of $5,000 at the end of months 1, 2, and 3; no cash inflows from the client in any month; and a market value at the end of month 3 of $110,000. Then

$$V_0 = \$100{,}000;\ N = 3;\ C_1,\ C_2, \text{ and } C_3 = \$5{,}000;\ V_3 = \$110{,}000.$$

R_D is the interest rate that satisfies the following equation:

$$\$100{,}000 = \frac{\$5{,}000}{(1+R_D)} + \frac{\$5{,}000}{(1+R_D)^2} + \frac{\$5{,}000 + \$110{,}000}{(1+R_D)^3}$$

It can be verified that the interest rate that satisfies this expression is 8.1%. This, then, is the dollar-weighted return.

The dollar-weighted rate of return and the time-weighted rate of return will produce the same result if no withdrawals or contributions occur over the evaluation period and all investment income is reinvested. The problem with the dollar-weighted rate of return is that it is affected by factors that are beyond the control of the manager. Specifically, any contributions made by the client or withdrawals that the client requires will affect the calculated return. This makes it difficult to compare the performance of two managers.

To see this, suppose that a pension plan sponsor engaged two managers, A and B, with \$10 million given to A to manage and \$200 million given to B. Suppose that (1) both managers invest in identical portfolios (i.e., the two portfolios have the same securities and are held in the same proportion), (2) for the following 2 months the rate of return on the two portfolios is 20% for month 1 and 50% for month 2, and (3) the amount received in investment income is in cash. Also assume that the plan sponsor does not make an additional contribution to the portfolio of either manager. Under these assumptions, it is clear that the performance of both managers would be identical. Suppose, however, that the plan sponsor withdraws \$4 million from the amount given to manager A at the beginning of month 2. This means that manager A could not invest the entire amount at the end of month 1 and capture the 50% increase in the portfolio value. Manager A's net cash flow would be as follows: (1) in month 1 the net cash flow would be \$2 million because \$2 million is realized in investment income and \$4 million is withdrawn by the plan sponsor. The dollar-weighted rate of return is then calculated as follows:

$$\$10 = \frac{-\$2}{\left(1+R_D\right)} + \frac{\$12}{\left(1+R_D\right)^2} \Rightarrow R_D = 0\%.$$

For manager B, the cash inflow for month 1 is \$40 million (\$200 million × 20%), and the portfolio value at the end of month 2 is \$360 million (\$240 million × 1.5). The dollar-weighted rate of return is

$$\$200 = \frac{\$40}{\left(1+R_D\right)} + \frac{\$330}{\left(1+R_D\right)^2} \Rightarrow R_D = 38.8\%.$$

These are quite different results for two managers we agreed had identical performance. The withdrawal by the plan sponsor and the size of the withdrawal relative to the portfolio value had a significant effect on the calculated return. Notice also that even if the plan sponsor had withdrawn \$4 million from the amount given to manager B at the beginning of month 2, this would not have had as significant an impact. The problem also would have occurred if we assumed that the return in month 2 is 50% and that instead of manager A realizing a withdrawal of \$4 million, the plan sponsor contributed \$4 million.

Despite this limitation, the dollar-weighted rate of return does provide information. It indicates information about the growth of the fund that a client will find useful. This growth, however, may not be attributable to the performance of the manager because of contributions and withdrawals.

ANNUALIZING RETURNS

The evaluation period may be less than or greater than 1 year. Typically, return measures are reported as an average annual return. This requires annualization of the subperiod returns. The subperiod returns are typically calculated for a period of less than 1 year for the reasons explained earlier. The subperiod returns are then annualized using the following formula:

$$\text{Annual return} = (1 + \text{average period return})^{\text{number of periods in year}} - 1.$$

Thus, for example, suppose that the evaluation period is 3 years and a monthly period return is calculated. Suppose further that the average monthly return is 2%. Then the annual return would be

$$\text{Annual return} = (1.02)^{12} - 1 = 26.8\%.$$

TRAILING RETURNS AND ROLLING RETURNS

A historical return can be calculated over any length of time (e.g., year to date, 1 year, 3 years, 5 years, or 10 years). Historical returns calculated for any length of time are called *trailing returns* or *point-to-point returns*. A problem with trailing returns is that the consistency with which the portfolio returns were realized is hidden because of the averaging process. That is, periods where performance was extremely poor can be offset by a period where performance was extremely good.

One way of identifying the variation in returns is to calculate some measure of return volatility, which we will discuss in Chapter 11. Another way to get some idea of the variation in returns is by calculating *rolling returns*. Rolling returns are used to study the return behavior of a fund or an investment strategy.

The calculation of rolling returns involves several steps:

Step 1. Determine the starting date, ending date, and the interval over which the rolling returns are to be calculated. For example, a 3-year rolling return can be calculated from April 1, 2016, to March 31, 2022, using monthly returns.

Step 2. Calculate the first average annual rolling return using the starting date in Step 1 and as the ending date the length of the time interval specified in Step 1. For example, for the first 3-year rolling return, the starting date is April 1, 2016, and 36 monthly returns are used. Therefore the ending date for the first 3-year rolling return would be March 31, 2019. So the first 3-year rolling return would be computed using monthly returns from April 1, 2016, to March 31, 2019.

Step 3. Eliminate the beginning date used in Step 2 and calculate the 3-year rolling return by increasing the ending date to obtain a 3-year time period. In our example, the starting date for the second 3-year rolling return would be May 1, 2016, and the ending date would be April 30, 2019.

Step 3 is then repeated to compute the 3-year average returns from the beginning to the ending dates.

CONTRIBUTION TO PORTFOLIO RETURN

The contribution to the portfolio return of each bond or each sector of the bond, or characteristics of a bond such as credit rating, can be calculated and provides information to the portfolio manager and client as to the sources of portfolio return. The formula is

Contribution of a bond holding to portfolio return
= weight of a bond in the portfolio × bond return.

A sector contribution to portfolio return is computed as follows:

Sector weight in the portfolio × sector return.

Illustration 10–5. Suppose that an \$800 million hypothetical bond fund's 1-year return was 6.69%. Exhibit 10–1 shows the portfolio's allocation to the bond sectors (shown in the second column) and the return for each sector (shown in the fourth column). The last column, which is the product of the percent allocated to a sector and the sector return, shows the contribution to portfolio return from each sector. As can be seen, the largest contributor to the hypothetical fund's 6.69% was generated from the agency mortgage-backed securities sector.

EXHIBIT 10–1

Illustration of the Calculation of Sector Contribution to Portfolio Return

Sector	Market Value ($ millions)	Allocation (%)	Sector Return (%)	Contribution to Portfolio Return (%)
U.S. Treasury Obligations	95	11.9	4.9	0.58
Investment-Grade Corporate Bonds	120	15.0	7.5	1.13
High-Yield Corporate Bonds	20	2.5	8.7	0.22
Non-U.S. Sovereign Bonds	80	10.0	9.1	0.91
Agency Mortgage-Backed Securities	270	33.8	9.8	3.31
Private Label Mortgage-Backed Securities	90	11.3	–1.6	–0.18
Asset-Backed Securities	85	10.6	6.4	0.68
Short-Term Instruments	40	5.0	0.9	0.05
Total	**800**	**100**		**6.69**

ABSOLUTE VERSUS RELATIVE RETURNS

A computed return can be viewed in isolation or relative to some other return. When it is viewed in isolation, it is referred to as an *absolute return.* The problem with looking at an absolute return is that it does not tell us much about a portfolio manager's performance. For example, is an 8% return a good return that indicates that the portfolio manager performed well? Of course, it does not. That assessment of the performance of the portfolio manager would depend not only on the performance relative to some target or reference portfolio but also on the risk that was incurred in order to generate the return. Basically, we want to be able to evaluate performance of a portfolio manager; an absolute return does not say much about that. In fact, in Chapter 11 when we describe basic return/risk performance measures—which are ratios in which some return measure is used in the numerator and some risk measure is used in the denominator—one of these measures is used in the numerator of the calculation.

Therefore, the performance of portfolio managers is evaluated based on relative performance. That is, it is the *relative return* and comes in the following forms: excess return, active return, abnormal return, and residual return.

Excess return is the general term for relative return that measures the difference between the actual portfolio return and some client-mandated reference return. That is,

Excess return = portfolio return – return on some target or reference portfolio.

When the return on some target or reference portfolio is the portfolio manager's benchmark, then the resulting excess return is called the *active return*. That is,

Active return = portfolio return – benchmark return.

Sometimes, in practice, the terms *active return* and *excess return* are used interchangeably despite the fact that the excess return can have as its target return or reference portfolio something other than the benchmark return. In practice, the active return is often referred to as *alpha* and reflects the value added over the benchmark. However, this is not technically correct. The proper definition of alpha is that it is the risk-adjusted return over a benchmark. The risk adjustment is based on market risk. That is,

Alpha = risk-adjusted active return
= portfolio return – risk-adjusted benchmark return.

In the equity market, the market adjustment is based on the portfolio's beta. Similar type measures can be used in the bond market. Therefore, despite the fact that alpha is technically not equivalent to the active return, we use the two terms interchangeably in this book.

Another excess return measure uses as the reference return the rate on a risk-free asset. In the U.S. market, the risk-free asset is some Treasury rate. The maturity of the Treasury security depends on the investment period. For example, if performance is measured on a monthly basis, the 1-month Treasury bill is used.

The *abnormal return* is an excess return measure that uses the target return or expected return. When regression analysis is used to calculate the expected (or predicted) return, the difference between the actual portfolio return and the expected return is referred to as the *residual return***.**

CFA INSTITUTE PERFORMANCE PRESENTATION STANDARDS

As just demonstrated, there are subtle issues in calculating the return over the evaluation period. There are also industry concerns as to how managers should present results to clients and how managers should disclose performance data and records to prospects from whom they are seeking funds to manage.

In 1993, the Committee for Performance Presentation Standards (CPPS) of the Association for Investment Management and Research (AIMR), now the CFA Institute, adopted standards that represent "a set of guiding ethical principles intended to promote full disclosure and fair representation by investment managers in reporting their investment results."[1] A secondary objective of the standards was to ensure uniformity in the presentation of results so that it is easier for clients to compare the performance of managers. The standards set forth how returns should be calculated.

In our illustrations of the various ways to measure portfolio return, we used the same length of time for the subperiod (e.g., a month or a quarter). The subperiod returns were averaged, with the preferred method being geometric averaging. The presentation standards require that the return measure minimize the effect of contributions and withdrawals so that cash flows beyond the control of the manager are minimized. If the subperiod return is calculated daily, the impact of contributions and withdrawals will be minimized. The time-weighted return measure then can be calculated from the daily returns.

From a practical point of view, the problem is that calculating a daily return requires that the market value of the portfolio be determined at the end of each day. While this does not present a problem for a mutual fund that must calculate the net asset value of its portfolio each business day, it is a time-consuming administrative problem for other managers. Moreover, there are fixed-income sectors in which the determination of daily prices would be difficult (e.g., esoteric derivative mortgage-backed securities and structured notes).

An alternative to the time-weighted rate of return has been suggested. This is the dollar-weighted rate of return, which as we noted earlier is less desirable in comparing the performance of managers because of the effect of withdrawals and contributions beyond the control of the manager. The advantage of this method from an operational perspective is that market values do not have to be calculated daily. The effect of withdrawals and contributions is minimized if they are small

1. *Performance Presentation Standards: 1993* (Charlottesville, VA: Association for Investment Management and Research, 1993).

relative to the length of the subperiod. However, if the cash flow is more than 10% at any time, the presentation standards require that the portfolio be revalued on that date.[2]

Once the subperiod returns in an evaluation period are calculated, they are compounded. The presentation standards specify that for evaluation periods of less than 1 year, returns should *not* be annualized. Thus, if the evaluation period is 7 months and the subperiod returns are calculated monthly, the 7-month return should be reported by calculating the compounded 7-month return instead.

2. For a further discussion of the implementation of the AIMR Standards, see Deborah H. Miller, "How to Calculate the Numbers According to the Standards," in *Performance Reporting for Investment Managers: Applying the AIMR Performance Presentation Standards* (Charlottesville, VA: Association for Investment Management and Research, 1991).

CHAPTER

11

RISK-ADJUSTED RETURNS/ REWARD-RISK RATIOS

The measurement of historical performance by examining returns (absolute or relative returns) is not sufficient for assessing the performance of a bond portfolio manager, investment strategy, or sectors of the bond market. Performance evaluation must consider the risks to which the portfolio or investment strategy was exposed in generating the return. In this chapter we describe measures that look at risk-adjusted returns for assessing performance and potential performance.[1] In the case of potential performance, they are used in conjunction with backtesting of strategies.

For these single metric performance measures, the numerator is the reward that can be measured on an absolute or relative basis. Risk-adjusted return measures that use *absolute* rewards in the numerator are measured as the difference between the realized return and the risk-free rate or zero. When the reward is measured on a *relative* basis, the numerator of the risk-adjusted return is the difference between the realized return and some benchmark return. In the denominator of these measures is some measure of risk. The three risk-adjusted measures we discuss in this chapter are the Sharpe ratio, the Sortino ratio, and the information ratio.

The problem with these types of measures in evaluating the performance of a bond manager is that they do not identify how a manager generated the return. We know that there are several decisions made by a portfolio manager that impact the return on a bond portfolio, and risk-adjusted return measures provide no insight for that purpose. For this reason, a more in-depth analysis of performance requires a different analytical approach called *return attribution analysis*, the subject of Chapters 28 and 29.

SHARPE RATIO

Probably the most popular risk-adjusted return measure is the *Sharpe ratio*, introduced in 1966 by William Sharpe, the 1990 corecipient of the Nobel Prize in Economic Science. Sharpe referred to the ratio as the "reward-to-variability ratio"

1. For a comprehensive discussion of the wide range of reward-risk ratios, see Patrick Cheridito and Eduard Kromer, "Reward-Risk Ratios," *Journal of Investment Strategies*, Vol. 3, No. 1, 2013, pp. 1–16.

and applied it to evaluate the performance of mutual funds. The numerator of the Sharpe ratio is the excess return, as measured by the difference between the portfolio return and the risk-free rate of return. The risk-free rate is typically the interest rate on a Treasury bill whose maturity is equal to the length of the time horizon over which the portfolio return is calculated. The denominator is the standard deviation of the returns, which is the most common measure of risk.

The Sharpe ratio is therefore

$$\text{Sharpe ratio} = \frac{\text{portfolio return} - \text{risk-free rate of return}}{\text{standard deviation of the portfolio return}}.$$

The Sharpe ratio can be interpreted as follows: it is the value added by the portfolio manager above the risk-free asset taking into account the extra volatility accepted for investing in a risky asset rather than holding the risk-free asset.

Illustration 11–1. Calculation of the Sharpe ratio involves computing the numerator for each time period and then calculating the average excess return. This is illustrated in Exhibit 11–1 using the hypothetical portfolio return for 30 months. The realized returns are shown in the second column, and the third column shows the monthly risk-free rate. The average excess monthly portfolio return is 0.359%, or 3.59 basis points. Annualized by multiplying by 12 gives 4.31% and is the numerator of the Sharpe ratio. The standard deviation of the realized monthly returns is 1.01%. Annualizing by multiplying 1.01% by the square root of 12 gives 3.50%, which is the denominator of the Sharpe ratio. Therefore, the Sharpe ratio over the 30 months is

$$\text{Sharpe ratio} = \frac{4.31\%}{3.50\%} = 1.23.$$

Sharpe used yearly returns in his illustrations of the calculation of the ratio. We annualized the Sharpe ratio when using monthly returns by annualizing the monthly returns by multiplying the average return by 12 and the standard deviation by the square root. In general, annualizing based on returns that are less than annual is done as follows:

$$\text{Annualized Sharpe ratio} = \sqrt{F} \times (\text{Sharpe ratio based on frequency}),$$

where frequency refers to the time period over which the return is calculated. That is, when monthly returns are used, the frequency is 12. When weekly returns are used, the frequency is 52. When working with daily returns, the number of days in a year is not used. Instead, typically the frequency is 252 because that is the assumed number of trading days in a year. Notice in our illustration that multiplying the numerator by 12 and the denominator by the square root of 12 is equivalent to multiplying the monthly Sharpe ratio by the square root of 12.

EXHIBIT 11–1

Sharpe Ratio: Data and Calculation

Month	Portfolio (%)	Risk Free (%)	Excess (%)
1	1.65	0.37	1.28
2	0.97	0.33	0.64
3	2.10	0.32	1.78
4	–0.85	0.33	–1.18
5	1.02	0.34	0.68
6	1.19	0.41	0.78
7	–0.59	0.42	–1.01
8	–0.81	0.43	–1.24
9	1.22	0.53	0.69
10	–0.05	0.45	–0.50
11	2.11	0.48	1.64
12	0.92	0.50	0.42
13	1.77	0.41	1.36
14	1.01	0.48	0.53
15	1.20	0.41	0.79
16	–0.42	0.33	–0.75
17	–0.87	0.45	–1.32
18	–0.60	0.37	–0.97
19	1.12	0.45	0.67
20	0.83	0.37	–0.46
21	1.72	0.32	1.40
22	1.91	0.40	1.51
23	–0.63	0.37	–1.00
24	0.34	0.45	–0.11
25	1.73	0.37	1.36
26	1.71	0.52	1.19
27	1.40	0.43	0.97
28	1.02	0.52	0.50
29	–0.71	0.41	–1.12
30	1.72	0.41	1.31
Average monthly excess return			0.359
Standard deviation of monthly realized returns			1.01
Annualizing			
Annual average excess return = montly average excess return × 12			4.31
Annual standard deviation (SD) = monthly SD × 12^.5			3.50
Sharpe ratio			1.23

Limitations of the Sharpe Ratio

The Sharpe ratio has come under attack by researchers and practitioners for the following three reasons. The first limitation requires an understanding of probability distributions, the subject of Chapter 30. Probability distributions are characterized in terms of their measure of central tendency measured by the mean, variation as measured by the standard deviation, skewness, and tail fatness. When a probability distribution is assumed to be normally distributed, the only two measures that are important are the mean and the standard deviation. The distribution is symmetric around its mean. When the standard deviation is used as a measure of risk, as it is in the Sharpe ratio, it means that returns both below and above the mean are treated as a form of risk. Obviously, this is inconsistent with the way in which investors view risk. Returns in excess of the mean are viewed as favorable outcomes.

The question is whether bond and bond portfolio returns can be fairly described as normally distributed. The empirical evidence consistently shows that these returns are not normally distributed. Moreover, there are strategies that are unlikely to exhibit a normal distribution. For example, a strategy may involve periods of virtually no return and then a huge payoff. In the next section we will see how this has led to adjustments to the Sharpe ratio and will introduce other measures to overcome this limitation.

The investment objective of many funds is not just to outperform the return on the risk-free asset but to outperform some client-designated benchmark. Thus, using the risk-free rate as the benchmark does not indicate that risk-adjusted performance was accomplished from the client's perspective.

Finally, the portfolio manager can manipulate the Sharpe ratio. One way is by increasing the measurement period in such a way as to obtain a lower volatility, which is used in the denominator of the Sharpe ratio. Another way is by simply selecting the time period that the manager knows will show better performance. For example, if a 3-year Sharpe ratio does not indicate good performance but the 5-year Sharpe ratio does, given that performance was good in years 4 and 5, then the manager may elect to report the longer-term Sharpe ratio. Finally, consider a situation where the manager's compensation is tied to the Sharpe ratio. Suppose that in the early part of the evaluation period the portfolio manager realized an attractive return relative to the risk-free rate and therefore is likely to obtain an attractive bonus. The risk that the manager faces is that for the balance of the evaluation period, performance may deteriorate, and the bonus will be lost. What the manager can do is rebalance the portfolio by placing the assets into a risk-free asset. This will enhance the Sharpe ratio because the standard deviation will decline given that the volatility of the risk-free asset is zero, thereby locking in any bonus for the manager.

The Sharpe ratio can be modified to deal with the limitations just noted. The first way is by using a better measure of volatility and using a client-designated benchmark. The information ratio and the Sortino ratio, described later, do just that. The second way is to adjust the Sharpe ratio to consider skewness and kurtosis. We discuss this next.

Adjusting the Sharpe Ratio for Nonnormality

If portfolio returns are not normally distributed, then it is difficult to compare the Sharpe ratios of two different bond portfolio managers or bond portfolio strategies without considering the skewness and kurtosis of the return distribution. Two approaches have been suggested to deal with this issue.

One approach, proposed by Pezier and White, is to adjust the Sharpe ratio to deal with the nonnormality of portfolio returns by taking into account skewness and kurtosis.[2] The adjustment they derived is

$$\text{Adjusted Sharpe ratio} = \text{Sharpe ratio}\,[1 + (S/6)\,(\text{Sharpe ratio}) - (E/24)\,(\text{Sharpe ratio})^2],$$

where
S = a measure of skewness;
E = excess kurtosis.

If portfolio returns are normally distributed, then S and E are equal to zero, and we have the unadjusted Sharpe ratio. The adjusted Sharpe ratio takes into account that investors prefer positive skewness (which increases the adjusted Sharpe ratio) and negative excess kurtosis (which increases the adjusted Sharpe ratio). In contrast, the adjusted Sharpe ratio is penalized for negative skewness and a positive excess kurtosis.

Another approach is the *probabilistic Sharpe ratio*, proposed by Bailey and López de Prado, which involves evaluating whether the probability is greater than a given threshold when returns are nonnormally distributed.[3] This involves calculating a confidence interval[4] for the Sharpe ratio, which the authors derived.

Bailey and López de Prado demonstrated that the probabilistic Sharpe ratio has two important applications in evaluating manager performance. First, it can be used to establish how long of a track record is needed in order to reject the hypothesis that a fund's Sharpe ratio is below a certain threshold with a given confidence level. Second, the probabilistic Sharpe ratio can be applied to model the tradeoff between the length of the track record and statistical aspects of return distributions

2. Jacques Pezier and Anthony White, "The Relative Merits of Investible Hedge Funds Indices and of Funds of Hedge Funds in Optimal Passive Portfolios," *Journal of Alternative Investments*, Vol. 10, No. 4, 2008, pp. 37–49.
3. David H. Bailey and Marcos López de Prado, "The Sharpe Ratio Efficient Frontier," *Journal of Risk*, Vol. 15, No. 2, 2012–2013.
4. Confidence intervals are explained in Chapter 30.

that investors view as unfavorable, such as negative skewness and positive excess kurtosis. Therefore, despite the limitations of the Sharpe ratio, the probabilistic Sharpe ratio can identify skill if the evaluator can identify the required length of time for the track record.

SORTINO RATIO

A well-known risk-adjusted return that measures reward on a relative return basis is the Sortino ratio, as described in Sortino and van der Meer[5] and Sortino and Price.[6] The *Sortino ratio* addresses the criticism of the Sharpe ratio of using the standard deviation of realized returns. Instead, the Sortino ratio uses, as the measure of relative performance, a client-specified return that these authors refer to as the *minimum acceptable return* (MAR). The risk measure is not the standard deviation of the realized returns but the standard deviation of the realized returns that are below the client-specified MAR. That is, instead of calculating total volatility (i.e., standard deviation), the risk measure only considers "bad" volatility, which is calculated using returns below the MAR. The Sortino ratio is

$$\text{Sortino ratio} = \frac{\text{realized portfolio return} - \text{MAR}}{\text{standard deviation of the realized returns below MAR}}.$$

Illustration 11–2. We illustrate the calculation using the portfolio realized returns in Exhibit 11–1. We use a MAR of 0% in our illustration—basically an absolute return target for the fund manager. The data for calculation of the Sortino ratio are provided in Exhibit 11–2. From this exhibit it can be seen that the annual realized return is 9.25%. Because in our illustration the numerator of the Sortino ratio, the MAR, is assumed to be zero, the annual realized return is equal to the annual realized return minus the MAR. Note that care should be taken in calculating the denominator. The third column of Exhibit 11–2 shows the returns below the MAR. There are only 9 months where the fund manager failed to earn a positive return. It is those months that are used to calculate the standard deviation. The annual standard deviation of the realized return below the MAR is 2.29%. Therefore,

$$\text{Sortino ratio} = \frac{9.25\%}{2.29\%} = 4.05.$$

5. Frank A. Sortino and Robert van der Meer, "Downside Risk," *Journal of Portfolio Management*, Vol. 17, No. 4, 1991, pp. 27–31.
6. Frank A. Sortino and Lee N. Price, "Performance Measurement in a Downside Risk Framework," *Journal of Investing* 3(3), 1994:50–58.

EXHIBIT 11–2

Sortino Ratio: Data and Calculation

Minimum acceptable return (MAR) = 0%

Month	Portfolio (%)	Port—MAR (%)	Squared
1	1.65	0.00	0.000
2	0.97	0.00	0.000
3	2.10	0.00	0.000
4	–0.85	–0.85	0.723
5	1.02	0.00	0.000
6	1.19	0.00	0.000
7	–0.59	–0.59	0.348
8	–0.81	–0.81	0.656
9	1.22	0.00	0.000
10	–0.05	–0.05	0.003
11	2.11	0.00	0.000
12	0.92	0.00	0.000
13	1.77	0.00	0.000
14	1.01	0.00	0.000
15	1.20	0.00	0.000
16	–0.42	–0.42	0.176
17	–0.87	–0.87	0.757
18	–0.60	–0.60	0.360
19	1.12	0.00	0.000
20	0.83	0.00	0.000
21	1.72	0.00	0.000
22	1.91	0.00	0.000
23	–0.63	–0.63	0.397
24	0.34	0.00	0.000
25	1.73	0.00	0.000
26	1.71	0.00	0.000
27	1.40	0.00	0.000
28	1.02	0.00	0.000
29	–0.71	–0.71	0.504
30	1.72	0.00	0.000
Number negative			9
Average monthly return – MAR			0.77
Standard deviation (SD) of monthly MAR =			0.66
Annualizing			
Annual realized return = monthly realized return × 12			9.25
Annual SD = monthly SD × 12^.5			2.29
Sortino ratio			4.05

INFORMATION RATIO

Sharpe proposed a measure that takes into account performance versus a benchmark designated by a client using the tracking error of the actively managed portfolio as a measure of risk.[7] The ratio Sharpe proposed has been labeled the *information ratio*, calculated as follows:

$$\text{Information ratio} = \frac{\text{alpha}}{\text{backward tracking error}}.$$

The reward is *alpha*, which is measured by the average of the active return over a time period. The risk is the backward-looking tracking error described in Chapter 32 and is the standard deviation of a portfolio's active return. The higher the information ratio, the better the asset manager performed relative to the risk assumed.

Illustration 11–3. To illustrate calculation of the information ratio, consider the active returns for the hypothetical portfolio shown in the first column of Exhibit 11–1. The annual average active return or alpha as shown in Exhibit 11–3 is 1.292%. Because the backward-looking tracking error is 2.138%, the information ratio is therefore

$$\text{Information ratio} = \frac{1.292\%}{2.138\%} = 0.66$$

Obviously, the higher the information ratio, the better is the performance. According to Zephyr StyleADVISOR, which specializes in performance analysis software, generally an information ratio in the range of 0.40 to 0.60 is viewed as good performance, and it is rare to have an information ratio of 1.00 for long evaluation periods.[8] A negative information ratio means that the active manager was not able to earn an excess return at all. StyleADVISOR looked at the range of 10-year information ratios for peer groups of managed accounts for several asset classes, including investment-grade and high-yield bonds for the period January 2003 to December 2012 to demonstrate how rare it is to find managers with an information ratio of 1 or more over long time periods. Grinold and Kahn state that investment managers who are in the top quartile typically have an information ratio of about 0.5.[9]

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7. William F Sharpe, "The Sharpe Ratio," *Journal of Portfolio Management*, Vol. 21, No. 1, 1994, pp. 49–58.
8. "Information Ratio," Zephyr StyleADVISOR, New York, n.d. Available at https://www.styleadvisor.com/resources/statfacts/information-ratio.
9. Richard C. Grinold and Ronald N. Kahn, *Active Portfolio Management*, 2nd ed. (New York: McGraw-Hill, 1999), p. 14.

EXHIBIT 11–3

Information Ratio Data and Calculation

Month	Portfolio (%)	Benchmark (%)	Active (%)
1	1.65	1.40	0.25
2	0.97	1.09	–0.12
3	2.10	1.85	0.25
4	–0.85	–0.61	–0.24
5	1.02	1.23	–0.21
6	1.19	1.91	–0.72
7	–0.59	–0.28	–0.31
8	–0.81	–1.16	0.35
9	1.22	1.57	–0.35
10	–0.05	0.43	–0.48
11	2.11	2.42	–0.31
12	0.92	0.71	0.21
13	1.77	1.25	0.52
14	1.01	–0.37	1.38
15	1.20	0.98	0.22
16	–0.42	–1.33	0.91
17	–0.87	–0.20	–0.67
18	–0.60	–0.32	–0.28
19	1.12	0.95	0.17
20	0.83	1.09	–0.26
21	1.72	1.92	–0.20
22	1.91	1.89	0.02
23	–0.63	–1.55	0.92
24	0.34	0.90	–0.56
25	1.73	–0.25	1.98
26	1.71	0.88	0.83
27	1.40	1.96	–0.56
28	1.02	1.03	–0.01
29	–0.71	–0.95	0.24
30	1.72	1.46	0.26
Average monthly active return			0.108
Standard deviation (SD) of monthly active returns			0.617
Annualizing			
Annual average active return = monthly average active return × 12 = alpha			1.292
Annual SD = monthly SD × 12^.5 = tracking error			2.138
Information ratio			0.604

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PART

FOUR

PRICE VOLATILITY FOR OPTION-FREE BONDS

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CHAPTER

12

PRICE VOLATILITY PROPERTIES OF OPTION-FREE BONDS

To implement effective portfolio trading and investment strategies, it is necessary to understand the price volatility characteristics of bonds. Chapters 12–15 in this part of the book focus on bond price volatility. In this chapter general price volatility properties of option-free bonds are discussed, as well as bond characteristics that determine price volatility. At the end of this chapter we discuss one way to measure a bond's price volatility.

A CLOSER LOOK AT THE PRICE/YIELD RELATIONSHIP FOR OPTION-FREE BONDS

In Chapter 5 we demonstrated a fundamental principle of all option-free bonds (a bond that does not have an embedded option): *the price of a bond changes in the opposite direction of the change in the required yield for the bond.* This principle follows from the fact that the price of an option-free bond is equal to the present value of its expected cash flows. An increase (decrease) in the required return decreases (increases) the present value of its expected cash flows and therefore the bond's price.

Exhibit 12–1 illustrates this property for nine hypothetical bonds: three bonds with the same coupon rate but different maturities (5, 15, and 30 years) and three bonds with the same maturity but different coupon rates (0%, 8%, and 10%). These nine bonds are used in this and the following three chapters to demonstrate the bond price volatility of option-free bonds and measures of bond price volatility.

For each bond in Exhibit 12–1, the price of the bond (with par equal to 100) is shown for 13 yield levels. The top panel shows the price for yield levels from 10% to 13%; the bottom panel, the price for yield levels from 7% to 10%.

If the price/yield relationship for any of the nine bonds in Exhibit 12–1 is graphed, the shape of the graph would be as shown in Exhibit 12–2. As the required yield rises, the price of the option-free bond declines. However, the relationship is not linear (i.e., it is not a straight line). The price/yield relationship for all option-free bonds takes this nonlinear shape, referred to as *convex*.

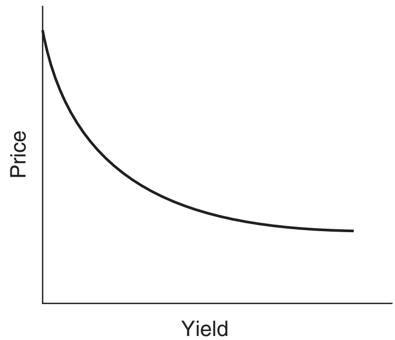
EXHIBIT 12–1

Price/Yield Relationship for Nine Hypothetical Bonds

		Yield Level						
Coupon (%)	**Term (years)**	**10.00%**	**10.01%**	**10.10%**	**10.50%**	**11.00%**	**12.00%**	**13.00%**
0.00	5	$61.39	$61.36	$61.10	$59.95	$58.54	$55.84	$53.27
0.00	15	23.14	23.10	22.81	21.54	20.06	17.41	15.12
0.00	30	5.35	5.34	5.20	4.64	4.03	3.03	2.29
8.00	5	92.28	92.24	91.91	90.46	88.69	85.28	82.03
8.00	15	84.63	84.56	83.95	81.32	78.20	72.47	67.35
8.00	30	81.07	80.99	80.29	77.30	73.83	67.68	62.42
10.00	5	100.00	99.96	99.61	98.09	96.23	92.64	89.22
10.00	15	100.00	99.92	99.24	96.26	92.73	86.24	80.41
10.00	30	100.00	99.91	99.06	95.46	91.28	83.84	77.45
Coupon (%)	**Term (years)**	**10.00%**	**9.99%**	**9.90%**	**9.50%**	**9.00%**	**8.00%**	**7.00%**
0.00	5	$61.39	$61.42	$61.68	$62.87	$64.39	$67.56	$70.89
0.00	15	23.14	23.17	23.47	24.85	26.70	30.83	35.63
0.00	30	5.35	5.37	5.51	6.18	7.13	9.51	12.69
8.00	5	92.28	92.32	92.65	94.14	96.04	100.00	104.16
8.00	15	84.63	84.70	85.31	88.13	91.86	100.00	109.20
8.00	30	81.07	81.15	81.87	85.19	89.68	100.00	112.47
10.00	5	100.00	100.04	100.39	101.95	103.96	108.11	112.47
10.00	15	100.00	100.08	100.77	103.96	108.14	117.29	127.57
10.00	30	100.00	100.09	100.95	104.94	110.32	122.62	137.42

EXHIBIT 12–2

Shape of Price/Yield Relationship for an Option-Free Bond



While all option-free bonds will have the convex shape shown in Exhibit 12–2, the curvature of every option-free bond will be different. As we will see in this chapter and the following three, it is this convex shape that holds the key to assessing the performance of a bond and a portfolio of bonds when interest rates change.

It is important to keep in mind that the price/yield relationship that we have discussed refers to an instantaneous change in the yield. As we explained in Chapter 2, assuming no change in the perceived credit risk of the issuer as a bond moves toward maturity, there are two factors that influence the price of any option-free bond. First, the bond's price will change as the required yield changes, as we know. Second, for discount and premium bonds, the bond's price will change even if yields remain the same. In particular, the price of a discount bond will increase as it moves toward maturity, reaching par value at the maturity date; for a premium bond, the bond's price will decrease as it moves closer to maturity, finally declining to the par value at the maturity date.

PRICE VOLATILITY CHARACTERISTICS OF OPTION-FREE BONDS

To investigate bond price volatility in terms of percentage price change, let's assume that the prevailing yield in the market is 10% for all nine bonds. The dollar price change per $100 of par value for various changes in yield is shown in Exhibit 12–3. Exhibit 12–4 shows the corresponding percentage change in each bond's price. The percentage price change shown in Exhibit 12–4 is found by dividing the dollar price change in Exhibit 12–3 by the price of the bond at a 10% yield, as shown in Exhibit 12–1.

For example, consider the 8%, 15-year bond. The price for this bond if the required yield is 10% is $84.63. If the required yield increases 100 basis points to 11%, the price of this bond would fall to $78.20. The dollar price change per $100

EXHIBIT 12–3

Dollar Price Change per $100 of Par Value as Yield Changes for Nine Hypothetical Bonds

		Change in Basis Points from 10%					
		1	***10***	***50***	***100***	***200***	***300***
		New Yield Level					
Coupon (%)	**Term (years)**	**10.01%**	**10.10%**	**10.50%**	**11.00%**	**12.00%**	**13.00%**
0.00	5	$–0.03	$–0.26	$–1.15	$–1.41	$–2.70	$–2.57
0.00	15	–0.03	–0.33	–1.59	–3.07	–5.73	–8.02
0.00	30	–0.02	–0.15	–0.71	–1.33	–2.32	–3.07
8.00	5	–0.04	–0.37	–1.81	–3.58	–7.00	–10.25
8.00	15	–0.07	–0.68	–3.31	–6.43	–12.16	–17.27
8.00	30	–0.08	–0.78	–3.78	–7.25	–13.39	–18.65
10.00	5	–0.04	–0.39	–1.91	–3.77	–7.36	–10.78
10.00	15	–0.08	–0.76	–3.74	–7.27	–13.76	–19.59
10.00	30	–0.09	–0.94	–4.54	–8.72	–16.16	–22.55
		Change in Basis Points from 10%					
		–1	***–10***	***–50***	***–100***	***–200***	***–300***
		New Yield Level					
Coupon (%)	**Term (years)**	**9.99%**	**9.90%**	**9.50%**	**9.00%**	**8.00%**	**7.00%**
0.00	5	$0.03	$0.29	$1.48	$3.00	$6.17	$9.50
0.00	15	0.03	0.33	1.72	3.56	7.69	12.49
0.00	30	0.02	0.16	0.82	1.78	4.15	7.34
8.00	5	0.04	0.37	1.86	3.77	7.72	11.88
8.00	15	0.07	0.69	3.51	7.23	15.37	24.57
8.00	30	0.08	0.79	4.12	8.61	18.93	31.40
10.00	5	0.04	0.39	1.95	3.96	8.11	12.47
10.00	15	0.08	0.77	3.96	8.14	17.29	27.59
10.00	30	0.09	0.95	4.94	10.32	22.62	37.42

EXHIBIT 12–4

Percentage Price Change as Yield Changes for Nine Hypothetical Bonds

		Change in Basis Points from 10%					
		1	10	50	100	200	300
		New Yield Level					
Coupon (%)	**Term (years)**	**10.01%**	**10.10%**	**10.50%**	**11.00%**	**12.00%**	**13.00%**
0.00	5	–0.05%	–0.47%	–2.35%	–4.64%	–9.04%	–13.22%
0.00	15	–0.14	–1.42	–6.89	–13.28	–24.75	–34.66
0.00	30	–0.29	–2.82	–13.30	–24.80	–43.38	–57.30
8.00	5	–0.04	–0.40	–1.97	–3.88	–7.58	–11.11
8.00	15	–0.08	–0.80	–3.91	–7.60	–14.37	–20.41
8.00	30	–0.10	–0.96	–4.66	–8.94	–16.52	–23.01
10.00	5	–0.04	–0.39	–1.91	–3.77	–7.36	–10.78
10.00	15	–0.08	–0.76	–3.74	–7.27	–13.76	–19.59
10.00	30	–0.09	–0.94	–4.54	–8.72	–16.16	–22.55

		Change in Basis Points from 10%					
		–1	–10	–50	–100	–200	–300
		New Yield Level					
Coupon (%)	**Term (years)**	**9.99%**	**9.90%**	**9.50%**	**9.00%**	**8.00%**	**7.00%**
0.00	5	0.05%	0.48%	2.41%	4.89%	10.04%	15.48%
0.00	15	0.14	1.44	7.41	15.40	33.25	53.98
0.00	30	0.29	2.90	15.38	33.16	77.57	137.10
8.00	5	0.04	0.40	2.02	4.08	8.37	12.87
8.00	15	0.08	0.81	4.14	8.54	18.16	29.03
8.00	30	0.10	0.98	5.08	10.62	23.35	38.73
10.00	5	0.04	0.39	1.95	3.96	8.11	12.47
10.00	15	0.08	0.77	3.96	8.14	17.29	27.59
10.00	30	0.09	0.95	4.94	10.32	22.62	37.42

of par value is –$6.43, as shown in Exhibit 12–3. The percentage price change is then

$$\frac{\$78.20 - \$84.63}{\$84.63} = -0.076 \text{ or } -7.60\%.$$

This is shown in Exhibit 12–4.

An examination of Exhibits 12–3 and 12–4 reveals the following four properties about price volatility of option-free bonds.

Property 1: Price Volatility Is Not the Same for All Bonds Although the prices of all option-free bonds move in the opposite direction of the change in yield, for a given change in the yield, the price change is not the same for all bonds. A natural question is, what characteristics of a bond determine its price volatility? We'll focus on this question in the next section.

Property 2: Price Volatility Is Approximately Symmetric for Small Yield Changes For very small changes in yield, the percentage price change for a given bond is roughly the same whether the yield increases or decreases. For example, look at the top panel of both Exhibits 12–3 and 12–4. For a 10 basis point change in the required yield, the 10%, 30-year bond's price would increase by $0.94 per $100 of par value or 0.94% of the initial price. Now in the bottom panel of the two exhibits note that the 10%, 30-year bond's price would increase by $0.95 per $100 of par value or 0.95% of the initial price for a 10 basis point decrease in the required yield.

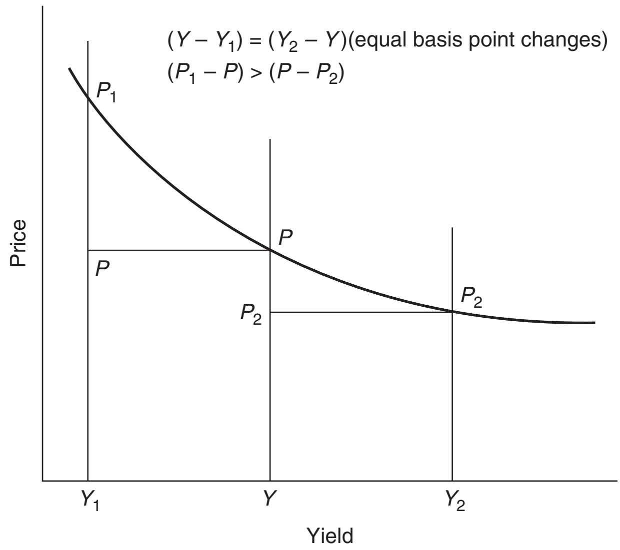
Property 3: Price Volatility Is Not Symmetric for Large Yield Changes For large changes in yield, the percentage price change is not the same for an increase in yield as it is for a decrease in yield. Once again, look at the 10%, 30-year coupon bond. Note that if the required yield increases by 300 basis points (from 10% to 13%), the price decline per $100 of par value would be $22.55 or 22.55% of the initial price; for a 300 basis point decrease in the yield, the price increase per $100 of par value would be $37.42 or 37.42%, of the initial price.

Property 4: For Large Yield Changes, Price Increases Are Greater Than Price Decreases For a given large change in yield, the price increase is greater than the price decrease. This can be seen in Exhibits 12–3 and 12–4 by comparing the upper panel (which shows price volatility for increases in yield) to the lower panel (which shows price volatility for decreases in yield). For each bond, in absolute terms the price change is larger in the lower panel than in the upper panel for a given large change in yield.

The reason for Property 4 lies in the convex shape of the price/yield relationship. This is illustrated graphically in Exhibit 12–5. Suppose the initial required

EXHIBIT 12–5

Illustration of Property 4



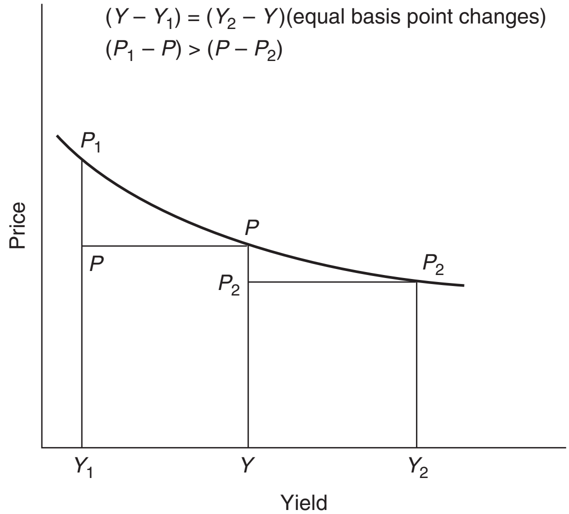
yield is Y in Exhibit 12–5. The corresponding initial price is P. Consider large equal basis point changes in the initial yield. Y_1 and Y_2 indicate the lower and higher yields, respectively; the corresponding prices are denoted P_1 and P_2. The vertical distance from the yield axis to the price/yield relationship represents the price. The distance between P_1 and P measures the price increase if the yield decreases; the distance between P and P_2 measures the price decrease if the yield increases. The diagram clearly demonstrates that the price increase will be greater than the price decrease for an equal change in basis points.

This will always be true when the curve is convex, and because all option-free bonds have a convex price/yield relationship, this property will always hold for option-free bonds. Exhibit 12–6 shows a less convex price/yield relationship than is shown in Exhibit 12–5. While the property still holds, the price gain when the yield falls is not that much greater than the price loss for an equal basis point increase in yield.

An implication of Property 4 is that if an investor owns a bond (i.e., is long a bond), the price appreciation that will be realized if the yield decreases is greater than the loss that will be realized if the yield rises by the same number of basis points. For an investor who is short a bond, the reverse is true; the potential loss is greater than the potential gain if the yield changes by a given number of basis points.

EXHIBIT 12–6

The Impact of Less Convexity on Property 4



CHARACTERISTICS OF A BOND THAT AFFECT ITS PRICE VOLATILITY

Property 1 states that price volatility is not the same for each bond. There are two characteristics of an option-free bond that determine its price volatility: (1) coupon and (2) term to maturity.

Characteristic 1: Price Volatility Is Greater the Lower the Coupon Rate[1] For a given term to maturity and initial yield, the price volatility of a bond is greater the lower the coupon rate.

This characteristic can be seen by comparing the 0%, 8%, and 10% coupon bonds with the same maturity.[2] The investment implication is that bonds selling at a deep discount will have greater price volatility than bonds selling near or above par. Zero-coupon bonds will have the greatest price volatility for a given maturity.

Characteristic 2: Price Volatility Increases with Maturity For a given coupon rate and initial yield, the longer the term to maturity the greater the price volatility.

This can be seen in Exhibit 12–4 by comparing the 5-year bonds to the 30-year bonds with the same coupon. The investment implication is that an investor expecting interest rates to fall, all other factors constant, should hold bonds with long

1. This property does not necessary hold for long-term deep-discount coupon bonds.
2. Notice that while the percentage price change is greater the lower the coupon rate, the dollar price change is smaller the lower the coupon rate.

maturities in the portfolio. To reduce a portfolio's price volatility in anticipation of a rise in interest rates, bonds with short maturities should be held in the portfolio.

The Effect of the Yield Level on Price Volatility

So far we have described how the characteristics of a bond affect its price volatility. But we cannot ignore the fact that credit considerations cause different bonds to trade at different yields, even if they have the same coupon and maturity. How, then, holding other factors constant, does the yield to maturity affect a bond's price volatility? As it turns out, the higher the yield to maturity at which a bond trades, the lower is its price volatility.

To see this, compare a 10%, 15-year bond trading at various yield levels, as shown in Exhibit 12–7. The first column shows the yield level each bond is trading at and the second column the initial price. The third column indicates the bond's price if yields change by 100 basis points. The fourth and fifth columns show the dollar price change and the percentage price change, respectively. As can be seen from these last two columns, the higher the initial yield, the less is the price volatility.

This can also be seen in Exhibit 12–8. When the yield level is high (Y_H in the exhibit), a given change in yield will not produce as large a change in the price of a bond as when the yield level is low (Y_L in the exhibit).

An implication of this is that for a given change in yields, price volatility is greater when yield levels in the market are low, and price volatility is lower when yield levels are high. For example, in the early 1980s when yields on long-term Treasury bonds were in the neighborhood of 16%, a yield change of 25 basis points produced a greater price change than in early 2022 when long-term Treasury yields were about 2.5%. Also, another implication is that high-yield bonds (more popularly referred to as *junk bonds*) will have less price volatility for a given change in

EXHIBIT 12–7

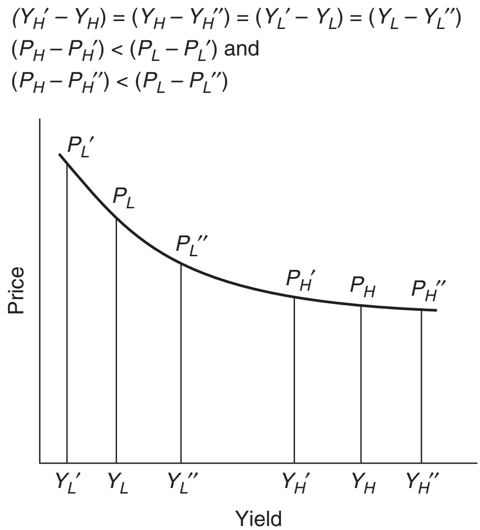
Price Change for a 100 Basis Point Change in Yield for 10%, 15-Year Bonds Trading at Different Yield Levels

Yield Level (%)	Initial Price ($)	New Price ($)*	Price Decline ($)	Percent Decline (%)
7	127.57	117.29	10.28	8.1
8	117.29	108.14	9.15	7.8
9	108.14	100.00	8.14	7.5
10	100.00	92.73	7.27	7.3
11	92.73	86.24	6.49	7.0
12	86.24	80.41	5.83	6.8
13	80.41	75.18	5.23	6.5

*As a result of a 100 basis point increase in yield.

EXHIBIT 12–8

The Effect of the Yield Level on Price Volatility



yield than Treasury bonds because the former trade at a higher yield level than the latter.

MEASURING PRICE VOLATILITY USING THE PRICE VALUE OF A BASIS POINT

Portfolio managers, arbitrageurs, hedgers, and traders need to have a way to measure a bond's price volatility in order to implement strategies. The three measures commonly used are (1) price value of a basis point, (2) duration, and (3) duration/convexity. The first measure is explained here and the last two measures in the two chapters that follow.

The *price value of a basis point* (denoted by PV01), also referred to as the *dollar value of a basis point* (denoted by DV01), is the change in the price of the bond if the yield changes by one basis point. Note that this measure of price volatility indicates *dollar* price volatility as opposed to price volatility as a *percentage* of the initial price. The price value of a basis point is expressed as the absolute value of the change in price.

Property 2 of the price/yield relationship explained earlier in this chapter states that for small changes in yield, price volatility is the same regardless of the direction of the change in yield. Therefore, it does not make any difference if we increase or decrease the yield by one basis point to compute the price value of a basis point.

We will show how to calculate the price value of a basis point using the nine bonds in Exhibit 12–1. For each bond, the initial price, the price after increasing the yield by one basis point (from 10.00% to 10.01%), and the price value of a basis point per \$100 of par value (the difference between the two prices) are shown in the top panel of Exhibit 12–9. Similarly, if we decrease the yield by one basis point, from 10.00% to 9.99%, we would find approximately the same price value of a basis point for the nine bonds, as shown in the bottom panel of Exhibit 12–9 on page 154.

The Price Value of More than One Basis Point

In implementing a strategy, some investors will calculate the price value of a change of more than one basis point. The principle of calculating the price value of any number of basis points is the same. The price value of *x* basis points is found by computing the difference between the initial price and the price if the yield is changed by *x* basis points. For example, the price value per $100 of par value for several values of *x* for the 10% coupon, 15-year bond is found as follows if the yield is increased:

Change (*x*, in Basis Points)	Initial Price (10% yield; $)	Price of 10% + *x* yield ($)	Price Value of *x* Basis Points ($)
10	100.00	99.2357	0.7643
50	100.00	96.2640	3.7360
100	100.00	92.7331	7.2669

For small changes in yield (such as 10 basis points), the price value of *x* basis points is roughly that found by multiplying the price value of one basis point by *x*. For example, the price value of 10 basis points computed by multiplying the price value of a basis point, namely $0.08, by 10 gives $0.80, which is very close to the price value of 10 basis points computed above. In fact, the approximation is much better than indicated here because we computed the price value of a basis point to only two decimal places.

For a decrease in the yield, the price value of an *x* basis point change per $100 of par value is

Change (*x*, in Basis Points)	Initial Price (10% Yield; $)	Price at 10% – *x* Yield ($)	Price Value of *x* Basis Points ($)
10	100.00	100.7730	0.7730
50	100.00	103.9551	3.9551
100	100.00	108.1444	8.1444

Once again, note the approximate symmetry of the price change for small changes (10 basis points) in yield (Property 2 of the price/yield relationship). For larger changes in yield, however, there will be a difference between the price value of an *x* basis point movement depending on whether the yield is increased or decreased. Many investors who use the price value of a large basis point movement in implementing a strategy will compute the average of the two price values. For example, the price value of 100 basis points would be approximated by averaging $7.2669 and $8.1444 to give $7.7057.

EXHIBIT 12–9

Computation of the Price Value of a Basis Point

Bond Coupon (%)	Team (years)	Price ($) 10.00%	Price ($) 10.01%	Price Value of a Basis Point ($)*
0.00	5	61.3913	61.3621	0.0292
0.00	15	23.1377	23.1047	0.0330
0.00	30	5.3636	5.3383	0.0153
8.00	5	92.2783	92.2415	0.0367
8.00	15	84.6275	84.5595	0.0681
8.00	30	81.0707	80.9920	0.0787
10.00	5	100.0000	99.9614	0.0386
10.00	15	100.0000	99.9232	0.0768
10.00	30	100.0000	99.9054	0.0946

Bond Coupon (%)	Team (years)	Price ($) 10.00%	Price ($) 9.99%	Price Value of a Basis Point ($)
0.00	5	61.3913	61.4206	0.0292
0.00	15	23.1377	23.1708	0.0331
0.00	30	5.3536	5.3689	0.0153
8.00	5	92.2783	92.3150	0.0367
8.00	15	84.6275	84.6957	0.0681
8.00	30	81.0707	81.1496	0.0788
10.00	5	100.0000	100.0386	0.0386
10.00	15	100.0000	100.0769	0.0769
10.00	30	100.0000	100.0947	0.0947

*Absolute value per $100 of par value.

The Price Value of a Basis Point for a Portfolio

Suppose that a portfolio manager wants to know the price value of a basis point for a portfolio. Let's assume the following portfolio, comprised of three of our hypothetical bonds, all selling to yield 10%:

Bond	Par Amount Owned ($)	Price ($)
10%, 5-year	4 million	4,1000,000
8%, 15-year	5 million	4,231,375
14%, 30-year	1 million	1,378,586

The price of a basis point per $100 of par value for each of these bonds assuming an increase of 1 basis point is

		Price Value for	
Bond	**Par Amount Owned ($)**	**$100 par ($)**	**Amount Owned ($)**
10%, 5-year	4 million	0.0386	1,544
8%, 15-year	5 million	0.0681	3,405
14%, 30-year	1 million	0.1263	1,263

Thus the portfolio's exposure to a one basis point movement is $6,212, that is, $1,544 + $3,405 + $1,263.

The same analysis can be used to determine the exposure of a bond dealer's inventory.

Price Change versus Percentage Change

So far we have dealt with the dollar price change. If we want to know the percentage price change, we can find it by dividing the price value of a basis point by the initial price. That is,

$$\text{Percentage price change} = \frac{\text{price value of a basis point}}{\text{initial price}}.$$

Application to Hedging

To hedge a portfolio or bond position, a portfolio manager or trader wants to take an opposite position in a cash market security (or securities) or in a derivative instrument (option or futures contract) so that any loss in the position held is offset by a gain in the opposite position. To do so, the *dollar* price volatility of the position used to hedge must equal the *dollar* price volatility of the position to be hedged. That is, the objective in hedging is

$$\begin{gathered}\text{Dollar price change in position to be hedged}\\ = \text{dollar price change of hedging vehicle.}\end{gathered}$$

The hedger encounters two problems. First, for a given change in yield, the dollar price volatility of the bond to be hedged will not necessarily be equal to the dollar price volatility of the hedging vehicle. For example, if yields change by 50 basis points, the price of the bond to be hedged may change by $*X*, while the price of the hedging vehicle may change by more or less than $*X*. The second problem is that factors that result in a change in the yield of a given number of basis points for the bond to be hedged may not result in a change of the same number of basis points for the hedging vehicle. That is, if yields change by *x* basis points for the bond to be hedged, the yield for the hedging vehicle may change by more or less than *x* basis points.

Consequently, a portfolio manager or trader attempting to hedge a $10 million long position in some bond will not necessarily take a $10 million short position in the vehicle used for hedging because the relative dollar price volatility of the two positions will not necessarily be the same. The hedger would have to consider the relative dollar price volatility of the two positions in constructing the hedge. As a result, the objective in hedging can be restated a follows:

$$\begin{aligned}&\text{Dollar price change of bond to be hedged}\\&\quad = \text{dollar price change of hedging vehicle}\\&\qquad \times \frac{\text{dollar price volatility of bond to be hedged}}{\text{dollar price volatility of hedging vehicle}}.\end{aligned}$$

The last ratio is commonly referred to as the *hedge ratio*. The hedge ratio can be computed from the price value of a basis point by the following formula:

$$\frac{\text{Price value of a basis point for bond to be hedged}}{\text{Price value of a basis point for hedging vehicle}} \times \frac{\text{change in yield for bond to be hedged}}{\text{change in yield for hedging vehicle}}.$$

The first ratio shows the price change for the bond to be hedged relative to the price change for the hedging vehicle, both based on a yield change of one basis point. The second ratio indicates the relative change in the yield of the two instruments. This ratio is commonly referred to as the *yield beta* and is estimated using the statistical technique of regression analysis, which is described in Chapter 31. Therefore, the hedge ratio can be rewritten as

$$\frac{\text{Price value of a basis point for bond to be hedged}}{\text{Price value of a basis point for hedging vehicle}} \times \text{yield beta.}$$

To illustrate this relationship, we'll look at how to hedge a $10 million long position in a 15-year, 8% coupon bond selling to yield 10% with a 10% coupon bond with the same maturity and selling at the same yield. The following information is known:

Bond to be hedged = 15-year, 8% bond selling to yield 10%;

Hedging vehicle = 15-year, 10% bond selling to yield 10%;

For bond to be hedged,

Price value of a basis point per $100 of par = 0.0681;

For hedging vehicle,

Price value of a basis point per $100 of par = 0.0768.

For this illustration, we shall assume that the yield beta is 1. The hedge ratio is then

$$\frac{0.0681}{0.0768} \times 1 = 0.8867.$$

The hedge ratio of 0.8867 means that for every \$1 of par value of the 15-year, 8% coupon bond to be hedged, the hedger should short \$0.8867 of \$1 par of the hedging vehicle. Because this illustration involves a long position of \$10 million of the 15-year, 8% coupon bond, a short position of \$8.867 million of the 15-year, 10% bond should be taken.

To demonstrates that this hedge ratio will result in equal dollar price changes for the bond to be hedged and the hedging vehicle, suppose that interest rates rise by 10 basis points. The price of the bond to be hedged will decline from \$84.6275 per \$100 of par to \$83.9505, resulting in a loss for the \$10 million position of \$67,700. The short position, however, will gain. The price of the 15-year, 10% coupon bond will decline from \$100 per \$100 of par to \$99.2357, resulting in a gain of \$67,770 for the \$8.867 million position. The gain is almost identical to the loss.

CHAPTER

13

DURATION AS A MEASURE OF PRICE VOLATILITY

In this chapter we discuss the most commonly used measure of bond price volatility: duration. We illustrate some applications of duration in portfolio strategies and discuss its limitations.

Duration as explained in this chapter is called *model duration* because it is based on prices that are generated from a bond pricing model. In Chapter 17 we will discuss another measure of a bond's price volatility called *empirical duration*, where historical price movements when interest rates change are used to estimate duration employing regression analysis.

MACAULAY DURATION

In 1938, Frederick Macaulay constructed a measure that he could use as a proxy for the length of time a bond investment was outstanding.[1] He referred to this number as the *duration* of a bond and defined it as a weighted-average term to maturity of the bond's cash flows. The weights in this weighted average are the present value of each cash flow as a percent of the present value of all the bond's cash flows (i.e., the weights are the present value of each cash flow as a percent of the bond's price). As we will see, Macaulay's measure is linked to the price volatility of a bond. First, let's look at how this measure, referred to as *Macaulay duration*, is computed.

Computing Macaulay Duration

Mathematically, Macaulay duration is computed as follows:

$$\text{Macaulay duration (in periods)} = \frac{(1)PVCF_1 + (2)PVCF_2 + (3)PVCF_3 + \cdots + (n)PVCF_n}{PVTCF},$$

1. Frederick Macaulay, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1865* (Cambridge, MA: National Bureau of Economic Research, 1938).

where

$PVCF_t$ = present value of the cash flow in period t discounted at the prevailing period yield (in the case of a semiannual-pay bond, one-half the yield to maturity);

t = period when the cash flow is expected to be received ($t = 1, \ldots, n$);

n = number of periods until maturity (specifically, number of years to maturity times k, rounded down to the nearest whole number[2]);

k = number of periods, or payments, per year (i.e., $k = 2$ for semiannual-pay bonds, $k = 12$ for monthly-pay bonds);

$PVTCF$ = total present value of the cash flow of the bond, where the present value is determined by using the prevailing yield to maturity.

For an *option-free bond* with semiannual payments, the cash flow for periods 1 to $n - 1$ is one-half the annual coupon interest. The cash flow in period n is the semiannual coupon interest plus the maturity value. Because the bond's price is equal to its cash flow discounted at the prevailing yield to maturity, *PVTCF* is the current market price *plus* accrued interest.

The formula for Macaulay duration gives a value in terms of periods. Dividing by the number of payments per year converts Macaulay duration to years. That is,

$$\text{Macaulay duration (in years)} = \frac{\text{Macaulay duration (in periods)}}{k}.$$

Exhibit 13–1 shows the calculation of Macaulay duration for a 10%, 5-year bond assuming that the bond is selling to yield 10%. The Macaulay duration in years for our nine hypothetical bonds is given below:

Bond	**Macaulay Duration (years)**
0%, 5-year	5.00
0%, 15-year	15.00
0%, 30-year	30.00
8%, 5-year	4.18
8%, 15-year	8.45
8%, 30-year	10.20
10%, 5-year	4.05
10%, 15-year	8.07
10%, 30-year	9.94

Note in this summary that the Macaulay duration of a coupon bond is less than its maturity. It should be obvious from the formula that the Macaulay duration

2. For example, if there are 5.2 years to maturity for a semiannual-pay bond (i.e., $k = 2$), then $n = 5.2 \times 2 = 10.4$ down to the nearest whole number gives a value of 10 for n.

EXHIBIT 13–1

Calculation of Macaulay Duration and Modified Duration for a 10%, 5-Year Bond Selling to Yield 10%

Coupon rate = 10.00%; Maturity (years) = 5; Initial yield = 10.00%.

Period (*t*)	Cash Flow ($)*	Present Value of $1 at 5%	Present Value of Cash Flow (PVCF; $)	$t \times$ PVCF ($)
1	5.00	0.952380	4.761905	4.761905
2	5.00	0.907029	4.535147	9.070295
3	5.00	0.863837	4.319188	12.957563
4	5.00	0.822702	4.113512	16.454049
5	5.00	0.783526	3.917631	19.588154
6	5.00	0.746215	3.731077	22.386461
7	5.00	0.710681	3.553407	24.873846
8	5.00	0.676839	3.384197	27.073574
9	5.00	0.644608	3.223045	29.007401
10	105.00	0.613913	64.460892	644.608920
Total			$100.000000	$810.782168

*Cash flow per $100 of par value.

$$\text{Macaulay duration (in half years)} = \frac{810.782168}{100.000000} = 8.11.$$

$$\text{Macaulay duration (in years)} = \frac{8.11}{2} = 4.05.$$

$$\text{Modified duration (in years)} = \frac{4.05}{1.0500} = 3.86.$$

of a zero-coupon bond is equal to its maturity. The lower the coupon, generally, the greater is the duration of the bond.[3]

Notice the consistency between the properties of bond price volatility that we discussed in the previous chapter and the properties of duration. We showed that, all other factors constant, the greater the maturity, the greater is the price volatility. A property of duration is that, all other factors constant, the greater the maturity, the greater is the duration. We also showed that the lower the coupon rate, all other factors constant, the greater is the bond price volatility. As we just noted, generally the lower the coupon rate, the greater is the duration. It appears that duration is telling us something about bond price volatility.

3. This property does not necessarily hold for some long-maturity, deep-discount coupon bonds, as explained below.

Short-Cut Formulas for Macaulay Duration

Mathematically, Macaulay duration for a semiannual-pay bond can be shown to be equivalent to[4]

$$\text{Macaulay duration (in 6-month periods)} = \left[\frac{1+y}{y}\right]H + \left[\frac{y-c}{y}\right]n(1-H),$$

where

y = one-half the yield;
H = ratio of the present value of the annuity of the semiannual coupon payments to the price of the bond;
c = one-half the annual coupon *rate*.

From Chapter 3, the present value of the semiannual coupon payments is

$$100c\left\{\frac{1-\left[\dfrac{1}{(1+y)^n}\right]}{y}\right\}.$$

Illustration 13–1. We can use the formula above to calculate Macaulay duration for the 14% coupon bond with 5 years to maturity, selling to yield 10%, by inserting the values

$y = 0.05\,(0.10/2)$; $c = 0.07\,(0.14/2)$; $n = 10\,(5 \times 2)$.

The price of this bond is \$115.44 (see Exhibit 12–1 of Chapter 12). The present value of the coupon payments is

$$\$100(0.07)\left\{\frac{1-\left[\dfrac{1}{(1.05)^{10}}\right]}{0.05}\right\}$$

$$= \$7\left[\frac{1-0.613913}{0.05}\right] = \$54.05.$$

Then

$$H = \frac{\$54.05}{\$115.44} = 0.4682$$

and

$$\text{Macaulay duration (in 6-month periods)}$$
$$= \frac{1.05}{0.05}(0.4682) + \left(\frac{0.05-0.07}{0.05}\right)(10)(1-0.4682) = 7.705.$$

4. The mathematical proof is given in Frank J. Fabozzi, *Bond Markets, Analysis and Strategies*, 2nd ed. (Englewood Cliffs, NJ: Prentice-Hall, 1993), pp. 70–71.

Dividing by 2 to give Macaulay duration in years, we have 3.85.

When a bond is selling at par, c is equal to y. The formula for Macaulay duration then reduces to

$$\text{Macaulay duration for a bond selling at par (in 6-month periods)} = \left(\frac{1+y}{y}\right)H.$$

Illustration 13–2. In the formula for the Macaulay duration for a bond selling at par, the numbers needed to calculate Macaulay duration for the 10% coupon bond with 5 years to maturity selling to yield 10% are

$$y = 0.05\,(0.10/2);\ c = 0.05\,(0.10/2);\ n = 10\,(5 \times 2).$$

The present value of the coupon payments is

$$\$100(0.05)\left\{\frac{1-\left[\frac{1}{(1.05)^{10}}\right]}{0.05}\right\} = \$38.61.$$

Then

$$H = \frac{\$38.61}{\$100} = 0.3861$$

and

$$\begin{aligned}\text{Macaulay duration (in 6-month periods)} &= \frac{1.05}{0.05}(0.3861)\\ &= 21(0.3861) = 8.1081.\end{aligned}$$

Dividing by 2 gives 4.05, the Macaulay duration in years. This result agrees with the calculation of Macaulay duration in Exhibit 13–1.

For a zero-coupon bond, the present value of the coupon payments is obviously zero; thus H is zero. The Macaulay duration for a zero-coupon bond in 6-month periods is

$$\left(\frac{1+y}{y}\right)H + \left(\frac{y-c}{y}\right)n(1-H) = \left(\frac{1+y}{y}\right)0 + \left(\frac{y-0}{y}\right)n(1-0) = n.$$

Thus we see that the Macaulay duration of a zero-coupon bond in periods is just the number of 6-month periods. Dividing by 2 gives the term of the bond in years.

LINK BETWEEN DURATION AND BOND PRICE VOLATILITY

Now that we know how to compute Macaulay duration, how do we interpret it? Some investors continue to think of duration in the context in which it was developed by Macaulay—as a measure of the length of time a bond investment is outstanding. Forget it! The significance and interpretation of Macaulay duration lie in its link to bond price volatility.

The link between bond price volatility and Macaulay duration can be shown to be[5]

$$\textit{Approximate} \text{ percentage change in price}$$
$$= -\left[\frac{1}{(1+y)}\right] \times \text{Macaulay duration} \times \text{yield change},$$

where

y = one-half the yield to maturity.

Generally, the first two terms are combined, and the resulting measure is called *modified duration*; that is,

$$\text{Modified duration} = \frac{\text{Macaulay duration}}{(1+y)}.$$

The relationship can then be expressed as

$$\textit{Approximate} \text{ percentage change in price}$$
$$= -\text{modified duration} \times \text{yield change}.$$

Illustration 13–3. Consider the 8%, 15-year bond selling at 84.63 to yield 10%. The Macaulay duration for this bond is 8.45. Modified duration is 8.05:

$$\text{Modified duration} = \frac{8.45}{(1.05)} = 8.05.$$

If the yield increases instantaneously from 10.00% to 10.10%, a yield change of +0.0010, the *approximate* percentage change in price is

$$-8.05 \times (+0.0010) = -0.00805 = -0.81\%.$$

Exhibit 12–4 in Chapter 12 shows that the actual percentage change in price is –0.80%. Similarly, if the yield decreases instantaneously from 10.00% to 9.90% (a 10 basis point decrease), the approximate change in price would be +0.81%. Exhibit 12–4 shows that the actual percentage price change would be +0.80%. This example illustrates that for small changes in yield, modified duration provides a good approximation of the percentage change in price.

Illustration 13–4. Instead of a small change in yield, let's assume that yields increase by 300 basis points, from 10% to 13% (a yield change of +0.03). The approximate percentage change in price estimated using modified duration would be

$$-8.05 \times (+0.03) = -0.2415 = 24.15\%.$$

5. Mathematically, this relationship is derived from the first term of a Taylor series of the price function.

How good is this approximation? As can be seen from Exhibit 12–4 in Chapter 12, the actual percentage change in price is only −20.41%. Moreover, if the yield decreases by 300 basis points, from 10% to 7%, the approximate percentage change in price based on duration would be +24.15%, compared to an actual percentage change in price of +29.03%. Thus not only is the approximation off, but we can see that modified duration estimates a symmetric percentage change in price, which, as we pointed out in Chapter 12, is not a property of the price/yield relationship for bonds for a large change in interest rates.

Exhibit 13–2 shows the estimated percentage change in price using modified duration for the nine hypothetical bonds used in Chapter 12. The difference between the actual percentage price change, as reported in Exhibit 12–4, and the estimated percentage change in price using modified duration, as reported in Exhibit 13–2, is shown in Exhibit 13–3. The property that we just illustrated—that modified duration gives a good estimate of the percentage price change for small changes in yield but a poor estimate for large changes—can be seen in Exhibit 13–3. The percentage price change not explained by modified duration increases the greater the change in yield. For some bonds, however, the approximation is in error by less than others.

The reason that modified duration provides a good approximation for small changes in yield but a poor approximation for large changes in yield lies in the convex shape of the price/yield relationship. This point is examined fully in the next chapter.

Modified Duration as a Measure of Percentage Price Change per 100 Basis Point Yield Change

Ignoring the direction of the change in yield, for a 100 basis point change in yield, the percentage change in the bond's price is

$$\text{Modified duration} \times 0.01.$$

For example, if the modified duration is 6, the approximate percentage change in the bond's price for a 100 basis point change in yield is

$$6 \times 0.01 = 0.06 \text{ or } 6\%.$$

Thus a bond with a modified duration of 6 would change by approximately 6% for a 100 basis point (1%) change in yield. Similarly, a bond with a modified duration of X would change by approximately $X\%$ for a 100 basis point change in yield. For this reason, some investors refer to modified duration as the approximate percentage price change of a bond per 100 basis point change in yield. So, for example, a bond with a modified duration of 5.4 will change by approximately 5.4% for a 100 basis point change in yield. For a 50 basis point change in yield, the approximate percentage price change would then be half of 5.4%, or 2.7%. For a 10 basis point change in yield, the approximate percentage price change would be one-tenth of 5.4%, or 0.54%.

Interpretations of Duration

Previously, we noted that duration can be defined as the approximate percentage change in price for a 100 basis point change in rates. If you understand this

EXHIBIT 13–2

Percentage Change in Price Estimated Using Modified Duration

			Yield Change (in Basis Points) from 10%					
			1	*10*	*50*	*100*	*200*	*300*
		Modified	***New Yield Level***					
Coupon (%)	**Term (years)**	**Duration (years)**	**10.01%**	**10.10%**	**10.50%**	**11.00%**	**12.00%**	**13.00%**
0.00	5	4.78	–0.05%	–0.48%	–2.38%	–4.76%	–9.52%	–14.28%
0.00	15	14.29	–0.14	–1.43	–7.15	–14.29	–28.58	–42.87
0.00	30	28.57	–0.29	–2.86	–14.29	–28.57	–57.14	–85.71
8.00	5	3.98	–0.04	–0.40	–1.99	–3.98	–7.96	–11.94
8.00	15	8.05	–0.08	–0.81	–4.03	–8.05	–16.10	–24.15
8.00	30	9.72	–0.10	–0.97	–4.86	–9.72	–19.44	–29.16
10.00	5	3.86	–0.04	–0.39	–1.93	–3.86	–7.72	–11.58
10.00	15	7.69	–0.08	–0.77	–3.85	–7.69	–15.38	–23.07
10.00	30	9.46	–0.09	–0.95	–4.73	–9.46	–18.92	–28.38

EXHIBIT 13–2

Percentage Change in Price Estimated Using Modified Duration (*Continued*)

			Yield Change (in Basis Points) from 10%					
			1	*10*	*50*	*100*	*200*	*300*
			New Yield Level					
Coupon (%)	**Term (years)**	**Modified Duration (years)**	**9.99%**	**9.90%**	**9.50%**	**9.00%**	**8.00%**	**7.00%**
0.00	5	4.76	0.05%	0.48%	2.38%	4.76%	9.52%	14.28%
0.00	15	14.29	0.14	1.43	7.15	14.29	28.58	42.87
0.00	30	28.57	0.29	2.86	14.29	28.57	57.14	85.71
8.00	5	3.98	0.04	0.40	1.99	3.98	7.96	11.94
8.00	15	8.05	0.08	0.81	4.03	8.05	16.10	24.15
8.00	30	9.72	0.10	0.97	4.86	9.72	19.44	29.16
10.00	5	3.86	0.04	0.39	1.93	3.86	7.72	11.58
10.00	15	7.69	0.08	0.77	3.85	7.69	15.38	23.07
10.00	30	9.46	0.09	0.95	4.73	9.46	18.92	28.38

EXHIBIT 13–3

Percentage Price Change Not Explained by Modified Duration

			Yield Change (in Basis Points) from 10%					
			1	10	50	100	200	300
			New Yield Level					
Coupon (%)	Term (years)	Modified Duration (years)	10.01%	10.10%	10.50%	11.00%	12.00%	13.00%
0.00	5	4.76	0.00%	0.00%	0.03%	0.12%	0.48%	1.06%
0.00	15	14.29	0.00	0.01	0.26	1.01	3.83	8.21
0.00	30	28.57	0.00	0.04	0.99	3.77	13.76	28.41
8.00	5	3.98	0.00	0.00	0.02	0.10	0.38	0.83
8.00	15	8.05	0.00	0.00	0.12	0.45	1.73	3.74
8.00	30	9.72	0.00	0.01	0.20	0.78	2.92	6.15
10.00	5	3.86	0.00	0.00	0.02	0.09	0.36	0.80
10.00	15	7.69	0.00	0.00	0.11	0.42	1.62	3.48
10.00	30	9.46	0.00	0.01	0.19	0.74	2.76	5.83

EXHIBIT 13–3

Percentage Price Change Not Explained by Modified Duration (*Continued*)

			Yield Change (in Basis Points) from 10%					
			1	10	50	100	200	300
			New Yield Level					
Coupon (%)	Term (years)	Modified Duration (years)	9.99%	9.90%	9.50%	9.00%	8.00%	7.00%
0.00	5	4.76	0.00%	0.00%	0.03%	0.13%	0.52%	1.20%
0.00	15	14.29	0.00	0.01	0.27	1.11	4.67	11.11
0.00	30	28.57	0.00	0.04	1.09	4.59	20.43	51.39
8.00	5	3.98	0.00	0.00	0.03	0.10	0.41	0.93
8.00	15	8.05	0.00	0.00	0.12	0.49	2.06	4.88
8.00	30	9.72	0.00	0.01	0.22	0.90	3.91	9.57
10.00	5	3.86	0.00	0.00	0.02	0.10	0.39	0.89
10.00	15	7.69	0.00	0.00	0.11	0.45	1.91	4.52
10.00	30	9.46	0.00	0.01	0.21	0.86	3.70	9.04

definition, there is actually no need to use the equation for the approximate percentage price change, and you can easily calculate the change in a bond's value. For example, suppose that we want to know the approximate percentage change in price for a 50 basis point increase in yield for a 5% coupon, 20-year bond selling at 113.6777. The duration for this bond can be shown to be 13.09. A 100 basis point change in yield would change the price by about 13.09%. For a 50 basis point increase in yield, the price will decrease by approximately 6.545% (13.09%/2). Thus, if the yield increases by 50 basis points, the price will decrease by 6.545% from 113.6777 to 106.2375.

Now let's look at some other definitions or interpretations of duration that have been used.

Duration Is the First Derivative Sometimes a market participant will refer to duration as the "first derivative of the price/yield function" or simply the "first derivative." *Derivative* here has nothing to do with derivative instruments (i.e., futures, swaps, options, etc.). A derivative as used in this context is obtained by differentiating a mathematical function. There are first derivatives, second derivatives, and so on. When market participants say that duration is the first derivative, here is what they mean. If it were possible to write a mathematical equation for a bond in closed form, the first derivative would be the result of differentiating that equation the first time. While this is a correct interpretation of duration, it is an interpretation that in no way helps us understand what the interest-rate risk is of a bond. That is, it is an operationally meaningless interpretation.

Why is this an operationally meaningless interpretation? Suppose that a portfolio manager has a $10 million bond position with a duration of 6. Suppose further that a client is concerned with the exposure of the bond to changes in interest rates. Now tell that client that the duration is 6 and that it is the first derivative of the price function for that bond. What have you told the client? Not much. In contrast, tell that client that the duration is 6 and that duration is the approximate price sensitivity of a bond to a 100 basis point change in rates, and you've told the client a great deal with respect to the bond's interest-rate risk.

Duration Is Some Measure of Time When the concept of duration was introduced by Macaulay in 1938, he used it as a gauge of the time that the bond was outstanding. More specifically, Macaulay defined *duration* as the weighted average of the time to each coupon and principal payment of a bond. Subsequently, duration has too often been thought of in temporal terms, that is, years. This is most unfortunate for two reasons.

First, in terms of dimensions, there is nothing wrong with expressing duration in terms of years because that is the proper dimension for this metric. But the proper interpretation is that duration is the price volatility of a zero-coupon bond with that number of years to maturity. Thus, when a manager says that a bond has a duration of 4 years, it is not useful to think of this measure in terms of time but rather that the bond has the price sensitivity to rate changes of a 4-year zero-coupon bond.

Second, thinking of duration in terms of years makes it difficult for portfolio managers and their clients to understand the duration of some complex securities. Here are a few examples. For a mortgage-backed security that is an interest-only security, the duration is negative. What does a negative number of, say, −4 mean? In terms of our interpretation as a percentage price change, it means that when rates change by 100 basis points, the price of the bond changes by about 4%, but the change is in the same direction as the change in rates.

As a second example, consider the duration of an interest-rate futures contract that matures in 1 year. Suppose that it is reported that its duration is 25. What does this mean? To someone who interprets duration in terms of time, does that mean 25 years, 25 days, or 25 seconds? It does not mean any of these. It simply means that the value of the futures tends to have the price sensitivity to rate changes of a 25-year zero-coupon bond.

The bottom line is that one should not care if it is technically correct to think of duration in terms of years (volatility of a zero-coupon bond) or in terms of first derivatives. There are even some who interpret duration in terms of the *half-life* of a security. Subject to the limitations that we will describe later, duration is used as a measure of the sensitivity of a security's price to changes in yield. We will fine-tune this definition as we move along.

Users of this interest-rate risk measure are interested in what it tells them about the price sensitivity of a bond (or a portfolio) to changes in rates. Duration provides the portfolio manager and client with a feel for the dollar price exposure or the percentage price exposure to potential rate changes.

Dollar Duration

The modified duration of a bond can be used to approximate the percentage change in price for a given change in yield. Similarly, the dollar price change can be approximated given the bond's modified duration, the bond's price, and a specified number of basis points. The dollar price change is referred to as *dollar duration*.

The dollar duration of a basis point change in yield can be calculated as follows:

$$\text{Dollar duration of a basis point} = \frac{\text{modified duration} \times \text{price}}{10{,}000}.$$

For example, consider an 8% coupon, 15-year bond selling to yield 10%. The price of this bond per \$100 par value is \$84.6275. The bond's modified duration is 8.05. Therefore,

$$\text{Dollar duration of a basis point} = \frac{8.05 \times \$84.6275}{10{,}000} = \$0.068125.$$

To find the dollar duration for more than one basis point, the dollar duration of a basis point is simply multiplied by the number of basis points. For example, for a 50 basis point change in yield, the dollar duration would be \$3.40625.

If the dollar duration for a par value different from $100 is sought, then the dollar duration per $100 par value must be scaled appropriately. For example, in our previous example, if $1 million is the par value, then the market value of the bond is $8,462,750. The dollar duration of a basis point is $6,812.51.

It should be noted that there is no standard terminology for dollar duration. More specifically, some market participants refer to "dollar duration" when they mean the "dollar duration of a basis point." In our example, the dollar duration would then be $0.0681 because this is the dollar duration of one basis point. The dollar duration of a basis point is equivalent to the price value of a basis point (PV01) that we discussed in the previous chapter. Some market participants refer to the dollar duration in terms of the dollar duration of 100 basis points. In our example, the dollar duration would then be $6.81.

How good an estimate of the dollar price change is provided by dollar duration? For small changes in yield, it will be good, as we saw in Illustration 13–3, where modified duration is used to estimate the percentage price change. For large changes in yield, the approximation is not as good. Recall from Illustration 13–4 that when the yield increases by 300 basis points, the estimated percentage decline in price is greater than the actual decline in price. This means that the estimated dollar price decline will be greater than the actual price decline, so the estimated new price is less than the actual price at a higher yield. Also in Illustration 13–4, the estimated percentage price increase is less than the actual price increase when the yield declines by 300 basis points. The actual price increase, therefore, is underestimated using modified duration, and the estimated price at a lower yield will be less than the actual price. Thus, whether the yield rises or falls by a large number of basis points, modified duration and dollar duration will underestimate what the new price will be. A graphical explanation for this fact is given in the next chapter.

PORTFOLIO DURATION

A portfolio's modified duration can be obtained by calculating the weighted average of the modified duration of the bonds in the portfolio. The weight is the proportion of the portfolio that a security comprises. Mathematically, a portfolio's modified duration can be calculated as

$$w_1 D_1 + w_2 D_2 + w_3 D_3 + \cdots + w_K D_K,$$

where

w_i = market value of bond *i*/market value of the portfolio;
D_i = modified duration of bond *i*;
K = number of bonds in the portfolio.

Each term of an individual bond's weight and its modified duration has a special meaning. It is the bond's contribution to portfolio duration. That is,

$$\text{Contribution to portfolio duration for bond } i = w_i D_i.$$

In the same way that a portfolio's modified duration can be computed using each bond in the portfolio, it can also be computed from the market value in each sector.

To illustrate the calculation of a portfolio's modified duration, consider the hypothetical portfolio shown in Exhibit 13–4. The hypothetical portfolio has 20 bonds and a market value of $300 million. Column (4) shows the weights and the last column shows the contribution to portfolio duration of each bond. The total shown in the last row of the last column is the portfolio duration, 5.10.

A portfolio modified duration of 5.10 means that for a 100 basis point change in the yield for *all* 20 bonds, the market value of the portfolio will change by approximately 5.10%. But keep in mind that the yield on all 20 bonds must change by 100 basis points for the modified duration measure to be useful. This is a critical

EXHIBIT 13–4

Illustration of the Calculation of Portfolio Duration

(1) Bond	(2) Mark Value ($)	(3) Duration *D*	(4) Weight *W*	(5) Contribution to Portfolio Duration $D \times W$
1	25,000,000	5.21	0.0833	0.4342
2	15,533,000	6.26	0.0518	0.3241
3	18,294,000	4.27	0.0610	0.2604
4	29,426,000	7.15	0.0981	0.7013
5	2,892,000	2.14	0.0096	0.0206
6	8,971,000	3.16	0.0299	0.0945
7	29,760,000	3.75	0.0992	0.3720
8	3,290,000	8.26	0.0110	0.0906
9	12,389,000	6.85	0.0413	0.2829
10	2,905,000	9.11	0.0097	0.0882
11	8,129,000	2.87	0.0271	0.0778
12	19,788,000	6.42	0.0660	0.4235
13	22,800,000	10.17	0.0760	0.7729
14	16,549,000	4.55	0.0552	0.2510
15	14,236,000	2.58	0.0475	0.1224
16	24,500,000	4.77	0.0817	0.3896
17	1,908,000	1.13	0.0064	0.0072
18	23,987,000	3.75	0.0800	0.2998
19	11,200,000	1.65	0.0373	0.0616
20	8,443,000	0.95	0.0281	0.0267
Total	$300,000,000	—	1.0000	5.1013

assumption, and its importance cannot be overemphasized. We shall return to this point in Chapter 15.

Portfolio Duration for a Global Portfolio

Many global money managers calculate a duration for a portfolio consisting of bonds with cash flows denominated in several currencies. For example, a U.S. portfolio manager who has a portfolio of U.S. Treasury securities denominated in U.S. dollars, German government bonds denominated in euros, Japanese government bonds denominated in yen, and British government bonds denominated in pounds sterling might calculate a modified duration for the portfolio. The question is what does a portfolio modified duration in such cases mean?

Because modified duration is a measure of the percentage price volatility of a bond to changes in yield, to what yield is modified duration measuring price sensitivity? U.S. yields, German yields, Japanese yields or British yields? There is no answer to this question. Using the modified duration measure as discussed thus far in this chapter for a global portfolio is of questionable value.

APPROXIMATING DURATION

Because modified duration is a measure of the price sensitivity of a bond to interest-rate changes, it can be approximated by simply changing yields by a small amount up and down and then looking at how the price changes. More specifically, let

V_0 = initial value or price of the bond;

Δy = change in the yield of the bond;

V_- = estimated value of the bond if the yield is decreased by Δy;

V_+ = estimated value of the bond if the yield is increased by Δy.

Then, for a decrease in yield, the percentage price change is

$$\frac{V_- - V_0}{V_0}.$$

The percentage price change per basis point change is found by dividing the percentage price change by the number of basis points ($\Delta y \times 100$). That is,

$$\frac{V_- - V_0}{V_0(\Delta y)100}.$$

Similarly, the percentage price change per basis point increase in yield is

$$\frac{V_0 - V_0}{V_0(\Delta y)100}.$$

As explained in the previous chapter, the percentage price change for an increase and decrease in interest rates will not be the same. Consequently, the average percentage price change per basis point change in yield can be calculated. This is done as follows:

$$\frac{1}{2}\left[\frac{V_{-}-V_{0}}{V_{0}(\Delta y)100}+\frac{V_{0}-V_{+}}{V_{0}(\Delta y)100}\right]$$

or, equivalently,

$$\frac{V_{-}-V_{+}}{2V_{0}(\Delta y)100}.$$

The approximate percentage price change for a 100 basis point change in yield is found by multiplying the previous formula by 100:

$$\text{Approximate duration} = \frac{V_{-}+V_{+}}{2V_{0}(\Delta y)}.$$

To illustrate this formula for approximating modified duration, consider a 20-year, 7% coupon bond selling at \$74.26 to yield 10%. Suppose that we evaluate the price changes for a 20 basis point change up and down. Then

$$V_{-} = 75.64;\ V_{+} = 72.92;\ V_{0} = 74.26;\ \Delta y = 0.002.$$

Substituting into the formula,

$$\frac{75.64-72.92}{2(74.26)(0.002)} = 9.16.$$

The modified duration for this bond calculated using the formula presented earlier in this chapter is 9.18.

In calculating duration using this short-cut formula, it is necessary to shock interest rates (yields) up and down by the same number of basis points to obtain the values for V_{-} and V_{+}. In our illustration, 100 basis points was arbitrarily selected. But how large should the shock be? That is, how many basis points should be used to shock the rate? All vendors of analytical systems and proprietary models of asset management firms select their own interest-rate shock based on their view of historical interest-rate changes. Often if the range is fewer than 100 basis points, there is not much difference in the approximate duration in the case of option-free bonds. However, when we deal with more complicated bonds, small rate shocks that do not reflect the types of rate changes that may occur in the market do not permit determination of how prices can change because expected cash flows may change when dealing with bonds with embedded options. In comparison, if large rate shocks are used, the asymmetry is caused by convexity (described in the next chapter). Moreover, large rate shocks may cause dramatic changes in the expected cash flows for bonds with embedded options that may be far different from how the expected cash flows will change for smaller rate shocks.

There is another potential problem with using small rate shocks for complicated securities. The prices that are inserted into the approximate duration formula are derived from a valuation model. In Chapters 22 and 30 we will discuss various valuation models and their underlying assumptions. The duration measure depends crucially on a valuation model. If the rate shock is small and the valuation model employed to obtain the prices used to calculate duration is poor, dividing poor price estimates by a small shock in rates in the denominator will have a significant effect on the duration estimate.

The approximation formula above is important when we analyze bonds that are not option free in later chapters, that is, when the bonds are putable or callable. Modified duration is not a good measure of the price sensitivity of such bonds to yield changes because it assumes that yield changes will not change the expected cash flows of the bond. For example, when a bond is callable, a decline in market yields may change the expected cash flows of the bond because it may increase the likelihood that the bond is called. In the case of mortgage-backed securities—the subject of Part Seven of this book—a change in market yields will change the prepayments of borrowers.

In such instances where the cash flow is interest-rate sensitive (interest-rate dependent), the price sensitivity of a bond to yield changes must take into consideration how yield changes affect the expected cash flows and, in turn, the price of the bond. A duration measure that does allow for changes in cash flows as yields change is called an *effective duration*. The formula we gave above for approximating the modified duration is the formula that is used to calculate the effective duration of a bond. We shall see how this formula is used for bonds with interest-rate-sensitive cash flows in later chapters.

APPLICATIONS

Dollar Duration Weighting of Yield-Spread Swap Strategies

A common active bond portfolio strategy is to position a portfolio to capitalize on expectations regarding the yield spread between sectors in the market. It is critical when assessing yield-spread strategies to compare positions that have the same dollar duration. To understand why, consider two bonds, X and Y. Suppose that bond X has a price of $80 and a modified duration of 5, while bond Y has a price of $90 and a modified duration of 4. Because modified duration is the approximate percentage change per 100 basis point change in yield, a 100 basis point change in yield for bond X would change its price by about 5%. Based on a price of $80, its price will change by about $4 per $80 of market value. Thus its dollar duration for a 100 basis point change in yields is $4 per $80 of market value. Similarly, for bond Y, its dollar duration for a 100 basis point change in yield per $90 of market value can be determined. In this case it is $3.6. So, if bonds X and Y are being considered as alternative investments in some strategy other than one based on anticipating interest-rate movements, the amount of each bond in the strategy should be such that they will both have the same dollar duration.

To illustrate this, suppose that a portfolio manager owns $10 million of par value of bond X that has a market value of $8 million. The dollar duration of bond X per 100 basis point change in yield for the $8 million market value is $400,000. Suppose further that this portfolio manager is considering exchanging bond X that it owns in its portfolio for bond Y. If the portfolio manager wants to have the same interest-rate exposure (i.e., dollar duration) for bond Y that she currently has for bond X, she will buy a market value amount of bond Y with the same dollar duration. If the portfolio manager purchased $10 million of *par value* of bond Y and therefore $9 million of *market value* of bond Y, the dollar price change per 100 basis point change in yield would be only $360,000. If, instead, the portfolio manager purchased $10 million of *market value* of bond Y, the dollar duration per 100 basis point change in yield would be $400,000. Because bond Y is trading at $90, $11.11 million of par value of bond Y must be purchased to keep the dollar duration of the position for bond Y the same as for bond X.

Mathematically, this problem can be expressed as follows:

Let

$\$D_X$ = dollar duration per 100 basis point change in yield for bond X for the market value of bond X held;

MD_Y = modified duration for bond Y;

MV_Y = market value of bond Y needed to obtain the same dollar duration as bond X.

Then the following equation sets the dollar duration for bond X equal to the dollar duration for bond Y:

$$\$D_X = (MD_Y/100)MV_Y.$$

Solving for MV_Y,

$$MV_Y = \$D_X/(MD_Y/100).$$

Dividing by the price per $1 of par value of bond Y gives the par value of bond Y that has an approximately equivalent dollar duration as bond X.

In our illustration, $\$D_X$ is $400,000 and MD_Y is 4, so

$$MY_Y = \$400{,}000/(4/100) = \$10{,}000{,}000.$$

Because the market value of bond Y is $90 per $100 of par value, the price per $1 of par value is $0.9. Dividing $10 million by 0.9 indicates that the par value of bond Y that should be purchased is $11.11 million.

Failure to adjust a portfolio repositioning based on some expected change in yield spread so as to hold the dollar duration the same means that the outcome of the portfolio will be affected by not only the expected change in the yield spread but also a change in the yield level. Thus a manager would be making a conscious yield spread bet and possibly an undesired bet on the level of interest rates.

Using Dollar Duration to Compute the Hedge Ratio

Modified duration is related to percentage price volatility. As explained in the previous chapter, in hedging we are interested in dollar price changes. Consequently, to compute the hedge ratio using duration, dollar duration should be used, not modified duration. The hedge ratio is computed from dollar duration as follows:

$$\frac{\text{Dollar duration for bond to be hedged}}{\text{Dollar duration for hedging vehicle}} \times \text{yield beta.}$$

Recall that the dollar duration per basis point per \$100 of par value is the same as the price value of a basis point. Therefore, dollar duration will produce the same hedge ratio as the price value of a basis point.

Role of Duration in Immunization Strategies

In Chapter 9 we explained that the return that is realized by investing in a coupon bond will depend on the interest rate earned on the reinvestment of the coupon payments. When interest rates rise, interest on interest from the reinvestment of the coupon payments will be higher, but if the investment horizon is shorter than the maturity of the bond, a loss will be realized on the sale of the bond. The reverse is true if interest rates fall: price appreciation will be realized when the bond is sold, but interest on interest from reinvesting the coupon payments will be lower. Because of these two risks, the investor cannot be assured of locking in the yield at the time of purchase.

Because interest-rate risk and reinvestment risk offset each other, is it possible to select a bond or bond portfolio that will lock in the yield at the time of purchase regardless of interest-rate changes in the future? That is, is it possible to *immunize* the bond or bond portfolio against interest-rate changes? Fortunately, under certain circumstances, it is. This can be accomplished by constructing a portfolio so that its Macaulay duration is equal to the length of the investment horizon. Thus a portfolio manager with an investment horizon of 5 years who wants to lock in a return over that time period should select a portfolio with a Macaulay duration of 5 years. This is demonstrated using an example.

Suppose that a portfolio manager knows that a liability of \$17,183,033 must be paid in 5.5 years. Also suppose that interest rates are currently 12.5% on a bond-equivalent basis. The present value of the \$17,183,033 liability 5.5 years from now assuming interest can be earned at a rate of 6.25% per 6-month period is \$8,820,262. Thus, if the portfolio manager invested \$8,820,262 at the current rate of 6.25% per 6-month period for the next eleven 6-month periods, the accumulated value would be sufficient to satisfy the liability.

Suppose that the portfolio manager buys \$8,820,262 par value of a bond selling at par with a 12.5% yield to maturity that matures in 5.5 years. The Macaulay duration for this bond is 4.14 years, which is shorter than the length of the investment horizon. Will the portfolio manager be assured of realizing the target yield of 12.5% or, equivalently, a target accumulated value of \$17,183,033? As we explained in Chapter 6, the portfolio manager will realize a 12.5% yield only if the coupon interest payments can be reinvested at 6.25% every 6 months. That is, the accumulated value will depend on the reinvestment rate.

To illustrate this, suppose that immediately after investing the $8,820,262 in the 12.5% coupon, 5.5-year maturity bond, yields in the market change and stay at the new level for the remainder of the 5.5 years. Exhibit 13–5 illustrates what happens at the end of 5.5 years. The first column shows the new yield. The second column shows the total coupon interest payments. The third column gives the interest on interest over the entire 5.5 years if the coupon interest payments are reinvested at the new yield shown in the first column. The price of the bond at the end of 5.5 years, shown in the fourth column, is the par value. The fifth column is the accumulated value from all three sources: coupon interest, interest on interest, and price of bond. The total return is shown in the last column.

When yields do not change so that the coupon payments can be reinvested at 12.5% (6.25% every 6 months), the target accumulated value will be achieved by the portfolio manager. If market yields rise, an accumulated value (total return) higher than the target accumulated value (target yield) will be achieved because the coupon interest payments can be reinvested at a rate higher than the initial yield to maturity. Contrast this circumstance with what happens when the yield declines. The accumulated value (total return) will be less than the target accumulated value (target yield). *Therefore, investing in a coupon bond with a yield to maturity equal to the target yield and a maturity equal to the investment horizon does not ensure that the target accumulated value will be achieved.*

Suppose that instead of investing in a bond maturing in 5.5 years, the portfolio manager invests in a 15-year bond with a coupon rate of 12.5% and selling at par to yield 12.5%. The Macaulay duration for this bond is 7.12 years, which is longer than the 5.5-year investment horizon. The accumulated value and total return if the market yield changes immediately after the bond is purchased and remains at the new yield are presented in Exhibit 13–6. The fourth column in Exhibit 13–6 is the market price of a 12.5% coupon, 9.5-year bond (because 5.5 years have passed), assuming that the market yield is as shown in the first column. If the market yield increases, the portfolio will fail to achieve the target accumulated value; the opposite is true if the market yield decreases—the accumulated value (total return) will exceed the target accumulated value (target yield).

The reason for this result can be seen in Exhibit 13–7, which summarizes the change in interest on interest and the change in price resulting from a change in the market yield. For example, if the market yield rises instantaneously by 200 basis points, from 12.5% to 14.5%, interest on interest will be $454,336 greater; the market price of the bond, however, will decrease by $894,781. The net effect is that the accumulated value will be $440,445 less than the target accumulated value. The reverse is true if the market yield decreases: the change in the price of the bond will more than offset the decline in the interest on interest, resulting in an accumulated value that exceeds the target accumulated value. Exhibit 13–7 clearly demonstrates the tradeoff between interest-rate (or price) risk and reinvestment risk.

EXHIBIT 13–5

Accumulated Value and Total Return

5.5-Year, 12.5% Coupon Bond Selling to Yield 12.5%

Investment horizon (years) = 5.5; Coupon rate = 12.5%; Maturity (years) = 5.50; Yield to maturity = 12.5%; Price = 100.00000; Par value purchased = $8,820.262; Purchase price = $8.820,262; Target accumulated value = $17,183,033.

	Over 5.5 Years				
New Yield (%)	**Coupon ($)**	**Interest on Interest ($)**	**Price of Bond ($)**	**Accumulated Value ($)**	**Total Return (%)**
16.0	6,063,930	3,112,167	8,820,262	17,996,360	13.40
15.5	6,063,930	2,990,716	8,820,262	17,874,908	13.26
14.5	6,063,930	2,753,177	8,820,262	17,637,369	13.00
14.0	6,063,930	2,637,037	8,820,262	17,521,230	12.88
13.5	6,063,930	2,522,618	8,820,262	17,406,810	12.75
13.0	6,063,930	2,409,894	8,820,262	17,294,086	12.62
12.5	6,063,930	2,298,840	8,820,262	17,183,033	12.50
12.0	6,063,930	2,189,433	8,820,262	17,073,625	12.38
11.5	6,063,930	2,081,648	8,820,262	16,965,840	12.25
11.0	6,063,930	1,975,462	8,820,262	16,859,654	12.13

EXHIBIT 13–5

Accumulated Value and Total Return (*Continued*)

	Over 5.5 Years				
New Yield (%)	**Coupon ($)**	**Interest on Interest ($)**	**Price of Bond ($)**	**Accumulated Value ($)**	**Total Return (%)**
10.5	6,063,930	1,870,852	8,820,262	16,755,044	12.01
10.0	6,063,930	1,767,794	8,820,262	16,651,986	11.89
9.5	6,063,930	1,666,266	8,820,262	16,550,458	11.78
9.0	6,063,930	1,566,246	8,820,262	16,450,438	11.66
8.5	6,063,930	1,476,712	8,820,262	16,351,904	11.54
8.0	6,063,930	1,370,642	8,820,262	16,254,834	11.43
7.5	6,063,930	1,275,014	8,820,262	16,159,206	11.32
7.0	6,063,930	1,180,808	8,820,262	16,065,000	11.20
6.5	6,063,930	1,088,003	8,820,262	15,972,195	11.09
6.0	6,063,930	996,577	8,820,262	15,880,769	10.96
5.5	6,063,930	906,511	8,820,262	15,790,703	10.87
5.0	6,063,930	817,785	8,820,262	15,701,977	10.77

EXHIBIT 13–6

Accumulated Value and Total Return

15-Year, 12.5% Coupon Bond Selling to Yield 12.5%

Investment horizon (years) = 5.5; Coupon rate = 12.5%; Maturity (years) = 15; Yield to maturity = 12.5%; Price = $100,00000;
Par value purchased = $8,820,262; Purchase price = $8,820,262; Target accumulated value = $17,183,033.

	Over 5.5 Years				
New Yield (%)	**Coupon ($)**	**Interest on Interest ($)**	**Price of Bond ($)**	**Accumulated Value ($)**	**Total Return (%)**
13.0	6,063,930	2,409,894	8,583,555	17,057,379	12.36
12.5	6,063,930	2,298,840	8,820,262	17,183,033	12.50
12.0	6,063,930	2,189,433	9,066,306	17,319,669	12.65
11.5	6,063,930	2,081,648	9,322,113	17,467,691	12.82
11.0	6,063,930	1,975,462	9,588,131	17,627,523	12.99
10.5	6,063,930	1,870,852	9,864,831	17,799,613	13.18
10.0	6,063,930	1,767,794	10,152,708	17,984,432	13.38
9.5	6,063,930	1,666,266	10,452,281	18,182,477	13.59
9.0	6,063,930	1,566,246	10,764,095	18,394,271	13.82
8.5	6,063,930	1,467,712	11,088,723	18,620,366	14.06
8.0	6,063,930	1,370,642	11,426,770	18,861,342	14.31
7.5	6,063,930	1,275,014	11,778,867	19,117,812	14.57

EXHIBIT 13–7

Change in Interest on Interest and Price Due to Interest-Rate Change

15-Year, 12.5% Coupon Bond Selling to Yield 12.5%

New Yield (%)	Change in Interest on Interest ($)	Change in Price ($)	Total Change in Accumulated Value ($)
13.5	223,778	–464,485	–240,707
13.0	111,054	–236,707	–125,654
12.5	0	0	0
12.0	–109,407	246,044	136,636
11.5	–217,192	501,851	284,659
11.0	–323,378	767,869	444,491
10.5	–427,989	1,044,569	616,581
10.0	–531,046	1,332,446	801,400
9.5	–632,574	1,632,019	999,445
9.0	–732,594	1,943,833	1,211,239
8.5	–831,128	2,268,461	1,437,333
8.0	–928,198	2,606,508	1,678,309
7.5	–1,023,826	2,958,605	1,934,779

Consider an 8-year, 10.125% coupon bond selling at 88.20262 to yield 12.5%, which has a Macaulay duration of 5.5 years. Suppose that $10,000,000 of par value of this bond is purchased for $8,820,262. For this bond, Exhibit 13–8 provides the same information as Exhibits 13–5 and 13–6. Looking at the last two columns, we see that the accumulated value and the total return are never less than the target accumulated value and target yield. Thus the target accumulated value is assured regardless of what happens to the market yield. Exhibit 13–9 shows why. When the market yield rises, the change in the interest on interest more than offsets the decline in price. In contrast, when the market yield declines, the increase in price exceeds the decline in interest on interest.

Notice that the last bond, which ensures that the target accumulated value will be achieved regardless of what happens to the market yield, has a Macaulay duration equal to the length of the investment horizon. This is the key. To immunize a portfolio's target accumulated value (target yield) against a change in the market yield, a portfolio manager must invest in a bond (or a bond portfolio) such that (1) the Macaulay duration is equal to the investment horizon,[6] and (2) the present value

6. This is equivalent to equating the modified duration of the portfolio to the modified duration of the investment horizon. This is because the Macaulay duration of the liability is the length of the investment horizon and dividing by one plus one-half the yield to maturity gives a modified duration that is the same as the portfolio.

EXHIBIT 13–8

Accumulated Value and Total Return

8-Year, 10.125% Coupon Bond Selling to Yield 12.5%

Investment horizon (years) = 5.5; Coupon rate = 10.125%; Maturity (years) = 8; Yield to maturity = 12.5%; Price = $88.20262;
Par value purchased = $10,000,000; Purchase price = $8,820,262; Target accumulated value = $17,183,033.

	Over 5.5 Years				
New Yield (%)	**Coupon ($)**	**Interest on Interest ($)**	**Price of Bond ($)**	**Accumulated Value ($)**	**Total Return (%)**
13.5	5,568,750	2,316,621	9,303,435	17,188,807	12.51
13.0	5,568,750	2,213,102	9,402,621	17,184,473	12.50
12.5	5,568,750	2,111,117	9,503,166	17,183,033	12.50
12.0	5,568,750	2,010,644	9,605,091	17,184,485	12.50
11.5	5,568,750	1,911,661	9,708,420	17,188,831	12.51
11.0	5,568,750	1,814,146	9,813,175	17,196,071	12.51
10.5	5,568,750	1,718,078	9,919,380	17,206,208	12.53
10.0	5,568,750	1,623,436	10,027,059	17,219,245	12.54
9.5	5,568,750	1,530,199	10,136,236	17,235,185	12.56
9.0	5,568,750	1,438,347	10,246,936	17,254,033	12.58
8.5	5,568,750	1,347,859	10,359,184	17,275,793	12.60
8.0	5,568,750	1,258,715	10,473,006	17,300,472	12.63
7.5	5,568,750	1,170,897	10,588,428	17,328,075	12.66

EXHIBIT 13–9

Change in Interest on Interest and Price Due to Interest-Rate Change

8-Year, 10.125% Coupon Bond Selling to Yield 12.5%

New Yield (%)	Change in Interest on Interest ($)	Change in Price ($)	Total Change in Accumulated Value ($)
13.5	205,504	–199,730	5,774
13.0	101,985	–100,544	1,441
12.5	0	0	0
12.0	–100,473	101,925	1,452
11.5	–199,456	205,254	5,798
11.0	–296,971	310,010	13,038
10.5	–393,039	416,215	23,176
10.0	–487,681	523,894	36,212
9.5	–580,918	633,071	52,153
9.0	–672,770	743,771	71,000
8.5	–763,258	856,019	92,760
8.0	–852,402	969,841	117,439
7.5	–940,221	1,085,263	145,042

of the cash flow from the bond (or bond portfolio) equals the present value of the liability.

In this example we assume a one-time instantaneous change in the market yield. In practice, the market yield will fluctuate over the investment horizon. As a result, the Macaulay duration of the portfolio will change as the market yield changes. In addition, the Macaulay duration will change with the passage of time. In the face of changing market yields, a portfolio manager can still immunize a portfolio by rebalancing it so that the Macaulay duration of the portfolio is equal to the remainder of the investment horizon. For example, if the investment horizon is initially 5.5 years, the initial portfolio should have a Macaulay duration of 5.5 years. After 6 months, the remaining investment horizon is 5 years. The Macaulay duration then will probably be different from 5 years. Thus the portfolio must be rebalanced so that its Macaulay duration is equal to 5 years. Six months later, the portfolio must be rebalanced so that its Macaulay duration is equal to 4.5 years, etc.

Constructing a portfolio so that its Macaulay duration is equal to the investment horizon will ensure that the target yield will be realized only if the yields on all bonds change by the same amount. For example, if short-term yields fall but long-term yields rise, then the offsetting of interest-rate risk and reinvestment risk by matching duration will not work. A decline in short-term yields will reduce reinvestment income, while a rise in long-term rates will result in a capital loss of bonds with a maturity greater than the investment horizon. This risk of failing to immunize can be reduced, however.[7]

Duration of an Interest-Rate Swap

An interest-rate swap is an agreement whereby two parties (called *counterparties)* agree to exchange periodic interest payments. The dollar amount of the interest payments exchanged is based on some predetermined dollar principal, which is called the *notional amount*. The dollar amount each counterparty pays to the other is the agreed-upon periodic interest rate times the notional amount. The only dollars that are exchanged between the parties are the interest payments, not the notional amount. In the most common type of interest-rate swap, one party agrees to pay the other party fixed interest payments at designated dates for the life of the contract. This party is referred to as the *fixed-rate payer*. The other party agrees to make interest-rate payments that float with some reference rate and is referred to as the *fixed-rate receiver*.

For example, suppose that for the next 5 years party X agrees to pay party Y 10% per year, while party Y agrees to pay party X 6-month LIBOR. Party X is a fixed-rate payer, while party Y is a fixed-rate receiver. Assume that the notional amount is $50 million and that payments are exchanged every 6 months for the next 5 years. This means that every 6 months party X (the fixed-rate payer) will pay party Y $2.5 million (10% times $50 million divided by 2). The amount that party Y (the fixed-rate receiver) will pay party X will be 6-month LIBOR times $50 million divided by 2. For example, if 6-month LIBOR is 7%, party Y will pay party X $1.75 million (7% times $50 million divided by 2). Note that we divide by 2 because one-half year's interest is being paid.

Interest-rate benchmarks commonly used for the floating rate in an interest-rate swap are those on various money market instruments: Treasury bills, London interbank offered rate (LIBOR), commercial paper, secured overnight financing rate, federal funds rate, and prime rate.

One way to interpret an interest-rate swap position is as a package of cash flows from buying and selling cash market instruments. To understand why, consider the following. Suppose that an investor enters into the following transaction:

- Buys $50 million par of a 5-year floating-rate bond that pays 6-month LIBOR every 6 months;
- Finances the purchase of the 5-year floating-rate bond by borrowing $50 million for 5 years with the following terms: 10% annual interest rate paid every 6 months.

7. See H. Gifford Fong and Oldrich Vasicek, "A Risk Minimizing Strategy for Multiple Liability Immunization," *Journal of Finance* (December 1984), pp. 1541–1546. For a less technical description, see H. Gifford Fong and Frank J. Fabozzi, *Fixed Income Portfolio Management* (Homewood, IL: Dow Jones-Irwin, 1985), pp. 133–136.

EXHIBIT 13–10

Cash Flow for the Purchase of a 5-Year Floating-Rate Bond Financed by Borrowing on a Fixed-Rate Basis

Transaction: Purchase for \$50 million a 5-year floating-rate bond: floating rate = LIBOR, semiannual pay;

Borrow \$50 million for 5 years: fixed rate = 10%, semiannual payments.

	Cash Flow (in millions of dollars) from		
6-Month Period	**Floating-Rate Bond**	**Borrowing Cost**	**Net**
0	–\$50	+\$50.0	\$0
1	+($LIBOR_1$/2) × 50	–2.5	+($LIBOR_1$/2) × 50 – 2.5
2	+($LIBOR_2$/2) × 50	–2.5	+($LIBOR_2$/2) × 50 – 2.5
3	+($LIBOR_3$/2) × 50	–2.5	+($LIBOR_3$/2) × 50 – 2.5
4	+($LIBOR_4$/2) × 50	–2.5	+($LIBOR_4$/2) × 50 – 2.5
5	+($LIBOR_5$/2) × 50	–2.5	+($LIBOR_5$/2) × 50 – 2.5
6	+($LIBOR_6$/2) × 50	–2.5	+($LIBOR_6$/2) × 50 – 2.5
7	+($LIBOR_7$/2) × 50	–2.5	+($LIBOR_7$/2) × 50 – 2.5
8	+($LIBOR_8$/2) × 50	–2.5	+($LIBOR_8$/2) × 50 – 2.5
9	+($LIBOR_9$/2) × 50	–2.5	+($LIBOR_9$/2) × 50 – 2.5
10	+($LIBOR_{10}$/2) × 50 + 50	–52.5	+($LIBOR_{10}$/2) × 50 – 2.5

Note: The subscript for LIBOR indicates the 6-month LIBOR as per the terms of the floating-rate bond at time *t*.

The cash flow of this transaction is presented in Exhibit 13–10. The second column of the exhibit sets out the cash flow from purchasing the 5-year floating-rate bond. There is a \$50 million cash outlay and then cash inflows. The amount of the cash inflows is uncertain because they depend on future LIBOR. The third column shows the cash flow from borrowing \$50 million on a fixed-rate basis. The last column shows the net cash flow from the entire transaction. As can be seen in the last column, there is no initial cash flow (no cash inflow or cash outlay). In all ten 6-month periods the net position results in a cash inflow of LIBOR and a cash outlay of \$2.5 million. This net position, however, is identical to the position of a fixed-rate payer.

It can be seen from the net cash flow in Exhibit 13–10 that a fixed-rate payer has a cash market position that is equivalent to a long position in a floating-rate bond and borrowing the funds to purchase the floating-rate bond on a fixed-rate basis. But the borrowing can be viewed as issuing a fixed-rate bond or, equivalently, being short a fixed-rate bond. Consequently, the position of a fixed-rate payer can be viewed as being long a floating-rate bond and short a fixed-rate bond.

What about the position of a floating-rate payer? It can be demonstrated that the position of a fixed-rate receiver is equivalent to purchasing a fixed-rate bond and financing that purchase at a floating rate, with the floating rate being the reference interest rate for the swap. That is, the position of a fixed-rate receiver is equivalent to a long position in a fixed-rate bond and a short position in a floating-rate bond.

As with any fixed-income instrument, the dollar value of a swap will change as interest rates change. Duration is a measure of the sensitivity of a bond's price to a change in interest rates. From the perspective of the party who pays floating and receives fixed, the position can be viewed as follows:

Long a fixed-rate bond + short a floating-rate bond.

This means that the dollar duration of an interest-rate swap from the perspective of a fixed-rate receiver is just the difference between the dollar duration of the two bond positions that make up the swap. That is,

Dollar duration of a swap
= dollar duration of a fixed-rate bond – dollar duration of a floating-rate bond.

Most of the interest-rate sensitivity of a swap will result from the dollar duration of the fixed-rate bond because the dollar duration of the floating-rate bond will be small. It will always be less than the length of time to the next reset date. Therefore, the duration of a floating-rate bond for which the coupon rate resets every 6 months will be less than 6 months. The dollar duration of a floating-rate bond is smaller, the closer the swap is to its reset date.

FURTHER THOUGHTS ON DURATION

This chapter provided a first look at the most commonly used measure of how sensitive a bond or bond portfolio is to changes in interest rates. There is much more to be said about this measure in the chapters ahead. We conclude this chapter with a brief description of what will be discussed about duration in those chapters.

First, duration is only a first approximation of how a bond's price or portfolio's value will change when interest rates change. In the next chapter, we introduce another measure, convexity, to improve upon the estimation.

Second, the underlying assumption in calculating duration is that all interest rates change by the same number of basis points (i.e., a parallel shift in the yield curve). Measures for assessing the sensitivity of a bond or a bond portfolio to a nonparallel shift in interest rates is the subject of Chapter 15. Actually, for Treasury, corporate, and municipal bonds, the assumption of a parallel shift in the yield curve is not important. However, for bond portfolios and for certain types of mortgage-backed securities, the assumption can be critical.

Third, as explained in this chapter, there is a difference between model (analytical) duration and empirical duration. Our focus in this chapter was on model duration. Empirical duration will be covered in Chapter 16. Moreover, model duration requires that a valuation model be used to determine the values to insert into the duration equation when interest rates are changed up and down. It was very easy to get those values in this chapter because we focused only on option-free bonds. When we cover bonds with embedded options such as fixed-rate callable and putable bonds in Chapter 21, we will see that there are assumptions that are required to value such bonds, and these assumptions impact model duration. One of the assumptions in valuation modeling of bonds with embedded options is how interest rates can change over time. The reason is that how interest rates move over time until a bond matures will determine whether the issuer will exercise the call provision if interest rates decline or whether the bondholder will exercise the put provision if interest rates rise. In Chapter 19, where we discuss interest rate modeling, we will see how the use of different interest rate models can impact the duration and convexity of fixed-rate bonds with an embedded option.

Finally, it is not merely duration that should be looked at in evaluating the risk exposure of a portfolio to changes in interest rates. To assess the potential interest rate risk of a portfolio, it is necessary to consider the volatility of yields themselves. For example, high-yield bonds (i.e., non-investment-grade bonds) trade at higher yield levels than same-maturity Treasury bonds. Given the properties of duration discussed in this chapter, this means that a Treasury bond would have a higher duration than a high-yield bond. Does this mean that high-yield bonds expose investors to less interest-rate risk? The answer is no. How much price volatility a portfolio will be exposed to depends on how volatile yields are. Because high-yield bonds have greater yield volatility (i.e., spreads to Treasuries change even when Treasury yields are unchanged), it would not be correct to say that there is less potential price volatility despite a lower duration. The key to understanding the price volatility of a portfolio is to look not only at duration but also at yield volatility, as discussed in Chapter 18.

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CHAPTER

14

COMBINING DURATION AND CONVEXITY TO MEASURE PRICE VOLATILITY

We're now ready to tie together the price/yield relationship and several of the properties about bond price volatility that we discussed in previous chapters. Recall that the shape of the price/yield relationship is convex. It is the convex shape that gives rise to the properties.[1]

ESTIMATING PRICE WITH DURATION: A GRAPHICAL DEPICTION

In the previous chapter we explained how duration (modified or dollar) can be used to estimate price change when yield changes. In Exhibit 14–1, a line is drawn tangent to the curve depicting the price/yield relationship at point y^*.[2] The tangent line shows the rate of change of price with respect to a change in interest rates at that point (yield). Consequently, the tangent line is directly related to the dollar duration of the bond.[3]

How can the tangent line be used to approximate the new price if yield changes? If we draw a vertical line from any yield (on the horizontal axis), as we do in Exhibit 14–2, the distance between the horizontal axis and the tangent line represents the price as estimated using duration. Notice that the approximation will always underestimate the actual price at the new yield. This agrees with what we illustrated in the previous chapter: duration leads to an underestimate of the new price.

1. The formulas shown in this chapter are presented without proof.
2. In nontechnical terms, a tangent line is defined as a line that touches the price/yield relationship at the point y^* and does not touch the price/yield relationship at any other point.
3. Technically, the slope of the tangent line is the change in price for a change in yield, sometimes referred to as *dP/dy*, a term adopted from calculus because the slope is the first derivative of the price function for a bond. The modified duration would be the slope of the tangent line if the price/yield relationship is drawn with the *natural logarithm* of the bond's price, rather than price, on the vertical axis. To simplify the discussion below, we shall refer to the slope of the tangent line as the *dollar duration*.

EXHIBIT 14–1

Tangent to the Price/Yield Relationship

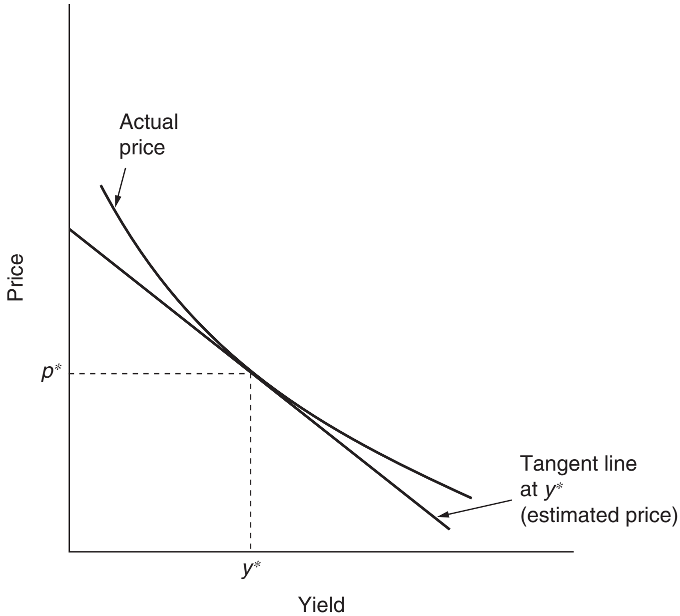


EXHIBIT 14–2

Estimating the Price Using Duration

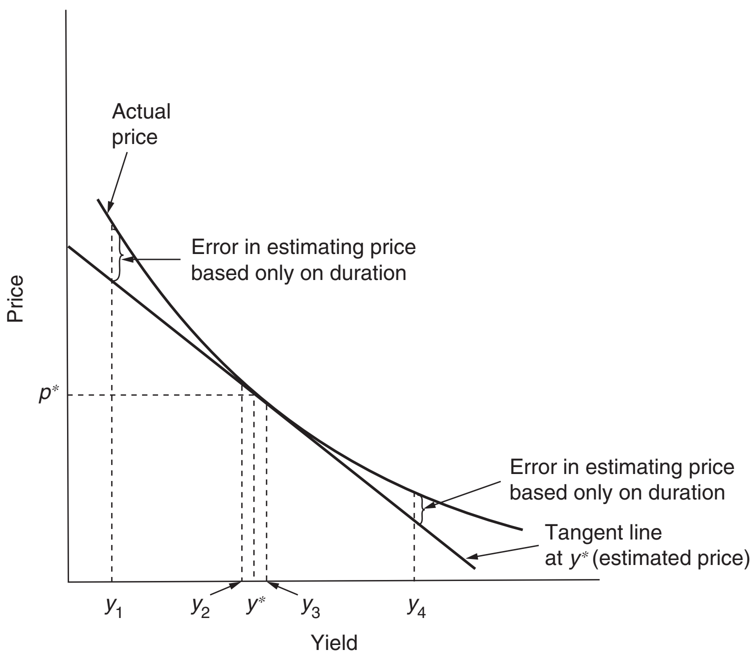
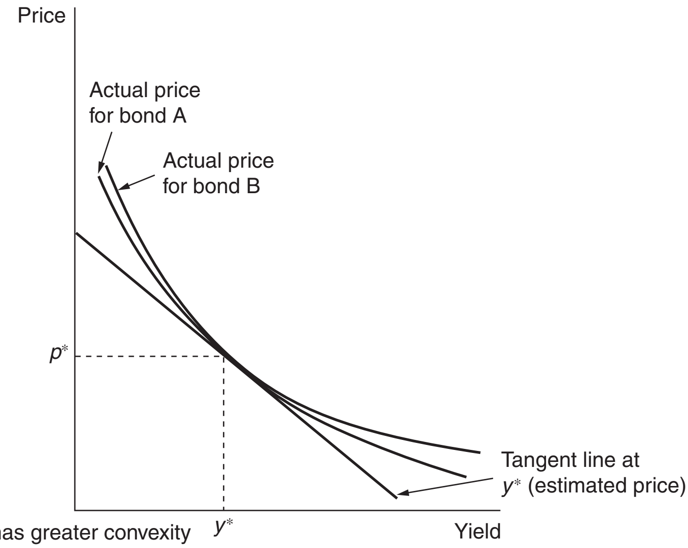


EXHIBIT 14–3

Comparison of Two Bonds with Different Convexities but the Same Duration



Bond B has greater convexity than bond A

For small changes in yield, the tangent line gives a satisfactory estimate of the actual price. However, the further we move from the initial yield, y^*, the worse is the approximation. It should be apparent that the accuracy of the price approximation from using duration (the tangent line) depends on the convexity of the price/yield relationship for the bond. Exhibit 14–3 shows the convexity of two hypothetical bonds. Both have the same duration at y^*, but bond A has less convexity than bond B. As a result, the duration-based approximation of the price is better for bond A than for bond B.

MEASURING CONVEXITY

Exhibit 14–3 indicates that how well we can approximate the new price will depend on the convexity[4] of a bond. In this section we provide a formula for measuring the convexity of a bond at a given yield level. In the next section we show how the price change due to convexity can be estimated.

The convexity of an option-free bond at a given yield level is measured as follows:

Convexity (in periods)

$$= \frac{1(2)\,PVCF_1 + 2(3)\,PVCF_2 + 3(4)\,PVCF_3 + \cdots + n(n+1)\,PVCF_n}{(1+y)^2 \times PVTCF}, \qquad (14\text{–}1)$$

4. In some books, convexity is referred to as the *convexity measure*.

where

$PVCF_t$ = present value of the cash flow in period t discounted at the prevailing period yield (in the case of a semiannual-pay bond, one-half the yield to maturity);
t = period when the cash flow is expected to be received ($t = 1, \ldots, n$);
n = number of periods until maturity;
y = one-half the yield to maturity;
$PVTCF$ = Total present value of the cash flow of the bond, where the present value is determined by using the prevailing yield to maturity.

For a zero-coupon bond, the convexity in periods reduces to

$$\text{Convexity for a zero-coupon bond (in periods)} = \frac{n(n+1)}{(1+y)^2}.$$

To convert a bond's convexity from periods to years, the following formula is used:

$$\text{Convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2},$$

where k = number of payments per year (i.e., $k = 2$ for semiannual-pay bonds, $k = 12$ for monthly pay bonds).

For an option-free bond, the convexity measure will always be positive.

Exhibit 14–4 shows the calculation of convexity for an 8%, 5-year bond whose Macaulay duration is calculated in Exhibit 13–1 of Chapter 13, assuming that each bond is selling to yield 10%. For the 5-year, zero-coupon bond, $n = 10$ (number of periods) and y is 0.05. Therefore,

$$\text{Convexity} = \frac{10(10+1)}{(1.05)^2} = 24.94.$$

The convexities in years for the nine hypothetical bonds we examined in the previous chapter are

Coupon (%)	Maturity (years)	Convexity
0.00	5	24.94
0.00	15	210.88
0.00	30	829.94
8.00	5	19.58
8.00	15	94.36
8.00	30	167.56
10.00	5	18.74
10.00	15	87.62
10.00	30	158.70

EXHIBIT 14–4

Worksheet for Computation of Convexity for a 5-Year, 8% Coupon Bond Selling to Yield 10%

Coupon rate = 8%; Term (years) = 5; Initial yield = 10%; Price = 92.27826.

Period (t)	Cash Flow ($)	*PVCF* ($)	$t(t+1)$	*PVCF* × $t(t+1)$ ($)
1	4.00	3.8095	2	7.6190
2	4.00	3.6281	6	22.7687
3	4.00	3.4554	12	41.4642
4	4.00	3.2908	20	65.8162
5	4.00	3.1341	30	94.0231
6	4.00	2.9849	42	125.3642
7	4.00	2.8427	56	159.1926
8	4.00	2.7074	72	194.9297
9	4.00	2.5784	90	232.0592
10	104.00	63.8470	110	7,023.1676
			Total	$7,965.4046

$$\text{Convexity (in half years)} = \frac{7{,}965.4046}{(1.05)^2 92.27826} = 78.2942.$$

$$\text{Convexity (in years)} = \frac{78.2942}{2^2} = 19.58.$$

What do these convexity numbers mean? How can they be used? We will answer these questions in the sections that follow.

PERCENTAGE PRICE CHANGE DUE TO CONVEXITY

Modified duration provides a first approximation to the percentage change in price. Convexity provides a second approximation, based on the following relationship[5]:

$$\begin{aligned}&\text{Approximate percentage change in price due to convexity}\\&\quad = (0.5) \times \text{convexity} \times (\text{yield change})^2. \qquad (14\text{–}2)\end{aligned}$$

Because the convexity measure is always positive for an option-free bond, the approximate percentage change in price due to convexity is positive for either an increase or a decrease in yield.

5. In some books, approximate percentage change in price due to convexity is referred to as the *convexity adjustment*.

Illustration 14–1. Consider the 8%, 15-year bond selling to yield 10%. If the yield increases from 10% to 13% (a 300 basis point or 0.03 yield change), then the approximate percentage price change due to convexity is

$$0.5 \times 94.36 \times (0.03)^2 = 0.0425 = 4.25\%.$$

If the yield decreases by 300 basis points, from 10% to 7%, the approximate percentage price change due to convexity is 4.25%.

Exhibit 14–5 shows the approximate percentage price change due to convexity for various changes in yield for each of the nine hypothetical bonds.

Alternative Formulations for Measuring Convexity and Percentage Price Change Due to Convexity

There is no standard definition for measuring convexity. The one given by equation (14–1) is one possibility. The reason is that equation (14–1) can be scaled in different ways because what is important is not the measurement of convexity but the approximate percentage price change due to convexity as given by equation (14–2). For example, in some publications the measurement for convexity would include in equation (14–1) a 2 in the denominator. That is, the computed value will be one-half the value computed in equation (14–1). This is not a problem because in that case, equation (14–2) would be changed by eliminating 0.5. The approximate percentage price change due to convexity will then be the same. Some vendors of fixed-income analytics would actually divide convexity given by equation (14–1) by 100. Equation (14–2) would then be adjusted by multiplying by 100.

The important point here is that, unlike duration, it is sometimes difficult to compare the measures of convexity from different vendors because of the way convexity can be scaled. Hence one vendor might calculate convexity for a Treasury bond to be 80, while another vendor reports it as 4. Both can be correct. What is important is that if a portfolio manager wants to use the convexity computed by either vendor, he must know how the equivalent of equation (14–1) is computed by the two vendors. This is needed to adjust equation (14–2) in order to calculate the approximate percentage price change due to convexity. If done properly, the percentage price change will be the same.

PERCENTAGE PRICE CHANGE DUE TO DURATION AND CONVEXITY

The approximate percentage change in price resulting from both duration and convexity is found by simply adding the two estimates.

Illustration 14–2. For the 8%, 15-year bond selling to yield 10% if yields change from 10% to 13%, we have

Price Change Based on	Approx. % Price Change
Duration	–24.15
Convexity	+4.25
Total	–19.90

EXHIBIT 14–5

Percentage Price Change Due to Convexity

			Change (in Basis Points)					
Coupon (%)	**Term (years)**	**Convexity**	**1**	**10**	**50**	**100**	**200**	**300**
0	5	24.94	0.00%	0.00%	0.03%	0.12%	0.50%	1.12%
0	15	210.88	0.00	0.01	0.26	1.05	4.22	9.49
0	30	829.94	0.00	0.04	1.04	4.15	16.60	37.35
8	5	19.58	0.00	0.00	0.02	0.10	0.39	0.88
8	15	94.36	0.00	0.00	0.12	0.47	1.89	4.25
8	30	167.56	0.00	0.01	0.21	0.84	3.35	7.54
10	5	18.74	0.00	0.00	0.02	0.09	0.37	0.84
10	15	87.62	0.00	0.00	0.11	0.44	1.75	3.94
10	30	158.70	0.00	0.01	0.20	0.79	3.17	7.14

The actual percentage change in price would be –20.41%.

For a decrease of 300 basis points, from 10% to 7%:

Price Change Based on	Approx. % Price Change
Duration	+24.15
Convexity	+4.25
Total	28.40

The actual percentage price change would be +29.03%. Consequently, for large yield movements, a better approximation for bond price volatility is obtained by combining duration and convexity.

Exhibit 14–6 shows the percentage price changes due to both duration and convexity for our nine hypothetical bonds. The percentage price change not explained by duration and convexity is shown in Exhibit 14–7. As can be seen from this exhibit, most of the change in price is explained by using duration and convexity.

Dollar Convexity

In the previous chapter we explained that dollar duration can be obtained by multiplying modified duration by the initial price. Dollar convexity can be obtained by multiplying convexity by the initial price:

$$\text{Dollar convexity} = \text{convexity} \times \text{initial price}.$$

To determine the dollar price change, the following formula is used:

$$\begin{aligned}&\text{Dollar price change due to convexity}\\&\quad = (0.5) \times \text{dollar convexity} \times (\text{yield change})^2.\end{aligned}$$

Illustration 14–3. For the 8%, 15-year bond selling to yield 10%, the dollar convexity per $100 of par value is

$$94.36 \times \$84.63 = \$7{,}985.69.$$

The dollar price change due to convexity per $100 of par value for a 100 basis point change is

$$(0.5) \times \$7{,}985.69 \times (0.01)^2 = \$0.399.$$

Thus, for a 100 basis point change, the price of the bond will change by approximately $0.40 per $100 par value due to convexity.

For a 200 basis point change, the dollar price change per $100 of par value due to convexity is

$$(0.5) \times \$7{,}985.69 \times (0.02)^2 = \$1.597.$$

For a 200 basis point change, the dollar price change due to convexity is approximately $1.60 per $100 of par value—quadruple the $0.40 for a 100 basis point

EXHIBIT 14–6

Estimated Percentage Price Change Using Duration and Convexity

		Yield Change (in Basis Points) from 10%					
		1	**10**	**50**	**100**	**200**	**300**
		New Yield Level					
Coupon (%)	**Term (years)**	**10.01%**	**10.10%**	**10.50%**	**11.00%**	**12.00%**	**13.00%**
0	5	–0.05%	–0.47%	–2.35%	–4.64%	–9.02%	–13.16%
0	15	–0.14	–1.42	–6.88	–13.24	–24.36	–33.38
0	30	–0.29	–2.82	–13.25	–24.42	–40.54	–48.36
8	5	–0.04	–0.40	–1.97	–3.88	–7.57	–11.06
8	15	–0.08	–0.80	–3.91	–7.58	–14.21	–19.90
8	30	–0.10	–0.96	–4.65	–8.88	–16.09	–21.62
10	5	–0.04	–0.39	–1.91	–3.77	–7.35	–10.74
10	15	–0.08	–0.76	–3.74	–7.25	–13.63	–19.13
10	30	–0.09	–0.94	–4.53	–8.67	–15.75	–21.24

EXHIBIT 14–6

Estimated Percentage Price Change Using Duration and Convexity (*Continued*)

		Yield Change (in Basis Points) from 10%					
		–1	–10	–50	–100	–200	–300
		New Yield Level					
Coupon (%)	**Term (years)**	**9.99%**	**9.90%**	**9.50%**	**9.00%**	**8.00%**	**7.00%**
0	5	0.05%	0.48%	2.41%	4.88%	10.02%	15.40%
0	15	0.14	1.44	7.41	15.34	32.80	52.36
0	30	0.29	2.90	15.32	32.72	73.74	123.06
8	5	0.04	0.40	2.01	4.08	8.35	12.82
8	15	0.08	0.81	4.14	8.52	17.99	28.40
8	30	0.10	0.98	5.07	10.56	22.79	36.70
10	5	0.04	0.39	1.95	3.95	8.09	12.42
10	15	0.08	0.77	3.95	8.13	17.13	27.01
10	30	0.09	0.95	4.93	10.25	22.09	35.52

EXHIBIT 14–7

Estimated Percentage Price Change Not Explained by Using Both Duration and Convexity

		Yield Change (in Basis Points) from 10%					
		1	**10**	**50**	**100**	**200**	**300**
		New Yield Level					
Coupon (%)	**Term (years)**	**10.01%**	**10.10%**	**10.50%**	**11.00%**	**12.00%**	**13.00%**
0	5	0.00%	0.00%	0.00%	0.00%	–0.02%	–0.07%
0	15	0.00	0.00	0.00	–0.05	–0.39	–1.28
0	30	0.00	0.00	–0.05	–0.38	–2.83	–8.94
8	5	0.00	0.00	0.00	0.00	–0.02	–0.05
8	15	0.00	0.00	0.00	–0.02	–0.15	–0.51
8	30	0.00	0.00	–0.01	–0.06	–0.43	–1.39
10	5	0.00	0.00	0.00	0.00	–0.01	–0.05
10	15	0.00	0.00	0.00	–0.01	–0.14	–0.46
10	30	0.00	0.00	–0.01	–0.06	–0.42	–1.31

EXHIBIT 14–7

Estimated Percentage Price Change Not Explained by Using Both Duration and Convexity (*Continued*)

		Yield Change (in Basis Points) from 10%					
		–1	**–10**	**–50**	**–100**	**–200**	**–300**
		New Yield Level					
Coupon (%)	**Term (years)**	**9.99%**	**9.90%**	**9.50%**	**9.00%**	**8.00%**	**7.00%**
0	5	0.00%	0.00%	0.00%	0.00%	0.02%	0.07%
0	15	0.00	0.00	0.00	0.05	0.46	1.62
0	30	0.00	0.00	0.05	0.44	3.83	14.05
8	5	0.00	0.00	0.00	0.00	0.02	0.05
8	15	0.00	0.00	0.00	0.02	0.18	0.64
8	30	0.00	0.00	0.01	0.06	0.56	2.03
10	5	0.00	0.00	0.00	0.00	0.02	0.05
10	15	0.00	0.00	0.00	0.02	0.16	0.58
10	30	0.00	0.00	0.01	0.07	0.53	1.90

change. Consequently, unlike dollar duration in which the dollar price change due to duration is proportionate to the change in yield—doubling the yield change from 100 basis points to 200 basis points, for example, doubles the dollar price change due to duration—the dollar price change due to convexity changes more than proportionately.

SUMMARY OF PROPERTIES OF CONVEXITY

The convexity properties of all option-free bonds are summarized below.

> ***Property 1*** As the yield increases (decreases), the dollar duration of a bond decreases (increases). We demonstrated this in the previous section.
>
> ***Property 2*** For a given yield and maturity, the lower the coupon, the greater is the convexity of a bond. This can be seen from the computed convexity of our nine hypothetical bonds. An implication of this property is that for two bonds with the same maturity, a zero-coupon bond has greater convexity than a coupon bond.
>
> ***Property 3*** For a given yield and modified duration, the lower the coupon, the smaller is the convexity. The investment implication of this property is that zero-coupon bonds have the lowest convexity for a given modified duration.
>
> ***Property 4*** The convexity of a bond increases at an increasing rate as duration increases. Doubling duration, for example, will more than double convexity.

VALUE OF CONVEXITY

Look again at Exhibit 14–3, where bonds A and B have the same duration but different convexities. We stated that bond B would be preferred to bond A because both have the same price (same yield) and the same duration, but bond B has greater convexity than bond A. Thus it offers better price performance if yield changes.

Generally, the market will take the greater convexity of bond B compared to bond A into account in pricing the two bonds. That is, the market prices convexity. Consequently, while there may be times when a situation such as depicted in Exhibit 14–3 may exist, generally the market will require investors to "pay up" (accept a lower yield) for the greater convexity offered by bond B.

How much should the market want investors to pay for convexity? Look again at Exhibit 14–3. Notice that if investors expect that yields will change by very little—that is, they expect low interest-rate volatility—the advantage of owning bond B over bond A is insignificant because the two bonds will offer approximately the same price performance for small changes in yield. Thus investors should not be willing to pay much for convexity. In fact, if the market is pricing convexity high, which means that bond A will be offering a higher yield than bond B, then investors with expectations of low interest-rate volatility would probably be willing to "sell convexity"—sell bond B if they own it, and buy bond A. In contrast, if investors expect substantial interest-rate volatility, bond B would probably sell at a much lower yield than bond A.

Illustration 14–4. To see how two portfolios with the same dollar duration but with different convexities can perform differently, consider the three bonds shown in Exhibit 14–8 and the following two portfolios.[6] One portfolio consists of only bond C, the 10-year bond, and shall be referred to as the *bullet portfolio*. A second portfolio consists of 50.2% of bond A and 49.8% of bond B; this portfolio shall be referred to as the *barbell portfolio*. The dollar duration per 100 basis point change in yield of the bullet portfolio is $6.43409. Recall from Chapter 13 that dollar duration is a measure of the dollar price sensitivity of a bond or a portfolio. As shown in Exhibit 14–8, the dollar duration per 100 basis point change in yield of the barbell—which is just the weighted average of the dollar duration of the two bonds—is the same as for the bullet portfolio. In fact, the barbell portfolio was designed to produce this result. The dollar convexity of the two portfolios, shown in Exhibit 14–8, is not equal. The dollar convexity of the bullet portfolio is less than that of the barbell portfolio.

The "yield" for the two portfolios is not the same. The yield (yield to maturity) for the bullet portfolio is simply the yield to maturity of bond C, 9.25%. The traditional yield calculation for the barbell portfolio, which is found by taking a weighted average of the yield to maturity of the two bonds included in the portfolio, is 8.998%. This would suggest that the "yield" of the bullet portfolio is 25.2 basis points greater than that for the barbell portfolio. Alternatively, a cash flow yield can be approximated for the barbell portfolio by calculating the dollar-duration market-weighted yield of the portfolio. As shown in Exhibit 14–8, the cash-flow yield of the barbell portfolio is 9.187%, suggesting that the "yield" of the bullet portfolio is 6.3 basis points greater than that for the barbell portfolio. Thus both portfolios have the same dollar duration but, using either yield measure, the yield of the bullet portfolio is greater than the yield of the barbell portfolio. However, the dollar convexity of the barbell portfolio is greater than that of the bullet portfolio. The difference in the two yields is sometimes referred to as the *cost of convexity*.

Exhibit 14–9 shows the difference in the total return over a 6-month investment horizon for the two portfolios, assuming that the yields for all three bonds change by the same number of basis points shown in the first column.[7] The total return reported in the second column of Exhibit 14–9 is

$$\text{Bullet portfolio's total return} - \text{barbell portfolio's total return.}$$

6. This illustration is adapted from Ravi E. Dattatreya and Frank. J. Fabozzi, *Active Total Return Management of Fixed Income Portfolios* (Chicago: Probus Publishing, 1989).
7. Note that no assumption is needed for the reinvestment rate because the three bonds shown in Exhibit 14–9 are assumed to be trading right after a coupon payment has been made and therefore there is no accrued interest.

EXHIBIT 14–8

Barbell-Bullet Analysis

Three Bonds Used in Analysis

Bond	Coupon (%)	Maturity (years)	Price Plus Accrued	Yield (%)	Dollar Duration*	Dollar Convexity*
A	8.50	5	100	8.50	4.00544	19.8164
B	9.50	20	100	9.50	8.88151	124.1702
C	9.25	10	100	9.25	6.43409	55.4506

Bullet portfolio = bond C;
Barbell portfolio = bonds A and B;
Composition of barbell portfolio = 50.2% of bond A; 49.8% of bond B.

Dollar duration of barbell

$$= 0.502 \times 4.00544 + 0.498 \times 8.88151 = 6.434.$$

Average yield of barbell

$$= 0.502 \times 8.50 + 0.498 \times 9.5 = 8.998.$$

Cash-flow yield of barbell**

$$= \frac{(8.5 \times 0.502 \times 4.00544) + (9.5 \times 0.498 \times 8.88151)}{6.434} = 9.187.$$

Yield pickup = yield on bullet – cash-flow yield of barbell
= 9.24 – 9.187 = 0.063, or 6.3 basis points.

Analysis based on duration, convexity, and average yield:

Dollar convexity of barbell

$$= 0.502 \times 19.8164 + 0.498 \times 124.1702 = 71.7846.$$

Yield pickup = yield on bullet = average yield of barbell
= 9.25 – 8.998 = 0.252, or 25.2 basis points

Convexity give-up = convexity of barbell – convexity of bullet
= 71.7846 – 55.4506 = 16.334.

*Per 100 basis point change in yield.

**The calculation shown is actually a dollar-duration weighted yield, a very close approximation to cash-flow yield.

Thus a positive sign in the second column means that the bullet portfolio outperformed the barbell portfolio, while a negative sign means that the barbell portfolio outperformed the bullet portfolio.

Which portfolio is the better investment alternative if the yields of all three bonds change by the same amount *and* the investment horizon is 6 months? The

EXHIBIT 14–9

Relative Performance of Bullet Portfolio and Barbell Portfolio over a 6-Month Investment Horizon

Yield Change	Difference in Total Return	Yield Change	Difference in Total Return
–5.000	–7.19	0.250	0.24
–4.750	–6.26	0.500	0.21
–4.500	–5.44	0.750	0.16
–4.250	–4.68	1.000	0.09
–4.000	–4.00	1.250	0.01
–3.750	–3.38	1.500	–0.08
–3.500	–2.62	1.750	–0.19
–3.250	–2.32	2.000	–0.31
–3.000	–1.88	2.250	–0.44
–2.750	–1.49	2.500	–0.58
–2.500	–1.15	2.750	–0.73
–2.250	–0.85	3.000	–0.88
–2.000	–0.59	3.250	–1.05
–1.750	–0.38	3.500	–1.21
–1.500	–0.20	3.750	–1.39
–1.250	–0.05	4.000	–1.57
–1.000	0.06	4.250	–1.75
–0.750	0.15	4.500	–1.93
–0.500	0.21	4.750	–2.12
–0.250	0.24	5.000	–2.31
0.000	0.25		

Note: Performance is based on the difference in total return over the 6-month investment horizon. Specifically,

Bullet portfolio's total return – barbell portfolio's total return.

Therefore, a negative value means that the barbell portfolio outperformed the bullet portfolio.

answer depends on the amount by which yields change. Notice in the second column that if yields change by less than 100 basis points, the bullet portfolio will outperform the barbell portfolio. The reverse is true if yields change by more than 100 basis points.

The key point here is that looking at measures such as yield (yield to maturity or some type of portfolio yield measure), duration, or convexity tells us little about performance over some investment horizon because performance depends on the magnitude of the change in yields. Moreover, as we shall see in Chapter 15, the relative performance will also depend on the relative change in yields of each bond.

APPROXIMATING CONVEXITY: EFFECTIVE CONVEXITY

The convexity measure of any bond can be approximated using the following formula:

$$\text{Convexity} = \frac{V_+ + V_- - 2V_0}{V_0(\Delta y)^2},$$

where

V_0 = initial value or price of the bond;
Δy = change in the yield of the bond;
V_- = estimated value of the bond if the yield is decreased by Δy;
V_+ = estimated value of the bond if the yield is increased by Δy.

These are the same values used for approximating duration.

For example, consider the 20-year, 7% coupon bond selling at 74.26 to yield 10% that we used in the previous chapter to illustrate how to approximate modified duration. Suppose that we evaluate the price change for a 20 basis point change up and down. Then

$$V_+ = 72.92;\ V_- = 75.64;\ V_0 = 74.26;\ \Delta y = 0.002.$$

Substituting these values in the formula, we get

$$\frac{72.92 + 75.64 - 2(74.26)}{(74.26)(0.002)^2} = 134.66.$$

The convexity is equal to 132.08, so the approximate convexity (134.66) has proven itself to be a good approximation of convexity (132.08).

Effective Convexity

Just as in the case of modified duration, the convexity measure discussed in this chapter does not take into consideration how the cash flows may change if yield changes. In later chapters we can use the formula above to calculate the convexity of a bond when there are interest-rate-sensitive cash flows. When the formula above is used in this way, the resulting convexity is called *effective convexity* because it allows the cash flow to change as yields change.

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CHAPTER
15

DURATION AND THE YIELD CURVE

As we explained in Chapter 7, the yield curve describes the relationship between maturity and yield on U.S. Treasury securities. The shape of the yield curve changes over time. Because a portfolio consists of bonds of different maturities, changes in the shape of the yield curve will have different effects on each bond's price and on the portfolio.

Our objective in this chapter is twofold. First, we illustrate how the duration measure discussed in Chapter 13 when applied to a portfolio of bonds does not capture the effect of changes in the shape of the yield curve. Second, we describe several approaches for measuring exposure to a shift in the yield curve.

DURATION AND NONPARALLEL YIELD-CURVE SHIFTS

Two portfolios that have the same duration may perform quite differently if the yield curve does not shift in a parallel fashion. To illustrate this point, consider the three bonds in Illustration 14–4 of Chapter 14 and summarized in Exhibit 14–8.[1] Two portfolios were created from these three bonds: a bullet portfolio consisting of bond C and a barbell portfolio consisting of roughly equal amounts of bonds A and B. As explained in Chapter 14, both portfolios have the same dollar duration. However, the dollar convexity of the two portfolios is not equal. The dollar convexity of the bullet portfolio is less than that of the barbell portfolio. Also, the "yield" for the two portfolios is not the same. The yield for the bullet portfolio is simply the yield to maturity of bond C, 9.25%. For the barbell portfolio, the traditional yield calculation is found by taking a weighted average of the yield to maturity of the two bonds included in the portfolio, which is 8.998%. Thus the yield of the bullet portfolio is 25.2 basis points higher than that of the barbell portfolio.

Exhibit 15–1, column 2, shows the difference in the total return over a 6-month investment horizon for the two portfolios assuming that the yield curve shifts in a parallel fashion.[2] By parallel we mean that the yield for the short-term bond (A), the intermediate-term bond (C), and the long-term bond (B) change by

1. This illustration is adapted from Ravi E. Dattatreya and Frank J. Fabozzi, *Active Total Return Management of Fixed Income Portfolios* (Chicago: Probus Publishing, 1995).
2. Note that no assumption is needed for the reinvestment rate because the three bonds are assumed to be trading right after a coupon payment has been made and therefore there is no accrued interest.

EXHIBIT 15–1

Relative Performance of Bullet Portfolio and Barbell Portfolio over a 6-Month Investment Horizon*

Yield Change (%)	Parallel Shift (%)	Flattened Yield Curve (%)**	Steepened Yield Curve (%)***
–5.000	–7.19	–10.69	–3.89
–4.750	–6.28	–9.61	–3.12
–4.500	–5.44	–8.62	–2.44
–4.250	–4.68	–7.71	–1.82
–4.000	–4.00	–6.88	–1.27
–3.750	–3.38	–6.13	–0.78
–3.500	–2.82	–5.44	–0.35
–3.250	–2.32	–4.82	0.03
–3.000	–1.88	–4.26	0.36
–2.750	–1.49	–3.75	0.65
–2.500	–1.15	–3.30	0.89
–2.250	–0.85	–2.90	1.09
–2.000	–0.59	–2.55	1.25
–1.750	–0.38	–2.24	1.37
–1.500	–0.20	–1.97	1.47
–1.250	–0.05	–1.74	1.53
–1.000	0.06	–1.54	1.57
–0.750	0.15	–1.38	1.58
–0.500	0.21	–1.24	1.57
–0.250	0.24	–1.14	1.53
0.000	0.25	–1.06	1.48
0.250	0.24	–1.01	1.41
0.500	0.21	–0.98	1.32
0.750	0.16	–0.97	1.21
1.000	0.09	–0.98	1.09
1.250	0.01	–1.00	0.96
1.500	–0.08	–1.05	0.81
1.750	–0.19	–1.10	0.66
2.000	–0.31	–1.18	0.49
2.250	–0.44	–1.26	0.32
2.500	–0.58	–1.36	0.14
2.750	–0.73	–1.46	–0.05
3.000	–0.88	–1.58	–0.24
3.250	–1.05	–1.70	–0.44
3.500	–1.21	–1.84	–0.64
3.750	–1.39	–1.98	–0.85
4.000	–1.57	–2.12	–1.06

EXHIBIT 15–1

Relative Performance of Bullet Portfolio and Barbell Portfolio over a 6-Month Investment Horizon* (*Continued*)

Yield Change (%)	Parallel Shift (%)	Flattened Yield Curve (%)**	Steepened Yield Curve (%)***
4.250	–1.75	–2.27	–1.27
4.500	–1.93	–2.43	–1.48
4.750	–2.12	–2.58	–1.70
5.000	–2.31	–2.75	–1.92

*Performance is based on the difference in total return over a 6-month investment horizon. Specifically,

Bullet portfolio's total return – barbell portfolio's total return.

Therefore, a negative value means that the barbell portfolio outperformed the bullet portfolio.

**Change in yield for bond C. Nonparallel shift as follows (flattening of yield curve):

Yield change bond A = yield change bond C + 25 basis points;
Yield change bond B = yield change bond C – 25 basis points.

***Change in yield for bond C. Nonparallel shift as follows (steepening of yield curve):

Yield change bond A = yield change bond C – 25 basis points;
Yield change bond B = yield change bond C + 25 basis points.

the same number of basis points, as shown in the first column of the exhibit. The total return reported in the second column of Exhibit 15–1 is

Bullet portfolio's total return – barbell portfolio's total return.

Thus a positive value in the second column means that the bullet portfolio outperforms the barbell portfolio, while a negative sign means that the barbell portfolio outperforms the bullet portfolio. As explained in Illustration 14–4, the better investment alternative depends on the amount by which yields change. If the yields change by less than 100 basis points, the bullet portfolio will outperform the barbell portfolio. The reverse is true if yields change by more than 100 basis points.

Now let's look at what happens if the yield curve does not shift in a parallel fashion. The third and fourth columns of Exhibit 15–1 show the relative performance of the two portfolios for a nonparallel shift of the yield curve. Specifically, the third column assumes that if the yield on bond C (the intermediate-term bond) changes by the amount shown in the first column, the yield on bond A (the short-term bond) will change by the same amount plus 25 basis points, while the yield on bond B (the long-term bond) will change by the same amount shown in the first column less 25 basis points. In this case, the nonparallel shift assumed is a flattening of the yield curve. For this yield-curve shift, the barbell will always outperform the bullet.

In the fourth column the nonparallel shift assumes that for a change in bond C's yield, the yield on bond A will change by the same amount less 25 basis points, while that on bond B will change by the same amount plus 25 points. That is, this nonparallel shift means that the yield curve will steepen. In this case, the bullet portfolio would outperform the barbell portfolio so long as the yield on bond C does not rise by more than 250 basis points or fall by more than 325 basis points.

The key point here is that measures such as yield (yield to maturity or some type of portfolio yield measure), duration, or convexity tell us little about performance over some investment horizon because performance depends on the magnitude of the change in yields and how the yield curve shifts.

Yield-Curve Shifts and the Value of Convexity

In Chapter 14 it was stated that there is a tradeoff between convexity and yield. This statement is not correct once nonparallel shifts in the yield curve are considered. This should not be surprising because as argued in earlier chapters, the yield measure is not a good indicator of the potential return. To illustrate this, consider the three hypothetical bonds shown in Exhibit 15–2. A barbell portfolio with the same dollar duration as the bullet portfolio was constructed. At the bottom of the exhibit are the yield, dollar duration, and dollar convexity. Notice that the average yield and the dollar convexity are greater for the barbell portfolio than for the bullet portfolio.

Thus it would seem that the barbell portfolio in our illustration would perform better than the bullet portfolio over a 6-month investment horizon. This is, in fact, the case for a parallel shift in the yield, as can be seen in Exhibit 15–3. However, this is not the case for nonparallel yield-curve shifts. Exhibit 15–4 shows that if the yield curve steepens as assumed in the exhibit, the bullet outperforms the barbell over the 6-month investment horizon.

EXHIBIT 15–2

Three Hypothetical Bonds to Illustrate the Lack of Tradeoff Between Yield and Convexity

Bond	Coupon Rate (%)	Price ($)	Yield to Maturity (%)	Maturity (years)
X	7.900	100	7.900	2
Y	8.800	100	8.800	7
Z	8.200	100	8.200	4

Bullet portfolio = bond Z;
Barbell portfolio = bonds X and Y;
Composition of barbell portfolio = 53.86% of bond X; 46.14% of bond Y.

Summary of parameters

Portfolio	Yield (%)	Dollar Duration	Dollar Convexity
Bullet	8.20	3.35253	6.90699
Barbell	8.32	3.35253	8.89010

EXHIBIT 15–3

Performance of Bullet and Barbell Portfolios over a 6-Month Horizon Assuming Parallel Yield-Curve Shifts

Yield Change (in Basis Points)	*Price Plus Coupon ($)*			*Total Return (%)*		
	X	Y	Z	Bullet	Barbell	Difference*
–300	108.2382	120.4549	113.5879	27.18	27.75	–0.57
–250	107.5063	117.5671	111.9321	23.86	24.30	–0.43
–200	106.7813	114.7679	110.3069	20.61	20.93	–0.32
–150	106.0633	112.0545	108.7117	17.42	17.66	–0.23
–100	105.3522	109.4239	107.1459	14.29	14.46	–0.17
–50	104.6477	106.8733	105.6089	11.22	11.35	–0.13
–25	104.2980	105.6272	104.8510	9.70	9.82	–0.12
0	103.9500	104.4000	104.1000	8.20	8.32	–0.12
25	103.6036	103.1915	103.3559	6.71	6.83	–0.12
50	103.2589	102.0013	102.6187	5.24	5.36	–0.12
100	102.5742	99.6748	101.1644	2.33	2.47	–0.14
150	101.8960	97.4180	99.7365	–0.53	–0.34	–0.19
200	101.2242	95.2286	98.3345	–3.33	–3.08	–0.25
250	100.5587	93.1044	96.9579	–6.08	–5.76	–0.32
300	99.8993	91.0431	95.6061	–8.79	–8.37	–0.41

*A positive sign indicates that the bullet portfolio outperformed the barbell portfolio; a negative sign indicates that the barbell portfolio outperformed the bullet portfolio.

EXHIBIT 15–4

Performance of Bullet and Barbell Portfolios over a 6-Month Horizon Assuming a Steepening of the Yield Curve

Yield Change for Z (in Basis Points)	Price Plus Coupon ($)			Total Return (%)		
	X	Y	Z	Bullet	Barbell	Difference*
–0.0300	108.6807	118.7114	113.5879	27.18	26.62	0.56
–0.0250	107.9446	115.8771	111.9321	23.86	23.21	0.65
–0.0200	107.2155	113.1298	110.3069	20.61	19.89	0.72
–0.0150	106.4933	110.4664	108.7117	17.42	16.65	0.77
–0.0100	105.7780	107.8842	107.1459	14.29	13.50	0.79
–0.0050	105.0696	105.3802	105.6089	11.22	10.43	0.79
–0.0025	104.7179	104.1568	104.8510	9.70	8.92	0.78
0.0000	104.3678	102.9520	104.1000	8.20	7.43	0.77
0.0025	104.0195	101.7655	103.3559	6.71	5.96	0.75
0.0050	103.6728	100.5969	102.6187	5.24	4.51	0.73
0.0100	102.9842	98.3125	101.1644	2.33	1.66	0.67
0.0150	102.3022	96.0964	99.7365	–0.53	–1.12	0.60
0.0200	101.6265	93.9464	98.3345	–3.33	–3.83	0.50
0.0250	100.9572	91.8602	96.9579	–6.08	–6.48	0.40
0.0300	100.2942	89.8356	95.6061	–8.79	–9.06	0.28

Assumptions:

Change in yield of bond Z (column 1) results in a change in the yield of bond X minus 30 basis points;

Change in yield of bond Z (column 1) results in a change in the yield of bond Y plus 30 basis points.

*A positive sign indicates that the bullet portfolio outperformed the barbell portfolio; a negative sign indicates that the barbell portfolio outperformed the bullet portfolio.

TYPES OF YIELD-CURVE SHIFTS AND APPROACHES TO MEASURING YIELD-CURVE RISK

It is clear from our illustrations thus far that duration alone is not sufficient for controlling interest-rate risk because duration assumes a parallel shift in the yield curve. What types of yield-curve shifts have been observed historically? Empirical studies of major bond markets throughout the world have found that three types of shifts dominate:

- A parallel shift;
- A change in the shape of the yield curve;
- A change in the curvature of the yield curve.

We will discuss these studies in Chapter 31 when we explain how the statistical technique of principal component analysis is used to analyze interest-rate movements in order to describe the way the yield curve changes.

There are several approaches that have been suggested and applied to measuring interest-rate risk that go beyond duration or equivalently that allow for a more general shift in the yield curve. These include

- Key rate durations;
- Level, slope, and curvature durations;
- Yield-curve-reshaping durations;
- Value-at-risk.

Value-at-risk has been increasingly used to measure the risk exposure of a trading position, portfolio, or financial institution to changes in interest rates. This measure draws on probability theory and statistics, and therefore, we postpone a discussion of this measure until Chapter 30, where we cover probability theory. We discuss the first three measures below. These measures are interrelated. Three of the approaches—key rate durations; level, slope, and curvature durations; and value-at-risk—have been tied together by Golub and Tilman, who also present them within a risk-management framework.[3]

While the measures described below estimate what the interest-rate exposure is for a given shift in the yield curve, the issue that portfolio managers and traders face is what are reasonable interest-rate yield-curve shifts that should be used in scenario analysis and stress testing. To address that question, Golub and Tilman[4]

3. Bennett W. Golub and Leo M. Tilman, "Measuring Yield Curve Risk Using Principal Components Analysis, Value at Risk, and Key Rate Durations," *Journal of Portfolio Management* (Summer 1997), pp. 72–84.
4. Bennett W. Golub and Leo M. Tilman, "Measuring Plausibility of Hypothetical Interest Rate Shocks," Chapter 6 in Frank J. Fabozzi (ed.), *Managing Fixed Income Portfolios* (Hoboken, NJ: John Wiley & Sons, 1997).

use the principal components technique for estimating the probability distribution[5] of hypothetical shifts in the yield curve.

KEY RATE DURATIONS

Probably the most popular approach to measuring the exposure of a portfolio or a position to a change in the yield curve is *key rate duration*. This measure, introduced by Ho,[6] begins with calculation of the sensitivity of a change in the value of a security or a portfolio for a change in the spot rate for a given maturity. This measure is called the *rate duration* for that maturity. A j rate duration is interpreted as follows: it is the approximate percentage change in the value of a security or portfolio for a 100 basis point change in the spot rate for maturity j holding all other spot rates constant. Thus a 2-year rate duration for a security of 0.2 means that if the 2-year spot rate changes by 100 basis points and all other spot rates are unchanged, the value of the security will change by approximately 0.2%.

In theory, there is a rate duration for every point on the spot-rate curve. However, to make this approach practical, market participants focus on the rate duration for a limited number of key maturities. The resulting set of rate durations is referred to as *key rate durations*. The key rate durations that are typically calculated by vendors are those for 11 maturities: 3 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 15 years, 20 years, 25 years, and 30 years.

The total change in value of a security or a portfolio if all spot rates change by the same number of basis points is simply the duration of a security or portfolio to a parallel shift in rates.

LEVEL, SLOPE, AND CURVATURE DURATIONS

As noted earlier, a statistical technique known as *principal component analysis* has been used to identify the types of shifts in the yield curve, and those shifts are a parallel shift, a change in the shape, and a change in the, curvature. The first two changes are also referred to as a *level shift* and a *yield-curve slope change*. Willner has suggested an approach to measuring exposure to changes in the yield curve using these three yield-curve shifts.[7] The approach involves representing the yield

5. The concept of a probability distribution is described in Chapter 30.
6. Thomas S. Y. Ho, "Key Rate Durations: Measures of Interest Rate Risk," *Journal of Fixed Income* (September 1992), pp. 29–44. The rate duration approach was suggested by Chambers and Carleton, who called it "duration vectors." (See Donald Chambers and Willard Carleton, "A Generalized Approach to Duration," *Research in Finance* , Vol. 7 (1988).) Reitano suggested a similar approach in a series of papers and referred to these durations as "partial durations." (See Robert R. Reitano, "Non-Parallel Yield-Curve Shifts and Durational Leverage," *Journal of Portfolio Management* (Summer 1990), pp. 62–67, and "A Multivariate Approach to Duration Analysis," *ARCH*. vol. 2 (1989)).
7. Ram Willner, "A New Tool for Portfolio Managers: Level, Slope, and Curvature Duration," *Journal of Fixed Income* (June 1996), pp. 48–59. See also Andrea J. Heuson, Thomas F. Gosnell Jr., and W. Brian Barrett, "Yield-Curve Shifts and the Selection of Immunization Strategies," *Journal of Fixed Income* (September 1995), pp. 53–64.

curve by a mathematical function that is described in terms of level, slope, curvature, and location of the yield curve hump (i.e., the maximum point of curvature). Given the estimated yield curve, exposure to changes in the parameters—level, slope, and curvature—can be calculated. This approach is referred to as the *level, slope, and curvature duration approach*. Because the approach is based on the factors provided by principal component analysis, it is sometimes referred to as the *principal component duration approach*.

The mathematical representation of the yield curve used by Willner is

$$Y = L + (S + C)\frac{(1 - e^{-M/H})}{M/H} - C(e^{-M/H}),$$

where

Y = yield to maturity;
H = constant associated with curve hump positioning;
M = maturity of security (in years);
e = 2.71828… .

The parameters L, S, and C have to be estimated statistically using historical data.

It can be demonstrated that

- L represents the level of rates;
- S represents the spread between the long-term rate and short-term rate;
- S/M represents the slope of the yield curve;
- C represents the curvature of the yield curve.

YIELD-CURVE-RESHAPING DURATIONS

While the total return framework clearly demonstrates the performance potential of a bond portfolio, money managers still seek measures of the interest-rate sensitivity of their portfolio, that is, duration and convexity. The weakness of the traditional modified duration, as we have illustrated, is that it assumes that the yield curve shifts in a parallel fashion.

The sensitivity of a portfolio to changes in the shape of the yield curve can be approximated. There have been several approaches suggested in the literature to accomplish this.[8] The methodology described here is the one proposed by Klaffky, Ma, and Nozari.[9] They refer to the sensitivity of a portfolio to changes in the shape of the yield curve as *yield-curve-reshaping durations*, or simply *reshaping durations*.

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8. For a description of these measures, see Frank J. Fabozzi, *Measuring and Controlling Interest Rate Risk* (New Hope, PA: Frank J. Fabozzi, 1996), Chapter 4.
9. Thomas E. Klaffky, Y. Y. Ma, and Ardavan Nozari, "Managing Yield Curve Exposure: Introducing Reshaping Durations," *Journal of Fixed Income* (December 1992).

EXHIBIT 15–5

Two Portfolios with Similar Durations but Different Yield-Curve Exposures

		Market Weight (%)	
On-the-Run Treasury Issue	**Modified Duration**	**Portfolio A**	**Portfolio B**
5.125% of 5/31/94	1.9	20	39
7.50% of 5/12/02	7.1	60	20
8.0% of 11/15/21	11.1	20	41
Portfolio modified duration		7	7

Source: Adapted from Thomas E Klaffky, Y. Y. Ma, and Ardavan Nozari, "Managing Yield Curve Exposure: Introducing Reshaping Durations," *Journal of Fixed Income* (December 1992).

To explain reshaping durations, we can use the illustration developed by Klaffky, Ma, and Nozari. Exhibit 15–5 shows two Treasury portfolios, portfolio A and portfolio B, constructed from three on-the-run Treasury securities: the 2-year, 10-year, and 30-year. The analysis is based on the yield curve as of the close of May 29, 1992. The yield for each issue on that date was 5.18% for the 2-year issue, 7.33% for the 10-year issue, and 7.84% for the 30-year issue. Also shown in the exhibit is the modified duration for each Treasury issue.[10] The modified duration for both portfolios is 7. Consequently, for a 50 basis point change for all three maturities of the yield curve, each portfolio should change by approximately 3.5%.

Let's look at the actual percentage price change for the two portfolios under three different yield-curve scenarios. In the first scenario, suppose that the yield curve shifts in a parallel fashion so that the yield for all Treasury issues declines by 50 basis points. Under this scenario, the actual percentage price increase is 3.52% for portfolio A and 3.56% for portfolio B. Thus the two portfolios would have similar performance. Note that the percentage price change is what would be projected using modified duration (3.5%).

In the next two scenarios, assume a reshaping of the yield curve. In the second scenario, suppose that the 2-year yield declines by 50 basis points, while the 10-year and 30-year yields do not change (i.e., the yield curve steepens). In contrast to the first scenario, portfolio A's price change will be 0.19%, while portfolio B's will be 0.36%. Thus portfolio A will outperform portfolio B by 0.17%.

10. Klaffky, Ma, and Nozari present a generalized approach in which Treasury and non-Treasury securities are included in the portfolio. The latter includes bonds with embedded options (i.e., callable and putable bonds). The valuation of these bonds is discussed in Part Five of this book. As will be explained, for bonds with embedded options, a more appropriate measure of price sensitivity is effective duration, a measure we introduced in Chapter 13. Thus, in their article, Klaffky, Ma, and Nozari refer to the effective duration rather than modified duration. For a noncallable Treasury issue, the two are almost identical, so the modified duration is used in our explanation of their methodology.

In the third scenario, assume that the 2-year and 10-year yields do not change, but the 30-year yield declines by 50 basis points (i.e., the yield curve flattens). In this scenario, portfolio B will outperform portfolio A by 1.25% (2.42% versus 1.17%).

What we seek is a duration measure that takes into account how the reshaping of the yield curve affects price changes. In the measures discussed below, we focus on three points on the yield curve—2-year, 10-year, and 30-year—and the spread between the 10-year and 2-year issues and between the 30-year and 10-year issues. The former spread is referred to as the *short end of the yield curve*, and the latter spread, as the *long end of the yield curve*. Klaffky, Ma, and Nozari refer to the sensitivity of a portfolio to changes in the short end of the yield curve as *short-end duration* (SEDUR) and to changes in the long end of the yield curve as *long-end duration* (LEDUR). These concepts, however, are applicable to other points on the yield curve.

Calculating Reshaping Duration

To calculate the SEDUR of each bond in the portfolio, the percentage change in the bond's price is calculated for (1) a steepening of the yield curve at the short end by 50 basis points and (2) a flattening of the yield curve at the short end of the yield curve by 50 basis points. Then the bond's SEDUR is computed as follows:

$$\text{SEDUR} = \frac{P_s - P_f}{P_0} \times 100,$$

where

P_s = bond's price if the short end of the yield curve steepens by 50 basis points;
P_f = bond's price if the short end of the yield curve flattens by 50 basis points;
P_0 = bond's current market price.

To calculate the LEDUR, we use the same procedure for each bond in the portfolio: calculate the price for (1) a flattening of the yield curve at the long end by 50 basis points and (2) a steepening of the yield curve at the long end of the yield curve by 50 basis points. Then the bond's LEDUR is computed in the following manner:

$$\text{SEDUR} = \frac{P_f - P_s}{P_0} \times 100.$$

The SEDUR and LEDUR calculations are equivalent to the formula for approximating duration given in Chapter 13 when a 50 basis point change is used in the formula. Because in our illustration we are using Treasury issues, the modified duration for each bond is the appropriate measure. For example, the SEDUR for the 2-year Treasury in Exhibit 15–7 is 1.9, and its LEDUR is zero. The SEDUR for the 30-year bond is zero, and its LEDUR is 11.1. The analysis proceeds from the shifting of the yield curve holding the 10-year yield constant, so the SEDUR and LEDUR for the 10-year issue are zero.

The portfolio SEDUR and LEDUR are the weighted average of the corresponding durations for each bond in the portfolio. For example, the SEDUR for the two portfolios is calculated as follows.

$$\text{Portfolio A: } 0.2\,(1.9) + 0.4\,(0) + 0.2\,(0) = 3.8;$$

$$\text{Portfolio B: } 0.39\,(1.9) + 0.2\,(0) + 0.41\,(0) = 7.4.$$

A SEDUR of 3.8 means that if the short end of the yield curve shifts by 50 basis points, the portfolio value will change by approximately 1.9%. A similar shift for portfolio B will result in a 3.7% change in the portfolio's value.

The LEDUR for the two portfolios is calculated in the following manner:

$$\text{Portfolio A: } 0.2\,(0) + 0.4\,(0) + 0.2(11.1) = 2.2;$$

$$\text{Portfolio B: } 0.39\,(0) + 0.2\,(0) + 0.41(11.1) = 4.6.$$

So far we have looked at only bonds with maturities at the three points on the yield curve that are used to define the short end and long end. The methodology can be generalized to other maturities as follows.

1. The shift in yields begins with the 10-year Treasury.
2. At the short end, the steepening or flattening of the yield curve for any maturity other than 2 years is proportionate to the 10-year to 2-year spread. For example, suppose that the 10-year to-2-year spread is 200 basis points and that the 10-year to-7-year spread is 120 basis points. Assume that the yield spread widens by 50 basis points, a 25% increase in the spread. The yield spread for the 10-year to-7-year spread is then assumed to widen by 20 basis points (50 basis points –0.25 × 120). The SEDUR for the 7-year spread is then calculated assuming a 20 basis point change in the spread, not 50 basis points.
3. At the long end, the steepening or widening for any bond with a maturity of greater than 10 years is assumed to be proportionate to the change in the 30-year to 10-year spread.

CHAPTER

16

EMPIRICAL DURATION

Duration is a measure of interest-rate risk. We described the measures of duration (modified duration and effective duration) in Chapter 14. Both of these measures were computed based on two key inputs: the estimated price if interest rates increase and the estimated price if interest rates decrease by the same number of basis points. The two estimated prices were obtained from a valuation model. If the valuation model to obtain the two estimated prices is poor, then the computed duration is a poor measure of the sensitivity of the bond's price to changes in interest rates. Calculation of duration in this way is referred to as *analytical duration* or *model duration*. Another method for estimating a bond's duration is to take a market-based approach. In this approach, historical prices and relevant interest rates are used. Typically, this is done using regression analysis, and the resulting duration measure is referred to as *empirical duration* (or, less commonly, *implied duration*).

To see how empirical duration and analytical duration can differ, Columbia Threadneedle Investments estimated the two measures for different sectors of the bond market based on 10 years of data as of December 31, 2016. The results are reported in Exhibit 16–1. Note that for 10-year Treasuries there is virtually no difference between the two duration estimates. However, the differences are substantial for the other bond sectors. A clear pattern can be seen from the duration estimates reported in the exhibit: the lower the credit rating, the lower is the empirical duration relative to the analytical duration. Note that for two of the bond sectors the analytical duration is positive, whereas the empirical duration is negative. As explained in this chapter, a negative duration means that credit risk is more important than interest-rate risk for a bond sector.

In this chapter we will explain how empirical duration is calculated. Empirical duration is computed for corporate bonds, mortgage-backed securities,[1]

1. Several studies have demonstrated that empirical duration is less than effective duration. See, for example, Scott M. Pinkus and Marie A. Chandoha, "The Relative Price Volatility of Mortgage Securities," *Journal of Portfolio Management*, Vol. 12, No. 44 (1986), pp. 9–22; Paul DeRosa, Laruie S. Goodman, and Mike Zazzarino, "Duration Estimates on Mortgage-Backed Securities," *Journal of Portfolio Management*, Vol. 19, No. 2 (1993), pp. 32-38; Lakhbir Hayre and Hubert Chang, "Effective and Empirical Durations for Mortgage Securities," *Journal of Fixed Income*, Vol. 6, No. 4 (1997), pp. 17–33; and Bennett W. Golub, "Approaches for Measuring the Duration of Mortgage-Backed Securities," Chapter 34 in Frank J. Fabozzi (ed.), *The Handbook of Mortgage-Backed Securities*, 6th ed. (New York: McGraw-Hill, 2006).

EXHIBIT 16–1

Analytical Versus Empirical Duration for Bond Sectors

Sector	Analytical Duration	Empirical Duration
10-Year U.S. Treasuries	8.7	8.6
Investment-grade corporate bonds	6.7	3.1
High-yield corporate bonds	4.2	–2.9
Emerging-market bonds	6.2	1.9
Mortgage-backed securities	3.5	2.3
Floating-rate loans	0.3	–3.8

Source: Columbia Threadneedle Investments, based on 10 years of data ending December 31, 2016, as reported in Colin J. Lundgren, "Empirical Duration: A Better Way to Compare Interest Rates Sensitivity," Seeking Alpha, November 19, 2017. Available at https://seekingalpha.com/article/4126242-empirical-duration-better-way-to-compare-interest-rate-sensitivity.

emerging-market bonds, floating-rate securities, and Treasury futures.[2] We will explain the empirical evidence regarding empirical duration for corporate bonds. At the end of the chapter we discuss a market-based duration measure used for agency mortgage passthrough securities called *coupon-curve duration*.

ANALYTICAL (MODEL) DURATION

As explained in Chapter 14, a bond's analytical duration can be estimated as follows:

$$\text{Duration} = \frac{V_- - V_+}{2(V_0)(\Delta y)}, \tag{16–1}$$

where

$\Delta y =$ change in the bond's yield (in decimal form);
$V_0 =$ initial price of the bond (per \$100 of par value);
$V_+ =$ estimated value of the bond per \$100 of par value if the yield is increased by Δy;
$V_- =$ estimated value of the bond per \$100 of par value if the yield is decreased by Δy.

The two unknowns in equation (16–1) are the prices when the yield is increased (V_+) and decreased (V_-). The method used to determine the new prices if yields change is what distinguishes the different types of duration measures. In applying equation (16–1) it is necessary to change the bond's yield by some number

2. For a discussion of empirical duration for U.S. Treasury futures, see David Boberski, "Empirical Duration: Measuring the Price Sensitivity of U.S. Treasury Futures," CME Group, Chicago, 2009. Available at https://www.cmegroup.com/trading/interest-rates/files/Empirical_Duration.pdf .

of basis points (Δy). With model duration, the two prices used in the numerator of the analytical duration formula are obtained from some valuation model.

EMPIRICAL DURATION

In contrast to analytical duration, empirical duration uses observed market prices rather than prices derived from a valuation model. Empirical duration is calculated as follows. First, the following regression relationship is estimated using regression analysis:

$$\text{Change in security's price} = a + b(\text{change in relevant yield}).$$

Given the estimate of b, the empirical duration is then calculated as follows:

$$\text{Empirical duration} = \frac{b\left(\text{change in security price/change in relevant yield}\right)}{\text{full price of security}}. \tag{16–2}$$

From an implementation perspective, there is no standardization as to the frequency of the data that should be used (i.e., daily, weekly, monthly), the length of the time period that should be used, and even the appropriate relevant yield that should be used.

There are at least three advantages of empirical duration over analytical duration:

1. Empirical duration estimation does not rely on any theoretical formulas or analytical assumptions;
2. Estimation of the required parameters is easy to do using regression analysis;
3. The only inputs that are needed are a reliable price series and the appropriate interest-rate series.

However, empirical duration has the following four disadvantages:

1. A reliable price series may not be available. For example, there may be no price series for a thinly traded corporate bond or mortgage derivative security, or the prices may be matrix priced rather than actual transaction prices;
2. An empirical relationship does not impose a structure for the options embedded in the case of a callable corporate bond or mortgage-backed security (MBS), and this can distort the market-based duration estimation;
3. The price history may lag current market conditions (this may occur after a sharp and sustained shock to interest rates has been realized);
4. The volatility of the spread to Treasury yields can distort how the price of a corporate bond or MBS reacts to yield changes.

EMPIRICAL DURATION FOR CORPORATE BONDS

The price sensitivity of a corporate bond has two components. The first component is attributable to changes in interest rates. The second component is the price sensitivity of a corporate bond to changes in the issuer's credit risk. Although the interest-rate sensitivity dominates in explaining a bond's price movement, the second component is important, particularly for lower-rated corporate bonds. That is, the lower the credit rating of a corporate bond, the more important is the price sensitivity due to the credit risk, and this is why lower-credit-rated corporate bonds behave similar to equity-like securities.

Consequently, for high-yield corporate bonds (i.e., non-investment-grade corporate bonds), the analytical model for calculating price sensitivity to interest-rate changes may not be an accurate estimate of a corporate bond's price sensitivity. This is where empirical duration is used to adjust the analytical duration to account for the credit-risk aspect of high-yield corporate bonds. We describe how this is done in this section.

Before discussing the methodology, let's discuss a heuristic rule of thumb that has been used by corporate bond portfolio managers to adjust the analytical duration of a high-yield corporate bond to recognize the credit risk. This rule simply reduces the computed analytical duration for a high-yield bond by a percentage that the portfolio manager believes is appropriate. For example, suppose that an analytically computed duration for a high-yield corporate bond issue is 7 and the portfolio manager believes that it should be reduced by 60%. Then the duration used by the portfolio manager in computing the portfolio's duration is 2.80 (= 7 × 0.40).

The Barclays Research Group used daily data for the period August 1998 to November 2009 for six credit-rating categories from the Barclays Capital Corporate Investment Grade Index (for the Aaa/Aa, A, and Baa credit ratings) and the Barclays High Yield Index (Ba, B, and Caa). Changes in the 10-year Treasury yield were used as a measure of the level of interest rates.[3] There were two major findings of the study. First, for lower-rated corporate bonds, the bond's price movement was less sensitive to the change in the level of interest rates. The second major finding was that within each credit-rating category there was clearly a dependence on spreads; that is, bonds with higher spreads were found to have lower empirical durations. This second finding is consistent with the duration-times-spread metric described in Chapter 23. However, even after adjusting for their higher spreads, the empirical durations of high-yield corporate bonds were reported to be systematically lower than those of investment-grade bonds. The fact that high-yield corporate bonds are less interest-rate sensitive than investment-grade corporate bonds may be because they sell in the marketplace primarily based on default/recovery expectations.

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3. See Madhur Ambastha, Arik Ben Dor, Lev Dynkin, Jay Hyman, and Vadim Konstantinovsky, "Empirical Duration of Corporate Bonds and Credit Market Segmentation," *Journal of Fixed Income*, Vol. 20, No. 1 (Summer 2010), pp. 5–27.

Duration Multipliers

The research by Barclays Research Group has shown that the true duration of a portfolio for a corporate bond requires adjusting the analytical duration as follows[4]:

$$M = \left[1 + \text{correlation}\left(\frac{\text{SD of Treasury yield change}}{\text{SD of corporate bond credit-spread change}}\right)\right], \quad (16\text{–}3)$$

where

M = adjustment factor, sometimes referred to as the *duration multiplier*;
correlation = correlation between the change in corporate bond yields and change in Treasury yields;
SD = standard deviation.

The empirical duration then is

$$\text{Empirical duration} = MD, \quad (16\text{–}4)$$

where D is the model effective duration computed using equation (16-1).

Historically, the correlation between the change in corporate bond yields and the change in Treasury yields has been negative. Looking at equation (16–3), it can be seen that if the correlation is negative, the duration multiplier (M) will be less than 1, and therefore, the empirical duration will be less than the analytical duration.

Let's use some actual estimates for the three inputs in equation (16–3) to illustrate calculation of the duration multiplier and the estimate of empirical duration. Barclays' risk model estimates that the standard deviation of the change in 10-year Treasury yields is 24 basis points per month. The standard deviation for the credit spread varies by credit rating as follows, according to Barclays' risk model:

10.1 basis points per month for AAA/A-rated corporate bonds;

18.2 basis points per month for A-rated corporate bonds;

29.6 basis points per month for BBB-rated corporate bonds.

A study by Barclays found that the correlation for AAA/AA-, A-, and BBB-rated bonds in the consumer cyclicals sector are as follows[5]:

AAA/AA-rated corporate bonds = −0.38;

A-rated corporate bonds = −0.34;

BBB-rated corporate bonds = −0.30.

Substituting these values into equation (16–3) gives the following duration multipliers for consumer cyclical bonds:

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4. Arthur Berd, Elena Ranguelova, and Antonio B. Silva, "Credit Portfolio Management in a Turning Rates Environment," *Journal of Investment Strategies*, Vol. 3, No. 1 (December 2013).
5. Ibid.

$$M \text{ for AAA/A-rated corporate bonds} = \left[1 + (-0.38)\frac{0.00101}{0.00240}\right] = 0.84;$$

$$M \text{ for A-rated corporate bonds} = \left[1 + (-0.34)\frac{0.00182}{0.00240}\right] = 0.74;$$

$$M \text{ for BBB-rated corporate bonds} = \left[1 + (-0.30)\frac{0.00296}{0.00240}\right] = 0.63.$$

As can be seen, the lower the credit rating, the greater is the reduction that must be made to the effective duration (D) in equation (16–4).

Coupon Effect of Corporate Bonds on Empirical Duration

In calculating analytical duration for corporate bonds, the projected cash flows make no distinction between cash flows that come from principal and cash flows that come from interest payments. Moreover, there is no explicit modeling of default and recovery rates. Consequently, there is no reason to expect that the behavior of prices will be different for premium, par, and discount corporate bonds.

In a study by Barclays' researchers, empirical duration is estimated considering the role of default and recovery rates in the pricing of premium, par, and discount corporate bonds. Their approach impacts corporate bond empirical duration. The Barclays researchers looked at how the coupon level impacts empirical duration of a corporate bond when measured relative to changes in Treasury yields.

They analyzed the noncallable bonds in the Barclays U.S. Investment-Grade Corporate Index from January 1992 through April 2014. In their regression model for empirical duration, they used additional explanatory variables to account for discount and premium bonds. They found the interest-rate sensitivity of premium bonds to be higher when yields rise and lower when yields fall. The interest-rate sensitivity for discount bonds is lower when yields rise and higher when yields fall. In addition, they found that empirical duration is slightly less than model duration in the case of corporate bonds with low spreads and continues to decrease as spreads increase. The implication is that corporate bonds trading at a discount exhibit a highly desirable price behavior whereby these bonds react more strongly to favorable changes in yield but less strongly to unfavorable changes. In contrast, premium bonds exhibit the opposite behavior.

COUPON-CURVE DURATION FOR AGENCY MORTGAGE PASSTHROUGH SECURITIES

Another duration measure that is used solely for agency mortgage passthrough securities using market prices is the *coupon-curve duration.*[6] This is an easier

6. This duration measure was first suggested in Douglas T. Breeden and Michael J. Giarla, "Hedging Mortgage-Backed Securities," Chapter 31 in Frank J. Fabozzi (ed.), *The Handbook of Mortgage-Backed Securities*, rev. ed. (Chicago: Probus Publishing, 1988).

approach to duration estimation than empirical duration but is limited in its applications for the reasons explained below.

Coupon-curve duration starts with the coupon curve of prices for generic agency mortgage passthrough securities. By rolling up and down the coupon curve of prices, the duration can be obtained. Because of the way it is estimated, empirical duration estimation by this approach is referred to as the *roll-up, roll-down approach*. The prices obtained from rolling up and rolling down the coupon curve of prices are substituted into the effective duration formula given by equation (16–1).

To illustrate the calculation of coupon-curve duration, let's use the following coupon curve of prices for generic agency passthrough securities as of a particular date:

Coupon (%)	Price ($ per $100 par)
3.0	101.0000
3.5	103.9688
4.0	106.4063
4.5	108.3750
5.0	110.5625

Suppose that the coupon-curve duration for the 4's is sought. If the yield declines by 50 basis points, the assumption is that the price of the 4's will increase to the price of the 4.5's. Thus the price will increase from 106.4063 to 108.3750. Similarly, if the yield increases by 50 basis points, the assumption is that the price of the 4's will decline to the price of the 3.5's (103.9688). Using the model duration formula, the corresponding values are

$$V_0 = 106.4063;\ V_- = 108.3750;\ V_+ = 103.9688;\ \Delta y = 0.005.$$

Then

$$\text{Current coupon duration} = \frac{108.3750 - 103.9688}{2(106.4063)(0.005)} = 4.14.$$

Note that if a 100 basis-points rate shock is used, the current coupon duration would be

$$\text{Current coupon duration} = \frac{110.5625 - 101.0000}{2(106.4063)(0.01)} = 4.49.$$

While two advantages of the coupon-curve duration are the simplicity of its calculation and the fact that current prices embody market expectations, there are disadvantages. The approach is limited to generic agency mortgage passthrough securities and to-be-announced (TBA) securities but difficult to use for mortgage derivatives such as bond classes of collateral mortgage obligations and stripped MBS.

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PART

FIVE

HISTORICAL RETURN AND YIELD VOLATILITY

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CHAPTER
17

MEASURING HISTORICAL RETURN VOLATILITY

To evaluate the historical performance of a bond portfolio, an individual bond, a bond sector, or a bond portfolio strategy, the next step after computing the historical return is to calculate return volatility. We explained how to properly calculate historical returns in Chapter 10.

Basically, historical return volatility involves describing the variation in historical returns and therefore analyzing the return distribution. Describing the return distribution requires a basic understanding of the different types of probability distributions and their properties. Here we look at three measures of historical return volatility, and in the last section we describe a measure of return volatility based on a comparison to a benchmark backward-looking tracking error.

HISTORICAL RETURN DISTRIBUTIONS

Once calculations for a fund's return are completed for the time periods covering the evaluation period, the collection of returns can be described by a probability distribution and by various summary measures. While we describe probability distributions in Chapter 34, here we describe some of the basic concepts.

There are three measures used to measure historical return volatility:

- Dispersion measures;
- Skewness measures;
- Kurtosis measures.

We describe these measures in this chapter using two actual exchange-traded funds' monthly historical performance over the period January 1, 2016, to December 31, 2020 (60 historical returns). We refer to these two exchange-traded funds (ETFs) as ETF A and ETF B. Both ETFs are taxable bond funds that are categorized as intermediate core bond funds. Exhibit 17–1 provides the historical monthly returns for both ETFs obtained from Portfolio Visualizer.

Over the 60-month time period, the mean returns for both ETF A and ETF B are summarized below:

Return	ETF A	ETF B
Mean monthly return (%)	0.37	0.46
Arithmetic annual return (monthly × 12) (%)	4.44	5.52
Geometric annual return (%)	4.53	5.66

EXHIBIT 17–1

Historical Monthly Returns for Two Actual ETFs: January 1, 2016, to December 31, 2020

Panel A: ETF A

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2012	0.63%	0.04%	–0.51%	1.10%	0.93%	0.07%	1.23%	0.16%	0.18%	–0.10%	0.27%	–0.88%
2013	–0.69%	0.54%	0.08%	1.01%	–1.93%	–1.65%	0.38%	–0.86%	1.11%	0.86%	–0.29%	–0.64%
2014	1.55%	0.47%	–0.17%	0.80%	1.05%	0.08%	–0.28%	1.14%	–0.57%	0.72%	0.83%	0.06%
2015	2.40%	–1.31%	0.55%	–0.32%	–0.50%	–1.11%	0.88%	–0.25%	0.81%	0.03%	–0.39%	–0.18%
2016	1.20%	0.85%	0.88%	0.40%	–0.01%	2.02%	0.60%	–0.31%	0.11%	–0.94%	–2.57%	0.33%
2017	0.19%	0.62%	–0.04%	0.80%	0.71%	0.05%	0.40%	0.86%	–0.47%	–0.03%	–0.11%	0.54%
2018	–1.24%	–1.04%	0.69%	–0.85%	0.68%	–0.04%	–0.04%	0.67%	–0.55%	–0.86%	0.64%	1.87%
2019	1.11%	–0.09%	1.94%	–0.02%	1.83%	1.25%	0.15%	2.77%	–0.57%	0.31%	–0.04%	–0.07%
2020	1.98%	1.67%	–1.43%	2.76%	0.67%	0.67%	1.45%	–0.94%	–0.10%	–0.56%	1.21%	0.15%

Panel B: ETF B

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2016	0.63%	0.62%	1.93%	1.11%	–0.04%	1.87%	1.08%	0.45%	0.06%	–0.56%	–2.34%	0.45%
2017	0.47%	0.71%	–0.10%	0.79%	0.64%	–0.08%	0.55%	0.81%	–0.33%	0.05%	–0.07%	0.42%
2018	–0.94%	–0.98%	0.45%	–0.72%	0.45%	0.07%	0.21%	0.47%	–0.44%	–0.95%	0.52%	1.53%
2019	1.55%	0.10%	1.93%	0.24%	1.51%	1.48%	0.23%	2.33%	–0.39%	0.45%	–0.04%	0.06%
2020	1.92%	1.50%	–1.72%	2.71%	1.35%	1.55%	1.77%	–0.76%	–0.11%	–0.38%	1.41%	0.32%

Source: Portfolio Visualizer, https://www.portfoliovisualizer.com/fund-performance.

MEASURES OF VARIATION

There are many types of risk that are described in investment management. Intuitively, one way of thinking about the risk associated with an investment strategy is how spread out are the possible returns. This is where *measures of variation*, also referred to as *measures of dispersion*, come into play. The more dispersed the realized returns, the more risky investors view a strategy to be. When analyzing the performance of a bond portfolio manager, the greater the dispersion of historical returns, the greater is the risk that investors incurred in generating the returns.

Range

The simplest measure of dispersion is the *range* because it just involves calculation of the difference between the largest historical return and the smallest historical return over the evaluation period. A wider range would suggest greater risk. However, the range is impacted by extreme values. For example, suppose that the two largest historical returns are 15% and 7% and that the minimum return is 1%. The range in this case would be 14% (15% − 1%). Because the next-to-largest historical return is 5%, this would suggest that the 15% return may have been a rare occurrence or an *extreme* or *outlier* historical return. The true range might be better described as 4% (5% − 1%). Consequently, the range may be of limited value as a measure of dispersion because it is sensitive to shifts in either of the two extreme historical returns (maximum and minimum historical returns) while the other returns are unchanged.

Exhibit 17–2 shows the monthly returns for ETF A from the lowest monthly return to the highest monthly return. It can be seen from the ordering that the smallest monthly return is −2.57% and the largest monthly return is 2.77%. Therefore, the range for ETF A is 2.77% minus −2.57% or 5.534%. To see the sensitivity of the range, consider that the smallest monthly return is −2.57%, while the second-smallest monthly return is −1.43%. Using the latter value to calculate the range (viewing the smallest monthly return as an extreme value), the range is 4.20%. The three largest returns are 2.02%, 2.76%, and 2.77%. One might view that the two largest monthly returns are not extreme values.

The range tells us how the values extend over some interval that is wider than the mean value. However, one must be careful about interpreting a range given that it is impacted by extreme values at both ends of the range. That is, the range is sensitive to shifts in the two extreme values while the rest of the data in between may remain unchanged. The same range is obtained from two data sets that have the same extremes but very different structures within the data. Consequently, the range as a measure of variation is limited in its usefulness because it offers limited insight into the data structure.

Interquartile Range

The *interquartile range* (IQR) is a solution for dealing with the range's sensitivity to extreme values. The IQR does so by giving the range based on elimination of the most extreme 25% of the data on both ends. That is, the IQR is the difference between the upper (i.e., 25%) and lower (i.e., 75%) quartiles, respectively. Although the IQR basically represents the body of the distribution by removing the influence of rare or extreme values, it still uses only a fraction of the historical returns. It conveys little about the entire variation. Naturally, if the IQR is large, the outer segments of historical returns are bound to be further away from the center of the historical returns than would be feasible if the IQR were narrow. However, as with the range of historical returns, the same historical return for the IQR can be easily obtained analytically by many different dispersions of the historical returns.

The IQR is calculated as follows:

Step 1. Order the data from lowest to highest values.

Step 2. Calculate the median value.

Step 3. Calculate the median value for the first half of the data set. We denote this value by Q1.

Step 4. Calculate the median value for the second half of the data set quartile. We denote this value by Q3.

Step 5. The IQR is equal to Q3 – Q1.

The median value is the value that is in the middle of the data set when the data set is arranged in ascending order. When there is an odd number of observations, the median will be one of the values in the data set. When there is an even number of observations, an average must be determined of the two middle observations.

To illustrate the IQR calculation, we will use the monthly returns shown in Exhibit 17–1:

Step 1. Exhibit 17–2 shows the monthly returns ordered from lowest to highest (i.e., in ascending order).

Step 2. There is an even number of observations (60), so the median is the average of the 30th monthly return (0.31%) and the 31st monthly return (0.33%) and is equal to 0.32%.

Step 3. The median monthly return for the first half of the data set is –0.09% and is therefore Q1.

Step 4. The median monthly return for the second half of the data set is 0.85% and is therefore Q3.

Step 5. The IQR is equal to Q3 – Q1 = 0.85% − (–0.09%) = 0.94%. Annualizing the return by simply multiplying by 12 gives an annualized IQR of 11.28%.

EXHIBIT 17–2

Monthly Returns for ETF A Ordered from Lowest to Highest

Month	Return	Month	Return
1	–2.57%	31	0.33%
2	–1.43%	32	0.40%
3	–1.24%	33	0.40%
4	–1.04%	34	0.54%
5	–0.94%	35	0.60%
6	–0.94%	36	0.62%
7	–0.86%	37	0.64%
8	–0.85%	38	0.67%
9	–0.57%	39	0.67%
10	–0.56%	40	0.67%
11	–0.55%	41	0.68%
12	–0.47%	42	0.69%
13	–0.31%	43	0.71%
14	–0.11%	44	0.80%
15	–0.10%	45	0.85%
16	–0.09%	46	0.86%
17	–0.07%	47	0.88%
18	–0.04%	48	1.11%
19	–0.04%	49	1.20%
20	–0.04%	50	1.21%
21	–0.04%	51	1.25%
22	–0.03%	52	1.45%
23	–0.02%	53	1.67%
24	–0.01%	54	1.83%
25	0.05%	55	1.87%
26	0.11%	56	1.94%
27	0.15%	57	1.98%
28	0.15%	58	2.02%
29	0.19%	59	2.76%
30	0.31%	60	2.77%

Mean Absolute Deviation

There are measures that overcome the shortcomings of the range and IQR as measures of variation. One such measure that accounts for all historical returns is the *mean absolute deviation* (MAD). For historical returns, MAD is the average deviation of all historical returns from some reference return. The reference return could be the mean of the historical returns, the median of the historical returns, the risk-free rate or return, or some minimum acceptable return specified by a client. The MAD for historical returns is computed as follows:

$$\text{MAD} = \frac{1}{T}\left|\sum_{1}^{T} r_t - \text{ reference return}\right|,$$

where

r_t = return for period t;
T = number of time periods in the sample.

For ETF A, the mean monthly return is 0.37%. The MAD can be shown to be 0.77%.

A problem with using MAD as a measure of risk is that it penalizes differences from historical returns that exceed the mean return. This is not the way investors view risk. For example, in the case of ETF A, the mean monthly return is 0.37%. There are months where the monthly return significantly exceeds 0.37%. Such return outcomes are favorable return outcomes, not returns that should be used to penalize the performance of the portfolio manager by showing a higher risk than there actually was. However, there is a caveat. If the historical distribution is symmetric (a property we describe later in this chapter), then there is no issue with using MAD as a measure of risk.

Variance and Standard Deviation

The *variance* is the measure of dispersion used in finance. Unlike MAD, which averages absolute deviations, variance uses squared deviations in the calculation of dispersion. The deviations are measured from the mean. The squaring of the deviations has the effect that larger deviations contribute even more to the dispersion measure than smaller deviations, as would be the case with the MAD.

The variance of historical returns is computed by the following formula:

$$\text{Variance} = \frac{1}{T}\sum_{1}^{T}\left(r_t - \text{ mean return}\right)^2.$$

Related to the variance is the even more commonly stated measure of dispersion, the *standard deviation*. The reason is that the units of the standard deviation correspond to the original units of the data, whereas the units are squared in the case of the variance. In the case of historical returns, variance measures the unit of measurement in returns squared, while standard deviation is measured in terms of return. The standard deviation is defined to be the positive square root of the variance.

For ETF A, the standard deviation of the monthly returns is 1.01%. Note that it is higher than for MAD. To annualize the standard deviation for the return, the following formula is used:

$$\text{Annual standard deviation of returns} = \text{monthly standard deviation} \times \sqrt{12} = 3.49\%.$$

For ETF B, the standard deviation of the monthly returns is 0.98%, and the annual standard deviation is then 3.39% (= 0.98% × $\sqrt{12}$).

As with MAD, because the variance penalizes favorable returns, it is a drawback as a measure of risk except if the historical returns are symmetric. A disadvantage of the variance is that the dispersion is measured relative to the mean historical return, and there is no reason for that return to be the basis for an investor to use as a measure of risk.

Semi-Variance and Semi-Standard Deviation

A way to correct the problem of the variance that penalizes returns above the mean historical return is to ignore in the calculation the squared deviations of historical returns above the mean and is called the *semi-variance*. That is,

$$\text{Semi-variance} = \frac{1}{T}\sum_{1}^{T}\left(r_{t*} - \text{ mean return}\right)^2,$$

where r_{t*} indicates an observation where the historical return is less than the mean return. The square root of the semi-variance is the *semi-standard deviation*.

For ETF A it can be shown that there are 31 monthly returns that were less than the mean monthly return. The monthly semi-standard deviation for ETF A is 0.52%. Annualizing by multiplying by the square root of 12, we obtain 1.80%. For ETF B the monthly semi-standard deviation is 0.47%, and therefore, the annualized semi-annual standard deviation is 1.62%.

Because the semi-variance does not penalize volatility for historical returns above the mean, it is referred to as a dispersion measure of *downside risk*. While the semi-variance deals with the problem of penalizing favorable historical returns above the mean, the problem of treating the mean as the acceptable return remains.

Lower Partial Moment

Another measure of downside risk or shortfall risk that overcomes the problem of using the mean historical return from which to measure deviation is the *lower partial moment* (LPM). The formula for the LPM is

$$\text{LPM} = \frac{1}{T}\sum_{1}^{T}\left(r_{\#} - \text{ reference return}\right)^p,$$

where $r_{\#}$ denotes returns below the reference return.

The parameter p in the LPM formula refers to the order of the LPM. This measure exhibits different sensitivities to extreme losses depending on the value of the order selected. Any order can be used. When the order is 1, the LPM is then just the sum of the deviations below the reference return. When the reference return is the mean historical return and the order is 2, then the LPM is computed as the semi-variance; therefore, the semi-variance is a special case of the LPM. Typically, the order of 2 is used. From the MLP formula it can be seen that the higher the order, the greater is the importance given to the extreme historical returns.

For ETF A, for $p = 2$, the LPM for various reference returns is summarized below:

Reference Return (%)	LPM (%)
0.00	0.51
0.37	0.52
0.50	0.52
0.60	0.53

SKEWNESS MEASURES

Skewness is a measure of the asymmetry of the distribution of a data set. The most popular measure of skewness is the *Pearson coefficient of skewness*, which is defined as three times the difference between the median and the mean divided by the standard deviation.[1] Using historical returns, the Pearson coefficient of skewness is computed as follows:

$$\text{Pearson coefficient of skewness} = 3\left(\frac{\text{median historical return} - \text{mean historical return}}{\text{standard deviation of return}}\right).$$

Note the following for the Pearson coefficient of skewness:

- A property of a symmetric distribution is that the mean and median are equal. So in the case of a symmetric distribution, Pearson skewness is zero.
- For historical returns where the mean is different from the median, then on a graph of the return distribution the mean is located on either the left or right half of the return distribution, and the historical return is said to be *skewed*.
- If the mean return is in the left half of the return distribution, the distribution is said to be *skewed to the left* (or *left skewed*) because there are more extreme values on the distribution's left side than on the distribution's right side. The opposite (i.e., skewed to the right or right skewed)

1. This is just one measure of Pearson skewness. Another formulation of Pearson skewness uses the mode instead of the mean.

is true for a return distribution because the mean return is further to the right than the median return.

- In contrast to the MAD and variance, skewness can obtain positive as well as negative values. This is because not only is some absolute deviation of interest but the direction is of interest as well.

For ETF A, the historical mean monthly return and median monthly return are 0.37% and 0.32%, respectively. The standard deviation of the monthly returns is 1.01%; therefore,

$$\text{Pearson coefficient of skewness} = 3\left(\frac{0.32\% - 0.37\%}{1.01\%}\right) = -0.144.$$

Note that this calculation is in terms of monthly returns; to annualize, the monthly Pearson coefficient must be multiplied by the square root of 12.[2]

KURTOSIS MEASURES

An investor is concerned with how heavy or fat the tails of a probability distribution are. It is in the tails that the extreme values are found. In the case of return distributions, it is the left tail of the distribution that can be viewed as the downside risk.

Kurtosis measures indicate whether tails of a return distribution are heavy. By heavy, we mean how the distribution compares to a normal distribution. In a normal distribution, the kurtosis measure is 3. The following kurtosis measure is called *excess kurtosis* because it shows kurtosis relative to the normal distribution:

$$\text{Excess kurtosis} = \left\{\frac{1}{T}\frac{\left[\sum_{1}^{T}\left(r_t - \text{ historical mean return}\right)^4\right]}{\left(\text{standard deviation}\right)^4}\right\} - 3.$$

A positive value for excess kurtosis means that the return distribution exhibits a heavy tail relative to the normal distribution. A negative value for the excess kurtosis means that the return distribution is light relative to a normal distribution.

For ETF A,

$$\text{Excess kurtosis} = \left[\frac{1}{60}\frac{(0.002116\%)}{\left(1.02987e^{-8}\right)}\right] - 3 = 0.424.$$

Thus there is positive excess kurtosis, suggesting that the return distribution exhibits a heavy tail.

2. Multiplying the numerator in the Pearson coefficient of skewness by 12 to annualize the monthly returns and multiplying the denominator by the square root of 12 to annualize the standard deviation means simply multiplying the Pearson coefficient of skewness by the square root of 12.

BACKWARD-LOOKING TRACKING ERROR

The measures for the dispersion use either the mean return or some constant reference return in determining deviations. In bond portfolio management, performance is typically measured relative to the returns on a benchmark, which is typically some bond index. The measure of dispersion relative to some bond index is referred to as *tracking error*.

Specifically, tracking error measures the dispersion of a portfolio's returns relative to the returns of its benchmark. That is, tracking error is the standard deviation of the portfolio's active return, where active return as explained in Chapter 11 is defined as

$$\text{Active return} = \text{portfolio actual return} - \text{benchmark actual return}.$$

When tracking error is calculated using historical active returns, it is referred to as *backward-looking tracking error* (or *ex post tracking error*) and is the measure we describe here. A portfolio created to match the benchmark index (i.e., an index fund) that regularly has zero active returns (i.e., it always matches its benchmark's actual return) would have a tracking error of zero. The closer the tracking error is to zero, the closer the risk profile of the portfolio matches the risk profile of the benchmark. In constructing a portfolio based on tracking error, which we describe in Chapter 32, statistical models are used to predict the tracking error, and it is referred to as *forward-looking tracking error, predictive tracking error*, or *ex ante tracking error*.

As explained in Chapter 11, backward-looking tracking error is the denominator in the information ratio (a risk-adjusted return measure). For ETF A, the prospectus indicated that the benchmark is the Bloomberg Barclays U.S. Aggregate Float Adjusted Total Return Index Value Unhedged. This benchmark is a broad-based benchmark that measures investment-grade U.S. dollar–denominated fixed-rate taxable bonds. The backward-looking tracking error for ETF A when this benchmark is used is 16.04%. For ETF B, the backward-looking tracking error is 15.50%.[3]

3. As computed by Portfolio Visualizer in its fund performance analysis tools.

CHAPTER

18

MEASURING AND FORECASTING YIELD VOLATILITY

An input to valuation models and interest-rate modeling is the estimated yield volatility. In this chapter we describe the various ways for estimating yield volatility (historical and implied volatility) and a method for forecasting yield volatility.

HISTORICAL VOLATILITY

The most commonly used measure of volatility is the standard deviation of the percentage change in yield between two dates over some time period. For example, suppose that we are interested in the daily percentage change in yields. Let X_t denote the percentage change in yield from day t and the prior day $t - 1$. Letting y_t denote the yield on day t and y_{t-1} denote the yield on day $t - 1$, then X_t, which is the natural logarithm (ln) of the percentage change in yield between two days, can be expressed as

$$X_t = 100\left(\ln \frac{y_t}{y_{t-1}}\right).$$

For example, suppose that on day 1 the yield is 2.92% and on the next day it is 2.77%. Therefore, the natural logarithm of X, the daily percentage yield change, is

$$X_t = 100\left(\ln \frac{2.77}{2.92}\right) = -5.27363.$$

The value for X can be calculated for each day for the interval of time that the investor wants to calculate yield volatility. Using the historical daily yield change, the standard deviation of the daily percentage yield change can be computed using the standard formula for the standard deviation. The daily standard deviation will vary depending on the number of days selected to compute yield volatility. Selection of the number of observations can have a significant effect on the calculated daily standard deviation.

Annualizing the Standard Deviation

The daily standard deviation can be annualized by multiplying it by the square root of the number of days in a year. That is,

$$\text{Daily standard deviation} = \sqrt{\text{number of days in a year}}.$$

Market practice varies with respect to the number of days in the year that should be used in this annualizing formula. Typically, either 250 days, 260 days, or 365 days are used. Thus, in calculating an annual standard deviation, the investor must decide on (1) the number of daily observations to use and (2) the number of days in the year to use to annualize the daily standard deviation.

When using the above formula to annualize the daily standard deviation, an assumption is made. For any probability distribution, it is important to assess whether the value of a random variable in one period is affected by the value that random variable took in a prior period. Casting this in terms of yield changes, it is important to know whether the yield today is affected by the yield in a prior period. The term *serial correlation* is used to describe the correlation between the yield in different periods. Annualizing the daily yield by multiplying the daily standard deviation by the square root of the number of days in a year assumes that serial correlation is not significant.

Interpreting the Standard Deviation

How does one interpret a yield volatility? Suppose that the historical yield volatility is *a*%. This means that if the prevailing yield is *b*%, then the annual standard deviation is *a*% × *b*% basis points. So, for example, if the historical yield volatility is 20% and the prevailing yield is 3%, then the annual standard deviation is 60 basis points (20% × 3%).

Assuming that the yield volatility is approximately normally distributed and that an investor believes that yield volatility going forward is constant at 20%, then an investor can use the properties of the normal probability distribution discussed in Chapter 30 to construct an interval or range for what the future yield will be. The interval or range constructed is called a *confidence interval*. For example, as will be explained in Chapter 30, for a normal distribution there is a 68.3% probability that the yield will be between one standard deviation below and above the mean. Assuming that the mean is the prevailing yield, then if the annual standard deviation is 60 basis points and the prevailing yield is 3%, there is a 68.3% probability that the yield next year will be between 2.4% (3% – 60 basis points) and 3.6% (3% + 60 basis points). That is, the confidence interval is 2.4% to 3.6%. For three standard deviations below and above the prevailing yield, there is a 99.7% probability that the yield next year will be in the computed interval. If the standard deviation is 60 basis points, then three standard deviations is 180 basis points. The confidence interval is then 1.2% (3% – 120 basis points) to 4.8% (3% + 180 basis points).

HISTORICAL VERSUS IMPLIED VOLATILITY

Investors estimate yield volatility in one of two ways. The first way is by estimating historical yield volatility, the method that we have just described. The resulting volatility is referred to as *historical volatility.* The second way is to estimate yield volatility based on the observed prices of interest-rate derivatives, specifically interest-rate caps and floors. Yield volatility calculated in this way is called *implied volatility.*

The implied volatility is based on some option pricing model. One of the inputs to any option pricing model in which the underlying is an interest rate or an interest-rate instrument is *expected yield volatility*. If the observed price of an option is assumed to be the fair price and the option pricing model is assumed to be the model that would generate that fair price, then the implied yield volatility is the yield volatility that when used as an input into the option pricing model would produce the observed option price.

There are several problems with using implied volatility. First, it is assumed that the option pricing model is correct. Second, option pricing models typically assume that volatility is constant over the life of the option. Therefore, interpreting an implied volatility becomes difficult. Third and perhaps most important, implied volatilities of options on the same underlying instrument should be the same regardless of the type of option (i.e., call or put), the time to expiration, and the strike price. In practice, implied volatilities do differ by the type of option, time to expiration, and strike price.

There is a well-established relationship between the implied volatility and the strike price of an option known as the *volatility skew* that has been documented for options where the underlying is individual stocks, a stock index, or foreign exchange. Specifically, when implied volatility is plotted against the strike price, there is a U-shaped curve, as shown in Exhibit 18–1 and referred to as the *volatility smile*. This pattern indicates the following[1]:

- For call options,
  - As the option moves in the money, implied volatility increases (i.e., moves to the left of the implied volatility for an at-the-money call option in Exhibit 18–1);
  - As the option moves out of the money, implied volatility increases (i.e., moves to the right of the implied volatility for an at-the-money call option in Exhibit 18–1).
- For put options,
  - As the option moves in the money, implied volatility increases (i.e., moves to the right of the implied volatility for an at-the-money put option in Exhibit 18–1);

1. An at-the-money option is an option where the strike price is equal to the current market price. An in-the money-option is an option that has intrinsic value if exercised while an out-of-the-money option has no intrinsic value if exercised.

- As the option moves out of the money, implied volatility increases (i.e., moves to the left of the implied volatility for an at-the-money call option in Exhibit 18–1).

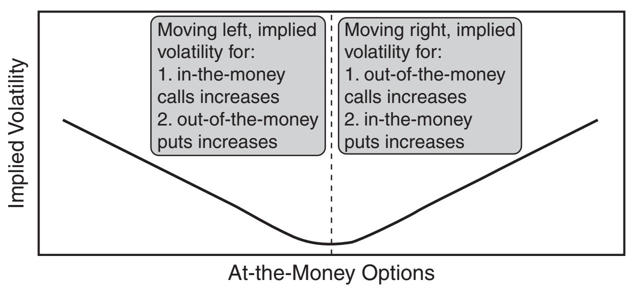
Therefore, implied volatility is greater for in-the-money and out-of-the-money options than for at-the-money options. The option pricing model used in these studies for computing implied volatility is the Black–Scholes option pricing model.[2]

While there are numerous studies documenting the volatility smile for stocks, stock indexes, and foreign exchange, there are only two empirical studies that investigate the pattern of implied volatility for interest-rate options, specifically using interest-rate caps. The first study was by Jarrow, Li, and Zhao, who found using the Black option pricing model[3] to compute implied volatility that there is a volatility smile for U.S. dollar interest-rate cap (and floor) prices.[4] They reported that implied volatility for interest-rate caps exhibits an asymmetric smile that they referred to as a "sneer." Specifically, they found that for in-the-money caps there was a stronger skew than for out-of-the-money caps. Analyzing the euro interest-rate caps, Deuskar, Gupta, and Subrahmanyam observed volatility smiles for all maturities in their study.[5]

This begs the question of which of the many implied volatilities should be used. To help answer this question, many practitioners construct a three-dimensional plot of implied volatility against time to expiration and strike price that is called the *implied volatility surface*. The implied volatility surface represents the constant value of volatility that equates each traded option's model price to its market price.

EXHIBIT 18–1

Volatility Smile



2. Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, Vol. 81 (1973), pp. 637–654.
3. Fischer Black, "The Pricing of Commodity Contracts," *Journal of Financial Economics*, Vol. 3 (1976), pp. 167–179.
4. Robert Jarrow, Haitao Li, and Feng Zhao, "Interest Rate Caps 'Smile' Too! But Can the LIBOR Market Models Capture It?" *Journal of Finance*, Vol. 62, No. 1 (2007), pp. 345–382.
5. Prachi Deuskar, Anurag Gupta, and Marti G. Subrahmanyam, "The Economic Determinants of Interest Rate Option Smiles," *Journal of Banking and Finance*, Vol. 32, No. 5 (2008), pp. 714–728.

FORECASTING YIELD VOLATILITY

Yield volatility as measured by the standard deviation typically varies based on the time period selected and the number of observations. Now we turn to the issue of forecasting yield volatility. There are several methods. Before describing these methods, let's address the question of what mean value should be used in the calculation of forecasted standard deviation.

Suppose that today an investor is interested in a forecast for volatility using the 30 most recent days of trading and updating that forecast at the end of each trading day. What mean value should be used? The investor can calculate a 30-day moving average of the daily percentage yield change. To calculate a moving average of the daily percentage change today, the investor would use the 30 prior trading days. On the next day, the investor would calculate the 30-day average by using the percentage yield change over the 30 prior trading days and would exclude the percentage yield change on the thirty-first prior trading day.

Rather than using a moving average, it is more appropriate to use an expectation of the average. It has been argued that it would be more appropriate to use a mean value of zero.[6] In that case, the variance is calculated by computing deviations from zero squared.

There are two methods commonly used for forecasting daily yield volatility:

- Moving averages and
- Autoregressive conditional heteroscedasticity.

Forecasting Using Moving Averages

Moving averages are used to forecast yield volatility. A *moving average* involves the use of a specified number of days to compute average yield volatility today and then in subsequent days using information for the same number of days to compute a moving average for the next day. For example, suppose that a trader wants to calculate 20-day yield volatility today. The last 20 days are used to calculate the estimated yield volatility today. On the next day, the estimated yield volatility is obtained by dropping the earliest day and using the prior day to calculate the average. A moving average can be calculated in one of two ways: equally weighted moving-average method or exponentially weighted moving-average method.

The equally weighted moving-average method assigns an equal weight to all observations (i.e., the daily percentage yield change). So, if an investor is calculating volatility based on the most recent 30 days of trading, each day is given a weight of 1/30. To give greater importance to more recent information, observations further in the past should be given less weight. That is, a weighted average

6. Jacques Longerstacey and Peter Zangari, *Five Questions About RiskMetrics* (New York: JP Morgan Research, 1995).

is calculated. For example, suppose that 30 trading days are used. Recalling that X_t is the daily percentage change in the yield, the weighted average for the 30-day trading period is computed as follows:

$$W_1 X_1 + W_2 X_2 + \ldots + W_{30} X_{30},$$

where W_t ($t = 1, \ldots, 30$) is the weight assigned to the ith observation, and the sum of the weights must be equal to 1.

The weights are also such that the observations closest to the current date are assigned a larger weight or, equivalently, the early observations are given a smaller weight. One approach suggested by RiskMetrics[7] is to use an *exponential moving average* (or *exponentially weighted moving average*) given by

$$W_t = \frac{1-\beta}{\beta^t} \times 100,$$

where β is a value between 0 and 1. The observations are arrayed so that the closest observation is $t = 0$, the second closest is $t = 1$, and so on. For example, if β is 0.87, then the weight for the closest six observations ($t = 0$ to $t = 4$) is

$$W_0 = [(1 - 0.87)/(0.87)^0] \times 100 = 13.00\%;$$
$$W_1 = [(1 - 0.87)/(0.87)^1] \times 100 = 11.31\%;$$
$$W_2 = [(1 - 0.87)/(0.87)^2] \times 100 = 9.84\%;$$
$$W_3 = [(1 - 0.87)/(0.87)^3] \times 100 = 8.56\%;$$
$$W_4 = [(1 - 0.87)/(0.87)^4] \times 100 = 7.45\%.$$

The parameter β is measuring how quickly the information contained in past observations is *decaying* and hence is referred to as the *decay factor*. The smaller the β, the faster is the decay. What decay factor to use depends on how fast the mean value for the random variable changes over time. A random variable whose mean value changes slowly over time will have a decay factor close to 1. A discussion of how the decay factor should be selected is beyond the scope of this chapter.[8]

Forecasting Using Econometric Methods

The field of econometrics has developed several methods for forecasting volatility using time-series data. These methods take into account the behaviors of volatility that have been observed in real-world financial markets. The patterns, or stylized facts, about volatility include persistence and mean reversion.[9]

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7. RiskMetrics, part of JP Morgan in the early 1990s, focused on the bank's risk exposure across the firm. It was spun off from JP Morgan in 1994 and subsequently purchased by MSCI in 2000.
8. A technical description is provided in *RiskMetrics—Technical Document*, RiskMetrics, New York, December 17, 1996, pp. 77–79. Available at https://www.msci.com.
9. Robert F. Engle and Andrew J. Patton, "What Good Is a Volatility Model?" *Quantitative Finance*, Vol. 1, No. 2 (2001), pp. 237–245.

Persistence means the clustering of large movements in the percentage yield changes and the clustering of small movements in the percentage yield changes. This results in *volatility clustering*, wherein a period of high volatility is followed by a period of high volatility. Furthermore, a period of relative stability in returns appears to be followed by a period that can be characterized in the same way. Volatility clustering has implications for modeling volatility because it says that volatility changes today can be expected to impact future volatility.

With volatility clustering, a period where volatility is high will eventually give way to a period where volatility decreases to some "normal" level. Similarly, if there is a period where volatility is low, eventually volatility will increase and return to a period where there is a "normal" level of volatility. The return of a high and low period to some normal level is referred to the *mean reversion of volatility*. The issue in forecasting is determining what that normal level of volatility is and how that normal level may change over time. The change can be the result of institutional changes or the structure of the market.

ARCH Model

The first econometric model used to forecast volatility was developed by Engle and called an *autoregressive conditional heteroscedasticity* (ARCH) model.[10] *Autoregressive* refers to the fact that past volatility impacts current volatility. The term *conditional* means that the value of the variance (the measure of volatility) depends on or is conditional on the value of the random variable. In terms of yield volatility forecasting, this means that the variance (volatility) depends on the yield level. The term *heteroscedasticity* means that the variance (volatility) is not equal for all values of the random variable. The estimate of the variance in an ARCH model is based on a long-run average variance using n observations. In the weighting of observations, less weight is given to older observations.

The ARCH model is

$$\sigma_t^2 = a + b(x_{t-1} - \overline{x})^2, \tag{18–1}$$

where

σ_t^2 = variance on day t;

$x_{t-1} - \overline{x}$ = deviation from the mean on day $t-1$.

a and b are the parameters to be estimated typically using regression analysis.

Equation (18–1) can be interpreted as follows. The estimate of the variance on day t depends on how much the observation on day $t - 1$ deviates from the mean. This is where the "conditional" aspect comes into play because the variance on day t is conditional on the deviation from day $t - 1$. The squaring of the deviation is done because what is important for forecasting the interest is the magnitude, not

10. Robert F. Engle, "Autoregressive Conditional Heteroskedasticity with Estimates of Variance of U.K. Inflation," *Econometrica*, Vol. 50 (1982), pp. 987–1008.

the direction, of the deviation. Use of the deviation on day $t - 1$ means that recent information (as measured by the deviation) is being considered when forecasting volatility.

Generalized ARCH Model

The ARCH model given by equation (18–1) can be generalized in several ways. First, information for days prior to $t - 1$ are ignored in the ARCH model. If information is believed to be provided by older data, information about days prior to $t - 1$ can be incorporated into equation (18–1) by using the squared deviations for several prior days. For example, suppose that three prior days are believed to provide information. Then equation (18–1) can be generalized to

$$\sigma_t^2 = a + b_1(x_{t-1} - \overline{x})^2 + b_2(x_{t-2} - \overline{x})^2 + b_3(x_{t-3} - \overline{x})^2, \qquad (18\text{–}2)$$

where a, b_1, b_2, b_3, and b_4 are parameters to be estimated using regression.

A second way that the ARCH model has been generalized is to include not only squared deviations from prior days as a random variable on which the variance is conditional but also the estimated variance for prior days. For example, the following equation generalizes equation (18–1) for the case where the variance at time t is conditional on the deviation squared at time $t - 1$ and the variance at time $t - 1$:

$$\sigma_t^2 = a + b(x_{t-1} - \overline{x})^2 + c\sigma_{t-1}^2. \qquad (18\text{–}3)$$

Now there is a third parameter, c, to be estimated by regression analysis in addition to a and b.

The ARCH model can be further generalized by combining further lags (i.e., use information for days prior to day $t - 1$) for both the squared deviations (as done in equation (18–2)) and the prior variances (as done in equation (18–3)). For example, suppose that the variance at time t is assumed to be conditional on three prior periods of squared deviations and two prior variances; then equation (18–1) can be generalized as follows:

$$\sigma_t^2 = a + b_1(x_{t-1} - \overline{x})^2 + b_2(x_{t-2} - \overline{x})^2 + b_3(x_{t-3} - \overline{x})^2 + c_1\sigma_{t-1}^2 + c_2\sigma_{t-1}^2. \qquad (18\text{–}4)$$

The models given by equations (18–2)–(18–4) are referred to as the *generalized ARCH* or *GARCH* model that was developed by Bollerslev.[11] Because the GARCH models depend on the number of prior squared deviations and the number of prior variances, the convention in the market is to use the notation GARCH(i,j) to describe the model where i indicates the number of prior squared deviations and j the number of prior variances.

11. Tim Bollerslev, "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, Vol. 31 (1986), pp. 307–327.

PART
SIX

ANALYZING BONDS WITH EMBEDDED OPTIONS

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CHAPTER

19

INTEREST-RATE MODELING

An interest-rate model is a probabilistic description of how interest rates can change over time. Given the information available today, we do not know what interest rates will be in the future. The characterization of that uncertainty about future interest rates is mathematically described by an interest-rate model. Interest-rate models are used to value fixed-income securities and option-type derivatives (i.e., options on bonds, swaptions, and caps). We will see how interest-rate models are used to value bonds with embedded options and mortgage-backed securities in Chapters 20 and 27, respectively. The reason is that these securities have a cash flow for each period that depends on the interest rate for that period. With respect to option-type derivatives such as options, caps, and swaptions, the exercise and payoff depend on the interest rate on the exercise date.

In this chapter we provide an overview of the types of interest-rate models and the most popular arbitrage-free interest-rate models. As we will see, these models can be classified as no-arbitrage and equilibrium models. No-arbitrage models are the preferred framework to value interest-rate derivatives. This is so because they minimally ensure that the market prices for option-free bonds are exact (i.e., consistent with the term structure of interest rates). Equilibrium models do not price bonds exactly, which can have significant effects on the corresponding contingent claims. Consequently, our focus will be on no-arbitrage models. Moreover, no-arbitrage models can be classified as one-factor and multifactor models. The most commonly used models are one-factor models, and the factor is the short-term interest rate. Therefore, we will focus on one-factor no-arbitrage models. The five models we discuss are the Ho–Lee model, the Kalotay–Williams–Fabozzi model, the Black–Derman–Toy model, the Hull–White model, and the Black–Karasinski model. We show how each of these interest-rate models is used to create a lattice that is used for valuing a bond or interest-rate derivative. Because the interest-rate model employed affects the values of bonds with option features, it also impacts the bond's effective duration, effective convexity, and option-adjusted spread, so we conclude the chapter with empirical evidence that addresses this issue.

This chapter is coauthored with Gerald W. Buetow, Jr., CEO of BFRC Services, LLC; Bernd Hanke, Director of Research at BFRC Services, LLC; and Brian J. Henderson, Associate Professor of Finance, School of Business, George Washington University.

Ultimately, a portfolio manager or analyst must select the interest-rate model to use. Cheyette offers an excellent review of how to select the appropriate model[1]; he suggests that one must consider the characteristics of the security to be evaluated in order to select the best model. Cheyette also illustrates empirically that some models may better capture actual interest-rate dynamics, but he readily notes that the empirical evidence is far from conclusive.

NO ARBITRAGE VERSUS EQUILIBRIUM INTEREST-RATE MODELS

Interest rates are typically modeled using a mathematical tool known as *stochastic differential equations* (SDEs). A discussion of this topic is well beyond of the scope of this chapter, so we will limit our discussion to the general features of interest-rate models.

Two approaches are used to implement the same SDE: equilibrium and no arbitrage. While these two approaches begin with a given SDE, they differ as to how each approach applies the SDE to bonds with embedded options and contingent claims (i.e., options, caps, and swaptions). *Equilibrium models* begin with an SDE model and develop pricing mechanisms for bonds under an equilibrium framework. *No-arbitrage models*, also referred to as *arbitrage-free models*, start with the same or similar SDE models as equilibrium models. However, arbitrage-free models use observed market prices to generate an interest-rate lattice. The interest-rate lattice represents the short rate in such a way as to ensure that there is a no-arbitrage relationship between the observed market price and the model-derived value. We will discuss and illustrate an interest-rate lattice and how it is used to value bonds with embedded options in Chapter 21.

Practitioners prefer arbitrage-free models to value options on bonds with embedded options and contingent claims because such models minimally ensure that the prices observed in the market for underlying bonds match the current term structure of interest rates. As a result, bonds and interest-rate derivatives will be valued in a consistent framework. Equilibrium models, in contrast, are not parameterized to produce bond prices that are consistent with the term structure, and therefore, there is not a consistent framework for valuing options on bonds and the underlying bonds. Therefore, our focus in this chapter is on arbitrage-free models.

ONE-FACTOR VERSUS MULTIFACTOR INTEREST-RATE MODELS

The most common model used to describe the behavior of interest rates assumes that short-term interest rates follow some statistical process and that other interest rates in the term structure are related to short-term rates. The short-term rate

1. Oren Cheyette, "Interest Rate Models," Chapter 1 in Frank J. Fabozzi (ed.), *Interest Rate, Term Structure, and Valuation Modeling* (New York: John Wiley & Sons, 2002).

(i.e., short rate) is the only one that is assumed to drive the rates of all other maturities. Hence these interest-rate models are referred to as *one-factor models*. The other rates are not randomly determined once the short rate is specified. The rate for all other maturities is determined using arbitrage arguments. The multifactor models that have been proposed in the literature include the short rate as one of the factors. Multifactor models typically specify a long-term rate as the second factor.

In practice, however, one-factor models are used because of the difficulty of applying even a two-factor model. The high correlation between rate changes for different maturities provides some support for the use of a one-factor model and evidence that supports the position that a level shift in interest rates accounts for the major portion of the change in the yield curve.[2]

NO-ARBITRAGE INTEREST-RATE MODELS

The interest-rate models we examine assume that the short-term interest rate follows a certain process that can be represented by a stochastic differential equation. All the interest-rate models are special cases of the general form of changes in the short-term rate:

$$df[r(t)] = \{\theta(t) + \rho(t)g[r(t)]\}dt + \sigma[r(t), t]dz. \qquad (19\text{–}1)$$

Equation (19–1) is a one-factor interest-rate model that considers only the short-term interest rate (one factor). Let's look carefully at each of the elements in equation (19–1).

The variables f and g are suitably chosen functions of the short-term rate and are the same for most models we describe in this chapter. The letters r and t represent the short-term interest rate and time, respectively. The letter d means instantaneous rate of change. So dr and dt are the instantaneous rate of change of the short-term interest rate and the instantaneous rate of change of time. The term σ is the local volatility of the short-term rate.

The letter z is a random variable that is a normally distributed Wiener process. It is the Wiener process that captures the randomness of future changes in the short-term interest rate. The Wiener process, the most common process used for building random-walk models in finance, is a stochastic process characterized by stationary independent normally distributed increments. We will explain what this means shortly.

The Greek letter θ is the expected or average change in the short-term rate. The notation $\theta(t)$ is the expected or average change in the short-term rate over a short period of time. The mean reversion to an equilibrium short-term rate is denoted by the Greek letter ρ, and $\rho(t)$ means that it depends on time. In the context of interest-rate modeling, mean reversion means that short-term interest rates will

2. A one-factor model should not be used in valuing financial instruments when the payoff depends on the shape of the spot-rate curve rather than simply the level of interest rates.

eventually revert to a long-run average (mean). So, if short-term interest rates are above the mean, they will eventually decline to the mean value; if short-term interest rates are below the mean, they will eventually rise to the mean value.

Equation (19–1) has two components:

First component: $\{\theta(t) + \rho(t)g[r(t)]\}dt$ is the expected or average change in the short-term rate over a short period of time. The first component consists of a drift term, $\theta(t)$, and mean reversion term, $\rho(t)g[r(t)]$.

Second component: $\sigma[r(t), t]dz$ is the risk term because it includes the random component dz.

All the interest-rate models described in the chapter are special cases of equation (19–1). Before we describe the most commonly used one-factor short-term interest-rate models, let's describe the Wiener process.

WIENER PROCESS

The second component of the general one-factor interest-rate model given by equation (19–1) is the risk term. A Wiener process is one of the models used for modeling *Brownian motion*. First discovered by the botanist Robert Brown in 1827, Brownian motion is a phenomenon used to describe the random movement of particles in a fluid. In 1905, Albert Einstein explained the behavior physically by demonstrating that the particles were constantly being bombarded by the molecules of the water. This explanation actually helped Einstein in establishing the atomic theory of matter. It was not until 1918 that the mathematician Norbert Wiener (who started the science of cybernetics) began to study Brownian motion as a mathematical random process. For this reason, the Brownian motion process is also known as the *Wiener process*. Now it is used in finance to model randomness for prices, returns, and interest rates. Although technically a Wiener process is a special case of Brownian motion, often a Wiener process is said to be a standard Brownian motion. The purpose of the Wiener process is to generate paths of possible values for the random variable. In the case of interest-rate modeling, a path of short-term interest rates is generated.

As stated earlier, a Wiener process is stationary independent normally distributed increments. What do the terms *stationary*, *independent*, and *normally distributed increments* mean? Each term has a specific meaning in probability theory.

By a *stationary process* or, more specifically, a *stationary time series*, it is meant that a property of the series is that it does *not* depend on the time that the series is observed. In contrast, when a time series for a random variable is not stationary, this means that there are trends that affect the value of the random variable at different times. The implication of a time series that is said to be stationary is that there is no predictable pattern in the long term. Increments being *independent* means that the changes in the random variable do not depend on each other.

A Wiener process is a *Markov process*. This means that current and future behaviors of a Wiener process do not depend on past behavior. Therefore, in terms of interest-rate modeling, it means that past interest-rate changes do not determine how interest-rate changes will behave today or in the future. A Wiener process is also said to be a *martingale*, which means that knowledge of past changes in interest rates is not needed to predict future changes in interest rates.

By *normally distributed increments* it is meant that the changes in the random variable follow a normal distribution. We describe the properties of a normal distribution in Chapter 30.

COMMONLY USED NO-ARBITRAGE INTEREST

The Ho–Lee Model

The Ho–Lee (HL) model assumes that changes in the short-term interest rate can be modeled using equation (19–1) by setting $f(r)$ equal to r and ρ equal to zero.[3] By doing so, the process for the short-term rate is

$$dr = \theta(t)dt + \sigma(t)dz. \tag{19–2}$$

Because dz is a normally distributed Wiener process, the HL process is a normal process for the short-term rate, and the model is said to be a *normal model*. As can be seen from equation (19–2), the short-term interest rate may become negative if the random term is large enough to dominate the drift term (dt). At one time, this was considered a serious shortcoming of the HL model when the view that negative interest rates were unlikely was prominent. Today, that is not the case because negative interest rates have prevailed in the government debt market in many countries. However, despite the criticism of the HL model when negative interests were unlikely, it was still a popular model because its proponents argued that as long as the model provides good prices for bonds with embedded options, it does not matter if some of its assumptions are unrealistic. A possible drawback of the model, however, is that the volatility of the short-term rate does not depend on the level of the short-term interest rate, and the short-term rate does not mean revert to a long-term equilibrium rate, as many practitioners believe would hold in reality.

The Kalotay–Williams–Fabozzi Model

The Kalotay–Williams–Fabozzi (KWF) model assumes that changes in the short-term rate can be modeled using equation (19–1) by setting $f(r) = \ln(r)$ (where ln is the natural logarithm) and $\rho = 0$.[4] Making these adjustments to equation (19–1) produces the short-term rate process:

$$d\ln(r) = \theta(t)dt + \sigma(t)dz. \tag{19–3}$$

3. Thomas Ho and Sang Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, Vol. 41, No. 5 (1986), pp. 1011–1029.
4. Andrew Kalotay, George Williams, and Frank J. Fabozzi, "A Model for the Valuation of Bonds and Embedded Options," *Financial Analysts Journal,* Vol. 49, No. 3 (1993), pp. 35–46. The Kalotay–Williams–Fabozzi model we present is a variation of the actual model in that we allow for a nonzero drift rate.

Comparing the KWF model given by equation (19–3) to the HL model given by equation (19–2), it can be seen that the KWF model is directly analogous to the HL model except that now the change in the natural logarithm of the short-term rate is modeled instead of the change in the short-term rate itself. Because $\ln(r)$ follows a normal process, r itself follows a log-normal process, and the KWF model is therefore referred to as a *log-normal model*. Hence, although $\ln(r)$ may become negative if the risk component in equation (19–3) dominates the drift component, r itself will never be negative because $r = e \ln(r)$ will always be positive. At one time viewed as an advantage of the KWF model relative to the HL model, today it is a drawback of the model given the need to allow for the possibility of negative short-term interest rates. As with the HL model, the KWF model does not allow for mean reversion.

The Black–Derman–Toy Model

One of the main advantages of the Black–Derman–Toy (BDT) model is that it is a log-normal model that is able to capture a realistic term structure of interest-rate volatilities.[5] To accomplish this feature, the short-term interest-rate volatility is allowed to vary over time, and the drift in interest-rate movements depends on the level of rates. While interest-rate mean reversion is not modeled explicitly, this property is introduced through the term structure of volatilities. Hence the extent to which the drift term depends on the level of rates depends on the local volatility process. In other words, mean reversion is endogenous to the model (i.e., a random variable that is changed or determined by its relationship with other variables in the model).

The BDT model is obtained from equation (19–1) by setting $f(r) = \ln(r)$ and $g(r) = \ln(r)$. Therefore, the short-term rate in the BDT model follows the log-normal process given by

$$d \ln(r) = [\theta(t) + \rho(t) \ln(r)]dt + \sigma(t)dz. \tag{19–4}$$

The mean reversion term, $\rho(t)$, depends on the interest-rate local volatility as follows:

$$\rho(t) = \frac{d}{dt}\ln\left[\sigma(t)\right] = \frac{\sigma'(t)}{\sigma(t)}.$$

which gives

$$d\ln(r) = \left\{\theta(t) + \left[\frac{\sigma'(t)}{\sigma(t)}\right]\ln(r)\right\}dt + \sigma(t)dz. \tag{19–5}$$

5. Fischer Black, Emanuel Derman, and William Toy, "A One Factor Model of Interest Rates and Its Application to the Treasury Bond Options," *Financial Analysts Journal,* Vol. 46, No. 3 (1990), pp. 33–39.

Comparing the BDT model given by equation (19–5) with the KWF model given by equation (19–3), we observe that if the volatility term structure is flat so that $\sigma(t)$ is constant, then $\sigma'(t) = 0$ and $\rho(t) = 0$, so the BDT model reduces to the KWF model. In this sense, the KWF model is a special case of the BDT model for constant local volatility. When the local volatility term structure is decreasing (i.e., if $\sigma'(t) < 0$), the BDT model will exhibit mean reversion. If $\sigma'(t) > 0$ (i.e., if the local volatility term structure is increasing), the BDT model will not exhibit mean reversion. Hence mean reversion depends entirely on the shape of the local volatility term structure.

While some researchers believe that the mean reversion in the BDT model will be more representative of the market because it is endogenous to the model, others argue that it might be more appropriate to model mean reversion independently of the volatility process. This can be accomplished only in the framework of a binomial model (discussed later in this chapter) through the use of varying time steps (as in the Hull–White and Black–Karasinski binomial trees that will be discussed later), which complicates both the numerical solution and the applicability of the model substantially.

The Hull–White Model

Similar to the HL model, the Hull–White (HW) model assumes a normal process for the short-term rate.[6] The model can be obtained from equation (19–1) by setting $f(r) = g(r) = r$ and $\rho = -\theta$. The process for the short-term rate is thus:

$$dr = (\theta - \varphi r)dt + \sigma dz, \tag{19–6}$$

where

θ = long-term equilibrium mean rate;
φ = mean reversion term.

Note that if $\varphi = 0$, the HW process reduces to the HL process. The HL model is therefore a special case of the HW model when there is no mean reversion.

The HW model explicitly models mean reversion by specifying a central tendency for the short-term rate and by specifying the speed at which the short-term rate reverts to that central tendency. The mean reversion coefficient allows correction for uncontrolled growth or decline in the HW model. The coefficient therefore reduces the probability of negative interest rates, although it does not completely rule out negative interest rates.

6. John Hull and Alan White, "Pricing Interest Rate Derivative Securities," *Review of Financial Studies*, Vol. 3, No. 4 (1990), pp. 573–592.

The Black–Karasinski Model

In order to obtain the Black–Karasinski (BK) short-term rate process, we set $f(r) = \ln(r)$, $\rho = -\theta$, and $g(r) = \ln(r)$ in equation (19–1), which results in the short-term rate process[7]

$$d\ln(r) = [\theta - \varphi \ln(r)]dt + \sigma dz. \tag{19–7}$$

Inspection of the BK model given by equation (19–7) shows that it is simply the logarithmic analogue of the HW model given by equation (19–6). In the BK model, $\ln(r)$ has the same properties as r in the HW model. As in the KWF model, however, r cannot become negative because $r = e^{\ln(\mathrm{r})}$, which is always positive. This is a disadvantage of the BK model over the HW model.

Therefore, we see that the BK model is an extension of the KWF process in the same way as the HW process is an extension of the HL process. In fact, because $\varphi = 0$, the KWF model is obtained.

The BK model explicitly models mean reversion by specifying a central tendency for the short-term rate and the speed at which the short-term rate reverts to that central tendency.

BINOMIAL INTEREST-RATE LATTICE

In this section we present binomial lattice representations of the no-arbitrage interest-rate models. In the binomial lattice, the interest rate may make one of two possible moves over discrete points in time. For our purposes, we present lattices where the length of each time step is six months. Exhibit 19–1 illustrates a four-period binomial tree. Note that at each node the interest rate takes either an up step or a down step. The size of each step is determined by the properties of the interest-rate model. Additionally, notice that the tree "recombines," meaning that an up step followed by a down step produces the same rate as a down step followed by an up step. Recombination is a common assumption in binomial trees and results from the imposition of additional algebraic constraints. The numerical methods required to fit the interest-rate models to binomial trees are beyond the scope of this chapter, but we present binomial lattice representations of the model to illustrate the important features of the interest-rate models.[8]

The no-arbitrage property of the interest-rate models presented in this chapter comes from the fact that the model rates match the properties of the current term structure. For example, denoting the current one-period (6-month) spot rate as $r_{1,0}$, the two-period spot rate as z, and the implied forward rate as f_1, we can illustrate the no-arbitrage property using the binomial lattice. The two possible

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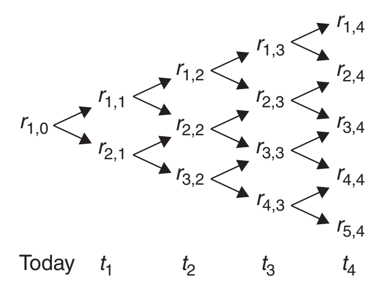
7. Fischer Black and Piotr Karasinski, "Bond and Option Pricing when Short Rates Are Lognormal," *Financial Analysts Journal,* Vol. 47, No. 4 (1991), pp. 52–59.
8. A thorough discussion of the numerical methodologies involved in fitting the models to binomial trees is provided in Gerald W. Buetow and James Sochacki, *Binomial Interest Rate Models* (Charlottesville, VA: CFA Institute Research Foundation, 2001).

EXHIBIT 19–1

Binomial Lattice of Forward Rates



values for the interest rate next period in the binomial lattice are $r_{1,1}$ and $r_{2,1}$, and the no-arbitrage property is satisfied by the following constraint, which requires the one-period spot rate, followed by the one-period rate at the next time step, $r_{1,1}$ or $r_{2,1}$, to be equal to the two-period spot rate:

$$\left(1+\frac{1}{2}z\right)^2=\left(1+\frac{1}{2}r_{1,0}\right)\left[p_u\left(1+\frac{1}{2}r_{1,1}\right)+p_d\left(1+\frac{1}{2}r_{2,1}\right)\right], \qquad (19\text{–}8)$$

where p_u and p_d are the probabilities of the up and down move, respectively. Imposing the no-arbitrage constraint ensures that pricing option-free bonds using the interest-rate lattice generates prices that are consistent with the observed spot curve produced from market prices. Because of this property, the no-arbitrage models have practical appeal and are useful for pricing and risk-management purposes.

Ho–Lee Binomial Lattice

Exhibit 19–2 presents the HL binomial lattice where the term structure is flat at 2% and volatility is constant. In panel A of the exhibit, volatility is 1%, and in panel B, volatility is 10%. There are several important features of the HL model evident in the binomial lattices. First, the one-period interest rate may be negative, which results from the fact that in the HL process the rate is distributed normally and is unbounded.[9] Additionally, the level of volatility drives the spread. To illustrate, panel B of Exhibit 19–2 presents the HL lattice where volatility is 10%. The greater

9. In fact, it can be demonstrated that the spread between the high and low rates in the HL lattice equals $2 \times k \times \sigma t$, where k is time (in years) and t is the length of the time step. This algebraic relation demonstrates that the spread, or distance, between the highest possible rate and the lowest at each time step is an increasing function of time and often produces negative rates.

EXHIBIT 19-2

Ho–Lee Binomial Interest-Rate Lattice

Panel A: Flat Term Structure at 2% and 1% Constant Volatility

									8.48%
								7.75%	
							7.02%		7.06%
						6.29%		6.33%	
					5.57%		5.60%		5.65%
				4.85%		4.88%		4.92%	
			4.14%		4.16%		4.19%		4.23%
		3.42%		3.44%		3.47%		3.50%	
	2.71%		2.72%		2.74%		2.78%		2.82%
2.00%		2.01%		2.02%		2.05%		2.09%	
	1.30%		1.31%		1.33%		1.36%		1.40%
		0.59%		0.61%		0.64%		0.67%	
			–0.11%		–0.08%		–0.05%		–0.01%
				–0.80%		–0.78%		–0.74%	
					–1.50%		–1.47%		–1.42%
						–2.19%		–2.15%	
							–2.88%		–2.84%
								–3.57%	
									–4.25%
Time in Years	0.5	1	1.5	2	2.5	3	3.5	4	4.5

EXHIBIT 19-2

The Ho-Lee Binomial Interest Rate Lattice (*Continued*)

Panel B: Flat Term Structure at 2% and 10% Constant Volatility

									76.72%
								67.44%	
							58.40%		62.58%
						49.61%		53.30%	
					41.06%		44.26%		48.44%
				32.75%		35.46%		39.16%	
			24.70%		26.92%		30.12%		34.30%
		16.88%		18.61%		21.32%		25.01%	
	9.32%		10.55%		12.77%		15.98%		20.15%
2.00%		2.74%		4.47%		7.18%		10.87%	
	–4.82%		–3.59%		–1.37%		1.83%		6.01%
		–11.40%		–9.67%		–6.96%		–3.27%	
			–17.73%		–15.51%		–12.31%		–8.13%
				–23.82%		–21.10%		–17.41%	
					–29.65%		–26.45%		–22.27%
						–35.25%		–31.55%	
							–40.59%		–36.41%
								–45.70%	
									–50.56%
Time in Years	0.5	1	1.5	2	2.5	3	3.5	4	4.5

dispersion of rates resulting from the greater volatility is immediately evident and illustrates that the rates in this model may be very large and frequently negative.

The HL process models the short rate as distributed normally. This is evident in the binomial lattice representation because the rates are symmetrical around the mean rate. For example, the distance between rates $r_{1,2}$ and $r_{2,2}$ is identical to the distance between rates $r_{2,2}$ and $r_{3,2}$.

Kalotay–Williams–Fabozzi Lattice

Recall that the KWF binomial lattice is analogous to a log-normal version of the HL model. It is also a special case of the BDT model when volatility is assumed to be positive and constant. Panel A of Exhibit 19–3 presents the KWF binomial lattice for the scenario where the term structure is flat at 2% and volatility is a constant 10%. Although the scenario is directly comparable to the scenario in panel B of Exhibit 19–2 for the HL model, there are two important distinctions between the two lattices. First, the log-normal distribution in the KWF model restrains the interest-rate paths from becoming negative. Because the HL model is distributed normally, negative rates are possible, but the log-normal distribution of the KWF model restricts rates to be positive. Second, the spread of possible rates at the same volatility level is smaller than in the HL model. Whereas the rates are distributed normally around the center nodes in the HL model, in the KWF model the rates are distributed asymmetrically around the center node and are skewed toward higher rates. This property illustrates the importance of the distributional assumptions stemming from the model's differential processes.

Black–Derman–Toy Binomial Lattice

For the BDT binomial lattice, as noted earlier, the differential process in this model is log-normal and incorporates endogenous mean reversion in which the slope of the volatility curve drives mean reversion in the model. Exhibit 19–4 presents the binomial lattice for the normal-interest-rate tree scenario where the current rate is 2% and the rate increases by 0.15 basis points each period. Panels A, B, and C present three scenarios for a normal term structure across different volatility structures. In panel A, the volatility structure is decreasing from 20% by 0.5% each period, in panel B it is increasing from 20% by 0.5% each period, and in panel C it is constant. When volatility is constant, the model reduces to the KWF model. Note that the rates are all positive and are not extreme, as in the normally distributed HL model.

The shape of the volatility curve drives the model's mean reversion. This is evident in panels A, B, and C of Exhibit 19–4 because the only change across the panels is the shape of the volatility structure. Comparing panel A to panel C, it is clear that the decreasing volatility structure has the effect of increasing mean reversion.

EXHIBIT 19-3

Kalotay–Williams–Fabozzi Binomial Interest-Rate Lattice

Panel A: Flat Term Structure at 2% and 10% Constant Volatility

									3.71%
								3.46%	
							3.23%		3.22%
						3.02%		3.00%	
					2.82%		2.80%		2.79%
				2.63%		2.62%		2.61%	
			2.45%		2.44%		2.43%		2.42%
		2.29%		2.28%		2.27%		2.26%	
	2.14%		2.13%		2.12%		2.11%		2.10%
2.00%		1.99%		1.98%		1.97%		1.96%	
	1.86%		1.85%		1.84%		1.83%		1.83%
		1.73%		1.72%		1.71%		1.70%	
			1.61%		1.60%		1.59%		1.58%
				1.49%		1.49%		1.48%	
					1.39%		1.38%		1.38%
						1.29%		1.28%	
							1.20%		1.19%
								1.11%	
									1.04%
Time in Years	0.5	1	1.5	2	2.5	3	3.5	4	4.5

(*Continued*)

EXHIBIT 19-3

Kalotay–Williams–Fabozzi Binomial Interest-Rate Lattice (*Continued*)

Panel B: Flat Term Structure at 2% and 20% Constant Volatility

									6.64%
								5.80%	
							5.07%		5.00%
						4.43%		4.36%	
					3.88%		3.81%		3.76%
				3.39%		3.34%		3.28%	
			2.97%		2.92%		2.87%		2.83%
		2.60%		2.56%		2.51%		2.47%	
	2.28%		2.24%		2.20%		2.16%		2.13%
2.00%		1.96%		1.93%		1.89%		1.86%	
	1.72%		1.69%		1.66%		1.63%		1.60%
		1.48%		1.45%		1.42%		1.40%	
			1.27%		1.25%		1.22%		1.20%
				1.09%		1.07%		1.05%	
					0.94%		0.92%		0.90%
						0.81%		0.79%	
							0.69%		0.68%
								0.60%	
									0.51%
Time in Years	0.5	1	1.5	2	2.5	3	3.5	4	4.5

EXHIBIT 19-4

Black–Derman–Toy Binomial Lattice: Normal Term Structure and Varying Volatility Structures

Panel A: Normal Term Structure and Decreasing Volatility

										10.61%
									9.44%	
								8.28%		8.79%
							7.15%		7.74%	
						6.08%		6.71%		7.29%
					5.08%		5.73%		6.35%	
				4.17%		4.82%		5.44%		6.04%
			3.35%		3.98%		4.59%		5.20%	
		2.62%		3.22%		3.82%		4.41%		5.01%
	2.00%		2.56%		3.11%		3.68%		4.26%	
		1.98%		2.49%		3.02%		3.58%		4.15%
			1.95%		2.44%		2.95%		3.50%	
				1.93%		2.39%		2.90%		3.44%
					1.91%		2.37%		2.87%	
						1.90%		2.35%		2.86%
							1.90%		2.35%	
								1.90%		2.37%
									1.93%	
										1.96%
Time in Years		0.5	1	1.5	2	2.5	3	3.5	4	4.5

(*Continued*)

EXHIBIT 19-4

Black–Derman–Toy Binomial Lattice: Normal Term Structure and Varying Volatility Structures (*Continued*)

Panel B: Normal Term Structure and Increasing Volatility										23.66%
									17.55%	
								13.16%		16.03%
							9.96%		12.06%	
						7.59%		9.17%		10.86%
					5.81%		7.04%		8.29%	
				4.46%		5.43%		6.39%		7.36%
			3.42%		4.21%		4.97%		5.70%	
		2.62%		3.27%		3.89%		4.46%		4.98%
	2.00%		2.55%		3.05%		3.51%		3.92%	
		1.98%		2.40%		2.78%		3.11%		3.38%
			1.89%		2.21%		2.48%		2.69%	
				1.76%		1.99%		2.16%		2.29%
					1.60%		1.75%		1.85%	
						1.42%		1.51%		1.55%
							1.24%		1.27%	
								1.05%		1.05%
									0.87%	
										0.71%
Time in Years		0.5	1	1.5	2	2.5	3	3.5	4	4.5

EXHIBIT 19-4

Black–Derman–Toy Binomial Lattice: Normal Term Structure and Varying Volatility Structures (*Continued*)

Panel C: Normal Term Structure and Constant Volatility

										15.91%
									12.92%	
								10.47%		11.94%
							8.46%		9.71%	
						6.80%		7.87%		8.96%
					5.44%		6.37%		7.29%	
				4.31%		5.12%		5.92%		6.72%
			3.39%		4.09%		4.79%		5.48%	
		2.62%		3.25%		3.86%		4.45%		5.04%
	2.00%		2.55%		3.08%		3.60%		4.12%	
		1.98%		2.45%		2.90%		3.35%		3.78%
			1.92%		2.32%		2.71%		3.09%	
				1.84%		2.19%		2.52%		2.84%
					1.75%		2.04%		2.32%	
						1.65%		1.89%		2.13%
							1.54%		1.74%	
								1.42%		1.60%
									1.31%	
										1.20%
Time in Years		0.5	1	1.5	2	2.5	3	3.5	4	4.5

Note that the upward drift and spread from high to low rates at each time step are different. The upward drift is checked by the mean reversion when the volatility structure is decreasing, and similarly, the spread is lower. Comparing panel B to panel C, it is clear that under the positively sloped volatility structure, the upward drift is larger, and the spread at each time step is greater.

Black–Karasinski Binomial Lattice

Exhibit 19–5 presents the interest-rate lattice for the BK model for a flat term structure of rates when volatility is initially 20% but decreases by 0.5% each period. To illustrate the importance of mean reversion in this model, the exhibit contains two panels; panel A illustrates the lattice with the mean reversion parameter equal to 0.015, and panel B illustrates the lattice when the mean reversion parameter equals 0.005. The exhibit illustrates that larger mean reversion impacts the rates by narrowing the spread between the rates at each time step. Additionally, it must be noted that in order to fit the model to a binomial lattice, the time step must be allowed to vary for this model. Interpolation is necessary to adjust this tree to equally spaced time steps. The larger mean reversion leads to a decreasing time step. One way to circumvent the uneven time step in this model is to use a trinomial lattice because it allows for an extra degree of freedom.

TRINOMIAL INTEREST-RATE LATTICE

Turning to the HW model, Exhibit 19–6 presents the HW trinomial lattice. The trinomial structure is identical to the binomial lattice except that there are three possible time steps from each node instead of two. Similar to the binomial lattice, the solutions impose restrictions to ensure that the trinomial lattice recombines and that the rates in the tree satisfy the necessary conditions. Exhibit 19–6 presents the HW trinomial lattice when the term structure is flat at 2%, 10% constant volatility, and zero mean reversion. The important properties of the HW process are evident in the lattice: Interest rates become negative for the bottom nodes, the spread between high and low rates at each time step is large, and the rates are distributed normally. Additionally, an upward drift is evident in the middle nodes.

Because the HW model incorporates mean reversion, Exhibit 19–7 presents the HW model with the same term structure and volatility as Exhibit 19–6 but incorporates 5% mean reversion. We expect that the mean reversion will have the effect of pulling rates back toward the mean or that the lattice tree should be "pruned." The mean reversion property is evident as the spread at each time step is reduced and each rate is pulled back toward the target rate compared to the tree in Exhibit 19–6.

In summary, the lattice representations of the no-arbitrage interest-rate models discussed in this chapter demonstrate the importance of the model assumptions about the short-rate process on the model output. It is of critical importance

EXHIBIT 19-5

Black–Karasinski Binomial Lattice: Inverted Term Structure

Panel A: Flat Term Structure, Mean Reversion = 0.015

									3.71%
								3.68%	
							3.59%		3.21%
						3.46%		3.13%	
					3.28%		3.01%		2.78%
				3.06%		2.85%		2.67%	
			2.82%		2.66%		2.52%		2.41%
		2.67%		2.44%		2.35%		2.27%	
	2.22%		2.20%		2.15%		2.11%		2.08%
2.00%		1.98%		1.94%		1.93%		1.93%	
	1.72%		1.72%		1.74%		1.77%		1.80%
		1.47%		1.54%		1.59%		1.65%	
			1.35%		1.41%		1.48%		1.56%
				1.23%		1.31%		1.40%	
					1.14%		1.24%		1.35%
						1.08%		1.19%	
							1.04%		1.17%
								1.02%	
									1.01%
Time in Years	0.50	0.99	1.48	1.96	2.43	2.89	3.35	3.81	4.25

(*Continued*)

EXHIBIT 19-5

Black–Karasinski Binomial Lattice: Inverted Term Structure (*Continued*)

Panel B: Flat Term Structure, Mean Reversion = 0.005

									4.01%
								3.92%	
							3.78%		3.41%
						3.60%		3.29%	
					3.37%		3.12%		2.91%
				3.12%		2.92%		2.76%	
			2.85%		2.70%		2.58%		2.47%
		2.65%		2.46%		2.38%		2.31%	
	2.24%		2.21%		2.17%		2.13%		2.10%
2.00%		1.97%		1.94%		1.94%		1.94%	
	1.74%		1.72%		1.74%		1.76%		1.79%
		1.46%		1.53%		1.58%		1.62%	
			1.34%		1.39%		1.45%		1.52%
				1.21%		1.28%		1.36%	
					1.11%		1.20%		1.30%
						1.04%		1.14%	
							0.99%		1.10%
								0.96%	
									0.94%
Time in Years	0.50	1.00	1.49	1.99	2.48	2.96	3.45	3.93	4.41

EXHIBIT 19-6

Hull-White Trinomial Lattice: Flat Term Structure, 10% Volatility, No Mean Reversion

									123.63%
								109.06%	111.38%
							94.77%	96.81%	99.13%
						80.74%	82.52%	84.57%	86.89%
					66.98%	68.50%	70.27%	72.32%	74.64%
				53.48%	54.73%	56.25%	58.03%	60.07%	62.39%
			40.23%	41.23%	42.49%	44.00%	45.78%	47.82%	50.14%
		27.24%	27.99%	28.99%	30.24%	31.75%	33.53%	35.58%	37.90%
	14.50%	14.99%	15.74%	16.74%	17.99%	19.51%	21.28%	23.33%	25.65%
2.00%	2.25%	2.74%	3.49%	4.49%	5.75%	7.26%	9.04%	11.08%	13.40%
	–10.00%	–9.50%	–8.76%	–7.76%	–6.50%	–4.99%	–3.21%	–1.17%	1.15%
		–21.75%	–21.00%	–20.00%	–18.75%	–17.24%	–15.46%	–13.41%	–11.09%
			–33.25%	–32.25%	–31.00%	–29.48%	–27.71%	–25.66%	–23.34%
				–44.50%	–43.24%	–41.73%	–39.95%	–37.91%	–35.59%
					–55.49%	–53.98%	–52.20%	–50.16%	–47.84%
						–66.23%	–64.45%	–62.40%	–60.08%
							–76.70%	–74.65%	–72.33%
								–86.90%	–84.58%
									–96.83%
Time in Years	0.5	1	1.5	2	2.5	3	3.5	4	4.5

EXHIBIT 19-7

Hull-White Trinomial Lattice: Flat Term Structure, 10% Volatility, 5% Mean Reversion

									121.55%
								107.60%	109.30%
							93.78%	95.35%	97.05%
						80.12%	81.54%	83.10%	84.81%
					66.62%	67.88%	69.29%	70.85%	72.56%
				53.30%	54.38%	55.63%	57.04%	58.61%	60.31%
			40.16%	41.05%	42.13%	43.38%	44.79%	46.36%	48.06%
		27.22%	27.91%	28.80%	29.88%	31.13%	32.55%	34.11%	35.82%
	14.50%	14.97%	15.67%	16.56%	17.63%	18.89%	20.30%	21.86%	23.57%
2.00%	2.25%	2.73%	3.42%	4.31%	5.39%	6.64%	8.05%	9.62%	11.32%
	–10.00%	–9.52%	–8.83%	–7.94%	–6.86%	–5.61%	–4.20%	–2.63%	–0.93%
		–21.77%	–21.08%	–20.19%	–19.11%	–17.86%	–16.44%	–14.88%	–13.17%
			–33.32%	–32.43%	–31.36%	–30.10%	–28.69%	–27.13%	–25.42%
				–44.68%	–43.60%	–42.35%	–40.94%	–39.37%	–37.67%
					–55.85%	–54.60%	–53.19%	–51.62%	–49.91%
						–66.85%	–65.43%	–63.87%	–62.16%
							–77.68%	–76.12%	–74.41%
								–88.36%	–86.66%
									–98.90%
Time in Years	0.5	1	1.5	2	2.5	3	3.5	4	4.5

that users of these models understand the model assumptions and the impact those assumptions have on any results (pricing or risk metrics) based on model outputs.

IMPACT ON EFFECTIVE DURATION, EFFECTIVE CONVEXITY, AND OPTION-ADJUSTED SPREAD

When the commonly used no-arbitrage models are used to price bonds with embedded options, they often provide different values for bonds with embedded options. Not surprisingly, then, they generate different sensitivity measures such as effective duration, effective convexity, and option-adjusted spreads. Buetow, Hanke, and Fabozzi document the differences in these measures based on the different interest-rate models described in this chapter and shed some light on why there are differences.[10] They analyze four types of bonds with embedded options: a callable bond, a putable bond, a callable range note, and a putable range note. As explained in Chapter 22, range notes are floating-rate securities whose coupon is equal to a reference rate as long as the reference rate is within a certain range at the reset date. If the reference rate is outside the range, the coupon rate is set equal to zero for that period. Their analysis is performed using 5-year maturities. Binomial and trinomial interest-rate lattices with 6-month time steps were used to price each bond.

Impact on Effective Duration

With respect to effective duration, Buetow, Hanke, and Fabozzi found that the HL, KWF, and BDT models produced values that were no different for any yield change of less than 100 basis points. When the original term structure was then shifted up and down in a parallel manner by ±250 basis points and by ±500 basis points, the KWF and BDT effective duration estimates were very similar, whereas the HL model sometimes produced substantially different estimates. This is to be expected because the HL model is a normal model, whereas the KWF and BDT models are log-normal models. Although the KWF model does not incorporate interest-rate mean reversion, the BDT model includes an implicit mean reversion term introduced through the term structure of local volatilities. The differences in their effective duration estimates were relatively minor, especially for the more extreme interest-rate term structures. Therefore, the impact of the implicit mean reversion term on the effective duration estimate in the BDT model is minor.

For the HL model, the effective duration does not vary much with the level of interest rates. The HL model produces effective duration estimates that are not as representative of the price-yield properties as the estimates resulting from the lognormal models. For example, when interest rates are extremely low, a callable bond should have a very short effective duration. In fact, the one-year delayed call should dictate an effective duration close to 1.0. The KWF and BDT models were

10. Gerald W. Buetow, Jr., Bernd Hanke, and Frank J. Fabozzi, "Impact of Different Interest Rate Models on Bond Value Measures," *Journal of Fixed Income*, Vol. 11, No. 3 (2001), pp. 41–53.

in line with this estimate, but the HL model produced an effective duration estimate of more than two years (more than a 100% difference). The same was true for the HL model estimate for the putable bond in an environment of extremely high interest rates.

The study also demonstrated how the estimate for effective duration was affected using a trinomial model. The HW model, as a normal model, produced effective duration estimates that vary less with interest-rate levels than the BK effective duration estimates. As for the HL model, it produces an effective duration estimate for the callable bond that is high at very low interest rates and low at very high interest rates. The reverse is true for the putable bond. The BK effective duration estimates are more variable for different interest-rate levels and appear to be more in line with theoretical reasoning. A comparison of effective duration for the HW model and the BK model using the trinomial model indicated that the effective duration estimates for the two models can be substantially different, especially for more complex securities such as the range notes.

Effective Convexity

Investigating the impact on effective convexity, for the callable bond studied, once again the three log-normal models (KWF, BDT, and BK) showed a distinctive effective convexity pattern. The effective convexity tended to be relatively high and positive for high interest rates; it then turned into concavity for medium interest rates and was fairly linear for low interest rates (i.e., the effective convexity is close to zero). Therefore, the possibility that the callable bond might be called if interest rates drop is already anticipated at medium interest rates (for the original term structure), and the effective convexity therefore becomes negative (i.e., concave) at this interest-rate level.

The HL model (as a normal-interest-rate model) showed a somewhat different effective convexity pattern with concavity at high interest rates (+250 basis points) and very low interest rates (–500 basis points). It suffered from effective convexity estimates that did not match the necessary pricing behavior. Effective convexity estimates for putable bonds were all positive, as required by the pricing behavior. Most normal-interest-rate models tended to produce effective convexity estimates that are relatively high at all interest-rate levels. The log-normal models produced effective convexity patterns that are more representative of the pricing behavior. The effective convexity is generally highest for intermediate interest-rate levels but in all cases becomes close to zero (i.e., a linear relationship between yields and bond prices) for very high interest rates, which again is due to the putability at high interest rates.

The effective convexity pattern for range notes was substantially more volatile than for regular callable and putable bonds. This is due to the boundary effects of the interest-rate range used in the study that determined the amount of the coupon and whether a coupon will be paid or not.

In general, Buetow, Hanke, and Fabozzi found that the convexity estimates were strongly influenced by the interest-rate model used. The differences were significant. In some cases, different interest-rate models produced convexity estimates for the same bond that were of large absolute value but opposite sign. This is true not only for more complex structures such as the range notes that they examined but also for the regular bonds with embedded options. The BDT and BK (HW) models seem to generate estimates that are consistent with the pricing behavior.

Impact on Option-Adjusted Spread

The option-adjusted spread (OAS) is the constant spread that when added to every rate in an interest-rate lattice used to price a security will make the model price equal to the market price of the security. Buetow, Hanke, and Fabozzi compared the OAS that results from using different interest-rate models. As with the effective duration and effective convexity estimates, the OAS estimates obtained using different interest-rate models differ substantially. The estimates differ in some cases by more than 100%! In general, the OAS estimates obtained from the normal-interest-rate models (HL and HW) were higher for almost all the bond types than the estimates obtained from the log-normal interest rate models (KWF, BDT, and BK). This is due to the distributional differences between the types of models. The HL and HW models allow for very low and even negative interest rates, whereas the log-normal models do not. This resulted in higher OAS estimates.

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CHAPTER

20

CALL OPTIONS: INVESTMENT AND PRICE CHARACTERISTICS

In previous chapters we described the investment characteristics of option-free bonds. To establish a basis for understanding the price/yield relationship and price volatility characteristics of bonds with embedded options, we review the investment and price characteristics of options in this chapter. Our focus is on *call* options because most corporate and municipal bonds that have an embedded option have an embedded call option (i.e., are callable bonds). All mortgage passthrough securities have embedded call options.

WHAT IS AN OPTION?

An option is an agreement granting the buyer of the option the right to purchase from or sell to the writer of the option a designated instrument at a specified price within a specified period of time. The writer, also referred to as the *seller*, grants this right to the buyer in exchange for a sum of money called the *option price* or *option premium.* The price at which the instrument may be bought or sold is called the *exercise price* or *strike price.* The date after which an option is void is called the *expiration date* or *maturity date.* An option is called an *American option* if the buyer may exercise the option at any time up to and including the expiration date. An option is called a *European option* if the buyer may exercise the option on only the expiration date.

When an option grants the buyer the right to purchase the designated instrument from the writer, it is called a *call option.* When the option buyer has the right to sell the designated instrument to the writer (seller), the option is called a *put option.* The buyer of an option is said to be *long the option*; the writer (seller) is said to be *short the option*.

The most popular exchange-traded interest-rate options are options in which the designated instrument is an interest-rate futures contract. In this chapter we focus on options on cash-market bonds because we will be applying the principles and concepts in this chapter to bonds with embedded options in which the cash-market bond is the underlying instrument.

PAYOFFS FROM BUYING AND SELLING OPTIONS

The maximum amount that an option buyer can lose is the option price. The maximum profit that the option writer (seller) can realize is the option price. The option buyer has substantial upside return potential, while the option writer has substantial downside risk.

We describe the profit/loss or payoff profile for four basic option positions: (1) long call (buying a call option), (2) short call (selling or writing a call option), (3) long put (buying a put option), and (4) short put (selling or writing a put option). The discussion assumes that a position in only one option is taken. That is, no position is taken in another option or bond. *The profit/loss profiles presented assume that each option position is held to the expiration date and not exercised early.* Also, to simplify the illustrations, we assume that there are no transaction costs in implementing the position.

Long Call Position (Buying a Call Option)

An investor who purchases a call option is said to be in a *long call position*. This is the most straightforward option for allowing the investor to participate in an anticipated decrease in interest rates (increase in the price of bonds).

To illustrate this strategy, assume that there is a call option on a particular 10% coupon bond with a par value of $100 and 15 years and 2 months to maturity. The call option expires in 2 months, the strike price is $100, and the option price is $5.

Suppose that the current price of the bond is $100 (i.e., the bond is selling at par), which means that the yield on this bond is 10%. The payoff from this position will depend on the price of the bond at the expiration date. The price, in turn, will depend on the yield on 15-year bonds with 10% coupons because in 2 months the bond will have only 15 years to maturity.

If the price of the bond at the expiration date is less than or equal to $100 (which means that the market yield is greater than or equal to 10%), then the investor would not exercise the option. The option buyer will lose the entire option price of $5. Notice, however, that this is the maximum loss that the option buyer will realize, no matter how far the price of the bond declines.

If the price of the bond is higher than $100 (i.e., the market yield at expiration is less than 10%), the option buyer will exercise the option. To exercise, the option buyer purchases the bond for $100 (the strike price) and then can sell it in the market for a higher price. The option buyer will realize a loss at expiration if the price of the bond is more than $100 but less than $105 (which corresponds to a market yield at expiration of 10% and approximately 9.35%, respectively). The option buyer will break even if the price of the bond at expiration is $105. This is the break-even price because it costs the option buyer $5 to acquire the call option and $100 to exercise the option to purchase the bond. A profit will be realized if the price of the bond at expiration is more than $105 (i.e., the market yield declines to at least 9.35%).

Short Call Position (Selling or Writing a Call Option)

An investor who sells or writes a call option is said to be in a *short call position*. Investors who believe that interest rates will rise or change very little can, if their expectations are realized, generate income by writing (selling) a call option.

To illustrate this option position, we can use the call option used to illustrate the long call position. The profit/loss profile of the short call position is the mirror image of the payoff of the long call strategy. That is, the profit (loss) of the short call position for any given price of the bond at the expiration date is the same as the loss (profit) of the long call position. Consequently, the maximum profit that the short call position can produce is the option price; the maximum loss is limited only by how much the price of the bond can increase (i.e., how far the market yield can fall) by the expiration date less the option price.

Long Put Position (Buying a Put Option)

The most straightforward option position an investor can take to benefit from an expected increase in interest rates is to buy a put option. This investment position is called a *long put position*.

To illustrate the payoff profile for this position, we'll assume a hypothetical put option for a 10% coupon bond with a par value of $100, 15 years and 2 months to maturity, and a strike price of $100 selling for $5.50. The current price of the bond is $100 (yield of 10%). The payoff for this strategy at the expiration date depends on the market yield at the time.

If the price of the bond is equal to or more than $100 because the market yield has fallen below 10%, the buyer of the put option will not exercise it, thereby incurring a loss of $5.50 (the option price). The investor will exercise the option when the price of the bond at expiration is less than $100 (the market yield at expiration is above 10%). If the market yield is higher than approximately 10.75%, the price of the bond will be lower than $94.50, resulting in a profit from the position. For market yields between 10% and about 10.75%, the investor will realize a loss, but the loss is less than $5.50. The break-even market yield is approximately 10.75% because this will result in a bond price of $94.50.

As with all long option positions, the loss is limited to the option price. The profit potential, however, is substantial; the theoretical maximum profit is generated if the bond price falls to zero.

Short Put Position (Selling or Writing a Put Option)

The *short put position* involves the selling (writing) of a put option. This position in taken if the investor expects interest rates to fall or stay flat so that the price of the bond will increase or stay the same. The profit/loss profile for a short put position is the mirror image of the long put option. The maximum profit from this strategy is the option price. The maximum loss is limited only by how low the price of the bond can fall by the expiration date less the option price received for writing the option.

Considering the Time Value of Money

Our discussion of the four basic option positions has neglected the time value of money. Specifically, the buyer of an option must pay the seller the option price at the time the option is purchased. Thus the buyer must finance the purchase of the option or, if the funds do not have to be borrowed, the buyer loses the interest that could be earned by investing the option price. In contrast, assuming that the seller does not have to use the option price as margin for the short position, the seller has the opportunity to invest the option price.

The time value of money changes the profit profile of the option positions. The break-even price for the buyer and the seller of an option will not be the same as in our discussion. For example, the break-even price for the underlying instrument at the expiration date is higher for the buyer of a call option; for the seller, it is lower.

THE INTRINSIC VALUE AND TIME VALUE OF AN OPTION

The cost to the buyer of an option is primarily a reflection of the option's *intrinsic value* and any additional amount over its intrinsic value. The premium over intrinsic value is often referred to as *time value.* This expression *time value* should not be confused with our earlier use of the term to describe the valuation of cash flows.

Intrinsic Value of an Option

The intrinsic value of an option is the economic value of the option if it is exercised immediately. Because the buyer of an option need not exercise the option and, in fact, will not do so if no economic value will result from exercising, the intrinsic value cannot be less than zero.

The intrinsic value of a call option on a bond is the difference between the current bond price and the strike price. For example, if the *strike price* for a call option is $100 and the *current bond price* is $107, the intrinsic value is $7. That is, if the option buyer exercised the option and simultaneously sold the bond, the option buyer would realize $107 from the sale of the bond, which would be covered by acquiring the bond from the option writer for $100, thereby netting $7.

When a call option has intrinsic value, it is said to be *in the money*. Our call option with a strike price of $100 is in the money when the price of the underlying bond is greater than $100. When the strike price of a call option exceeds the current bond price, the call option is said to be *out of the money* and has no intrinsic value. A call option for which the strike price is equal to the current bond price is said to be *at the money*.

These relationships are summarized below for a *call* option:

If the current bond price is higher than the strike price, then the

1. Intrinsic value is the difference between the current bond price and strike price;
2. Option is said to be in the money.

If the current bond price equals the strike price, then the

1. Intrinsic value is zero;
2. Option is said to be at the money.

If the current bond price is lower than the strike price, then the

1. Intrinsic value is zero;
2. Option is said to be out of the money.

For a put option, the intrinsic value is equal to the amount by which the current bond price is below the strike price. For example, if the strike price of a put option is $100 and the current bond price is $88, the intrinsic value is $12. That is, if the buyer of the put option exercises it and simultaneously buys the bond, he or she will net $12. The bond will be sold to the writer for $100 and purchased in the market for $88.

When the put option has intrinsic value, the option is said to be "in the money." For our put option with a strike price of $100, the option will be in the money when the bond price is less than $100. A put option is "out of the money" when the current bond price exceeds the strike price. A put option is "at the money" when the strike price is equal to the current bond price.

These relationships are summarized below for a *put* option:

If the current bond price is lower than the strike price, then the

1. Intrinsic value is the difference between the strike price and current bond price;
2. Option is said to be in the money.

If the current bond price equals the strike price, then the

1. Intrinsic value is zero;
2. Option is said to be at the money.

If the current bond price is higher than the strike price, then the

1. Intrinsic value is zero;
2. Option is said to be out of the money.

Time Value of an Option

The time value of an option is the amount by which the option price exceeds the intrinsic value. That is,

$$\text{Time value of an option} = \text{option price} - \text{intrinsic value.}$$

For example, if the price of a call option with a strike price of \$100 is \$18 when the current bond price is \$107, then for this option

$$\text{Intrinsic value} = \$107 - \$100 = \$7;$$
$$\text{Time value of option} = \$18 - \$7 = \$11.$$

If the current bond price is \$88 instead of \$107, then the time value of this option is \$18 because the option has no intrinsic value. Notice that for an at-the-money or out-of-the-money option, the time value of the option is equal to the option price because the intrinsic value is zero.

At the expiration date, the time value of the option will be zero. The option price at the expiration date will be equal to its intrinsic value.

Why would an option buyer be willing to pay a premium over the intrinsic value for an option? The reason is that the option buyer believes that at some time prior to expiration, changes in the market yield will increase the value of the rights conveyed by the option.

THE OPTION PRICE

In the next chapter we shall see that the price of a bond with an embedded option is determined by the price of the underlying bond and the value of the option. While we can easily determine the value of an option at the expiration date and the intrinsic value of an option at any time prior to the expiration date, the fair value or price of the option at any time prior to the expiration date must be estimated. In this section we discuss the factors that influence the fair or "theoretical" value of an option.[1]

Factors That Influence the Option Price

Six factors influence the option price:

1. Current price of the underlying bond;
2. Strike price;
3. Time to expiration;
4. Short-term risk-free interest rate over the life of the option;

1. For a more detailed discussion of the impact of these factors on the price of an option, see Mark Pitts and Frank J. Fabozzi, *Interest Rate Futures and Options* (Chicago: Probus Publishing, 1989).

5. Coupon rate;
6. Expected interest-rate volatility over the life of the option.

The effect of each of these factors depends on whether (1) the option is a call or a put and (2) the option is an American option (an option that may be exercised up to and including the expiration date) or a European option (an option that may be exercised only at the expiration date).[2]

Current Price of the Underlying Bond

For a call option, as the current price of the underlying bond increases (decreases), the option price increases (decreases). For a put option, as the current price of the bond decreases (increases), the option price increases (decreases).

Strike Price

All other factors constant, the higher the strike price the lower is the price of a call option. For a put option, the opposite is true: the higher the strike price, the higher is the price of a put option.

Time to Expiration

For American options, all other factors constant, the longer the time to expiration, the higher is the option price. No general statement can be made for European options.

Short-Term Risk-Free Interest Rate over the Life of the Option

Holding all other factors constant, the price of a call option on a bond will increase as the short-term risk-free interest rate rises. For a put option, the opposite is true: an increase in the short-term risk-free interest rate will reduce the price of a put option.

Coupon Rate

Coupons for options on bonds tend to decrease the price of a call option because the coupons make it more attractive to hold the bond than the option. Thus call options on coupon-bearing bonds will tend to be priced lower than other similar call options on non-coupon-bearing bonds. For put options, coupons tend to increase their price.

Expected Interest-Rate Volatility over the Life of the Option

As the expected interest-rate volatility over the life of the option increases, the price of an option increases. The reason is that the more the expected volatility, as measured by the standard deviation or variance of interest rates,[3] the greater is the probability that the price of the underlying bond will move in the direction that will benefit the option buyer.

2. The option price also will depend on whether the underlying instrument is a cash-market bond or an interest-rate futures contract. Our focus in this chapter is on options on cash-market bonds.
3. These statistical measures are explained in Chapter 30.

Option-Pricing Model

Several models have been developed to estimate the theoretical or fair price of an option. These models are based on an arbitrage or riskless hedge valuation model. Our purpose here is not to describe option-pricing models but instead to mention some of the models that are commercially available from software vendors. Most of the dealer firms have developed their own option-pricing models, which typically are not available to clients.

The most popular option-pricing model for American call options on common stock is the Black–Scholes option-pricing model.[4] The key insight of the Black-Scholes model is that a synthetic option can be created by taking an appropriate position in the underlying common stock and borrowing or lending funds at the riskless interest rate.[5]

There are several problems in applying the Black–Scholes model to price interest-rate options. To illustrate these problems, consider a 3-month European call option on a 3-year zero-coupon bond.[6] The maturity value of the underlying bond is $100, and the strike price is $120. Suppose further that the current price of the bond is $75.13, the 3-year risk-free rate is 10% annually, and expected price volatility is 4%. What would be the fair value for this option? Do you really need an option-pricing model to determine the value of this option?

Think about it. This zero-coupon bond will never have a price above $100 because that is the maturity value. Because the strike price is $120, the option will never be exercised; its value is therefore zero. If you can get anyone to buy such an option, any price you obtain will be free money. Yet an option buyer armed with the Black–Scholes option-pricing model will input the variables we assume above and come up with a value for this option of $5.60! Why is the Black–Scholes model off by so much? The answer lies in the underlying assumptions.

There are three assumptions underlying the Black–Scholes model that limit its use in pricing options on interest-rate instruments. First, the probability distribution for the prices assumed by the Black–Scholes option-pricing model is a lognormal distribution, which permits some probability—no matter how small—that the price can take on any positive value. In the case of the zero-coupon bond, this means that the price can take on a value above $100. In the case of a coupon bond, we know that the price cannot exceed the sum of the coupon payments plus the maturity value. For example, for a 5-year, 10% coupon bond with a maturity value of $100, the price cannot be more than $150 (five coupon payments of $10 plus

4. Fischer Black and Myron Scholes, "The Pricing of Corporate Liabilities," *Journal of Political Economy* (May–June 1973), pp. 637–659.
5. The appropriate position in the underlying common stock depends on how the price of the option will change when the price of the stock changes. This relationship between the change in price of the option and the change in price of the underlying common stock is called the *delta* of the option and is discussed later in this chapter.
6. Lawrence Dyer and David Jacob, "Guide to Fixed Income Option Pricing Models," in Frank J. Fabozzi (ed.), *The Handbook of Fixed Income Options* (Chicago: Probus Publishing, 1989).

the maturity value of $100). The only way a bond's priced can exceed the maximum value is if negative interest rates are permitted. This is not likely to occur, so any probability distribution for prices assumed by an option-pricing model that permits bond prices to be more than the maximum bond value can generate nonsensical option prices. The Black–Scholes model does allow bond prices to exceed the maximum bond value (or, equivalently, allows negative interest rates). This is one of the reasons we obtained the nonsensical option price for the 3-month European call option on the 3-year zero-coupon bond.

The second assumption of the Black–Scholes option-pricing model is that the short-term interest rate is constant over the life of the option. Yet the price of an interest-rate option does change as interest rates change. A change in the short-term interest rate changes the rates along the yield curve. Therefore, to assume that the short-term rate will be constant is inappropriate for interest-rate options.

The third assumption is that the variance of prices is constant over the life of the option. Recall from Chapter 5 that as a bond moves closer to maturity, its price volatility declines. Therefore, the assumption that price variance is constant over the life of the option is inappropriate.

A binomial option-pricing model based on the price distribution of the underlying bond suffers from the same problems as the Black–Scholes model. A way around the problem of negative interest rates is to use a model based on the distribution of yields rather than prices.

While the binomial option-pricing model based on yields is superior to models based on prices, it still has a theoretical drawback. All option-pricing models to be valid theoretically must satisfy the put–call parity relationship (explained later in the chapter). The problem with the binomial model based on yields is that it does not satisfy this relationship. It violates the relationship in that it fails to take into consideration the yield curve, thereby allowing arbitrage opportunities.

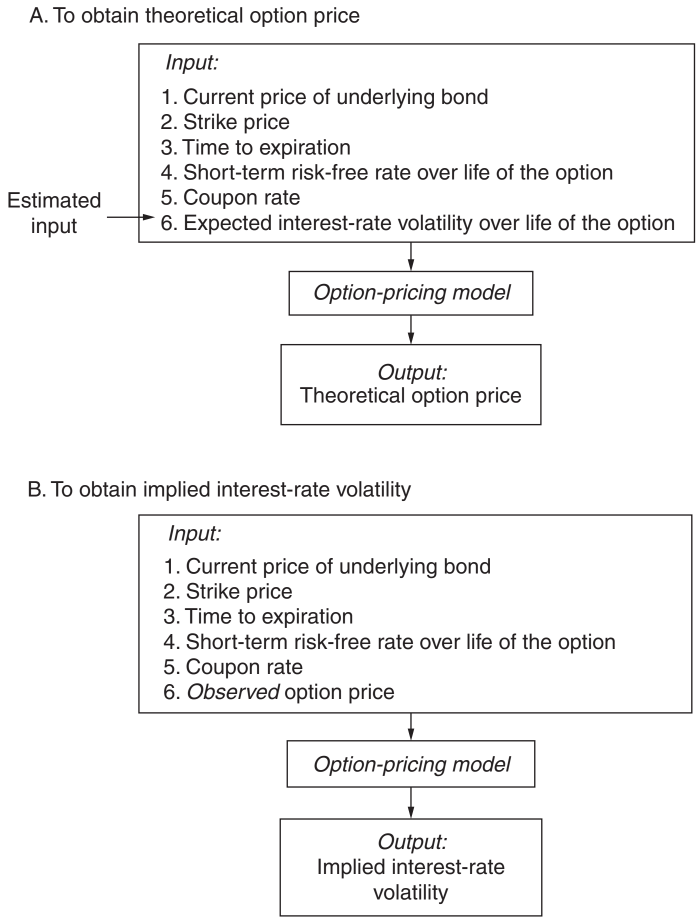
The most elaborate models that take the yield curve into consideration and as a result do not permit arbitrage opportunities are called *arbitrage-free option-pricing models* or *yield-curve option-pricing models.* These models can incorporate different volatility assumptions along the yield curve. The binomial model discussed in the next chapter to value a bond with an embedded option is based on an arbitrage-free option-pricing model.

Implied Interest-Rate Volatility

Option-pricing models provide a theoretical option price based on the six factors that we discussed earlier. Of the six factors, the only one that is not known and must be estimated is the expected interest-rate volatility over the life of the option. A popular methodology for assessing whether an option is fairly priced is to assume that the option is priced correctly and to estimate the interest-rate volatility that is implied by an option-pricing model. Exhibit 20–1 describes the process for computing implied interest-rate volatility.

EXHIBIT 20–1

The Process of Obtaining Implied Interest-Rate Volatility



For example, suppose that a money manager—using some option-pricing model, the current price of the option, and the five factors that determine the price of an option—computes an implied interest-rate volatility of 12%. If the money manager expects that the interest-rate volatility over the life of the option will be greater than the implied interest-rate volatility of 12%, the option is undervalued. If the money manager's expected interest-rate volatility over the life of the option is less than the implied interest-rate volatility, the option is overvalued.

While we have focused on the option price, the key to understanding the options market is that trading and investment strategies in this market involve buying and selling interest-rate volatility. Looking at the implied interest-rate volatility and comparing it to the trader's or money manager's expectations of future interest-rate volatility is just another way of evaluating options.

SENSITIVITY OF THE THEORETICAL CALL OPTION PRICE TO CHANGES IN FACTORS

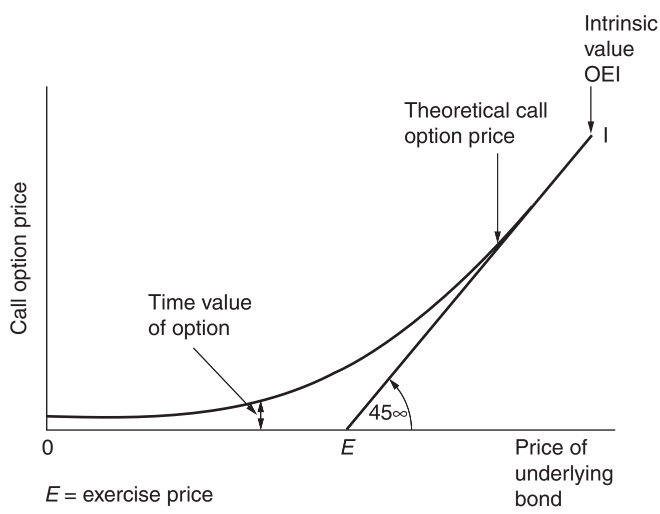
When any one of the six factors that affect the price of an option changes, the option price will change. Because the price of a bond with an embedded option will be affected by how the price of the embedded option changes, we will look at the sensitivity of the option price to three of the factors—the price of the underlying bond, time to expiration, and expected interest-rate volatility. We focus our attention on call options.[7]

The Call Option Price and the Price of the Underlying Bond

Exhibit 20–2 shows the theoretical price of a call option based on the price of the underlying bond. The horizontal axis is the price of the underlying bond at any time. The vertical axis is the option price. The shape of the curve representing the

EXHIBIT 20–2

Theoretical Call Price and the Price of the Underlying Bond



7. For a detailed discussion of the role of these measures in option strategies, see Mark Pitts and Frank J. Fabozzi, *Interest Rate Futures and Options*; and Richard M. Bookstaber, *Option Pricing and Investment Strategies* (Chicago: Probus Publishing, 1987), Chapter 4.

theoretical price of a call option, given the price of the underlying bond, would be the same regardless of the actual option-pricing model used. Specifically, the relationship between the price of the underlying bond and the theoretical call option price is convex. Thus option prices also exhibit convexity.

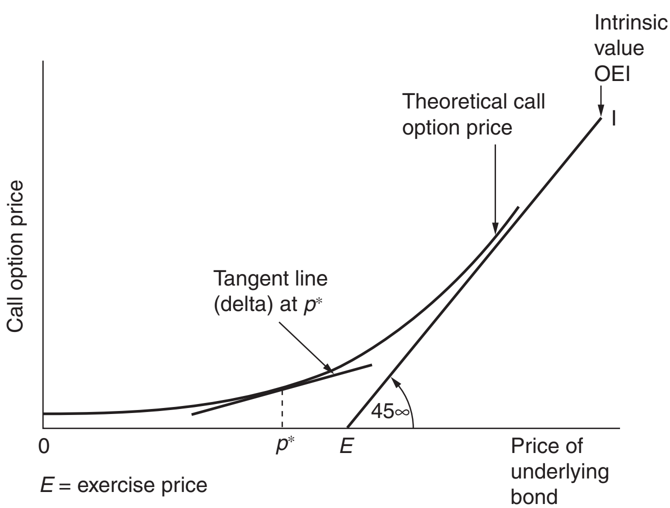
The line from the origin to the strike price on the horizontal axis in Exhibit 20–2 is the intrinsic value of the call option when the price of the underlying bond is less than the strike price because the intrinsic value is zero. The 45-degree line extending from the horizontal axis is the intrinsic value of the call option once the price of the underlying bond exceeds the strike price. The reason is that the intrinsic value of the call option will increase by the same amount as the increase in the price of the underlying bond. For example, if the exercise price is \$100 and the price of the underlying bond increases from \$100 to \$101, the intrinsic value will increase by \$1. If the price of the bond increases from \$101 to \$110, the intrinsic value of the option will increase from \$1 to \$10. Thus the slope of the line representing the intrinsic value after the strike price is reached is 1.

Because the theoretical call option price is shown by the convex line, the difference between the theoretical call option price and the intrinsic value at any given price for the underlying bond is the time value of the option.

Exhibit 20–3 shows the theoretical call option price but with a tangent line drawn at the price of p^*. Recall from Chapter 13 that the tangent line was used to estimate the new price of a bond at a new yield level. Here we have an analogous situation. The tangent line in Exhibit 20–3 can be used to estimate what the new option price will be (and therefore what the change in the option price will be) if

EXHIBIT 20–3

Delta of a Call Option



the price of the underlying bond changes. Once again, because of the convexity of the relationship between the option price and the price of the underlying bond, the tangent line closely approximates the new option price for a small change in the price of the underlying bond. For large changes, however, the tangent line does not provide as good an approximation of the new option price.

The slope of the tangent line shows how the theoretical call option price will change for small changes in the price of the underlying bond. The slope of the tangent line is commonly referred to as the *delta* of the option.[8] Specifically,

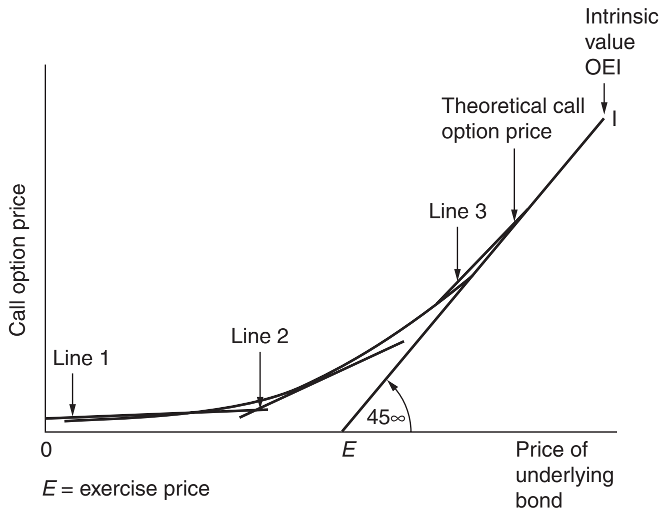
$$\text{Delta} = \frac{\text{change in price of call option}}{\text{change in price of underlying bond}}.$$

For example, a delta of 0.5 means that a \$1 change in the price of the underlying bond will change the price of the call option by \$0.50.

Exhibit 20–4 shows the curve of the theoretical call option price with three tangent lines drawn. The steeper the slope of the tangent line, the greater is the delta. When an option is deep out of the money (i.e., the price of the underlying bond is substantially below the strike price), the tangent line is nearly flat (see line 1 in Exhibit 20–4). This means that the delta is close to 0. To understand why, consider a call option with a strike price of \$100 and 2 months to expiration. If the

EXHIBIT 20–4

Delta of a Call Option at Three Prices for the Underlying Bond



8. Delta is also referred to as the *hedge ratio*.

price of the underlying bond is \$20, the option price would not increase by much if the price of the underlying bond increased by \$1, from \$20 to \$21.

For a call option that is deep in the money, the delta will be close to 1. That is, the call option price will increase almost dollar for dollar with an increase in the price of the underlying bond. In terms of Exhibit 20–4, the slope of the tangent line approaches the slope of the intrinsic value line after the strike price. As we stated earlier, the slope of that line is 1.

Thus the delta for a call option varies from 0 (for call options deep out of the money) to 1 (for call options deep in the money). The delta for a call option at the money is approximately 0.5.

Related to the delta is the *lambda* of an option, which measures the percentage change in the price of the option for a 1% change in the price of the underlying bond. That is,

$$\text{Lambda} = \frac{\text{percentage change in the price of the call option}}{\text{percentage change in the price of the underlying bond}}.$$

For example, a lambda of 1.5 indicates that if the price of the call option changes by 1%, the call option's price will change by 1.5%. The lambda of a call option will be greater than 1 because of the leverage offered by an option. As the price of the underlying bond increases, the option's lambda decreases.

In Chapter 14 we measured the convexity of an option-free bond. We also can measure the convexity of a call option. Recall from Chapter 14 that the convexity of a bond measures the rate of change in dollar duration. For call options, convexity measures the rate of change in delta. The measure of convexity for options is commonly referred to as *gamma*, defined as follows:

$$\text{Gamma} = \frac{\text{change in delta}}{\text{change in price of underlying bond}}.$$

The Call Option Price and Time to Expiration

All other factors constant, the longer the time to expiration, the higher is the option price. Because each day the option moves closer to the expiration date, the time to expiration decreases. The *theta* of an option measures the change in the option price as the time to expiration decreases. That is,

$$\text{Theta} = \frac{\text{change in price of option}}{\text{decrease in time to expiration}}.$$

Assuming that the price of the underlying bond does not change so that the intrinsic value of the option does not change, theta measures how quickly the time value of the option changes as the option moves toward expiration.

The Call Option Price and Expected Interest-Rate Volatility

All other factors constant, a change in the expected interest-rate volatility will change the option price. The *kappa* (or *vega*) of an option measures the dollar change in the price of the option for a 1% change in the expected interest-rate volatility. That is,

$$\text{Kappa (vega)} = \frac{\text{change in option price}}{\text{1\% change in expected interest-rate volatility}}.$$

DURATION OF AN OPTION

The modified duration of a bond is an indicator of its price sensitivity to changes in interest rates. We can similarly define a modified duration for an option as follows:

$$\begin{aligned}&\text{Modified duration for an option}\\&\quad = \left(\begin{array}{c}\text{modified duration of the}\\ \text{underlying instrument}\end{array}\right) \times \text{delta}\\&\quad\quad \times \left(\frac{\text{price of underlying instrument}}{\text{price of option}}\right)\end{aligned}$$

Therefore, the modified duration for an option depends on three factors: (1) the modified duration of the underlying instrument, (2) the delta of the option, and (3) the ratio of the price of the underlying instrument to the price of the option. This last factor can be thought of as a measure of the "leverage" in the option.

Put–Call Parity Relationship

There is a relationship between the price of a call option and the price of a put option on the same underlying bond with the same strike price and the same expiration date. This relationship is commonly referred to as the *put–call parity relationship*. For coupon-bearing bonds it is

$$\begin{aligned}\text{Put price} &= \text{call price} + \text{present value of strike price}\\&\quad + \text{present value of coupon} - \text{price of underlying bond}\end{aligned}$$

This relationship is one form of the put–call parity relationship for European options. It is approximately true for American options. The relationship is based on arbitrage arguments.

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CHAPTER

21

VALUATION AND PRICE VOLATILITY OF BONDS WITH EMBEDDED OPTIONS

Now that we understand the fundamental characteristics of options, in this chapter we turn to the analysis of bonds with embedded options. By an embedded option we mean that either the issuer or the bondholder has the right to alter the bond's cash flow. Bonds with embedded options include callable bonds, putable bonds, range notes, floaters with restrictions on the coupon rate (i.e., cap and/or floor), and mortgage-backed securities. In each case the cash flow depends on the future level of interest rates. Bonds with embedded options also include convertible bonds in which the bondholder can convert the bond into common stock and foreign-currency bonds in which either the issuer or bondholder has the option to select the currency in which a coupon and/or the principal are paid.

The analysis of a bond with an embedded option involves determining the fair value of the bond (i.e., its theoretical value) and its price volatility. Our focus will be on the most popular type of bond with an embedded option, callable bonds. The principles are applicable to other bonds whose cash flows are sensitive to interest rates.

The valuation model that will be described in this chapter is the *lattice model*. This model is based on a consistent framework for valuing both option-free bonds and bonds with embedded options. The valuation principles that we have discussed so far in this book are used here. Specifically, we saw in Chapter 7 two important things. First, it is inappropriate to use a single rate to discount all the cash flows of a bond. Second, the correct rate to use to discount each cash flow is the spot rate. This is equivalent to discounting at a series of forward rates. What we have to add to the valuation process is how interest-rate volatility affects the value of a bond through its effects on the embedded options.

An alternative valuation model used by some dealers and vendors is the Monte Carlo model. Because this model is more commonly used for the valuation of mortgage-backed securities, we postpone discussion of this model until Chapter 27 where the analysis of these securities is explained.

The price volatility of a bond with an embedded option can then be assessed given the valuation model. More specifically, we will see how to determine the effective duration and convexity using the lattice model.

PRICE/YIELD RELATIONSHIP FOR A CALLABLE BOND

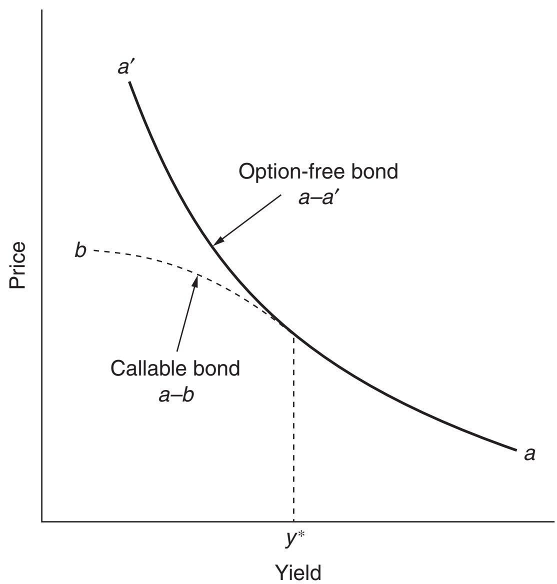
As explained in Chapter 4, the price/yield relationship for an option-free (i.e., noncallable/nonputable) bond is convex. Exhibit 21–1 shows the price/yield relationship for both an option-free bond and the same bond if it is callable. The convex curve *a–á* is the price/yield relationship for the noncallable (option-free) bond. The unusual-shaped curve denoted by *a–b* is the price/yield relationship for the callable bond.

The reason for the shape of the price/yield relationship for the callable bond is as follows. When the prevailing market yield for comparable bonds is higher than the coupon interest on the bond, it is unlikely that the issuer will call the bond. For example, if the coupon rate on a bond is 8% and the prevailing yield on comparable bonds is 16%, it is highly improbable that the issuer will call an 8% coupon bond so that it can issue a 16% coupon bond. Because the bond is unlikely to be called, the callable bond will have the same price/yield relationship as an option-free bond. However, even when the coupon rate is just below the market yield, investors may not pay the same price for the callable bond had it been noncallable because there is still the chance that the market yield may drop further, making it beneficial for the issuer to call the bond.

As yields in the market decline, the likelihood that yields will decline further so that the issuer will benefit from calling the bond increases. The exact yield level

EXHIBIT 21–1

Price/Yield Relationship for an Option-Free Bond and a Callable Bond



at which investors begin to view the issue likely to be called may not be known, but we do know that there is some level. In Exhibit 21–1, at yield levels below y^*, the price/yield relationship for the callable bond departs from the price/yield relationship for the option-free bond. For example, suppose the market yield were such that an option-free bond was selling for $109. Suppose instead that the bond is callable with a call price of $104. Investors would not be willing to pay $109 for this callable bond. If they did and the bond were called, investors would receive $104 (the call price) for a bond they purchased for $109. Notice that for a range of yields below y^*, there is price compression; that is, there is limited price appreciation as yields decline. The portion of the callable bond price/yield relationship below y^* is said to be *negatively convex.*

Negative convexity means that the price appreciation will be less than the price depreciation for a large change in yield of a given number of basis points. For a bond that is option free and exhibits positive convexity, the price appreciation will be greater than the price depreciation for a large change in yield. The price changes resulting from bonds exhibiting positive convexity and negative convexity can be expressed as follows:

	Absolute Value of Percentage Price Change for	
Change in Interest Rates	**Positive Convexity**	**Negative Convexity**
–100 basis points	*X*%	Less than *Y*%
+100 basis points	Less than *X*%	*Y*%

THE COMPONENTS OF A BOND WITH AN EMBEDDED OPTION

To develop an analytical framework for valuing a bond with an embedded option, it is necessary to decompose a bond into its component pairs. A callable bond, for example, is a bond in which the bondholder has sold the issuer an option (more specifically, a call option) that allows the issuer to repurchase the contractual cash flows of the bond from the time the bond is first callable until the maturity date.

Consider the following two bonds: (1) a callable bond with an 8% coupon, 20 years to maturity, and callable in 5 years at $104 and (2) a 9% coupon, 10-year bond callable immediately at par. For the first bond, the bondholder owns a 5-year option-free bond and has sold a call option granting the issuer the right to call away from the bondholder 15 years of cash flows 5 years from now for a price of $104. The investor who owns the second bond has a 10-year option-free bond and has sold a call option granting the issuer the right to immediately call the entire 10-year contractual cash flows or any cash flows remaining at the time the issue is called, for $100.

Effectively, the owner of a callable bond is entering into two separate transactions. First, she buys an option-free bond from the issuer for which she pays some price. Then she sells the issuer a call option for which she receives the option price. Therefore, we can summarize the position of a callable bondholder as follows:

Long a callable bond = long an option-free bond + sold a call option.

In terms of price, the price of a callable bond is therefore equal to the price of the two components parts. That is,

Callable bond price = option-free bond price – call option price.

The reason the call option price is subtracted from the price of the option-free bond is that when the bondholder sells a call option, she receives the option price. Graphically this can be seen in Exhibit 21–1. The difference between the prices of the option-free bond and the callable bond at any given yield is the price of the embedded call option.

Actually, the position is more complicated than we just described. The issuer may be entitled to call the bond at the first call date and any time thereafter or at the first call date and any subsequent coupon anniversary. Thus the investor has effectively sold an American-type call option to the issuer, but the call price may vary with the date the call option is exercised. This is because the call schedule for a bond typically has a different call price depending on the call date. Moreover, the underlying bond for the call option is the remaining coupon payments that would have been made by the issuer had the bond not been called. For exposition purposes, it is easier to understand the principles associated with the investment characteristics of callable bonds by describing the investor's position as long an option-free bond and short a call option.

The same logic applies to putable bonds. In the case of a putable bond, the bondholder has the right to sell the bond to the issuer at a designated price and time. A putable bond can be broken into two separate transactions. First, the investor buys an option-free bond. Second, the investor buys an option from the issuer that allows the investor to sell the bond to the issuer. This type of option is called a *put* option. Therefore, the position of a putable bondholder can be described as

Long a putable bond = long an option-free bond + own a put option.

The price of a putable bond is then

Price of a putable bond = option-free bond price + put option price.

TRADITIONAL VALUATION METHODOLOGY

When a bond is callable, the practice has been to calculate a yield to call as well as a yield to maturity. The former yield calculation assumes that the issuer will call the bond at the first call date. As explained in Chapter 6, the procedure for calculating the yield to call is the same as for any yield calculation: determine the interest rate that will make the present value of the expected cash flows equal to the price.

In the case of yield to call, the expected cash flows are the coupon payments to the first call date and the call price.

According to the traditional approach, conservative investors should compute the yield to call and yield to maturity for a callable bond selling at a premium, selecting the lower of the two as a measure of potential return. The smaller of the two yield measures should be used to evaluate the relative value of a callable bond. More recently, the traditional approach has been extended to compute not just the yield to the first call date but also the yield to all possible call dates. Because most bonds can be called at any time after the first call date, the approach has been to compute the yield to every coupon anniversary date following the first call date. Then all the yields to calls calculated and the yield to maturity are compared. The lowest of these yields is called the *yield to worst*, which is the yield that the traditional approach has investors believing should be used in relative value analysis.

The limitations of the yield to call as a measure of the potential return of a security are given in Chapter 4. The yield to call does consider all three sources of potential return from owning a bond. However, as in the case of the yield to maturity, it assumes that all cash flows can be reinvested at the computed yield—in this case the yield to call—until the assumed call date. Moreover, the yield to call assumes that (1) the investor will hold the bond to the assumed call date, and (2) the issuer will call the bond on that date.

Often these underlying assumptions about the yield to call are unrealistic because they do not take into account how an investor will reinvest the proceeds if the issue is called. For example, consider two bonds, M and N. Suppose that the yield to maturity for bond M, a 5-year option-free bond, is 10%, while the yield to call for bond N is 10.5% assuming that the bond will be called in 3 years. Which bond is better for an investor with a 5-year investment horizon? It is not possible to tell from the yields cited. If the investor intends to hold the bond for 5 years and the issuer calls the bond after 3 years, the total dollars that will be available at the end of 5 years will depend on the interest rate that can be earned from investing funds from the call date to the end of the investment horizon.

LATTICE MODEL FOR VALUING BONDS WITH EMBEDDED OPTIONS[1]

The discussion in the previous section provides a useful way to conceptualize a bond with an embedded option. Specifically, the value of a callable bond equals the value of a comparable option-free bond less the value of the call option. This insight led to the first generation of valuation models for callable bonds. These early models attempted to directly estimate the value of the embedded call option, but without explicitly incorporating the shape of the yield curve.

1. This section is adapted from Andrew Kalotay, George O. Williams, and Frank J. Fabozzi, "A Model for the Valuation of Bonds and Embedded Options," *Financial Analysis Journal* (May–June 1993), pp. 35–46.

Instead of relying on an external option pricing model, the lattice model discussed here is based on an internally consistent framework appropriate to bonds with and without embedded options. The difference between the values of a bond with an embedded option and an otherwise identical bond without that option is the value of the option.

As we saw in Chapter 7, instead of discounting all cash flows at the same rate, one should discount each cash flow at its own spot rate. This is equivalent to discounting at a sequence of forward rates. Both the spot rates and the implied forward rates can be calculated by the bootstrapping methodology described in Chapter 7. What was not considered was how expected interest-rate volatility affects spot rates and forward rates and, in turn, how interest-rate volatility affects the value of a bond through its effects on the embedded options.

Valuation of Option-Free Bonds

We begin with a review of the valuation technique for bonds without any embedded options. The price of an option-free bond is the present value of the cash flows discounted at the spot rate. To illustrate this, we start with the on-the-run yield curve for the particular issuer whose bonds we want to value. The starting point is the Treasury's on-the-run yield curve. To obtain a particular issuer's on-the-run yield curve, an appropriate credit spread is added to each on-the-run Treasury issue. The credit spread need not be constant for all maturities. For example, the credit spreads may increase with maturity.

In our illustration, we use the following hypothetical on-the-run issue for an issuer:

Maturity (years)	Yield to Maturity (%)	Market Price ($)
1	3.50	100
2	4.00	100
3	4.50	100

Each bond is trading at par value ($100), so the coupon rate is equal to the yield to maturity. We will simplify the illustration by assuming annual-pay bonds.

Determined by using the bootstrapping methodology, the spot rates are given below:

Year	Spot Rate (%)
1	3.500
2	4.010
3	4.531

The corresponding 1-year forward rates are

Current 1-year forward rate = 3.500%;

1-year forward rate 1 year from now = 4.523%;

1-year forward rate 2 years from now = 5.580%.

Now consider an option-free bond with 3 years remaining to maturity and a coupon rate of 5.25%. The value of this bond can be calculated in one of two ways, both producing the same value. First, the coupon payments can be discounted at the zero-coupon rates as shown below:

$$\frac{\$5.25}{(1.035)}+\frac{\$5.25}{(1.0401)^2}+\frac{\$100+\$5.25}{(1.04531)^3}=\$102.075.$$

The second way is to discount by the 1-year forward rates as shown below:

$$\frac{\$5.25}{(1.035)}+\frac{\$5.25}{(1.035)(1.04523)}+\frac{\$100+\$5.25}{(1.035)(1.04523)(1.0558)}=\$102.075.$$

Lattice Interest-Rate Tree

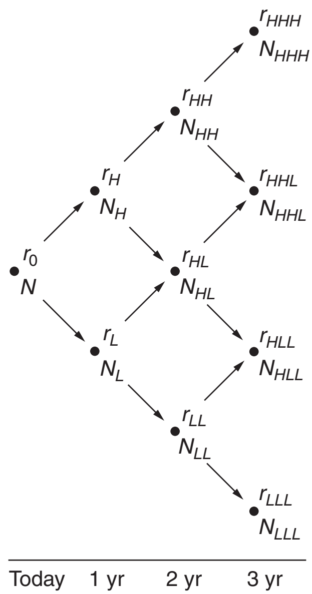
Once we allow for embedded options, consideration must be given to interest-rate volatility. This can be done by introducing a *lattice interest-rate tree*. This tree is nothing more than a graphical depiction of the 1-period forward rates over time based on some assumption about interest-rate volatility. How this tree is constructed is illustrated below. In our presentation in this chapter we will focus on a specific type of lattice model, the *binomial model*. In a binomial model, the interest rate can take on only two possible values in the next period. Extension to more than two possible changes in the interest rate in the next period is the same as for the binomial model.[2]

Exhibit 21–2 shows an example of a binomial interest-rate tree. In this tree each node (bold circle) represents a time period that is equal to 1 year from the node to its left. Each node is labeled with an N, representing node, and a subscript that indicates the path that 1-year forward rates took to get to that node. L represents the lower of the two 1-year forward rates, and H represents the higher of the two 1-year forward rates. For example, node N_{HH} means to get to that node, the following path for 1-year forward rates occurred: the 1-year forward rate realized is the higher of the two forward rates in the first year and then the higher of the 1-year forward rates in the second year.[3]

2. When there are three possible interest rates that can be realized in the next period, the model is referred to as a *trinomial model*.
3. Note that N_{HL} is equivalent to N_{LH} in the second year. Also, in the third year, N_{HHL} is equivalent to N_{HLH} and N_{LHH}, and N_{HLL} is equivalent to N_{LLH}. We have simply selected one label for a node rather than clutter up the figure with unnecessary information.

EXHIBIT 21–2

Three-Year Binomial Interest-Rate Tree



Look first at the point denoted by just *N* in Exhibit 21–2. This is the root of the tree and is nothing more than the current 1-year rate or, equivalently, the current 1-year forward rate, which we denote by $r_{0.}$ What we have assumed in creating this tree is that the 1-year forward rate can take on two possible values the next period and that the two forward rates have the same probability of occurring. One forward rate will be higher than the other. It is assumed that the 1-year forward rate can evolve over time based on a random process called a *log-normal random walk* with a certain volatility.[4]

We use the following notation to describe the tree in the first year. Let

σ = assumed volatility of the 1-year forward rate;

$r_{1,L}$ = lower 1-year forward rate 1 year from now;

$r_{1,H}$ = higher 1-year forward rate 1 year from now.

The relationship between $r_{1,L}$ and $r_{1,H}$ is as follows:

$$r_{1,H} = r_{1,L}(e^{2\sigma})$$

where e is the basis of the natural logarithm, 2.71828. For example, suppose that $r_{1,L}$ is 4.074% and σ is 10% per year. Then

$$r_{1,H} = 4.074\% \ (e^{2 \times 10}) = 4.976\%.$$

4. This is one type of interest-rate model. See Chapter 19 for a discussion of interest-rate models.

In the second year there are three possible values for the 1-year forward rate, which we will denote as follows:

$r_{2,LL}$ = 1-year forward rate in the second year assuming the lower forward rate in the first year and the lower forward rate in the second year;

$r_{2,HH}$ = 1-year forward rate in the second year assuming the higher forward rate in the first year and the higher forward rate in the second year;

$r_{2,HL}$ = 1-year forward rate in the second year assuming the higher forward rate in the first year and the lower forward rate in the second year or, equivalently, the lower forward rate in the first year and the higher forward rate in the second year.

The relationship between $r_{2,LL}$ and the other two forward rates is as follows:

$$r_{2,HH} = r_{2,LL}(e^{4\sigma})$$

and

$$r_{2,HL} = r_{2,LL}(e^{2\sigma}).$$

So, for example, if $r_{2,LL}$ is 4.53%, and assuming once again that σ is 10%, then

$$r_{2,HH} = 4.53\% \, (e^{4 \times 10}) = 6.757\%$$

and

$$r_{2,HL} = 4.53\% \, (e^{2 \times 10}) = 5.532\%.$$

Exhibit 21–2 shows the notation for the binomial interest-rate tree in the third year. We can simplify the notation by letting r_1 be the 1-year forward rate t years from now for the lower forward rate because all the other forward rates t years from now depend on that rate. Exhibit 21–3 shows the interest-rate tree using this simplified notation.

Before we go on to show how to use this binomial interest-rate tree to value bonds, let's focus on two issues here. First, what does the volatility parameter σ in the expression $e^{2\sigma}$ represent? Second, how do we find the value of the bond at each node?

Volatility and the Standard Deviation

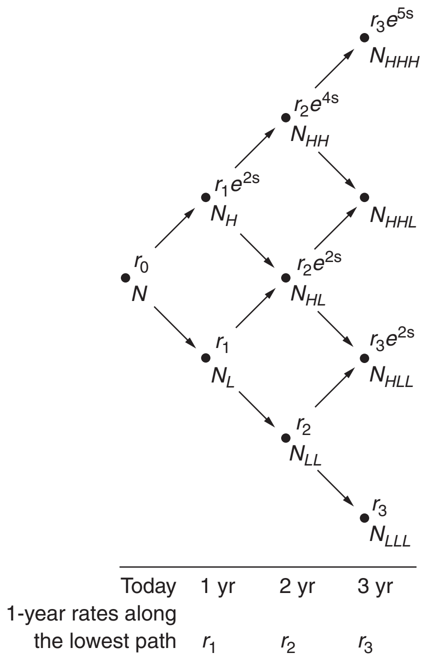
It can be shown that the standard deviation of the 1-year forward rate is equal to $r_0\sigma$.[5] The standard deviation is a statistical measure of volatility. In Chapter 26 it will be shown how the standard deviation of interest rates can be estimated. For now it is important to see that the process that we assumed generates the binomial interest-rate tree (or equivalently the forward rates) implies that volatility is measured relative to the current level of rates. For example, if σ is 10% and the 1-year

5. This can be seen by noting that $e^{2a} \approx 1 + 2\sigma$. Then the standard deviation of 1-period forward rates is

$$\frac{re^{2a} - r}{2} \approx \frac{r + 2\sigma r - r}{2} = \sigma r.$$

EXHIBIT 21–3

Three-Year Binomial Interest-Rate Tree with 1-Year Forward Rates



rate (r_0) is 4%, then the standard deviation of the 1-year forward rate is 4% × 10% = 0.4% or 40 basis points. However, if the current 1-year rate is 12%, the standard deviation of the 1-year forward rate would be 12% × 10% or 120 basis points.

Determining the Value at a Node

To find the value of the bond at a node, we first calculate the bond's value at the two nodes to the right of the node we are interested in. For example, in Exhibit 21–3, suppose that we want to determine the bond's value at node N_H. The bond's value at node N_{HH} and N_{HL} must be determined. Don't be concerned now with how we get these two values because as we will see the process involves starting from the last year in the tree and working backwards to get the final solution we want so that these two values will be known.

Effectively, what we are saying is that if we are at some node, then the value at that node will depend on the future cash flows. In turn, the future cash flows depend on (1) the bond's value 1 year from now and (2) the coupon payment 1 year from now. The latter is known. The former depends on whether the 1-year forward rate is the higher or lower rate. The bond's value if the rate is higher or lower is reported at the two nodes to the right of the node that is the focus of our attention. So the cash flow at a node will be either (1) the bond's value if the forward rate is the higher rate plus the coupon payments or (2) the bond's value if the forward rate is the lower rate plus the coupon payment. For example, suppose that we are interested in the bond's value at N_H. The cash flow will be either the bond's value at N_{HH} plus the coupon payment or the bond's value at N_{HL} plus the coupon payment.

To get the bond's value at a node, we follow the fundamental rule for valuation: the value is the present value of the expected cash flows. The appropriate discount rate to use is the 1-year forward rate at the node. Now there are two present values in this case: the present value if the 1-year forward rate is the higher rate and one if it is the lower rate. Because it is assumed that the probabilities of both outcomes are equal, an average of the two present values is computed. This is illustrated in Exhibit 21–4 for any node assuming that the 1-year forward rate is $r*$ at the node where the valuation is sought and letting

V_H = bond's value for the higher 1-year forward rate;

V_L = bond's value for the lower 1-year forward rate;

C = coupon payment.

Using our notation, the cash flow at a node is either

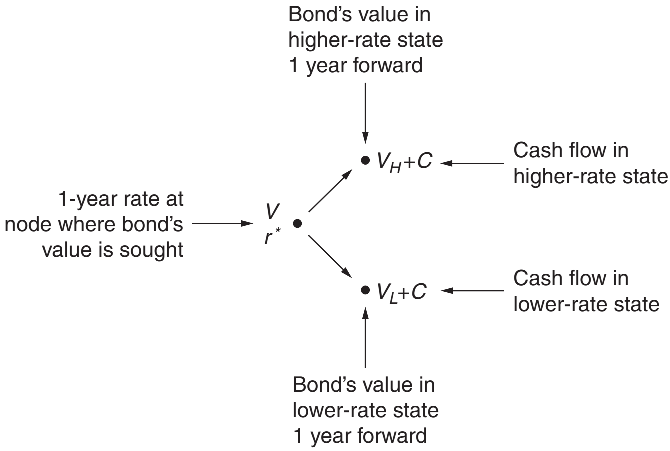
$$V_H + C \text{ for the higher 1-year forward rate}$$

or

$$V_L + C \text{ for the lower 1-year forward rate.}$$

EXHIBIT 21–4

Calculating a Value at a Node



The present value of these two cash flows using the 1-year forward rate at the node, r^*, is

$$\frac{V_H + C}{(1 + r^*)} = \text{present value for the higher 1-year forward rate;}$$

$$\frac{V_L + C}{(1 + r^*)} = \text{present value for the lower 1-year forward rate.}$$

Then the value of the bond at the node is found as follows:

$$\text{Value at a node} = \frac{1}{2}\left[\frac{V_H + C}{(1 + r^*)} + \frac{V_L + C}{(1 + r^*)}\right].$$

Constructing the Binomial Interest-Rate Tree

To see how to construct the binomial interest-rate tree, let's use the assumed on-the-run yields we used earlier. We will assume that volatility σ is 10% and construct a 2-year tree using the 2-year bond with a coupon rate of 4%.

Exhibit 21–5 shows a more detailed binomial interest-rate tree with the cash flow shown at each node. We'll see how all the values reported in the exhibit are obtained. The root rate for the tree r_0 is simply the current 1-year rate, 3.5%.

In the first year there are two possible 1-year forward rates, the higher forward rate and the lower forward rate. What we want to find are the two forward rates that will be consistent with the volatility assumption, the process that is assumed to generate the forward rates, and the observed market value of the bond. There is no simple formula for this. It must be found by an iterative process (i.e., trial and error). The steps are described and illustrated below.

Step 1. Select a value for r_t. Recall that r_1 is the lower 1-year forward rate. In this first trial we *arbitrarily* selected a value of 4.5%.

Step 2. Determine the corresponding value for the higher 1-year forward rate. As explained earlier, this rate is related to the lower 1-year forward rate as follows: $r_1 e^{2\sigma}$. Because r_1 is 4.5%, the higher 1-year forward rate is 5.496% (= 4.5%$e^{2 \times 10}$). This value is reported in Exhibit 21–5 at node N_H.

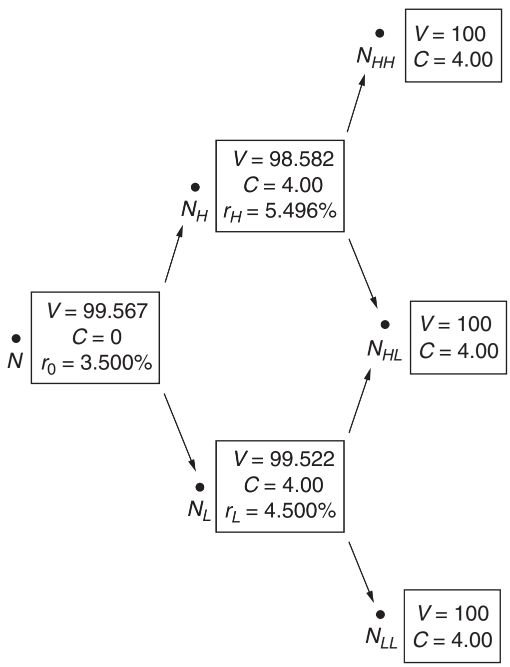
Step 3. Compute the bond value's 1 year from now. This value is determined as follows:

3a. Determine the bond's value 2 years from now. In our example, this is simple. Because we are using a 2-year bond, the bond's value is its maturity value ($100) plus its final coupon payment ($4). Thus it is $104.

3b. Calculate the present value of the bond's value found in Step 3a for the higher forward rate in the second year. The appropriate discount rate is the higher 1-year forward rate, 5.496% in our

EXHIBIT 21–5

Find the 1-Year Forward Rates for Year 1 by Using the 2-Year, 4% On-the-Run Issue: First Trial



example. The present value is \$98.582 (= \$104/1.05496). This is the value of V_H that we referred to earlier.

3c. Calculate the present value of the bond's value assumed in Step 3a for the lower forward rate. The discount rate assumed for the lower 1-year forward rate is 4.5%. The present value is \$99.522 (= \$104/1.045) and is the value of V_L.

3d. Add the coupon to both V_H and V_L to get the cash flow at N_H and N_L, respectively. In our example we have \$102.582 for the higher forward rate and \$103.522 for the lower forward rate.

3e. Calculate the present value of the two values using the 1-year forward rate $r*$. At this point in the valuation, $r*$ is the root rate, 3.50%. Therefore,

$$\frac{V_H + C}{1 + r^*} = \frac{\$102.582}{1.035} = \$99.113$$

and

$$\frac{V_L + C}{1 + r^*} = \frac{\$103.522}{1.035} = \$100.021.$$

Step 4. Calculate the average present value of the two cash flows in Step 3. This is the value we referred to earlier as

$$\text{Value at a node} = \frac{1}{2}\left[\frac{V_H + C}{(1 + r^*)} + \frac{V_L + C}{(1 + r^*)}\right].$$

In our example we have

$$\text{Value at a node} = \frac{1}{2}(\$99.113 + \$100.021) = \$99.567.$$

Step 5. Compare the value in Step 4 to the bond's market value. If the two values are the same, then the r_1 used in this trial is the one we seek. This is the 1-year forward rate that would then be used in the binomial interest-rate tree for the lower forward rate and the corresponding higher forward rate. If, instead, the value found in Step 4 is not equal to the market value of the bond, this means that the value r_1 in this trial is not the 1-year forward rate that is consistent with (1) the volatility assumption of 10%, (2) the process assumed to generate the 1-year forward rate, and (3) the observed market value of the bond. In this case the five steps are repeated with a different value for r_1.

When r_1 is 4.5%, a value of \$99.567 results in Step 4, which is less than the observed market price of \$100. Therefore, 4.5% is too large, and the five steps must be repeated, trying a lower rate for r_1.

Let's jump right to the correct rate for r_1 in this example and rework Steps 1 through 5. This occurs when r_1 is 4.074%. The corresponding binomial interest rate is shown in Exhibit 21–6.

Step 1. In this trial we select a value of 4.074% for r_1, the lower 1-year forward rate.

Step 2. The corresponding value for the higher 1-year forward rate is 4.976% (= 4.074%$e^{2 \times 10}$).

Step 3. The bond's value 1 year from now is determined as follows:

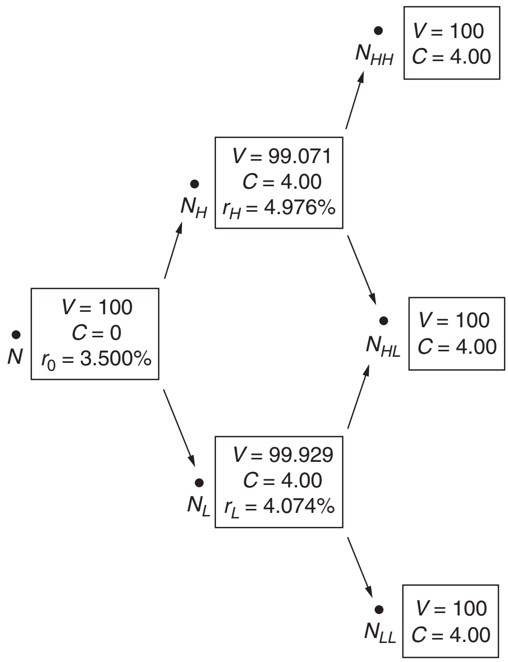
3a. The bond's value 2 years from now is \$104, just as in the first trial.

3b. The present value of the bond's value found in Step 3a for the higher 1-year forward rate, V_H, is \$99.071 (= \$104/1.04976).

3c. The present value of the bond's value found in Step 3a for the lower 1-year forward rate, V_L, is \$99.929 (= \$104/1.04074).

EXHIBIT 21–6

The 1-Year Forward Rates for Year 1 by Using the 2-Year, 4% On-the-Run Issue



3d. Adding the coupon to V_H and V_L, we get \$103.071 as the cash flow for the higher forward rate and \$103.929 as the cash flow for the lower forward rate.

3e. The present value of the two cash flows using the 1-year forward rate at the node to the left, 3.5%, gives

$$\frac{V_H + C}{(1 + r^*)} = \frac{\$103.071}{1.035} = \$99.586$$

and

$$\frac{V_L + C}{1 + r^*} = \frac{\$103.929}{1.035} = \$100.414.$$

Step 4. The average present value is \$100, which is the value at the node.

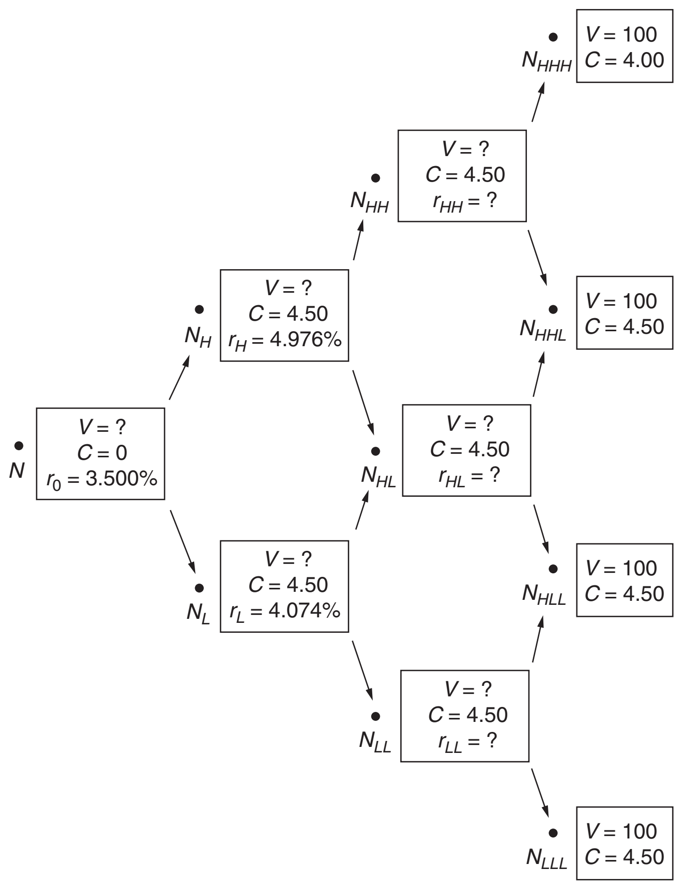
Step 5. Because the average present value is equal to the observed market price of \$100, r_1 or $r_{1,L}$ is 4.074% and $r_{1,H}$ is 4.976%.

We're not done. Suppose that we want to "grow" this tree for one more year—that is, we want to determine r_2. Now we will use the 3-year on-the-run issue, the 4.5% coupon bond, to get r_2. The same five steps are used in an iterative process to find the 1-year forward rates in the tree 2 years from now. Our objective now is to find the value of r_2 that will produce a bond value of $100 (because the 3-year on-the-run issue has a market price of $100) and is consistent with (1) a volatility assumption of 10%, (2) a current 1-year forward rate of 3.5%, and (3) the two forward rates 1 year from now of 4.074% (the lower forward rate) and 4.976% (the higher forward rate).

We explain how this is done using Exhibit 21–7. Let's look at how we get the information in the exhibit. The maturity value and coupon payment 3 years

EXHIBIT 21–7

Information for Deriving the 1-Year Forward Rates for Year 2 by Using the 3-Year, 4.5% On-the-Run Issue

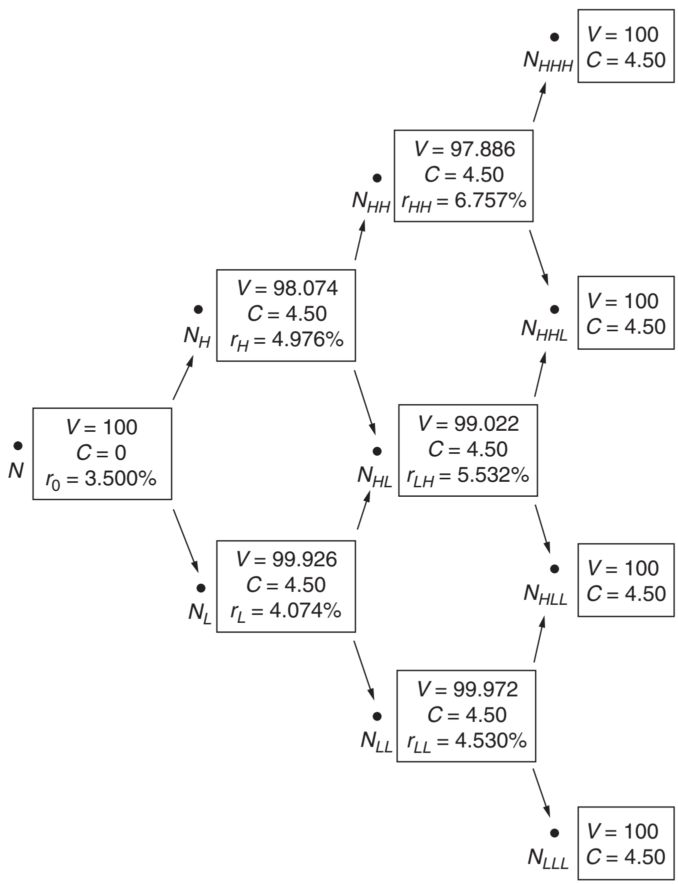


from now are shown in the boxes at the four nodes. Because the 3-year on-the-run issue has a maturity value of \$100 and a coupon payment of \$4.5, these values are the same in the box shown at each node. For the three nodes 2 years from now, the coupon payment of \$4.5 is shown. Unknown at these three nodes are (1) the three forward rates 2 years from now and (2) the value of the bond 2 years from now. For the two nodes 1 year from now, the coupon payment is known, as are the 1-year forward rates 1 year from now. These are the forward rates found earlier. The value of the bond, which depends on the bond values at the nodes to the right, is unknown at these two nodes.

Exhibit 21–8 is the same as Exhibit 21–7 complete with the values previously unknown. As can be seen from Exhibit 21–8, the value of r_2 or, equivalently, $r_{2,LL}$,

EXHIBIT 21–8

The 1-Year Forward Rates for Year 2 by Using the 3-Year, 4.5% On-the-Run Issue



which will produce the desired result, is 4.53%. We showed earlier that the corresponding forward rates $r_{2,HL}$ and $r_{2,HH}$ would be 5.532% and 6.757%, respectively. To verify that these are the 1-year forward rates 2 years from now, work backwards from the four nodes at the right of the tree in Exhibit 21–8. For example, the value in the box at N_{HH} is found by taking the value of \$104.5 at the two nodes to its right and discounting at 6.757%. The value is \$98.886. (Because it is the same value for both nodes to the right, it is also the average value.) Similarly, the value in the box at N_{HL} is found by discounting \$104.50 by 5.532% and at N_{LL} by discounting at 4.53%. The same procedure used in Exhibits 21–5 and 21–6 is used to get the values at the other nodes.

Valuing an Option-Free Bond

Exhibit 21–9 shows the 1-year forward rates or binomial interest-rate tree that can then be used to value any 1-year, 2-year, or 3-year bond for this issuer. To illustrate how to use the binomial interest-rate tree, consider a 5.25% option-free bond of this issuer with 3 years remaining to maturity. Also assume that the issuer's on-the-run yield curve is the one given earlier and hence that the appropriate binomial interest-rate tree is the one in Exhibit 21–9. Exhibit 21–10 shows the various values in the discounting process and produces a bond value of \$102.075.

It is important to note that this value is identical to the bond value found earlier when we discounted at either the spot rates or the 1 year forward rates. We should expect to find this result because our bond is option-free. This clearly demonstrates that the valuation model is consistent with the standard valuation model for an option-free bond.

EXHIBIT 21–9

Binomial Interest-Rate Tree for Valuing up to a 3-Year Bond for Issuer

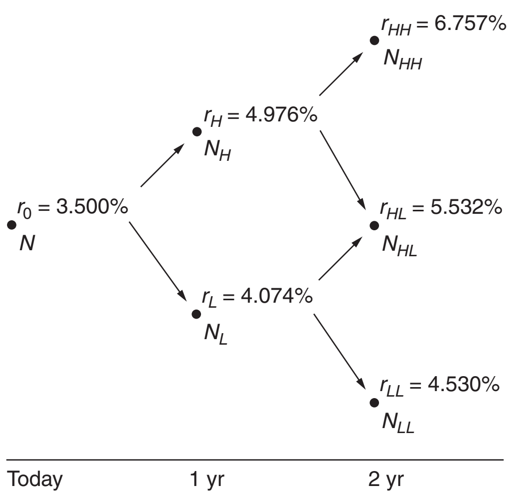
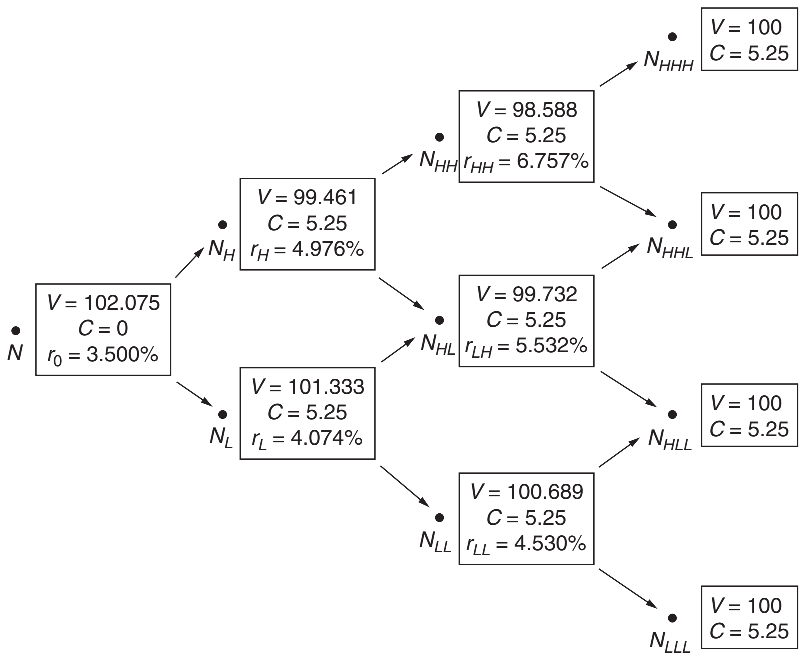


EXHIBIT 21–10

Valuing an Option-Free Bond of Issuer with 3 Years to Maturity and a Coupon Rate of 5.25%



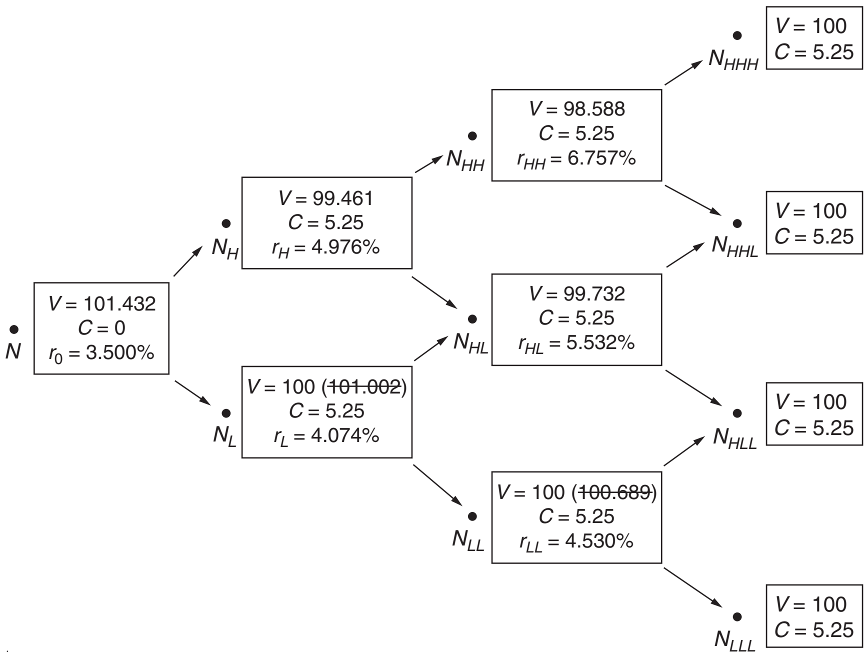
Valuing a Callable Bond

Now we will demonstrate how the binomial interest-rate tree can be applied to value a callable bond. The valuation process proceeds in the same fashion as in the case of an option-free bond but with one exception: when the call option may be exercised by the issuer, the bond value at a node must be changed to reflect the lesser of its values if it is not called (i.e., the value obtained by applying the recursive valuation formula described above) and the call price.

For example, consider a 5.25% bond with 3 years remaining to maturity that is callable in 1 year at \$100. To simplify the illustration, let's assume that the issuer will call the bond if its price exceeds \$100. Exhibit 21–11 shows the values at each node of the binomial interest-rate tree. The discounting process is identical to that shown in Exhibit 21–10 except that at two nodes N_L and N_{LL} the values from the recursive valuation formula (\$101.002 at N_L and \$100.689 at N_{LL}) exceed the call price (\$100) and therefore have been struck out and replaced with \$100. Each time a value derived from the recursive valuation formula has been replaced, the process for finding the values at that node is reworked starting with the period to the right. The value for this callable bond is \$101.432.

EXHIBIT 21–11

Valuing a Callable Bond with 3 Years to Maturity, a Coupon Rate of 5.25%, and Callable in 1 Year at $100*



*Bond assumed to be called if value exceeds $100.

The question that we have not addressed in our illustration but which is nonetheless important is under which circumstances the issuer will call the bond. A detailed explanation of the call rule is beyond the scope of this chapter. Basically, it involves determining when it is economical for the issuer on an after-tax basis to call the issue.

Because the value of a callable bond is equal to the value of an option-free bond minus the value of the call option, this means that

Value of the call option
= value of the option-free bond – value of the callable bond.

But we have just seen how the value of an option-free bond and the value of a callable bond can be determined. The difference between the two values is therefore the value of the call option.

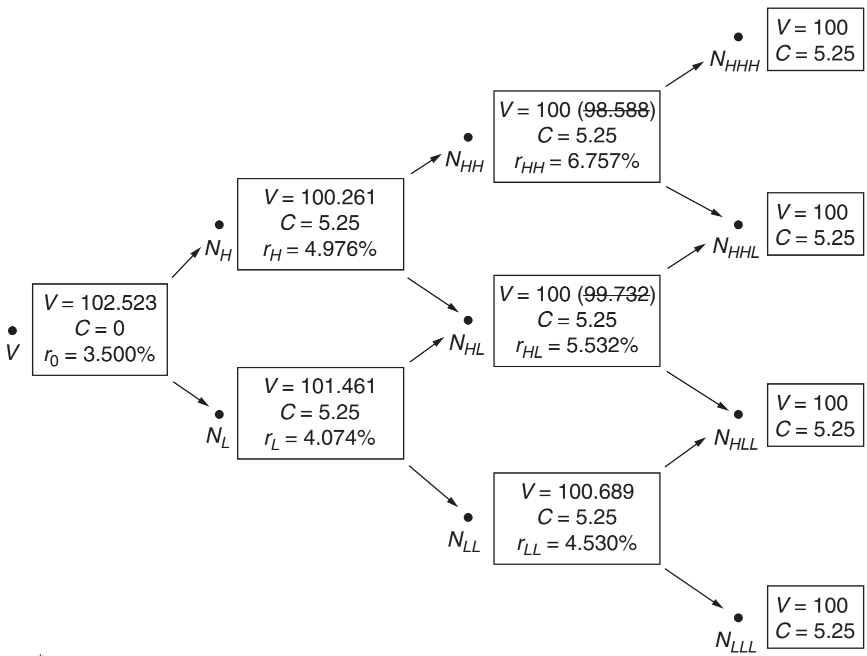
In our illustration, because the value of the option-free bond is $102.075 and the value of the callable bond is $101.432, the value of the call option is $0.643.

Extension to Other Embedded Options

The bond-valuation framework presented here can be used to analyze other embedded options such as put options, caps and floors on floating-rate notes, step-up

EXHIBIT 21–12

Valuing a Putable Bond with 3 Years to Maturity, a Coupon Rate of 5.25%, and Putable in 1 Year at $100*



*Bond assumed to be called if value exceeds $100.

notes, range notes, and the optional accelerated redemption granted to an issuer in fulfilling its sinking-fund rquirement. This application of the lattice method will be illustrated in the next chapter. Here let's consider a putable bond.

Suppose that a 5.25% bond with 3 years remaining to maturity is putable in 1 year at par ($100). Also assume that the appropriate binomial interest-rate tree for this issuer is the one in Exhibit 21–9. Exhibit 21–12 shows the binomial interest-rate tree with the bond values altered at two nodes (N_{HH} and N_{HL}) because the bond values at these two nodes fall below $100, the assumed value at which the bond can be put. The value of this putable bond is $102.523. Because the value of an option-free bond can be expressed as the value of a putable bond minus the value of a put option on that bond, this means that

Value of the put option

= Value of the option-free bond – value of the putable bond.

In our example, because the value of the putable bond is $102.523 and the value of the corresponding option-free bond is $102.075, the value of the put option is –$0.448. The negative sign indicates that the issuer has sold the option or, equivalently, the investor has purchased the option.

The framework can also be used to value a bond with multiple or interrelated embedded options. The bond values at each node are altered based on whether one of the options is exercised.

Interest-Rate Volatility and the Theoretical Value

In our illustration, interest-rate volatility was assumed to be 10%. The volatility assumption has an important impact on the theoretical value. More specifically, the higher the expected volatility, the higher is the value of an option. The same is true for an option embedded in a bond. Correspondingly, this affects the value of the bond with an embedded option.

For example, for a callable bond, a higher interest-rate volatility assumption means that the value of the call option increases, and because the value of the option-free bond is not affected, the value of the callable bond must be lower. For a putable bond, higher interest-rate volatility means that its value will be higher.

OPTION-ADJUSTED SPREAD

The valuation model gives the theoretical value of a bond. For example, if the observed price of the 3-year, 5.25% callable bond is $101 and the theoretical value is $101.432, this means that this bond is cheap by $0.432 according to the valuation model. Bond market participants, however, prefer to think not in terms of a bond's price being cheap or expensive in dollar terms but rather in terms of a yield spread—a cheap bond trades at a higher yield spread and an expensive bond at a lower yield spread.

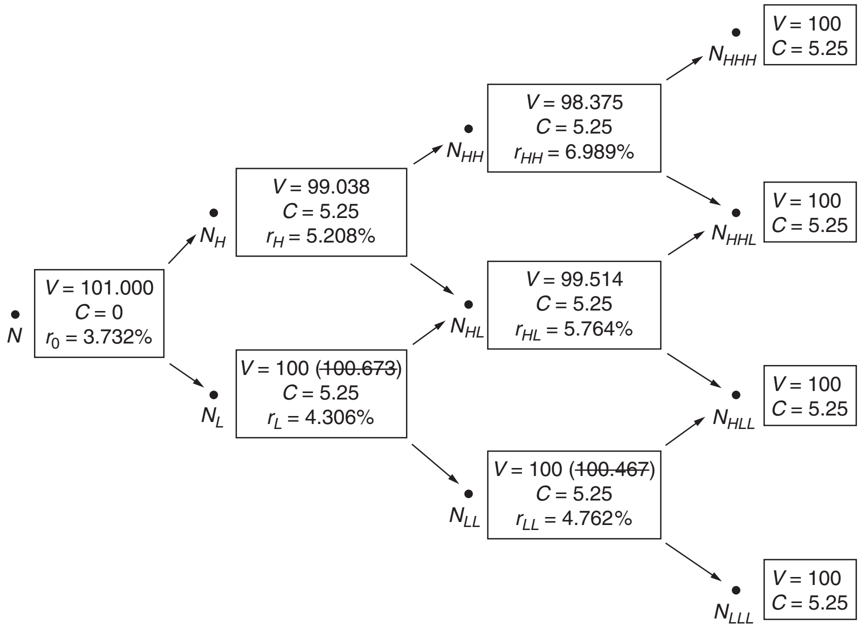
The market convention is to think of a yield spread as the difference between the yield to maturity on a particular bond and the yield on a comparable-maturity Treasury. However, this is inappropriate because, as we have explained, there is not one rate at which all cash flows should be discounted but a set of spot rates or, equivalently, forward rates. Given that this is the correct procedure for discounting, market participants determine the spread over the issuer's spot-rate curve or forward rates. In terms of our binomial interest-rate tree, it is the constant spread that when added to all the forward rates on the tree will make the theoretical value equal to the market price. The constant spread that satisfies this condition is called the *option-adjusted spread* (OAS). The spread is referred to as an option-adjusted spread because the spread takes into consideration the option embedded in the issue.

Returning to our illustration, if the observed market price is $101, the OAS would be the constant spread added to every forward rate in Exhibit 21–9 that will make the theoretical value equal to $101. In this case, that spread is 23.2 basis points, as can be verified in Exhibit 21–13.

As with the value of a bond with an embedded option, the OAS will depend on the volatility assumption. For a given bond price, the higher the interest-rate volatility assumed, the lower is the OAS for a callable bond and the higher is the OAS for a putable bond.

EXHIBIT 21–13

Demonstration of the Option-Adjusted Spread



The interpretation of the OAS depends on the benchmark on-the-run issue used in the mode. For example, some dealers use the Treasury on-the-run issue to generate the binomial tree. In this case, the OAS captures the credit spread, a liquidity premium, and any richness or cheapness of the bond being valued after adjusting for the embedded option. In contrast, some dealers use the issuer's on-the-run issue to construct the binomial tree, and as a result, the spread has already accounted for the credit risk. Thus the OAS reflects a liquidity premium and the richness or cheapness of the issue after considering the embedded option.

It is critical to understand that OAS is not a valuation model. Rather, it is product of a valuation model. If the valuation model is poor, the OAS will be meaningless.

PRICE VOLATILITY OF BONDS WITH EMBEDDED OPTIONS

In Chapter 12 the price volatility characteristics of option-free bonds were explained. In Chapter 13 duration was introduced as a measure of interest-rate risk. More specifically, the concept of modified duration was explained. Modified duration is a measure of the sensitivity of a bond's price to interest-rate changes, *assuming that the expected cash flow does not change with interest rates*. Consequently, modified duration may not be an appropriate measure for bonds with embedded options because the expected cash flows change as interest rates change. For example, when interest rates fall, the binomial tree changes, resulting in a change in the expected cash flow for a bond with an embedded option.

While modified duration may be inappropriate as a measure of a bond's price sensitivity to interest-rate changes, there is a duration measure that is more appropriate for bonds with embedded options. Because duration measures price responsiveness to changes in interest rates, the duration for a bond with an embedded option can be estimated by letting interest rates change by a small number of basis points above and below the prevailing yield and seeing how the prices change. As explained and illustrated in Chapter 13, the duration for *any* bond can be *approximated* as follows:

$$\text{Approximate duration} = \frac{V_- - V_+}{2V_0(\Delta y)},$$

where

V_0 = initial value or price of the security;
V_- = estimated value of the security if the yield is decreased by Δy;
V_+ = estimated value of the security if the yield is increased by Δy;
Δy = change in the yield of a security.

Effective Duration

Application of this formula to an option-free bond gives the modified duration because the cash flows do not change when yields change. When the approximate duration formula is applied to a bond with a embedded option, the new prices at the higher and lower yield levels should reflect the change in the cash flow. Duration calculated in this way is called *effective duration* or *option-adjusted duration*.

The difference between modified duration and effective duration for fixed-income securities with an embedded option can be quite dramatic. For example, a callable bond could have a modified duration of 4 but an effective duration of only 2. For certain mortgage-backed securities, the modified duration could be 7 and the effective duration 40! Thus using modified duration as a measure of the price sensitivity of a security to a parallel shift in the yield curve would be misleading. The more appropriate measure for any security with an embedded option is effective duration.

Calculating the Effective Duration by Using the Binomial Model

The procedure for calculating the values to be substituted into the duration formula by using the binomial model is described below. First, V_+ is determined as follows:

Step 1. Calculate the option-adjusted spread (OAS) for the issue.

Step 2. Shift the on-the-run yield curve up by a small number of basis points.

Step 3. Construct a binomial interest-rate tree based on the new yield curve in Step 2.

Step 4. Add the OAS to each of the forward rates in the binomial interest-rate tree to obtain an "adjusted tree."

Step 5. Use the adjusted tree found in Step 4 to determine the value of the security, which is V_+.

To determine the value of V_-, the same five steps are followed except that in Step 2 the on-the-run yield curve is shifted down by a small number of basis points.

Let's return to the example of the callable 5.25% 3-year bond. Given the yield curve and volatility assumptions we have been using, V_0 is \$101.432. Following the five steps above for a shift of the yield curve by 10 basis points down and up gives values for V_- and V_+ of \$101.628 and \$101.234, respectively. The effective duration is then

$$\text{Effective duration} = \frac{\$101.628 - \$101.234}{2(101.432)(0.001)} = 1.94.$$

Effective Convexity

In the same manner, the standard convexity measure explained in Chapter 14 may be inappropriate for a bond with embedded options because it does not consider the effect of a change in interest rates on the bond's cash flow. The convexity of any bond can be approximated using the following formula:

$$\text{Convexity} = \frac{V_+ + V_- - 2V_0}{V_0(\Delta y)^2}.$$

The values in the formula are the same values used in the duration formula.

When the prices used in this formula assume that the cash flows do not change when yields change, the resulting convexity is a good approximation of the standard convexity for an option-free bond. When the prices used in the formula are derived by changing the cash flows when yields change, the resulting convexity is called *effective convexity* or *option-adjusted convexity*. In our illustration, the effective convexity is

$$\text{Effective convexity} = \frac{\$101.628 + \$101.234 - 2(101.432)}{(101.432)(0.001)^2} = 6.77.$$

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CHAPTER
22

ANALYSIS OF FLOATING-RATE SECURITIES

Floating-rate securities, or simply *floaters*, have a coupon interest rate that varies over the security's life. Floaters are issued in every sector of the bond market—government, agency, corporate, municipal, mortgage, and asset backed—in the United States and in markets throughout the world. Although a floater's coupon formula may depend on a wide variety of economic variables (e.g., foreign exchange rates or commodity prices), a floater's coupon payments usually depend on the level of a money-market interest rate (e.g., the London Interbank Offered Rate [LIBOR], the Secured Overnight Financing Rate, or Treasury bills). A floater's coupon rate can be reset semiannually, quarterly, monthly, weekly, or daily. The terms *adjustable rate* and *variable rate* typically refer to securities with coupon rates reset not more than annually or based on a longer-term interest rate. However, this is a distinction without a difference, and we will refer to both floating-rate securities and adjustable-rate securities as floaters.

In this chapter we first review the general features of floaters. Most market participants use *spread* or margin measures (e.g., adjusted simple margin or discount margin) to assess the relative value of a floater, and we will explain these measures and their limitations. We will also explain the price volatility of floaters. As will be explained, there are floaters that have embedded options. The valuation of such floaters requires the use of the recursive method or lattice method described in Chapter 21. We apply this methodology to the valuation of three floating-rate structures. In the last section of this chapter we provide the analytics necessary to understand inverse floaters.

GENERAL FEATURES OF FLOATERS

A floater is a debt security whose coupon rate is reset at designated dates based on the value of some designated reference rate. The coupon formula for a pure floater (i.e., a floater with no embedded options) can be expressed as follows:

$$\text{Coupon rate} = \text{reference rate} \pm \text{quoted margin}$$

The *quoted margin* is the adjustment (in basis points) that the issuer agrees to make to the reference rate. The *reference rate* is the interest rate or index that appears in a floater's coupon formula, and it is used to determine the coupon payment on each reset date within the boundaries designated by any embedded caps and/or floors.

A floater often imposes limits on how much the coupon rate can float. Specifically, a floater may have a restriction on the maximum coupon rate that will be paid on any reset date. This is called a *cap* and is an unattractive feature from the investor's perspective but an important one for the borrower. In contrast, a floater also may specify a minimum coupon rate called a *floor*. A floor is an attractive feature from the investor's perspective. When a floater possesses both a cap and a floor, this feature is referred to as a *collar*. Thus a collared floater's coupon rate has a maximum value and a minimum value.

There is a wide variety of floaters that have special features that may appeal to certain types of investors. For example, *step-up notes* are floaters whose coupon rate is increased (i.e., *stepped up*) at designated times. When the coupon rate is increased only once over the security's life, it is said to be a *single-step-up note*. A *multiple-step-up note* is a step-up callable note whose coupon is increased more than one time over the life of the security. Moreover, step-up notes may be callable. A *range note* is a floater where the coupon payment depends on the number of days that the specified reference rate stays within a preestablished collar. There are also floaters whose coupon formula contains more than one reference rate. A *dual-indexed floater* is one such example. The coupon-rate formula is typically a fixed percentage plus the difference between two reference rates.

Although the reference rate for most floaters is an interest rate or an interest-rate index, numerous kinds of reference rates appear in coupon formulas. This is especially true for structured notes. Potential reference rates include movements in foreign-exchange rates, the price of a commodity (e.g., gold), movements in an equity index (e.g., the Standard & Poor's 500 Index), or an inflation index (e.g., the Consumer Price Index). Financial engineers are capable of structuring floaters with almost any reference rate.

Call and Put Provisions

Just like fixed-rate issues, a floater may be callable. The call provision gives the issuer the right to buy back the issue prior to the stated maturity date. The call option may have value to the issuer sometime in the future for two reasons. First, market interest rates may fall so that the issuer can exercise the option to retire the floater and replace it with a fixed-rate issue. Second, the required margin may decrease so that the issuer can call the issue and replace it with a floater with a lower quoted margin.[1] The issuer's call option is a disadvantage to the investor because the proceeds received must be reinvested either at a lower interest rate or at a lower margin. Consequently, an issuer who wants to include a call feature when issuing a floater must compensate investors by offering a higher quoted margin.

Floaters also may include a *put provision* that gives the security holder the option to sell the security back to the issuer at a specified price on designated dates.

1. The *required margin* is the spread (either positive or negative) the market requires as compensation for the risks embedded in the issue. If the required margin equals the quoted margin, a floater's price will be at par on coupon reset dates.

The specified price is called the *put price.* The put's structure can vary across issues. Some issues permit the holder to require the issuer to redeem the issue on any coupon payment date. Others allow the put to be exercised only when the coupon is adjusted. The time required for prior notification to the issuer or its agent varies from as little as 4 days to as long as a couple of months. The advantage of the put provision to the holder of the floater is that if after the issue date the margin required by the market for a floater to trade at par rises above the issue's quoted margin, the investor can force the issuer to redeem the floater at the put price and then reinvest the proceeds in a floater with a higher quoted margin.

SPREAD MEASURES

There are several yield-spread measures or margins that are used routinely to evaluate floaters. The four margins commonly used are (1) spread for life, (2) adjusted simple margin, (3) adjusted total margin, and (4) discount margin.

Spread for Life

When a floater is selling at a premium (discount) to par, a potential buyer of the floater will consider the premium or discount as an additional source of dollar return. *Spread for life* (also called *simple margin*) is a measure of potential return that accounts for the accretion (amortization) of the discount (premium) as well as the constant index spread over the security's remaining life.

Adjusted Simple Margin

The *adjusted simple margin* (also called *effective margin*) is an adjustment to spread for life. This adjustment accounts for a one-time cost-of-carry effect when a floater is purchased with borrowed funds. Suppose that a security dealer has purchased $10 million of a particular floater. Naturally, the dealer has a number of alternative ways to finance the position—borrowing from a bank, repurchase agreement, and so on. Regardless of the method selected, the dealer must make a one-time adjustment to the floater's price to account for the cost of carry from the settlement date to the next coupon reset date.

Adjusted Total Margin

The *adjusted total margin* (also called *total adjusted margin*) adds one additional refinement to the adjusted simple margin. Specifically, the adjusted total margin is the adjusted simple margin plus the interest earned by investing the difference between the floater's par value and the carry-adjusted price.[2]

2. When the floater's adjusted price is greater than 100, the additional increment is negative and represents the interest forgone.

Discount Margin

One common method of measuring potential return that employs discounted cash flows is *discount margin.* This measure indicates the average spread or margin over the reference rate the investor can expect to earn over the security's life given a particular assumed path that the reference rate will take to maturity. The assumption that the future levels of the reference rate are equal to today's level is the current market convention. The procedure for calculating the discount margin is as follows:

1. Determine the cash flows assuming that the reference rate does not change over the security's life;
2. Select a margin (i.e., a spread above the reference rate);
3. Discount the cash flows found in (1) by the current value of the reference rate plus the margin selected in (2);
4. Compare the present value of the cash flows as calculated in (3) with the price. If the present value is equal to the security's price, the discount margin is the margin assumed in (2). If the present value is not equal to the security's price, go back to (2) and select a different margin.

For a floater selling at par, the discount margin is simply the quoted margin. Similarly, if the floater is selling at a premium (discount), then the discount margin will be below (above) the quoted margin.

Practitioners use the spread measures presented above to gauge the potential return from holding a floater. Much like conventional yield measures for fixed-income securities, the yield or margin measures discussed here are, for the most part, relatively easy to calculate and interpret. However, these measures reflect relative value only under several simplifying assumptions (e.g., reference rates do not change).

One of the key difficulties in using the measures described in this chapter is that they do not recognize the presence of embedded options. As discussed earlier, there are callable/putable floaters and floaters with caps and/or floors. However, the recognition of embedded options is critical for valuing floaters properly. If an issuer can call an issue when presented with the opportunity and refund at a lower spread, the investor must then reinvest at the lower spread. With this background, it should not be surprising that sophisticated practitioners value floaters using arbitrage-free binomial interest-rate trees and Monte Carlo simulations. These models are designed to value securities whose cash flows depend on interest rates.

PRICE VOLATILITY CHARACTERISTICS OF FLOATERS

The change in the price of a fixed-rate security when market rates change occurs because the security's coupon rate differs from the prevailing rate for new comparable bonds issued at par. Thus an investor in a 10-year, 7% coupon bond purchased at par, for example, will find that the bond's price will decline below par if the market requires a yield greater than 7% for bonds with the same risk and maturity. In contrast, a floater's coupon resets periodically, thereby reducing its sensitivity to changes in rates. For this reason, floaters are said to be more *defensive* securities. This does not mean, of course, that a floater's price will not change.

Factors That Affect a Floater's Price

A floater's price will change depending on the following factors:

Factor 1. Time remaining to the next coupon reset date.

Factor 2. Changes in the market's required margin for that specific issuer.

Factor 3. Whether or not the cap or floor is reached or is close to being reached.

Below we discuss the impact of each of these factors.

Time Remaining to the Next Coupon Reset Date

The longer the time to the next coupon reset date, the more a floater behaves like a fixed-rate security, and the greater is a floater's potential price fluctuation. Conversely, the shorter the time between coupon reset dates, the smaller is the floater's potential price fluctuation.

To understand why this is so, consider a floater with 5 years remaining to maturity whose coupon formula is the 1-year Treasury rate plus 50 basis points, and the coupon is reset today when the 1-year Treasury rate is 3.5%. The coupon rate will remain at 4% for the year. One month hence an investor in this floater effectively would own an 11-month instrument with a 4% coupon. Suppose that at that time the market requires a 4.2% yield on comparable issues with 11 months to maturity. Then our floater would be offering a below-market rate (4% versus 4.2%). The floater's price must decline below par to compensate the investor for the submarket yield. Similarly, if the yield that the market requires on a comparable instrument with a maturity of 11 months is less than 4%, the floater will trade at a premium. For a floater in which the cap is not binding and for which the market does not demand a margin different from the quoted margin, a floater that resets daily will trade at par.

Changes in the Market's Required Margin

At the initial offering of a floater, the issuer will set the quoted margin based on market conditions so that the security will trade near par. Subsequently, if the market requires a higher (lower) margin, the floater's price will decrease (increase) to reflect the current margin required. We refer to the margin that is demanded by the market as the *required margin*. For example, consider a floater whose coupon formula is the 1-month LIBOR plus 40 basis points. If market conditions change such that the required margin increases to 50 basis points, this floater would be offering a below-market margin. As a result, the floater's price would decline below par. By the same token, the floater would trade above its par value if the required margin is less than the quoted margin—less than 40 basis points in our example.

The required margin for a particular issue depends on

- The margin available in competitive funding markets;
- The credit quality of the issue;
- The presence of any embedded call or put options;
- The liquidity of the issue.

An alternative source of funding to floaters is a syndicated loan. Consequently, the required margin will be driven, in part, by margins available in the syndicated loan market.

The portion of the required margin attributable to credit quality is referred to as the *credit spread*. The risk that an increase in the credit spread will be required by the market is referred to as *credit spread risk*. The concern for credit spread risk applies not only to an individual issue but also to a sector or the economy as a whole. For example, credit spreads may increase because of a financial crisis (e.g., the Global Financial Crisis) while the individual issuer's condition and prospects remain essentially unchanged.

A portion of the required margin reflects the call risk if the floater is callable. Because the call feature imposes hazards on the investor, the greater the call risk, the higher is the quoted margin at issuance, other things being equal. After issuance, depending on how interest rates and required margins change, the perceived call risk and the margin required as compensation for that risk will change accordingly. In contrast to call risk owing to an embedded call option, a put provision provides benefits to the investor. If a floater is putable at par, all else being equal, its price should trade at par near the put date.

Finally, a portion of the quoted margin at issuance reflects the issue's perceived liquidity. *Liquidity risk* is the threat of an increase in the required margin because of a perceived deterioration in an issue's liquidity. Investors in nontraditional floater products are particularly concerned with liquidity risk.

Whether or Not the Cap or Floor Is Reached

For a floater with a cap, once the coupon rate as specified by the coupon formula rises above the cap rate, the floater then offers a below-market coupon rate, and the floater will trade at a discount. The floater will trade more and more like a fixed-rate security the further the capped rate is below the prevailing market rate.

Simply put, if a floater's coupon rate does not float, it is effectively a fixed-rate security and will assume the duration risk of a fixed-rate bond. *Cap risk* is the risk that the floater's value will decline because the cap is reached. The situation is reversed if the floater has a floor. Once the floor is reached, all else being equal, the floater will trade either at par value or at a premium to par if the coupon rate is above the prevailing rate offered for comparable issues.

Duration of Floaters

We have just described how a floater's price responds to a change in the required margin, holding all other factors constant. As explained in Chapter 13, the measure used by market participants to quantify the sensitivity of a security's price to changes in interest rates is duration. A security's duration tells us the approximate percentage change in its price for a 100 basis point change in rates. The procedure for computing a security's duration was explained in Chapter 13.

Two measures are employed to estimate a floater's sensitivity to each component of the coupon formula. *Index duration* is a measure of the floater's price sensitivity to changes in the reference rate holding the quoted margin constant. Correspondingly, *spread duration* measures a floater's price sensitivity to a change in the *quoted margin* or *spread* assuming that the reference rate remains unchanged. Note that the term *spread duration* can have two possible meanings. Spread duration for a fixed-rate bond is the sensitivity of the bond's price to changes in the spread relative to Treasuries. In the case of a floater, spread duration is the sensitivity of the bond's price to changes in the quoted margin.

LATTICE METHOD/TREE METHOD FOR VALUING COMPLEX FLOATERS

In Chapter 21 we explained the underlying principles and concepts involved in the valuation of bonds using the lattice method. There we showed how to value a callable bond and a putable bond. Here we will use the method to value a range note where there are no embedded options but the value does depend on future interest rates. Then we will see how to use the methodology to value three floater structures with embedded options: a step-up callable note, a capped floater, and a callable capped floater. Recall that in valuing a bond with an embedded option, the volatility of interest rates must be assumed.

In our illustration we will use the on-the-run issue introduced in Chapter 7, and the calculated spot rates are reproduced below.

Maturity (yrs)	Yield to Maturity (%)	Market Price ($)	Spot Rate (%)
1	3.5	100	3.5000
2	4.2	100	4.2147
3	4.7	100	4.7345
4	5.2	100	5.2707

We will simplify the illustration by assuming annual-pay bonds. The corresponding 1-year forward rates are

Current 1-year forward rate = 3.500%;

1-year forward rate 1 year from now = 4.935%;

1-year forward rate 2 years from now = 5.784%;

1-year forward rate 3 years from now = 6.893%.

We will use the same notation as in Chapter 21, which we repeat below:

- Each node is labeled with an *N*, representing node, and a subscript;
- The higher *branch* of the tree is denoted with an *H* and the lower with an *L*.

The *root* of the tree is denoted by *N* and is nothing more than the current 1-year spot rate or, equivalently, the current 1-year rate, which we denote by R_0. In the binomial model, what is assumed in creating this tree is that the 1-year rate can take on two possible values the next year and that the two rates have the same probability of occurring. One rate will be higher than the other. We assumed that the 1-year rate evolved over time based on a log-normal random walk with a certain volatility.

Valuing a Range Note

As explained earlier in this chapter, a range note is a floater that pays the reference rate only if the rate falls within a specified band. If the reference rate falls outside the band, whether the lower or upper boundary, no coupon is paid. Typically, the band increases over time.

To illustrate, suppose that the reference rate is, again, the 1-year rate. Suppose further that the band (or coupon schedule) is defined as follows:

Limit	Year 1 (%)	Year 2 (%)	Year 3 (%)
Lower	3.00	4.00	5.00
Upper	5.00	6.25	8.00

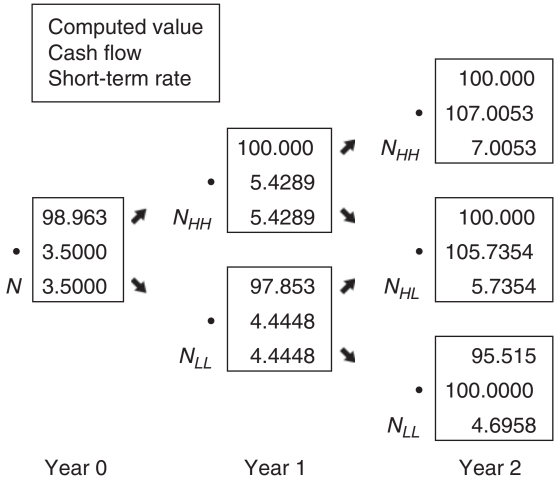
Exhibit 22–1 shows the binomial tree and the cash flows expected at each node. Either the 1-year reference rate is paid at each node or nothing. In the case of this 3-year range note, there are a number of states in which no coupon is paid. Using the lattice model, we can work back through the tree to the current value, $98.963.

Valuing a Step-up Callable Note

Step-up callable notes are callable instruments whose coupon rate is increased (i.e., *stepped up*) at designated times. Valuation using the lattice model is similar to that for valuing a callable bond described earlier except that the cash flows are altered at each node to reflect the coupon characteristics of the step-up note.

EXHIBIT 22–1

Valuation of a Range Note



Suppose that a 4-year step-up callable note pays 4.25% for 2 years and then 7.5% for 2 more years. Assume that this note is callable at par at the end of years 2 and 3. We will use the binomial tree given in Exhibit 22–2 to value this note.

Exhibit 22–2 shows the value of the note if it were not callable. The coupon in the box at each node reflects the step-up terms. The value is \$102.082. Exhibit 22–3 shows that the value of the single step-up callable note is \$100.031. The value of the embedded call option is equal to the difference in the optionless step-up note value and the step-up callable note value, \$2.051

Valuing Capped Floating-Rate Bonds

Consider a floater with a coupon indexed to the 1-year rate (the reference rate) plus a spread. For our purposes, assume a 25 basis point spread (i.e., quoted margin) to the reference rate. In Exhibit 22–4 we use the tree from Exhibit 22–2, and as was the case with the option-free fixed-rate coupon bond, at each node we have entered the cash flow expected at the end of each period. Using the same valuation method as before, we can find the value at each node. Consider N_{HLL}:

$$V_3[N_{HLL}] = 0.5\left(\frac{100+6.416}{1.06166}\right) + 0.5\left(\frac{100+6.416}{1.06166}\right) = \$100.235.$$

Stepping back one period,

$$V_2[N_{LL}] = 0.5\left(\frac{100.235+4.9458}{1.046958}\right) + 0.5\left(\frac{100.238+4.9458}{1.0469586}\right) = \$100.465.$$

EXHIBIT 22–2

Valuing a Single Step-up Noncallable Note with 4 Years to Maturity (10% Volatility Assumed)

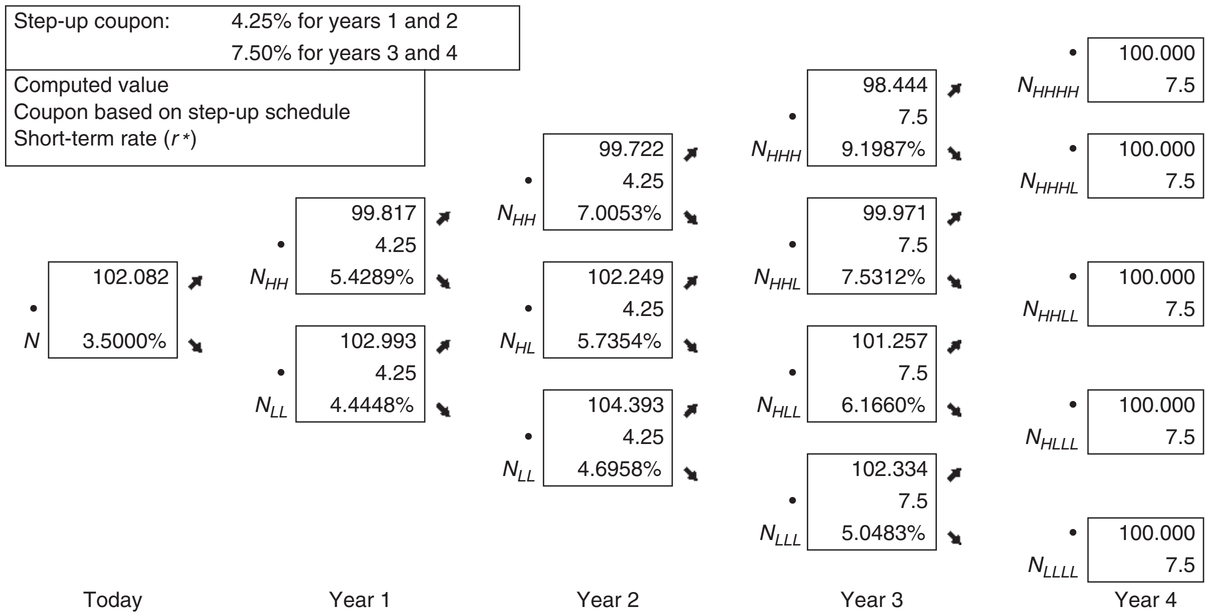
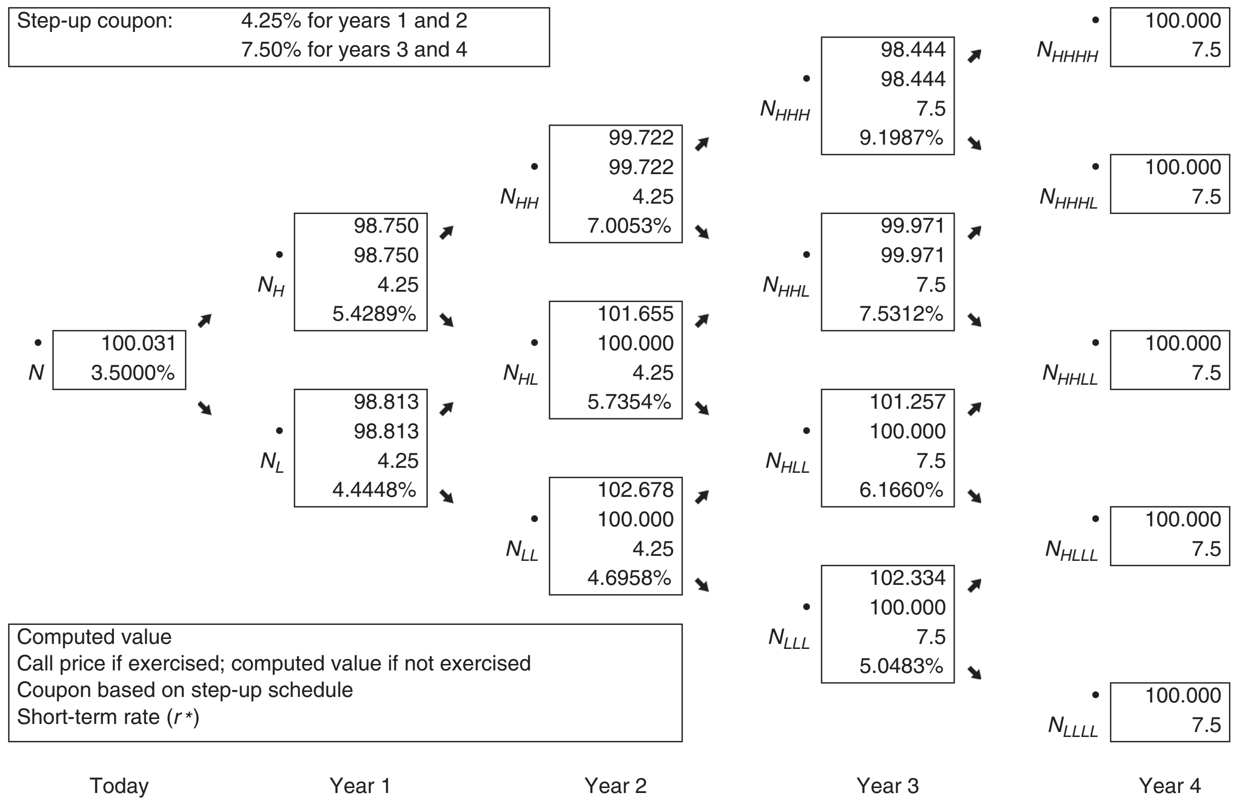


EXHIBIT 22–3

Valuing a Single Step-up Callable Note with 4 Years to Maturity, Callable in 2 Years at 100 (10% Volatility Assumed)



Following this same procedure, we arrive at the price of \$100.893. How would this change if the interest rate on the bond were capped?

Assume that the cap is 7.25%. Exhibit 22–5 provides a picture of the effects of the cap on the value of the bond. As rates move higher, there is a possibility that the current reference rate exceeds the cap. Such is the case at N_{HHH} and N_{HHL}. The coupon is subject to the following constraint:

$$C_t = \max[R_t, 7.25\%].$$

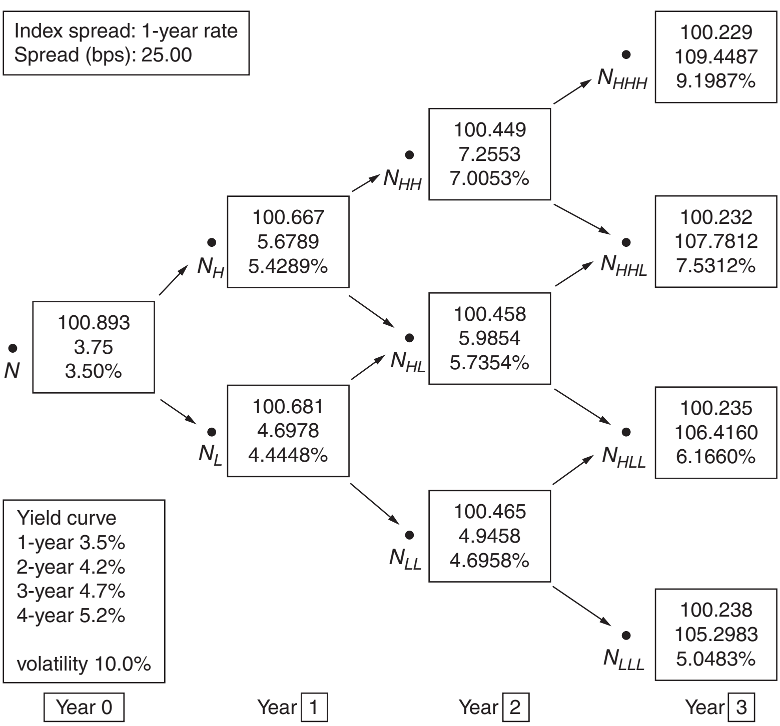
As a result of the cap, the value of the bond in the upper nodes at $t = 4$ falls below par. Explicitly,

$$V_3[N_{HHH}] = 0.5\left(\frac{100+7.25}{1.09187}\right) + 0.5\left(\frac{100+7.25}{1.09187}\right) = \$98.25.$$

Valuing recursively through the tree, we arrive at the current value of the capped floater, \$100.516. This last calculation gives us a means for pricing the cap. Without a cap, the bond is priced at \$100.893. With the cap, it is priced at \$100.516. The difference between these two prices is the value of the cap, \$0.377. It is

EXHIBIT 22–4

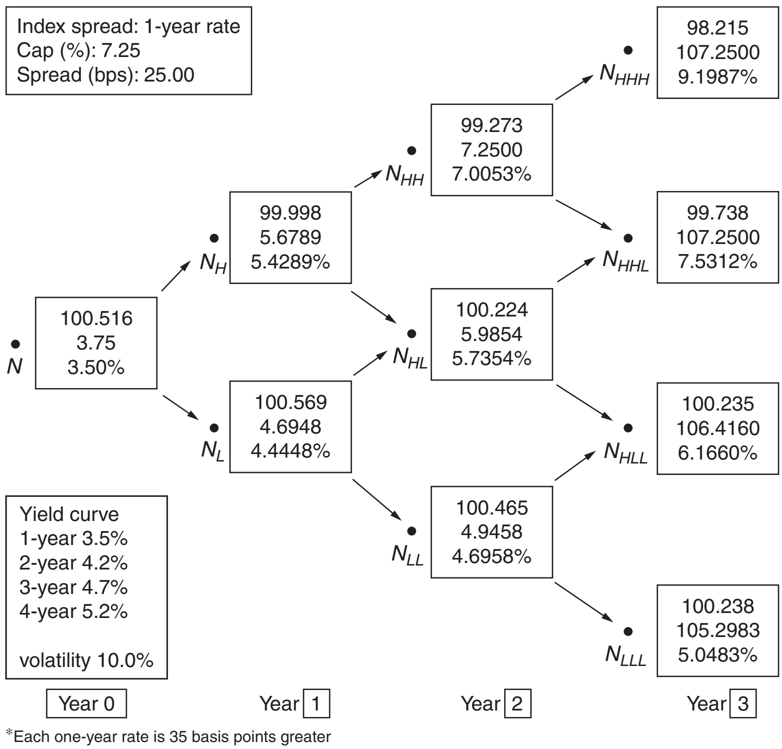
Valuation of a Floating-Rate Bond with No Cap



*Each one-year rate is 35 basis points greater

EXHIBIT 22–5

Valuation of a Capped Floating-Rate Bond



important to note that the price of the cap depends on volatility. Any change in the volatility would result in a different valuation for the cap. The greater the volatility, the higher is the price of the option, and vice versa.

What if an issuer wanted to offer this bond at par? In such a case, an adjustment has to be made to the coupon. To lower the price from \$100.516 to par, a lower spread over the reference rate need only be offered to investors. Suppose that the issuer decides that the coupon will be the 1-year rate plus 5 basis points. It turns out that this is not enough. For example, at a spread of 8.70 basis points over the 1-year reference rate, it can be shown that the bond will be priced at par. The spread of 8.7 basis points also depends on volatility.

Valuing Callable Capped Floating-Rate Bonds

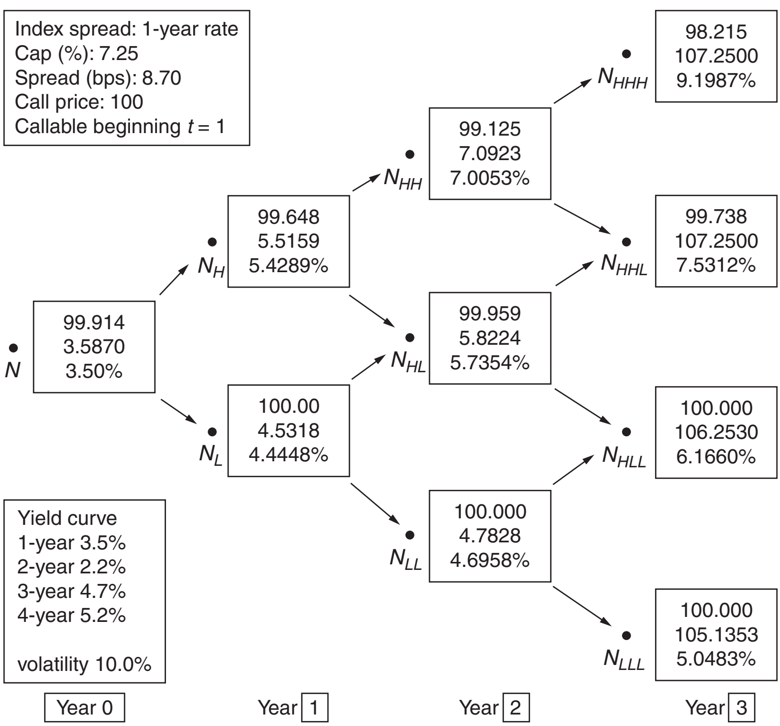
Now consider a callable capped floater. We must be careful to specify the appropriate rules for calling the bond on the valuation tree. For our purposes, any time the bond has a value above par, the bond will be called. (Here we assume a par call to simplify the illustration.)

Before we get into the details, it is important to motivate the need for a call on a floating-rate bond. The value of a cap to the issuer increases as market rates near the cap, and there is the potential for rates to exceed the cap prior to maturity. As rates decline, so does the value of the cap, eventually approaching zero. The problem for the issuer in this case is the additional basis point spread it is paying for a cap that now has no value. Thus when rates decline, a call has value to the issuer because it can be called and reissued at current rates without the spread attributable to the cap.

Suppose that the capped floater is callable at par any time after the first year. Exhibit 22–6 provides details on the effect of the call option on valuation of the capped floater. Again, it is assumed that for this callable bond, when the market price exceeds par in a lattice model, the bond is called. In the case of our 4-year bond, you can see that the price of the bond at the previously mentioned nodes N_{LL}, N_{LLH}, and N_{LLL} is now 100 in Exhibit 22–6, the call price. The effect of the call option on the value is also evident with today's value for the bond moving to $99.914.

EXHIBIT 22–6

Valuation of a Callable Floating-Rate Bond with a 7.25% Cap



As with the case of a cap, the by-product of this analysis is the value of the call option on a capped floater. We now have the fair value of the capped floater versus the callable capped floater. So the call option has a value of 100 – 99.914 = $0.086.

How would one structure the issue so that it is priced at par? Again, the issuer would have to offer the bondholder additional spread over the reference rate and the 8.7 basis points the holder is already receiving for accepting the cap. In this case, an additional 4.67 basis points is required, moving the total spread over the 1-year reference rate to 13.37 basis points. As before, we arrive at this 4.67 basis point spread through an iterative process described earlier for the structuring of a cap.

To summarize, the callable capped floater offers benefits to the issuer, but there is a cost. To avoid increasing payments as rates rise, the issuer can put a cap on the bond. However, investors will demand compensation in the form of higher coupon payments, that is, a higher spread to the reference rate.

This spread is a burden that the bond issuer would like to avoid paying if rates decline and the cap has no value due to the low rates. A call option allows the issuer to retire the capped floater and reissue at lower rates. Again, the bondholder recognizes that the benefit to the issuer is a detriment to the bondholder and demands additional compensation, a higher spread.

Once the spread is determined, valuation of the callable capped floater is an application of the recursive valuation process. The coupon payments are defined at each node, and the call option is exercised at nodes where the market price exceeds the call price. All that is left is the discounting of the cash flow period by period back to the present to arrive at a value for the instrument.

INVERSE FLOATER ANALYTICS

Thus far in this chapter we have described the analytics for floaters linked to a reference interest rate, where the reset interest rate on a reset date was positively related to the change in the reference rate. That is, if the reference interest rate increased (decreased), the reset interest rate was higher (lower) than the prior interest rate on the floater. However, not all floaters have this structure. There are floaters whose interest rate at a reset date changes in the opposite direction of the change in the reference rate. That is, if the reference interest rate increases (decreases), the reset interest rate declines (increases). Because the reset interest rate changes inversely with the change in the reference rate, a floater with this structure is referred to as an *inverse floating-rate security* or, simply, an *inverse floater*. In this section we describe inverse floater analytics.

An inverse floater is typically created by splitting of a bond issue with a fixed interest rate into an inverse floater and a floater. The creation of an inverse floater can be done by an investment banker when a bond is issued or by a dealer purchasing a bond with a fixed interest rate.

Fundamental Relationships for an Inverse Floater

The fixed-rate bond from which the inverse floater and the floater are created is referred to as the *collateral*. It is the creator of the inverse floater (investment banker or dealer) who makes the decision as to how much of the par value of the fixed-rate bond should be allocated to a floater and an inverse floater.

The general coupon formula for the inverse floater is

$$\text{Fixed rate} - (\text{leverage} \times \text{reference rate}).$$

The leverage in the coupon formula is the ratio of the par value of the floater to the par value of the inverse floater. For example, suppose that an investment banker decides to split a planned fixed-rate bond with a $400 million par value (i.e., the collateral) as follows: $300 million floater and $100 million inverse floater. The leverage, also referred to as *coupon leverage*, in this example is 3 ($300 million/$100 million). The splitting decision is important because, as explained later, it determines the interest-rate sensitivity of the inverse floater's price (i.e., duration).[3]

Regardless of how the collateral's par value is split between the floater and inverse floater, the combined interest paid to the floater and inverse floater for every period cannot exceed the interest received by the collateral. For example, consider a 10-year, 4.5% coupon semiannual-pay bond. Suppose that the $400 million par value of the bond is used as collateral to create a floater with a principal of $300 million and an inverse floater with a principal of $100 million. Suppose that the coupon rate for the floater and the inverse floater are reset every 6 months based on the following formula:

Floater coupon: Reference rate + 1%;

Inverse floater coupon: 15% − 3 × reference rate.

In the inverse floater, the fixed rate is 15% and the leverage is 3.

Notice that the total principal of the floater and inverse floater equals the principal of the collateral, $400 million. The weight of the floater and inverse floater based on the collateral par value is 75% ($300 million/$400 million) and 25% ($100 million/$400 million), respectively. The weighted-average coupon must equal the coupon rate for the collateral, 4.5% in our example. This is indeed the case, as can be seen from the following equation, which is the weighted-average coupon from the floater and inverse floater:

$$0.75(\text{reference rate} + 1\%) + 0.25(15\% - 3 \times \text{reference rate}) = 4.5\%.$$

There is a major problem with using the coupon formula for the inverse floater if no restriction is imposed on the coupon rate for the inverse floater. In our example, if the reference rate exceeds 15%, the inverse floater's coupon rate will

3. The leverage decision is not determined arbitrarily by creators of inverse floaters. The market is the final arbiter. Specifically, the amount of leverage is dictated by client inquiries and/or market demand for recently created issues.

be negative. To prevent a negative interest rate for the inverse floater, a restriction or floor is placed on the inverse floater's coupon rate. Typically, the floor is set at zero. However, by imposing a floor on the inverse floater, a maximum interest rate must be imposed on the floater, recalling that the weighted-average coupon must be 4.5%. The maximum interest rate for the floater is called a *cap*. In our hypothetical structure, the maximum coupon rate that must be imposed on the floater is 15%. Notice that the maximum coupon rate for the floater is the fixed rate in the inverse floater's coupon rate. Because the floater has a maximum interest rate, it is a *capped* floater. The inverse floater has a cap that is the fixed rate in the coupon formula for the inverse floater.

Valuing an Inverse Floater

The value of any financial asset is the present value of its expected cash flows. As with a floater, applying this valuation principle to inverse floaters is problematic because of the uncertainty about future values for the reference rate. Fortunately, the valuation of an inverse floater is not complex.

We can express the relationships among the collateral value, the floater value, and the inverse floater value as follows:

Collateral value = floater value + inverse floater value.

If this relationship does not hold, opportunities for arbitrage arise. An alternative way to express the relationship is

Inverse floater value = collateral value − floater value.

This expression states that the value of an inverse floater can be found by first valuing the collateral and valuing the floater, and then calculating the difference between these two values. In this case, the value of an inverse floater is not found directly but is instead derived from the value of the collateral and the value of the floater. We discussed how to value the floater earlier in his chapter.

Given that the floater created from the collateral is a capped floater, the value of an inverse floater can be expressed as

Inverse floater value = collateral value – capped floater value.

The factors that affect the value of an inverse floater are the same factors that affect the collateral's value and the capped floater's value.

Economic Interpretation of a Position in an Inverse Floater

Because the capped floater and inverse floater are created from the fixed-rate collateral, the following relationship holds:

Long a fixed-rate collateral = long a capped floater + long an inverse floater.

Recasting this relationship in terms of an inverse floater (i.e., put the inverse floater on the left-hand side of the equation), we can write

Long an inverse floater = long a fixed-rate collateral − long a capped floater.

This relationship can be equivalently expressed as

Long an inverse floater = long a fixed-rate collateral + short a capped floater.

Thus the owner of an inverse floater has effectively purchased a fixed-rate bond (i.e., the collateral) and sold a capped floater (i.e., shorted a capped floater). Shorting a floater is equivalent to borrowing funds at an uncertain rate. Therefore, shorting a floater is equivalent to borrowing at the reference rate plus the spread. Consequently, the owner of an inverse floater has effectively purchased a fixed-rate bond with borrowed funds (i.e., a levered long position).[4]

Duration of an Inverse Floater

Duration is a measure of a security's price sensitivity to a change in required yield. Because valuations are additive (i.e., the value of collateral is the sum of the floater and inverse floater values), durations (properly weighted) are additive as well. Accordingly, the duration of the inverse floater is related to the duration of the collateral and the duration of the floater. Specifically, the duration of an inverse floater will be a multiple of the duration of the collateral from which it is created. The multiple is the leverage.

To understand this, suppose that a 10-year fixed-rate bond with a market value of \$400 million is split into a floater and an inverse floater with market values of \$300 million and \$100 million, respectively. Assume also that the duration of the collateral (i.e., the 10-year fixed-rate bond) is 7. Given this information, we know that for a 100 basis point change in required yield, the collateral's value will change by approximately 7%, or \$28 million (7% × \$400 million). Because the floater and inverse floater are created from the underlying collateral, the combined change in value of the floater and the inverse floater must be \$28 million given a

4. Given this interpretation, an interest-rate swap and an inverse floater possess similar characteristics. An interest-rate swap is a contract between two counterparties who agree to exchange periodic interest payments based on some notional principal. One party—the fixed-rate payer—agrees to pay the other party fixed interest-rate payments at designated dates for the contract's tenor. The other party—the floating-rate payer—agrees to make interest-rate payments that float with some reference rate. An interest-rate swap can be interpreted as a package of cash-market instruments. The fixed-rate payer has a cash-market position that is equivalent to a long position in a floating-rate bond and a short position in a fixed-rate bond—the short position being the equivalent of borrowing by issuing a fixed-rate bond. Conversely, the floating-rate payer's position is equivalent to purchasing a fixed-rate bond and financing that purchase at a floating rate, where the floating rate is the swap's reference rate. In other words, the position of a floating-rate payer is equivalent to a long position in a fixed-rate bond and a short position in a floating-rate bond. With the exception of the cap on the floater, the owner of an inverse floater receives fixed and pays floating.

100 basis point change in required yield. The question becomes how we partition the change in value between the floater and the inverse floater. If the duration of the floater is small, as explained in this chapter, then the inverse floater must experience the full force of the \$28 million change in value. For this to occur, the duration of the inverse floater must be approximately 28. A duration of 28 will mean a 28% change in the inverse floater's value for a 100 basis point change in required yield and a change in value of approximately \$28 million (28% × \$100 million).

Notice from our illustration that the duration of an inverse floater is greater the collateral's term to maturity. For individuals who interpret duration in terms of years (i.e., Macaulay duration), this presents something of a puzzle. After all, how can a fixed-income security have a duration greater than the maturity of the collateral from which it is created? Of course, there is no puzzle. The confusion is the residue from continuing to think about duration in the context in which it was developed by Frederick Macaulay in 1938—as a measure of the average time taken by a bond, on a discounted basis, to return the original investment. The significance and interpretation of Macaulay duration lie in its link to bond price volatility.

In general, assuming that the duration of the floater is close to zero, it can be shown that the duration of an inverse floater is

$$\text{Duration of an inverse floater} = (1 + \text{leverage}) \times \text{duration of collateral.}$$

In our illustration, leverage is 3, and the duration of the collateral is assumed to be 7. Therefore,

$$\text{Duration of an inverse floater} = (1 + 3) \times 7 = 28.$$

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PART

SEVEN

CREDIT AND LIQUIDITY CONCEPTS

CHAPTER
23

CREDIT RISK CONCEPTS AND MEASURES

The analytics for managing the credit risk of a portfolio of credit-risky securities has changed dramatically. At one time, credit risk assessment of a portfolio basically concentrated on traditional credit risk analysis and the distribution of the bonds in the portfolio in terms of their credit rating. Today, several models have been developed to measure, monitor, and control a portfolio's credit risk more effectively; the same models are employed in the pricing of credit-risky debt instruments. In this chapter we discuss basic measures relevant to managing the credit risk of a portfolio of credit-risky debt instruments.

TYPES OF CREDIT RISK

While reference is commonly made to *credit risk*, it is important to understand that there are two main types of credit risk to which a portfolio or position is exposed:

- Credit default risk and
- Credit spread risk.

Credit Default Risk

Credit default risk is the risk that an issuer of debt (obligor) is unable to meet its financial obligations. When a borrower defaults, an investor generally incurs a loss equal to the amount owed by the borrower less any recovery amount that the investor receives as a result of foreclosure, liquidation, or restructuring of the defaulted obligation. All portfolios with credit-risky debt obligations are exposed to credit default risk. The credit default risk exposure to an issuer is described by the issuing firm's *credit rating*, which is a formal opinion given by a rating agency of the credit default risk faced by investing in a particular issue of debt securities. For long-term debt obligations, a credit rating is a forward-looking assessment of the probability of default and the relative magnitude of the loss should a default occur. For short-term debt obligations, a credit rating is a forward-looking assessment of the probability of default.

The three major ratings agencies (also referred to as *nationally recognized statistical rating organizations*) in the United States—Moody's, Standard & Poor's, and Fitch Ratings—undertake a formal analysis of the borrower, after which a rating is announced. The credit ratings are summarized in Exhibit 23–1. The ratings systems use similar symbols. In addition to the generic rating category, Moody's employs a numerical modifier of 1, 2, or 3 to indicate the relative standing of a particular issue within a rating category. This modifier is called a *notch*. Both Standard & Poor's and Fitch Ratings use a plus (+) and a minus (–) to convey the same information. Bonds rated triple B or higher are referred to as *investment-grade bonds*. Bonds rated below triple B are referred to as *non-investment-grade bonds*, *speculative bonds*, or, more popularly, *high-yield bonds* or *junk bonds*.

EXHIBIT 23–1

Bond Credit Ratings

Fitch	Moody's	S&P	Summary Description
			Investment Grade
AAA	Aaa	AAA	Gilt-edged, prime, maximum safety, lowest risk, and when sovereign borrower considered "default-free"
AA+	Aa1	AA+	
AA	Aa2	AA	High-grade, high credit quality
AA–	Aa3	AA–	
A+	A1	A+	
A	A2	A	Upper-medium grade
A–	A3	A–	
BBB+	Baa1	BBB+	
BBB	Baa2	BBB	Lower-medium grade
BBB–	Baa3	BBB–	
			Speculative Grade
BB+	Ba1	BB+	
BB	Ba2	BB	Low-grade; speculative
BB–	Ba3	BB–	
B+	B1		
B	B	B	Highly speculative
B–	B3		
			Predominantly Speculative, Substantial Risk or in Default
CCC+		CCC+	
CCC	Caa	CCC	Substantial risk, in poor standing
CC	Ca	CC	May be in default, very speculative
C	C	C	Extremely speculative
		CI	Income bonds—no interest being paid
DDD			
DD			Default
D	D		

Credit Spread Risk

Credit spread is the excess yield over the government or risk-free rate required by the market for taking on a certain assumed credit exposure. The benchmark is typically the on-the-run or *active* U.S. Treasury issue for the given maturity. The relationship between the credit spread and maturity is called the *term structure of credit spreads*. The higher the credit rating, the smaller is the credit spread for a given maturity.

Credit spread risk is the risk of financial loss resulting from changes in the levels of credit spreads used in the marking-to-market of a debt obligation. Changes in observed credit spreads affect the value of the portfolio and can lead to losses for traders or underperformance for portfolio managers. An estimate of this risk is spread duration and duration times spread, two measures we discuss at the end of this chapter.

MEASURING CREDIT DEFAULT RISK

In this section we focus on the measures used in assessing credit default risk.

Measuring Default Rates

In computing a default rate, the first step is obviously to define a default. In studying defaults, Moody's defines a *default* as "any missed or delayed disbursement of interest and/or principal." Once a default is defined, a default rate can be measured from historical data in two ways.

The first and simplest way to measure a default rate is to use the issuer as the unit of observation. A default rate is then measured as the number of issuers that default divided by the total number of issuers at the beginning of the year. This default rate measure is referred to as the *issuer default rate*. This measure gives no recognition to the amount defaulted nor the total amount of issuance. Moody's, for example, has employed this default-rate statistic in its study of default rates.[1] The rationale for ignoring dollar amounts is that the credit decision of an investor does not increase with the size of the issuer.

The second measure is to define the default rate as the par value of all bonds that defaulted in a given calendar year divided by the total par value of all bonds outstanding during that year. This measure is referred to as the *dollar default rate*. Edward Altman, who has performed extensive analyses of default rates for corporate bonds, measures default rates in this way.

With either default-rate statistic, one can measure the default rate for a given year or an average annual default rate over a certain number of years. Researchers

1. Moody's Investors Service, "Corporate Bond Defaults and Default Rates: 1970–1994," *Moody Special Report*, January 1995, p. 13.

who have defined dollar default rates in terms of an average annual default rate over a certain number of years have measured it as

$$\frac{\text{Cumulative \$ value of all defaulted bonds}}{\text{Cumulative \$ value of all issuance} \times \text{ weighted average number of years outstanding}}.$$

Alternatively, some researchers report a cumulative annual default rate. This is done by not normalizing by the number of years. For example, a cumulative annual dollar default rate is calculated as

$$\frac{\text{Cumulative \$ value of all defaulted bonds}}{\text{Cumulative \$ value of all issuance}}.$$

Understanding the different ways in which a default rate is computed is important for interpreting the default rate reported in studies.

Recovery Rate and Loss Given Default

It is possible for a portfolio of credit-risky bonds to suffer defaults but yet outperform Treasuries. This can occur if the yield spread of the portfolio is sufficiently high to offset the losses from default. Consequently, from a portfolio performance perspective, default rates by themselves are not of paramount significance. Furthermore, because holders of defaulted bonds typically recover a percentage of the face amount of their investment, concentrating solely on default rates merely highlights the worst possible outcome that a diversified portfolio of credit-risky bonds would suffer, assuming that all defaulted bonds would be totally worthless.

In assessing default risk, two measures are computed: (1) exposure at default and (2) loss given default. *Exposure at default* (EAD) is the amount of the outstanding obligation if a default occurs. The fraction of EAD that will not be recovered if a default occurs is called *loss given default* (LGD). The *recovery rate* in the case of a default is simply one minus the LGD.

Several studies have focused on recovery rates or LGDs for corporate debt. Measuring the amount recovered is not a simple task. The final distribution to claimants when a default occurs may consist of cash and securities. Often it is difficult to track what was received and then determine the present value of any noncash payments received. Moody's, for example, in its study of recovery rates, uses the trading price at the time of default as a proxy for the amount recovered. The recovery rate is the trading price at that time divided by the par value.

Approaches to Credit-Default-Risk Modeling for Corporate Bonds

Modeling credit default risk is a difficult task. Credit default is a rare event. Corporate default data are considerably less in comparison with the data available for the modeling of interest-rate risk, for which historical U.S. Treasury prices are available on a daily basis for many decades. The sheer diversity of the corporations involved in terms of industry, sector, size, and leverage, coupled with the lack of complete information regarding corporate practices, makes it extremely difficult to use default data to draw any meaningful and possibly predictive conclusions about the likelihood of default of a corporate issuer. Moreover, default has many different causes, ranging from microeconomic factors (such as poor management) to macroeconomic factors (such as high interest rates and recession). These various causes make default very hard to predict. In these cases, default is a result of an inability to pay corporate debtors.

In addition, default is not a universal concept. Different countries have different laws dealing with defaults. In the United States, for example, the Bankruptcy Act of 1978 as amended sets forth the rights of the parties involved in a bankruptcy proceeding. Even where there are laws setting forth the priority of payments in a bankruptcy, the courts have not always followed them. For example, when a company is liquidated, creditors receive distributions based on the *absolute priority rule* to the extent that assets are available. The absolute priority rule is the principle that senior creditors are paid in full before junior creditors are paid anything. For secured creditors and unsecured creditors, the absolute priority rule guarantees seniority to stockholders. In liquidations, the absolute priority rule generally holds. In contrast, there is a good body of literature that argues that strict absolute priority has not been upheld by the courts in the case of corporate reorganizations.

Despite these difficulties, credit risk models have long been employed in the finance and insurance industries. The early models concentrated on default rates, credit ratings, and credit spreads. These traditional models focused on diversification and assumed that credit default risks are idiosyncratic and hence can be diversified away in large portfolios. For single, isolated credits, the models calculate credit spreads as markups onto the risk-free rate.

Today, credit risk models are much more sophisticated, although the theoretical foundation for some of these models dates back to the early 1970s, and the statistical tools employed to estimate the models date back even further.

From credit default risk models, one can obtain metrics such as a *default probability*, which is the likelihood that a borrower will default at some time over the life of the debt obligation. A *rating transition table* can be used for estimating the probability of default. Table 23-2 is an example of a 1-year rating transition table. As will be explained, the cells in the rating transition table can be used to estimate the probability of an issue with a particular rating at the beginning of the period defaulting 1 year later. When calculated for a 1-year horizon, this probability is sometimes referred to as an *expected default frequency*.

Types of Credit Default Risk Models

Credit default risk models can be divided into two types:

- Structural models and
- Reduced-form models.

There are also models that are hybrids of both types.

EXHIBIT 23–2

Hypothetical 1-Year Rating Transition Table

From	AAA	AA	A	BBB	BB	B	CCC/C	D	NR
AAA	85.71	14.29	0	0	0	0	0	0	0
AA	0	91.16	3.87	0.55	0	0	0	0	4.42
A	0	1.48	89.64	4.28	0.16	0	0	0	4.44
BBB	0	0	1.04	90.22	3.13	0.26	0	0	5.35
BB	0.18	0	0.18	3.37	81.53	6.22	0.36	0.18	7.99
B	0	0	0	0.16	5.92	72.64	4.48	1.60	15.20
CCC/C	0	0	1.15	0	2.30	14.94	49.43	16.09	16.09

The basic idea behind all *structural models*, first formulated by Merton,[2] is that a company defaults on its debt if the value of the assets of the company falls below a certain default point. For this reason, these models are also known as *firm-value models*. In these models, it has been demonstrated that default can be modeled as an option. Consequently, researchers were able to apply the same principles used for option pricing to the valuation of corporate bonds. The application of option-pricing theory avoids use of the risk premium and tries to use other marketable securities to price the option. More specifically, the data for a given corporation to value its bonds are its stock price and its balance sheet. Use of the option-pricing theory set forth by Black–Scholes–Merton[3] (BSM) hence provided a significant improvement over traditional methods for valuing corporate bonds. There have been many modifications and extensions of the BSM structural model.[4]

The second type of credit-default-risk models, *reduced-form models*, are more recent. These models, most notably the Jarrow–Turnbull[5] and Duffie–Singleton[6] models, do not look into the microeconomic factors of a company. Rather, they model directly the default probability or downgrade risk using the information in the rating transition table that we describe in more detail later in this chapter and bond market prices and credit derivatives.

2. Robert Merton, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, Vol. 29 (1974), pp. 449–470.
3. Option-pricing theory from which the Merton proposal draws is set forth in Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, Vol. 81 (1973), pp. 637–654; and Robert Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics* (Spring 1973), pp. 141–183.
4. For a review of these models, see Tim Backshall, Kay Giesecke, and Lisa Goldberg "Credit-Risk Modeling," Chapter 42 in Frank J. Fabozzi (ed.), *Handbook of Fixed Income Securities*, 9th ed. (New York: McGraw-Hill, 2021).
5. Robert Jarrow and Stuart Turnbull, "Pricing Derivatives on Financial Securities Subject to Default Risk," *Journal of Finance,* Vol. 50 (1995), pp. 53–86.
6. Darrel Duffie and Kenneth Singleton, "Modelling the Term Structure of Defaultable Bonds," *Review of Financial Studies*, Vol. 12 (1999), pp. 687–720.

Both approaches assume that the information reported by the issuing corporations is accurate. However, recent corporate bankruptcies due to fraud and opaque/inaccurate accounting data reported in financial statements (e.g., Enron, Tyco, and WorldCom) have made practitioners aware that when modeling credit default risk, there must be consideration of the fact that there is imperfect information. A model that combines the structural and reduced-form approaches but incorporates incomplete information has been proposed by Giesecke and Goldberg.[7]

A discussion of how the parameters of credit risk models are estimated is beyond the scope of this chapter.

Default Correlation and Copula

In assessing a portfolio's credit default risk there is the risk that an event that triggers a default of one corporation whose debt obligation is in a portfolio will also adversely impact another corporation whose debt obligation is also in the portfolio, increasing the likelihood of the default of that second corporation. A commonly used statistical concept to quantify the dependence between two random variables is correlation. In credit risk management, this type of risk is referred to as *correlation risk*. The correlation in this case is called *default correlation*. As expected, for firms in the same industry sector, default correlation is high.

Developers of credit risk models need an estimate of the default correlations. Moreover, rating agencies have developed models to estimate default correlation for assessing the credit risk of collateralized debt obligations. The techniques used to estimate the default correlation vary. For example, one way is using Monte Carlo simulation[8] of historical data on rating changes, and defaults are employed in the analysis. Another way is to use correlations based on equity prices changes.

While we just noted that correlation quantifies the dependence between two variables, it should be noted that correlation is often used incorrectly to mean any notion of dependence between two random variables. However, correlation is only one of many measures of dependence between two random variables. There are several reasons why correlation is not a good measure of dependence. We won't dwell on them here. Rather, we'll just focus on one, and that is that the independence of two random variables implies a correlation that is equal to zero. Generally speaking, however, the opposite is not necessarily true: a correlation of zero does not imply independence. To see the relevance for credit default risk modeling, suppose that company A supplies parts to the automotive industry and that there are numerous potential companies that can supply the same parts as company A. From the perspective of company A, defaults of firms in the automotive industry are likely to have severe adverse economic consequences for company A, potentially

7. Kay Giesecke and Lisa Goldberg, "Forecasting Default in the Fact of Uncertainty," *Journal of Derivatives*, Vol. 12 (2004), pp. 14–25.
8. Monte Carlo simulation is discussed in Chapter 33.

leading to bankruptcy of company A. Hence, from company A's perspective or the holder of a debt obligation of company A, there is high default correlation between company A and the automotive industry. However, for the holder of a debt obligation of firms in the automotive industry, the default of company A is unlikely to have any impact on other firms in the automotive industry. Thus, from the perspective of the automotive industry, the default correlation between the automotive industry and company A is likely to be zero.

Because of this and other drawbacks of correlation as a measure of risk, many developers of credit default risk models use a different measure of dependence, *copulas*. The underlying statistical theory to understand copulas is beyond the scope of this book. Suffice it to say that the advantages of using copulas rather than default correlations are that the nature of the dependency between two variables that can be modeled is more general (i.e., it need not be linear), and it can better deal with the modeling of extreme events.

Statistical Models for Predicting Corporate Bankruptcy

An important tool in credit risk management is a *credit-scoring model*. The statistical techniques that have been employed to construct credit-scoring models fall into two categories. The first category seeks to classify borrowers into high- and low-default-risk groups. This category does not attempt to assign a probability to the borrowers in each group. While there are several statistical techniques that are used for classification analysis, the most commonly used in credit-scoring systems is multiple discriminant analysis. The second category seeks to estimate the probability of a borrower defaulting. The three statistical techniques commonly used for this purpose are (1) linear probability models, (2) logit models, and (3) probit models. These models are machine learning models, the subject of Chapter 35.

Below we discuss each of these statistical techniques. All the models have a similar structure, given in general linear form by

$$Y = a + b_1X_1 + b_2X_2 + \cdots + b_KX_K + e,$$

where

Y = dependent variable;
X_k = independent variable $k = 1, 2, \ldots, K$;
K = number of dependent variables;
b_k = coefficient of the independent variable;
e = error term

The independent variables are the factors that have been found using historical data to drive defaults. The parameters b_k ($k = 1, 2, \ldots, K$) are estimated from historical data.

The difference in the four statistical techniques described below is (1) the method used to estimate these parameters and (2) the underlying assumptions that are made.

Multiple Discriminant Analysis

Multiple discriminant analysis (MDA) is a statistical classification technique that is helpful in distinguishing between or among groups of objects and in identifying the characteristics of objects responsible for their inclusion in one or another group. One of the chief advantages of MDA is that it permits a simultaneous consideration of a large number of characteristics and does not restrict the investigator to a sequential evaluation of each individual attribute. For example, MDA permits a credit analyst studying ratings of corporate bonds to examine, at one time, the total and joint impact on ratings of multiple financial ratios, financial measures, and qualitative factors. Thus the analyst is freed from the cumbersome and possibly misleading task of looking at each characteristic in isolation from the others. MDA seeks to form groups that are internally as similar as possibly but that are as different from one another as possible.

From the above description of MDA, it can be seen why it has been applied to the question of why bonds get the ratings they do and what variables seem best able to account for a bond's rating. Moreover, MDA has been used as a predictor of bankruptcy. While the steps involved in MDA for predicting bond ratings and corporate bankruptcies are a specialist topic, we will discuss the results of the work by Edward Altman, the innovator of MDA, for predicting corporate bankruptcy.[9] The models of Altman and others involved in this area are updated periodically. Our purpose here is only to show what an MDA model looks like.

In one of Altman's earlier models, referred to as the *Z-score model*, he found that the following MDA could be used to predict corporate bankruptcy[10]:

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 1.0X_5,$$

where

X_1 = working capital/total assets (in decimal);
X_2 = retained earnings/total assets (in decimal);
X_3 = earnings before interest and taxes/total assets (in decimal);
X_4 = market value of equity/total liabilities (in decimal);
X_5 = sales/total assets (number of times);
Z = Z-score.

Given the value of the five variables for a given firm, a Z-score is computed. It is the Z-score that is used to classify firms with respect to whether or not there is potentially a serious credit problem that would lead to bankruptcy. Specifically, Altman found that a Z-score less than 1.81 indicated a firm with serious credit problems, whereas a Z-score in excess of 3.0 indicated a healthy firm.

9. See Chapters 8 and 9 in Edward I. Altman, *Corporate Financial Distress and Bankruptcy: A Complete to Predicting and Avoiding Distress and Profiting from Bankruptcy* (Hoboken, NJ: John Wiley & Sons, 1993).
10. Edward I. Altman, "Financial Bankruptcies, Discriminant Analysis and the Prediction of Corporate Bankruptcy," *Journal of Finance,* Vol. 23, No. 4 (1968), pp. 589–609.

Subsequently, Altman and his colleagues revised the *Z*-score model based on more recent data. The resulting model, referred to as the *zeta model*, found that the following seven variables were important in predicting corporate bankruptcies and were highly correlated with bond rating[11]:

1. Earnings before interest and taxes (EBIT)/total assets;
2. Standard error of estimate of EBIT/total assets (normalized) for 10 years;
3. EBIT/interest charges;
4. Retained earnings/total assets;
5. Current assets/current liabilities;
6. Five-year average market value of equity/total capitalization;
7. Total tangible assets, normalized.

While credit-scoring models have been found to be helpful to analysts and bond portfolio managers, they do have limitations as a replacement for human judgment in credit analysis. Marty Fridson, for example, provides the following sage advice about using MDA models:

> [Q]uantitative models tend to classify as troubled credits not only most of the companies that eventually default, but also many that do not default. Often, firms that fall into financial peril bring in new management and are revitalized without ever failing in their debt service. If faced with a huge capital loss on the bonds of a financially distressed company, an institutional investor might wish to assess the probability of a turnaround—an inherently difficult-to-quantify prospect—instead of selling purely on the basis of a default model.[12]

Fridson then goes on to explain that credit analysts must bear in mind that "companies can default for reasons that a model based on reported financials cannot pick up" and provides several actual examples of companies that filed for bankruptcy for such reasons.

Linear Probability Model

A *linear probability model* is simply multiple regression analysis in which the dependent variable is a binary variable. A variable that can take on a value of "yes" or "no" or "default on loan" or "nondefault on loan" are examples of binary variables. We focus on application of the linear probability model for predicting the probability of default. Consequently, the binary dependent variable will be *default on loan* or *nondefault on loan*.

11. Edward I. Altman, Robert G. Haldeman, and Paul Narayann, "Zeta Analysis: A New Model to Identify Bankruptcy Risk of Corporations," *Journal of Banking and Finance,* Vol. 1, No. 1 (1977), pp. 29–54.
12. Martin S. Fridson, *Financial Statement Analysis: A Practitioner's Guide*, 2nd ed. (Hoboken, NJ: John Wiley & Sons, 1995), p. 195.

With respect to the underlying assumptions, the linear probability model assumes that there is a linear relationship between the probability of default and the factors that the modeler has determined affect default. The parameters of the model can be obtained by using the method of ordinary least squares and are simply an application of the estimation of a multiple regression model discussed in Chapter 31.[13] Once the parameters of the model are estimated, the predicted value for Y can be interpreted as the event probability or, in our application, the probability of default.

A major drawback of the linear probability model is that the predicted value may be negative. In the two models described next, the predicted probability is forced to be between 0 and 1.

Probit Regression Model

The *probit regression model* is a nonlinear regression model in which the dependent variable is a binary variable. The predicted values are between 0 and 1 because what is being predicted is the standard normal cumulative probability distribution. The general form for the probit regression model is

$$\text{Probability}(Y = 1 \mid \mathrm{X}_1, X_2, \ldots, X_K) = N(a + b_1X_1 + b_2X_2 + \cdots + b_KX_K),$$

where N is the cumulative standard normal distribution function. The method for estimating the parameters is beyond the scope of this chapter.

To illustrate the model, suppose that there are three factors that have been found historically to explain defaults. The probit regression model is then

$$\text{Probability}(Y = 1 \mid \mathrm{X}_1, X_2, X_3) = N(a + b_1X_1 + b_2X_2 + b_3X_3).$$

Suppose that the following parameters are estimated: $a = –2.1$; $b_1 = 1.9$; $b_2 = 0.3$; and $b_3 = 0.8$. Then

$$N(a + b_1X_1 + b_2X_2 + b_3X_3) = N(–2.1 + 1.9X_1 + 0.3X_2 + 0.8X_3).$$

Now suppose that the probability of default of a borrower with the following values for the independent variables is sought: $X_1 = 0.2$; $X_2 = 0.9$; and $X_3 =1.0$. Substituting these values, we get

$$N[–2.1 + 1.9(0.2) + 0.3(0.9) + 0.8(1.0)] = N(–0.65).$$

The standard normal cumulative probability for $N(–0.65)$ is 25.8%. Therefore, the probability of default for a borrower with these characteristics is 25.8%.

Logit Regression Model

As with the probit regression model, the *logit regression model* is a nonlinear regression model in which the dependent variable is a binary variable and the predicted values are between 0 and 1. The predicted value is also a cumulative probability distribution. However, rather than being a standard normal cumulative

13. Note that when using a linear probability model, the coefficient of determination is not used as described in Chapter 31 unless all the independent variables are also binary variables.

probability distribution, it is the standard cumulative probability distribution of a distribution called the *logistic distribution.*

The general formula for the logit regression model is

$$\text{Probability } (Y = 1 \mid X_1, X_2, \ldots, X_K) = F(a + b_1X_1 + b_2X_2 + \ldots + b_KX_K)$$
$$= 1/(1 + e^{-W}),$$

where

$$W = a + b_1X_1 + b_2X_2 + \cdots + b_KX_K.$$

Using our previous illustration, $W = -0.65$. Therefore,

$$1/(1 + e^{-W}) = 1/(1 + e^{-(-0.65)}) = 34.3\%.$$

Hence the probability of default for the borrower with these characteristics is 34.3%.

MEASURING CREDIT SPREAD RISK

Several measures have been developed to measure the exposure of a corporate bond portfolio to changes in the credit spread (i.e., measures of *credit spread risk*). The two most common measures are spread duration and duration times spread, the latter being the market standard measure for credit volatility. One way of assessing the likely change in credit spreads is by looking at the likelihood of an upgrade or downgrade in a bond's credit rating. Because this information can be obtained from a rating transition table, we begin with an explanation of rating transition tables.

Rating Transition Table

One reason why credit spreads change over time is because of the risk that an issuer may be downgraded. A *downgrade* of an issue means that the rating is changed to a lower credit rating. The risk that an issue will be downgraded is referred to as *downgrade risk*. An *upgrade* is a change in a credit rating such that the issue receives a better credit rating. A portfolio manager will benefit if an upgrade can be anticipated before it is announced by a rating agency or anticipated by the marketplace.

To see how ratings change over time, the rating agencies periodically publish this information in the form of a table. As noted earlier in this chapter, this table is called a *rating transition table*. The table is useful for investors to assess potential downgrades and upgrades. A rating transition table is available for different transition periods. Exhibit 23–2 shows a hypothetical 1-year rating transition table. The first column shows the rating at the start of the 1-year period, and the subsequent columns show the rating one year later.

Let's interpret one of the numbers in Exhibit 23–2. Look at the cell that shows the rating at the beginning of the 1-year period is A and the rating 1 year later is A. This cell indicates the percentage of issues rated A at the beginning of the 1-year period that did not change their rating over the 1-year period—that is, issues for which there were no downgrades or upgrades. As can be seen, 89.64% of the issues rated A at the start of the 1-year period were rated A 1 year later. Now look at the cell where the rating at the beginning of the 1-year period is A and 1 year later is BBB. This shows the percentage of issues rated A at the beginning of the 1-year period that were downgraded to BBB 1 year later. In the 1-year rating transition table, this percentage is 4.28%. One can view this figure as a probability. It is the probability that an issue rated A will be downgraded to BBB by the end of the year.

A rating transition table also shows the potential for upgrades. Again, look at the row that shows issues rated A at the beginning of the 1-year period. Looking at the cell shown in the column AA rating 1 year later, there is the figure 1.48%. This figure represents the percentage of issues rated A at the beginning of the 1-year period that were upgraded to AA 1 year later.

In general, the following hold for rating transition tables reported by rating agencies. First, the probability of a downgrade is much higher than for an upgrade for investment-grade bonds. Second, the longer the transition period, the lower is the probability that an issuer will retain its original rating. That is, a 1-year rating transition table will have a lower probability of a downgrade for a particular rating than a 5-year rating transition table for that same rating.

Measuring Changes in Credit Spreads

The credit spread of a credit-risky bond is the difference between the yield offered on that bond and the yield offered on a comparable Treasury security. The change in the credit spread from one period to another period can be calculated simply by computing either the basis point difference in the credit spread or the percentage change in the credit spread.

Letting s_t denote the credit spread of a credit-risky bond at time t and Δst the change in the credit spread for the same bond, then

Basis point change in the credit spread = Δst

Percentage change in the credit spread = $\Delta st\ /st.$

By measuring the change in the credit spread in terms of percentage change, the assumption is that the change in the credit spread is proportional to the level of the credit spread. The implication of this assumption is that credit-risky bonds that have wider credit spreads are likely to have larger spread changes than those with tighter credit spreads. There is empirical support that in the U.S. bond market credit spread changes are proportional to the level of the credit spread.[14]

14. Arik Ben Dor, Lev Dynkin, Jay Hyman, Patrick Houweling, Erik van Leeuwen, and Olaf Penninga, "DTS[SM] (Duration Times Spread)," *Journal of Portfolio Management,* 33(2), 2007: 77–100.

Credit Spread Duration

In earlier chapters we described various duration measures (e.g., modified, effective, and empirical). These measures quantified a bond or bond portfolio's sensitivity to changes in Treasury interest rates. A general definition of these duration measures is that they are the approximate percentage change in the value of a bond, bond sector, or bond portfolio to a 100 basis point change in interest rates. However, to be more specific, it is sensitivity to changes in Treasury rates.

We know, however, that the sensitivity of the price of a bond, value of a bond sector, or value of a bond portfolio is also attributable to changes in spreads in addition to changes in the Treasury rate. *Spread duration* measures the sensitivity of a bond or bond portfolio to changes in any factor that causes the spread to change. It can be interpreted as the approximate percentage change in the price of a bond or bond portfolio for a 100 basis point change in the spread. For example, a spread duration for a bond of 2.8 means that for a 100 basis point change in the bond's spread, the bond's price will change by approximately 2.8%.

A spread duration can be further qualified based on the risk factor that causes the yield spread. For example, for a particular corporate bond issue, the spread may be due primarily to credit risk. Hence the spread duration is labeled *credit spread duration*. However, a portion of the spread could also be due to an issue being callable. One of the objectives of fixed-income analysis is to separate the spread duration attributable to changes in the credit spread and that due to the spread change because of the call option embedded in an issue, referred to as an *option-adjusted spread*.

Duration Times Spread Measure

A superior credit spread sensitivity measure to credit spread duration is a measure developed jointly in the early 2000s by researchers at the asset-management firm Robeco and researchers at the then Lehman Brothers.[15] The measure, *duration times spread* (DTS), is simply the product of the credit spread duration (*duration* in the name of the metric) and credit spread (*spread* in the name of the metric). For example, suppose that a bond has a credit spread duration of 3 and a credit spread of 200 basis points (0.02 in decimal form). Then

$$\text{DTS} = 3 \times 0.020 = 0.060 = 6.0\%$$

Let's look at how DTS is linked to the sensitivity of a bond to a change in credit spreads. First, let's compute the percentage change in price due to a change in the credit spread. To do this, we'll use the following notation:

D_{spread} = credit spread duration;
P = price of a corporate bond;
ΔP_{spread} = price change for a corporate bond or sector due to a change in the credit spread;
Δs = change in the credit spread.

15. This joint research led to the publication of Ben Dor, Dynkin, Hyman, Houweling, van Leeuwen, and Penninga, "DTS (Duration Times Spread)."

The absolute percentage change in price due to a change in the spread is

$$\left|\frac{\Delta P_{\text{spread}}}{P}\right|.$$

Then

$$\left|\frac{\Delta P_{\text{spread}}}{P}\right| = D_{\text{spread}}\,(\Delta s)$$

If we multiply the numerator and the denominator on the right-hand side of this equation by the credit spread s, we get

$$\left|\frac{\Delta P_{\text{spread}}}{P}\right| = D_{\text{spread}}\, s\,\frac{\Delta s}{s}.$$

This equation tells us that the percentage change in price is the product of three terms: credit spread duration (D_{spread}), credit spread (s), and percentage change in the spread ($\Delta s/s$). The product of the first two terms is DTS.

It is important to recognize that DTS only measures the credit risk associated with changes in credit spreads, not default risk. For example, consider the following two bonds with different credit ratings:

Bond	Credit Rating	Credit Spread	Credit Spread Duration
X	BB/Ba	400 basis points	1.5
Y	A	100 basis points	6.0

The DTS for both bonds is 600 and would be expected to have the same credit spread volatility. However, there is more credit default risk with bond X because of its lower credit default risk rating.

In Chapter 14 we explained how to compute the contribution to duration of a bond (or a bond sector) to a portfolio. In the same way, the contribution of a bond or bond sector to a portfolio can be computed as the product of the DTS and the weight of a bond or bond sector in the portfolio. For example, suppose that the bond whose DTS is 6% has a 10% weight in a portfolio, then its contribution to portfolio DTS is

Contribution to portfolio DTS = $0.10 \times 0.06 = 0.006$ = 6 basis points

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CHAPTER

24

MEASURING BOND LIQUIDITY

Liquidity risk is one of the major risks associated with investing in bonds. There are concerns about the deteriorating liquidity in the bond market and how liquidity should be measured. There are a good number of studies by market professionals and academics that have proposed metrics for measuring the liquidity of individual financial instruments and portfolios. While there is an extensive literature on liquidity measures for stocks, the infrequency of trading in most sectors of the bond market makes it difficult to apply these measures to the bond market.

Having reliable liquidity measures that can be used to gauge liquidity risk is critical in risk-management systems. Traditional liquidity measures described in this chapter and the reliance on the Trade Reporting and Compliance Engine (TRACE) data developed by the Financial Industry Regulatory Authority (FINRA) to facilitate the mandatory reporting of over-the-counter fixed-income trades have serious limitations to be relied on in a risk-management system.

There are four critical roles that liquidity measures play in bond portfolio management. The first is its role in risk management. These measures can be used to estimate the time it will take to liquidate positions when seeking to achieve certain targeted parameters for the portfolio under stressed and normal market conditions. This information can be integrated into the risk-management system provided to clients and senior management and, if required, to regulators. For institutional investors who lever their portfolio, liquidity measures can provide guidance in terms of funding needs based on various scenarios, particularly during distressed market conditions. The second role of liquidity measures is that they provide insights when implementing bond investment strategies that use liquidity as a selection or portfolio construction criterion. For a bond portfolio strategy that employs securities lending or repo transactions to provide financing, knowing the liquidity of a bond is critical. The fourth role is that liquidity measures can help in the design of trading strategies, especially in determining targets for holding periods, turnover constraints, and so on. Finally, liquidity measures are important for tail-risk models. When a bond becomes distressed, the direct loss is usually exacerbated by illiquidity.

Fortunately, more data have become available to measure bond liquidity as a result of electronic trading. Coupled with data-based analytics that make the information available to investors, better measures for liquidity, while not perfect, are available to investors. In this chapter we explain the difficulties of measuring liquidity in the bonds and then describe various measures of liquidity.

Asset-management firms seeking to quantify liquidity measures to monitor liquidity risk for individual bond holdings in their portfolios have developed their own measures based on the metrics proposed in the literature. Over time, data vendors have developed metrics for quantifying liquidity measures not only for individual bonds but also for entire portfolios. While at one time asset-management firms used liquidity measures basically without any influence from regulators, this changed in the United States and Europe as liquidity-related regulations were promulgated. At the end of this chapter we briefly describe the reporting of liquidity for certain institutional funds that is now required in the United States.

LIQUIDITY DEFINED

Before quantifying a bond's or a bond portfolio's liquidity, the starting point is defining liquidity. Despite liquidity being an important attribute of any financial market, liquidity is a difficult concept to define precisely. Consequently, there is no uniform definition for liquidity. Here is a sample of some definitions that have appeared in the literature.

In describing the properties of assets, James Tobin, the 1981 recipient of the Nobel Prize in Economic Sciences, described the liquidity property in terms of how much sellers stand to lose if they wish to sell immediately against engaging in a costly and time-consuming search.[1]

The European Union in its regulation of financial markets provided in Article 2 of the Markets in Financial Instruments Regulation (MiFIR) the following definition for a liquid market:

> (a) for the purposes of Articles 9, 11, and 18, a market for a financial instrument or a class of financial instruments, where there are ready and willing buyers and sellers on a continuous basis, and
> where the market is assessed in accordance with the following criteria, taking into consideration the specific market structures of the particular financial instrument or of the particular class of financial instruments:
>
> (i) the average frequency and size of transactions over a range of market conditions, having regard to the nature and life cycle of products within the class of financial instrument;
> (ii) the number and type of market participants, including the ratio of market participants to traded financial instruments in a particular product;
> (iii) the average size of spreads, where available[2]

The Bank for International Settlement (BIS) broadly defines market liquidity as "the ability to rapidly execute large financial transactions with a limited price impact."[3] The Basel Committee of the BIS suggests that the characteristics of what is defined in

1. James Tobin, "Properties of Assets," n.d., Yale University, New Haven, CT.
2. European Securities and Markets Authority, "Article 4—Definitions," Paris, available at https://www.esma.europa.eu/databases-library/interactive-single-rulebook/clone-mifid-ii/article-4-0.
3. Committee on the Global Financial System, "Market-Making and Proprietary Trading: Industry Trends, Drivers and Policy Implications," Bank for International Settlements, Basel, CGFS Papers No. 52, November 2014, p. 4. Available at https://www.bis.org/publ/cgfs52.pdf.

Basel III to be high-quality liquid assets (HQLAs) as assets that "can be easily and immediately converted into cash at little or no loss of value" and goes to the fundamental characteristics and market characteristics of HQLAs, as shown in Exhibit 24–1.

EXHIBIT 24–1

Fundamental and Market Characteristics of High-Quality Liquid Assets: Basel Committee

Fundamental Characteristics

- **Low risk:** assets that are less risky tend to have higher liquidity. High credit standing of the issuer and a low degree of subordination increase an asset's liquidity. Low duration, low legal risk, low inflation risk and denomination in a convertible currency with low foreign exchange risk all enhance an asset's liquidity.
- **Ease and certainty of valuation:** an asset's liquidity increases if market participants are more likely to agree on its valuation. Assets with more standardised, homogeneous and simple structures tend to be more fungible, promoting liquidity. The pricing formula of a high-quality liquid asset must be easy to calculate and not depend on strong assumptions. The inputs into the pricing formula must also be publicly available. In practice, this should rule out the inclusion of most structured or exotic products.
- **Low correlation with risky assets:** the stock of HQLA should not be subject to wrong-way (highly correlated) risk. For example, assets issued by financial institutions are more likely to be illiquid in times of liquidity stress in the banking sector.
- **Listed on a developed and recognised exchange:** being listed increases an asset's transparency.

Market-Related Characteristics

- **Active and sizable market:** the asset should have active outright sale or repo markets at all times. This means that:
  - o There should be historical evidence of market breadth and market depth. This could be demonstrated by low bid-ask spreads, high trading volumes, and a large and diverse number of market participants. Diversity of market participants reduces market concentration and increases the reliability of the liquidity in the market.
  - o There should be robust market infrastructure in place. The presence of multiple committed market makers increases liquidity as quotes will most likely be available for buying or selling HQLA.
- **Low volatility:** Assets whose prices remain relatively stable and are less prone to sharp price declines over time will have a lower probability of triggering forced sales to meet liquidity requirements. Volatility of traded prices and spreads are simple proxy measures of market volatility. There should be historical evidence of relative stability of market terms (eg prices and haircuts) and volumes during stressed periods.
- **Flight to quality:** historically, the market has shown tendencies to move into these types of assets in a systemic crisis. The correlation between proxies of market liquidity and banking system stress is one simple measure that could be used.

Source: Reproduced from Basel Committee on Banking Supervision, "Basel III: The Liquidity Coverage Ratio and Liquidity Risk Monitoring Tools," Bank for International Settlements, Basel, Switzerland, January 2013, pp. 7–8. Available at https://www.bis.org/publ/bcbs238.pdf.

In a 2015 study by PricewaterhouseCoopers on global liquidity, the following definition is provided: "Liquidity is a multi-dimensional concept, generally referring to the ability to execute large transactions with limited price impact, and tends to be associated with low transaction costs and immediacy in execution."[4]

A 2002 working paper by the International Monetary Fund[5] describes the five dimensions of liquid assets:

- *Tightness:* low transaction costs (bid–ask spreads as well as implicit costs);
- *Immediacy:* speed with which orders can be executed;
- *Depth:* the existence of abundant actual orders or orders that can be easily identified;
- *Breadth:* the number of orders of varying size that can be executed at minimal impact on prices;
- *Resiliency:* the flow at which new orders arrive to correct market imbalances.

We suggest that one can think of liquidity as having four main dimensions: (1) cost of trading, (2) market (price) impact, (3) trade size, and (4) execution speed.

LIQUIDITY AND TRANSACTION COSTS

Liquidity and transaction costs are interrelated. A market that is said to be a *highly liquid market* is one where large transactions can be immediately executed without realizing high transaction costs. Because a bond's liquidity will impact the cost of trading, we briefly review the different components of trading/transactions costs. The two most general categories of trading/transaction costs are (1) fixed versus variable transaction costs and (2) explicit versus implicit transaction costs.

Fixed transaction costs are the costs that are independent of factors such as the size of a trade and prevailing market conditions. *Variable transaction costs*, in contrast, depend on some or all of these independent factors, and bond portfolio managers can pursue strategies to reduce, optimize, and efficiently manage variable transaction costs.

Explicit transaction costs are the costs that are observable and known up front. Hence explicit transaction costs are commonly referred to as *observable transaction costs*. Although the largest component in equity portfolio management is commissions, other such costs include transfer fees and any taxes that may be triggered by a trade such as a capital gains tax. In the case of bond trading (as well as equities traded in the over-the-counter market), it is the bid–ask spreads: the

4. PricewaterhouseCoopers, "Global Financial Markets Liquidity Study," London, August 2015, p. 8. Available at https://www.pwc.se/sv/pdf-reports/global-financial-markets-liquidity-study.pdf.
5. Abdourahmane Sarr and Tonny Lybek, "Measuring Liquidity in Financial Markets," IMF Working Paper WP/02/232, International Monetary Fund, Washington, DC, 2002.

difference between the quoted sell and buy orders is called the *bid–ask spread*. This spread is the immediate transaction cost that the market charges for the privilege of trading, with high immediate liquidity corresponding to small spreads. It is effectively the price charged by bond dealers for supplying immediacy and short-term price stability in the presence of short-term order imbalances.

Unlike explicit transaction costs, *implicit transaction costs* are nonobservable and are not known prior to the trade. Typically, implicit transaction costs make up the largest component of the total transaction costs. Also referred to as *nonobservable transaction costs*, the three major ones are investment delay cost, opportunity cost, and market (price) impact cost.

Investment Delay Cost

There is typically a delay between the time when a bond portfolio manager makes a decision to trade (buy/sell a bond) and when the actual trade, communicated to the portfolio team's trader, is brought to the market by a trader. If the price of the bond to be traded changes during this time period, the price change (possibly adjusted for general market moves) represents the *investment delay cost*, or the cost of not being able to execute immediately.

Opportunity Cost

Opportunity cost refers to the cost of not transacting. For example, when a bond trade fails to be executed, the portfolio manager misses an opportunity associated with the expectation when the decision was to make the trade. Typically, this cost is defined as the difference in performance between a portfolio manager's desired investment and the actual investment after transaction costs.

Market Impact Cost

The *market impact cost* or *price impact cost* is the difference between the transaction price and the estimated market price that would have prevailed had the trade not occurred, the so-called no-trade price. The price movement is the cost, the market impact cost, of liquidity. The price impact of a trade is positive when a trade occurs above the no-trade price and negative when the trade occurs at below the no-trade price.

The market impact cost can be decomposed into temporary and permanent impact costs. A *temporary market impact cost* is of transitory nature and is the result of the additional liquidity concession necessary for the market maker to take the order due to inventory imbalances or price incentives to induce market participants to acquire the bond. The *permanent market impact cost* is the persistent price change that occurs because the market adjusts to the information content of the trade. The view here is that a sell trade may suggest to the market that the bond may be overvalued, whereas a buy trade may signal that the bond may be undervalued. Bond prices change when market participants adjust their views and perceptions as they observe news and the information implied by new trades during the trading day.

Another type of market impact cost is *impact cost due to crowding* or, simply, *crowding cost*. The basic view of capacity is that as the number of assets under management increases for a given investment strategy, it becomes more costly to implement that strategy.[6] The adverse impact on investment performance is attributable to an increase in transaction costs. Because many bond managers may employ the same active management strategy and use the same model for valuing bonds, they all may enter the market simultaneously to execute trades on the same bond. This results in an adverse impact on the prices of the bonds that are traded.

Several quantitative models are commonly used in the equity markets to measure market impact. However, as noted by Sommer and Pasquali, because of different levels of transparency in equity and markets, the models used in the equity market to measure market impact costs are not as easily extended to the bond market.[7]

CHALLENGES IN MEASURING LIQUIDITY IN THE BOND MARKET

In previous chapters we described how to measure interest-rate risk and credit risk. Measuring and modeling liquidity risk for bonds has been far more challenging. There are quantitative and qualitative aspects of computing liquidity measures.

The liquidity of a bond is always observed after a trade. The portfolio manager can see the cost of trading a position and the delay in execution. However, despite the importance of knowing the delay in order to measure the speed of execution, while the portfolio manager can observe this, this cannot be observed by the market because it is an over-the-counter trade. It also is difficult to use trade data to measure market depth (i.e., the ability of the market to accommodate large orders to buy or sell). Another factor impacting the liquidity measures is whether actual trades are used or quoted bid–ask spreads.

Although one associates liquidity with the ability to trade, using trading activity of a bond also is insufficient. For example, suppose that just a few institutional investors own an asset-backed security that was purchased at an attractive yield level and that these institutions have no intention of selling the security. Consequently, there are no trading data for this bond, but this does not mean that the bond is illiquid. If any of these institutions decides to sell its position, the bond

6. For an explanation of *capacity* in the equity market and how portfolio managers pursuing an active strategy can deal with it, see Marco Vangelisti, "The Capacity of an Equity Strategy," *Journal of Portfolio Management*, Vol. 32, No. 2 (2006), pp. 44–50.
7. Philip Sommer and Stefano Pasquali, "Liquidity—How to Capture a Multidimensional Beast," *Journal of Trading*, Vol. 11, No. 2 (2016), pp. 21–39.

might be easily sold when it is put out for bids. Therefore, a bond can rarely trade but be highly liquid.

Moreover, looking at how the price moves cannot be relied on to measure liquidity without considering how long it took to trade a position. So a liquidity measure should consider the time it takes to liquidate a position, but it is only the portfolio manager who would know how long it took!

Measuring market impact is particularly difficult in the bond market. On any particular day, relatively few bonds out of the full universe of bonds trade. Round trips with little time between buys and sells are even rarer. So the challenge is to disentangle true price impact from unrelated price moves.

In using market data, a distinction must be made between whether a quote is an indicative quote or an actual transaction price. Unlike the price at which an actual trade takes place, an *indicative quote* provides information at which a market maker may be willing to buy or sell a security. Indicative quotes are subject to change. Yet quotes are important measures because significantly more bonds are quoted on a particular day than traded, offering an advantage of the quote-based approach for measuring liquidity relative to using actual trades. Besides, a quote-based approach for measuring liquidity makes it possible to quantify liquidity a priori, which is more useful for bond portfolio managers.

Moreover, there is a distinction between backward-looking measures and forward-looking measures. Trading volume and measures constructed from those data are backward-looking measures. What is not reported, however, is trades that are not executed because a party interested in transacting could not find a counterparty with whom to trade at the target price and quantity. It is in such cases that indicative prices from dealers offer some insight that can potentially serve as a forward-looking measure.

It should be noted that trade data for U.S. credit products come from TRACE. However, there is a problem with using TRACE data because TRACE imposes a cap on large trades. Yet it is precisely large trades that cause a material price impact that provides insightful information about liquidity to bond portfolio managers.

Given the dimensions of liquidity described earlier, what should be apparent is that each dimension cannot be measured in isolation but rather is a moving part that is strongly interconnected. Touch one, and the measures of the other dimensions move. This means that coming up with one measure of liquidity is pretty much impossible with any degree of certainty.

BOND LIQUIDITY MEASURES

We classify bond liquidity measures into three types: (1) raw data measures of liquidity, (2) adjusted single-datum measures of liquidity, and (3) liquidity scores. Raw data measures of liquidity are simply a single measure based on trade or other market information. Examples include bid/offer spreads, number of quoting sources, number of trades per day, and number of quotes per day. Adjusted single-datum measures of liquidity use the raw data and adjust those data to obtain a better picture of a bond's liquidity. Examples that we briefly describe below are the Amihud index, lambda measure, and Roll's price reversal measure. The need for a liquidity score follows from the fact that in computing a liquidity measure it is probably unreasonable to expect that just one raw measure or adjusted measure can be used to capture all the dimensions of liquidity, but instead multiple sources are needed. Liquidity scores employ statistical analysis to assess the relative liquidity of comparable bonds using multiple sources of liquidity trading information and can be integrated into a portfolio-management system.

Raw Data Measures of Liquidity

The raw data that have been used to measure liquidity are classified in terms of quantity-based versus price-based measures. *Quantity-based measures* use bond trading activity as an indicator of the intensity of bond trading. Specifically, these measures indicate the average trading of a bond where the higher the average trading, the higher the bond's liquidity is believed to be.

Price-based measures are calculated using actual bond trades. These liquidity measures seek to quantify both the price impact and transaction cost dimensions. Corporate bonds with high transactions costs are considered less liquid than bonds with low transactions costs. The concern with price-based measures is that the change in price may not be representative of liquidity should the time period between the two prices be so large that there may be some other factor or factors that impact the price, such as a downgrade of the issuer or change in market spreads. In such instances, it is difficult to distinguish between a measure of liquidity and a change in market conditions.

Examples of raw data measures for liquidity are those provided by Markit and include data such as bid–offer spreads, number of quoting sources, number of quotes per day, number of end-of-day sources, and number of unique price points.

Adjusted Single-Datum Measures of Liquidity

Given raw trading information, the data can be refined to measure different aspects of liquidity. Based on data from the U.S. bond market for the period October 2004 to September 2012, Schestag, Schuster, and Uhrig-Homburg[8] investigated the most appropriate liquidity measures when using benchmark intraday data and proxies based on daily data. Liquidity measures that are based on intraday data fall into two categories:

- Measures that attempt to capture the size of spreads based on prices;
- Measures that try to quantify the notion of the price impact of a trade.

8. Raphael Schestag, Philipp Schuster, and Marliese Uhrig-Homburg, "Measuring Liquidity in Bond Markets," *Review of Financial Studies*, Vol. 29, No. 5 (2016), pp. 1170–1219.

In the first category are the following measures:

- Roll's price reversal measure, a measure of bid–ask spreads, is an implicit spread measure computed using the autocorrelation of observed prices.[9] This measure seeks to exploit the fact that orders to buy and sell arrive randomly, resulting in prices moving between bid and ask prices.
- Round-trip transaction cost estimates of a dealer's cost in market making obtained from the difference between the maximum and minimum purchase price sequences by a client followed by the sale to another dealer or client.[10] This measure serves as a proxy for the bid–ask spread and consequently an estimate of transaction costs.
- A regression-based approach that estimates the transaction cost based on the difference between the trade price and the bid price at a given point in time.[11]
- The interquartile ranges calculated by first computing the difference between the 25th and 75th percentiles of the price distribution and then dividing the difference by the average price over the entire trading day.
- A measure based on the difference between actual and theoretical prices where an econometric model is used to estimate the theoretical price.[12]

The second category of measures looks at the relationship between the size of an order and the price change caused by that order.

- The Amihud index involves first calculating the average of the absolute values of the price changes between two consecutive trades and then dividing the difference by the volume.[13] Because the Amihud index measures the price impact of a trade, it is therefore a measure of the depth and resilience of the market for the specific bond. The intuition is that in liquid markets prices should react relatively little when large quantities are traded.
- The lambda measure involves the use of econometric models as described in Chapter 31 between the return on trades and the signed volumes for all trades for a specific bond on a given day.

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9. Richard R. Roll, "A Simple Implicit Measure of the Effective Bid–Ask Spread in an Efficient Market," *Journal of Finance*, Vol. 39, No. 4 (1984), 1127–1139.
10. Peter Feldhütter, "The Same Bond at Different Prices: Identifying Search Frictions and Selling Pressure," *Review of Financial Studies*, Vol. 25, No. 4 (2012), pp. 1155–1206. Roll reports that this measure has performed better at representing cross-country liquidity effects than do volume-based liquidity measures.
11. This measure was first proposed in Paul Schultz, "Corporate Bond Trading Costs: A Peek Behind the Curtain," *Journal of Finance*, Vol. 56, No. 2 (2001), 677–698.
12. This measure was proposed in Amy K. Edwards, Laurence E. Harris, and Michael S. Piwowar, "Corporate Bond Market Transaction Costs and Transparency," *Journal of Finance*, Vol. 62, No. 3 (2007), pp. 1421–1451.
13. Yakov Amihud, "Illiquidity and Stock Returns: Cross-Section and Time-Series Effects," *Journal of Financial Market*, Vol. 5, No. 1 (2002), pp. 31–56.

Schestag, Schuster, and Uhrig-Homburg also propose measures derived from the above measures that are estimated on a daily frequency based on the proportion of days in a given month during which there is no trading in a bond (referred to as *zero-return days*). For this measure, the higher the zero-return days, the less liquid is a bond.

Because of the difficulty of obtaining data to calculate quantity- and price-based liquidity measures, alternative measures have been developed. Ben Slimane and de Jong describe two alternative ways to measure the liquidity of bonds.[14] The first method is the one proposed by Lesmond, Ogden, and Trzcinka, referred to as the *line optimization technique* (LOT) *method*.[15] This measure counts the number of idle trading days over a trailing time window. The methods proposed by Helwege, Huang, and Wang[16] involve pairwise comparisons between two bonds that have a similar credit risk profile. They attribute the differences in price behavior they observe to liquidity.

Liquidity Scores

In 2005, Houweling, Mentink, and Vorst proposed using the individual characteristics of a bond to generate a bond liquidity score.[17] Several such measures have been developed by third-party data analytics providers and asset managers. Below we describe four of them.

Barclays' Liquidity Cost Score

The proprietary product developed by Barclays, called the *Liquidity Cost Score* (LCS), was introduced in 2009.[18] It is defined as the cost of a standard institutional-sized round-trip transaction. Hence a lower LCS signifies better liquidity. LCS is expressed as a percentage of the bond's price and can be aggregated across bonds in a portfolio and compared over time. LCS is based on Barclays' market activity and as of mid-2021 is computed for about 25,000 bonds with a total market value exceeding $50 trillion.

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14. Mohamed Ben Slimane and Marielle de Jong, "Bond Liquidity Scores," *Journal of Fixed Income*, Vol. 27, No. 1 (2017), pp. 77–82.
15. David A. Lesmond, Joseph P. Ogden, and Charles A. Trzcinka, "A New Estimate of Transaction Costs," *Review of Financial Studies*, Vol. 12, No. 5 (1999), pp. 1113–1141.
16. Jean Helwege, Hing Zhi Huang, and Yuan Wang, "Liquidity Effects in Corporate Bond Spreads," *Journal of Banking and Finance*, Vol. 45, No. 1 (2014), pp. 105–116.
17. Patrick Houweling, Albert Mentink, and Tom Vorst, "Comparing Possible Proxies of Corporate Bond Liquidity," *Journal of Banking and Finance*, Vol. 29, No. 6 (2005), pp. 1331–1358.
18. The LCS description and empirical tests of the model appeared in several articles as well as in books authored by the Barclays Quantitative Portfolio Strategy team. For example, see Vadim Konstantinovsky, Kwok Yuen Ng, and Bruce D. Phelps, "Measuring Bond-Level Liquidity," *Journal of Portfolio Management*, Vol. 42, No. 4 (2016), pp. 116–128; and Siddhartha Dastidar, Ariel Edelstein, and Bruce D. Phelps, "Liquidity Cost Scores (LCS) for Pan-European Credit Bonds," Barclays Research publication, London, 2010.

Calculation of the LCS for a particular bond depends on whether there are real-time bid–ask quotes available from the firm's trading desks, or the bid–ask spreads need to be estimated. If bid–ask quotes are available, LCS is computed in one of two conceptually similar ways depending on whether the bond is quoted on spread to Treasuries (e.g., U.S. dollar investment-grade credit) or price (e.g., U.S. dollar high yield and essentially all non-U.S. bond markets):

$$\text{LCS} = (\text{bid spread} - \text{ask spread}) \times \text{OASD}$$

if the bond is spread quoted, and

$$\text{LCS} = \frac{\text{ask price} - \text{bid price}}{\text{bid price}}$$

if the bond is price quoted, where OASD is the *option-adjusted spread duration*.

As mentioned earlier, a trader's quote may be indicative. For example, it may be issued without considering the size of a potential trade or the urgency with which the client will want to execute. For quotes deemed *indicative*, bid–ask spreads are adjusted higher by a variable factor determined by an objective algorithm that takes into account certain bond and issuer attributes as well as the actual trading volume. LCS is expressed as a percentage of the bond's price and can be aggregated across bonds in a portfolio and compared over time.

Large bond markets include thousands of securities. For example, Bloomberg Barclays USD IG Credit Index has almost 7,500 bonds as of March 2021. It is unrealistic to expect real-time quotes for all of them. Hence LCS needs to be estimated for a certain number of bonds. A cross-sectional multiple linear regression is used to estimate a statistical relationship between bonds' attributes and the observed LCS of quoted bonds. It is assumed that the same relationship holds for nonquoted bonds, and their LCS values are calculated accordingly. Finally, LCS is adjusted upward because a bond without a single trader quote in a month is likely to be less liquid than a quoted bond with similar attributes.

Cross-sectional models vary across markets. Some attributes that are important in one market may not matter or indeed even exist in other markets. Regardless of the market, all independent variables are observable attributes of the bond. Among universally important variables are (1) the bond's age, (2) monthly trading volume, (3) issuer size (amount outstanding, number of bonds from the issuer), (4) duration times spread (DTS),[19] (5) credit rating, (6) currency, and (6) industry sector.[20]

All explanatory variables in these cross-sectional regressions tend to be statistically significant, and their signs are consistent with what market practitioners would expect. For example, LCS is higher (and liquidity lower) for older and smaller issues. Liquidity is better for less risky bonds (lower DTS) and actively traded securities (higher volume). Most of the time, DTS proves to be the most

19. See Chapter 23 for a discussion of this measure.
20. Some variables (e.g., industry sector) may be binary dummies in the regression. Dummy variables are explained in Chapter 31.

significant of the explanatory variables. Trading volume tends to be among the least.

According to Ben Slimane and de Jong, the three main strengths of Barclays' LCS measure are (1) actual market data are used, (2) the rules for assigning an LCS are transparent, and (3) market participants use them.[21]

Ben Slimane–de Jong Liquidity Score

Ben Slimane and de Jong, both on the Amundi asset-management team at the time, proposed a scoring model that uses multiple regression analysis to make cross-sectional comparisons between bid–ask spreads on one day and relates those spreads to a bond's characteristics.[22] Their model, liquidity score (LS), differs from Barclays' LCS model in that all bid–ask spread quotes are replaced by their modeled equivalent values without distinguishing between reliable and unreliable quotes, and a price-dependent explanatory variable is not used (i.e., the score is disconnected from bond prices).

The explanatory variables are classified as group dummy variables and individual bond features. The dummy variables include the classification of the individual bond in terms of Treasury, investment grade, inflation linked, subordinated debt, zero coupon, major currency, financial corporate, and supranational. The individual bond features include issue size, age, time to maturity, coupon, size of issuing group (in U.S. dollars), size of issuing country (in U.S. dollars), numerical credit rating, and carry.

For the estimated model, Ben Slimane and de Jong found that all the explanatory variables were highly statistically significant, and all signs were consistent with expectations. As expected, they found that

- Treasuries are more liquid than non-Treasuries;
- Investment-grade bonds are more liquid than high-yield bonds;
- Inflation-linked securities, subordinated debt, and zero-coupon bonds are relatively illiquid;
- With respect to size, the size of the bond issue is the most dominant determinant of liquidity, followed by the size of the bond issuer;
- The higher the coupon rate for a given credit rating, the more liquid the bond is;
- The higher the credit rating, the more liquid is the bond;
- The greater the bond's age and time to maturity, the better is the liquidity (i.e., on-the-run issues are more liquid than off-the-run issues).

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21. Ben Slimane and de Jong, "Bond Liquidity Scores," *Journal of Fixed Income*, Vol. 27, No. 1 (2017), pp. 77–82.
22. Ibid.

The LS model was tested out of sample by Ben Slimane and de Jong using a database consisting of 200,000 bond trades from year-end 2014 to year-end 2016 created by Amundi's trading desk. They found that their LS model generated fewer volatility scores than Barclays' LCS. They concluded that their model is a "best effort" considering the poor access to data they had and that when market conditions evolve, their model would evolve with them.

Markit's Liquidity Score

IHS Markit, a vendor that combines data and analytics, provides a liquidity score. The firm provides a composite measure of market liquidity that is an ordinal approximation of the many characteristics of liquidity based on observable and trade data. Markit estimates *market breadth* (which it defines as the number of participants in a market) and implied liquidity when trading information is incomplete or securities do not trade. The lower Markit's liquidity score, the higher is the liquidity. There is a liquidity score per period and a number representing the merits per liquidity score input that it uses (i.e., depth, bid–ask, maturity, etc). The score ranges from 1 to 5, and the data used in computing the score are also provided.

Intercontinental Exchange Indicators

The Intercontinental Exchange (ICE) provides bond liquidity metrics that incorporate data from ICE Data Services.[23] ICE uses two approaches to obtain a liquidity indicator: a quantitative model-based approach and a heuristics-driven approach.

For the quantitative model-based approach, ICE uses its proprietary liquidity model that incorporates market data to generate a broad array of liquidity metrics. A large number of cross-sectional multifactor regression models are estimated that result in the assignment of each instrument to a liquidity bucket based on a "reasonable trade size, stress settings, settlement days, and the target market price impact assumption."

The heuristics-driven approach involves determining liquidity buckets and days to liquidate metrics that can be defined with a relatively homogeneous liquidity profile. Then ICE's rules-based approach analyzes the underlying bond characteristics to assign to a bond an indicative liquidity bucket based on certain predefined logic. The reason for pursuing a heuristics-driven approach is that because it is more qualitative in nature, this approach provides an alternative for securities that may exhibit liquidity characteristics that are less suitable for the quantitative model-based approach.

REGULATIONS DEALING WITH LIQUIDITY MEASURES

Because of the importance of liquidity risk, regulators have mandated liquidity requirements and stress testing of liquidity. Here we briefly describe these efforts

23. https://www.theice.com/market-data/pricing-and-analytics/analytics/liquidity.

for the U.S. and European Union markets. There are similar efforts for liquidity stress testing in other countries, many focusing primarily on bank liquidity.

SEC Liquidity Measuring Rules for Open-End Funds and Exchange-Traded Funds

From October 2016 to February 2019, the U.S. Securities Exchange Commission (SEC) adopted a series of rules for open-end investment companies for providing disclosure about a fund's liquidity risks. The focus was on open-end funds because of their need to have sufficient liquidity in their portfolio to satisfy shareholder redemptions while at the same time minimizing any adverse impact on shareholders who have not redeemed their shares. SEC Rule 223-F requires open-end funds (which also cover exchange-traded funds [ETFs]) to establish a liquidity risk-management program. One of the requirements is that for the risk-management program classification of the liquidity of each holding in the portfolio be done at least on a monthly basis.

Rather than assigning a numerical value to each investment holding, SEC Rule 223-F requires classifying them into one of four liquidity categories: (1) highly liquid investments, (2) moderately liquid investments, (3) less liquid investments, and (4) illiquid investments. The classification is based on the "number of days within which it determined that it reasonably expects an investment would be convertible to cash (or, in the case of the less-liquid and illiquid categories, sold or disposed of) without the conversion (or, in the case of the less-liquid and illiquid categories, sale or disposition) significantly changing the market value of the investment."[24] Analytical systems for categorizing a bond holding into each of the four SEC categories are available. One example is BlackRock's Aladdin system.

ESMA Liquidity Stress Tests for Investment Funds

Guidance for liquidity stress testing (LST) of investment funds is provided by the European Securities and Markets Authority (ESMA), the European Union's securities market regulator. The guidelines, published in September 2019 and applicable to alternative investment funds (AIFs) and undertakings for the collective investment in transferable securities (UCITS), mandate that asset managers stress-test the asset liabilities of the funds they manage.

LST provides information on the liquidity of both assets and liabilities in terms of the risk based on scenarios. In establishing LST, managers must make decisions about (1) the risk factors that are expected to impact a fund's liquidity,

24. Securities and Exchange Commission, "Investment Company Liquidity Risk Management Programs," Release No. IC-33010, File No. S7-03-18, Washington, DC, February 22, 2018. Available at https://www.sec.gov/rules/interim/2018/ic-33010.pdf.

(2) the scenarios that are to be used, (3) the severity of the scenario tested, and (4) the measures that should be reported and how they should be used and reported in the fund's risk-management framework. LST should be based on historical and hypothetical scenarios but should not overly rely on historical scenarios. ESMA suggests that the outcomes resulting from the LST should assist fund manager in identifying the potential liquidity weaknesses so as take action to reduce liquidity risk.

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PART

EIGHT

ANALYZING SECURITIZED PRODUCTS

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CHAPTER

25

CASH-FLOW CHARACTERISTICS OF FIXED-RATE AMORTIZING MORTGAGE LOANS

A loan that includes scheduled principal payments as part of the loan payment is called an *amortizing loan*. A loan that requires the borrower to only make interest payments over the term of the loan and repay the entire principal at the end of the loan term is called an *interest only loan*. Another characteristic of a loan is the interest rate. It can be a fixed rate or a floating rate or it can be fixed for a specified period and then float thereafter.

Because of the importance of the residential mortgage market, in this chapter we will focus on the most popular residential mortgage design: the fixed-rate, level-payment, fully amortizing loan. The principles can be applied to other types of mortgage loan designs, as well as other types of assets that have been securitized.

RESIDENTIAL FIXED-RATE, LEVEL-PAYMENT, FULLY AMORTIZING MORTGAGE LOANS

A *mortgage* is a loan secured by the collateral of specified real estate property, which obliges the borrower to make a predetermined series of payments. The mortgage gives the lender (the mortgagee) the right of foreclosure on the loan if the borrower (the mortgagor) defaults. That is, if the borrower fails to make the contracted payments, the lender can seize the property to ensure that the debt is paid off.

A characteristic that mortgages share with callable bonds is that their cash flow is not known with certainty. For callable bonds, this happens because the bondholder has effectively granted the issuer the option to call the issue. For mortgages, the lender has effectively granted the homeowner the right to repay *all* or *any part* of the mortgage balance at any time.

The types of real estate properties that can be mortgaged are divided into two broad categories: *residential* and *nonresidential* properties. The former category includes houses, condominiums, cooperatives, and apartments. Residential real estate can be subdivided into single-family (one- to four-family) structures and multifamily structures (apartment buildings in which more than four families reside). Nonresidential property includes commercial and farm properties. Our primary focus in this section is on one- to four-family residential mortgage loans.

The contract defines the interest rate (i.e., contract rate), the term of the loan, and the frequency of payment. Typically, the latter is monthly. There are many types of mortgage loans from which a borrower can select. We begin with the most common type of mortgage loan: the fixed-rate, level-payment, fully amortized type. By *level payment* it is meant that the monthly payment does not change over the term of the loan. By *fully amortizing* it is meant that the mortgage payments are determined such that the monthly payment includes repayment of the principal as well as interest and that when the last scheduled principal payment is made, there is no mortgage balance outstanding.

DETERMINING THE MONTHLY MORTGAGE PAYMENT

In Chapter 3 we explained the calculation of the present value of an ordinary annuity. To compute the monthly mortgage payment for a level-payment fixed-rate mortgage requires application of the formula for the present value of an ordinary annuity formula, which is

$$PV = A\left\{\frac{1-\left[\frac{1}{(1+i)^n}\right]}{i}\right\},$$

where

PV = present value of an annuity ($);
A = amount of the annuity ($);
i = periodic interest rate;
n = number of periods.

We can redefine these terms to yield a formula for a level-payment fixed-rate mortgage as follows:

$$MB_0 = MP\left\{\frac{1-\left[\frac{1}{(1+i)^n}\right]}{i}\right\},$$

where

MB_0 = original mortgage balance ($);
MP = monthly mortgage payment ($);
i = simple monthly interest rate (annual interest rate/12);
n = number of months.

Solving for the monthly mortgage payment (MP) gives

$$MP = \frac{MB_0}{\left\{\dfrac{1-\left[\dfrac{1}{(1+i)^n}\right]}{i}\right\}}.$$

This can be expressed in a simplified form as follows:

$$MP = MB_0\left\{\frac{i(1+i)^n}{[(1+i)^n-1]}\right\}. \tag{25–1}$$

Illustration 25–1. Suppose that a homeowner enters into a mortgage for \$100,000 for 360 months (30 years) at a mortgage rate of 9.5%. The monthly mortgage payment is determined as follows:

$$\begin{aligned} MB_0 &= \$100{,}000; \\ i &= 0.0079167\ (= 0.095/12); \\ n &= 360. \end{aligned}$$

The monthly mortgage payment is then

$$\begin{aligned} MP &= \$100{,}000\left\{\frac{0.0079167(1.0079167)^{360}}{[(1.0079167)^{360}-1]}\right\} \\ &= \$100{,}000\left\{\frac{0.0079167(17.095)}{(17.095-1)}\right\} \\ &= \$840.85. \end{aligned}$$

CASH FLOW OF A LEVEL-PAYMENT FIXED-RATE MORTGAGE

Each monthly mortgage payment for a level-payment fixed-rate mortgage due on the first of each month consists of

1. Interest of one-twelfth of the fixed annual interest rate times the amount of the outstanding mortgage balance at the beginning of the previous month (interest *in arrears*);
2. Repayment of a portion of the outstanding mortgage balance (principal).

The difference between the monthly mortgage payment and the portion of the payment that represents interest equals the amount that is applied to reduce the outstanding mortgage balance. The monthly mortgage payment is designed so that after the last scheduled monthly payment of the loan is made, the amount of the outstanding mortgage balance is zero (i.e., the mortgage is fully repaid).

Illustration 25–2. Consider the mortgage in Illustration 25–1. Exhibit 25–1 shows how each monthly mortgage payment is divided between interest and repayment of principal. At the beginning of month 1, the mortgage balance is \$100,000, the amount of the original loan. The mortgage payment for month 1 includes interest for the month on the \$100,000 borrowed. Because the interest rate is 9.5%, the monthly interest rate is 0.0079167 (0.095 divided by 12). Interest for month 1 is therefore \$791.67 (\$100,000 times 0.0079167). The portion of the monthly mortgage payment that represents repayment of principal is the difference between the monthly mortgage payment of \$840.85 and the interest of \$791.67. Thus the scheduled principal repayment is \$49.18, and the mortgage balance is reduced by this amount.[1]

The mortgage balance at the end of month 1 (beginning of month 2) is then \$99,950.81 (\$100,00 minus \$49.19). The interest for the second month is \$791.28, the monthly interest rate (0.0079167) times the mortgage balance at the end of month 1 (\$99,950.81). The difference between the \$840.85 monthly mortgage payment and the \$791.28 interest is \$49.57, representing the amount of the mortgage balance paid off with that monthly mortgage payment.

The last line of Exhibit 25–1 shows the last monthly mortgage payment is sufficient to pay off the remaining mortgage balance. When a loan repayment schedule is structured so that the payment made by the borrower will completely pay off the interest and principal, the loan is said to be *self-amortizing*. Exhibit 25–1 is referred to as an *amortization schedule*.

As Exhibit 25–1 clearly shows, *the portion of the monthly mortgage payment applied to interest declines each month, and the portion that goes to reducing the mortgage balance increases*. The reason for this is that as the mortgage balance is reduced with each monthly mortgage payment, the interest on the mortgage balance declines. Because the monthly mortgage payment is fixed, a larger part of the monthly payment is applied to reduce the mortgage balance each month.

It is not necessary to construct an amortization schedule such as Exhibit 25–1 in order to determine the remaining mortgage balance for any month. The following formula can be used:

$$MB_t = MB_0 \left\{ \frac{[(1+i)^n - (1+i)^t]}{[(1+i)^n - 1]} \right\}, \qquad (25\text{–}2)$$

where

MB_t = mortgage balance after t months (\$);
MB_0 = original mortgage balance (\$);
i = simple monthly interest rate (annual interest rate/12);
n = original number of months of mortgage.

1. Because Exhibit 25–1 is computer generated, rounding resulted in the value of \$49.19 shown in the exhibit.

EXHIBIT 25–1

Amortization Schedule for a Level-Payment Fixed-Rate Mortgage

Mortgage loan = $100,000;
Mortgage rate = 9.5%;
Monthly payment = $840.85;
Term of loan = 30 years (360 months).

Month	Beginning Mortgage Balance ($)	Monthly Mortgage Payment ($)	Interest for Month ($)	Principal Repayment ($)	Ending Mortgage Balance ($)
1	100,000.00	840.85	791.67	49.19	99,950.81
2	99,950.81	840.85	791.28	49.58	99,901.24
3	99,901.24	840.85	790.88	49.97	99,851.27
4	99,851.27	840.85	790.49	50.37	99,800.90
5	99,800.90	840.85	790.09	50.76	99,750.14
6	99,750.14	840.85	789.69	51.10	99,698.97
7	99,698.97	840.85	789.28	51.57	99,647.40
8	99,647.40	840.85	788.88	51.90	99,595.42
9	99,595.42	840.85	788.46	52.39	99,543.03
10	99,543.03	840.85	788.05	52.81	99,490.23
11	99,490.23	840.85	787.63	53.22	99,437.00
12	99,437.00	840.85	787.21	53.64	99,383.36
13	99,383.36	840.85	786.78	54.07	99,329.29
14	99,329.29	840.85	786.36	54.50	99,274.79
15	99,274.79	840.85	785.93	54.93	99,219.86

EXHIBIT 25–1

Amortization Schedule for a Level-Payment Fixed-Rate Mortgage (*Continued*)

Month	Beginning Mortgage Balance ($)	Monthly Mortgage Payment ($)	Interest for Month ($)	Principal Repayment ($)	Ending Mortgage Balance ($)
..	....	....	....	....	....
..	....	....	....	....	....
..	....	....	....	....	....
110	91,537.52	840.85	724.67	116.18	91,421.34
111	91,421.34	840.85	723.75	117.10	91,304.24
112	91,304.24	840.85	722.83	118.03	91,186.21
113	91,186.21	840.85	721.89	118.96	91,067.25
114	91,067.25	840.85	720.95	119.91	90,947.34
115	90,947.34	840.85	720.00	120.85	90,826.49
...	....	....	....	....	....
...	....	....	....	....	....
...	....	....	....	....	....
209	74,177.40	840.85	587.24	253.62	73,923.78
210	73,923.78	840.85	585.23	255.62	73,668.16
211	73,668.16	840.85	583.21	257.65	73,410.51

212	73,410.51	840.85	581.17	259.69	73,150.82
213	73,150.82	840.85	579.11	261.74	72,889.08
214	72,889.08	840.85	577.04	263.82	72,625.26
...	....	....	....	....	....
...	....	....	....	....	....
...	....	....	....	....	....
356	4,106.24	840.85	32.51	808.35	3,297.89
357	3,297.89	840.85	26.11	814.75	2,483.14
358	2,483.14	840.85	19.66	821.20	1,661.95
359	1,661.95	840.85	13.16	827.70	834.25
360	834.25	840.85	6.60	834.25	0.00

Illustration 25–3. For the mortgage in Illustration 25–1, the mortgage balance after the 210th month is

$$t = 210;\ MB_0 = \$100{,}000;\ i = 0.0079167;\ n = 360.$$

$$MB_{210} = \$100{,}000\left\{\frac{[(1.0079167)^{360} - (1.0079167)^{210}]}{[(1.0079167)^{360} - 1]}\right\}$$

$$= \$100{,}000\left\{\frac{(17.095 - 5.2381)}{(17.095 - 1)}\right\} = \$73{,}668.$$

This agrees with the ending mortgage balance for month 210 shown in Exhibit 25–1.

Another formula can be used to determine the amount of the scheduled principal repayment in month t:

$$P_t = MB_0\left\{\frac{[i(1+i)^{t-1}]}{[(1+i)^n - 1]}\right\}, \qquad (25\text{–}3)$$

where P_t is scheduled principal repayment for month t.

Illustration 25–4. The scheduled principal repayment for the 210th month for the mortgage in Illustration 25–1 is

$$\$100{,}000\left\{\frac{[0.0079167(1.0079167)^{210-1}]}{[(1.0079167)^{360} - 1]}\right\}$$

$$= \$100{,}000\left\{\frac{[0.0079167(5.19696)]}{(17.095 - 1)}\right\} = \$255.62.$$

Once again, this agrees with Exhibit 25–1.

To compute the interest paid for month t, a final formula can be used:

$$I_t = MB_0\left\{\frac{i[(1+i)^n - (1+i)^{t-1}]}{[(1+i)^n - 1]}\right\},$$

where I_t is interest for month t.

Illustration 25–5. For the 210th month, the interest for the mortgage in Illustration 25–1 is

$$I_{210} = \$100{,}000\left\{\frac{0.0079167[(1.0079167)^{360} - (1.0079167)^{210-1}]}{[(1.0079167)^{360} - 1]}\right\}$$

$$= \$100{,}000\left\{\frac{0.0079167(17.095 - 5.19696)}{(17.095 - 1)}\right\} = \$585.23.$$

This result can be confirmed by examination of Exhibit 25–1.

CASH FLOW AND SERVICING FEE

An investor who owns a mortgage receives the monthly mortgage payment as the *scheduled* cash flow. A mortgage, however, requires monitoring to ensure that the borrower complies with the terms of the mortgage. In addition, the investor must periodically supply certain information to the borrower. These activities are referred to as *servicing* the mortgage. More specifically, servicing of the mortgage involves collecting monthly payments from mortgagors, sending payment notices to mortgagors, reminding mortgagors when payments are overdue, maintaining records of mortgage balances, furnishing tax information to mortgagors, administering an escrow account for real estate taxes and insurance purposes, and, if necessary, initiating foreclosure proceedings. Part of the contract rate includes the cost of servicing the mortgage. The servicing fee is a fixed percentage of the outstanding mortgage balance.

An investor who acquires a mortgage may either service the mortgage or sell the right to service the mortgage. In the former case, the investor's cash flow is the entire cash flow from the mortgage. In the latter case, it is the cash flow net of the servicing fee.

The monthly cash flow from the mortgage therefore can be decomposed into three parts:

1. The amount to service the mortgage;
2. The interest payment net of the servicing fee;
3. The scheduled principal repayment.

Illustration 25–6. Consider once again the 30-year mortgage loan for \$100.000 with a mortgage rate of 9.5%. Suppose that the servicing fee is 0.5% per year. Exhibit 25–2 shows the cash flow for the mortgage assuming this servicing fee. The monthly mortgage payment is unchanged from Exhibit 25–1, and so is the amount of the principal repayment. The difference is that the interest is reduced by the amount of the servicing fee. The dollar amount of the servicing fee, just like the dollar interest, declines each month because the mortgage balance declines.

EXHIBIT 25–2

Cash Flow for a Mortgage with Servicing Fee

Mortgage loan = $100,000;
Mortgage rate = 9.5%;
Servicing fee = 0.5%;
Monthly payment = $840.85;
Term of loan = 30 years (360 months).

Month	Beginning Mortgage Balance ($)	Monthly Mortgage Payment ($)	Net Interest for Month ($)	Servicing Fee ($)	Principal Repayment ($)	Ending Mortgage Balance ($)
1	100,000.00	840.85	750.00	41.67	49.19	99,950.81
2	99,950.81	840.85	749.63	41.65	49.58	99,901.24
3	99,901.24	840.85	749.26	41.63	49.97	99,861.27
4	99,851.27	840.85	748.88	41.60	50.37	99,800.90
5	99,800.90	840.85	748.51	41.58	50.76	99,750.14
6	99,750.14	840.85	748.13	41.56	51.17	99,698.97
7	99,698.97	840.85	747.74	41.54	51.57	99,647.40
8	99,647.40	840.85	747.36	41.52	51.98	99,595.42
9	99,595.42	840.85	746.97	41.50	52.39	99,543.03
10	99,543.03	840.85	746.57	41.48	52.81	99,490.23
11	99,490.23	840.85	746.18	41.45	53.22	99,437.00
12	99,437.00	840.85	745.78	41.43	53.64	99,383.36
13	99,383.36	840.85	745.38	41.41	54.07	99,329.29
14	99,329.29	840.85	744.97	41.39	54.50	99,274.79

15	99,274.79	840.85	744.56	41.36	54.93	99,219.86
..	....	....	....	....	....	....
..	....	....	....	....	....	....
..	....	....	....	....	....	....
110	91,537.52	840.85	686.53	38.14	116.18	91,421.34
111	91,421.34	840.85	685.66	38.09	117.10	91,304.24
112	91,304.24	840.85	684.78	38.04	118.03	91,186.21
113	91,186.21	840.85	683.90	37.99	118.96	91,067.25
114	91,067.25	840.85	683.00	37.94	119.91	90,947.34
115	90,947.34	840.85	682.11	37.89	120.85	90,826.49
..	....	....	....	....	....	....
..	....	....	....	....	....	....
..	....	....	....	....	....	....
201	76,135.92	840.85	571.02	31.72	238.11	75,897.80
209	74,177.40	840.85	556.33	30.91	253.62	73,923.78
210	73,923.78	840.85	554.43	30.80	255.62	73,668.16

EXHIBIT 25–2

Cash Flow for a Mortgage with Servicing Fee (*Continued*)

211	73,668.16	840.85	552.51	30.70	257.65	73,410.51
212	73,410.51	840.85	550.58	30.59	259.69	73,150.82
213	73,150.82	840.85	548.63	30.48	261.74	72,889.08
214	72,889.08	840.85	546.67	30.37	263.82	72,625.26
..	....	....	....	....	....	....
..	....	....	....	....	....	....
..	....	....	....	....	....	....
356	4,106.24	840.85	30.80	1.71	808.35	3,297.89
357	3,297.89	840.85	24.73	1.37	814.75	2,483.14
358	2,483.14	840.85	18.62	1.03	821.20	1,661.95
359	1,661.95	840.85	12.46	0.69	827.70	834.25
360	834.25	840.85	6.60	0.38	834.25	0

CHAPTER

26

CASH-FLOW CHARACTERISTICS OF MORTGAGE-BACKED SECURITIES

A mortgage-backed security (MBS) is created by pooling mortgage loans and using this pool of mortgage loans as collateral for the security. There are three types of mortgage-backed securities: a mortgage passthrough security, a collateralized mortgage obligation, and a stripped MBS. The cash flow of any MBS depends on the cash flow of the underlying mortgage pool. Because the cash flow of an individual mortgage loan is uncertain owing to the potential for prepayments, the same is true for MBS.

In this chapter we focus on the various conventions for projecting the cash flow of an MBS. We show how to project the cash flow for a mortgage passthrough security as well as for the other products—collateralized mortgage obligations and a stripped MBS. Most of the chapter is devoted to agency MBS. We then look at the convention for constructing cash flow for nonagency MBS, for which defaults must be considered, and asset-backed securities backed by a pool of home equity loans.

THE PREPAYMENT OPTION AND THE CASH FLOW

In our illustration of the cash flow from a level-payment fixed-rate mortgage in the previous chapter we assumed that the homeowner would not pay any portion of the mortgage balance off prior to the scheduled due date. But homeowners do pay off some or all of their mortgage balance prior to the maturity date. Payments made in excess of the scheduled principal repayments are called *prepayments.*

Prepayments occur for one of several reasons. First, homeowners pay off the entire mortgage when they sell their home. The sale of a home may be due to (1) a change of employment that necessitates moving, (2) the purchase of a more expensive home (“trading up”), or (3) a divorce settlement requiring sale of the marital residence. Second, borrowers have an incentive to prepay a mortgage loan as mortgage rates in the market fall below the contract rate on the current mortgage. Third, a borrower may want to equitize (i.e., cash out on) the appreciation in the value of a home, and this would require paying off the existing loan and obtaining a new one. A fourth reason occurs when a homeowner cannot meet the contractual mortgage obligation.

In such instances, the property will be repossessed and sold, with the proceeds from the sale used to pay off the mortgage loan. Finally, if the property is destroyed by fire or another insured catastrophe occurs, the insurance proceeds are used to pay off the mortgage. In the next chapter we will discuss prepayment models and provide a more structured discussion of the reasons for prepayments.

The effect of the right to prepay is that the cash flow from a mortgage is not known with certainty. This is true for all mortgage loans, not just level-payment fixed-rate mortgages.

OVERVIEW OF AGENCY MORTGAGE-BACKED SECURITIES

Agency MBS are those issued by Ginnie Mae, Fannie Mae, or Freddie Mac. The securities issued by these three entities represent the largest sector of the MBS market. Ginnie Mae (Government National Mortgage Association) is part of the U.S. Department of Housing and Urban Development, and the securities it issues are backed by the full faith and credit of the U.S. government. Hence there is no credit risk. Fannie Mae (Federal National Mortgage Association) and Freddie Mac (Federal Home Loan Mortgage Corporation) are government-sponsored enterprises (GSEs).[1] While the securities issued by these two GSEs do not carry the full faith and credit of the U.S. government, market participants view them as securities with minimal credit risk. In contrast to agency MBS, there are securities issued by private entities. These securities expose investors to greater credit risk than the securities issued by the GSEs and are referred to as *nonagency MBS* and *residential-backed asset-backed securities*. We will refer to these securities as *credit-sensitive MBS*. Our focus in this section is on agency MBS.

We illustrate the creation of agency MBS. Exhibit 26–1 shows 10 mortgage loans (each loan depicted as a home) and the cash flows from these loans. For the sake of simplicity, we assume that the amount of each loan is $100,000 so that the aggregate value of all 10 loans is $1 million. The cash flows are monthly and consist of three components:

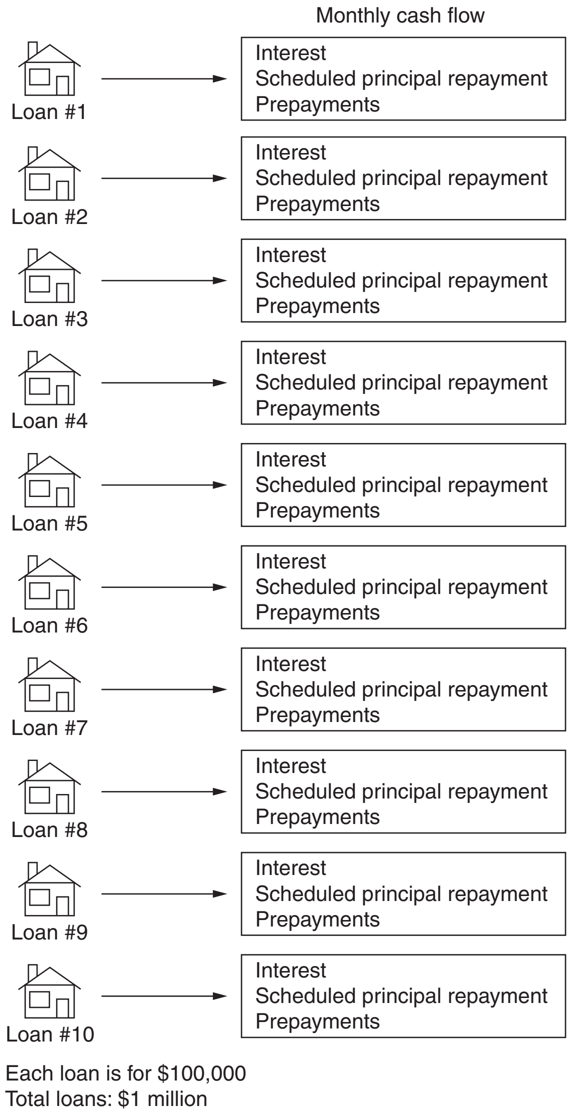
1. Interest;
2. Scheduled principal repayment;
3. Payments in excess of the regularly scheduled principal repayment.

How (1) and (2) are determined is explained in Chapter 25. The third component, payments in excess of the scheduled principal payment, is what we said is a prepayment. It is the amount and timing of this component of the cash flow from a mortgage that make the entire cash flow (and therefore analysis of mortgages and MBS) complicated. This is referred to as *prepayment risk.* Creation of mortgage-backed securities does not alter the total amount of prepayment risk. The distribution of that risk among investors, however, can be altered.

1. Many market participants refer to the securities issued by these two entities as *conventional MBS*.

EXHIBIT 26–1

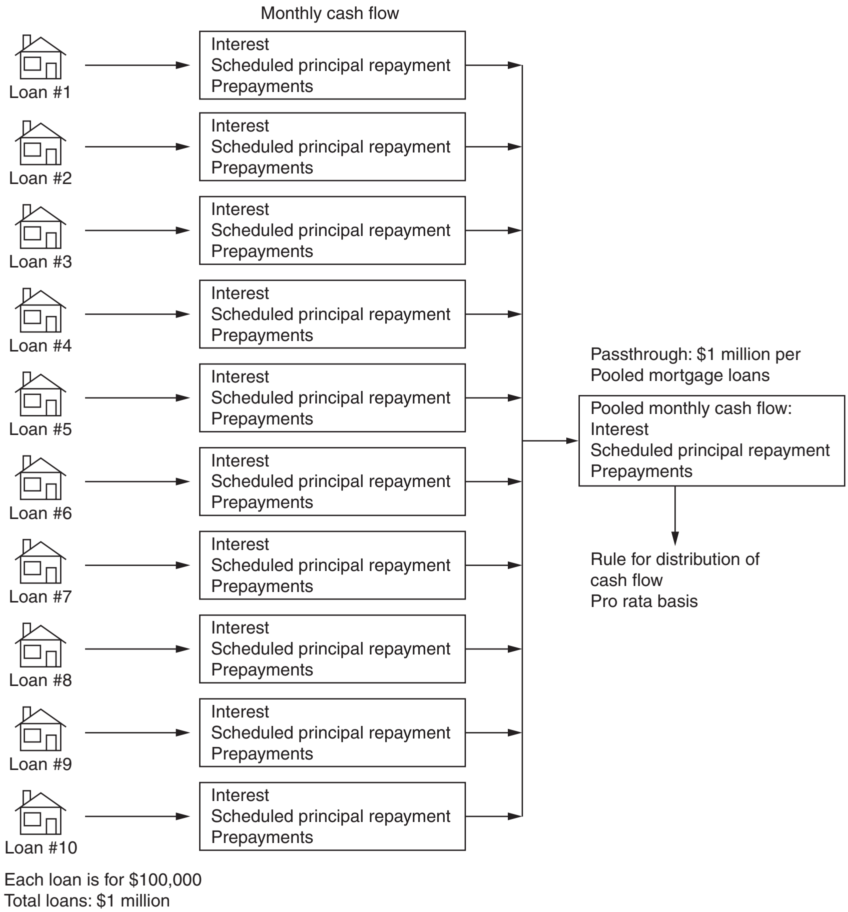
Ten Mortgage Loans



An investor who owns any one of the mortgage loans shown in Exhibit 26–1 faces prepayment risk. For an individual loan it may be difficult to predict prepayments, but if an individual investor purchased all 10 loans, there could be a way to predict prepayments better. In fact, if there were 500 mortgage loans in Exhibit 26–1 rather than 10, we could use historical prepayment experience to improve predictions

EXHIBIT 26–2

Creation of a Passthrough Security



about prepayments. But an investor would have to invest $1 million to buy 10 loans and $50 million to buy 500 loans assuming that each loan is for $100,000.

Agency Mortgage Passthrough Securities

Suppose, instead, that some entity purchases all 10 loans in Exhibit 26–1 and pools them. The 10 loans can be used as collateral for the issuance of a security whose cash flow would reflect the cash flow from the 10 loans, as depicted in Exhibit 26–2. Suppose that there are 40 units of this security issued. Thus each unit is initially worth $25,000 ($1 million divided by 40). Each unit would be entitled to 2.5%

$\left(\frac{1}{40}\right)$ of the cash flow. The security created is called a *mortgage passthrough security* or, simply, a *passthrough.*

Let's see what has been accomplished by creating the passthrough. The total amount of prepayment risk has not changed. Instead, with an investment of less than \$1 million, the investor is now exposed to the total prepayment risk of all 10 loans rather than facing the risk of an individual mortgage loan.

So far this financial "engineering" has not resulted in the creation of a totally new instrument because an individual investor could have achieved the same outcome by purchasing all 10 loans. The passthrough does reduce the \$1 million requirement and increases the liquidity of the security. Moreover, by selling a passthrough, the investor can dispose of all 10 loans rather than having to dispose of each loan one by one. Thus a passthrough can be viewed as a more transactionally efficient vehicle for investing in mortgages than the purchase of individual mortgages.

Mortgage loans that are included in a pool to create a passthrough are said to be *securitized.* The process of creating a passthrough is referred to as the *securitization* of mortgage loans.

Agency Collateralized Mortgage Obligations

An investor in a passthrough is still exposed to the total prepayment risk associated with the pool of mortgage loans underlying the security. A way to change this is instead of distributing the monthly cash flow on a pro rata basis, as in the case of a passthrough, to distribute the principal (both scheduled and prepayments) on some prioritized basis. How this may be done is illustrated in Exhibit 26–3.

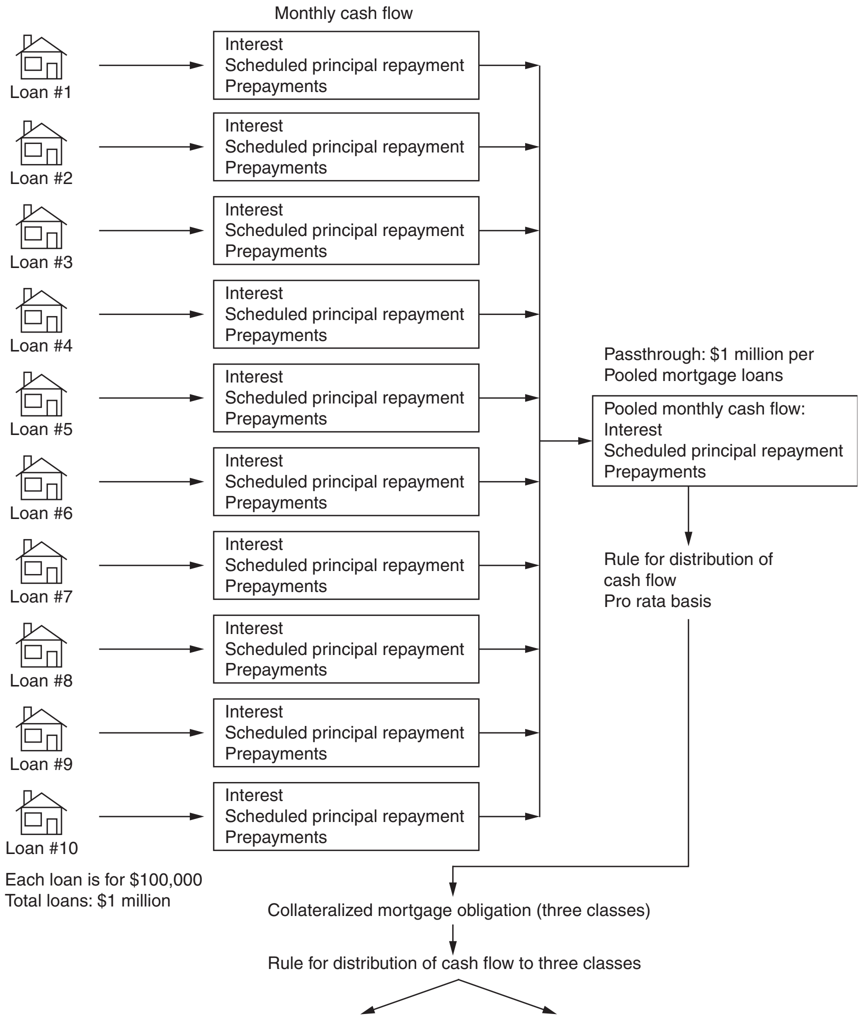
Exhibit 26–3 shows the cash flow of our original 10 mortgage loans and the passthrough. Also shown are three classes of bonds of different par values, with a set of rules indicating how the principal from the passthrough is to be distributed to each. The sum of the par value of the three classes is equal to \$1 million. While it is not shown in the exhibit, for each of the three classes there are units representing a proportionate interest in the class. Each unit then receives a proportionate share of what is received by the class. For example, suppose that for class A, which has a par value of \$400,000, there are 50 units issued. Class A holders then receive proportionate shares of 2% of the distribution of principal.

The rule for the distribution of principal shown in Exhibit 26–3 is that class A will receive all principal (both scheduled and prepayments) until that class receives its entire par value of \$400,000. Then class B receives all principal payments until it receives its par value of \$350,000. Finally, after class B is completely paid off, class C receives principal payments. The rule for the distribution of interest in Exhibit 26–3 indicates that each of the three classes will receive interest based on the amount of par value outstanding.

The MBS that has been created is called a *collateralized mortgage obligation* (CMO). The collateral for a loan may be either one or more passthroughs or a pool of mortgage loans that have not been securitized. The ultimate source for the CMO's cash flow is the pool of mortgage loans.

EXHIBIT 26–3

Creation of a Collateralized Mortgage Obligation



Class (par value)	Interest	Principal
A ($400,000)	Pay each month based on par amount outstanding	Receives all monthly principal until completely paid off
B ($350,000)	Pay each month based on par amount outstanding	After class A paid off receives all monthly principal until completely paid off
A ($250,000)	Pay each month based on par amount outstanding	After class B paid off receives all monthly principal

Let's evaluate what has been accomplished. Once again, the total prepayment risk for the CMO remains the same as the total prepayment risk for the 10 mortgage loans. Creation of three classes, however, means that the prepayment risk is distributed differently among the three classes of the CMO. Class A absorbs prepayments first, next class B, and last of all class C. The result is that class A effectively holds a shorter-term security than the other two classes; class C has the longest maturity. Institutional investors will be attracted to the different classes given the varying nature of their liability structure and the effective maturity of the CMO class. Moreover, the cash-flow distribution rules mitigate the uncertainty about the maturity of each class of the CMO.

Thus, redirection of the cash flow from the underlying mortgage pool creates classes of bonds that may be more attractive to institutional investors to satisfy asset/liability objectives than a passthrough. In the online supplement, where we explain structuring, we will see how different bond classes can be created using different rules for the allocation of principal and interest.

Agency Stripped MBS

The CMO in Exhibit 26–3 specified a set of rules for prioritizing the distribution of the principal payments among the various bond classes. A stripped MBS calls for dividing all the principal and all the interest between two bond classes. One class is entitled to receive all the principal; the other class, all the interest. The former class is called the *principal-only* or *PO* class and the latter the *interest-only* or *IO* class. This is depicted in Exhibit 26–4.

It may be less clear why stripped mortgage-backed securities are created. Suffice it to say that the risk/return characteristics of stripped MBS make them attractive for purposes of hedging a portfolio of passthroughs, creating synthetic securities, and hedging a portfolio of mortgage-related products such as mortgage servicing rights.

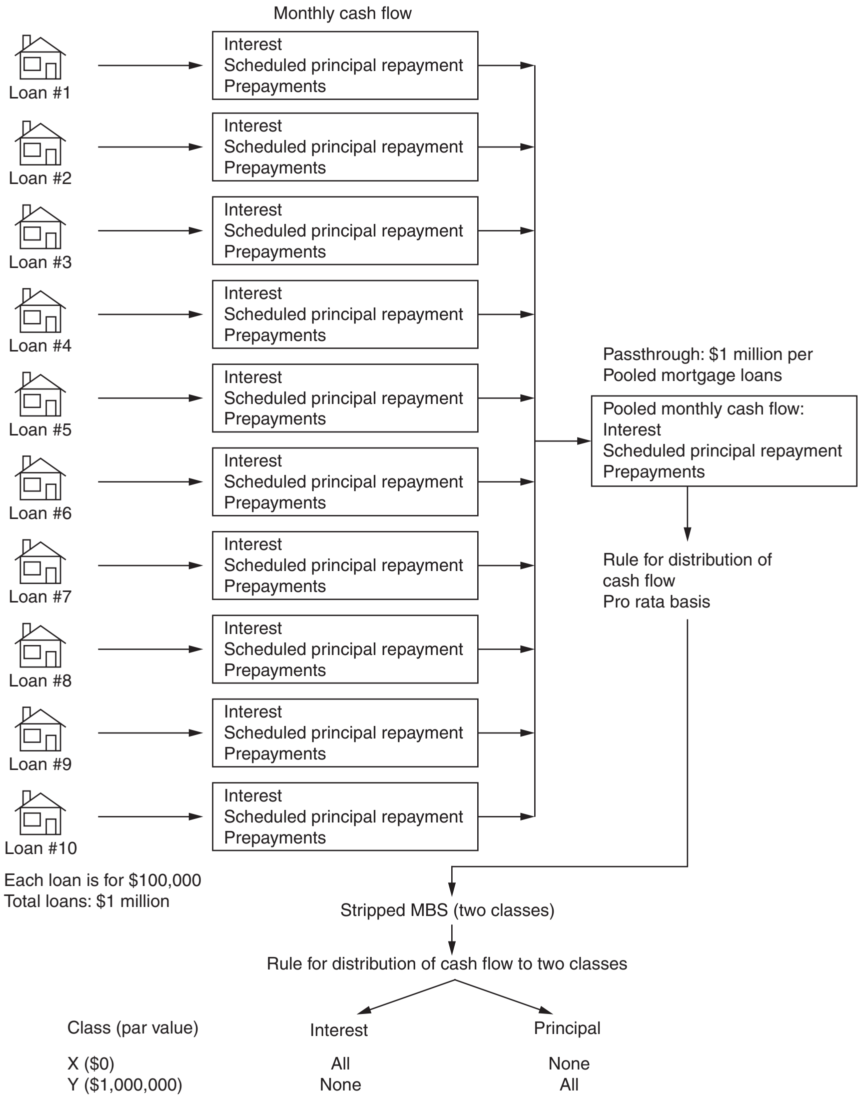
Mortgage Servicing Rights

There is yet another asset whose value depends on the cash flow of a mortgage pool. Consider that the cash flow to investors in Exhibits 26–1 through 26–4 omits an important component: a fee paid to service a mortgage loan. If an investor in an MBS were forced to maintain a staff to service the underlying pool of mortgage loans, these securities would have much less appeal to institutional investors.

Fortunately, the servicing of mortgage loans can be separated from the investment in those loans. This is depicted in Exhibit 26–5, which is similar to Exhibit 26–2 but now shows that the cash flow is separated into two parts: (1) a servicing fee (which is a fixed percentage of the outstanding mortgage balance) and (2) the cash flow minus the servicing fee. The right to service the mortgage loans is an asset whose cash flow is uncertain because of the uncertainty about prepayments, as well as the future costs associated with servicing the mortgages.

EXHIBIT 26–4

Creation of a Stripped Mortgage-Backed Security



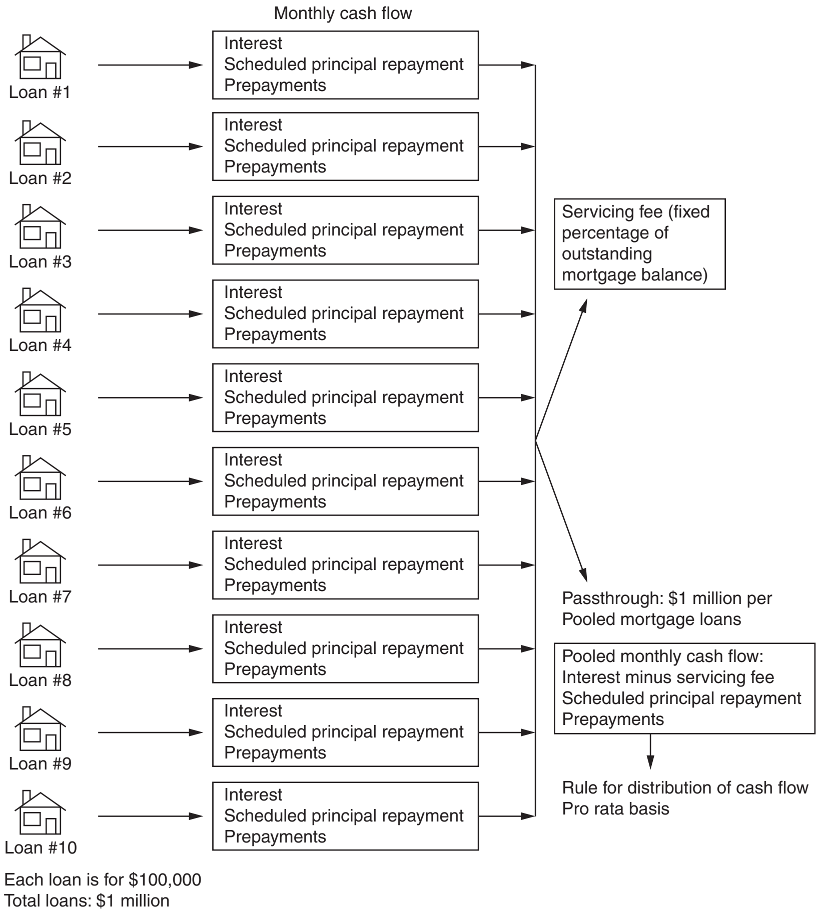
Cash Flow for an Agency Passthrough

An agency passthrough is commonly used as collateral for an agency CMO and an agency stripped MBS. Given the estimated cash flow of the passthrough, the cash flow of CMO bond classes and PO and IO classes can then be estimated.

The cash flow of a passthrough depends on the cash flow of the underlying mortgages. It consists of monthly mortgage payments representing interest, the

EXHIBIT 26–5

Cash Flow When Servicing Is Considered



scheduled repayment of principal, and any prepayments. Payments are made to security holders each month. Neither the amount nor the timing, however, of the cash flow from the pool of mortgages is identical to that of the cash flow passed through to investors. The monthly cash flow for a passthrough is less than the monthly cash flow of the underlying mortgages by an amount equal to servicing and other fees. The other fees are those charged by the issuer or guarantor of the passthrough for guaranteeing the issue.[2] The coupon rate on a passthrough, called the *passthrough rate*, is less than the contract rate on the underlying pool of mortgage loans by an amount equal to the servicing and guaranteeing fees.

2. Actually, the servicer pays the guarantee fee to the issuer or guarantor.

The timing of the cash flow is also different. The monthly mortgage payment is due from each mortgagor on the first day of each month. There is a delay in passing through the corresponding monthly cash flow to the security holders, which varies by the type of agency passthrough. Because of prepayments, the cash flow of a passthrough is not known with certainty.

Prepayment Benchmark Conventions

Estimating the cash flow from a passthrough requires forecasting prepayments. Current practice is to use the Public Securities Association (PSA) prepayment benchmark. This benchmark is based on a series of conditional prepayment rates.

Conditional Prepayment Rate

One way to project prepayments and cash flow is to assume that some fraction of the remaining principal in the mortgage pool is prepaid *each* month for the remaining term of the mortgages. The prepayment rate assumed for a pool, called the *conditional prepayment rate* (CPR), is based on the characteristics of the pool (including its historical prepayment experience) and the current and expected economic environment.

The CPR is an annual prepayment rate. To estimate monthly prepayments, the CPR must be converted into a monthly prepayment rate, commonly referred to as the *single monthly mortality rate* (SMM). The following formula can be used to determine the SMM for a given CPR:

$$\text{SMM} = 1 - (1 - \text{CPR})^{1/12}.$$

Illustration 26–1. Suppose that the CPR used to project prepayments is 6%. The corresponding SMM is

$$\begin{aligned}\text{SMM} &= 1 - (1 - 0.06)^{1/12} \\ &= 1 - (0.94)^{0.083333} = 0.005143.\end{aligned}$$

An SMM of *w%* means that approximately *w%* of the remaining mortgage balance at the beginning of the month after subtracting the scheduled principal payment will prepay that month. That is,

$$\begin{aligned}\text{Prepayment for month} = \text{SMM} \times (&\text{beginning mortgage balance} \\ &- \text{scheduled principal for month}).\end{aligned}$$

Illustration 26–2. Suppose that an investor owns a passthrough whose remaining mortgage balance at the beginning of some month is $50,525. Assuming that the SMM is 0.5143% and that the scheduled principal payment is $67, the estimated prepayment for the month is

$$0.005143 \times (\$50{,}525 - \$67) = \$260.$$

PSA Standard Prepayment Benchmark

The PSA standard prepayment benchmark is expressed as a monthly series of annual CPRs. The PSA benchmark model assumes that prepayment rates will be low for newly originated mortgages and then will speed up as the mortgages become seasoned.

More specifically, the PSA standard prepayment benchmark assumes the following prepayment rates for 30-year mortgages:

1. A CPR of 0.2% for the first month, increased by 0.2% per month for the next 30 months, when it reaches 6% per year;
2. A 6% CPR for the remaining years.

This benchmark is referred to as *100% PSA* and can be expressed as follows:

$$\text{If } t \leq 30, \text{then CPR} = \frac{6\%t}{30};$$
$$\text{if } t > 30, \text{then CPR} = 6\%;$$

where t is the number of months since the mortgage originated.

Slower or faster speeds than the benchmark are then referred to as some percentage of PSA. For example, 50% PSA means one-half the CPR of the PSA prepayment rate; 150% PSA means one-and-a-half the CPR of the PSA prepayment rate. The CPR is converted to an SMM using the formula presented earlier.

Illustration 26–3. The SMM for month 5, month 20, and months 31 through 360, assuming 100% PSA, are calculated as follows:

For month 5:

$$\text{CPR} = \frac{6\%(5)}{30} = 1\% = 0.01;$$
$$\text{SMM} = 1-(1-0.01)^{1/12}$$
$$= 1-(0.99)^{0.083333} = 0.000837.$$

For month 20:

$$\text{CPR} = \frac{6\%(20)}{30} = 4\% = 0.04;$$
$$\text{SMM} = 1-(1-0.04)^{1/12}$$
$$= 1-(0.96)^{0.083333} = 0.003396.$$

For months 31–360:

$$\text{CPR} = 6\%;$$
$$\text{SMM} = 1-(1-0.06)^{1/12}$$
$$= 1-(0.94)^{0.08333} = 0.005143.$$

Illustration 26–4. The SMM for month 5, month 20, and months 31 through 360, assuming 150% PSA, are computed as follows:

For month 5:

$$\begin{aligned}
\text{CPR} &= \frac{6\%(5)}{30} = 1\% = 0.01;\\
150\% \text{ PSA} &= 1.5(0.01) = 0.015;\\
\text{SMM} &= 1 - (1 - 0.015)^{1/12}\\
&= 1 - (0.985)^{0.083333} = 0.001259.
\end{aligned}$$

For month 20:

$$\begin{aligned}
\text{CPR} &= \frac{6\%(20)}{30} = 4\% = 0.04;\\
150\% \text{ PSA} &= 1.5(0.04) = 0.06;\\
\text{SMM} &= 1 - (1 - 0.06)^{1/12}\\
&= 1 - (0.94)^{0.083333} = 0.005143.
\end{aligned}$$

For months 31–360:

$$\begin{aligned}
\text{CPR} &= 6\% = 0.06;\\
150\% \text{ PSA} &= 1.5(0.06) = 0.09;\\
\text{SMM} &= 1 - (1 - 0.09)^{1/12}\\
&= 1 - (0.91)^{0.083333} = 0.007828.
\end{aligned}$$

Notice that the SMM assuming 150% PSA is not just 1.5 times the SMM assuming 100% PSA. It is the CPR that is a multiple of the CPR assuming 100% PSA.

Illustration 26–5. The SMM for month 5, month 20, and months 31 through 360, assuming 50% PSA, are as follows:

For month 5:

$$\begin{aligned}
\text{CPR} &= \frac{6\%(5)}{30} = 1\% = 0.01;\\
50\% \text{ PSA} &= 0.5(0.01) = 0.005;\\
\text{SMM} &= 1 - (1 - 0.005)^{1/12}\\
&= 1 - (0.995)^{0.083333} = 0.000418.
\end{aligned}$$

For month 20:

$$\begin{aligned}
\text{CPR} &= \frac{6\%(20)}{30} = 4\% = 0.04;\\
50\% \text{ PSA} &= 0.5(0.04) = 0.02;\\
\text{SMM} &= 1 - (1 - 0.02)^{1/12}\\
&= 1 - (0.98)^{0.083333} = 0.001682.
\end{aligned}$$

For months 31–360:

$$\begin{aligned} \text{CPR} &= 6\% = 0.06; \\ 50\%\,\text{PSA} &= 0.5\,(0.06) = 0.03; \\ \text{SMM} &= 1-(1-0.03)^{1/12} \\ &= 1-(0.97)^{0.083333} = 0.002535. \end{aligned}$$

Once again, notice that the SMM assuming 50% PSA is not just half the SMM assuming 100% PSA. It is the CPR that is a multiple of the CPR assuming 100% PSA.

Constructing the Projected Cash Flow

We can construct a cash-flow schedule for a mortgage passthrough security on the basis of some assumed prepayment rate (or rates) using several formulas. First, the formula to obtain the projected monthly mortgage payment for any month is

$$\overline{MP}_t = \overline{MB}_{t-1}\left[\frac{i(1+i)^{n-t+1}}{(1+i)^{n-t+1}-1}\right],$$

where

$\overline{MP}_t$ = projected monthly mortgage payment for month t;

$\overline{MB}_{t-1}$ = projected mortgage balance at the end of month t given that prepayments have occurred in the past (which is the projected mortgage balance at the beginning of month t);

i = simple monthly interest rate (annual interest rate/12);

n = original number of months of mortgage.

Second, to compute the portion of the projected monthly mortgage payment that is interest, the formula is

$$\overline{I}_t = \overline{MB}_{t-1}\, i,$$

where $\overline{I}_t$ is projected monthly interest for month t.

This formula states that the projected monthly interest is found by multiplying the mortgage balance at the end of the previous month by the monthly interest rate. The projected monthly interest rate can be divided into two parts: (1) the projected net monthly interest rate after the servicing fee and (2) the servicing fee. The formulas are as follows:

$$\overline{NI}_t = \overline{MB}_{t-1}\,(i-s),$$

$$\overline{S}_t = \overline{MB}_{t-1}\, s,$$

where

$\overline{NI}_t$ = projected interest net of servicing fee for month t;
s = servicing fee rate;
$\overline{S}_t$ = projected servicing fee for month t.

The projected monthly scheduled principal payment is found by subtracting from the projected monthly mortgage payment the projected monthly interest. In terms of our notation,

$$\overline{SP}_t = \overline{MP}_t - \overline{I}_t,$$

where $\overline{SP}_t$ is projected monthly scheduled principal payment for month t.

As explained earlier, the projected monthly principal prepayment is found by multiplying the SMM by the difference between the outstanding balance at the beginning of the month (the ending balance in the previous month) and the projected scheduled principal payment for the month. That is,

$$\overline{PR}_t = SMM_t(\overline{MB}_{t-1} - \overline{SP}_t),$$

where

$\overline{PR}_t$ = projected monthly principal prepayment for month t;

SMM_t = assumed single monthly mortality rate for month t.

The cash flow to the investor is then the sum of (1) the projected monthly interest net of the servicing fee, (2) the projected monthly scheduled principal payment, and (3) the projected monthly principal prepayment. That is,

$$\overline{CF}_t = \overline{NI}_t + \overline{SP}_t + \overline{PR}_t,$$

where $\overline{CF}_t$ is projected cash flow for month t. Alternatively, this can be expressed as

$$\overline{CF}_t = \overline{I}_t + \overline{SP}_t + \overline{PR}_t - \overline{S}_t.$$

The next three illustrations demonstrate the application of these formulas.

Illustration 26–6. Suppose that an investor owns a passthrough with an original mortgage balance of $100 million, mortgage rate of 9.5%, a 0.5% servicing fee, and 360 months to maturity. The passthrough rate is therefore 9%. Suppose that the PSA prepayment benchmark is used to project prepayments for the passthrough. In particular, assume that the investor believes that the mortgages will prepay at 100% PSA.

Using our notation:

$MB_0 = \$100{,}000{,}000$; $n = 360$; $i = 0.0079167\ (0.095/12)$; $s = 0.0004167\ (0.005/12)$.

Exhibit 26–6 shows the cash flow for the passthrough for selected months. The SMMs shown in the third column agree with those computed earlier.

Let's look at the components of the cash flow for the first month. The initial monthly mortgage payment for this passthrough, shown in column (5), is \$841,000. The monthly interest is the monthly interest rate of 0.0079167 (0.095 divided by 12) multiplied by the original mortgage balance, \$100 million. The regularly scheduled principal is the difference between the monthly mortgage payment for the month, \$841,000 in the first month, and the monthly interest of \$792,000. The difference, \$49,000, is shown in column (6).

The prepayment for the month is found by multiplying the SMM for the month by the difference between the mortgage balance at the beginning of the month and the regularly scheduled principal repayment. In the first month, because the SMM is 0.000166 and the difference between the beginning mortgage balance of \$100 million and the projected scheduled principal payment of \$49,000 is \$99,510,000, the projected prepayment for the month is \$17,000, shown in column (8).

The amount of the monthly servicing fee is found by multiplying the mortgage balance at the beginning of the month by the servicing fee. For the first month in our illustration it is 0.0004166 (0.005 divided by 12) multiplied by \$100 million. The product is \$42,000, which is shown in column (9).

The monthly cash flow is then the projected monthly mortgage payment (\$841,000) plus the projected monthly prepayment (\$17,000) minus the amount of the servicing fee (\$42,000). Alternatively, the monthly cash flow is the monthly interest net of servicing (\$792,000 minus \$42,000) plus the projected principal repayment, which consists of the projected monthly regularly scheduled principal repayment (\$49,000) and the monthly projected prepayment (\$17,000). The projected monthly cash flow of \$816,000 is shown in column (10).

Finally, the last column shows the end-of-month mortgage balance, found by subtracting the projected principal repayment from the mortgage balance at the beginning of the month. In the first month, the ending mortgage balance is \$100,000,000 minus \$66,000 (\$49,000 + \$17,000), or \$99,934,000. This amount is then the beginning mortgage balance for month 2. From this amount, the cash flow can be calculated for the second month.

Notice that the monthly mortgage payment declines over time. This is because mortgages in the pool are assumed to be prepaying.

Illustration 26–7. Now suppose that instead of 100% PSA, 150% PSA is assumed. Exhibit 26–7 shows the cash flow for the mortgage passthrough security for selected months. The SMMs shown in column (3) are the same as those computed in Illustration 26–6.

Illustration 26–8. Consider the passthrough we have been using in all previous illustrations. Assuming a prepayment speed of 100% PSA, the projected monthly mortgage payment for month 8 is

$$\overline{MP}_8 = \bar{b}_7 MP.$$

EXHIBIT 26–6

Projected Cash Flow Assuming 100% PSA

Original balance = $100 million – dollar values shown per $100,000 loan;
Mortgage rate = 9.5%;
Servicing fee = 0.5%;
Term = 360 months.

(1)	(2) $\overline{MB}_{t-1}$ (\$)	(3) SMM	(4) $\overline{b}_{t-1}$	(5) $\overline{MP}_t$ (\$)	(6) $\overline{SP}_t$ (\$)	(7) $\overline{I}_t$ (\$)	(8) $\overline{PR}_t$ (\$)	(9) $\overline{S}_t$ (\$)	(10) $\overline{CF}_t$ (\$)	(11) $\overline{MB}_t$ (\$)
1	100,000	0.000166	1.00000	841	49	792	17	42	816	99,934
2	99,934	0.000333	0.99983	841	50	791	33	42	832	99,851
3	99,851	0.000501	0.99950	840	50	790	50	42	849	99,751
4	99,751	0.000669	0.99900	840	50	790	67	42	865	99,634
5	99,634	0.000837	0.99833	839	51	789	83	42	881	99,500
6	99,500	0.001005	0.99749	839	51	788	100	41	897	99,349
7	99,349	0.001174	0.99649	838	51	787	117	41	913	99,181
8	99,181	0.001343	0.99532	837	52	785	133	41	929	98,996
9	98,996	0.001512	0.99398	836	52	784	150	41	994	98,795
10	98,795	0.001682	0.99248	835	52	782	166	41	959	98,576
11	98,576	0.001852	0.99081	833	53	780	182	41	975	98,341
..	....	....	....	...	..	...	...	..	....	....
98	60,735	0.005143	0.65403	550	69	481	312	25	837	60,354
99	60,354	0.005143	0.65067	547	69	478	310	25	832	59,975
100	59,975	0.005143	0.64732	544	70	475	308	25	827	59,597

...	....	....	....	...	..	...	...	..	....	....
209	27,372	0.005143	0.36901	310	94	217	140	11	439	27,138
210	27,138	0.005143	0.36711	309	94	215	139	11	436	26,905
211	$26,905	0.005143	0.36522	$307	$94	$213	$138	$11	$434	$26,673
...	....	....	....	...	..	...	...	..	....	....
358	425	0.005143	0.17115	144	141	3	1	0	145	283
359	283	0.005143	0.17027	143	141	2	1	0	144	141
360	141	0.005143	0.16939	142	141	1	0	0	142	0

Key:

$\overline{MB}_{t-1}$ = projected mortgage balance at the end of month $t - 1$;

SMM_t = single monthly mortality rate;

$\bar{b}_{t-1}$ = $(1 - \text{SMM}_{t-1})(1 - \text{SMM}_{t-2}) \cdots (1 - \text{SMM}_2)(1 - \text{SMM}_1)$;

$\overline{MP}_t$ = projected monthly mortgage payment for month t;

$\overline{SP}_t$ = projected monthly scheduled principal payment for month t;

$\bar{I}_t$ = projected monthly interest for month t;

$\overline{PR}_t$ = projected monthly principal prepayment for month t;

$\bar{S}_t$ = projected servicing fee for month t;

$\overline{CF}_t$ = projected cash flow for month t.

EXHIBIT 26–7

Projected Cash Flow Assuming 150% PSA*

Original balance = $100 million – dollar values per $100,000 loan;
Mortgage rate = 9.5%;
Servicing fee = 0.5%;
Term = 360 months;

(1) t	(2) $\overline{MB}_{t-1}$ ($)	(3) SMM_t	(4) $\bar{b}_{t-1}$	(5) $\overline{MP}_t$ ($)	(6) $\overline{SP}_t$ ($)	(7) $\bar{I}_t$ ($)	(8) $\overline{PR}_t$ ($)	(9) $\bar{S}_t$ ($)	(10) $\overline{CF}_t$ ($)	(11) $\overline{MB}_t$ ($)
1	$100,000	0.000250	1.00000	$841	$49	$792	$25	$42	$824	$99,926
2	99,926	0.000501	0.99975	841	50	791	50	42	849	99,826
3	99,826	0.000753	0.99925	840	50	790	75	42	874	99,701
4	99,701	0.001005	0.99850	840	50	789	100	42	898	99,551
5	99,551	0.001258	0.99749	839	51	788	125	41	923	99,375
6	99,375	0.001512	0.99624	838	51	787	150	41	947	99,174
7	99,174	0.001767	0.99473	836	51	785	175	41	970	98,947
8	98,947	0.002022	0.99297	835	52	783	200	41	994	98,695
9	98,695	0.002278	0.99096	833	52	781	225	41	1,017	98,419
10	98,419	0.002535	0.98871	831	52	799	249	41	1,040	98,117
11	98,117	0.002792	0.98620	829	52	777	274	41	1,062	97,791
..	....	....	....	...	..	...	...	..	....	....
98	48,647	0.007828	0.52386	400	55	385	380	20	801	48,211
99	48,211	0.007828	0.51976	437	55	382	377	20	794	47,779
100	47,779	0.007828	0.51569	434	55	378	374	20	787	47,350

...	....	....	....	...	..	...	...	..	..	....
209	16,241	0.007828	0.21896	184	56	129	127	7	304	16,059
210	16,059	0.007828	0.21724	183	56	127	125	7	301	15,878
211	15,878	0.007828	0.21554	181	56	126	124	7	299	15,699
...	....	....	....	...	..	...	...	..	..	....
358	169	0.007828	0.06789	57	56	1	1	0	58	112
359	112	0.007828	0.06735	57	56	1	0	0	57	56
360	56	0.007828	0.06683	56	56	0	0	0	56	0

*See the key in Exhibit 26–6.

The monthly mortgage payment assuming no principal prepayments is \$840,850. The SMM for each month is shown in column (3) of Exhibit 26–6. Then

$$\bar{b}_8 = (1-0.000166)(1-0.000333)(1-0.000501)(1-0.000669)$$
$$\times(1-0.000837)(1-0.001005)(1-0.001174) = 0.99532.$$

So

$$\overline{MP}_8 = (0.99532)\$840,850 = \$836,915.$$

The projected scheduled principal payment is

$$\overline{SP}_8 = \bar{b}_7 P_8.$$

The scheduled principal payment in month 8 assuming no prepayments is \$52,000. The projected scheduled principal payment is then

$$\overline{SP}_8 = (0.99532)\$52,000 = \$51,757.$$

Notice that both the values computed here agree with the values for month 8 shown in Exhibit 26–6 (\$837,000 for $\overline{MP}_8$ and \$52,000 for $\overline{SP}_8$).

WAC, WAM, WALA, and CAGE

In our illustrations we assumed that all the mortgage loans in the pool have the same mortgage rate (9.5%), the same servicing fee (0.5%), and the same number of months remaining to maturity (360). In practice, a pool of mortgage loans can have different mortgage rates, servicing fees, and remaining months to maturity.

To construct the cash flow for a passthrough, the parameters used for the mortgage rate and the remaining months to maturity are the *weighted-average contract rate* (WAC) and the *weighted-average maturity* (WAM), respectively. The WAC is the weighted average of all the contract rates in the pool, where the weight used for each mortgage loan is the amount of the balance outstanding. The WAM is the weighted average of the number of months to maturity of all the mortgage loans in the pool, where the weight for each mortgage loan is the amount of the balance outstanding.

Once a mortgage is seasoned, additional information is needed about the underlying mortgage pool for a passthrough. Remaining average maturity changes. The WAM after the issuance date is sometimes referred to as the *weighted-average remaining maturity* (WARM) or *weighted-average remaining time* (WART).

For Freddie Mac passthroughs, the remaining number of months are reported. This measure is called the *weighted-average loan age* (WALA). Because of partial prepayments, called *curtailments*, the WALA is not simply the original term of the mortgages less the WARM. A measure similar to WALA to measure the remaining number of months for the underlying mortgages is reported by Fannie Mae. This measure is called the *calculated loan age* (CAGE).

Beware of Prepayment Conventions

The PSA prepayment benchmark is simply a market convention, originally introduced to provide a standard measure for pricing mortgage-backed securities backed by 30-year fixed-rate fully amortizing mortgages. It is a product of a study by the PSA that evaluated the payoff rates of residential loans insured by the FHA. Data that the PSA committee examined seemed to suggest that mortgages became *seasoned* (i.e., prepayment rates tended to level off) after 29 months, at which time the CPR tended to hover at approximately 6%. How did the PSA come up with the CPRs used for months 1 through 29? It was not based on recent empirical evidence of FHA mortgages. Instead, a linear increase from month 1 to month 30 was arbitrarily selected so that at month 1 the CPR assumption is 0.2% and at month 30 the CPR assumption is 6%.

Astute money managers recognize that the CPR is a convenient convention, useful for quoting yield and/or price, but that it also has many limitations in determining the value of a passthrough. The message is that analysts must take care in using any measure that is based on the PSA prepayment benchmark. It is simply a market convention. In fact, there is empirical evidence that suggests that the benchmark may no longer be appropriate. However, despite this evidence, the PSA benchmark continues to be used in the analysis of the cash flow of agency MBS.

CASH FLOW FOR AN AGENCY CMO AND STRIPPED MBS

Given the cash flow for an agency passthrough, we can determine the cash flow for an agency CMO and an agency stripped MBS. This is done by simply following the rules for how the principal repayments (scheduled plus any prepayments) and net interest are to be distributed among the bond classes. In the online supplement we will look at these rules more closely when we discuss the structuring of transactions.

Illustration 26–9. Consider a stripped MBS whose collateral is the passthrough in Illustrations 26–6 and 26–7. Then, for a principal-only security with this collateral, the cash flow assuming 100% PSA can be determined from Exhibit 26–6. The cash flow is only the principal. Because the principal is equal to the sum of the regularly scheduled principal repayment and prepayments, the cash flow for this PO security is the sum of columns (6) and (8) in the exhibit. The cash flow for the corresponding interest-only security assuming the same prepayment speed of 100% PSA is the net interest. In Exhibit 26–6, this is the difference between the gross interest reported in column (7) and the servicing fee in column (9).

Illustration 26–10. Suppose that the same passthrough is used as collateral for a CMO with two bond classes, A and B. Suppose further that the coupon rate for both bond classes is 9.0% (the same as the passthrough in our illustration) and that the par value for class A is $54,333,000 and for class B is $45,667,000. The rules for distributing the cash flow to the two bond classes are as follows.

For the principal:

Pay all principal to class A until it is fully paid off; then pay class B all the principal.

For the interest:

Pay interest to both classes based on the balance outstanding at the beginning of the month.

According to these disbursement rules, and assuming a prepayment speed of 150% PSA, the cash flow for each bond class can be determined from Exhibit 26–7. At the end of month 104 (not shown in the exhibit), the remaining mortgage balance is $45,667,000. This means that $54,333,000 of principal has been repaid. Because this is the amount of the par value assumed for class A, the principal payments to this bond class will be paid off by month 104. The principal payments for class A are therefore those shown in months 1 through 104 in Exhibit 26–7. The principal payment in each month is the sum of columns (6) and (8).

The interest payment is determined by first calculating the balance for class A at the beginning of the month and then multiplying by the passthrough rate, 0.75% (9%/12). For example, at the beginning of month 1, the balance for class A is $54,333,000. The principal payment to class A in month 1 is $74,000 ($49,000 + $25,000). The monthly interest for class A is then $407,498 ($54,333,000 × 0.75%). The balance at the beginning of month 2 is then $54,259,000 ($54,333,000 –$74,000). In month 2, the interest for class A is then $406,943 ($54,259,000 × 0.75). The cash flow for class A is then the principal payments and monthly interest for months 1 through 104.

For class B, the principal payments are as shown in Exhibit 26–7 for months 105 through 360. For months 1 through 104, the monthly interest is simply the difference between the interest to class A and the net interest shown in the exhibit. For months 105 through 360, the interest is the net interest in the exhibit.

CASH FLOW FOR CREDIT-SENSITIVE MBS

For credit-sensitive MBS, it is rare for passthroughs to be created and then subsequently used as collateral for CMOs or stripped MBS. In fact, credit-sensitive stripped MBS are not issued. Credit-sensitive CMO are created when the underlying mortgages loans are the collateral, not passthroughs. Hence credit-sensitive CMOs are also referred to as *whole-loan CMOs*.

For agency MBS, no adjustment for defaults is taken into account in projecting the cash flow because these securities are viewed as having no or minimal credit risk. For credit-sensitive CMOs, the cash flow is distributed to different bond classes based on the level of seniority within the structure. As explained in the online supplement, typically a structure will have senior and subordinate bond classes. The prospectus supplement will describe how any losses are to be absorbed by the bond classes. In addition, unlike agency CMOs, which do not make a distinction between principal repayment from regularly scheduled principal payments (amortization) and prepayments, for credit-sensitive CMOs there are different rules for the distribution to bond classes for each type of principal payment. We discuss this further in the online supplement.

Consequently, for credit-sensitive CMOs, a prepayment model and a default model must be used to project cash flows. With respect to prepayments, in the early years of the market, the practice was to use some multiple of the PSA benchmark. The market soon realized that there are differences in the prepayment behavior for the types of loans used as collateral in credit-sensitive deals and those used for agency CMOs. Wall Street firms involved in the underwriting and market making of credit-sensitive CMOs developed prepayment benchmarks based on the type of loan.

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CHAPTER

27

ANALYSIS OF AGENCY MORTGAGE-BACKED SECURITIES

With our understanding of the characteristics of mortgages and agency mortgage-backed securities (MBS), we can now look at the various methodologies for analyzing MBS. Specifically, we will look at two valuation models: the static cash-flow model and the Monte Carlo simulation model. We will also discuss how to estimate the effective duration of an MBS for each valuation model. As explained in the last section of this chapter, the principles and methodologies discussed in this chapter can also be applied to other securitized products.

STATIC CASH-FLOW MODEL

The static cash-flow model involves estimating the cash flow based on a single-prepayment assumption. No consideration is given to how changes in interest rates in the future will affect the cash flows (i.e., prepayments).

Cash-Flow Yield

Recall from Chapter 5 that the yield is the interest rate that will make the present value of the expected cash flows equal to the price (plus accrued interest). A yield computed in this manner is called a *cash-flow yield.*

The cash flow for an MBS is typically monthly. Thus the cash-flow yield as just described must be annualized. The convention is to compare the yield on an MBS with that of Treasury coupon securities by calculating the bond-equivalent yield of the MBS. Recall that the bond-equivalent yield for a Treasury coupon security is found by doubling the semiannual yield. It is not correct to do this for an MBS. The reason is that because an MBS pays monthly, the investor has the opportunity to generate greater interest by reinvesting the cash flows.

The market practice is to calculate a yield so as to make it comparable to the yield to maturity on a bond-equivalent yield basis. The formula for annualizing the monthly cash-flow yield on an MBS is

$$\text{Bond-equivalent yield} = 2\,[(1 + i_M)^6 - 1],$$

where i_M is the interest rate that will equate the present value of the projected monthly cash flow equal to the price of the MBS (plus accrued interest). The bond-equivalent semiannual yield for a monthly pay MBS is

$$(1 + i_M)^6 - 1.$$

Drawbacks of the Cash-Flow Yield Measure

As we noted several times throughout this book, the yield to maturity has two shortcomings as a measure of a bond's potential return: (1) the coupon payments must be reinvested at a rate equal to the yield to maturity, and (2) the bond must be held to maturity. These shortcomings are equally applicable to the cash-flow yield measure: (1) the projected cash flows are assumed to be reinvested at the cash-flow yield, and (2) the MBS is assumed to be held until the final payout based on some prepayment assumption.

The importance of reinvestment risk, the risk that the cash flows must be reinvested at a rate below the cash-flow yield, is particularly important for most MBS because payments are monthly or quarterly. In addition, the cash-flow yield is dependent on realization of the projected cash flows according to some prepayment rate. If the prepayment experience is different from the prepayment rate assumed, the cash-flow yield will not be realized.

Spread to Treasuries

It should be clear that at the time of purchase it is not possible to determine an exact yield for an MBS; the yield depends on the actual prepayment experience of the mortgages in the pool. Nevertheless, the convention in all fixed-income markets is to measure the yield on a non-Treasury security to that of a "comparable" Treasury security.

In comparing the yield of an MBS to a comparable Treasury, it is inappropriate to use the stated maturity of the MBS because of the potential for prepayments. Instead, market participants have used two measures: Macaulay duration and average life.

Macaulay Duration

Calculating Macaulay duration requires a projection of the cash flows, which, in turn, requires a prepayment-rate assumption. We can then use the projected cash flows, the price of the MBS, and the periodic interest rate (computed from the yield on a bond-equivalent basis) to compute Macaulay duration, as illustrated in Chapter 13. Macaulay duration is converted into years by dividing the periodic Macaulay duration by 12 in the case of a monthly-pay MBS.

Illustration 27–1. For the passthrough in Illustration 27–1 selling for $94,521 and yielding 10.21% (assuming 100% PSA), Macaulay duration is 6.17, as shown below.

The numerator of the Macaulay duration is the present value of the projected cash flows using a monthly interest rate of 0.8333% times the time period (the month). For our passthrough, the numerator is \$6,998,347. Macaulay duration is then found by dividing by the price of the passthrough.Thus

$$\text{Macaulay duration (in months)} = \frac{\$6{,}998{,}347}{\$94{,}521} = 74.04.$$

To convert the Macaulay duration in months to Macaulay duration in years:

$$\text{Macaulay duration (in years)} = \frac{74.04}{12} = 6.17.$$

Average Life

A second measure commonly used to compare Treasury securities and MBS is the average life (or weighted-average life), which is the average time to receipt of principal payments (projected scheduled principal and projected principal prepayments) weighted by the amount of principal expected divided by the total principal to be repaid.

Mathematically, the average life is expressed as follows:

$$\text{Average life} = \frac{1}{12}\sum_{t=1}^{n} \frac{t\ (\text{principal received at time } t)}{\text{total principal received}},$$

where n is the number of months remaining. (The *tail* is defined as the principal payments that extend from the average life to the last principal payment.)

Price

Given the required yield for an MBS, the price is simply the present value of the projected cash flows. Care must be taken, however, in determining the monthly interest rate that should be used to compute the present value of each cash flow.

To convert a bond-equivalent yield to a monthly interest rate, we can use the following formula:

$$i_M = [1 + (0.5)\text{bond-equivalent yield}]^{1/6} - 1.$$

Illustration 27–2. Suppose that the investor wants a yield of 12.30%. Also assume that the investor believes that a 25% PSA rate is appropriate to project the cash flow. The monthly interest rate is determined as follows:

$$\begin{aligned} i_M &= [1 + (0.5)0.1230]^{1/6} - 1 \\ &= (1.0615)^{0.16667} - 1 = 0.01. \end{aligned}$$

The cash flows of IO and PO securities depend on the cash flows of the underlying passthrough, which, in turn, depend on the cash flows of the underlying pool of mortgages. Thus, to determine the price of an IO or PO, a prepayment rate must be assumed. The price of an IO is the present value of the projected interest payments net of the servicing fee. The price of a PO is the present value of the projected principal payments (projected scheduled principal payments and projected principal prepayments).

Effective Duration and Convexity

Modified duration is a measure of the sensitivity of a bond's price to interest-rate changes assuming that the expected cash flows do not change with interest rates. Consequently, modified duration (as well as Macaulay duration) is not an appropriate measure for MBS because mortgage prepayments mean that the projected cash flows change as interest rates change. When interest rates decline (rise), prepayments are expected to rise (fall). As a result, when interest rates decline (increase), duration may decline (increase) rather than increase (decline). This property, as we explained in Chapter 21, is referred to as *negative convexity*.

To see this effect, consider the modified duration for the 9.5% passthrough (assuming 100% PSA) selling to yield 10.21%. If the required yield decreases to 8.14% instantaneously, and the prepayment rate is assumed not to change, modified duration will increase from 6.17 to 6.80. Suppose, however, that when the yield declines to 8.14%, the assumed prepayment rate changes to 150% PSA. The modified duration would decline to 5.91 rather than increase.

Negative convexity has the same impact on the price performance of an MBS as it does on a callable bond (discussed in Chapter 21). When interest rates decline, a bond with an embedded call option such as an MBS will not perform as well as an option-free bond.

For example, if the required yield decreases instantaneously from 10.21% to 8.14%, the price will increase from \$94,521 to \$107,596. If the prepayment rate increases to 150% PSA, however, the price will rise to only \$106,710.

Effective Duration

The proper measure to use is the effective duration. This measure allows for the cash flow to change when interest rates change. The formula to approximate duration is

$$\text{Approximate duration} = \frac{V_{-} - V_{+}}{2\, V_0 (\Delta y)},$$

where

V_0 = initial value or price of the security;
V_{-} = estimated value of the security if the yield is decreased by Δy;
V_{+} = estimated value of the security if the yield is increased by Δy;
Δy = change in the yield of a security.

The approximate duration is the effective duration when the two values V_{-} and V_{+} are obtained from a valuation model that allows for the cash flow to change due to prepayments when interest rates change. The values are obtained from a valuation model. The two models that we are discussing in this chapter are the static cash-flow model and the Monte Carlo simulation model. Here we focus on the values obtained from the former model.

When computing effective duration using the static cash-flow model, the price at the higher and lower interest rates will depend on the prepayment rate assumed. A higher prepayment rate is typically assumed at the lower interest rate than at a higher interest rate. Thus calculation of effective duration requires a prepayment model to determine how prepayments are expected to change as interest rates change.

Drawback of the Static Cash-Flow Model

For the static cash-flow yield model, the yield spread, which is referred to as the *nominal spread*, for an MBS is the difference between the cash-flow yield and the yield to maturity of a comparable Treasury. The latter is obtained from the yield curve. The drawback of this procedure is that neither the cash-flow yield for an MBS nor the Treasury security is calculated properly because it fails to take into consideration (1) the term structure of interest rates for Treasuries (i.e., the theoretical spot-rate curve) and (2) expected interest-rate volatility that will alter the expected cash flow for the passthrough due to changes in prepayments. The next valuation model discussed overcomes these drawbacks.

MONTE CARLO SIMULATION MODEL

Conceptually, the valuation of passthroughs using the Monte Carlo simulation model is simple. In practice, however, it is very complex. The simulation involves generating a set of cash flows based on simulated future mortgage refinancing rates, which, in turn, imply simulated prepayments rates.

Valuation modeling for CMOs is similar to valuation modeling for passthroughs, although the difficulties are amplified because the issuer has sliced and diced both the prepayment and interest-rate risk into smaller pieces called *bond classes* or *tranches*. The sensitivity of the passthroughs comprising the collateral to these two risks is not transmitted equally to every tranche. Some of the tranches wind up more sensitive to prepayments and interest-rate risk than the collateral, while some of them are much less sensitive.

Using Simulation to Generate Interest-Rate Paths and Cash Flows

The typical model that Wall Street firms and commercial vendors use to generate random interest-rate paths takes as input today's term structure of interest rates and a volatility assumption. The term structure of interest rates is the theoretical spot-rate (or zero-coupon) curve implied by today's Treasury securities, although any benchmark can be used (such as LIBOR), with the output of the model interpreted accordingly. In our discussion in this chapter we will assume that the benchmark is generated from the U.S. Treasury market. The volatility assumption determines the dispersion of future interest rates in the simulation. The simulations should be calibrated so that the average simulated price of a zero-coupon Treasury bond equals today's actual price.

Each model has its own model of the evolution of future interest rates[1] and its own volatility assumptions. Typically, there are no significant differences in the interest-rate models of dealer firms and vendors, although their volatility assumptions can be significantly different.

The random paths of interest rates should be generated from an arbitrage-free model of the future term structure of interest rates. By arbitrage free it is meant that the model replicates today's term structure of interest rates, an input of the model, and that for all future dates there is no possible arbitrage within the model.

The simulation works by generating many scenarios of future interest-rate paths. In each month of the scenario, a monthly interest rate and a mortgage refinancing rate are generated. The monthly interest rates are used to discount the projected cash flows in the scenario. The mortgage refinancing rate is needed to determine the cash flow because it represents the opportunity cost the mortgagor is facing at that time.

If the refinancing rates are high relative to the mortgagor's contract rate, the mortgagor will have less incentive to refinance. If the refinancing rate is low relative to the mortgagor's contract rate, the mortgagor has an incentive to refinance.

Prepayments are projected by feeding the refinancing rate and loan characteristics, such as age, into a prepayment model. Given the projected prepayments, the cash flow along an interest-rate path can be determined.

To make this more concrete, consider a newly issued mortgage passthrough security with a maturity of 360 months. Exhibit 27–1 shows N simulated interest-rate path scenarios. Each scenario consists of a path of 360 simulated 1-month future interest rates. Just how many paths should be generated is explained later. Exhibit 27–2 shows the paths of simulated mortgage refinancing rates corresponding to the scenarios shown in Exhibit 27–1. Assuming these mortgage refinancing rates, the cash flow for each scenario path is shown in Exhibit 27–3.

Calculating the Present Value for a Scenario Interest-Rate Path

Given the cash flow on an interest-rate path, its present value can be calculated. The discount rate for determining the present value is the simulated spot rate for each month on the interest-rate path plus an appropriate spread. The spot rate on a path can be determined from the simulated future monthly rates. The relationship that holds between the simulated spot rate for month T on path n and the simulated future 1-month rates is

$$z_T(n) = \{[1 + f_1(n)]\,[1 + f_2(n)] \cdots [1 + f_T(n)]\}^{1/T} - 1,$$

where

$z_T(n)$ = simulated spot rate for month T on path n;
$f_j(n)$ = simulated future 1-month rate for month j on path n.

1. Interest-rate models are described in Chapter 19.

EXHIBIT 27–1

Simulated Paths of 1-Month Future Interest Rates

	Interest-Rate Path Number						
Month	**1**	**2**	**3**	. . .	***n***	. . .	***N***
1	$f_1(1)$	$f_1(2)$	$f_1(3)$	. . .	$f_1(n)$	. . .	$f_1(N)$
2	$f_2(1)$	$f_2(2)$	$f_2(3)$	. . .	$f_2(n)$	. . .	$f_2(N)$
3	$f_3(1)$	$f_3(2)$	$f_3(3)$	. . .	$f_3(n)$	. . .	$f_3(N)$
t	$f_t(1)$	$f_t(2)$	$f_t(3)$	. . .	$f_t(n)$	. . .	$f_t(N)$
358	$f_{358}(1)$	$f_{358}(2)$	$f_{358}(3)$	. . .	$f_{358}(n)$	. . .	$f_{358}(N)$
359	$f_{359}(1)$	$f_{359}(2)$	$f_{359}(3)$	. . .	$f_{359}(n)$	. . .	$f_{359}(N)$
360	$f_{360}(1)$	$f_{360}(2)$	$f_{360}(3)$	. . .	$f_{360}(n)$		$f_{360}(N)$

Key:

$f_t(n)$ = 1-month future interest rate for month t on path n;

N = total number of interest-rate paths.

EXHIBIT 27–2

Simulated Paths of Mortgage Refinancing Rates

	Interest-Rate Path Number						
Month	**1**	**2**	**3**	. . .	***n***	. . .	***N***
1	$r_1(1)$	$r_1(2)$	$r_1(3)$	. . .	$r_1(n)$	. . .	$r_1(N)$
2	$r_2(1)$	$r_2(2)$	$r_2(3)$	. . .	$r_2(n)$	. . .	$r_2(N)$
3	$r_3(1)$	$r_3(2)$	$r_3(3)$	. . .	$r_3(n)$	. . .	$r_3(N)$
t	$r_t(1)$	$r_t(2)$	$r_t(3)$	. . .	$r_t(n)$	. . .	$r_t(N)$
358	$r_{358}(1)$	$r_{358}(2)$	$r_{358}(3)$	. . .	$r_{358}(n)$	. . .	$r_{358}(N)$
359	$r_{359}(1)$	$r_{359}(2)$	$r_{359}(3)$	. . .	$r_{359}(n)$	. . .	$r_{359}(N)$
360	$r_{360}(1)$	$r_{360}(2)$	$r_{360}(3)$	. . .	$r_{360}(n)$		$r_{360}(N)$

Key:

$r_t(n)$ = mortgage refinancing rate for month t on path n;

N = total number of interest-rate paths.

EXHIBIT 27–3

Simulated Cash Flow on Each of the Interest-Rate Paths

	Interest-Rate Path Number						
Month	**1**	**2**	**3**	...	***n***	...	***N***
1	$C_1(1)$	$C_1(2)$	$C_1(3)$	...	$C_1(n)$	...	$C_1(N)$
2	$C_2(1)$	$C_2(2)$	$C_2(3)$	...	$C_2(n)$	...	$C_2(N)$
3	$C_3(1)$	$C_3(2)$	$C_3(3)$	...	$C_3(n)$	...	$C_3(N)$
t	$C_t(1)$	$C_t(2)$	$C_t(3)$	...	$C_t(n)$	...	$C_t(N)$
358	$C_{358}(1)$	$C_{358}(2)$	$C_{358}(3)$	...	$C_{358}(n)$	...	$C_{358}(N)$
359	$C_{359}(1)$	$C_{359}(2)$	$C_{359}(3)$	...	$C_{359}(n)$	...	$C_{359}(N)$
360	$C_{360}(1)$	$C_{360}(2)$	$C_{360}(3)$	...	$C_{360}(n)$	...	$C_{360}(N)$

Key:

$C_t(n)$ = cash flow for month t on path n;
N = total number of interest-rate paths.

Consequently, the interest-rate path for the simulated future 1-month rates can be converted to the interest-rate path for the simulated monthly spot rates as shown in Exhibit 27–4.

Therefore, the present value of the cash flow for month T on interest-rate path n discounted at the simulated spot rate for month T plus some spread is

$$PV[C_T(n)] = \frac{C_T(n)}{[1 + z_T(n) + K]^{1/T}},$$

where

$PV[C_T(n)]$ = present value of cash flow for month T on path n;
$C_T(n)$ = cash flow for month T on path n;
$z_T(n)$ = spot rate for month T on path n;
K = spread.

The present value for path n is the sum of the present value of the cash flow for each month on path n. That is,

$$PV[\text{path}(n)] = PV[C_1(n)] + PV[C_2(n)] + \cdots + PV[C_{360}(n)],$$

where $PV[\text{path}(n)]$ is the present value of interest-rate path n.

Determining the Theoretical Value

The present value of a given interest-rate path can be thought of as the theoretical value of a passthrough if the path was actually realized. The theoretical value of

EXHIBIT 27–4

Simulated Paths of Monthly Spot Rates

	Interest-Rate Path Number						
Month	**1**	**2**	**3**	. . .	***n***	. . .	***N***
1	$z_1(1)$	$z_1(2)$	$z_1(3)$	. . .	$z_1(n)$	. . .	$z_1(N)$
2	$z_2(1)$	$z_2(2)$	$z_2(3)$	. . .	$z_2(n)$	. . .	$z_2(N)$
3	$z_3(1)$	$z_3(2)$	$z_3(3)$	. . .	$z_3(n)$	. . .	$z_3(N)$
t	$z_t(1)$	$z_t(2)$	$z_t(3)$	. . .	$z_t(n)$	. . .	$z_t(N)$
358	$z_{358}(1)$	$z_{358}(2)$	$z_{358}(3)$	. . .	$z_{358}(n)$	. . .	$z_{358}(N)$
359	$z_{359}(1)$	$z_{359}(2)$	$z_{359}(3)$	. . .	$z_{359}(n)$	. . .	$z_{359}(N)$
360	$z_{360}(1)$	$z_{360}(2)$	$z_{360}(3)$	. . .	$z_{360}(n)$	. . .	$z_{360}(N)$

Key:

$z_t(n)$ = spot rate for month t on path n;

N = total number of interest-rate paths.

the passthrough can be determined by calculating the average of the theoretical value of all the interest-rate paths. That is,

$$\text{Theoretical value} = \frac{PV[\text{path}(1)] + PV[\text{path}(2)] + \cdots + PV[\text{path}(N)]}{N},$$

where N is the number of interest-rate paths.

This procedure for valuing a passthrough is also followed for CMO bond classes and mortgage strips. The cash flow for each month on each interest-rate path is found according to the principal repayment and interest distribution rules of the deal. In order to do this for CMOs, a model for reverse engineering the deal structure is needed.

Distribution of Path Present Values

The Monte Carlo simulation method is a commonly used management science tool in business. (See Chapter 33.) It is employed when the outcome of a business decision depends on the outcome of several random variables. The product of the simulation is the average value and the probability distribution of the possible outcomes.

Unfortunately, the use of Monte Carlo simulation to value fixed-income securities has been limited to just the reporting of the average value, which is referred to as the *theoretical value* of the security. This means that all the other information about the distribution of the path present values is ignored, yet this information is quite valuable.

For example, consider a well-protected PAC bond. The distribution of the present value for the paths should be concentrated around the theoretical value. That is, the standard deviation should be small. In contrast, for a support tranche, the distribution of the present value for the paths could be wide or, equivalently, the standard deviation could be large.

Therefore, before using the theoretical value for an MBS generated from the Monte Carlo simulation model, a portfolio manager should ask for information about the distribution of the path's present values.

Option-Adjusted Spread

As explained in Chapter 21, the option-adjusted spread (OAS) is a measure of the yield spread that can be used to convert dollar differences between value and price. It represents a spread over the issuer's spot-rate curve or benchmark.

In the Monte Carlo simulation model, the OAS is the spread K that when added to all the spot rates on all interest-rate paths will make the average present value of the paths equal to the observed market price (plus accrued interest). Mathematically, OAS is the spread that will satisfy the following condition:

$$\text{Market price} = \frac{PV[\text{path}(1)] + PV[\text{path}(2)] + \cdots + PV[\text{path}(N)]}{N},$$

where N is the number of interest-rate paths.

Zero-Volatility Spread

The proper way to evaluate an MBS of the same duration but with different coupon rates is to compare them with a portfolio of zero-coupon Treasury securities that have the same cash flow stream as projected for the MBS. Assuming that the cash flows of the MBS are riskless, its value is equal to the portfolio value of all the zero-coupon Treasuries, with cash flows valued at the spot rates. A mortgage-backed security's value will be less than the portfolio of zero-coupon Treasury securities because investors demand a spread for the risk associated with holding an MBS rather than a riskless package (in the sense of default risk and certainty of the cash flows) of Treasury securities.

The *zero-volatility spread* (also called the *static spread*) to Treasuries is determined as follows. It is the spread that will make the present value of the projected cash flows from the MBS when discounted at the spot rate plus a spread equal to its market price. An iterative procedure is required to determine this spread.

Option Cost

Using the decomposition principle for a callable bond discussed in Chapter 20, we can obtain the implied cost of the option embedded in an MBS by calculating the difference between the option-adjusted spread at the assumed volatility of interest rates and the zero-volatility spread. That is,

$$\text{Option cost} = \text{zero-volatility spread} - \text{option-adjusted spread}.$$

The option cost measures the prepayment (or option) risk embedded in the security.

Suppose that the option cost is zero. Substituting zero into the above equation for the option cost, we see that the zero-volatility spread and the option-adjusted spread are equal. The implication is that for securitized products for which there is either no option or an option but it is not exercised to take advantage of refinancing opportunities, the zero-volatility spread can be used instead of the option-adjusted spread.

Some Technical Issues

In the lattice model for valuing bonds discussed in Chapter 21, the interest-rate tree is constructed so that it is arbitrage free. That is, if any on-the-run issue is valued, the value produced by the model is equal to the market price. This means that the tree is calibrated to the market. In contrast, in our discussion of the Monte Carlo simulation model, there is no mechanism that we have described that will ensure that the valuation model will produce a value for an on-the-run Treasury security (the benchmark in the case of an agency MBS) equal to the market price. In practice, this is accomplished by adding a *drift term* to the short-term return-generating process (Exhibit 27–1) so that the value produced by the Monte Carlo simulation model for all on-the-run Treasury securities is their market price.[2] A technical explanation of this process is beyond the scope of this chapter.[3]

There is also another adjustment made to interest-rate paths. Restrictions on interest-rate movements must be built into the model to prevent interest rates from reaching levels that are believed to be unreasonable (e.g., an interest rate of zero or an interest rate of 30%). This is done by incorporating *mean reversion* into the model. We discussed mean reversion in Chapter 19, where we explained interest-rate models.

The specification of the relationship between short-term rates and refinancing rates is necessary. Empirical evidence of the relationship is also necessary. More specifically, the correlation between the short- and long-term rates must be estimated.

The number of interest-rate paths determines how "good" the estimate is, not relative to the truth but relative to the valuation model used. The more paths there are, the more the theoretical value tends to settle down. It is a statistical sampling problem. Most Monte Carlo simulation models employ some form of *variance reduction* to cut down on the number of sample paths necessary to get a good statistical sample. Variance-reduction techniques allow us to obtain value estimates within a tick. By this we mean that if the model is used to generate more scenarios, value estimates from the model will not change by more than a tick. So, for example, if

2. This is equivalent to saying that the OAS produced by the model is zero.
3. For an explanation of how this is done, see Lakhbir S. Hayre and Kenneth Lauterbach, "Stochastic Valuation of Debt Securities," in Frank J. Fabozzi (ed.), *Managing Institutional Assets* (New York: Harper & Row, 1990), pp. 321–361.

1,024 paths are used to obtain the estimated value for a security, there is little more information to be had from the model by generating more than that number of paths. (For some very sensitive CMO tranches, more paths may be needed to estimate value within a tick.)

Effective Duration and Effective Convexity

With the Monte Carlo simulation model, effective duration and effective convexity can be computed by increasing and decreasing short-term Treasury rates by a small amount. When these interest rates are changed, the OAS is kept constant. This will produce two average total present values: one when short-term interest rates increase and one when short-term interest rates decrease. These average total present values can be viewed as the theoretical values under small interest-rate changes. These values are then substituted into the formula for effective duration and effective convexity.

When effective duration is calculated using the Monte Carlo simulation model to produce the two prices for a higher and lower yield, the result is commonly referred to as an *OAS duration*. Thus effective duration means in general that the duration is calculated after allowing for the cash flows to change as yields change. The OAS duration is a special case where price sensitivity is determined within the Monte Carlo simulation framework.

GENERAL APPROACH TO ABS VALUATION

As explained earlier in this chapter (as well as in earlier chapters), the value of a security can be found by discounting the expected cash flows by the appropriate spot rates. If some type of spread measure is sought, the zero-volatility spread is calculated. This is the spread that must be added to either the on-the-run Treasury spot rate or to the spot rate for some benchmark to obtain the market price (plus accrued interest). The zero-volatility spread differs from the nominal spread if (1) the yield curve is steep and/or (2) the security is an amortizing asset (i.e., principal is repaid periodically). Because most ABS are amortizing assets, the zero-volatility approach is superior to the nominal spread.

When there is an embedded option in a security, then either the lattice model described in Chapter 21, or the Monte Carlo simulation model as described in this chapter can be used. When an ABS has an embedded option, it is in the form of a call or prepayment option. Whether the lattice or Monte Carlo simulation model should be used depends on whether the cash flows are interest-rate-path independent or interest-rate-path-dependent. In the former case, at any point on an interest-rate path or node of an interest-rate tree, how the interest rate evolved to get to that point is unimportant and will not affect the cash flow at that point. In such cases, the lattice model is employed. This is why the lattice model is used to value agency, corporate, and municipal bonds. In contrast, because of prepayment burnout, the prepayments at a given point on an interest-rate path will depend on how the interest rate evolved to get to that point. For interest-rate-path-dependent securities, the Monte Carlo simulation model is employed. This is why this model is used to value mortgage-backed securities. With either model, an OAS can be calculated.

The decision as to which valuation model to employ depends on the particular type of ABS. Specifically, while an ABS may be backed by collateral that has a prepayment or call option whose exercise will depend on the prevailing level of interest rates versus the contract rate paid by the borrower, whether in practice that option will be exercised must be assessed empirically. When there is a prepayment option for the loans or receivables backing an ABS, but borrowers do not take advantage of that option when borrowing rates decline (e.g., ABS backed by auto loans), then the zero-volatility spread is used to discount the security to determine its value. When there is a prepayment option for the loans backing an ABS that borrowers appear to exercise when the prevailing borrowing rate declines below the contract rate, typically the cash flows are interest-rate-path-dependent. Thus, the Monte Carlo simulation model is used and OAS is used to discount the cash flows to determine their value.

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PART

NINE

PERFORMANCE ANALYSIS

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CHAPTER 28

HOLDINGS-BASED PERFORMANCE ATTRIBUTION ANALYSIS

Risk-adjusted return measures provide a starting point for assessing the performance of a bond portfolio manager. However, these measures fail to identify the reasons why the portfolio manager may have matched, outperformed, or underperformed a benchmark. That is, what active strategies or sources of return resulted in the realized performance? The decomposition of performance results to explain why the results were achieved is called *performance attribution analysis*.

There are three approaches to performance attribution models:

- Holdings-based attribution approach;
- Transaction-based attribution approach;
- Returns-based style attribution approach.

The holdings-based approach to performance attribution analysis relies solely on the portfolio's beginning-period holdings to determine the attribution effects (i.e., sources of returns). Thus the holdings-based approach effectively reflects a buy-and-hold strategy for the evaluation period. As the name suggests, the transaction-based approach seeks to capture any intraperiod transaction activity undertaken by the portfolio manager. That is, the transaction-based approach does not rely only on the initial portfolio holdings. A widely held view is that although the transaction-based approach is more accurate in attributing returns than the holdings-based approach, the data requirements are more costly while the marginal improvement in the accuracy is not worth considering transactions. The shorter the length of the time period for which an attribution model is applied and therefore the fewer the transactions, the less will be the difference in attribution using the two approaches. The third approach, returns-based style analysis, is a low-cost alternative to the first two approaches. Unlike holdings-based and transaction-based attribution analyses, which require the holdings of the portfolio, returns-based style analysis does not require information on portfolio holdings but is a statistical approach.

In this chapter we first discuss some general principles about performance attribution models.[1] Then we focus on the holdings-based approach, illustrate the approach, and describe the different types of attribution models that use this approach. In the next chapter we explain the returns-based style approach.

ABSOLUTE RETURN ATTRIBUTION VERSUS RELATIVE RETURN ATTRIBUTION

There are various ways of analyzing the sources of return. When the sources of a portfolio's actual return are analyzed without regard to some benchmark such as a client-designated benchmark or the risk-free rate, this is referred to as *absolute return attribution* (also referred to as *return contribution analysis*). With this approach to return attribution, the impact of the sectors, group of securities, or individual securities in which the portfolio manager invested held on the portfolio return is identified.

When the attribution considers performance relative to a benchmark, the approach is referred to as *relative return attribution*. In practice, however, it is simply referred to as *return attribution*, and we will follow that convention in the rest of this chapter. In relative return attribution, the weights of the holdings and the weights of the component members of the benchmark are used.

Calculation of the excess return can be in terms of arithmetic returns or geometric returns.[2] The return attribution model can be based on either arithmetic or geometric returns—the decision is based on the preference of the user (portfolio manager or client).

REQUIREMENTS OF A HOLDINGS-BASED RETURN ATTRIBUTION MODEL

As explained by Fabozzi and Fong,[3] there are three requirements of a bond performance and attribution process. First, the return must be measured properly. We explained the proper way for calculating return in Chapter 10. The second requirement is that the process must be informative. It should be capable of evaluating the skills associated with managing a bond portfolio. To do so, the attribution process must effectively address the key managerial skills and quantitatively measure how these skills are related to actual portfolio performance. The final requirement is that the results of the attribution process must be easily understood by the client and the manager.

In addition, the return attribution model should include all the active decisions that can be made by a portfolio manager in constructing the portfolio, and

1. For a further discussion of performance attribution, see Part Four in Bruce J. Feibel, *Investment Performance Measurement* (Hoboken, NJ: John Wiley & Sons, 2003).
2. If the analysis is over multiple valuation periods and is performed using arithmetic returns, then a multiperiod smoothing algorithm is required to scale attribution effects over time.
3. Frank J. Fabozzi and Gifford Fong, *Advanced Fixed Income Portfolio Management: The State of the Art* (New York, NY: McGraw-Hill, 1994), p. 281.

each of these active decisions should be quantified. One of the reasons why fixed-income return attribution analysis is more complicated than equity return attribution analysis is the fact that many more decisions must be made by an active fixed-income portfolio manager. Moreover, all the sources of returns used in the model should be fully reconciled. That is, if the portfolio excess return is 150 basis points, the sum of all the sources of excess return should total to 150 basis points.

The appropriate attribution model to use differs depending on the strategy to be evaluated and the audience. Different methods have been developed to add successive information, but at a cost of additional input data and the potential for confusion over interpreting the active factors. Models have been developed by practitioners to isolate currency and hedging, income, duration, and other effects.

If the single-period return attribution model fails to reconcile the excess return, then the shortfall is referred to as a *residual*. Different return attribution models that we describe later in this chapter treat the residual in different ways. For example, some models attribute any residual to security selection or classify it as an *interaction effect*. There is considerable debate about the residual. If the residual is large, it casts doubt on how good the return attribution model is.

PRINCIPLES OF PERFORMANCE ATTRIBUTION

To satisfy the three requirements for a successful performance attribution model, Barclays[4] suggests that the following principles should be satisfied by the model: (1) additivity, (2) completeness, and (3) fairness. To understand these three principles of a performance attribution system, it important to understand that the realized return is the result of a decision by two or more members of the portfolio management team. Performance evaluation methodology must be consistent with the decision-making process used to manage the strategy. *Additivity* means that a performance attribution model should reflect the contribution of all members of the portfolio management team. *Completeness* means that adding the contributions of all members of the portfolio management team should equal the realized portfolio return. *Fairness* means that the allocation of outperformance to the members of the portfolio management team should be done in such a way as to be viewed by all involved as being fair.

Other requirements that Barclays states are needed for an effective performance attribution model is that because it must reflect the decisions made by the portfolio management team, the model should[5]

- Quantify the risk factors associated with the investment decisions made;
- Independently quantify the contributions of each factor;
- Be available on a timely basis;
- Be flexible enough to explain short- and long-term evaluation periods.

4. Barclays, "Principles of Performance Attribution," in Frank J. Fabozzi (ed.), Chapter 70 in *The Handbook of Fixed Income Securities*, 9th ed. (New York: McGraw-Hill, 2021), p. 1712.
5. Ibid., p. 1790.

Campisi[6] describes the following characteristics of what he refers to as the "4R's" of a comprehensive fixed-income attribution model:

- *Representative.* The model should be consistent with how investment decisions are made by the portfolio management team, clearly demonstrating the return attributable to each systematic risk taken.
- *Rigorous.* The model should report what happened during the holding period, clearly explaining why there was over- or underperformance.
- *Reasonable.* The model should provide a balance between the cost and rigor.
- *Responsive.* The model should allow sufficient flexibility to permit the client to customize a benchmark to match the strategy pursued by the portfolio manager.

TYPES OF FIXED-INCOME PERFORMANCE ATTRIBUTION MODELS

Fixed-income performance attribution models fall into three categories: (1) sector based, (2) factor based, and (3) hybrid sector based/factor based. *Sector-based models* decompose the excess return (active return) into two sources. The first is the allocation to the bond sectors, and the second is the selection of individual securities within each sector. Basically, sector-based models are variants of the Brinson model with an adjustment made to take into consideration duration by weighting each security by its contribution to portfolio duration.

Factor-based models begin by identifying the risk factors that are the source of excess returns. The risk factors reflect the systematic risk sources of return. The basic factor models include yield-curve shape (which includes shift, twist, and butterfly effects), spread effects, and income. Yield-curve changes also can be assessed based on key rate duration. *Hybrid sector-based/factor-based models*, as the name suggests, decomposes the exces return using both sectors and factors.

FIXED-INCOME VERSUS EQUITY PERFORMANCE ATTRIBUTION MODELS

In equity portfolio management, the most popular performance attribution model is the Brinson model.[7] This model is a sector-based model that looks at sector allocation and security selection, and it was the first model applied in the fixed-income

6. Stephen Campisi, "Primer on Fixed Income Performance Attribution," *Journal of Performance Measurement,* Vol. 4, No. 4 (2000), pp. 14–25.
7. Gary P. Brinson and Nimrod Fachler, "Measuring Non-US Equity Portfolio Performance," *Journal of Portfolio Management*, Vol. 11, No. 3 (1985), pp. 73–76; and Gary P. Brinson, L. Randolph Hood, and Gilbert L. Beebower, "Determinants of Portfolio Performance," *Financial Analysts Journal*, Vol. 51, No. 1 (1995), pp. 133–136.

area. However, the complexity of fixed-income securities in terms of their attributes (e.g., coupons, maturity, spreads, embedded options, etc.) and the many more decisions required and more risk factors than in equities limits the use of equity performance models when applied to fixed-income portfolios. There are allocations based on expectations about the changes in the level of interest rates, changes in the shape of the yield curve, changes in credit spreads, and changes in prepayments for the different coupon sectors of the mortgage-backed securities market.

To appreciate some of the limitations of applying the traditional Brinson attribution model to a fixed-income portfolio, suppose that the model finds that the excess return is attributable to overweighting of the industrial sector. One cannot be sure that the excess return was due to the duration exposure of that sector rather than the allocation to that sector. As another example, it is important to look at the return generated by allocation to credit products, separating the credit-spread movements and spread carry. This is often simply attributed to active management skill, even if it is a buy-and-hold strategy that was pursued by the portfolio manager.[8]

Campisi[9] illustrated the shortcoming of applying the traditional Brinson equity attribution model to analyze the performance of a fixed-income portfolio. He compared the analysis using the traditional Brinson equity attribution model with analysis using the Campisi fixed-income attribution model that we will describe below. Using a sample fixed-income portfolio, the Campisi model showed that alpha for this portfolio was 5 basis points and that the contribution from security selection was 6 basis points. In stark contrast, the contribution from security selection using the traditional Brinson equity allocation model was 2 basis point. Consider the implications. Suppose that the portfolio manager of this sample portfolio had marketed the firm to a client as being particularly skilled at security selection. Had the client been presented with the output of this sample portfolio using the traditional Brinson equity model, the client may have viewed the portfolio manager as not being able to deliver on the claim that the firm had a special skill in security selection. However, using a performance attribution model tailored to the fixed-income market, the portfolio manager did indeed deliver on the claim because the bulk of the alpha was from security selection. It should be noted that application of the traditional Brinson equity attribution to fixed-income portfolios, while limited, is useful for some active fixed-income strategies such as when the focus is on credit-sector selection and issue selection.

In addition to the unsuitability of applying equity attribution models to fixed-income portfolios, there is another difficulty in applying fixed-income attribution models. For an equity portfolio, determining the sources' contribution to performance is far less complicated than for fixed-income securities. As a result, it is not simple to determine the contributions of a fixed-income benchmark that

8. Antonio F. Silva, Jr., "Performance Attribution for Fixed Income Portfolios in Central Bank of Brazil International Reserves Management," in *Risk Management for Central Bank Foreign Reserves* (Brussels: European Central Bank, 2004).

9. Stephen Campisi, "A Sector Based Approach to Fixed Income Performance Attribution." *Journal of Performance Measurement*, Vol. 15, No. 3 (2011), pp. 23–42.

can then be used to compare the performance of a fixed-income portfolio. Further complicating the analysis is that there are far more securities in a fixed-income benchmark than in an equity benchmark, and the sources for the prices used in the analysis of the individual issues in the benchmark and portfolio differ. The sources for the pricing of the securities in the benchmark are typically different from the prices used to price the securities in the portfolio. There is also a difference as to whether the sources use closing prices, transaction prices, and for less traded securities model prices. Pricing source differences can result in the residual.

FIXED-INCOME PEFORMANCE MODELS USED IN PRACTICE

Below we describe fixed-income performance attribution models that have been developed and have served as the basis for the development of many in-house models by asset-management firms.[10]

Fong–Pearson–Vasicek Attribution Model

In 1983, Fong, Pearson, and Vasicek[11] developed a model that decomposes the portfolio return into two levels. The first level decomposes the return so as to differentiate between

- The effect of the external interest-rate environment and
- The contribution of the management process.

Distinguishing these two effects allows for the identification of circumstances that are beyond the control of the portfolio manager. Next, the first component can be decomposed into two components:

- The interest-rate level and
- The interest-rate change.

By decomposing as just described, the effect of the interest-rate environment considers the return that would be realized if interest rates did not change and the return

10. One of the first such models is the duration-based attribution model developed by Wagner and Tito in 1977, which focuses on how the portfolio manager's skill in selecting the portfolio's duration contributes to return performance. Of course, positioning a portfolio on duration is not a complete explanation of how a portfolio is affected by changes in the yield curve. It only accounts for changes in a parallel shift in the yield curve. See Wayne H. Wagner and Dennis A. Tito, "Definitive New Measures of Bond Performance and Risk," in *CFA Reading in Financial Analysis: Security Analysis and Portfolio Management* (Homewood, IL: Richard D. Irwin, 1977), pp. 126–132.
11. Gifford Fong, Charles Pearson, and Oldrich Vasicek, "Bond Performance: Analyzing Sources of Return," *Journal of Portfolio Management*, Vol. 9, No. 3 (1983), pp. 46–50. Also see Frank J. Fabozzi and Gifford Fong, *Advanced Fixed Income Portfolio Management: The State of the Art* (New York: McGraw-Hill, 1994), pp. 298–301.

attributable to the actual interest-rate change. As for the contribution of the management process, the Fong–Pearson–Vasicek attribution model decomposes it into three components:

- Return from maturity management;
- Return from spread/quality management;
- Return attributable to the selection of the specific securities.

The first of the three components is basically management of the duration of the portfolio. The second component refers to the allocation to different sectors of the bond market or quality (i.e., credit) groups of the bond market. The purpose of this decision is to exploit the portfolio manager's view on changes in spread relationships in the market. The last component is selection of the specific securities based on the portfolio management's credit analyst's view on relative value of securities that are expected to outperform otherwise comparable bonds.

Kahn Multifactor Attribution Model

A multifactor attribution model was developed by Kahn.[12] The six sources of return in the Kahn model include market-wide factors and bond-specific features. The six effects in the model that contribute to the realized portfolio return include the following:

Effect 1: Appreciation of discount bonds and depreciation of premium bonds as the maturity shortens (i.e., *rolling down* the term structure of interest rates) plus accrued interest;[13]

Effect 2: Changes in bond values as a result of changes in the default-free term structure;

Effect 3: Changes in bond values because of changes in sector and quality spreads;

Effect 4: Changes in unexpected cash flows as a result of embedded options or mortgage prepayments that are different from the expected levels when the securities were purchased;

Effect 5: Changes in bond values owing to unexpected changes in quality ratings (i.e., upgrades and downgrades);

Effect 6: Changes in individual bond values that may have moved closer to or farther away from their fair market value, generating bond-specific price changes.

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12. Ronald N. Kahn, "Bond Performance Analysis: A Multifactor Approach," *Journal of Portfolio Management*, Vol. 18, No. 1 (1991), pp. 40–47.
13. This is also referred to as the *carry effect*.

Effects 1–3 are the multifactor-generated portfolio returns, whereas the last three effects are bond-specific portfolio returns. The second effect basically involves changes in the yield curve. Kahn provides a further breakdown of this effect, decomposing it into

- Parallel shifts in the term structure;
- Twists in the term structure that involve short rates increasing by Δ basis points, intermediate rates unchanged, and long-term rates decreasing by Δ basis points;
- Butterfly shift in the term structure, which involves both short and long rates increasing by Δ basis points while intermediate rates decrease by 2Δ basis points.

Dynkin–Hyman–Konstantinovsky Attribution Model

The 1998 model developed by Dynkin, Hyman, and Konstantinovsky at Lehman Brothers at the time takes into account yield-curve movements based on changes in bellwether yields.[14] The model uses fully option-adjusted spread techniques to give the most accurate treatment for securities with cash flows that depend on the yield curve. It was developed at the time to attribute returns based on the most popular fixed-income indices, at the time the Lehman Brothers indices.

Van Breukelen Attribution Model

Van Breukelen combines the Brinson (sector-based) model and the Wagner–Tito duration-based model.[15] The approach computes the duration contribution[16] and then corrects returns to calculate allocation and selection.

Silvia–De Carvalho–Ornelas Fixed-Income Attribution Model

The model proposed by Silvia, De Carvalho, and Ornelas, developed when these researchers were employed by the Executive Office for Risk Management of the Central Bank of Brazil, combines the duration-based model with the asset-selection

14. Lev Dynkin, Jay Hyman, and Vadim Konstantinovsky, "Return Attribution Model for Fixed Income Securities," Chapter 21 in Frank J. Fabozzi (ed.), *Handbook of Portfolio Management* (New Hope, PA: Frank J. Fabozzi Associates, 1998), pp. 427–458; Lev Dynkin and Jay Hyman, "Multi-Factor Fixed Income Models and Their Applications," in Frank J. Fabozzi and Harry M. Markowitz (eds.), *The Theory and Practice of Investment Management* (Hoboken, NJ: John Wiley & Sons, 2002), pp. 665–696.
15. Gerald van Breukelen, "Fixed Income Attribution," *Journal of Performance Measurement*, Vol. 4, No. 4 (2000), pp. 61–68.
16. See Chapter 14 for an explanation of the contribution to duration.

factor.[17] First, yield-curve fitting is used to analyze the effects of changes in the curve. Second, three hypothetical portfolios are constructed in order to classify "non–yield curve factors." One of the key factors in this model is the liquidity factor.

Campisi Fixed-Income Attribution Model

The simplest fixed-income attribution model is the one suggested by Campisi.[18] This model, which will be illustrated in the last section of this chapter, starts by decomposing the total return into two effects: the *income-return effect* and the *price-return effect*. The income-return effect is the return generated from interest income. The price-return effect is based on return resulting from three effects: the *Treasury effect*, the *spread effect*, and the *selection effect*. The Treasury effect is attributable to a change in the yield curve that encompasses both a parallel shift in the yield curve and a nonparallel shift in the yield curve. Thus the Treasury effect captures the manager's decision with respect to the portfolio's duration and yield-curve reshaping duration. The spread effect is the portion of the return attributable changes in the spread relative to the Treasury yield curve.

As noted earlier in this chapter, typically in return attribution models there is a *residual*. This component of an attribution model reflects the part of the total return that is not explained by the other effects. In the Campisi model, however, there is no residual component. Instead, the difference between the total return and the sum of the return due to the Treasury effect and the spread effect is treated as attributable to the selection effect. This effect is assumed to be due to the selection of specific securities and sectors of the bond market.

BlackRock Fixed-Income Attribution Model[19]

Laipply, Madhavan, Sobczyk, and Tucker of BlackRock describe an attribution model for fixed-income returns based on the approach suggested for attributing equity market returns by Lo[20] and Hsu, Kalesnik, and Myers,[21] who showed how,

17. Antonio Francisco A. Silva Jr., Pablo José Campos de Carvalho, and José Renato Haas Ornelas, "A Performance Attribution Methodology for Fixed Income Portfolios," December 2009. https://www.researchgate.net/publication/228317159_A_Performance_Attribution_Methodology_for_Fixed_Income_Portfolios.
18. Campisi, "Primer on Fixed Income Performance Attribution."
19. The information about this model was obtained solely from Stephen Laipply, Ananth Madhavan, Aleksander Sobczyk, and Matthew Tucker, "Sources of Excess Return and Implications for Active Fixed-Income Portfolio Construction," *Journal of Portfolio Management,* Vol. 46, No. 2 (2020), pp. 106–120. One of the coauthors of this book in on the board of directors of the BlackRock Fixed Income Complex and chair of the Performance Committee, but none of the information about BlackRock models for attribution is covered here other than what appears in the cited article.
20. Andrew Lo, "Where Do Alphas Come From? A Measure of the Value of Active Investment Management," *Journal of Investment Management*, Vol. 6, No. 2 (2008), pp. 1–29.
21. Jason C. Hsu, Vitali Kalesnik, and Brett W. Myers, "Performance Attribution: Measuring Dynamic Allocation Skill," *Financial Analysts Journal*, Vol. 66, No. 6 (2010), pp. 17–26.

using security-level holdings, a manager can identify factor and nonfactor components.

The BlackRock model attributes a fund's active return (alpha) to returns to

- Static factor exposures;
- Factor timing;
- Individual bond selection.

An example of returns attributable to static factor exposures would be a manager's decision to have a constant tilt to credit risk. A manager's decision to alter the exposure over time to a factor such as duration is an example of factor timing. Individual bond selection is simply the excess over the returns attributable to static factor exposure and factor timing.

ILLUSTRATION OF THE CAMPISI ATTRIBUTION MODEL[22]

We will use two illustrations to explain return attribution analysis.[23] Both illustrations are an implementation of a broadly applicable model developed by Campisi (mentioned earlier).[24] The first illustration shows the return attribution based on the effect of yield-curve positioning as measured by duration and changes in yield-curve spreads. The second adds to the analysis by decomposing the yield-curve effect into the effect due to a parallel shift in the yield curve and a twist in the yield curve.

The data for the two illustrations are provided in the two panels of Exhibit 28–1. Shown in panel A of the exhibit are the portfolio returns and benchmark returns for four sectors (corporate, government, mortgage backed, and asset backed) in which the manager has invested. The returns are decomposed into income return and principal return. Panel B shows the beginning weights for each of the four sectors for the portfolio and the benchmark.

Duration Plus Spread Return Attribution Illustration

The attribution of price return in the model is the result of three decisions:

- The impact of sector- or segment-level yield-curve positioning (referred to as the *duration effect* or the *Treasury effect*);
- Changes in the yield spread over the Treasury return, or *spread effect*;
- A *residual selection effect* generated by purchasing securities that are better performing within each sector or segment of the bond market relative to the performance of the benchmark constituents.

22. This illustration was provided by Bruce J. Feibel, CFA, Senior Vice President, State Street Corp.
23. The online supplement has another illustration of how a corporate bond/credit portfolio is presented in a typical attribution report. The illustration uses a hypothetical U.K. bond portfolio containing only pound sterling–denominated assets and run against a similar benchmark.
24. Campisi, "Primer on Fixed Income Performance Attribution."

EXHIBIT 28–1

Portfolio and Benchmark Returns and Weights

Panel A: Portfolio and Benchmark Returns

	Portfolio Returns			Benchmark Returns		
Sector	**Income**	**Principal**	**Total**	**Income**	**Principal**	**Total**
Corporate	1.00%	7.25%	8.25%	1.00%	7.00%	8.00%
Government	1.00%	8.00%	9.00%	1.00%	6.00%	7.00%
Mortgage backed	1.10%	6.00%	7.10%	1.00%	6.00%	7.00%
Asset backed	1.00%	7.00%	8.00%	1.00%	6.00%	7.00%
Total	1.03%	7.01%	8.04%	1.00%	6.25%	7.25%

Panel B: Initial Portfolio Weights

	Begin Weights			Total Returns		
Sector	**Portfolio**	**Benchmark**	**Difference**	**Portfolio**	**Benchmark**	**Difference**
Corporate	25.00%	25.00%	0.00%	8.25%	8.00%	0.25%
Government	25.00%	25.00%	0.00%	9.00%	7.00%	2.00%
Mortgage backed	30.00%	25.00%	5.00%	7.10%	7.00%	0.10%
Asset backed	20.00%	25.00%	–5.00%	8.00%	7.00%	1.00%
Total	100.00%	100.00%	0.00%	8.04%	7.25%	0.79%

The two panels in Exhibit 28–2 provide information about the beginning of the period duration for each sector and returns based on those durations and the actual change in Treasury yield. Panel A does this for the portfolio, whereas panel B does it for the benchmark. The third column in both panels shows the change in the duration-matched Treasury yield (DMT change).[25] For all four sectors, the DMT yield is assumed to have declined by 200 basis points (–2.00%). To obtain the Treasury return, the following formula is used:

$$\text{Treasury return} = -\text{DMT} \times \text{duration}.$$

For example, for the corporate sector,

$$\text{Treasury return} = -(-2.00\%) \times 3.1 = 6.2\%.$$

Also shown in Exhibit 28–2 is the spread return. To obtain the spread return, the following formula is used:

$$\text{Spread return} = -\text{portfolio duration} \times \text{benchmark spread change},$$

25. In practice, the duration-matched Treasury yield is interpolated from a yield curve at the begining and end of the period.

EXHIBIT 28–2

Beginning of Period Duration and Treasury and Spread Returns for Portfolio and Benchmark

Panel A: Beginning of the Period Portfolio Sector Durations and Treasury and Spread Returns

Portfolio Analytics

Sector	Duration	DMT Change	Treasury Return	Spread Return
Corporate	3.1	–2.00%	6.20%	1.03%
Government	4.0	–2.00%	8.00%	0.00%
Mortgage backed	3.0	–2.00%	6.00%	0.00%
Asset backed	3.0	–2.00%	6.00%	0.00%
Total	3.3		6.55%	0.25%

Panel B: Beginning of the Period Benchmark Sector Durations and Treasury and Spread Returns

Benchmark Analytics

Sector	Duration	DMT Change	Treasury Return	Spread Return
Corporate	3.0	–2.00%	6.00%	1.00%
Government	3.0	–2.00%	6.00%	0.00%
Mortgage backed	3.0	–2.00%	6.00%	0.00%
Asset backed	3.0	–2.00%	6.00%	0.00%
Total	3.0		6.00%	0.25%

where

Benchmark spread change

$$= \frac{-(\text{benchmark total return} - \text{benchmark income return} - \text{benchmark Treasury return})}{\text{benchmark duration}}.$$

For example, for the corporate sector,

Benchmark total return = 8.00%;

Benchmark income return = 1.00%;

Benchmark Treasury return = 6.00%;

Benchmark duration = 3.0.

Therefore,

$$\text{Benchmark spread change} = \frac{-(8.00\% - 1.00\% - 6.00\%)}{3.0} = -0.333\%.$$

Because the portfolio duration is 3.1,

$$\text{Spread return} = -3.1 \times -0.333\% = 1.03\%.$$

Panels A and B in Exhibit 28–3 provide the portfolio return and benchmark return attribution based on income, Treasury, and spread effects. Panel A provides this for the portfolio, whereas panel B, for the benchmark. Look at the portfolio return attribution (panel A) for the corporate sector. The return attributable to income is 1.00%, obtained from panel A of Exhibit 28–1. The Treasury and spread returns are obtained from panel A of Exhibit 28–2. Where does the selection return attribution come from? This is the residual of the actual portfolio return that cannot be explained by the other three effects. From Exhibit 28–2 we see that the total return of the portfolio for the corporate sector is 8.25%. The sum of the income, Treasury, and spread returns is 8.23% (1.00% + 6.20% + 1.03%). The difference between 8.25% and 8.23% is 0.02%, which is shown in Exhibit 28–3.

Performance attribution is shown in Exhibit 28–4. The four effects (income, Treasury, spread, and selection) are the difference between the corresponding effects in the two panels in Exhibit 28–3.

Now let's look at the return attribution for this portfolio relative the benchmark. Over the period, benchmark-relative total return value added equaled 0.79%,

EXHIBIT 28–3

Portfolio and Benchmark Attribution: Income, Treasury, Spread, and Selection Effects

Panel A: Portfolio Return Attribution

Portfolio Return Attribution

Sector	Income	Treasury	Spread	Selection	Total
Corporate	1.00%	6.20%	1.03%	0.02%	8.25%
Government	1.00%	8.00%	0.00%	0.00%	9.00%
Mortgage backed	1.10%	6.00%	0.00%	0.00%	7.10%
Asset backed	1.00%	6.00%	0.00%	1.00%	8.00%
Total	1.03%	6.55%	0.26%	0.20%	8.04%

Panel B: Benchmark Return Attribution

Benchmark Return Attribution

Sector	Income	Treasury	Spread	Selection	Total
Corporate	1.00%	6.00%	1.00%	0.00%	8.00%
Government	1.00%	6.00%	0.00%	0.00%	7.00%
Mortgage backed	1.00%	6.00%	0.00%	0.00%	7.00%
Asset backed	1.00%	6.00%	0.00%	0.00%	7.00%
Total	1.00%	6.00%	0.25%	0.00%	7.25%

EXHIBIT 28–4

Performance Attribution

Sector	Income	Treasury	Spread	Selection	Total
Corporate	0.00%	0.20%	0.03%	0.02%	0.25%
Government	0.00%	2.00%	0.00%	0.00%	2.00%
Mortgage backed	0.10%	0.00%	0.00%	0.00%	0.10%
Asset backed	0.00%	0.00%	0.00%	1.00%	1.00%
Total	0.03%	0.55%	0.01%	0.20%	0.79%

as shown in panel B of Exhibit 28–2. The 0.79% was comprised of a 0.03% income return (1.03% portfolio income return – 1.00% benchmark income return) and a 0.76% price return (7.01% portfolio price return – 6.25% benchmark price return). The income return is generated from coupon accruals. Except for the portfolio mortgage-backed sector (1.1%), each sector of the portfolio and benchmark generated 1% of income return. The interaction between the higher return and a sector overweight of 30% versus 25% delivered the 0.03% impact on total relative income return.

Relative price return was generated from multiple interacting active bond portfolio management decisions. The procedure to isolate the duration or Treasury effect requires the duration of each sector and the change in yield over the period for the DMT, as shown in Exhibit 28–2. Because a Treasury issue may not exist with a duration exactly equal to the sector duration, an interpolation methodology is employed to estimate the change in yield.

The price impact of the change in Treasury yields acts as a risk-free proxy to isolate the impact of the change in yields from the change in yield spreads over the Treasury for the period. For each sector of the portfolio and benchmark, a Treasury return is calculated as shown in Exhibit 28–2. The sector returns are duration weighted, and the contributions to Treasury returns are summed to derive a total portfolio and benchmark level Treasury return.

In our illustration, the portfolio manager earned a 0.55% higher Treasury return than the benchmark. While the change in yields was the same for each sector of the portfolio and benchmark (−2.00%), the portfolio was more positively impacted by the decline in yields because of the higher corporate- and government-sector duration exposures. The relative Treasury returns are weighted and then compared to derive a benchmark-relative Treasury or duration effect equal to 0.55%.

The next step is to quantify the impact on return by the change in sector-level yield spreads over the risk-free Treasury during the period. This is done as follows. First, isolate the benchmark sector return left after accounting for the income and Treasury returns, which is 1% (= 8% – 1% – 6%). Second, weight this residual by the benchmark sector duration multiplied by −1 (1%/3.0) and then multiply this result by the portfolio sector duration multiplied by −1 to determine the impact on the portfolio sector level return.

The corporate spread return is 1.03%. This is higher than the corresponding benchmark return of 1.00%. The spread return difference is weighted to calculate a contribution to relative return due to the spread effect of 0.01%.

In this attribution model, the selection effect is a residual left after accounting for the Treasury and spread effects. The portfolio's selection effect is equal to 1% from selecting better-performing securities relative to the benchmark in the asset-backed sector. The better returns could be earned via relative convexity or other security-specific factors. The weighed effect of the sector selection effects at the portfolio level is 0.2%. The sum of the active management effects (= the benchmark relative value added over the period) is 0.79%.

Decomposing the Treasury Return (Shift and Twist) Attribution Illustration

The basic Campisi model can be extended by decomposing the Treasury return into the effect of parallel shifts in the yield curve and twists in the yield curve. To illustrate how this is done, we use the same data as in Exhibits 28–1 and 28–2. To capture shifts and twists in the yield curve, we use the 5-year key-rate duration (KRD) changes. For our illustration, Exhibit 28–5 shows the 5-year KRD changes for the four sectors, panel A for the portfolio and panel B for the benchmark. While in our illustration we use the 5-year KRD, an alternative key rate can be used based on the situation.

Exhibit 28–6 shows the breakdown of the Treasury return portfolio (panel A) and benchmark (panel B) into the shift and twist returns. Let's look at the shift return and twist returns for the corporate sector for the portfolio (panel A). The shift return is found as follows:

$$\text{Shift return} = -\text{duration} \times \text{5-year KRD yield change.}$$

For example, look at panel A of Exhibit 28–5, which shows the shift return for the corporate sector of the portfolio. The shift return is 4.65%, which is found using the duration of 3.1 and the 5-year KRD yield change of –1.50% from Exhibit 28–5 to obtain

$$\text{Shift return} = -3.1 \times (-1.50\%) = 4.65\%.$$

The twist return is then the return that is not attributable to a shift in the 5-year KRD. In our illustration, the Treasury return for the corporate sector of the portfolio is 6.20% (see panel A of Exhibit 28–3), and the shift is 4.65%. Therefore, the twist return is 1.55% (= 6.20% − 4.65%)

Exhibit 28–7 provides the curve-position attribution for the portfolio relative to the benchmark. The values in the exhibit are simply the difference in the corresponding values in panels A and B of Exhibit 28–6. Notice that attribution of the

EXHIBIT 28–5

Duration and Key-Rate Duration

Panel A: Portfolio Duration and Key-Rate Duration

Portfolio Analytics

Sector	Duration	DMT Change	Treasury Return	Spread Return	KBD Change
Corporate	3.1	–2.00%	6.20%	1.03%	–1.50%
Government	4.0	–2.00%	8.00%	0.00%	–1.50%
Mortgage backed	3.0	–2.00%	6.00%	0.00%	–1.50%
Asset backed	3.0	–2.00%	6.00%	0.00%	–1.50%
Total	3.3		6.55%	0.26%	–1.50%

Panel B: Benchmark Duration and Key-Rate Duration

Portfolio Analytics

Sector	Duration	DMT Change	Treasury Return	Spread Return	KBD Change
Corporate	3.0	–2.00%	6.00%	1.00%	–1.50%
Government	3.0	–2.00%	6.00%	0.00%	–1.50%
Mortgage backed	3.0	–2.00%	6.00%	0.00%	–1.50%
Asset backed	3.0	–2.00%	6.00%	0.00%	–1.50%
Total	3.0		6.00%	0.25%	–1.50%

EXHIBIT 28–6

Decomposition of Treasury Return into Shift and Twist Returns

Panel A: Portfolio Shift, Twist, and Treasury Returns

Sector	Shift Return	Twist Return	Treasury Return
Corporate	4.65%	1.55%	6.20%
Government	6.00%	2.00%	8.00%
Mortgage backed	4.50%	1.50%	6.00%
Asset backed	4.50%	1.50%	6.00%
Total	4.91%	1.64%	6.55%

Panel B: Benchmark Shift, Twist, and Treasury Returns

Sector	Shift Return	Twist Return	Treasury Return
Corporate	4.50%	1.50%	6.00%
Government	4.50%	1.50%	6.00%
Mortgage backed	4.50%	1.50%	6.00%
Asset backed	4.50%	1.50%	6.00%
Total	4.50%	1.50%	6.00%

EXHIBIT 28–7

Yield-Curve Positioning Attribution

Corporate	0.15%	0.05%	0.20%
Government	1.50%	0.50%	2.00%
Mortgage backed	0.00%	0.00%	0.00%
Asset backed	0.00%	0.00%	0.00%
Total	0.41%	0.14%	0.55%

Treasury return over the benchmark is 0.55% in Exhibit 28–7, which agrees with the performance attribution in Exhibit 28–4. Of the 0.79% value added by the portfolio manager of this hypothetical bond portfolio, 0.55% is attributable to the duration effect. In turn, 0.41% of the Treasury return is attributable to a parallel shift in the yield curve.

CHAPTER

29

RETURNS-BASED STYLE ATTRIBUTION ANALYSIS

As explained in the previous chapter, there are three approaches to evaluating manager performance: (1) holdings-based attribution analysis, (2) transaction-based attribution analysis, and (3) returns-based style analysis (RBSA). The first two approaches are the subject of the previous chapter, and the third is the subject of this chapter. RBSA, developed by William Sharpe,[1] is a low-cost alternative to holdings- and transactions-based performance attribution. Unlike holdings- and transaction-based attribution analysis, which required the holdings of the portfolio, RBSA does not require that detailed information.

By investment style or, equivalently, investment philosophy or investment strategy, it is meant the dominant approach pursued by a portfolio manager in constructing a portfolio to meet a client-specified objective. The assets selected in constructing the portfolio will have the characteristics of the investment style. In the equity space, the most fundamental investment styles are value, growth, market capitalization, and momentum. In the fixed-income space there are even more styles based on sector, credit rating, and yield-curve positioning (duration and key-rate duration). In a global equity or fixed-income portfolio, style will cover whether currency is hedged or unhedged.

In general, the global fixed-income market can be described by a large number of factors. Diversity in spreads results because of broad sector- and country-related yield curves. Yield-curve positioning becomes a highly complex undertaking for bond managers because the different yield curves offer a broad spectrum of investment decisions and allocations. Therefore, the factors that apply to global fixed-income portfolios have to control for several risks not applicable to more narrow equity or fixed-income markets—for example, domestic equity or bond markets. Liquidity is specific for the international fixed-income markets. Although major indexes representing these markets allocate to the U.S., Japanese, and European markets, emerging markets are a significant part of these markets. Factors should capture the liquidity issues.

The key in using RBSA is identifying the factors that should be used to evaluate performance. The statistical tool used to link a portfolio return to the factors is *regression analysis*. The factors identified are the explanatory (independent) variables

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This chapter is coauthored with Gueorgui S. Konstantinov of LBBW Asset Management.

1. William F. Sharpe, “Determining a Fund’s Effective Asset Mix,” *Investment Management Review* (December 1998), pp. 59–69; and William F. Sharpe, “Asset Allocation: Management Style and Performance Measurement,” *Journal of Portfolio Management* (Winter 1992), pp. 7–19.

in the regression model. The explanatory variables in the regression model are factors that financial theory and empirical studies suggest drive returns for both the portfolio whose fund manager is being evaluated and a client-designated benchmark.

RBSA involves identifying candidate factors, selecting the factors, calculating the time-series data for the returns of the factors, and computing the time-series regression using the portfolio return as the dependent variable and the factors as the explanatory variables to compute the exposure of the portfolio to each factor. Finally, the regression results are analyzed to determine whether the portfolio manager added value after adjusting for the factor exposures.

Because RBSA uses the statistical technique of regression analysis, we discuss in the online appendix the econometric issues associated with the approach to performance attribution.

SELECTING FACTORS

The objective of factor selection is to identify factors that drive returns. The selection of factors is based on sound economic principles and robust statistical (empirical) evaluation. In other words, the relevant factors have to be meaningful and appropriate when explaining bond returns. To select appropriate factors, a close look into the underlying portfolio and benchmark constituents is necessary: instruments, sectors, and industries (and for global bond portfolios, foreign exchange and country exposure). These are among the first sources of information when applying RBSA. For example, running regression models with foreign-exchange factors on a U.S. domestic bond portfolio might be statistically sound but economically less meaningful and appropriate.

Of course, factor availability is a serious concern. Some factors do not exist for specific segments or exposures. For example, a factor seeking to gauge the quasi–government bond exposure might not be available and might not be statistically robust. The other concern might be that too narrowly defined factors might cause overfitting. That is, the factor might explain returns perfectly, creating too good an explanation. Therefore, a more broadly defined factor not only might be appropriate but also may make more sense economically.

Although some of the factors may be specific to the underlying bond benchmark or portfolio, other factors are more generally used and universally applicable. These factors reflect the bond dynamics in a broader sense and are not restricted by the specific exposure. Such factors are the spread and yield factors that explain the risk of bonds and should be included in most cases when investigating bond returns. Other factors are more generally specified and applied universally to all types of asset classes. Notable examples are the three factors in the Fama–French model[2] and some asset-based factors using indicies aimed at capturing returns.

2. Eugene F. Fama and Kenneth R. French, "The Cross-Section of Expected Stock Returns," *Journal of Finance*, Vol. 47, No. 2 (1992), pp. 427–465; and Eugene F. Fama and Kenneth R. French, "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics*, Vol. 33, No. 1 (1993), pp. 3–56.

Studies of mutual funds have used four types of factors in RBSA: market yield and currency factors, asset-class factors, equity-market factors, and strategy (style) factors.[3] However, in the application of RBSA to an individual fund or portfolio, typically only strategy factors are used, and strategy indexes are used to represent these factors.

Factors seek to explain the complex behavior of an underlying portfolio simplistically. Thus, to select the relevant factors, it is always better to follow a top-down approach. That is, a convenient way is to propose a wide range of factors and to select the few most important ones that economically reflect the underlying risks. Strategy (or style) factors exercise the role of explaining portfolio returns that might be affected by time-varying styles and risk premiums.

Financial research identified more than 300 factors used for performance analysis. The reason for choosing the strategy or style factors is straightforward and applies to the following. First, style factors have a major advantage over other types of factors in that they are provided as strategies by investment banks and brokerage houses. Thus the methodology of the factor construction is well documented and transparent for investors. Second, these factors are available as trading strategies that can be easily replicated. For example, the foreign-exchange (FX) carry factor can be viewed as, say, five long high-yielding currencies and short positions in, say, five low-yielding currencies. In other words, the strategy factors gauge trading styles that portfolio managers apply. Style factors should correlate or explain fund managers' style. Finally, the strategy factors have in general long out-of-sample time series available that allow for reliable econometric analysis. These important properties make the strategy factors reliable tools to explain portfolio returns.

Factors that seek to capture a specific management style are considered *strategy factors* or *style factors*. To use these style factors in an RBSA, indexes by third-party providers such as investment banks are used. Investment banks have identified a growing demand for such products that might be defined as *passive index strategies*—portfolios using long and short positions to generate a target return. These strategies or indexes use highly liquid instruments such as bonds, spot prices on foreign exchange, swaps, futures, and forward contracts that allow for low-cost turnover and flexibility. Therefore, the most liquid developed markets are usually part of the strategies. Because these strategies are offered to investors as passive vehicles, they have the properties of an index. That is, they underlie

3. This categorization of factors is suggested in a forthcoming book by Gueorgui Konstantinov, Frank J. Fabozzi, and Joseph Simonian, *Quantitative International Bond Portfolio Management*, to be published by the World Scientific Press. Briefly, these are factors that a portfolio manager can measure using market data or other observable variables such as market yield and currency factors (yield-curve exposure factors, interest-rate volatility factor, and foreign-exchange volatility factor). Factors that represent components of portfolio exposure such as bonds, equities, currencies, options, and commodities fall into the category of asset-class factors. For example, bond portfolio asset-class factors include default, term, and duration times spread. For equity markets, studies by Fama and French found that equity returns can be explained by the variability of value, as measured by book-to-market ratio, and size, as measured by market capitalization. These factors fall into the category of equity market factors.

index methodology. A major drawback of strategy (style) indexes is that they depend on third-party providers, are costly, and are subject to market demand and supply for the index.

Following a top-down approach, the easiest way to evaluate the economic relevance of the factors that should be considered is to follow the general index decomposition and sector, country, and currency allocations. Another important determinant of factor choice for RBSA is the liquidity of the underlying bonds. Whereas for investment-grade bonds a liquidity factor might be a matter of choice, the style effects of less liquid bonds require consideration of liquidity factors that gauge the liquidity premium associated with these bonds.

The following properties of the Bloomberg Barclays Global Aggregate Bond Index are important for the choice of the factors (of course, these properties can be associated with the underlying risks):

- Sovereign high-yield corporate bonds exposure;
- Developed- and emerging-markets exposure;
- Local currency and country exposure;
- Duration, maturity, and liquidity issues.

These important properties can be translated into a factor set that captures and reflects the underlying risks of a portfolio. Specifically, the choice might be made more broadly, such as capturing general bond risks—for example, using bond trend and carry strategy factors. Alternatively, the overall bond market risk can be captured using more granular factors that apply to sovereign, high-yield, and corporate bond exposure. In this case, the duration times spread (DTS)[4] and the 10-year U.S. Treasury bond level yield might be a suitable choice.

A CLOSER LOOK AT STYLE FACTORS: AN ILLUSTRATION

To help understand RBSA and the factor selection process, we will use an illustration of an actual index. In our illustration, bear in mind that we are not going to use the index as a benchmark but instead assume that the index is an actual portfolio that we want to analyze using RBSA. In our illustration, the portfolio to which we will apply RBSA is the Bloomberg Barclays Global Aggregate Bond Index. We use this index as our actual portfolio because it allows us to illustrate RBSA where currency decisions are involved, something we did not cover in the previous chapter when we discussed holdings-based attribution analysis. Therefore, when we refer to the portfolio that is the subject of the RBSA, we mean the Bloomberg Barclays Global Aggregate Bond Index (LEGATRUU Index).

The index is representative of global government, corporate, and quasi-sovereign fixed-income markets. It comprises both emerging and developed markets. The bonds included in the index are denominated in multiple currencies. However,

4. See Chapter 23.

as with every index, the Bloomberg Barclays Global Aggregate Bond Index can be expressed in U.S. dollars, among other currencies. This index is intended to represent a global fixed-income market portfolio that allocates to different bond market-related asset styles and comprises more than 20,000 bonds. Explaining the style of an index might be a fundamental part of what portfolio managers must do for their clients because their style might diverge from or replicate the index portfolio. Therefore, a desired task is to explain a complex portfolio by several factors, which is the purpose of RBSA and why we use it as the hypothetical portfolio in our illustration.

Because the Bloomberg Barclays Global Aggregate Bond Index represents the global fixed-income market, it is intuitive to suggest that this index can be explained using multiple factors. Specifically, the way the factors are combined in a model is decisive in determining the performance drivers. Because of the diversity of underlying variables that are intuitively related to international bonds, a regression model should explain, for example, interest-rate risk, currency risk, sector risk, and country risk, among others. Most important, investigating returns requires the use of factors that capture the market risk of bonds.

A set of strategy (style) factors that we believe (and empirical evidence supports) provides a good explanation of bond returns is the following: (1) bond trend, (2) bond carry, (3) FX volatility, (4) FX carry, and (5) rate volatility. Of course, these five factors represent just one of the possible solutions for the set of factors that could be used. Below we provide a brief explanation of factors that are necessary to understand the economic relevance when estimating the style of the Bloomberg Barclays Global Aggregate Bond Index (i.e., the hypothetical portfolio that will be analyzed).

BOND CARRY FACTOR

A bond's *carry* is the net cost of ownership of the bond, which is the difference between the interest received minus the cost of financing the bond. There are investment strategies that focus on carry strategies. Consequently, the *bond carry factor* is a strategy factor. This factor is measured by an index. Bond carry factor portfolios use long and short positions to generate target return using liquid instruments that capture interest-rate differentials. Several index providers offer investable carry strategies for bond investors. As mentioned earlier, these index strategies use liquid bond instruments such as futures contracts, options, and traded cash bonds. Again, these products are strategies that have specified rule-based techniques.

The bond carry index is a cost-efficient way to implement long positions in long-term bonds and short in a short-term bond by using futures contracts. The intuition behind a bond carry strategy is that it harvests the difference in short- and long-term yields, thus providing investors with a constant bond maturity exposure. Typically, a bond carry strategy goes long in 10-year bond futures and short in 2-year futures to capture bond carry. This strategy applies to international markets with sufficient future contract liquidity. Examples are the European, U.S., and

Japanese bond markets. For our illustration, we use the Barclays Bond Futures Carry Index (BXIIBFCU Index).

Bond Trend Factor

The *bond trend factor* seeks to capture the steepening trends in the bond market that result from central bank activities regarding interest-rate policy, changes in economic conditions, and changes in investor preferences (including risk appetites). From a macro perspective, this factor incorporates value and momentum in the bond market. The reliance on macroeconomic variables and central bank policies means that market instruments reflect economic conditions. Because yield-curve steepness reflects changing economic conditions, this factor captures economic trends: inverse yield curves, which have negative steepness, signal recession, and normal or positive steepness means a healthy economic environment. As we will see, both value and trends are incorporated within the steepness of the yield curve.

The strategy uses 2-year to 10-year steepener/flattener positioning in the U.S. or European markets, thus capturing the trends in the changing shape of the yield curve owing to economic activities. A major drawback of strategy (style) indexes is that they depend on third-party providers, are costly, and are subject to market demand and supply for the index. For our illustration, we use the Barclays Bond Trend Index (BXIIXTBP Index).

FX Carry Factor

The *FX carry factor* represents the difference between borrowing in low-yielding currencies and investing in high-yielding ones. In general, carry index strategies are long-short products that follow a rule-based investment process and rebalancing. The index consists of long positions in the several (in most cases three to five) high-yielding currencies and short positions in three to five low-yielding currencies. For example, a widely used carry style index is the Deutsche Bank Currency Harvest Index, which represents a global currency carry factor (Bloomberg ticker is DBHVG10U Index). An alternative is the Deutsche Bank Currency Harvest Index, which invests in global currencies (Bloomberg ticker is DBHVGGUI Index).

FX carry is a reliable and important factor for both bond and currency managers trying to exploit interest-rate differentials worldwide. The methodology for the currency carry factor might be based on the entire yield curve or on specific interest-rate differentials—3-month, 2-year, or even 10-year bond yields.

FX Value Factor

The *FX value factor* represents the under- or overvalued currencies that an investor may consider. This factor is typical for currency investors and is used to explain the fundamental currency valuation. A systematic investing process considers the ranking of the three currencies with the highest average spot-market return adjusted for the purchasing power parity (PPP) exchange rate published by the Organisation for Economic Co-operation and Development (OECD). Like the FX carry factor, there are several notable providers for currency value carry. Thus the index consists of a long position in the three undervalued currencies and short positions in the three overvalued currencies. The Deutsche Bank Currency Value Index is widely used in style evaluation.[5]

The systematic approach that underlies the strategy calculation is to select undervalued currencies against overvalued currency pairs. The intuition of the rule-based approach is to buy (long) the undervalued currencies and sell (short) the overvalued currencies. The trading might be held for a specific holding period of, say, 30 days. The rebalancing of the strategy and its success to some extent depend on the process and methodology to determine PPP by the index provider.

FX Trend Factor

The *FX trend factor* describes the short-term or some of the midterm tendencies in price movements. FX trend is a short-lived strategy that is often used as an overlay strategy that aims to gain from short-term movements in the FX markets. The Deutsche Bank Trend Index is an example of an index that can serve as a proxy for this strategy factor.[6] Alternative indexes are provided by Barclays, Citibank, and Deutsche Bank. The overall intuition behind strategies offered by brokerage houses is a rule-based algorithm that buys (long) several currencies that have positive returns over a specified past period and sells (short) currencies with negative returns in previous periods. One of the most widely used FX trend factors is the Deutsche Bank Currency Momentum factor (Bloomberg ticker is DBMOMUSF Index).

FX Volatility Factor

The FX volatility index is constructed using implied volatilities to proxy for the FX volatility factor. A widely used index that captures risk in foreign exchange is the Deutsche Bank FX Volatility Indicator (Bloomberg ticker CVIX Index). Like the Chicago Board Options Exchange (CBOE) Volatility Index (VIX Index), the Deutsche Bank FX Volatility Index measures the implied volatility of the foreign-exchange market. Thus the index gauges the expected future FX volatility.

Rate Volatility Factor

Investment banks provide data on interest-rate volatilities in the form of specific indexes. Notable examples are the Credit Suisse Interest Rate Volatility Estimate (Bloomberg ticker CIRVE Index) and the Nomura Interest Rate Volatility Risk Premium Index (Bloomberg ticker NMIVRU Index). Implied interest-rate volatility time series are inputs for option prices, and monitoring them thus provides information about the risk to be gained in both asset classes. Exhibit 29–1 shows the correlations between the style factors.

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5. The Bloomberg ticker for the Deutsche Bank Value Index is DBPPPUSF Index.
6. The Bloomberg ticker for the Deutsche Bank Trend Index is DBMOMUSF Index.

EXHIBIT 29–1

Style Factor Correlations from February 2, 2000, to December 27, 2019

Factor	Bond Carry	Bond Trend	FX Carry	FX Trend	FX Value	FX Volatility	Rate Volatility
Bond carry	1.00						
Bond trend	0.17	1.00					
FX carry	0.05	–0.22	1.00				
FX trend	–0.05	0.05	–0.21	1.00			
FX value	0.06	0.07	–0.08	–0.05	1.00		
FX volatility	–0.04	0.27	–0.48	0.17	0.09	1.00	
Rate volatility	0.04	–0.10	0.28	–0.17	–0.09	–0.35	1.00

FX = foreign exchange

REGRESSION MODEL AND ITS ESTIMATION

Applying the RBSA model suggested by Sharpe is typically done to identify the style exposure and return contribution of the specific factors.[7] The general RBSA model is

$$r_t^{\text{portfolio}} = \sum\nolimits_i \beta_i r_t^{\text{factor } i} + \varepsilon_t, \tag{29–1}$$

where

$r_t^{\text{portfolio}}$ = portfolio return for period t;
$r_t^{\text{factor } i}$ = factor i return for period t;
β_i = sensitivity of the portfolio return to the return of factor i (referred to at the beta exposure);
ε_t = random error term in period t.

A condition that must be satisfied by the regression model given by equation (29–1) is that the sum of the estimated betas is equal to 1; that is, $\Sigma_i \beta_i = 1$.

Usually, regression-style models apply over a prespecified historical period. In our illustration, we use weekly returns from February 2, 2000, to December 27, 2019. This corresponds to 935 weekly observations. In our illustration, there are seven style factors, so equation (29–1) can be rewritten as

$$r_t^{\text{portfolio}} = \beta_1 r_t^{\text{bond trend}} + \beta_2 r_t^{\text{bond carry}} + \beta_3 r_t^{\text{FX volatility}} + \beta_4 r_t^{\text{FX carry}} + \beta_5 r_t^{\text{FX trend}} + \beta_6 r_t^{\text{FX value}} + \beta_7 r_t^{\text{rate volatility}} + \varepsilon_t,$$

where $r_t^{\text{portfolio}}$ is the return on the portfolio that we are applying the RBSA to (the Bloomberg Barclays Global Aggregate Bond Index) at time t, and β_i is the exposure

7. Sharpe, "Asset Allocation: Management Style and Performance Measurement."

of the Bloomberg Barclays Global Aggregate Bond Index return associated with the related strategy (style) factor at time t. The estimated β for each style factor is reported in Exhibit 29–2.

Estimation of the style regression indicates that, on average, 30.36% of the portfolio (i.e., Bloomberg Barclays Global Aggregate Bond Index) returns are explained by the FX volatility style factor. That is, more than 30% of the bond returns are associated with FX volatility. Furthermore, bond carry and bond trend account for 46.38% of the style. In addition, approximately 48% of the portfolio return comprises currency-related factors, among which the FX carry style factor is the most important (10.37%). Note that the FX value has zero meaning in the long term.

The static picture, that is, the style over a long period of time from February 2, 2000, to December 27, 2019, provides insightful information indicating that, for example, currencies are responsible for roughly 50% of the return style of the global aggregate bond market. This is important information for portfolio managers who dynamically switch their styles and might time the market. However, bond portfolio managers might be interested in the time-varying and short-term variation of the factors. Therefore, rolling style analysis provides more valuable information. The time series that represents rolling estimations captures the style exposure and thus asset weights over the period $t - k$ through t. We calculated rolling exposure using weekly returns for a rolling sample period of 26 weeks. The results of the rolling-style regression in the period from February 2, 2000 to December 27, 2019 are reported in Exhibit 29–3.

Obviously, the FX volatility style factor has a large short-term impact on bond returns. Bond market trend and bond carry are the two factors that change their loadings through time. The most important takeaway from the style-based regression model in Exhibit 29–3 is that it provides an insightful and detailed picture of the changing style. In a straightforward way, the information shows which style was most important in the specific past period.

The online appendix discusses the economic issues associated with the regression model used for RBSA.

EXHIBIT 29–2

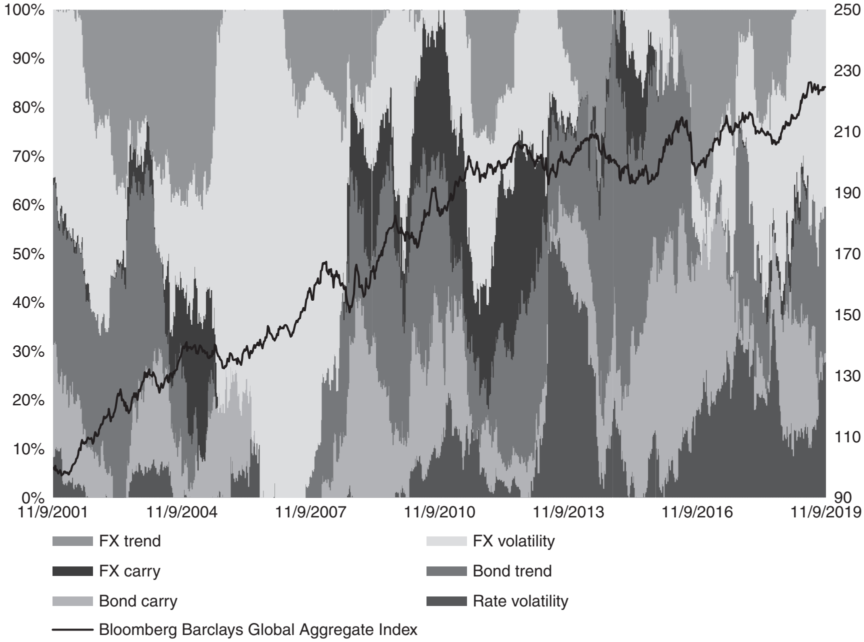
Estimated β for Each Style Factor from February 2, 2000, to December 27, 2019

Factor No.	Style Factor	
1	Bond trend	25.03%
2	Bond carry	21.35%
3	FX volatility	30.36%
4	FX carry	10.37%
5	FX trend	7.67%
6	FX value	0%
7	Rate volatility	5.21%

Source: Data from Bloomberg, LLC; the Bloomberg ticker is LEGATRUU Index.

EXHIBIT 29–3

Dynamic Style of Bloomberg Barclays Global Aggregate Bond Index: February 2, 2000, to December 27, 2019



Note: The style contribution is plotted on the left-hand side; the index level is plotted on the right-hand side. The composition of the index regarding the weight of the strategy (style) factor and the explanatory variables is estimated for equation (29–1). The equation is modified in accordance with the model proposed by Sharpe. The first 26 weekly observations are omitted due to estimations of the first point. The asset weights are calculated on a weekly basis; the index decomposition is aimed at representing the global fixed-income market portfolio.

PART

TEN

STATISTICAL AND OPTIMIZATION TECHNIQUES

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CHAPTER
30

PROBABILITY DISTRIBUTIONS AND STATISTICS

In most instances of fixed-income analysis or fixed-income portfolio management, a decision's outcome depends on the outcome of variables that are not known with certainty at the time the decision is made. Probability theory is used in these circumstances to aid in making decisions. This involves estimating probabilities of the various outcomes that may result from a decision.

Probabilities can be determined through objective or subjective analysis. The *objective approach* to probability theory basically asserts that probabilities relate to long-run frequencies of occurrence. That is, the objective probability of an outcome or event can be defined as the relative frequency with which an event would occur, given a large number of observations. An objectivist would assign a probability of $^1/_6$ (one in six), for example, to the outcome or event that a value of 1 would appear if a fair die is rolled and a probability of $^4/_{52}$ to the outcome or event that an ace of any suit would be drawn from a fair deck of 52 cards. The *subjective approach* basically maintains that probabilities measure the decision maker's degree of belief in the likelihood of a given outcome. It is a broad and flexible approach, allowing assignment of probabilities to outcomes for which no objective data may exist or in circumstances where there may be a combination of objective data and subjective belief.

In the online supplement we explain the basics of probability theory. The focus of this chapter is on the applications of probability theory and the different types of probability distributions.

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

A *random variable* is a variable for which a probability can be assigned to each possible outcome (value) that the variable can take. A *probability distribution* or *probability function* is a function that describes all the values that the random variable can take and the probability associated with each. A *cumulative probability distribution* is a function that shows the probability that the random variable will attain a value less than or equal to each value that the random variable can take on.

For example, suppose that a bond portfolio manager is considering buying a 20.5-year bond with a coupon rate of 10% and selling at par to yield 5%. The

EXHIBIT 30–1

Probability Distribution for the 6-Month Total Return for a 20.5-Year 5% Coupon Bond Selling Initially at Par

Horizon Yield	Probability	Cumulative Probability	Price at Horizon	Total Return	
1	7.00%	5%	5%	$78.64	–18.64%
2	6.50%	8%	13%	$83.34	–14.16%
3	6.00%	10%	23%	$88.44	–9.06%
4	5.50%	16%	39%	$93.98	–3.52%
5	5.00%	22%	61%	$100.00	2.50%
6	4.50%	16%	77%	$106.55	9.05%
7	4.00%	10%	87%	$113.68	16.18%
8	3.50%	8%	95%	$121.45	23.95%
9	3.00%	5%	100%	$129.92	32.42%

Note: Because the initial price is $100 and the 6-month coupon is $2.5 per $100 of par value, the 6-month total return is found as follows: (price at horizon – 100 + 2.5)/100.

next coupon payment is exactly 6 months from now. Suppose also that the portfolio manager has a 6-month investment horizon. The portfolio's total return will depend only on the yield for 20-year bonds 6 months from now. We refer to this yield as the *horizon yield*. The random variable in this illustration is the horizon yield. The portfolio manager believes that only the 9 possible horizon yields for 20-year bonds 6 months from now shown in the second column of Exhibit 30–1 will result. Also shown in the exhibit is the assumed probability of realizing each horizon yield. Notice that the sum of the probabilities is 1.

While the random variable is the horizon yield, the total return also can be considered the random variable. A total return corresponds to each horizon yield, with the probability distribution for the total return the same as the probability distribution for the horizon yield. The cumulative probability distribution for the total return is shown in the last column of Exhibit 30–1. The probability that the total return will be 16.18% is 10%. The probability that the total return will be less than 16.18% is the cumulative probability shown in the fourth column of Exhibit 30–1. The probability is 77%.

DISCRETE VERSUS CONTINUOUS PROBABILITY DISTRIBUTION

A probability distribution can be classified according to the values that a random variable can realize. When the random variable can only take on specific values, then the probability distribution is referred to as a *discrete probability distribution*. For example, in our illustration, we assumed only nine specific values for the random variable. Hence, to this point, we have been working with a discrete probability distribution. If, instead, the random variable can take on any possible value within the range of outcomes, then the probability distribution is said to be a *continuous probability distribution*.

When a random variable is either the price or the return on a traded financial asset, the distribution can be assumed to be a continuous probability distribution. This means that it is possible to obtain, for example, a price of \$95.43231 or \$109.34872 and any value in between. In practice, we know that financial assets are not quoted in such a way. Nevertheless, there is no loss in describing the distribution as continuous. However, what is important in using a continuous distribution is that in moving from one price to the next, there is no major jump. For example, if the price declines from \$95.14 to \$70.50, it is assumed that there are trades that are executed at prices at small increments below \$95.14 before getting to \$70.50. In contrast, if the price can just "jump" from \$95.14 to \$70.50, then the distribution is referred to as a *jump process*.

DESCRIBING A PROBABILITY DISTRIBUTION FUNCTION

There are measures that are used to describe the shape of a probability distribution. The five most common ones are measures of (1) location, (2) dispersion, (3) asymmetry, (4) concentration in tails, and (5) quantiles. We discussed the first three measures in previous chapters using observations from a data set. Here we explain these measures when a probability distribution is used. The first four measures are referred to as *statistical moments*, *central moments*, or simply *moments* of the probability distribution. In this section we discuss these measures and show how they are computed from a set of observations.

Measures of Location

The first moment used to describe a probability distribution is a "typical value" that best describes the data. This typical value is referred to as a measure of *location* or a measure of *central value*. The mean, median, and the mode are examples of measures used to measure location. There is a relationship among these three measures of location, and the relationship depends on another measure, asymmetry, which we will describe shortly. Specifically, for a symmetric distribution, all three measures are the same, but for an asymmetric distribution they differ. We will see the relationship among the three later.

The most commonly used measure of location is the *mean* or *average value*. For a probability distribution, the mean is referred to as the *expected value* and is simply the weighted-average value of the distribution. The weights in this case are the probabilities associated with the random variable X. The expected value of a random variable is denoted by $E(X)$ and is computed using the following expression:

$$E(X) = P_1X_1 + P_2X_2 + \cdots + P_nX_n,$$

where P_i is the probability associated with the outcome X_i.

The expected value for the 6-month total return for the probability distribution in Exhibit 30–1 is found as follows:

$$\begin{aligned} E(X) &= 0.05\,(-18.86\%) + 0.08\,(-14.16\%) + 0.10\,(-9.06\%) \\ &\quad + 0.16\,(-3.52\%) + 0.22\,(2.50\%) + 0.16\,(9.05\%) + 0.10\,(16.18\%) \\ &\quad + 0.08\,(23.95\%) + 0.05\,(32.4\%) = 3.61\%. \end{aligned}$$

The expected value for the 6-month total return is 3.61%.

Dispersion

The second measure of a probability distribution is how spread out the values are that the random variable can realize. This is referred to as the *dispersion* of the probability distribution and is the second moment. The range, variance, and mean absolute deviation are the most commonly used measures of dispersion. The dispersion measure that plays a central role in finance is the variance and its square root, the standard deviation.

The *variance* of a random variable X, denoted by var(X), is computed as

$$\text{var}(X) = [X_1 - E(X)]^2 P_1 + [X_2 - E(X)]^2 P_2 + \cdots + [X_n - E(X)]^2 P_n.$$

Notice that the variance is simply a weighted average of the deviations of each possible outcome from the expected value, where the weight is the probability of an outcome occurring. The greater the variance, the wider is the distribution of the possible outcomes for the random variable. The reason that the deviations from the expected value are squared is so that outcomes above and below the expected value will not cancel each other out.

One problem with using the variance as a measure of dispersion is that it is expressed in terms of squared units of the random variable. Consequently, we use the positive square root of the variance, called the *standard deviation*, as a better measure of the degree of dispersion. Mathematically, this can be expressed as

$$\text{std}(X) = \sqrt{\text{var}(X)},$$

where std(X) denotes the standard deviation of the random variable X.

The variance for the 6-month total return whose probability distribution is given in Exhibit 30–1 is calculated as follows:

$$\begin{aligned} \text{var}(X) &= 0.05(3.61\% - [-18.86\%])^2 + 0.08(3.61\% - [-14.16\%])^2 \\ &\quad + 0.10(3.61\% - [-9.06\%])^2 + 0.16(3.61\% - [-3.52\%])^2 \\ &\quad + 0.22(3.61\% - 2.50\%)^2 + 0.16(3.61\% - 9.05\%)^2 \\ &\quad + 0.10(3.61\% - 16.18\%)^2 + 0.08(3.61\% - 23.95\%)^2 \\ &\quad + 0.05(3.61\% - 32.42\%)^2 = 1.70\%. \end{aligned}$$

The standard deviation is 13.04%, which is the square root of the variance, 1.70%.

As can be seen, the variance is computed by squaring the deviations from the expected value and then weighting or averaging them. The *mean absolute deviation* computes the absolute value of the deviations from the expected value and then weights or averages them.

Asymmetry

A probability distribution may be symmetric or asymmetric around its expected value or mean and is the third moment. Three *symmetric distributions* are shown in Exhibit 30–2, while Exhibit 30–3 shows two skewed distributions.

The most popular asymmetry measure is *skewness*. A negative skewness measure indicates that the probability distribution is skewed to the left; that is, compared to the right tail, the left tail is elongated. This is depicted in panel A of Exhibit 30–3. A positive skewness measure indicates that the probability distribution is skewed to the right; that is, compared to the left tail, the right tail is elongated. Panel B of Exhibit 30–3 provides an illustration of a probability distribution that has positive skewness.

When a distribution is symmetric, the three most popular measures of location are equal (i.e., mean = median = mode). When a distribution has negative skewness,

EXHIBIT 30–2

Three Symmetric Distributions

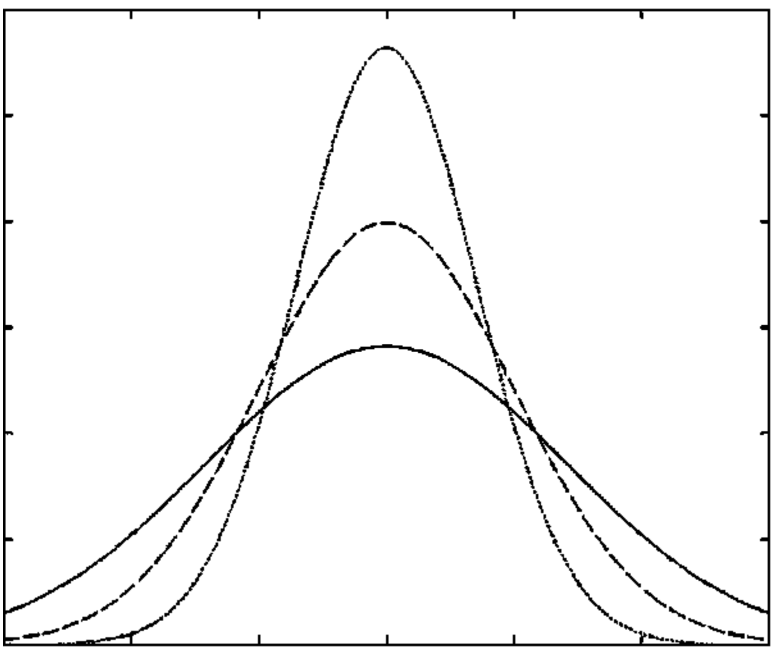
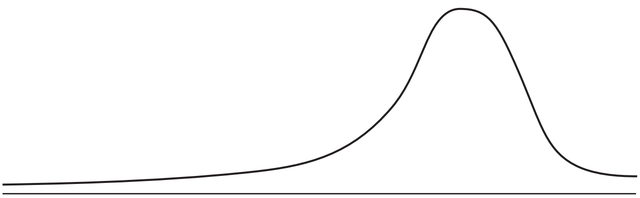


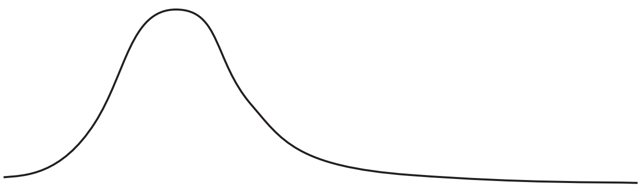
EXHIBIT 30–3

Two Asymmetric Probability Distributions: Skewed Distributions

(A) Distribution Skewed to the Left (Negatively Skewed)



(B) Distribution Skewed to the Right (Positively Skewed)



the relationship between these three measures is mean > median > mode. The reverse is true for a distribution that has positive skewness: mode > median > mean.

The skewness of a distribution can be measured. In the case of a discrete distribution for a random variable X, it is

$$\text{Skewness}(X) = \frac{[X_t - E(X)]^3 P_1 + [X_2 - E(X)]^3 P_2 + \cdots + [X_n - E(X)]^3 P_n}{[\text{std}(X)]^3}.$$

A normal distribution has a skewness equal to 0.

Concentration in Tails

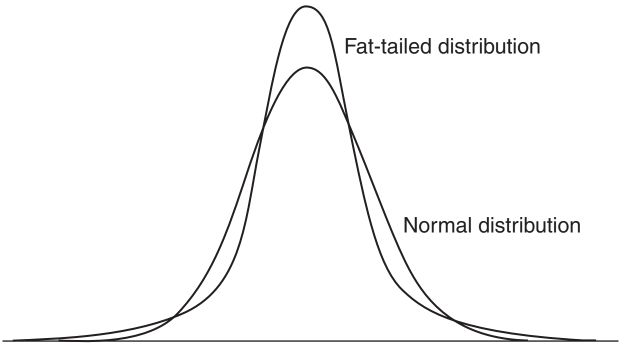
Another helpful measure to describe a probability distribution is the fourth moment, which measures the concentration of potential outcomes in its tails. It is in the tails of a probability distribution that the extreme values are found, thereby providing information about the likelihood for a financial fiasco or financial ruin. Extreme values in a set of data are called *outliers*.[1]

Consider the two symmetric probability distributions shown in Exhibit 30–4. Both probability distributions have the same measure of location or central value. However, the distribution labeled "Normal distribution" (a symmetric distribution

1. In recent years, portfolio managers have examined extreme values of a probability distribution closely. The area of probability theory that focuses on these values is called *extreme-value theory*.

EXHIBIT 30–4

Two Symmetric Distributions: Normal Distribution and Fat-Tailed Distribution



that we will discuss later in this chapter) has less mass in the tails than the distribution that is labeled "fat-tailed distribution."[2] A fat-tailed distribution, also called a *heavy-tailed distribution*, is viewed as "fat" relative to the normal distribution.[3] Notice that the mass in the tails is related to the peakedness of the distribution around the central value. For the fat-tailed distribution, because of the higher peakedness relative to the normal distribution, there is more concentration close to the central value but larger deviations leading to the fat tails.

The joint measure of peakedness and tail fatness is called *kurtosis*. Distributions with a high peak are said to be *leptokurtic*; distributions with a flat peak are said to be *platykurtic*. The kurtosis measure for a discrete probability distribution X is

$$\text{Kurtosis } (X) = \frac{[X_1 - E(X)]^4 P_1 + [X_2 - E(X)]^4 P_2 + \cdots + [X_n - E(X)]^4 P_n}{[\text{std}(X)]^4}.$$

A kurtosis value of 3 means that a distribution is neither leptokurtic (i.e., excessively peaked) nor platykurtic (i.e., excessively flat).

Quantiles

An alternative approach to using moments to describe a probability distribution is to use quantiles. *Quantiles*, also referred to as *fractiles*, are values that divide the distribution such that there is a given proportion of observations below the quantile. For example, the α-quantile is the value of the random variable where α% of the

2. The symmetric fat-tailed probability distribution shown in Exhibit 30–4 is the Cauchy distribution.
3. More precisely, a fat-tailed distribution is a probability distribution in which the extreme portion of the distribution spreads out farther relative to the width of the center of the distribution than is the case for the normal distribution, for which the extreme portion is the part farthest away from the median and the center is the middle 50%.

probability distribution falls below that value. Alternatively, it is the value for the random variable where $100\% - \alpha\%$ is greater than that value.

There are special names for some quantiles:

- The 25%, 50%, and 75% quantiles are referred to as the *first quartile*, *second quartile*, and *third quartile*, respectively;
- The 20%, 40%, 60%, and 80% quintiles are referred to as the *first quintile, second quintile, third quintile*, and *fourth quintile*, respectively;
- The 1%, 2%, …, 98%, 99% quantiles are called *percentiles*.

DISCRETE PROBABILITY DISTRIBUTIONS

Three of the most important discrete probability distributions used in fixed-income analysis are the Bernoulli distribution, the binomial distribution, and the Poisson distribution. These distributions are commonly used in credit-risk modeling. We provide a brief description of each below.

Bernoulli Distribution

The *Bernoulli distribution* is a discrete probability distribution that can take on only *two* possible outcomes, usually referring to the outcome of interest as a *success* and the other outcome as a *failure*. The probability of realizing a success is denoted by p and the probability of realizing a failure is therefore $1 - p$. Such experiments are referred to as *Bernoulli trials*. The classic example is coin tossing, where success can be defined as realizing a head, failure is then realizing a tail, and the probability of realizing a head is 0.50. In credit risk modeling, if what is of interest to a portfolio manager is the number of companies that will default over the next year, then the "default" of a company would be labeled a "success" and the survival of a company labeled a "failure."

The Bernoulli distribution is the building block for more complicated discrete probability distributions used in credit-risk modeling. One such distribution is the binomial distribution.

Binomial Distribution

The *binomial distribution* gives the discrete probability distribution of obtaining exactly k successful outcomes out of n independent Bernoulli trials when the probability of success is equal to p. Letting k ($k \le n$) equal a random variable representing the number of successes observed, then the probability of observing k successes is given by

$$B(k, n, p) = \binom{n}{k} p^k (1-p)^{n-k},$$

where $\binom{n}{k}$ means

$$\frac{n!}{(n-k)!k!}.$$

Recall that the factorial of a positive integer n is denoted by $n!$ and is equal to $n(n-1)(n-2)\cdot \cdots \cdot 2\cdot 1$. For example, if n is 10 and k is 7, then

$$\begin{aligned}\binom{10}{7} &= \frac{10!}{(10-7)!7!}\\ &= \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(3\cdot 2\cdot 1)(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}\\ &= \frac{3{,}628{,}800}{30{,}240} = 120.\end{aligned}$$

To illustrate the binomial distribution, suppose that the Bernoulli trial involves flipping 10 coins and we are interested in knowing the probability of realizing 7 heads, where a head is labeled a success. Then, we know that

$n = 10;$

$k = 7;$

$p = 0.5.$

Substituting these values into the formula for the binomial distribution:

$$\begin{aligned}B(7,10,0.5) &= \binom{10}{7}(0.5)^7(1-0.5)^{10-7}\\ &= 120(0.0078125)(0.125) = 0.1172 = 11.72\%.\end{aligned}$$

In credit-risk modeling, a portfolio manager is concerned not with just the default of a single company but also with the many companies whose debt obligations are included in the portfolio. Assuming that n companies have the same annualized probability of default p, the probability that k of these companies will default can be obtained from the binomial distribution. A situation in which a portfolio manager needs this information is if the portfolio manager has a position in a credit default swap on a basket of companies because payments by a credit-protection seller may depend on the number of companies that default.

To illustrate the binomial distribution, suppose that a portfolio manager wants to estimate the probability that of the 20 companies in his portfolio, 2 companies will default over the next year. Also assume that the probability of each company defaulting is the same and is estimated to be 3%. In this application of the binomial distribution, we have

$n = 20;\ k = 2;\ p = 0.03.$

Substituting these values into the formula for the binomial distribution:

$$\begin{aligned}B(2,20,0.03) &= \binom{20}{2}(0.03)^2(1-0.03)^{20-2}\\ &= 0.0988 = 9.88\%.\end{aligned}$$

Note that the assumption is that the Bernoulli trials are independent. When applying this to credit-risk modeling, this assumption must be carefully investigated because the default of one credit in a portfolio may be highly correlated with the default of other credits.

Poisson Distribution

The *Poisson distribution* depends on only one parameter, λ, and can be interpreted as an approximation to the binomial distribution. A Poisson-distributed random variable is usually used to describe the random number of events occurring over a certain time interval. We used this previously in terms of the number of defaults. One main difference compared with the binomial distribution is that the number of events that might occur is unbounded—at least theoretically. The parameter λ indicates the rate of occurrence of the random events; that is, it tells us how many events occur on average per unit of time and is referred to as the *intensity parameter*.

The probability distribution of a Poisson-distributed random variable N is described by the following equation:

$$P(N = k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda}, \qquad k = 0, 1, 2, \ldots.$$

The Poisson distribution occurs in the context of finance as a generic distribution of a stochastic process, called a *Poisson process*, that is used to model the time of default in some credit-risk models. To understand the use of the Poisson distribution in a credit-risk model, let's briefly describe the two general models that are in use today: structural models and reduced-form models.[4]

Structural models, also known as *firm-value models*, were pioneered by Black, Scholes, and Merton.[5] The basic idea, common to all structural-type models, is that a company defaults on its debt if the value of the assets of the company falls below a certain default point. In these models it has been demonstrated that default can be modeled as an option, and as a result, researchers were able to apply the same principles used for option pricing to the valuation of risky corporate securities. The application of option pricing theory avoids the use of risk premium and tries to use other marketable securities to price the option.

4. For a further discussion of these models, see Chapters 8 and 9 in Mark J. P. Anson, Frank J. Fabozzi, Moorad Choudhry, and Ren-Raw Chen, *Credit Derivatives: Instruments, Applications, and Pricing* (Hoboken, NJ: John Wiley, & Sons 2004).

5. Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, Vol. 81 (1973), pp. 637–654; Robert Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics* (Spring 1973), pp. 141–183; and Robert Merton, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, Vol. 29 (1974), pp. 449–470.

Reduced-form models, most notably the Jarrow–Turnbull and Duffie–Singleton models,[6] do not look inside the firm. Instead, they model directly the likelihood of default or downgrade.[7] Modeling a probability has the effect of making default a surprise—the default event is a random event that can occur suddenly at any time. All we know is its probability. The Poisson process is the theoretical framework for reduced-form models.

CONTINUOUS PROBABILITY DISTRIBUTIONS

We now turn to continuous probability distributions. We will discuss just two distributions: the normal probability distribution and the stable distribution. The normal distribution is by far the most often assumed distribution in applications in finance. The central theories in finance assume that the return distribution is normally distributed. However, the preponderance of evidence for all asset classes and countries that have examined whether return distributions are normal rejects this assumption.[8] An alternative distribution is the stable distribution, of which the normal distribution is a special case.

We begin with a description of the normal distribution, and it will become clear why it is convenient to use this distribution despite the overwhelming evidence that it fails to reflect what occurs in real-world financial markets.

Normal Probability Distribution

Exhibit 30–5 shows a graphical presentation of the normal distribution or, as it is also referred to, a *Gaussian distribution*. The area under the normal distribution (or normal curve) between any two points on the horizontal axis represents the probability of obtaining a value between those two values. For example, the probability of realizing a value for the random variable X that is between X_1 and X_2 in Exhibit 30–5 is shown by the shaded area. Mathematically, the probability of realizing a value for X between these two values can be written as

$$P(X_1 < X < X_2).$$

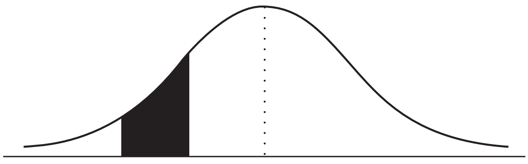
6. Robert Jarrow and Stuart Turnbull, "Pricing Derivatives on Financial Securities Subject to Default Risk," *Journal of Finance*, Vol. 50 (1995), pp. 53–86; and Darrell Duffie and Kenneth Singleton, "Modelling the Term Structure of Defaultable Bonds," *Review of Financial Studies,* Vol. 12 (1977), pp. 687–720.

7. Not only is the current probability of default modeled, some researchers attempt to model a "forward curve" of default probabilities, which can be used to price instruments of varying maturities.

8. For a discussion of this evidence, see Svetlozar T. Rachev, Christian Menn, and Frank J. Fabozzi, *Fat-Tailed and Skewed Asset Return Distributions: Implications for Risk Management, Portfolio Selection, and Option Pricing* (Hoboken, NJ: John Wiley & Sons, 2005).

EXHIBIT 30–5

Normal Distribution (or Normal Curve)



The entire area under the normal curve is equal to 1. This follows from the second rule of probability theory, which states that the sum of the probabilities of all possible simple events is one.

The normal distribution has the following properties:

1. The point in the middle of the normal curve is the expected value for the distribution.
2. The distribution is symmetric around the expected value. That is, half the distribution is to the left of the expected value and the other half is to the right. Thus the probability of obtaining a value less than the expected value is 50%. The probability of obtaining a value greater than the expected value is also 50%. Mathematically, this is expressed as

$$P[X < E(X)] = 0.5 \qquad \text{and} \qquad P[X > E(X)] = 0.5.$$

3. The probability that the actual outcome will be within a range of one standard deviation above the expected value and one standard deviation below the expected value is 68.26%, rounded to 68.3%.
4. The probability that the actual outcome will be within a range of two standard deviations above the expected value and two standard deviations below the expected value is 95.46%, rounded to 95.5%.
5. The probability that the actual outcome will be within a range of three standard deviations above the expected value and three standard deviations below the expected value is 99.74%, rounded to 99.7%.

Notice that the smaller the standard deviation, the narrower is the range of the possible outcome for a given probability.

Suppose, for example, that the 1-year total return for a portfolio has an expected value of 7% and a standard deviation of 4% and that the probability distribution is a normal distribution. The probability is 68.3% that the 1-year total return will be between 3% (the expected value of 7% minus one standard deviation of 4%) and 11% (the expected value of 7% plus one standard deviation of 4%).

The probability is 95.5% that the 1-year total return will be between −1% (the expected value minus two standard deviations) and 15% (the expected value plus two standard deviations).

Because the normal distribution is a continuous distribution, one cannot obtain a probability for a specific value. Rather, a range for the value of a random variable can be obtained. There are four ways such values can be obtained:

- Using a normal distribution table;
- Using a function in a spreadsheet like Excel;
- Using a scientific calculator;
- Using an online source.

Standard statistics textbooks demonstrate how to use a normal distribution to calculate the probability of realizing a value in a certain range. Here will discuss the use of Excel to determine probabilities.

Excel has the following function that provides the cumulative probability to a specified value given the mean and standard decision:

```
=norm.dist(x,mean,stand_dev,cumulative),
```

where x is the specified value for the random variable. For the input `cumulative`, there is a choice of `TRUE-cumulative probability distribution` and `FALSE-probability mass function`. Select `TRUE`.

For example, suppose that a life insurance company has estimated that the value of its high-yield corporate bond portfolio is normally distributed with an expected value of $10 million and a standard deviation of $7 million. What is the probability that the portfolio value will be less than or equal to $14 million? In this example, the mean is used for the expected value. The value for x in this example is $14 million. Then the entry would be

```
=norm.dist(14,10,4,TRUE).
```

The cumulative probability returned by Excel would be 71.61%. This is the probability of getting a value that is less than $14 million.

If the insurance company wants to know the probability of realizing a portfolio value greater than $14 million, then it is calculated by subtracting from 100% the probability of getting a value that is less than $14 million. That is, the probability of realizing a value greater than $14 million is 23.39% (100% – 71.61%).

Suppose that the insurance company wants to know the probability of realizing a value between $9 million and $14 million. This is done by subtracting the probability of realizing a value less than or equal to $14 million (71.61%) from the probability of realizing a value less than or equal to $9 million. To find the latter value, we use

```
=norm.dist(9,10,4,TRUE),
```

which gives a probability of 44.32%. Therefore, the probability of realizing a portfolio value between \$9 million and \$14 million is 27.29% (71.61% – 44.32%).

Stable Distribution

Corporate bonds are well known to have nonnormal probability distributions that have negative skewness attributable to downgrading (i.e., a lowering of a bond's or issuer's credit rating) and default events. For assets whose returns or prices exhibit fat-tail attributes, nonnormal distribution models are required to accurately model the tail behavior and compute the probabilities of extreme returns.

Various nonnormal distributions have been proposed for modeling extreme events.[9] The one we describe in this section is the *stable Paretian distribution* (also known as *Lévy stable* and *alpha stable distribution*), which will be referred to simply as the *stable distribution*. The reason we focus on stable distributions is that they have attractive enough mathematical properties to be a viable alternative to normal distributions in trading, optimization, and risk-management systems.[10]

The stable distribution was suggested by Benoit Mandelbrot in 1963.[11] Despite both the theoretical arguments presented by Mandelbrot in favor of using the stable distribution and the subsequent preponderance of evidence supporting that distribution and rejecting the normal distribution reported by researchers, market practitioners have failed to embrace it. One attack on the stable distribution was that there is no closed-form solution to obtain the necessary information about the distribution. While this may have been a valid criticism at one time, advances in computational finance make it fairly straightforward to fit observed returns to determine the parameters of a stable distribution. Thus this criticism is no longer valid.

What is important to understand about the stable distribution is the parameters and their meaning. They are

- α, which determines the tail weight or the distribution's kurtosis, with $0 < \alpha \leq 2$;
- β, which determines the distribution's skewness;
- σ, which is a scale parameter;
- μ, which is a location parameter.

---

9. These include, in addition to the distribution that we will describe, mixtures of two or more normal distributions, Student *t*-distributions, hyperbolic distributions, gamma distributions, and extreme-value distributions.
10. A major drawback of all alternative models is their lack of stability, where stability is a technical term that basically means that the distribution family of the returns does not depend on the time interval over which the returns are considered. The properties that make them attractive are described in Rachev, Menn, and Fabozzi, *Fat-Tailed and Skewed Asset Return Distributions*. A more technical discussion is provided in Svetlozar T. Rachev and Stefan Mittnik, *Stable Paretian Models in Finance* (Chichester, UK: John Wiley & Sons, 2000).
11. Benoit B. Mandelbrot, "The Variation of Certain Speculative Prices," *Journal of Business*, Vol. 36 (1963), pp. 394–419.

When the β of a stable distribution is zero, the distribution is symmetric around μ. Stable distributions allow for skewed distributions when $\beta \neq 0$ and fat tails; this means a high probability for extreme events relative to the normal distribution when $\alpha < 2$. The value of β can range from −1 to +1. When β is positive, a stable distribution is skewed to the right; when β is negative, a stable distribution is skewed to the left.

The value of α is greater than zero and does not exceed 2 (i.e., $0 < \alpha \leq 2$). As the value of α decreases, the distribution exhibits fatter tails and more peakedness at the origin. Increasing (decreasing) values of β result in skewness to the right (left).

There are three special cases of the stable distribution. The case where $\alpha = 2$ (and $\beta = 0$, which plays no role in this case) and with the reparameterization in scale, yields the normal distribution. Thus the normal distribution is one of the three special cases of the stable distribution. The case where $\alpha = 1$ and $\beta = 0$ yields the *Cauchy distribution*, with much fatter tails than the normal distribution. The Cauchy distribution, which is the fat-tailed distribution shown in Exhibit 30–4, is the second case of the stable distribution. The last special case is obtained for $\alpha = 0.5$ and $\beta = 1$. In this case, we have the *Lévy distribution*.

CONFIDENCE INTERVALS

When a range for the possible values of a random variable and a probability associated with that range are calculated, the range is referred to as a *confidence interval*. In general, for a normal distribution, the confidence interval is calculated as follows:

(expected value − standardized value × standard deviation)
to (expected value + standardized value × standard deviation).

The standardized value indicates the number of standard deviations away from the expected value and corresponds to a particular probability. For example, suppose that a portfolio manager wants a confidence interval of 95%. This means that there will be 2.5% in each tail. From Exhibit 34–7, we see that a standardized value with a 2.5% probability is 1.96. Thus a 95% confidence interval is

(Expected value − 1.96 × standard deviation)
to (expected value + 1.96 × standard deviation).

Suppose that a portfolio manager wants to construct a confidence interval for the rate of return on a portfolio. Assuming that the rate of return is normally distributed with an expected value of 7% and a standard deviation of 4%, then a 95% confidence interval would be

$$(7\% - 1.96 \times 4\%) \text{ to } (7\% + 1.96 \times 4\%) \qquad \text{or} \qquad -0.84\% \text{ to } 14.84\%.$$

Notice that the larger the standard deviation, the wider is the confidence interval. For a confidence interval with a smaller probability, the standardized value is smaller, and therefore the confidence interval is narrower for a given standard deviation.

HYPOTHESIS TESTING

The testing of hypotheses is an important application of probability theory. *Hypothesis testing* involves several steps.

Setting Up the Null and Alternative Hypotheses

The first step in hypothesis testing is to make an assertion about a parameter of interest. This hypothesis is referred to as the *null hypothesis* and typically is denoted by H_0. Here are a few examples of null hypotheses that a bond portfolio manager might be interested in testing:

- The recovery rate of a high-yield corporate bond is 35% (i.e., H_0 = 35% recovery rate);
- The percentage of high-yield corporate bonds upgraded to investment grade in 1 year is 10% (i.e., H_0 = 10% high-yield corporate bond upgrade to investment grade);
- The correlation between investment-grade bond returns and stock returns is 25% (i.e., H_0 = 25% correlation between investment grade and stock returns);
- The skew of the return distribution for the Bloomberg Barclays Global Aggregate Bond Index is 0 (H_0 = 0 skewness for the index return distribution).

In the next chapter we discuss the estimation of parameters of a statistical model such as that obtained from regression analysis. Hypothesis testing is used to evaluate some hypotheses for one or more estimated parameters. For example, suppose that a bond portfolio manager believes that there is a relationship between some financial accounting ratios and the credit rating assigned to a firm's debt. The null hypothesis would be that there is no relationship. As another example, in estimating a relationship between a bond's sensitivity to changes in Treasury yields, as is done in Chapter 16 to estimate empirical duration, suppose that the bond portfolio manager believes that the duration is 6.0, and suppose further that the duration of the benchmark is 6.4. Using hypothesis testing, the portfolio manager can test whether the computed empirical duration is statistically different from the benchmark duration.

Hypothesis testing is used in backtesting bond portfolio strategies. In such strategies, the objective might be to outperform a bond index by, say, at least 30 basis points after considering transaction costs and management fees. Suppose that via backtesting it is determined that the strategy produces an average return of 27 basis points. While 27 basis points is clearly below the minimum target of 30 basis point, there is variation in the backtested returns (i.e., there is a probability distribution generated from the strategy return). That variation must be taken into account in hypothesis testing to determine if the average of 27 basis points is statistically below 30 basis points.

The *alternative hypothesis* is the statement that the null hypothesis is not true and has a value different from what is asserted by the null hypothesis. Commonly denoted by H_1, the alternative hypothesis can be that the value that is being tested is not equal to the value assumed by the null hypothesis or that the value is either greater than or equal to the value assumed by the null hypothesis. Hypothesis testing in the former case is referred to as a *two-tail test*, while in the later case it is referred to as a *one-tail test*.

Calculate from Data the Value of the Parameter Tested

After specifying the null and alternative hypotheses, the second step is to calculate the value of the variable or parameter of interest. For example, consider the estimation of the recovery rate for a high-yield corporate bond. Suppose that the null hypothesis is 35% and the portfolio manager has accumulated data on a large sample of defaulted bond issues and found that the average value is 33%.

Decision Rule for Accepting or Rejecting the Null Hypothesis

Hypothesis testing is used to determine given the variation in the sample of default bonds whether the average recovery rate of 33% is different from 35%. In hypothesis testing, the terminology used is whether the 33% is "statistically significantly different" from 35%. Hypothesis testing involves taking into consider the variation (as measured by the standard deviation) when comparing the calculated value from the sample and the value assumed by the null hypothesis.

That is, is the difference statistically large enough to be considered a reason to reject the null hypothesis? This requires that a decision rule be established to determine whether to accept or reject the null hypothesis. To make this determination, a quantity called a *test statistic* is computed that depends on the observed and hypothesized parameters. In some cases, this test statistic follows a known probability distribution. Knowing that the test statistic follows a specific probability distribution, the portfolio manager can determine whether the computed test statistic obtained from the sample is "very far" from the center of the known probability distribution, which tells us whether the sample estimate we obtained would be a rare occurrence if the hypothesized value were true.

There are different ways to measure the rarity of the observed test statistic. Statistical software packages typically report a *p-value*, which is the probability that if the null hypothesis is true, a more extreme sample statistic than the one observed from the sample would be obtained. If the p-value is "small" (the actual cutoff is arbitrary, but the values typically used are 1%, 5%, or 10%), then the null hypothesis is rejected. This is because a small p-value means that there is a very small probability that a statistic as extreme as the one obtained from the sample would occur if the hypothesis is true.

Types of Errors in Hypothesis Testing

No matter how carefully a hypothesis test is designed, there is the risk of committing an error by making the wrong decision. The two possible errors that can occur are characterized by unintentionally deciding against the true hypothesis and are referred to as the *Type I error* and *Type II error*. The decisions that can be made can be summarized in the following four categories:

Decision About Null Hypothesis	Null Hypothesis Is True	Null Hypothesis Is False (i.e., Alternative Hypothesis Is True)
Accept	Correct	Error (Type II error)
Reject	Error (Type I error)	Correct

In two cases, the decision is correct. In the two other cases, there is an error that is commonly labeled Type I error or Type II error. That is,

Type I Error: The error resulting by rejecting the null hypothesis given that it is actually true. Type I error is also referred to as an α *error*.

Type II error: The error resulting from not rejecting the null hypothesis even though the alternative hypothesis is true. Type II error is also referred to as a β *error*.

Let's explain these errors in backtesting strategies. In evaluating an investment strategy, the null hypothesis (H_0) is that the strategy is not profitable. The errors made in hypothesis testing as they apply to evaluating investment strategies can be referred to as a *false discovery error* and a *missed discovery error*. A false discovery error is made when an investment strategy is found to be profitable when in fact it is not. This is a Type I error (the null hypothesis is rejected when it is in fact true). A missed discovery error results when an investment strategy is found to be unprofitable when in fact it is profitable. This is a Type II error (the null hypothesis is not rejected when it is actually false).

CHAPTER
31

REGRESSION AND PRINCIPAL COMPONENT ANALYSIS

Today, fixed-income portfolio analysis and management frequently draw on tools from the field of statistics. Within the field of statistics there are various specialty areas. The two most often used to deal with problems in the fixed-income space are econometrics, the subject of this chapter, and data science, the subject of Chapter 35. *Econometrics* is the branch of economics that draws heavily on statistics for testing and analyzing economic relationships. Further specialization within econometrics, and the area that directly relates to the topics covered in this book, is *financial econometrics*. Financial econometrics involves the modeling and forecasting of financial data such as asset prices, asset returns, interest rates, financial ratios, defaults and recovery rates on debt obligations, and risk exposure.[1]

The most commonly used tool in bond portfolio analysis that draws from the toolkit of financial econometrics is the linear regression model. This tool is used to estimate the relationship between two or more variables. We have already discussed two applications in earlier chapters. In Chapter 29 we discussed the use of regression analysis for returns-based style analysis. In Chapter 16 we explained what empirical duration is and how it is estimated using regression analysis. Other tools include multiple discriminant analysis, linear probability models, prohibit regression models, and logit regression models, which were described in Chapter 23.

Our objective here is to explain how regression analysis is used in fixed-income portfolio management. We will not provide the equations needed to estimate the parameters of a regression model because software is available on spreadsheets, in online software, and from commercial vendors. Instead, in addition to the application of regression analysis, we will discuss how to interpret the output of a regression model, the issues associated with estimation of a regression model, and the different types of regression models. Another econometric tool described in this chapter that has been applied in fixed-income analysis is principal component analysis.

1. Robert Engle and Clive Granger, two econometricians who shared the 2003 Nobel Prize in Economic Sciences, have contributed greatly to the field of financial econometrics. In Chapter 18 we discussed the autoregressive conditional heteroscedasticity (ARCH) model for estimating volatility, which was developed by Engle.

SIMPLE LINEAR REGRESSION MODEL

Suppose that a bond portfolio manager wants to estimate the relationship between the yield on A-rated medium-term industrial bonds and the yield on 10-year Treasuries. Assume that the portfolio manager believes the industrial bond/Treasury yield relationship can be expressed as

$$\text{Industrial yield} = b_0 + b_1(\text{Treasury yield}). \tag{31–1}$$

If the values for b_0 and b_1 can be estimated, the bond portfolio manager can use the industrial bond/Treasury yield relationship to estimate the yield on A-rated medium-term industrial bonds for a given yield level of 10-year Treasuries. The values b_0 and b_1 are called the *parameters* of the model. The parameter b_0 is referred to as the *intercept term* in the regression. The objective of regression analysis is to estimate the parameters.

There are several points to understand about this relationship. First, there are only two variables in the relationship—the yield on A-rated medium-term industrial bonds and the yield on 10-year Treasuries. Because there are only two variables and the relationship is linear, the regression model is called a *simple linear regression* or a *univariate regression*. Because it is assumed that the yield on the industrial bond depends on the yield on 10-year Treasuries, the former variable is referred to as the *dependent variable*. The yield on 10-year Treasuries is referred to as the *explanatory variable* or *independent variable* because it is what is assumed to explain the yield on the industrial bonds. When there is more than one explanatory or independent variable, the regression model is called a *multiple regression model*.

Second, it is highly unlikely that the estimated industrial bond/Treasury yield relationship above will describe the true relationship between the two yields exactly because other factors besides the level of 10-year Treasuries may influence the yield on new A-rated medium-term industrial bonds. Consequently, this industrial bond/Treasury yield relationship may be described more accurately by adding a random error term to the relationship. That is, the industrial bond/Treasury yield relationship can be expressed as

$$\text{Industrial yield} = b_0 + b_1(\text{Treasury yield}) + \text{random error term}. \tag{31–2}$$

This equation can be written in shorthand notation as

$$Y = b_0 + b_1X + e, \tag{31–3}$$

where

Y = yield on A-rated medium-term industrial bonds;

X = yield on 10-year Treasuries;

e = random error term.

Equation (31–3) is referred to as the *simple linear regression model*.

Given data on historical returns for A-rated medium-term industrial bonds and 10-year Treasuries, the values for the parameters b_0 and b_1 can be estimated using a methodology referred to as the *method of least squares*. As mentioned earlier, it is not necessary to provide the equations for calculated b_0 and b_1 because they are programmed in spreadsheets and available online and from commercial vendors of statistical packages. The key is understanding how to interpret the resulting values and determine how good the relationship is.

To illustrate this, we will use the relationship of 45 monthly observations in Exhibit 31–1. Because we have observations over time, we are estimating a *time-series regression model*. If we denote an observation using the subscript t, then equation (31–3) can be written as

$$Y_t = b_0 + b_1 X_t + e_t. \tag{31–4}$$

For example, for the third observation ($t = 3$), equation (31–4) is

$$9.800 = b_0 + b_1(8.983) + e_3.$$

For the sixth observation ($t = 6$), equation (31–4) is

$$9.950 = b_0 + b_1(9.057) + e_6.$$

The values for e_3 and e_6 are referred to as the *observed error terms for the observation.* Note that the value of the observed error term for both observations will depend on the values selected for b_0 and b_1. This suggests a criterion for selecting the two parameters: the parameters should be estimated so that the sum of the observed error terms for all observations is as small as possible.

Although this is a good standard, it presents one problem. Some observed error terms will be positive, and others will be negative. Consequently, positive and negative observed error terms will offset each other. To overcome this problem, each error term could be squared. On the basis of this criterion, the objective then would be to select parameters so as to minimize the sum of the square of the observed error terms. This is precisely the criterion used to estimate the parameters in regression analysis. Because of this property, regression analysis is referred to as the method of least squares.

This criterion is portrayed graphically in Exhibit 31–2, a generic diagram (referred to as a *scatter diagram*) with an arbitrarily estimated regression line drawn. Consider point L in Exhibit 31–2, which represents a specific observation. If a line is dropped from point L to the X axis at point N, the observed value for Y is LN, and the estimated value of Y based on the estimated regression line is MN. The observed error term is the difference between the observed value for Y and the estimated value for Y. In Exhibit 31–2, the difference is the vertical difference LM.

The estimated line in Exhibit 31–2 overestimates the observed Y for all observed values below the line and underestimates all observed points above the estimated line. Squaring the observed error term removes the possibility that the under- and overestimates will offset each other. By squaring and summing the observed errors, this cannot occur.

EXHIBIT 31–1

Data for Treasury and Industrial Bond Yields for 45 Months to Illustrate a Simple Linear Regression

	Yield on	
Observation. *I*	**Treasury *X***	**Industrial *Y***
1	9.057%	9.900%
2	9.140	10.000
3	8.983	9.800
4	9.298	10.250
5	9.279	10.100
6	9.057	9.950
7	8.598	9.550
8	8.079	9.000
9	7.808	8.700
10	8.256	9.150
11	8.298	9.250
12	7.913	9.000
13	7.833	8.950
14	7.924	9.100
15	8.418	9.380
16	8.518	9.550
17	8.636	9.650
18	9.028	10.050
19	8.599	9.550
20	8.414	9.400
21	8.341	9.200
22	8.854	9.875
23	8.800	10.000
24	8.620	9.850
25	8.252	9.500
26	8.069	9.350
27	8.011	9.200
28	8.036	9.100
29	8.059	9.050
30	8.013	8.900
31	8.059	8.875
32	8.227	9.050
33	8.147	9.000

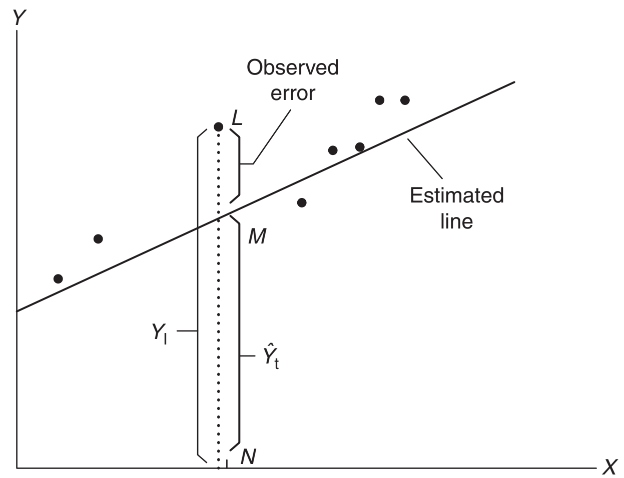
EXHIBIT 31–1

(*Continued*)

	Yield on	
Observation. ***I***	**Treasury** ***X***	**Industrial** ***Y***
34	7.814	8.600
35	7.448	8.250
36	7.462	8.250
37	7.378	8.125
38	6.700	7.600
39	7.281	8.000
40	7.257	8.050
41	7.530	8.230
42	7.583	8.300
43	7.325	8.000
44	7.123	7.750
45	6.709	7.300

EXHIBIT 31–2

Graphical Illustration of Error Term



The objective in estimation of the regression line is to select a value for the parameters— b_0 and b_1—so as to minimize the sum of the squares of the observed error terms. The formulas used to estimate the parameters on the basis of this criterion are derived using calculus and are provided in all statistics textbooks. When the data in Exhibit 31–1 are used, the estimated relationship for the industrial bond/Treasury yield relationship using the method of least squares is

$$Y = 0.0649 + 1.1052X$$

Goodness of Fit of the Relationship

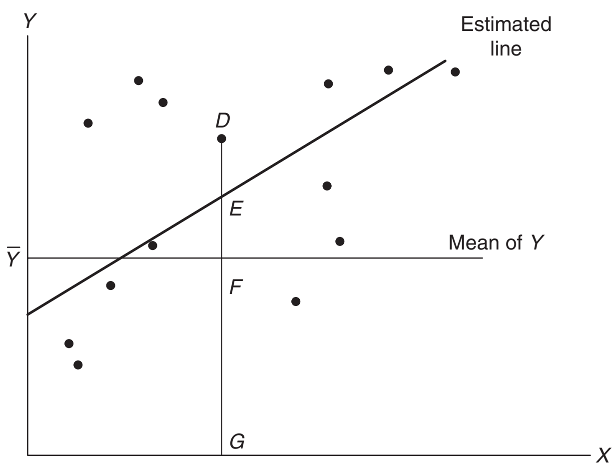
A bond portfolio manager will want to know how "good" an estimated relationship is before relying on it for any investment strategy, so statistical tests are needed to determine in some sense how good the relationship is between the dependent variable and the explanatory variable. A measure of the *goodness of fit* of the relationship is the *coefficient of determination*.

Exhibit 31–3 demonstrates the meaning and measurement of the coefficient of determination. We know that the explanatory or independent variable X is being used to try to explain movements in the dependent variable Y. That is, the variable X is trying to explain why the variable Y would deviate from its mean. If no explanatory variable is used to try to explain movements in Y, the method of least squares would give the mean of Y as its value estimate. Thus the ability of X to explain deviations of Y from its mean is the factor of interest.

In Exhibit 31–3 the observed value of Y at point D deviates from the mean of Y as measured by the vertical distance DF. It is the deviation of Y from its

EXHIBIT 31–3

Meaning and Measurement of the Coefficient of Determination



mean—that is, it is DF—that the variable X must explain. This DF difference is made up of two components. The first component is EF, which is the difference between the value estimated by the regression line and the mean value of Y. This distance is the amount of deviation from the mean of Y explained by X or, equivalently, explained by the regression line.

The second component is DE, which is the deviation of Y from its mean that is still unexplained. Consequently, the deviation of the observed Y from the mean of Y can be expressed as follows:

$$DF = EF + DE,$$

where

DF = deviation of the observed Y from the mean of Y;

EF = amount of deviation of the observed Y from the mean of Y explained by X;

DE = amount of deviation of Y from the mean of Y still unexplained.

If the deviations of the observed Y from the mean of Y are squared and all these deviations are summed for each observation, the resulting value indicates the total deviations of the observed Y from the mean of Y. This is referred to as the *total sum of squares*. The second component is the deviations of the observed Y from the mean of Y that are still unexplained. This component is referred to as the *unexplained sum of squares*. Thus

$$\text{Total sum of squares} = \text{explained sum of squares} + \text{unexplained sum of squares}.$$

How much of the total sum of squares is explained by X is what we are interested in—one way, therefore, to measure how good the relationship is to determine what percent of the total sum of squares is explained by X. That is, how good the relationship is can be measured by dividing the explained sum of squares by the total sum of squares. This ratio is the coefficient of determination:

$$\text{Coefficient of determination} = \frac{\text{explained sum of squares}}{\text{total sum of squares}}.$$

The coefficient of determination can take on a value between 0 and 1. If the total sum of squares is fully explained by X, the coefficient of determination is 1. When none of the total sum of squares is explained by X, the coefficient of determination is 0. Hence the closer the coefficient of determination is to 1, the stronger is the relationship between the variables.

Another interpretation of the coefficient of determination is that it measures how close the observed points are to the regression line. The nearer the observed points are to the regression line, the closer the coefficient of determination will be to 1. The farther the scatter of the observed points from the regression line, the closer the coefficient of determination will be to 0.

The coefficient of determination is commonly referred to as R^2. This value is provided in software that calculates the estimated parameters of the regression model. In our illustration, R^2 is 0.96. This means that approximately 96% of the total variation in the yield on new A-rated medium-term industrial bonds is explained by the yield on 10-year Treasuries.

A warning is in order about the coefficient of determination. A coefficient of determination close to 0 does not necessarily imply that there is no relationship between the two variables. There may in fact be a strong relationship, but it may not be a linear relationship.

Correlation Coefficient

The coefficient of determination is related to the *correlation coefficient*, which measures the association between two variables. No cause and effect are assumed in calculation of a correlation coefficient, so there is no assumption about which of two variables is the dependent variable and which is the explanatory variable. This is different from regression analysis, which assumes that one variable is dependent on the other.

The coefficient of determination turns out to be equal to the square of the correlation coefficient. Thus the square root of the coefficient of determination is the correlation coefficient. Because the coefficient of determination can be between 0 and 1, the correlation coefficient will be between −1 and 1. The sign of the correlation coefficient is the same as the sign of the slope of the regression b. In our illustration, because the coefficient of determination is 0.96, the correlation coefficient is 0.98. The *covariance* measures how the two random variables vary together.

Standard Error of Estimate and Confidence Interval for Forecasted Value

Knowing how to assess the strength of the relationship between two variables, we can now show how to use the estimated regression model to obtain a forecasted value for the dependent variable Y. Continuing estimation of the relationship between the yield on 10-year Treasuries (X) and the yield on A-rated medium-term industrial bonds (Y), suppose that the portfolio manager believes that the yield on 10-year Treasuries next month will be 7%. X is therefore 7. The forecasted yield for A-rated medium-term industrial bonds (Y) is

$$Y = 0.0649 + 1.1052\,(7) = 7.801.$$

The forecasted value of 7.801% is referred to as a *point estimate* because only one value is forecasted. In practice, a bond portfolio manager would want a range for the estimated dependent variable, along with a probability that the actual value will fall within that range. To accomplish this, the *standard error of the estimate* is provided by an output of the software, and that value is used to calculate what is called a *confidence interval* or *range* for the forecast for a given probability assigned to the interval. (In the previous chapter we discussed a confidence interval.) For example, in our illustration, a confidence interval for a 95% confidence level is 7.713% –7.801%.

The interpretation of a 95% confidence interval is as follows: if the process of constructing this confidence interval is repeated a large number of times, then 95% of those intervals will contain the true value of the estimate of the value sought. In our illustration, a 95% confidence interval means that if the confidence interval is constructed a large number of times, the true value of the yield on new A-rated medium-term industrial bonds will be in the interval 7.713%–7.801% at least 95% of the time.

MULTIPLE LINEAR REGRESSION MODEL

In many real-world applications, a dependent variable is best explained by more than one explanatory variable. When a regression model has more than one explanatory variable, it is said to be a *multiple linear regression model* and has the following general form:

$$Y = b_0 + b_1X_1 + b_2X_2 + \ldots + b_KX_K + e \tag{31–5}$$

where

Y = dependent variable;

X_k = independent variable k;

K = number of independent variables;

b_k = coefficient of the independent variable k ($k = 1, ..., K$);

b_0 = intercept term;

e = error term.

We will not explain how to compute the parameters of the model ($b_{0,}$ b_1, $b_2, \ldots, b_K$) here.

Binary Variables

To estimate a relationship, it is not necessary to restrict selection of explanatory variables to quantitative variables, that is, a variable that can be expressed numerically. Qualitative variables, those expressed by a nonnumerical property, also can be used.

Suppose that the bond portfolio manager believes that the industrial bond/Treasury yield relationship differs in times of economic recession and economic prosperity. Such a hypothesis can be tested by using a qualitative variable as follows: if the observation occurs in a period of economic recession, the value for the variable for the observation is equal to 1, and if the observation occurs in a period of economic prosperity, the value for the variable for the observation is equal to 0.

A qualitative variable used in regression analysis is commonly referred to as a *binary variable* or a *dummy variable*. In general, a binary variable takes on a value of 1 if the observation has the attribute assumed by the qualitative variable, and a value of 0 otherwise.

There are regression models in which the dependent variable is a binary variable. We discussed such models in Chapter 23.

Model Significance

Diagnosing the quality of some model is essential before employing it. Therefore, it is necessary to establish criteria for determining model quality. If, according to some criteria the fit is determined to be insufficient, we might have to redesign the model by including different independent variables or conclude that the regression has established no relationship.

We know from our discussion of the simple linear regression that the goodness-of-fit measure is the coefficient of determination (R^2). That measure is also used in multiple regression analysis. As with a simple linear regression, the coefficient of determination measures the percentage of variation in the dependent variable explained by all of the independent variables employed in the regression. The R^2 of the multivariate linear regression is referred to as the *multiple coefficient of determination*.

Before using the results of a regression model, one needs to verify the model by determining its statistical significance. (These statistical tests are described in basic statistics textbooks.) To do so, the software provides the overall model's significance and also the significance of the individual regression parameters (($b_{0,}$ $b_1, b_1, b_2, \ldots, b_K$). The estimated regression errors play an important role as well.

Testing for the Significance of the Model

Interpretation of the coefficient of determination is much the same for multiple regression as in simple linear regression. In the latter case, it is the total sum of squares explained by the explanatory variable X. In multiple regression, the coefficient of determination is the total sum of squares explained by all the explanatory variables. By adding an explanatory variable to a regression model, the belief is that the new explanatory variable will increase the explained sum of squares significantly.

For example, suppose that a simple linear regression is estimated and that the total sum of squares is 1,000 and the explained sum of squares is 600. Suppose that another explanatory variable is added to the regression model and that this inclusion of the explanatory variable increases the explained sum of squares from 600 to 750. Thus the coefficient of determination would rise from 60% to 75% (750/1,000). This new explanatory variable would appear to have contributed substantially to explaining the variation in the dependent variable. Had the explained sum of squares increased from 600 to 610, the coefficient of determination would increase from 60% to only 61%, in which case the new explanatory variable would not appear to do much to help explain the dependent variable.

There are statistical tests that are used to test whether a relationship estimated using regression analysis is statistically significant. There are also tests to determine whether an increase in the explained sum of squares attributable to the inclusion of an additional explanatory variable is statistically significant. Such tests are described in textbooks on regression analysis.

In many applications we can make the model more complex by adding more explanatory variables that financial theory suggests might explain the dependent variable. However, it is important to avoid creating a multiple regression model that is more complicated than necessary. A good guideline is to use the simplest model suitable. Complicated and refined models tend to be inflexible and fail to work with different samples. In most cases they are poor models for forecasting purposes. Thus the best R^2 is not necessarily an indicator of the most useful model. The reason is that one can artificially increase R^2 by including additional explanatory variables into the regression model, but the resulting seemingly better fit may be misleading. One will not know the true quality of the model if one evaluates it by applying it to the same data used for the fit. However, often if one uses the fitted model for a different set of data, the weakness of the overfitted model becomes obvious.

It is for this reason that a redefined version of the coefficient of determination is needed and is called the *adjusted* R^2 and is provided in the software output. This adjusted goodness-of-fit measure considers the number of observations as well as the number of explanatory variables. As long as the number of observations is very large compared with the number of parameters to be estimate, R^2 and adjusted R^2 will be approximately the same. However, if the number of explanatory variables included increases, the adjusted R^2 drops noticeably compared with the original R^2. This new measure of fit can be interpreted as penalizing excessive use of explanatory variables. Instead, in formulating the regression model it should be set up as parsimonious as possible. To take the most advantage of the set of possible explanatory variables, one should consider those that contribute a maximum of explanatory variation to the regression. That is, a balance must be found between the cost of additional explanatory variables and the reduction in the adjusted R^2.

Testing for the Significance of the Independent Variables

Suppose that the estimated multiple regression model has been found to be significant. Then it is necessary to determine (i.e., test) for the statistical significance of the coefficient of the individual explanatory variables. The appropriate statistical test is the Student t test or, simply, t test that is reported of each coefficient by the software. Accompanying the reported t test is what is referred to as a *p value*. This is the value used in hypothesis testing described in the previous chapter.

Test for the Inclusion of Additional Explanatory Variables

Suppose that a bond portfolio manager has estimated a multiple linear regression with K explanatory variables. Suppose that the bond portfolio manager wants to determine whether it is appropriate to add another explanatory variable to the regression model. There is a test statistic measuring the improvement in the goodness of fit due to the additional variable. Fortunately, there is a statistical test, called the *F* test, that can be used to determine if it is appropriate. This value and its statistical significance are provided by the software.

Diagnostic Tests for Multicollinearity

When discussing the suitability of a multiple linear regression model, there are several important tests that must be performed after the model's significance has been determined and all significant explanatory variables to be used in the final regression have been determined. One of these tests involves checking for the presence of multicollinearity.

Multicollinearity means that the explanatory variables in a multiple linear regression are highly correlated so that it becomes difficult to identify how a particular explanatory variable affects the dependent variable. Investigation for the presence of multicollinearity involves examining the correlation between the independent variables and the dependent variable.

Intuition is helpful in assessing whether the estimated regression coefficients make any sense. For example, one by one, every explanatory variable is investigated, excluding all other explanatory variables (i.e., estimate a simple linear regression), to see whether for this particular explanatory variable the regression coefficient seems unreasonable, because if its sign is counterintuitive or its value appears too small or large, one may want to consider removing that independent variable from the regression. It could very well be that this outcome is due to multicollinearity. Technically, the presence of multicollinearity is caused by explanatory variables in the regression model that contain common information. The explanatory variables are highly intercorrelated. As a consequence, the presence of multicollinear independent variables prevents us from obtaining insight into the true contribution to the regression from each explanatory variable.

Although it is not possible to provide a general rule to eliminate the problem of multicollinearity, there are some techniques that can be employed to mitigate the problem. Multicollinearity might be present if there appears to be a mismatch between the sign of the correlation coefficient and the regression coefficient of that particular independent variable. Thus the first place to always check is the correlation coefficient for each independent variable and the dependent variable. Other indicators of the presence of multicollinearity are the sensitivity of the regression coefficients to the inclusion of additional explanatory variables and changes from significance to insignificance of already included explanatory variables after new ones have been added. A consequence of this is that the regression coefficient estimates vary dramatically as a result of only minor changes in the data.

A commonly employed tool provided by regression analysis software to detect multicollinearity is the *variance inflation factor* (VIF). What the VIF does is estimate how much of a regression is increased (i.e., inflated) as a result of the presence of multicollinearity in the model. The VIF indicates the percentage of the variance for the estimate of an explanatory variable is inflated. The minimum value for the VIF is 1, with higher values indicating levels of multicollinearity. A VIF of 1 means that the explanatory variables are not correlated, a value between 1 and 5 means moderate correlation, and a value exceeding 5 means high correlation.

Diagnostic Tests of Violation of the Error Term

In addition to testing for multicollinearity, there are two tests used to determine whether two assumptions about the error term are violated: heteroscedasticity and autocorrelations. An assumption in applying the linear regression model is that the variance of the probability distribution for the error term does not depend on the level of any of the explanatory variables. That is, the variance of the error term is constant regardless of the level of the explanatory variable. If this assumption holds, the error terms are said to be *homoscedastic*; if this assumption is violated, the variance of the error term is said to be *heteroscedastic*.

Many time-series finance data such as bond returns exhibit heteroscedasticity, where the error terms may be expected to be larger for some observations or periods of the data than for others. There are several tests that have been used to detect the presence of heteroscedasticity, but we will not discuss them here.[2]

If heteroscedasticity is detected, the issue then is how to construct models that can deal with this feature of the residual variance so that valid regression coefficient estimates and models are obtained for the variance of the error term. There are two methodologies used for dealing with heteroscedasticity: weighted-least-squares estimation technique and autoregressive conditional heteroscedastic (ARCH) models. We described the second method in Chapter 18 because of its importance for not just testing for heteroscedasticity but for forecasting volatility.

With respect to autocorrelation, the relevant assumption is that the error terms are uncorrelated from observation to observation. When they are correlated, it said that the error terms are *autocorrelated*. In time-series analysis, this means there is significant correlation between the error term in two consecutive time periods. From an estimation perspective, correlation of the error terms is critical, as explained below. Autocorrelation is quite common in financial data.

Autocorrelation, which is also referred to as *serial correlation* and *lagged correlation* in time-series analysis, like any correlation, can range from −1 to +1. A positive autocorrelation means that if the computed error term in period t is positive (negative), then the computed error term that follows in period $t + 1$, tends to be positive (negative). Positive autocorrelation is said to exhibit *persistence*. A *negative autocorrelation* means that a positive (negative) computed error term in period t tends to be followed by a negative (positive) computed error term in period $t + 1$.

The presence of significant autocorrelation in a time series means that, in a probabilistic sense, the series is predictable because future values are correlated with current and past values. From an estimation perspective, the existence of

2. These tests include the White generalized heteroscedasticity test, the Park test, the Glejser test, the Goldfeld–Quandt test, and the Breusch–Pagan–Godfrey test (Lagrangian multiplier test).

autocorrelation complicates the testing of the statistical significance of the regression coefficients because variances (which are used for testing the statistical significance of the regression coefficients) may be significantly underestimated, and the resulting statistical test is questionable.

There are several tests for autocorrelation of residuals that can be used. Two such tests are the Durbin–Watson test and the Dickey–Fuller test. The software will provide the results of these tests. In the case of the Durbin–Watson test, the statistic that is reported is typically denoted by d. If the computed value for d is close to 2, this means that there is no autocorrelation. If the computed value is around 4, this means that there is negative autocorrelation; if it is near 0, there is positive autocorrelation.

There are models for dealing with the problem of autocorrelation in time-series data. These models are called *autoregressive moving average* (ARMA) models.

Application of Regression Analysis for Relative-Value Analysis

Bond portfolio managers and traders seek to identify a benchmark curve that can be used to determine the fair value of a bond. Then, based on the fair-value curve (or fair-value baseline), deviations from the curve represent return enhancement opportunities. Bonds, for example, that are above the fair-value curve are viewed as cheap, whereas bonds below the fair-value curve are viewed as rich.

Relative-value analysis as a tool for bond analysis has been around since the 1980s. One of the first such models was developed by Salomon Brothers, which at the time was the largest bond-trading house.[3] The Salomon Brothers' model, the "2+ term structure fair value model" was used to identify rich and cheap bonds trading in the U.S. Treasuries market. Other models were developed by third parties, most notably by Gifford Fong Associates.[4]

The fair-value curve is typically estimated using regression analysis. Typically, several estimated fair-value curves are used as a check for the robustness of the resulting identification of rich and cheap bonds. Identification of rich and cheap bonds involves more sophisticated statistical analysis than merely the position of the bonds above or below the fair-value curve. For example, one fixed-income quantitative developer, RiskVal Financial Solutions, uses a bond's Z-score in identifying rich and cheap analysis. The Z-score is basically the number of standard deviations of the current value from the mean value. The larger the absolute value of the Z-score, the greater is the mispricing. The assumption when using

3. Jordan Hu, Marc Seah, and Xu Gao, "Relative Value Trading," Chapter 35 in Frank J. Fabozzi (ed.), *The Handbook of Fixed Income Securities*, 9th ed. (New York: McGraw-Hill, 2021), p. 810.
4. See H. Gifford Fong and Frank J. Fabozzi, *Fixed Income Portfolio Management* (Homewood, IL: Dow Jones–Irwin, 1985).

the Z-score is that the current level of a bond will revert to the mean level (i.e., that the pricing will mean revert).

STEPS IN APPLYING REGRESSION ANALYSIS

There are three steps in applying regression analysis that help bond portfolio managers, analysts, and traders in formulating investment strategies:

Step 1. Model selection;

Step 2. Model estimation;

Step 3. Model testing.

Model Selection

Model selection involves the selection of a family of models with given statistical properties. This entails the mathematical analysis of the model properties as well as financial economic theory to justify the model choice. It is in this step that one or more members of the portfolio management team decide to use, for example, the tool or tools from the field of econometrics or machine learning (discussed in Chapter 35).

Generally, it is believed that one needs a strong economic intuition to choose models. For example, it is economic intuition that might suggest what bond risk factors are likely to produce good forecasting results or under what conditions one might expect to find processes that tend to revert to some long-run mean. One way to think about model selection is as an adaptive process where economic intuition suggests some family of models, but ultimately the models need to pass rigorous statistical testing.

In contrast, a portfolio manager or analyst might also use an approach purely based on data, an approach referred to as *data mining*. Although this approach might be useful in certain circumstances, it must be applied with great care. Data mining is based on using very flexible models that adapt to any type of data and letting statistics make the selection. The risk is that one might capture special characteristics of the sample that will not be repeated in the future. While judicious use of data mining might suggest true relationships that may be buried in the data, it could result in serious misrepresentations of risks and opportunities.

Criterion for Determining the Best Model

Model selection relies on goodness-of-fitness tests to evaluate the performance of an econometric model in terms of how well it explains the data. Suppose that a portfolio manager, trader, or analyst wants to select the best model from among several candidates. There is a criterion that can be applied for doing so. Given a data set, the objective is to determine which of the candidate models best approximates the data (accepting the fact that models only approximate reality). This involves trying to minimize the loss of information. The field of *information theory* is used to quantify or measure the expected value of information. Although the information-theoretic approach allows the derivation of more than one criterion to select the best model, the two most commonly used criteria are the Akaike information criterion and the Bayesian information criterion.[5]

The *Akaike information criterion* (AIC) is generally considered the first model-selection criterion that should be used in practice. While it is not important to present the formula for AIC because most econometric software provides an AIC score, it is important to understand that the calculation involves the number of explanatory (independent) variables used in the model and how well the model explains the data in out-of-sample prediction.[6] An AIC score is computed for each candidate model. The model that is the best fit is the one that explains the greatest percentage of the variation in the dependent variable with the least number of explanatory variables. In terms of the AIC, it is the model with the smallest AIC score. It is important to bear in mind that the AIC score is a relative measure of the quality of a model. Consequently, all the candidate models could be poor models, and therefore, the best model based on the AIC score may not be a good model.

The other criterion is the *Bayesian information criterion* (BIC), also referred to as the *Schwartz information criterion* and *Schwartz Bayesian information criterion.* It is based on information theory but set within the context of the Bayesian approach to probability theory. The difference between the BIC and the AIC is the greater penalty imposed for the number of explanatory variables by BIC versus AIC. As with the AIC, the best model is the one with the lowest BIC score from among the candidate models.

Model Estimation

Models are embodied in mathematical expressions that include a number of parameters that have to be estimated from sample data. Estimation of these parameters is the second step in financial modeling. Suppose that a bond analyst decides to model returns on a major bond market index such as the Bloomberg Barclays U.S. Aggregate Bond Index using regression analysis discussed in this chapter. This requires the estimation of the regression coefficients, identified using historical data. Estimation provides the link between reality and models. We choose a family of models in the model-selection phase and then determine the optimal model in the estimation phase.

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5. Other criteria such as the Vapnik–Chervonenkis (VC) theory of learning offer a solid theoretical foundation for selecting the best model. Practical applicability of the VC theory of learning is complex and has not yet found a broad following in finance.
6. The commonly used method to estimate the model is the maximum-likelihood method. Basically, the log-likelihood measure (which is used to calculate the AIC score) indicates how likely it is for a given model to see the observed data. The one with the maximum likelihood is considered the model that is the best fit for the data. It turns out that in the formula for the AIC score, the one with the largest maximum likelihood penalized for the number of explanator variables is the one with the lowest AIC.

Model estimation then involves (1) finding estimators and (2) understanding the behavior of estimators. Estimators are never really equal to the theoretical values of the parameters whose estimate is being sought. Estimators depend on the sample and only approximate the theoretical values. There are four estimation methods used in financial econometrics: the least-squares method (with ordinary-least-squares method, weighted-least-square method, and generalized-least-squares method), the maximum-likelihood method, the method of moments, and the Bayesian method. A discussion of these methods is beyond the scope of this chapter but is provided in all econometric books.

Model Testing

The first two steps, model selection and model estimation, are performed using historical data. For this reason, there is the risk that the model fitted to historical data captures characteristics that are specific to the sample data used but are not general and will not recur in future samples. For example, a model estimated in a period of particularly high returns for bond might give the wrong indication about the true average bond returns. Consequently, there is the need to test models on data different from the data on which the model was estimated, referred to as *out-of-sample data*. This is the third step in financial modeling, referred to as *backtesting a model*.

PRINCIPAL COMPONENT ANALYSIS

Principal component analysis (PCA) is a tool used to parsimoniously represent data. In PCA, the modeler begins with a variable that the modeler wishes to explain and a large number of variables that are believed to explain the movement in the variable of interest. PCA involves transforming the large number of explanatory variables into a smaller number of uncorrelated variables that are called *principal components* (PCs) and are a linear combination of the explanatory variables. It is also common to refer to the PCs as *factors*. Once the PCs are identified, the researcher seeks to give an economic interpretation for each PC.

More specifically, let

Y = dependent variable;

X_k = explanatory variable k ($k = 1, 2, \ldots, K$);

F_j = PC or factor j ($j = 1, 2, \ldots, J$);

$J < K$.

Then

$$F_j = c_{j1}X_1 + c_{j2}X_2 + \cdots + c_{jK}X_K.$$

The coefficients c_{jk} are the parameters to be estimated by PCA and are referred to as the *factor loadings*. The resulting PCs or factors are such that they are uncorrelated. The first PC is the principal component with the most explanatory power. The second PC is the principal component with the second-largest explanatory power, and so on.

Applications of PCA

There are two applications in bond portfolio management where PCA has been employed. The first application of PCA, which is quite common, is explaining the movements in the yield curve. The resulting PCs are then used to measure yield-curve risk. Once yield-curve risk is measured in terms of PCs, the factor loadings can be used to

- Construct hedges that neutralize exposure to changes in the direction of interest rates;
- Construct hedges that neutralize exposure to changes in nonparallel shifts in the yield curve;
- Structure yield-curve trades.

Axel and Vankudre illustrate how this is done.[7] They also present evidence that using PCA to measure and control interest-rate risk is superior to the traditional approaches using duration and key-rate duration. In out-of-sample hedges that they performed using PCA, they found significantly lower profit-and-loss variance than duration-neutral hedges.

The second application of PCA analysis is to identify risk factors beyond changes in the term structure. For example, given historical bond returns and variables that are believed to affect bond returns, PCA can be used to obtain PCs that are linear combinations of the variables that explain the variation in returns. We describe the empirical evidence based on PCA for both applications below.

Empirical Evidence Using PCA to Explain Yield-Curve Dynamics

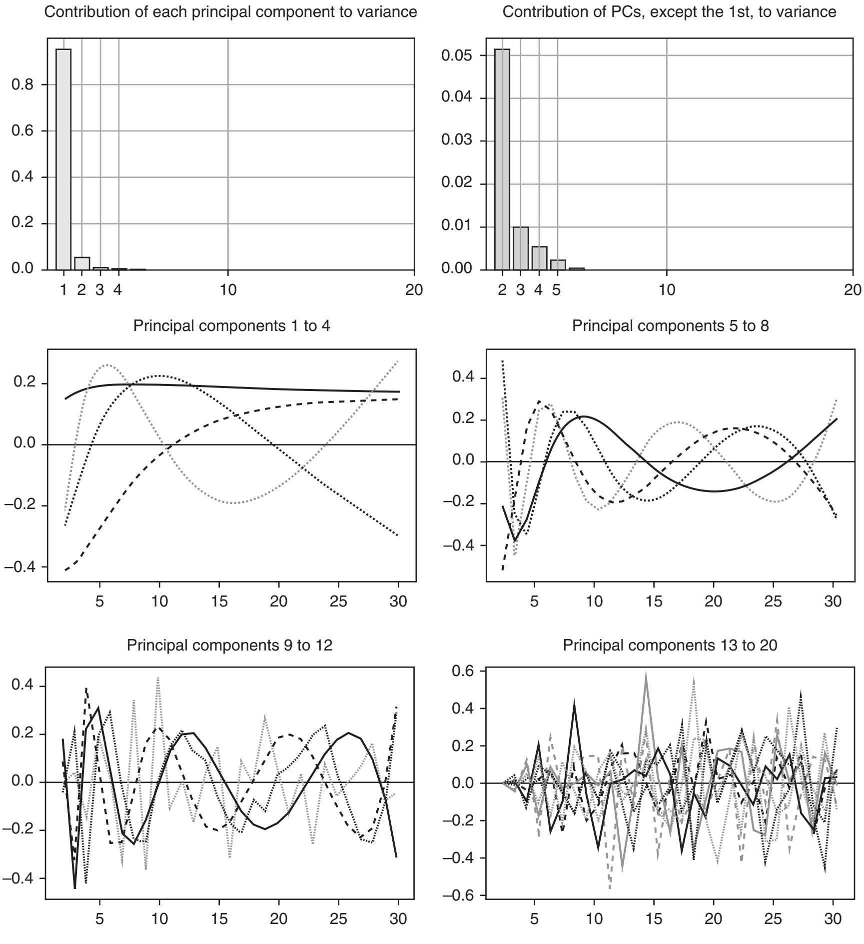
Empirical studies using PCA have investigated the factors that have affected the historical returns on Treasury portfolios. A study by Litterman and Scheinkman found that three factors explain historical bond returns for U.S. zero-coupon Treasuries.[8] The first PC is changes in the level of rates, the second PC is changes in the slope of the yield curve, and the third PC is changes in the curvature of the yield curve.

More recent analysis is provided by Phoa, who applied PCA to U.S. Treasury yields from 1984 to 2020.[9] The first 20 PCs are shown in Exhibit 31–4. The first

7. Ralph Axel and Prashant Vankudre, "Managing the Yield Curve with Principal Component Analysis," in Frank J. Fabozzi (ed.), *Professional Perspectives on Fixed Income Portfolio Management*, Vol. 3 (Hoboken, NJ: John Wiley & Sons, 2002).
8. Robert Litterman and Jose Scheinkman, "Common Factors Affecting Bond Returns," *Journal of Fixed Income* (June 1991), pp. 54–61.
9. Wesley Phoa, "Empirical Yield-Curve Dynamics and Yield-Curve Exposure," Chapter 33 in Frank J. Fabozzi (ed.), *Handbook of Fixed Income Securities*, 9th ed. (New York: McGraw-Hill, 2021), pp. 743–775.

EXHIBIT 31–4

Principal Component Analysis, U.S. Treasuries Yields, 1984–2020



Source: Reproduced from Wesley Phoa, "Empirical Yield-Curve Dynamics and Yield-Curve Exposure," Chapter 33 in Frank J. Fabozzi (ed.), *Handbook of Fixed Income Securities*, 9th ed. (New York: McGraw-Hill, 2021), p. 747.

four PCs explain most of the observed variance (98.5%): the first PC explains 92%, the second PC about 5%, the third PC about 1%, and the fourth PC about 0.5%. The remaining PCs explain the balance of 1.5%. Phoa attributes the first three PCs as follows: the first PC (shown by the solid line in the exhibit) is the change due to a parallel shift in the yield curve; the second PC (shown by the dashed line) is due to changes in the slope of the yield curve; the third PC (shown by the dark dotted line) is attributable to change in the hump of the yield curve, with the peak of the hump at around the 10-year maturity, and the fourth PC (shown by the light dotted line) is based less on intuition and is referred to by Phoa as a "snake shift." None of the other PCs seem to be basically noise.

Phoa also applied PCA to compare four other countries, Canada, the United Kingdom, Germany, and Japan, using government issues with 3-month to 10-year maturities. The results are shown in Exhibit 31–5. Notice that for all four countries, relative to the United States, the relative importance of the first level shift is less. The implication is that using duration as a measure of the sensitivity of a portfolio to a parallel shift in the yield curve in Canada, the United Kingdom, Germany, and Japan does a poorer job than in the United States.

Empirical Evidence on Bond Risk Factors

Using PCA, Gauthier and Goodman have empirically identified the risk factors that generate nominal excess returns for a popular bond index at the time, Salomon Smith Barney Broad Investment Grade Index (SSB BIG Index), for the period January 1992 to March 2003.[10] The results of their PCA for the first six PCs were as follows: the first three PCs explained 98.1% of the variation in nominal excess returns. However, these three components are different from those found in the studies that focused only on explaining the factors that explain yield-curve movements.

The first PC (factor 1) explains 92.7% of the variation. How do we know how to interpret the first principal component or risk factor? First, while we do not report the average duration of each sector of the SSB BIG Index here, it turns out that the order of magnitude of the factor loading on each of the sectors looks very much like the average duration for each sector. To confirm, Gauthier and Goodman did two things. First, they looked at a scatter plot of the return on factor 1 versus the change in the 10-year Treasury yield. Factor 1 had a very clear linear relationship with changes in interest rates. Second, they looked at the correlation of each of the first three PCs with various market measures (such as the slope of the 2–10 spread, the 5-year cap volatility, etc.). They found that the 10-year yield had a correlation of −89% to nominal returns.

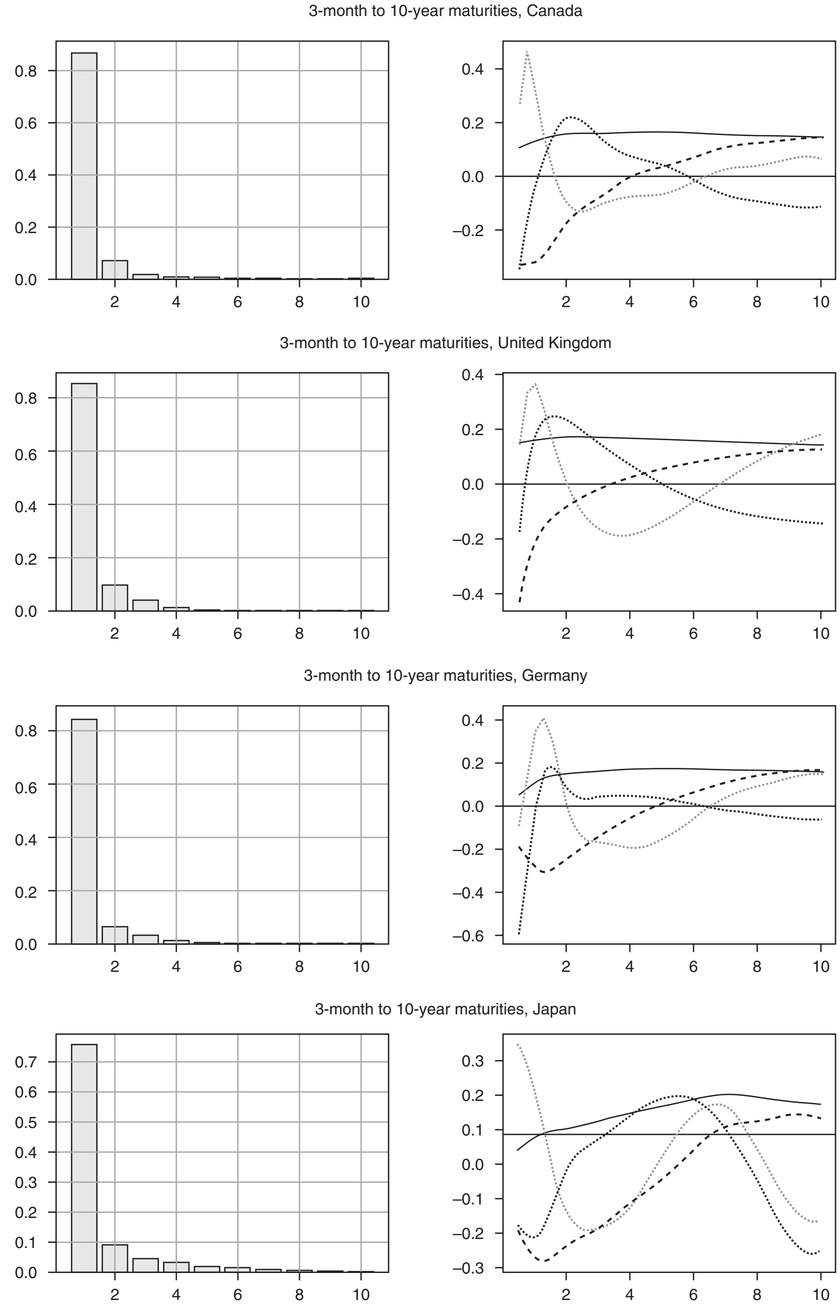
The second PC (factor 2) explained 3.1% of nominal excess returns. Gauthier and Goodman identify this factor as the credit-specific factor because of the high negative factor loadings on the credit index combined with a high positive weighting on Treasuries. They confirm this by looking at the correlation between factor 2 and the Standard & Poor's 500 Index. The correlation was –0.5. The weight on the credit index was –0.82, indicating that the lower the S&P 500, the lower corporate bond returns will be.

Gauthier and Goodman identify the third PC (factor 3) as an optionality factor. This can be supported by noting that the factor loading on the asset classes

10. Laurent Gauthier and Laurie Goodman, "Risk/Return Trade-Offs on Fixed Income Asset Classes," in Fabozzi (ed.), *Professional Perspectives on Fixed Income Portfolio Management*, Vol. 4 (Hoboken, NJ: John Wiley & Sons, 2003). In addition to nominal excess returns, Gauthier and Goodman also analyzed duration-adjusted excess returns. Only the results for the nominal excess returns are discussed here.

EXHIBIT 31–5

Principal Component Analysis, Other Countries



Source: Reproduced from Wesley Phoa, "Empirical Yield-Curve Dynamics and Yield-Curve Exposure," Chapter 33 in Frank J. Fabozzi (ed)., *Handbook of Fixed Income Securities*, 9th ed. (New York: McGraw-Hill, 2021), p. 752.

that have some optionality (callable agencies, mortgage-backed securities, and asset-backed securities) is positive, whereas the factor loading on the noncallable series (Treasuries, noncallable agencies, and credit) is negative. This third PC that represents optionality is consistent with studies of the movements of the yield curve discussed earlier because it reflects market factors such as the shape of the curve and volatility. Gauthier and Goodman show that there is a very positive correlation between the optionality factor and the slope of the yield curve but a negative relationship with 5-year cap volatility. This suggests that (1) the steeper the yield curve slope, the better a callable series should do,[11] and (2) the higher the volatility, the lower the return on the callable series will be.

11. This is because the options that have been implicitly call options are now more out of the money.

CHAPTER

32

MULTIFACTOR RISK MODELS AND THEIR APPLICATION TO PORTFOLIO CONSTRUCTION

In this chapter we describe models that can be used to construct a fixed-income portfolio so as to achieve some objective relative to a designed benchmark. These models take into consideration the various factors that drive the return of the benchmark and are generically referred to as *multifactor models*. However, such models are really of two types: multifactor asset-pricing models and multifactor risk models. In the case of the former, the model is used to determine the expected return from a portfolio based on the factors expected to impact the return. The most well-known single-factor model is the capital asset pricing model, where the only factor is the market. In the equity market, the most popular model is the Fama–French three-factor model combined with momentum.[1] The factors in addition to the market are size and value. In the bond market, the Fama–French asset-pricing model includes credit and term. Several studies suggest that factors used in equity factor models are also appropriate for fixed-income multifactor asset-pricing models.

Our focus in this chapter is on multifactor risk models. Multifactor risk models provide portfolio managers with information about the sources of risk in their portfolios. Hence such models are indispensable tools for constructing portfolios so as to realize the desired exposure to the risk factors where a portfolio manager may have a view. Moreover, these models can be used to monitor and control the risk exposure of the portfolio. The metric that takes into consideration all the relevant risks that impact the return on the benchmark is *tracking error*. There are two types of tracking error: backward-looking and forward-looking tracking error. In multifactor risk models, the metric that is used to control a portfolio's risk exposure relative to a benchmark is forward-looking (or predictive) tracking error.

Rather than present a pure description of fixed-income multifactor models and the formulas for the relevant metrics, we will illustrate them with three actual models.

1. Eugene F. Fama and Kenneth R. French, "The Cross-Section of Expected Stock Returns," *Journal of Finance*, Vol. 47 (1992), pp. 427–465; and Mark M. Carhart, "On Persistence in Mutual Fund Performance," *Journal of Finance*, Vol. 52 (1997), pp. 57–82.

RISK DECOMPOSITION IN FIXED-INCOME MULTIFACTOR RISK MODELS

Before we explain how to use a multifactor risk model to construct a fixed-income portfolio, let's look at how a multifactor risk model can be used to identify the risk exposure of a portfolio relative to a benchmark. We will illustrate this by using an actual 50-bond portfolio constructed on February 28, 2011, with a market value of $100 million. For now, it is not important to know how this portfolio was constructed. Later we will see that this portfolio was constructed using a multifactor risk model combined with an optimization model.[2]

The risk exposure for this portfolio will be measured in terms of forward-looking tracking error. It is therefore necessary to know what this portfolio's benchmark is. In this illustration let's suppose that the client established a composite index made up of the Barclays Capital U.S. Treasury Index (now the Bloomberg Barclays Capital U.S. Treasury Index), Barclays Capital U.S. Credit Index (now the Bloomberg Barclays Capital U.S. Credit Index), and Barclays Capital U.S. MBS Index (now the Bloomberg Barclays U.S. MBS Index) on an equally weighted basis (i.e., each index has a one-third weight). This is shown in the third column of Exhibit 32–1. The U.S. Credit Index includes corporate bonds and government-related securities. In Exhibit 32–1, the distribution of the U.S. Credit Index is shown in terms of corporate sector (industrials, utilities, and financials) and government-related sectors. The U.S. MBS Index includes agency passthrough securities.

The analysis of the portfolio begins with a comparison of the portfolio to the benchmark. Identification of the mismatches indicates where the portfolio manager has taken a view (unintentional or not). Exhibit 32–1 compares the portfolio and the benchmark in terms of the allocation to the major sectors of the benchmark. It is clear from the exhibit that the portfolio manager is taking a positive view on the corporate sector (more specifically industrials and financials) by overweighting it and that this is achieved by underweighting the other sectors.

Although the information contained in Exhibit 32–1 about the allocation based on percentage market value of a sector relative to the benchmark provides a good starting point for the analysis of portfolio risk, the information has limited value because it is not known how the exposures to the sectors are related to the exposures to the risk factors that drive the portfolio's return. Here are three examples. First, consider the Treasury sector. In Chapter 13 we discussed the concept of contribution to portfolio duration. It is possible that the specific Treasury securities included in the portfolio have a greater contribution to portfolio duration than the contribution to index duration of the Treasuries in the benchmark despite the underweighting of Treasuries in the portfolio. As a second example of why the portfolio manager must look beyond the percentage allocation to a sector, consider the corporate bonds in the financial sector. Corporate financials will have a contribution to spread duration in both the portfolio and the benchmark. It is possible to

2. The portfolio for this illustration, which is also discussed later in this chapter, was provided by Cenk Ural using at the time the Barclays Capital Global Risk Model.

EXHIBIT 32–1

Summary of Portfolio and Benchmark Sector Allocation as of February 28, 2011

Sector	Portfolio (%)	Benchmark (%)	Difference (%)
Treasury	29.5	33.3	–3.8
Government related	3.6	6.8	–3.2
Corporate industrials	15.1	13.9	1.2
Corporate utilities	2.9	3.0	–0.1
Corporate financials	19.9	9.7	10.2
Agency MBS	29.0	33.3	–4.3
Total	100.0	100.0	0.0

have an overweight of this sector in the portfolio and yet have a contribution to spread duration that is less than that of the benchmark. Finally, a portfolio's convexity relative to the benchmark will impact relative performance. It is possible, for example, to underweight the portfolio's exposure to agency mortgage-backed securities (MBS) so as to create a portfolio with large negative convexity while the benchmark has much lower negative convexity.

It is for this reason that the portfolio manager must look beyond a naive assessment of portfolio risk relative to the benchmark based on percentage allocation to sectors. Exhibit 32–2 provides information about the relative exposure to interest-rate risk as measured by duration, spread risk as measured by spread duration, and call/prepayment risk as measured by vega, as well as convexity. From Exhibit 32–2 we observe the following:

- The duration of the portfolio slightly exceeds that of the benchmark, so the portfolio has slightly more exposure to changes in the level of interest rates.
- Due to the underweighting of Treasuries, spread duration is higher.
- The slightly higher portfolio convexity compared with the benchmark means less exposure to call and prepayment risk that is attributable to the reduced exposure to agency MBS.
- Exposure to call/prepayment risk as a measure is small and about the same for the portfolio and the benchmark.

In addition, the spread of the portfolio is 107 basis points, while that of the benchmark is 57 basis points.

More information about the portfolio's relative risk exposure to interest-rate risk can be obtained by looking at the contribution to duration for the portfolio and the benchmark. This is shown in Exhibit 32–3. As can be seen, the major reason

EXHIBIT 32–2

Analytics for the 50-Bond Portfolio and the Composite Index (Benchmark)

Exposure	Portfolio	Benchmark	Difference
Duration	5.56	5.41	0.15
Spread duration	5.37	5.27	0.09
Convexity	0.11	0.06	0.05
Vega	–0.04	–0.03	–0.01

EXHIBIT 32–3

Contribution to Duration by Bond Sector for the 50-Bond Portfolio

Contribution to Duration	Portfolio	Benchmark	Difference
Treasury	2.14	1.77	0.37
Government related	0.07	0.40	–0.33
Corporate	1.84	1.73	0.12
Agency MBS	1.50	1.51	–0.01
Total	5.56	5.41	0.15

MBS = mortgage-backed securities

for the slightly longer duration of the portfolio relative to the benchmark is mainly attributable to the duration of the Treasury securities selected for the portfolio.

The analysis thus far, while helpful, is missing one important element. To understand why, suppose that a portfolio has more exposure to a risk factor than the benchmark. This would mean that if the risk factor moves, the portfolio will have a greater movement than the benchmark. But the question is, to what extent does that risk factor move? Another way of asking this is, how volatile is the risk factor? For example, from Exhibit 32–3 we know that the portfolio has greater exposure than the benchmark to changes in the level of interest rates (i.e., a higher duration) and more exposure to changes in spreads (i.e., a higher spread duration). But which exposure (i.e., risk factor) has greater volatility?

To address this, volatility must be taken into consideration. Exhibit 32–4 shows the monthly volatility of risk factor categories. Let's look at each one of these volatilities and what they mean. Isolated risk in this exhibit displays the tracking error/volatility of different exposures of the portfolio in isolation. Consider first the yield-curve risk of 3.9 reported in Exhibit 32–4. Yield-curve risk is the risk exposure to changes in interest rates. We know from Exhibit 32–2 that the portfolio duration is greater than the benchmark (5.56 versus 5.41), but how does that translate into what it will cost the manager in terms of additional risk? This is where the 3.9 is useful. Suppose that the portfolio only differs from the benchmark with respect to its exposure to changes in the yield curve. Then the 3.9 means that this mismatch relative to the benchmark creates a risk equal to 3.9 basis points per month of volatility. That is, if rates were the portfolio's only net exposure, this number would be the tracking-error volatility of that portfolio versus the benchmark.

Similarly, consider the risk factor securitized spread in Exhibit 32–4. This risk factor is the exposure to changes in the spreads in the agency MBS market. The value of 2.5 means that if the portfolio differs from the benchmark only with respect to its exposure to changes in the spread in the agency MBS sector, then this mismatch relative to the benchmark would result in a monthly isolated tracking error of 2.5 basis points.

Notice in Exhibit 32–5 that there is a risk-exposure category labeled "volatility." This risk factor is the risk associated with changes in interest-rate volatility and is critical for quantifying the exposure of a portfolio or benchmark to securities with embedded options such as callable bonds and agency MBS because they are impacted by changes in volatility. Hence the value of 1.3 is the exposure of the portfolio to the risk factor volatility. The value of 1.3 means that if the portfolio differs from the benchmark only with respect to its exposure to changes in volatility, then this mismatch relative to the benchmark would result in a monthly isolated tracking error of 1.3 basis points.

How can we determine the monthly tracking error for the portfolio given the monthly tracking error for each risk factor exposure in Exhibit 32–4? One might think that the solution is to just add up the monthly tracking errors, which would give 13.9 basis points (= 3.9 + 2.6 + 1.3 + 0.8 + 2.8 + 2.5 basis points). However, this would be incorrect because standard deviations are not additive. (That is, portfolio risk is not the sum of the variances of the returns for the securities in the portfolio.) Assuming a zero correlation between any pair of risk factors, the portfolio isolated tracking error attributable to systematic risk is found by squaring

EXHIBIT 32–4

Monthly Tracking Error for Risk Factors

Risk Factor Categories	Isolated Risk/Tracking Error
Yield-curve risk	3.9
Swap-spread risk	2.6
Volatility risk	1.3
Government-related spread risk	0.8
Corporate spread risk	2.8
Securitized spread risk	2.5

each isolated tracking error for each risk factor, summing them, and then taking the square root. That is, for the general case where there are K risk factors,

$$\text{Portfolio isolated systematic } TE = [(TE_1)^2 + (TE_2)^2 + \cdots + (TE_K)^2]^{1/2},$$

where TE denotes tracking error, and the subscript denotes the risk factor.

Thus, for the 50-security portfolio in this illustration whose monthly isolated tracking error for each risk factor is shown in Exhibit 32–4, the portfolio isolated systematic TE is 6.24 basis points per month:

$$\text{Portfolio isolated systematic } TE = [(3.9)^2 + (2.6)^2 + (1.3)^2 + (0.8)^2 + (2.8)^2 + (2.5)^2]^{1/2} = 6.24.$$

The assumption that there is zero correlation between every pair of factor risks must be investigated. Obviously, to address this, correlations or covariances must be brought into the analysis. Calculation of the portfolio risk then involves use of the variance–covariance matrix for the risk factors.[3] Let's consider the case where there are only two risk factors, F_1 and F_2. Then the portfolio tracking error can be shown to be equal to

$$\text{Portfolio } TE = [(TE_{F1})^2 + (\text{TE}_{F2})^2 + 2\,\text{cov}(F_1, F_2)]^{½},$$

where $\text{cov}(F_1,F_2)$ is the covariance between risk factor exposures 1 and 2.

In our illustration, there are six risk factors that we denote as follows:

F_1 = yield-curve risk; F_2 = swap-spread risk; F_3 = volatility risk;
F_4 = government-related spread risk; F_5 = corporate spread risk;
F_6 = securitized spread risk.

The variance–covariance matrix is then

$$\begin{bmatrix}
\text{var}(F_1) & \text{cov}(F_1,F_2) & \text{cov}(F_1,F_3) & \text{cov}(F_1,F_4) & \text{cov}(F_1,F_5) & \text{cov}(F_1,F_6) \\
\text{cov}(F_2,F_1) & \text{var}(F_2) & \text{cov}(F_2,F_3) & \text{cov}(F_2,F_4) & \text{cov}(F_2,F_5) & \text{cov}(F_2,F_6) \\
\text{cov}(F_3,F_1) & \text{cov}(F_3,F_2) & \text{var}(F_3) & \text{cov}(F_3,F_4) & \text{cov}(F_3,F_5) & \text{cov}(F_3,F_6) \\
\text{cov}(F_4,F_1) & \text{cov}(F_4,F_2) & \text{cov}(F_4,F_3) & \text{var}(F_4) & \text{cov}(F_4,F_5) & \text{cov}(F_4,F_6) \\
\text{cov}(F_5,F_1) & \text{cov}(F_5,F_2) & \text{cov}(F_5,F_3) & \text{cov}(F_5,F_4) & \text{var}(F_5) & \text{cov}(F_5,F_6) \\
\text{cov}(F_6,F_1) & \text{cov}(F_6,F_2) & \text{cov}(F_6,F_3) & \text{cov}(F_6,F_4) & \text{cov}(F_6,F_5) & \text{var}(F_6)
\end{bmatrix}$$

The diagonal terms in the variance–covariance matrix are the variance or, equivalently, the square of the tracking error.

Further information is available to describe the risk attributes of a portfolio. Here we will simply provide an example of the type of information that can be

3. When we discuss the Amundi asset-management fixed-income multifactor risk model at the end of this chapter, the risk factors were found to be statistically independent, justifying the use of the isolated tracking error.

EXHIBIT 32–5

Summary Report for Illustrative 50-Bond Portfolio

Panel A: Parameter	**Portfolio**	**Benchmark**	
Positions	50	5,693	
Issuers	25	780	
Currencies	1	1	
Market value ($ million)	100	13,615	
Notional ($ million)	95	12,851	
Panel B: Analytics	**Portfolio**	**Benchmark**	**Difference**
Coupon (%)	4.65	4.39	0.26
Average life (yr)	8.09	7.77	0.31
Yield to worst (%)	3.69	3.18	0.51
Spread (bsp)	107	57	50
Duration	5.56	5.41	0.15
Vega	–0.04	–0.03	–0.01
Spread duration	5.37	5.27	0.09
Convexity	0.11	0.06	0.05
Panel C: Volatility	**Portfolio**	**Benchmark**	**Tracking Error**
Systematic (bps/month)	125.4	123.5	4.6
Idiosyncratic (bps/month)	9.6	4.8	7.8
Total (bps/month)	125.7	123.6	9.0
Panel D: Portfolio Beta			1.01

bps = basis points

provided.[4] Specifically, we will provide more detailed information about the general exposure of the portfolio and then focus on exposure to yield-curve risk in more detail.

A summary report for the 50-bond portfolio is shown in Exhibit 32–5. The portfolio has only 50 positions and 25 issuers in contrast to 5,893 positions in the benchmark and 780 issuers. What this means is that the portfolio is not very well diversified, and as a result, the portfolio manager should expect a significant level of idiosyncratic risk. All the holdings are dollar-denominated, and the same is true

4. A more detailed description can be found in Anthony Lazanas, António Baldaque da Silva, Radu Găbudean, and Arne D. Staal, "Multifactor Fixed Income Risk Models and Their Applications," Chapter 21 in Frank J. Fabozzi and Harry M. Markowitz, *The Theory and Practice of Investment Management* (Hoboken, NJ: John Wiley & Sons, 2011).

for the benchmark. For this reason, there is only one currency shown in the summary report. From our previous discussion, we know about the last five metrics reported in the analytics section of the exhibit. Average life and yield to worst are measures discussed in earlier chapters.

Exhibit 32–5 gives a breakdown of the standard deviation of the returns for the portfolio and the benchmark in terms of systematic and idiosyncratic risk. The portfolio has greater systematic and idiosyncratic risk than the benchmark. For the total risk of the portfolio and the benchmark, because the systematic and idiosyncratic risks are constructed so as to be independent, the standard deviations of the portfolio and the benchmark can be calculated as follows:

$$\text{Total risk (volatility of returns)} = [(\text{systematic risk})^2 + (\text{idiosyncratic risk})^2]^{1/2}.$$

The total risk for the portfolio and the total risk for the benchmark using the values in Exhibit 32–5 are 125.7 and 123.6, respectively.

Notice that for the benchmark, the percentage of the total risk (123.6) that is explained by the systematic risk factors (123.5) is 99.99%. For the portfolio it is 99.76% (125.4/125.7). It would therefore seem that the idiosyncratic risk is not important. This, however, is not true when dealing with the tracking error of the portfolio (volatility of the net position, portfolio versus the benchmark). The systematic and idiosyncratic tracking errors (per month) are 4.6 and 7.8 basis points per month, respectively. The portfolio tracking error is

$$\text{Portfolio tracking error} = [(\text{systematic } TE)^2 + (\text{idiosyncratic } TE)^2]^{1/2}.$$

Therefore, the portfolio tracking error is 9 basis points per month. Consequently, although idiosyncratic risk is minimal for the portfolio on a standalone basis, when risk is assessed relative to a benchmark, there is tracking error risk of 9 basis points per month. The systematic risk is responsible for 7.8/9.0 or 87% of the total risk. (This tracking error is well within the risk budget of 15 basis points per month that we assume later in this chapter that the manager is permitted.)

This is an extremely important point: it is the tracking error not the idiosyncratic risk (as measured by the standard deviation of the idiosyncratic returns) that the portfolio manager must consider in portfolio construction and monitoring. In our illustration, the portfolio tracking error is small, only 9 basis points, but the illustration could just as easily have been constructed where the systematic risk relative to the total risk (as measured by the standard deviation of returns) for the portfolio was 99.76% with a tracking error per month of 200 basis points.

As with equities where a portfolio beta is computed that shows the movement of an equity portfolio in response to a movement in some equity market index (such as the S&P 500), a beta can be computed for a bond portfolio. As shown in Exhibit 32–5, the portfolio beta is 1.01. Because the benchmark is the Composite Index, a beta of 1.01 means that if that index increases by 10%, the portfolio will increase, on average, by 10.1%.

A beta-type measure can be estimated for each risk factor. For example, consider the risk factor measuring changes in the level of the yield curve, which is the portfolio's duration. A *duration beta* can be calculated as follows:

$$\text{Duration beta} = \frac{\text{portfolio duration}}{\text{benchmark duration}}$$

For our portfolio and benchmark, because the durations are 5.56 and 5.41, respectively (see Exhibit 32–3), the duration beta is 1.03.

While the information contained in Exhibit 32–5 provides a starting point for understanding the portfolio's risk relative to the benchmark, further insight can be gained by looking at how the portfolio risk (as measured by tracking error) is allocated across the different (1) categories of risk factors and (2) sectors of the benchmark.

Exhibit 32–6 provides information about the portfolio risk across the different categories of risk factors. Shown are the systematic risk and the idiosyncratic risk and the seven components of systematic risk. The second column shows the isolated tracking error. The contribution to tracking error for each group of risk factors is shown in the third column. As can be seen, the four major risk exposures of the 50-bond portfolio are (1) yield-curve risk, (2) spread risk, (3) corporate spread risk as measured by the swap spread, and (4) idiosyncratic risk. The fourth column gives a new metric, *liquidation effect* on tracking error. This metric indicates the impact

EXHIBIT 32–6

Detailed Monthly Tracking Error for the 50-Bond Portfolio by Risk Factor Group

Risk Factor Group	Isolated Tracking Error	Contributions to Tracking Error	Liquidation Effect on Tracking Error	Tracking Error Elasticity (%)
Total	9.0	9.0	–9.0	1.0
Systematic risk	4.6	2.3	–1.2	0.2
Yield-curve risk	3.9	0.9	–0.1	0.1
Swap-spread risk	2.6	0.6	–0.2	0.1
Volatility risk	1.3	0.1	0.0	0.0
Government-related spread risk	0.8	0.0	0.0	0.0
Corporate spread risk	2.8	0.7	–0.3	0.1
Securitized spread risk	2.5	0.0	0.4	0.0
Idiosyncratic risk	7.8	6.8	–4.2	0.7

on the portfolio's tracking error by hedging (i.e., eliminating) the exposure to the respective risk group. For example, consider the systematic risk. The liquidation effect on tracking error shown in the exhibit is −1.2 and is interpreted as follows: if the portfolio manager hedges the systematic risk, then the portfolio's tracking error will decline by 1.2 basis points per month. Because the portfolio's tracking error is 9 basis points per month, this means that hedging the systematic risk reduces the monthly tracking error for the portfolio to 7.8 basis points per month.

A detailed analysis of the systematic and idiosyncratic risks applied at the asset class level rather than at the individual risk factor level is provided in Exhibit 32–7. The five sectors of the benchmark are shown in the first column, and in the second column the under- or overweightings of each asset class (referred to as the *net market weight*) are shown. The last three columns report the contribution to tracking error for systematic risk, idiosyncratic risk, and total risk. The row labeled "Total" shows what we already know about the portfolio risk from earlier analysis: systematic risk is 2.3 basis points per month, idiosyncratic risk is 6.8 basis points per month, and total risk is 9 basis points per month. The exhibit then shows how each sector of the benchmark contributes to systematic, idiosyncratic, and total risk. Looking at the exhibit at a high level, idiosyncratic risk seems to be dominant compared with the systematic risk because of a small number of securities in the portfolio. It is interesting to note that the major contribution to systematic risk is coming from the Treasuries sector, although it has a smaller net market weight (in magnitude) than the corporate sector. This is mainly due to the duration mismatch between the Treasury component of the portfolio and the benchmark. In contrast, Treasuries have a negligible idiosyncratic risk contribution because a large proportion of variation in the return of these securities can be explained by systematic yield-curve factors.

Another important observation to take away from the analysis reported in Exhibit 32–7 is that corporate bonds are the major contributor to idiosyncratic risk because of the overweighting of this sector, carrying relatively higher idiosyncratic

EXHIBIT 32–7

Systematic and Idiosyncratic Monthly Tracking Errors for the 50-Bond Portfolio by Asset Class

Asset Class	Net Market Weight (%)	Contribution to Tracking Error: Systematic	Idiosyncratic	Total
Total	0.0	2.3	6.8	9.0
Treasuries	–3.8	2.5	0.2	2.7
Government agencies	–1.8	–0.3	0.0	–0.3
Government non-agencies	–1.4	–0.9	0.2	–0.7
Corporates	11.4	1.0	6.0	6.9
MBS	–4.4	0.0	0.4	0.4

MBS = mortgage-backed securities

risk at the individual security level. If any of these corporate bonds selected for the portfolio performs poorly (i.e., the idiosyncratic risk is realized), then this could have a substantial adverse impact on the portfolio's performance relative to the benchmark. This highlights the significant name risk to which the portfolio is exposed.

An analysis similar to the decomposition of risk shown in Exhibit 32–6 by sector of the bond market instead of risk factor group is shown in Exhibit 32–8. Notice that the isolated tracking error for both the Treasury and corporate asset classes exceeds that of the portfolio tracking error (10.6 and 9.3 basis points per month versus 9 basis points per month). How is this possible? This occurs because exposures to certain asset classes in the portfolio are acting as hedges to certain other asset classes (because Treasuries and corporate bonds also could be hedging each other). The hedging effect can be seen in the fourth column, where the liquidation effect on tracking error is shown for each asset class in Exhibit 32–8.

Now let's look at the exposure of the portfolio to yield-curve risk in more detail. In Chapter 15 we discussed the term structure of interest rates. Although our focus was on the Treasury yield curve, we explained that there are other interest-rate benchmarks that can be used, such as the swap curve. During noncrisis periods, the Treasury and swap curves tend to behave the same way. This is not the case during crisis periods in the financial markets. This was apparent in the global financial crisis of 2008. Consequently, in the model we describe here, a different yield curve is used for government products. With the exception of Treasuries, the other four bond sectors have exposure to the swap-spread factors on top of the Treasury curve. By decomposing the swap curve into the Treasury curve and swap spreads, the model used here gives portfolio managers the flexibility to analyze their spread risk over the Treasury curve or the swap curve depending on their preference.

EXHIBIT 32–8

Isolated Monthly Tracking Error and Liquidation Effect for the 50-Bond Portfolio by Asset Class

Asset Class	Net Market Weight (%)	Isolated Tracking Error	Liquidation Effect on Tracking Error	Tracking Error Elasticity (%)
Total	0.0	9.0	–9.0	1.0
Treasuries	–3.8	10.6	2.9	0.3
Government Agencies	–1.8	2.0	0.5	0.0
Government non-agencies	–1.4	7.2	2.9	–0.1
Corporates	11.4	9.3	–2.9	0.8
MBS	–4.4	3.4	0.3	0.0

MBS = mortgage-backed securities

There are different measures to look at the exposure to changes in the shape of the yield curve. The most common is key rate duration (described in Chapter 15). In the model used here, the six key rates are the 6-month, 2-year, 5-year, 10-year, 20-year, and 30-year rates. For the 50-bond portfolio and for the benchmark, these six key rate durations with respect to the U.S. Treasury curve, as well as the option-adjusted or effective convexity, are shown in Exhibit 32–9. Summing up the key rate durations for the portfolio and the benchmark gives a portfolio duration of 5.56 and a benchmark duration of 5.41, which agree with the values in Exhibit 32–6.

The fourth column shows the mismatch for the key rate duration and convexity between the portfolio and the benchmark. From Chapter 15 we know how to interpret the key rate duration. It is the approximate percentage change in the portfolio or benchmark value for a 100 basis point change in the rate for a particular maturity, holding all other rates constant. In terms of mismatch, it is the approximate differential percentage change in the portfolio relative to the benchmark for a 100 basis point change in the rate for a particular maturity, holding all other rates constant. For example, consider the net 5-year key rate duration of 0.09. The impact on return relative to the benchmark for a 100 basis point change in 5-year interest rates will be 0.09. The question then is how volatile is the 5-year rate. The fifth column, labeled "Factor Volatility," is the factor's forecasted volatility with a monthly horizon. The 5-year rate, for example, has a factor volatility of 29.48 basis

EXHIBIT 32–9

Treasury Curve Risk for 50-Bond Portfolio

Factor Name	Portfolio	Benchmark	Net*	Factor Volatility	*Return Impact of a Typical Movement* Isolated *TE*	Correlated *TE*	Marginal Contribution to Tracking Error
USD 6M key rate	0.11	0.11	0.01	22.20	–0.2	–0.9	2.2
USD 2Y key rate	0.59	0.64	–0.05	21.59	1.1	–1.4	3.3
USD 5Y key rate	1.53	1.43	0.09	29.48	–2.8	–1.7	5.3
USD 10Y key rate	1.39	1.51	–0.12	30.42	3.6	–2.0	6.6
USD 20Y key rate	0.98	0.91	0.07	27.84	–2.0	–2.1	6.3
USD 30Y key rate	0.96	0.82	0.14	27.18	–3.8	–2.2	6.4
USD Convexity	0.11	0.06	0.05	4.72	0.2	0.5	0.3

*Differences due to rounding.

points per month. Assuming that the factor volatility represents a typical movement for the factor (i.e., key interest rate), then the isolated impact of that movement on the return of our portfolio (versus the benchmark) can be found as follows:

$$\text{Return impact of a typical movement} = -(\text{net key rate duration}) \times \text{typical rate movement.}$$

For example, for the 5-year key rate duration, we have

$$\begin{aligned}\text{Return impact of a typical movement} &= -0.09 \times 29.48 \\ &= -2.8 \text{ basis points per month.}\end{aligned}$$

Looking at the 10-year key rate, we have

$$\begin{aligned}\text{Return impact of a typical movement} &= -(-0.12) \times 30.42 \\ &= 3.6 \text{ basis points per month.}\end{aligned}$$

Recall the difference between isolated and correlated return impact. Notice that for the 10-year key rate, the isolated return impact of a typical movement in the 10-year rate is positive, but when correlation between factors is considered, it is –2 basis points per month.

The last column in Exhibit 32–9, which shows the marginal contribution to tracking error, is useful for a portfolio manager who seeks to effectively reduce portfolio exposure to Treasury yield-curve risk. Basically, it shows that a change of one unit of exposure to the 10-year key rate changes the tracking error by 6.6 basis points per month.

Exhibit 32–10 shows the exposure of the portfolio to the change in the swap spread. The swap spread is the difference between the swap rate and the Treasury

EXHIBIT 32–10

Swap-Spread Risk for 50-Bond Portfolio

	Exposure (SS-KRD)					
Factor Name	**Portfolio**	**Benchmark**	**Net**	**Factor Volatility**	**Return Impact Correlated**	**Marginal Contribution to to *TE***
USD 6M swap spread	0.11	0.08	0.04	19.5	–0.9	2.0
USD 2Y swap spread	0.43	0.38	0.05	11.0	–0.6	0.7
USD 5Y swap spread	0.74	0.89	–0.15	6.7	–0.5	0.4
USD 10Y swap spread	1.16	1.08	0.09	8.5	1.2	–1.1
USD 20Y swap spread	0.66	0.72	–0.06	9.9	1.7	–1.8
USD 30Y swap spread	0.21	0.40	–0.19	12.5	1.9	–2.6

rate. All securities in the portfolio except Treasuries expose the portfolio to this risk. A comparison of the return impact based on correlated factors in the next-to-the-last columns of Exhibits 32–9 and 32–10 indicate that potential movements in swap-spread factors have less impact on the portfolio than the Treasury curve factors.

Portfolio Construction Using a Multifactor Risk Model and an Optimizer

Using the multifactor risk model just described, let's see how the information obtained from the exhibits can be applied to construct a bond portfolio. To illustrate, suppose that a bond portfolio manager has views on the various primary factors driving the return on the benchmark and wants to position the portfolio to capitalize on those expectations. The portfolio manager does this in the face of constraints in terms of the risk budget as well as any restrictions on maximum exposure to any sector, industry, or issuer. Of course, the portfolio manager also must consider transaction costs not only when constructing a portfolio from cash but also when rebalancing a current portfolio to revise the portfolio's risk exposure to factors.

A multifactor risk model is used in conjunction with an *optimization model* or, simply, *optimizer* to construct a portfolio.[5] Although we have discussed the 50-bond portfolio in terms of its exposure to factor risks, this portfolio was actually generated from an optimizer.[6]

In using an optimizer, the optimal value for all the variables that the decision maker seeks is the output for the model. The decision maker specifies the variables (i.e., decision variables), an objective function, and constraints. Given all of this information, the optimizer finds the optimal value for all the decision variables.

In the case of portfolio construction using a multifactor risk model, the decision variables are the amounts of each security to be held in the optimized portfolio. This requires that the portfolio manager specify the universe of securities that are acceptable for inclusion in the portfolio. But more is needed than just information about the securities that may be included. The price at which each candidate security can be purchased is needed. The tradable universe that will be used is the securities included in the Composite Index with a minimum amount outstanding of $300 million. There is no need for the tradable universe to be the same as the benchmark. If the portfolio manager is permitted to invest in nonindex securities,

5. For an explanation of optimization models, see Chapter 34. See Anuj Kumar, "The POINT Optimizer," Barclays Capital Publication, London, June 2010.
6. When this 50-bond hypothetical portfolio was optimized, the optimizer used was the Barclays Capital POINTR Optimizer. See ibid.

a larger tradable universe could be used. The imposition of a minimum issue size was to avoid inclusion of small issues into the portfolio that could potentially lead to a liquidity problem.

The portfolio manager must also specify the objective function. This is the measure or quantity that is to be minimized or maximized. In portfolio construction using a multifactor risk model, the measure to be optimized is the portfolio's forward-looking tracking error. The manager wants that measure to be minimized.

Optimization of the objective function is typically done subject to constraints. These constraints may be client imposed based on the investment guidelines, self-imposed by the portfolio manager, or imposed by regulations. The most obvious constraint is the risk budget, which imposes a maximum forward-looking portfolio tracking error. Other common constraints in the initial construction of a portfolio include

- Restrictions on short selling;
- A maximum deviation from the benchmark's duration;
- A maximum mismatch with any one sector in the benchmark;
- A maximum exposure to any one issuer or industry;
- A minimum size purchase of an issue to avoid the purchase of odd lots.

If the tradable universe includes securities not included in the benchmark, a typical constraint imposed on the portfolio manager is the percentage of the portfolio that may be allocated to nonindex securities. Of course, there must be a requirement on the market value of the portfolio and an upper limit on the number of securities.

In constructing the portfolio in our illustration, the following restrictions were imposed:

1. Market value of $100 million;
2. No more than 50 securities in the portfolio;
3. No short sales;
4. The portfolio's duration must exceed the benchmark's duration by at least 0.15 but no more than 0.30;
5. Spreads between 50 and 80 basis points higher than the benchmark;
6. Monthly tracking error not to exceed 15 basis points;
7. Maximum under-/overweight of 3% per issuer.

Notice constraint 4 dealing with the portfolio duration. It is this requirement that tilts the portfolio in the direction of the portfolio manager's view to have a mismatch with the benchmark with respect to the factor risk representing curve risk. Constraint 5 is the risk budget, which permits not only a duration mismatch but a mismatch on benchmark sector allocations. Constraint 7 is imposed for diversification purposes.

It is clear from Exhibit 32–1 that the portfolio manager is also taking a positive view on the corporate sector by overweighting it, and this is achieved by underweighting the three other sectors.

Portfolio Rebalancing

While it is common to illustrate portfolio construction starting with a position of cash and building a portfolio of securities, in practice, the more common task is to rebalance an existing portfolio. A multifactor risk model along with an optimizer can be used to efficiently rebalance a portfolio so as to realign it if it has drifted away from the characteristics of the benchmark over time (such as a change in the duration of the benchmark requiring a change in the duration of the portfolio or an upgrade or downgrade of some issues in the portfolio) and/or tilt it to reflect a manager's new views. Rebalancing is also required when a portfolio manager receives additional funds from a client or portfolio cash inflows or when a client withdraws funds.

Rebalancing must be done so as to minimize transaction costs by reducing the need to turn over current holdings unnecessarily. The optimizer is able to evaluate the tradeoff of replacing one issue held (i.e., a sale) with another issue (i.e., a purchase). The optimizer can identify a package of transactions (i.e., sells and buys) and identify the decrease (or increase) in risk that would result from execution of those transactions so that the portfolio manager can assess the risk-adjustment benefit relative to the cost of executing the transactions.

To illustrate, a detailed analysis of the portfolio's holdings shown in Exhibit 32–1 would indicate an overweighting of issuers in the banking sector. Suppose that the portfolio manager wants to limit the overweight to banks to 5% but wants to do so with no more than 15 trades. The optimizer then can be used where the inputs are the set of tradable securities and their current prices. A constraint must be added to restrict the overweight to banks to less than or equal to 5% and to restrict the number of trades to no more than 15.

Exhibit 32–11 shows the trades that would have been recommended by the optimizer at the time. The total market value of the trades was roughly $13 million. Almost half the sales from the portfolio were for banks, and they were replaced with various Treasury notes, a corporate bond, a sovereign bond, and an agency MBS.

Before the manager executes the package of trades proposed in Exhibit 32–11, there must be an evaluation of the change in risk exposure. The new systematic tracking error after rebalancing is 4.2 basis points (the original was 4.6 basis points), idiosyncratic tracking error is 7.8 basis points (same as before rebalancing), and total tracking error is 8.8 basis points (9.0 basis points before rebalancing). The decline in the total tracking error is before there are more than 50 securities in the portfolio after the rebalancing.

EXHIBIT 32-11

Trades for Portfolio Rebalancing

Identifier	Description	Position Amount	Market Value
	Buys		
912828LK	U.S. Treasury Notes	3,133,909	3,235,179
912828LS	U.S. Treasury Notes	2,814,967	2,924,353
489170AB	Kennametal Inc	1,959,720	2,087,886
94986EAA	Wells Fargo Capital XIII	1,286,097	1,360,888
912810QD	U.S. Treasury Bonds	1,118,189	1,111,380
465138ZR	Israel, State of	920,297	1,097,735
912810QB	U.S. Treasury Bonds	1,017,169	991,185
GNG03410	GNMA II Single- Family 15yr	117,277	119,672
Total			12,928,278
	Sells		
912828NV	U.S. Treasury Notes	–2,662,260	–2,586,183
16132NAV	Charter One Bank FSB	–2,203,358	–2,332,312
05946NAD	Banco Bradesco SA	–1,564,870	–1,828,328
827065AA	Silicon Valley Bank	–1,692,776	–1,770,613
912828NL	U.S. Treasury Notes	–1,603,631	–1,612,239
912810QC	U.S. Treasury Bonds	–1,462,336	–1,468,727
912810QE	U.S. Treasury Bonds	–1,298,352	–1,329,875
Total			–12,928,278

AXIOMA FIXED-INCOME MULTIFACTOR RISK MODEL[7]

Earlier in this chapter we showed the critical components of a multifactor risk model and how it could be used to identify the risks of a portfolio relative to a benchmark and how the model could be used to construct and rebalance a portfolio. In the last two sections of this chapter we will illustrate two actual multifactor risk models: the Axioma model, which we illustrate in this section, and the model by Amundi Asset Management, which we illustrate in the next section.

The Axioma model was developed by Qontigo's Analytics Research Group. This is a multifactor risk model that provides broad global coverage of fixed-income bonds, notes, loans, structured debt, and derivatives. As with all multifactor risk

7. The Axioma factor-based fixed-income risk model is available from Qontigo. The description of the model and the illustration were provided by Bill Morokoff, managing director and head of analytics research at Qontigo.

EXHIBIT 32.12

Types of Risk Factors in the Axioma Fixed-Income Multifactor Risk Model

Rates	Credit Spread	MBS Spread	Volatility
Treasury	Currency	Sector	Swaption implied volatility
Swap spread	Sector	Spread	Equity implied volatility
Inflation	Country	Refinance	
FX	Quality	Turnover	
	Market	Specific	
	Issuer specific		
	Issue specific		

MBS = mortgage-backed securities; FX = foreign exchange

models, each debt instrument in the portfolio has exposure to a common set of systematic risk factors and also may have an idiosyncratic specific risk component. A summary of the types of risk factors in the model is provided in Exhibit 32–12.

A debt instrument may have exposure to multiple categories of risk factors. For example, a callable corporate bond is modeled with Treasury rate and swap-spread risk factors, credit-spread risk factors, and swaption implied volatility risk factors. The instrument exposure to a risk factor, such as key rate duration, is computed as a price sensitivity to small risk factor return shocks through full instrument repricing.

Definition of the Risk Factors in the Axioma Model

The definition of risk factors and estimation of returns varies by risk factor type, as detailed here.

Rates

The rate risk factors capture the risk associated with changes in the level and shape of interest-rate term structures. These risk factors are defined as key rates on the term structure of sovereign zero-rate curves, swap-spread curves, and break-even inflation curves (nominal rate – real rate). The choice of key rates in the model is user configurable. The factor return is defined as the change in the level of the key rate, while the exposure is thekey rate duration. For foreign exchange (FX) rate risk, the risk factor is the spot rate, and the return is computed as the log change in the spot rate.

Credit Spread

The credit-spread risk factors capture the risk associated with changes in the option-adjusted spread over swap rates for a range of asset types including corporate, foreign currency sovereign, subsovereign/provincial, supranational, asset-backed, covered, and agency bonds. The various factor returns are estimated jointly through a cross-sectional regression of issuer spread curve–modified log returns[8] using an estimation universe of more than 4,500 issuer curves. The bond-spread return not explained by the systematic factor returns forms the basis for estimating issuer- and issue-specific risk. The systematic credit-spread risk factors are estimated separately for each currency and include the following factor types:

Sector risk: The risk associated with bonds in a specified sector or industry. The spread exposure for the cross-sectional regression is binary—one for bonds in the sector, zero for bonds not in the sector. The sectors cover industry groups for corporate and the various asset types, including agency, covered, foreign currency sovereign, supranational, and so on.

Country risk: The additional risk associated with bonds from issuers with a specified country of origin. The spread exposure is binary.

Quality risk: The additional risk associated with bonds at various spread levels. Quality factors are associated with average spread returns of bonds in a specified spread range corresponding roughly to rating-spread bands. Each bond has spread exposure to two quality risk factors defined by the two closest spread ranges such that the exposures sum to one.

Market risk: The additional risk associated with how the market prices bonds based on factors such as beta to the market, size, and so on. Spread exposure is based on a *Z*-scored rank ordering.

MBS Spread

The MBS spread risk factors capture the risk associated with changes in the option-adjusted spread over swap rates of standard agency MBS bonds (MBS generics) covering the various agency issuers, maturities, ranges of coupons, property types, and vintages. Six MBS sectors are defined by combinations of issuing agency (Fannie Mae and Freddie Mac, Ginnie Mae), maturity (15-year, 30-year), and property type (single-family, multifamily). For each sector, the factor returns are estimated jointly through a cross-sectional regression on spread returns computed as change in spread level based on a total universe of more than 300 MBS generics. The price exposure for an MBS bond is its effective spread duration. Bond spread returns not explained by the factor returns form the basis for the MBS bond specific risk. For each sector, the systematic MBS spread risk factors are

Spread risk: The risk associated with spread fluctuations of mortgage pools within a sector due to changing economic conditions reflected by overall average market conditions. The spread exposure is one for MBS generics in the sector.

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8. Modified log returns are defined as changes in the log of spread for spreads above a threshold $\theta = 100$ basis points, whereas for spreads below this threshold, the returns are simple changes in spread divided by θ. The corresponding price exposure is modified duration times or $D \times \max(S, \theta)$.

Refinance risk: The additional risk associated with mortgage borrowers prepaying the mortgage by refinancing the loan at a lower interest rate. The exposure is determined by option-adjusted spread sensitivity to changes in the refinance factor in the MBS pricing model.

Turnover risk: The additional risk associated with mortgages being prepaid driven by seasonal variations in housing turnover, defaults, and so on. The exposure is determined by option-adjusted spread sensitivity to changes in the turnover factor in the MBS pricing model.

Volatility

The volatility risk factors capture risk associated with variations in level and shape of swaption or option-implied equity volatility surfaces. The risk factors are user-selected nodes on the volatility surfaces, with factor returns defined as changes in the log of volatility level. Exposures are computed as price sensitivities to volatility factor shocks (vega). Callable bonds, MBS pools, and various fixed-income derivatives typically have exposure to the swaption volatility factors, whereas convertible bonds have sensitivity to equity implied volatility factors.

For model construction, rigorous statistical methods incorporating peer analysis and outlier downweighting are used to build the issuer curves that both serve as pricing factors for computing bond analytics and form the core of the factor model estimation. The factor selection process ensures that only factors that consistently explain statistically significant components of issuer spread returns are included, while robust regression helps control the impact of outlier data on factor return estimation. Shrinkage estimators are employed for computing specific risk based on the quality of the issuer curve data. The resulting risk models perform well in backtests in both total and active risk space.[9]

Illustration of the Axioma Model

To illustrate the capabilities of the Axioma fixed-income multifactor risk model, we analyze the active risk decomposition of a portfolio of 58 bonds constructed to minimize tracking error to a large, diversified U.S. dollar fixed-income benchmark portfolio consisting of around 9,600 bonds with market weights distributed as 40% U.S. Treasuries, 31% corporate bonds (agency, financial, industrial, and utilities), 28% agency MBS pools, and 1% U.S. dollar sovereign bonds (non-U.S.). The portfolio was constructed to constrain deviations of the portfolio from the benchmark for exposures to key rate durations, duration times spread (DTS), and sector, as well as to ensure that no single name was more than 5% or less than 0.05% of the portfolio weight.

The predictive or forward-looking tracking error, defined as the annualized predicted standard deviation of the portfolio return minus the benchmark return,

9. A discussion of the importance of issuer curves in fixed-income risk modeling can be found in the white paper, "A New Data-Driven Fixed-Income Risk Framework," available at Qontigo.com.

is computed based on the covariance matrix of the fixed-income risk model factors and the portfolio price exposures. The covariance matrix is estimated from an exponentially weighted moving average of factor return time series with weekly overlapping returns using a half-life of 1 year and a look-back period of 4 years.

The constructed portfolio is somewhat overweight in U.S. Treasuries and sovereign bonds and underweight in corporate and MBS bonds relative to the benchmark. The total tracking error of this portfolio relative to the benchmark is 50 basis points. The active risk decomposition is given in Exhibit 32–13. For various risk types, the exhibit shows the absolute level of active risk in basis points as well as the percentage contribution to variance, which at each level of the decomposition is additive to the next level. At the top level, the percentage contribution to variance of the systematic risk plus the percentage contribution to variance of the specific risk sums to 100%. Because systematic risk and specific risk are by construction uncorrelated, this corresponds to (40 basis points)^2 + (30 basis points)^2 = (50 basis points)^2.

EXHIBIT 32–13

Risk Decomposition of Total Tracking Error of 50 Basis Points for the Portfolio Relative to the Benchmark for the Axioma Model Illustration

Risk Type			Active Risk (bps)			% Contribution to Variance		
Systematic risk			40.0			64.0		
	Rates			45.3			46.9	
		Treasury			40.0			28.9
		Swap Spread			21.4			18.0
	Credit			26.3			–6.2	
		Sector			25.7			–2.5
		Quality			12.1			–0.2
		Market			16.0			–1.7
		Country			4.6			–1.8
	MBS			30.6			19.8	
		Spread			4.5			2.0
		Refinance			17.8			3.9
		Turnover			32.6			13.9
	Volatility			6.1			3.5	
Specific risk			30.0			36.0		
	Credit			28.6			32.7	
		Corporate			28.1			31.5
		Foreign Sovereign			5.4			1.2
	MBS			9.1			3.3	

bps = basis points; MBS = mortgage-backed securities

AMUNDI ASSET MANAGEMENT'S FIXED-INCOME MULTIFACTOR RISK MODEL[10]

Let's now look at how the Amundi Asset Management fixed-income multifactor risk model can be used to create an optimized portfolio that tracks the ICE BofA Euro Large Cap Corporates Index (the benchmark) between January 2016 until May 2021. The initial investment is €300 million.

The tracking is accomplished by minimizing for each sector the differences in the modified duration, duration times spread (DTS), and weights between the two portfolios. The optimization method is performed using genetic algorithms to build portfolios satisfying the bonds' constraints of minimum tradable and lot size.[11] In addition, for trading reasons, in this illustration €1 million is set as the maximum tradable amount per bond. Rebalancing is performed at the end of each month. The proceeds from the bonds removed from the portfolio and paid coupons over the past month are reinvested so as to match the benchmark risk metrics. Some additional sales of bonds may occur when the available cash is not sufficient.

Exhibit 32–14 shows the evolution of the portfolio's modified durations and DTS. Note that although the optimized portfolio exhibits the same DTS as the benchmark, the duration is slightly lower, allowing the portfolio to show a higher weighted spread.

Exhibit 32–15 illustrates the total return performance of the optimized portfolio and the benchmark. Exhibit 32–16 reports the statistics of the returns and their distribution. The portfolio exhibits a slightly higher return (2.54% versus 2.51%) and lower volatility (3.95% versus 4.14%) than the benchmark, thus resulting in a higher Sharpe ratio (0.64 versus 0.61). The negative values of skewness (−3.14 and −2.93) together with the high kurtosis values (21.09 and 20.09) indicate that the distribution of returns is heavy tailed and presents a mass concentrated at the right of the median.

Exhibit 32–17 depicts a summary of the risk factors to which the portfolio is exposed. As in all multifactor risk models, these risk factors are classified into systematic and nonsystematic risk factors. *Systematic risk* refers to the risk inherent in the entire market or a market segment. *Nonsystematic risk* is the risk that is not attributable to systematic risk factors. Nonsystematic risk factors are associated with exposures to particular issuers or particular issues and can be mitigated through diversification.

Again, as commonly found in these models, systematic risk factors consist of two categories: term-structure risk factors and non-term-structure risk factors. *Term-structure* risk factors are associated with shape changes in the term structure

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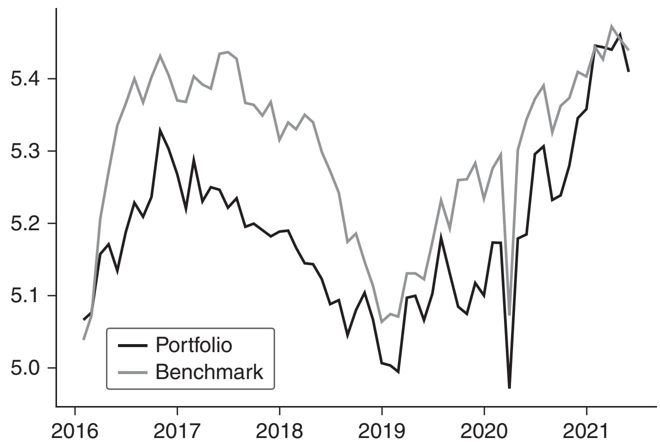
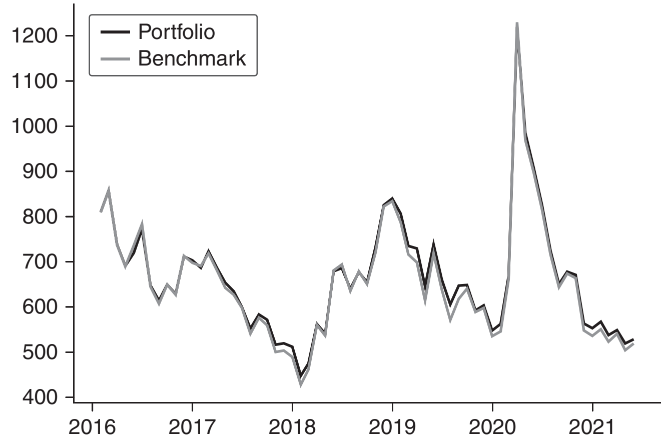
10. The description of the model and the illustration were provided by Amina Cherief and Mohamed Ben Slimane in the Quantitative Research Group of Amundi Asset Management, Paris.
11. A genetic algorithm is a search-based optimization method. The algorithm is based on the ideas of Charles Darwin's natural selection and genetics. This algorithm is described in Mohamed Ben Slimane, "Bond Index Tracking with Genetic Algorithms," Amundi Working Paper No. 108, Amundi Asset Management, Paris, 2021. Available at https://research-center.amundi.com/article/bond-index-tracking-genetic-algorithms.

EXHIBIT 32–14

Modified Duration and DTS for Amundi Model Illustration



(a) Portfolio Modified Duration



(b) Portfolio DTS

of interest rates and credit spreads. *Non-term-structure* risk factors include the sector, quality, optionality, and coupon risks. The sector risk is the risk associated with exposure to the 18 sectors of the benchmark index.[12] Quality, optionality, and coupon risks are the risks associated with the exposure, respectively, to credit-default risk in terms of credit ratings, embedded options, and different coupon rates.

12. These sectors are basic industry, energy, telecommunications, banking, healthcare, financial services, insurance, technology and electronics, utility, capital goods, consumer goods, automotive, media, real estate, services, transportation, retail, and leisure.

EXHIBIT 32–15

Total Return Performance for the Amundi Model Illustration

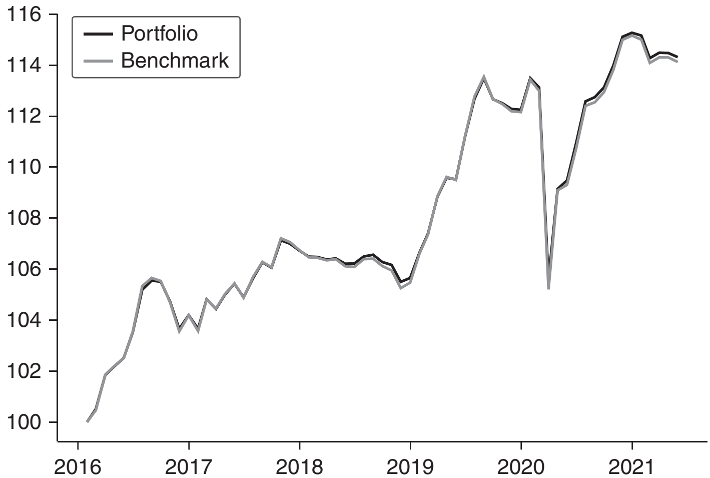


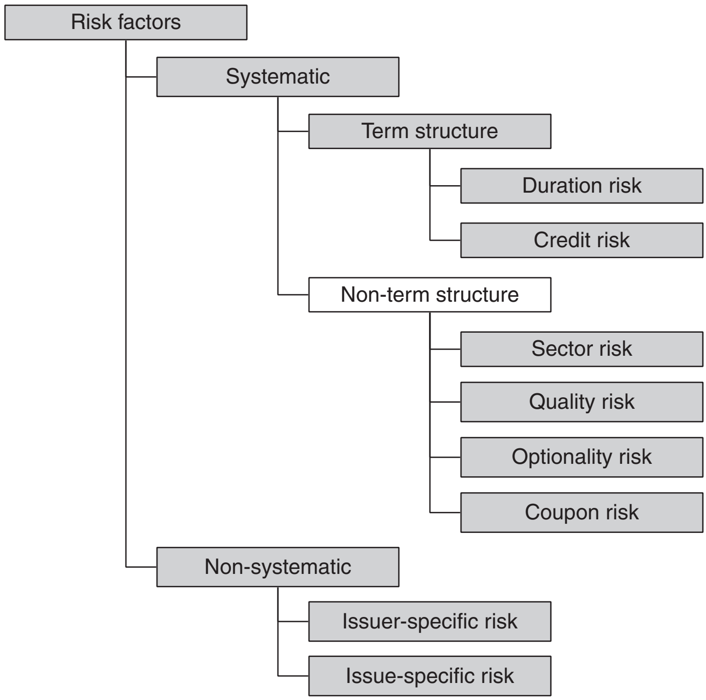
EXHIBIT 32–16

Statistics

Factor	Portfolio	Benchmark
Return (%)	2.54	2.51
Standard deviation (%)	3.95	4.14
Sharpe	0.64	0.61
Skewness	–3.13	–2.92
Kurtosis	21.09	20.09
Tracking error (%)	0.24	—
Correlation	0.999	1
Beta	0.954	1

EXHIBIT 32–17

Risk Factors in the Amundi Model



Exhibits 32–18 through 32–22 show the relevant correlations: correlations between systematic and nonsystematic risks, systematic risk correlations, term-structure risk correlations, non-term-structure risk correlations, and nonsystematic risk correlations, respectively.

The predicted or forward-looking tracking error is 24.1 basis points. The part due to systematic risk is 9.8 basis points, whereas the part due to nonsystematic risk is 22.2 basis points. These risks are almost orthogonal[13] because their correlation is −0.01 (see Exhibit 32–18). It can be verified that the square root of the sum of the squares of the two tracking errors [$(9.8^2 + 22.2^2)^{1/2} = 24.2$ basis points] does not differ much from the portfolio's tracking error.

13. Orthogonal means that two or more variables are unrelated (independent) to one another but both have an influence on the variable to be explained. In our context, it means that two or more risk factors are independent, but both influence the tracking error. Moreover, each risk factor separately contributes a distinct value to the tracking error.

EXHIBIT 32–18

Correlation Between Systematic and Nonsystematic Risks

	Active Return	Systematic Risk	Nonsystematic Risk
Active Return	1		
Systematic Risk	0.39	1	
Nonsystematic Risk	0.91	–0.01	1

EXHIBIT 32–19

Systematic Risk Correlations

	Active Return	Systematic Risk	Term-Structure Risk	Non-Term-Structure Risk
Active Return	1			
Systematic Risk	0.39	1		
Term-Structure Risk	0.41	0.75	1	
Non-Term-Structure Risk	0.02	0.45	–0.25	1

If we focus on the systematic risk, the tracking error due to the term-structure risk and the non-term-structure risk are respectively 9.0 and 6.7 basis points. Their correlation is –0.25[5]. If they were statistically independent, the tracking error for the systematic risk would be 11.2 basis points. The tracking errors associated with duration and credit risk, the components of term-structure risk, are, respectively, 7.4 and 4.4 basis points. With a correlation of zero between them, the term-structure risk would be 8.6 basis points, below the current level of 9.0 basis points, indicating that the correlation is slightly positive. Using the term-structure correlations shown in Exhibit 32–20, it can be verified that it equals 0.12.

Exhibit 32–23 displays the breakdown of the tracking error between the non-term-structure components. Optimization performed by sector ensures a relatively small sector risk. The coupon risk level can be explained by the fact that the optimizer prefers bonds with high spreads.

The tracking errors attributed to issuer- and issue-specific risks are, respectively, 10.5 and 15.8 basis points. Compared with the tracking errors of the risks seen above, they are relatively higher and arise from the differences in exposure to issuers and issues. Indeed, the sample portfolio does not include all the benchmark's

EXHIBIT 32–20

Term-Structure Risk Correlations

	Active Return	Systematic Risk	Term-Risk Structure	Duration Risk
Active Return	1			
Systematic Risk	0.39	1		
Term-Risk Structure	0.41	0.75	1	
Duration Risk	0.47	0.72	0.88	1

EXHIBIT 32–21

Non-Term-Structure Risk Correlations

	Active Return	Systematic Risk	Non-Term-Structure Risk	Sector Risk	Quality Risk	Optionality Risk	Coupon Risk
Active Return	1						
Systematic Risk	0.39	1					
Non-Term-Structure Risk	0.02	0.45	1				
Sector Risk	–0.48	0.01	0.59	1			
Quality Risk	0.68	0.08	–0.22	–0,6	1		
Optionality Risk	0.53	0.09	–0.11	–0.45	0.46	1	
Coupon risk	–0.18	0.37	0.70	0.27	–0.42	–0.53	1

EXHIBIT 32–22

Nonsystematic Risk Correlations

	Active Return	Nonsystematic Risk	Issuer-Specific Risk	Issue-Specific Risk
Active Return	1			
Nonsystematic Risk	0.91	1		
Issuer-Specific Risk	0.61	0.76	1	
Issue-Specific Risk	0.87	0.90	0.39	1

EXHIBIT 32–23

Tracking Errors for Non-Term-Structure Risks

Risk	Tracking Error (in Basis Points)
Sector risk	5.6
Quality risk	2.8
Optionality risk	3.9
Coupon risk	6.4

issues and issuers and thus has different weights than those of the benchmark. For instance, the weight mismatches per issuer and security have an average of 0.12% and 0.06%. As of May 31, 2021, the benchmark is composed of 3,277 securities from 646 different issuers.

Risk Factor Picking

We can use the Amundi model to explain how the risk factors are determined. Amundi uses the technique of the least absolute shrinkage and selection operator (LASSO) to identify the most pertinent risk factors. LASSO is a shrinkage and selection method for linear regressions as described in the next chapter.

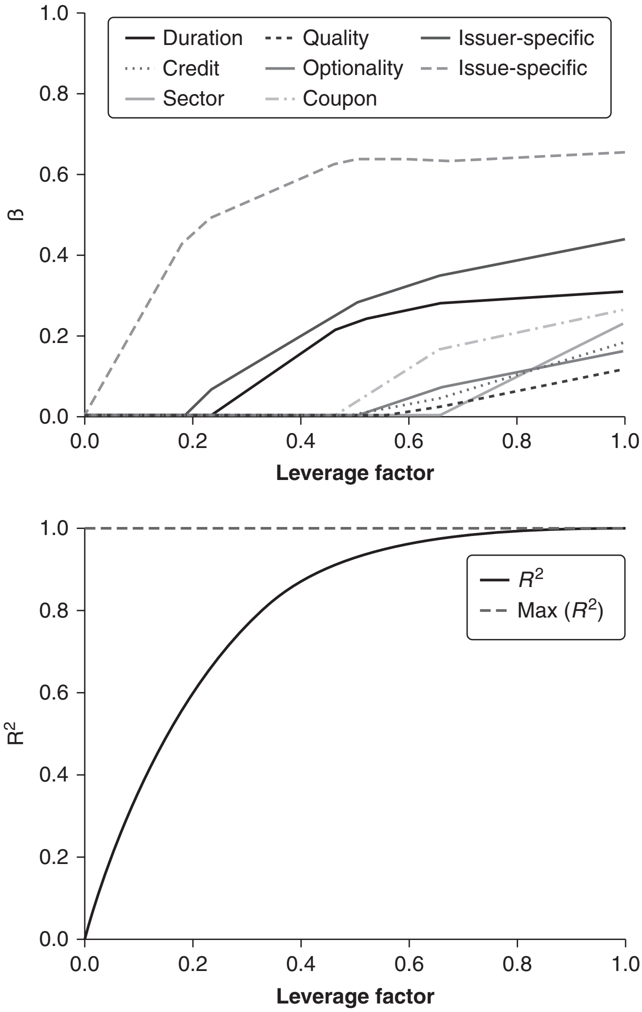
Exhibit 32–24 shows that the most relevant explanatory variable is the issue-specific component. The issuer-specific factor is the runner-up. The two nonsystematic risk components, taken together, explain 66.78% of the active return variance. This result is due to the high correlations of returns between the active return and nonsystematic risk (91%) and between the active return and the issue-specific risk (87%), as shown in Exhibit 32–22.

Among the systematic risks, duration appears third in the ranking. Coupon, credit, optionality, quality, and sector factors are picked in this order. The last position of the sector factor is a consequence of its negative return correlation (–0.48) with the active return (see Exhibit 32–21). Interestingly, the on-boarding of these last risk factors marks a stopping point in the beta progression of the issue-specific risk factor.

In Exhibit 32–25, the variance inflation factor (VIF) is reported. As explained in Chapter 31, the VIF measures how much the variance of an independent variable is influenced or inflated by its interaction/correlation with the other independent variables. When significant multicollinearity issues exist, the VIF will be very large

EXHIBIT 32–24

LASSO Factor Picking



(typically above 5) for the variables involved. The highest value for VIF reported in Exhibit 32–25 of 2.68 indicates a low dependence between the risk factors. With VIFs of, respectively, 2.50 and 2.68, sector and quality risks seem more related than the other risks. This result is consistent with their high correlation of returns (−0.60), as shown in Exhibit 32–21.

EXHIBIT 32–25

Variance Inflation Factor (VIF) for the Risk Factors

Risk Factor	VIF
Duration risk	1.16
Credit risk	1.57
Sector risk	2.50
Quality risk	2.68
Optionality risk	1.88
Coupon risk	1.79
Issuer-specific risk	1.61
Issue-specific risk	2.01

CHAPTER

33

MONTE CARLO SIMULATION

In fixed-income portfolio management the performance of a portfolio or strategy often depends on the outcome of many variables. For example, the performance of a fixed-income portfolio will be affected by the magnitude of the change in Treasury rates, the spread between non-Treasury and Treasury securities (or some other benchmark securities), changes in the shape of the yield curve, changes in the credit rating of individual corporate bond issues, changes in interest-rate volatility, and in the case of mortgage-backed securities (MBS), changes in prepayment speeds. Moreover, changes in every random variable may proceed along a substantial number of possible paths. These eventualities make it impractical to evaluate all possible combinations of outcomes in order to assess the risks associated with a portfolio.

The technique employed to deal with such problems faced by portfolio managers and analysts is *Monte Carlo simulation.* Some examples of where Monte Carlo simulation have been employed in fixed-income portfolio management include

- Valuing interest-rate-path-dependent cash-market fixed-income instruments such as MBS;[1]
- Valuing interest-rate-path-dependent fixed-income derivatives;[2]
- In credit-risk management, estimating recovery rates and credit exposures for a portfolio;[3]
- Evaluating MBS portfolio strategies;[4]
- Estimating risk measures for a portfolio such as value at risk and related measures for interest-rate risk and the integration of market and credit risk;[5]
- Backtesting a bond investment strategy.

1. This is illustrated in Chapter 27.
2. The use of simulation to value derivatives was first suggested in P. Boyle, "Options: A Monte Carlo Approach," *Journal of Financial Economics*, Vol. 4 (1977), pp. 323–338.
3. See Chapter 6 in Srichander Ramaswamy, *Managing Credit Risk in Corporate Bond Portfolios: A Practitioner's Guide* (Hoboken, NJ: John Wiley & Sons, 2004).
4. David E. Canuel and Charles F. Melchreit, "Total Return Analysis in CMO Portfolio Management," Chapter 4 in Frank J. Fabozzi (ed.), *Advances in the Valuation and Management of Mortgage-Backed Securities* (Hoboken, NJ: John Wiley & Sons, 1998); and Chapter 5 in Bennett W. Golub and Leo M. Tilman, *Risk Management: Approaches for Fixed Income Markets* (New York: John Wiley & Sons, 2000).
5. See, e.g., Jonathan Stein, "The Integration of Market and Credit Risk Measurement," *Financial Engineering News* (November 1998); and Farshid Jamshidian and Yu Zhu, "Scenario Simulation Model for Fixed Income Portfolio Risk Management," Chapter 11 in Frank J. Fabozzi, Lionel Martellini, and Philippe Priaulet (eds.), *Advanced Bond Portfolio Management: Best Practices in Modeling and Strategies* (Hoboken, NJ: John Wiley & Sons, 2006).

In this chapter we explain the basic elements of Monte Carlo simulation by describing the steps in a simulation using a hypothetical bond portfolio management situation. Critical to understanding simulation is a basic knowledge of probability distributions because they are the building blocks of simulation models. That is, simulation models are based on probability distribution assumptions about the random variables in the model, which are then used to generate scenarios (often referred to as *trials*) that happen with probabilities described by the probability distributions. Portfolio managers then record what happens to variables of interest over these scenarios and observe the characteristics of the probability distribution of the output of interest. At the end of this chapter we describe how simulation is used in backtesting investment strategies.

MOTIVATION FOR THE USE OF MONTE CARLO SIMULATION

Suppose that a portfolio manager wants to assess the performance of a portfolio over a 1-year investment horizon. Suppose further that the portfolio's performance will be determined by the actual outcome for each of eight random variables and that each of the eight random variables has 10 possible outcomes. There are thus more than 1 billion possible outcomes, representing all possible combinations of the eight random variables.[6] Furthermore, each of the more than 1 billion outcomes has a different probability of occurrence.

One approach for a portfolio manager is to take the *best guess* for each random variable and determine the impact on performance. The best guess value for each random variable is usually the expected value of the random variable.[7] There are serious problems with this shortcut approach. To understand its shortcomings, suppose that the probability associated with the best guess for each random variable is 50%. If the probability distribution for each random variable is independently distributed, the probability of occurrence for the best guess result would be less than one-half of 1%. Obviously, at this level of probability, no portfolio manager would have a great deal of confidence in this best guess result.

Between the extremes of enumerating and evaluating all possible combinations and the best guess approach is the *simulation approach*. Simulation is less a model than a procedure or algorithm. The solutions obtained do not represent an optimal solution to a problem. Rather, simulation provides information about a problem so that a portfolio manager can assess the risks of a particular course of action. Because it offers a flexible approach to dealing with business problems, simulation is probably one of the tools used most often.

There are many types of simulation techniques. When probability distributions are assigned to the random variables, the simulation technique is known as a *Monte Carlo simulation*—named after the famous gambling spot on the French Riviera. Monte Carlo simulation enables portfolio managers to determine the

6. More precisely, there are 1,073,741,824 (= 8^{10}) possible outcomes.
7. The expected value of a random variable is explained in Chapter 30.

statistical properties of a problem they face. Armed with this information, the portfolio manager can select the most prudent course of action.

STEPS FOR MONTE CARLO SIMULATION

There are 12 steps in a Monte Carlo simulation:

Step 1. The performance measure must be specified. This measure is referred to as the *output variable*. The performance measure may be, for example, total return over some investment horizon or net interest (spread) income.

Step 2. The problem under investigation must be expressed mathematically, including all important variables and their interactions. The variables in the mathematical model will be either *deterministic* or *random*, and they are referred to as the *input variables*. A deterministic variable can take on only one value; a random variable can take on more than one value.

Step 3. For input variables that are random variables, a probability distribution for each must be specified.

Step 4. For each input variable whose probability distribution must be specified, the probability distribution must be converted into a cumulative probability distribution.

Step 5. For each random variable that is an input variable, representative numbers must be assigned on the basis of the cumulative probability distribution to each possible outcome specified.

Step 6. A random number must be obtained for each random input variable.

Step 7. For each random number, the corresponding value of the random input variable must be determined.

Step 8. The corresponding value of each random input variable found in the previous step must be used to determine the value of the performance measure (i.e., the output variable) developed in Step 2.

Step 9. The value of the performance measure (output variable) found in Step 8 must be recorded.

Step 10. Steps 6–9 must be repeated many times, say, 100 to 1,000 times. The repetition of Steps 6–9 is known as a *trial*.

Step 11. The values for the performance measure for each trial recorded in Step 9 become the basis for construction of a probability distribution and cumulative probability distribution.

Step 12. The cumulative probability distribution constructed in Step 11 is analyzed/interpreted. This is done by calculating summary statistics such as the mean, standard deviation, range, and skewness.

To follow the 12-step process just described to apply Monte Carlo simulation to a particular problem, there are several decisions that must be made. First, a decision must be made to determine what probability distribution should be used for each of the random input variables used in the simulation model and the correlation between each pair of random input variables. Second, a decision must be made about how many scenarios (trials) should be used. Finally, how are random numbers generated efficiently. We describe how to deal with the three decisions below.

Determining the Probability Distribution and Correlation for the Random Input Variables

In formulating the simulation model to obtain possible future values for the random input variables, it is necessary to assme which distribution is appropriate for modeling the future values. Four approaches have been used. A commonly employed method, *bootstrapping*, uses a random input variable's historical distribution and assumes that the future will behave in the same way.[8]

A second approach is to assume a particular probability distribution for the future value of the random input variables and use historical data to estimate the parameters of this distribution (i.e., the parameters that determine the specific shape of the distribution). For example, if a normal distribution is assumed, the estimated sample mean and sample standard deviation are estimated from historical data.

A third approach is not to start out with a particular distribution but to use historical data to find a distribution for the random input variable that provides the best fit to the data.[9] A fourth approach is to ignore the past and look forward, constructing a probability distribution based on the portfolio manager's subjective guess about how the random input variables in the simulation model will behave.

With respect to the correlation between the random input variables, this is typically obtained from historical data. Statistical tests can be used to determine whether the correlations are statistically significant.

Monte Carlo simulation can be used to determine to how sensitive the results are to the probability distribution for the random input variables. That is, Monte Carlo simulation should be accompanied by sensitivity analysis for the assumed probability distributions.

Determining the Number of Trials

The product of a Monte Carlo simulation is the probability distribution for the random variable of interest. How do we know how many trials to perform? In many applications the portfolio manager or analyst might want one estimate of the

8. The statistical concept of bootstrapping is different from the concept of bootstrapping used in fixed-income analysis as described in Chapter 7, where we described how bootstrapping can be used to derive the Treasury spot-rate curve.
9. There are different goodness-of-fit tests that we described in Chapter 31.

outcome of the simulation. For example, for a financial instrument (security or derivative), the theoretical (model) price may be sought. The best estimate is the average value or mean from all the trials of the simulation.

The *mean standard error* is a commonly used measure of how good the estimate is. The mean standard error is defined as

$$\text{Mean standard error} = \sqrt{\frac{\text{variance of the trial value}}{\text{number of trials}}}.$$

A confidence interval for the estimate of the value sought can be constructed using the mean standard error. The smaller the mean standard error, the greater is the precision of the estimated value.

One way to reduce the mean standard error is to increase the number of trials. This can be costly in terms of the increase in the number of trials necessary to reduce the mean standard error to a satisfactory level. Alternatively, the mean standard error can be reduced by reducing the variance of the trial values. This approach to improving the precision of the estimated value is called *variance reduction*. Several variance-reduction methods have been used in finance, but a discussion of these methods is beyond the scope of this chapter.[10]

Generating Random Numbers

Before we illustrate these steps, we need to explain how random numbers are obtained in Step 6. Any procedure used must have the following property: for each number that may possibly be selected, the probability of selection must be equal.

There are several procedures that can be used to generate random numbers. First, suppose that two-digit random numbers from 00 to 99 are wanted. The portfolio manager could write each number on a piece of paper of equal size, put it in a bag, shake the bag, and then pick out one piece of paper. The number that turns up on that pick is the random number for the random variable. As long as each piece of paper has an equal chance of being selected, the procedure is acceptable for generating random numbers. For each trial, however, the paper picked in the previous trial must be put back into the bag. If it is not, each piece of paper would no longer have an equal chance of being selected in a subsequent trial. Because there are many instances where random numbers are needed, *random number tables*, spreadsheets, and other software include features that can be used to generate random numbers.

10. For a discussion of two variance-reduction methods used in pricing derivative instruments, the antithetic variates method and the control variates method, see John M. Charnes, "Sharper Estimates of Derivative Values," *Financial Engineering News* (June–July 2002). For a discussion of other variance-reduction methods, see Chapter 11 in Averill M. Law and W. David Kelton, *Simulation Modeling and Analysis*, 3rd ed. (New York: McGraw-Hill, 2002).

INTERPRETING THE OUTPUT OF A MONTE CARLO SIMULATION MODEL

Once the performance measure distribution is generated from a Monte Carlo simulation model, it is necessary to interpret the results. Again, the interpretation draws on probability and statistical concepts discussed in Chapter 30. Specifically, the statistical concept of confidence interval estimates is used. The main idea of confidence intervals in statistics is that when a decision maker wants to estimate a specific parameter of a distribution, such as the mean, random samples can be obtained, and the observed value of the parameter is in the sample. Rather than reporting a single value for the estimated value for the parameter of interest, the decision maker can look at an interval whose length is related to the probability that the true distribution parameter indeed lies in that interval.

Simulation is very similar to statistical sampling in that the decision maker represents the uncertainty by generating scenarios, that is, the *sampling values* for the output parameter of interest from an underlying probability distribution. When a decision maker estimates any parameter of interest from the sample of scenarios, the decision maker needs to worry about the accuracy of the estimate value. For example, suppose that the performance measure (which in this example is the "parameter" of interest) sought is the annual return from a proposed bond portfolio strategy. The portfolio manager selects a confidence interval. A commonly used one is the 95% confidence interval. Based on the number of samples (trials), the mean value for the sample annual return from the strategy, and its standard deviation, a confidence interval can be computed.

ILLUSTRATION OF THE STEPS OF MONTE CARLO SIMULATION

Consider a portfolio manager who has invested $15 million in three bonds. Exhibit 33–1 specifies the three hypothetical bonds and the relevant information about each issue. For purposes of our illustration, each bond is assumed to pay its next coupon 6 months from now. The portfolio manager wants to simulate the 6-month period total return assuming that the only two random variables are (1) the change in the

EXHIBIT 33–1

Hypothetical Three-Bond Portfolio for Simulation Illustration

Bond	Maturity (years)	Coupon (%)	Price ($)	Par ($)	YTM (%)
Treasury	5.5	6.0	100	5 million	6.0
BBB corporate	15.5	9.0	100	4 million	9.0
BBB corporate	25.5	10.5	100	6 million	10.5

EXHIBIT 33–2

Information for Monte Carlo Simulation of a Three-Bond Portfolio—Random Variable: Treasury Yield Curve

Treasury Yield Curve			Probability	Cumulative Probability	Representative
5-Year	15-Year	25-Year	Distribution	Distribution	Numbers Assigned
4%	6%	7%	0.20	0.20	0–19
5	8	9	0.15	0.35	20–34
6	7	7	0.10	0.45	35–44
7	8	8	0.10	0.55	45–54
9	9	9	0.20	0.25	55–74
10	8	8	0.25	1.00	75–99

EXHIBIT 33–3

Information for Monte Carlo Simulation of a Three-Bond Portfolio—Random Variable: BBB-Rated Corporate/Treasury Yield Spread

BBB/Treasury Yield Spread (in Basis Points)	Probability Distribution	Cumulative Probability Distribution	Representative Numbers Assigned
75	0.10	0.10	0–9
100	0.20	0.30	10–29
125	0.25	0.55	30–54
150	0.25	0.80	55–79
175	0.15	0.95	80–94
200	0.05	1.00	95–99

level and shape of the Treasury yield curve and (2) the change in the quality spread between Treasuries and BBB corporate bonds.

Exhibit 33–2 gives six possible Treasury yield curves 6 months from now that the portfolio manager believes may occur. Exhibit 33–3 gives six possible Treasury/corporate bond spreads (i.e., quality spreads) assuming that the spread is the same regardless of maturity. We use all this information to describe each of the 12 steps of the Monte Carlo simulation technique:

Step 1. The performance measure must be specified. The performance measure is the annualized 6-month total return for the three-bond portfolio.

Step 2. The problem under investigation must be expressed mathematically.

The 6-month total return can be expressed as follows: let

V_i = value of bond i at the end of 6 months (i = 1, 2, and 3, where 1 is the Treasury, 2 is the shorter BBB-rated corporate bond, and 3 is the longer BBB-rated corporate bond);

c_i = 6-month coupon payment for bond i.

The total future dollars from each bond at the end of the 6-month horizon is then $v_i + c_i$. The 6-month total return for the three-bond portfolio whose initial portfolio value is $15 million is then

$$\text{Total return} = \frac{(V_1 + c_1) + (V_2 + c_2) + (V_3 + c_3)}{\$15,000,000}$$

Doubling the total return gives an annualized total return on a bond-equivalent yield basis.

Step 3. For variables that are random, a probability distribution for each must be specified. In this case there are two random variables—the Treasury yield curve and the yield spread—and a probability distribution must be specified for both. Assume that the probability distributions for these two random variables are as shown in Exhibits 33–2 and 33–3. The coupon payment is a deterministic variable.

Step 4. For each random variable whose probability distribution must be specified, the probability distribution must be converted into a cumulative probability distribution. As we explained in Chapter 30, the cumulative probability of attaining a value that is less than or equal to a specified value is computed by summing the probabilities over the range of outcomes up to the specified value. The cumulative probability distribution for the two random variables in this illustration is shown in Exhibits 33–2 and 33–3.

Step 5. For each random variable, representative numbers must be assigned on the basis of the cumulative probability distribution to each possible outcome specified by the probability distribution. The representative numbers assigned for the two random variables in this illustration are shown in the last column of Exhibits 33–2 and 33-3.

Note that there are 100 assigned two-digit numbers ranging from 0 to 99. Each possible outcome (value for the random variable) is assigned enough numbers so that the ratio of the total numbers assigned to 100 will equal the probability of the outcome. For example, for the Treasury yield curve (the first random variable), the first yield-curve outcome in Exhibit 33–2 is assigned numbers from 0 to 19. There are 20 numbers in the range of 0 to 19, so 20 of the 100 numbers, or 20%, are assigned to the first possible Treasury yield-curve shape to equal its probability of 20%. A similar choice applies for the fifth possible Treasury yield curve, which also has a probability of

20%. Here the 20 numbers assigned are 55 to 74; the numbers 0 to 19 are not assigned to this outcome because they were assigned at the outset to the first Treasury yield curve.

Step 6: First Trial. Obtain a random number for each random variable. We can obtain a random number for a random variable either from a random number table or by computer generation. The first random number selected in Steps 6–9 will be for the random variable representing the Treasury yield curve. The second will be for the random variable representing the quality spread. Suppose that the two random numbers drawn are 91 and 12. Then, in the first trial, the first random number is 91, and the second random number is 12.

Step 7: First Trial. For each random number, determine the corresponding value of the random variable. Given the random number 91 for the Treasury yield curve and 12 for the quality spread, the corresponding outcome for the two random variables can be determined with the information in Exhibits 33–2 and 33–3. The Treasury yield curve in this trial would be

5-year Treasury = 10%;
15-year Treasury = 8%;
25-year Treasury = 8%.

The quality spread in this trial would be 100 basis points. Therefore, the horizon yield for the three bonds in this trial is

5-year Treasury = 10%;
15-year BBB-rated corporate bond = 9%;
25-year BBB-rated corporate bond = 9%.

Step 8: First Trial. Use the corresponding value of each random variable found in the previous step to determine the value of the performance measure developed in Step 1. The total future dollars from each bond for the first trial are shown in the eighth, ninth, and tenth columns of Exhibit 33–4. The next-to-last column shows the total future dollars for the portfolio, and the last column shows the total return percentage for the first trial.

Step 9: First Trial. Record the value for the performance measure found in Step 8. The value of the performance measure for the first trial is 10.16%.

Step 10: Second Trial. Repeat Steps 6–9. Here we describe repeating Steps 6–9 for only one more trial.

Step 6: Second Trial. Obtain a random number for each random variable. Suppose that the next two random numbers generated are 64 and 18.

Step 7: Second Trial. For each random number, determine the corresponding value of the random variable. Given the random number 64 for the Treasury yield curve and 18 for the quality spread, we can determine the corresponding outcome for the two random variables with the information in Exhibits 33–2 and 33–3. The Treasury yield curve in this trial would be

5-year Treasury = 9%;
15-year Treasury = 9%;
5-year Treasury = 9%.

The quality spread in this trial would be 100 basis points. Therefore, the horizon yield for the three bonds in this trial is

5-year Treasury = 9%;
15-year BBB-rated corporate bond = 10%;
25-year BBB-rated corporate bond = 10%.

Step 8: Second Trial. Use the corresponding value of each random input variable found in the previous step to determine the value of the performance measure. Future dollars for the portfolio produce the value shown in the last column of Exhibit 33–4 (0.24%).

Step 9: Second Trial. Record the 0.24% value for the performance measure found in Step 8. Step 10 involves repeating the trials. Exhibit 33–4 shows the results for the 20 trials.

Exhibit 33–5 shows summary statistics (average, standard deviation, skew, kurtosis, and return range) for trials of 500, 1,000, 2,000, and 4,000. The illustration can be enhanced by looking at the correlation between corporate yield spreads and Treasury rates.

EXHIBIT 33–4

Results of First 20 Trials for Total Return Simulation Illustration

Trial	Rand. No.	Treasury Yield Curve 5-Year	15-Year	25-Year	Rand. No.	Quality Spread (In Basis Points)	5.5-Year Treasury ($)	15.5-Year BBB Corp. ($)	25.5-Year BBB Corp. ($)	Portfolio ($)	Total Return (%)
1	91	10%	8%	8%	12	100	4,377,827	4,180,000	7,204,290	15,762,117	10.16
2	64	9	9	9	18	100	4,556,546	3,872,551	6,588,839	15,017,936	0.24
3	48	7	8	8	54	125	4,942,085	4,099,740	7,041,255	16,083,079	14.44
4	44	6	7	7	20	100	5,150,000	4,525,841	7,926,164	17,602,005	34.69
5	85	10	8	8	84	175	4,377,827	3,946,091	6,733,821	15,057,738	0.77
6	10	4	6	7	76	150	5,599,129	4,714,877	7,550,584	17,864,590	38.19
7	66	9	9	9	53	125	4,556,546	3,801,105	6,449,317	14,806,968	–2.57
8	38	6	7	7	64	150	5,150,000	4,347,790	7,550,584	17,048,374	27.31
9	42	6	7	7	14	100	5,150,000	4,525,841	7,926,164	17,602,005	34.69
10	23	5	8	9	81	175	5,368,802	3,948,091	6,185,646	15,500,539	6.67
11	61	9	9	9	62	150	4,556,546	3,731,683	6,315,000	14,603,329	–5.29
12	90	10	8	8	5	75	4,377,827	4,262,656	7,373,955	16,014,438	13.53
13	57	9	9	9	51	125	4,556,546	3,801,105	6,449,317	14,806,968	–2.57
14	29	5	8	9	49	125	5,368,802	4,099,740	6,449,317	15,917,858	12.24
15	6	4	6	7	18	100	5,599,129	4,915,682	7,926,164	18,440,975	45.88
16	67	9	9	9	81	175	4,556,546	3,664,220	6,185,646	14,406,412	–7.91
17	38	6	7	7	40	125	5,150,000	4,435,489	7,734,529	17,320,018	30.93
18	42	6	7	7	8	75	5,150,000	4,618,938	8,125,883	17,894,412	38.60
19	68	9	9	9	86	175	4,556,546	3,664,220	6,185,646	14,406,412	–7.91
20	86	10	8	8	71	150	4,377,827	4,021,796	6,884,531	15,284,154	3.79

EXHIBIT 33–5

Summary Statistics for Three-Bond Portfolio Illustration for 500, 1,000, 1,500, 2,000, and 4,000 Trials

Number of Trials	Average Return	Standard Deviation	Skew	Kurtosis	Minimum Return	Maximum Return
500	13.7465	0.6841	0.0637	0.069760	12.1244	15.7501
1,000	13.7927	0.4865	0.4027	–0.396535	12.8027	14.9272
1,500	13.8948	0.4222	–0.1055	–0.243479	12.7064	14.8488
2,000	13.8644	0.3216	–0.0517	–0.593209	13.1302	14.5096
4,000	13.8707	0.2674	–0.0504	–0.452237	13.2582	14.5536

APPLICATION OF MONTE CARLO SIMULATION TO BACKTESTING

Bond portfolio managers are constantly developing strategies for financial products that they believe will accomplish the investment objectives of investors. These strategies should be backtested prior to offering them to clients or creating collective investment products for investors. Monte Carlo simulation is one of three methods that can be used for backtesting bond investment strategies. The two other strategies are the walk-forward method and the resampling method. Below we describe the three methods for backtesting, their limitations, and the advantages of using Monte Carlo simulation.

The most common backtesting method is the *walk-forward* method. Using this method, the performance of a portfolio strategy is assessed assuming that history repeats itself exactly. Despite its popularity as a method for backtesting, it is challenging to carry out this method flawlessly. The walk-forward method has two advantages for backtesting. First, it has a clear historical interpretation. Second, it can be reconciled by paper trading (i.e., simulated trading to replicate the strategy without risking real money). Unfortunately, the walk-forward method has at least three major limitations. First, only the historical path is analyzed, and the method ignores other possible scenarios. Second, it does not inform the portfolio manager about the reasons why the proposed investment strategy may have generated an attractive return. Finally, because they can be biased by the particular sequence of data points, the results obtained from a walk-forward method will not necessarily be representative of future performance.

The *resampling method* addresses the first limitation of the walk-forward method that only a single historical path is tested. Unlike the walk-forward method, the resampling method assesses the performance of an investment strategy under the assumption that future paths can be simulated through the resampling of past observations. The objective for backtesting using the resampling method is not to derive a historically accurate view of the performance of the proposed investment strategy but rather to infer future performance from a number of out-of-sample scenarios. This method has the advantage of not being dependent on a particular (historical) scenario. Moreover, it offers the flexibility of allowing a portfolio manager to introduce stress scenarios.

The Monte Carlo simulation method, like the resampling method, overcomes the limitation of the walk-forward method by allowing assessment of the performance of an investment strategy using more than just one future path of possible outcomes. Moreover, there are four critical advantages to this method over the other two backtesting methods. Specifically, it allows (1) backtesting to be conducted based on randomized, controlled experiments, (2) development of tactical investment algorithms; (3) incorporation of the probability of an event before new data are collected, and (4) length of backtests to be expanded for as long as needed to achieve a targeted degree of confidence.[11]

When comparing two alternative decisions under uncertainty, it is technically correct (and fair) to evaluate them under the same set of scenarios. This procedure is referred to as *blocking*. Doing so eliminates circumstances in which the model happens to generate more favorable scenarios when evaluating one of the strategies than the other, which would lead the portfolio manager to conclude erroneously that the strategy evaluated over the more favorable set of scenarios is better.

Biases in Backtesting

Monte Carlo simulation for backtesting is not simple to implement. Not only are there issues regarding estimation, but there are also biases of the modeler that can have a major impact on the outcome of a simulation. Some of these biases may be innocent errors, whereas others may occur deliberately to influence the outcome. An example of the latter is when the individual or group proposing an investment strategy selects the time period over which the backtest is conducted and that time period is likely to produce the desired results. The application of backtesting that excludes time periods where the strategy performed poorly so as to generate a favorable performance for the investment strategy is referred to as *optimal-period bias*. Selecting the dates when the backtest begins and ends such that the results are sensitive to different time periods with respect to macroeconomic conditions, the level and shape of the yield curve, and interest-rate volatility is referred to as *time-period bias*.

Two other biases that may impact the outcome of a backtest that should be recognized are survivorship bias and look-ahead bias. When the backtest uses only surviving examples in assessing the performance of a proposed investment strategy, this bias is referred to as *survivorship bias*. This occurs when a portfolio manager uses only the current investment universe for backtesting an investment strategy, ignoring the fact that some companies or bonds may no longer be in the sample. This bias makes a proposed investment strategy appear better than it actually is.

11. For a further explanation of these advantages, see Chapters 12 and 13 in Marcos Lỏpez de Prado, *Advances in Financial Machine Learning* (Hoboken, NJ: Wiley, 2018).

When information is used that was not public at the time the investment decision was made it is referred to as *look-ahead bias* and could result in the false discovery of an investment strategy. For example, this can occur when data that would be unavailable if the strategy were live are used in the backtest. Another example of when look-ahead bias occurs is when financial statements filed with regulators are used without looking at subsequent revisions. This bias can be addressed by checking the timestamp for each data point and taking into account release dates, distribution delays, and backfill corrections.

CHAPTER 34

OPTIMIZATION MODELS

An *optimization model* prescribes the best course of action to be pursued in order to achieve an objective. The optimization models used in most fixed-income portfolio-management applications are mathematical programming models. We describe in this chapter how optimization models can be used in fixed-income portfolio management without delving into the nuances of these models or the algorithms for solving a mathematical programming problem. Any real-world problem confronting a portfolio manager will be complex enough to require the use of commercially available software to solve.

Optimization models are commonly used in structured portfolio strategies that do not rely on expectations of interest-rate movements or changes in yield spreads. Instead, the objective is the design of an optimal portfolio that will achieve the performance of a predetermined benchmark. The target could be (1) the return on a specific benchmark index, (2) sufficient dollars to satisfy a future single liability, or (3) sufficient dollars to satisfy each liability of a future liability stream. The structured bond portfolio strategy used when the target to be achieved is replication of a predetermined benchmark index is called an *indexing strategy.* When the objective is to generate sufficient funds to pay off predetermined future liabilities, the strategy is called a *liability-funding strategy.* Optimization models are also used in making asset-allocation decisions (i.e., decisions on allocation of funds among major asset classes and among different sectors of the bond market).

UNCONSTRAINED VERSUS CONSTRAINED OPTIMIZATION

In calculus, the first application a student is introduced to is an optimization problem. Given a mathematical function, say $y = f(x)$, the objective is to find the optimal value for y over some range of x, say x from a to b. The optimal value for y is the value for x that produces the largest value for y in a maximization problem and the lowest value in a minimization problem. To find that optimal value for y, the first derivative of the mathematical function is computed, set equal to zero, and solved for x. The value obtained for x is then used in the mathematical function to solve for y. However, it is unknown whether the value for y produced by the value for x is a maximum, minimum, or point of inflection. To determine which one it is, the second derivative of the mathematical function is computed. There is no need to go over the rules because this type of optimization does not occur in bond portfolio management. It is a problem where there are no constraints that must be satisfied. Consequently, it is referred to as an *unconstrained optimization problem*.

A *constraint* is a condition that must be satisfied by the optimal value. An optimization problem where there are constraints is appropriately referred to as a *constrained optimization problem* or a *mathematical programming problem*. One special case of a constrained optimization problem is when there are constraints, but they are equality constraints. In this special constrained optimization case, a technique in calculus known as the *Lagrangian method* for constrained optimization can be used. This method is commonly used in economics, but in bond portfolio management, a need for such a method is unlikely because most real-world problems have constraints, but one or more of those constraints will be inequalities.

SETTING UP A MATHEMATICAL PROGRAMMING MODEL

There are four steps in setting up a mathematical programming or optimization problem:

Step 1. Define the variables on which a decision must be made. These variables are called *decision variables*. In a structured portfolio problem, the decision variables are typically the amount of a security that should be purchased or sold. In an asset-allocation problem, the decision variables are the amounts to be allocated to each major asset class.

Step 2. Specify mathematically the objective the decision maker seeks to optimize. This mathematical expression is called the *objective function*. The linear or nonlinear nature of the objective function is what distinguishes the different types of mathematical programming models.

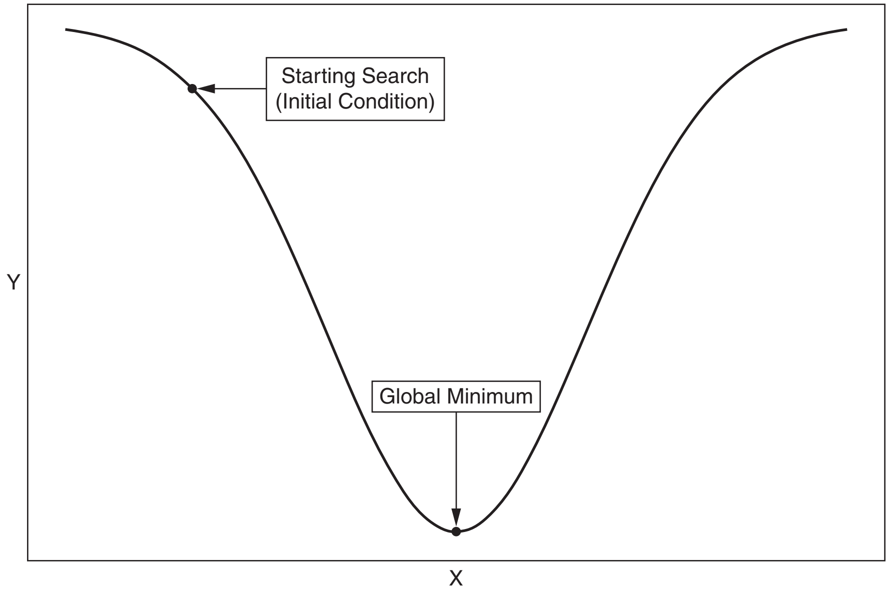
Step 3. Specify mathematically the constraints under which the objective function is to be optimized. One constraint, for example, might be that the portfolio's duration be equal to 4. Or a constraint may specify that the amount allocated to one sector of the bond market may not exceed a certain percentage of the portfolio or that the minimum amount allocated to liquid assets be a specified percentage. As with the objective function, mathematical characterization of the constraints determines the type of mathematical programming problem.

Step 4. Solve for the *optimal solution*.

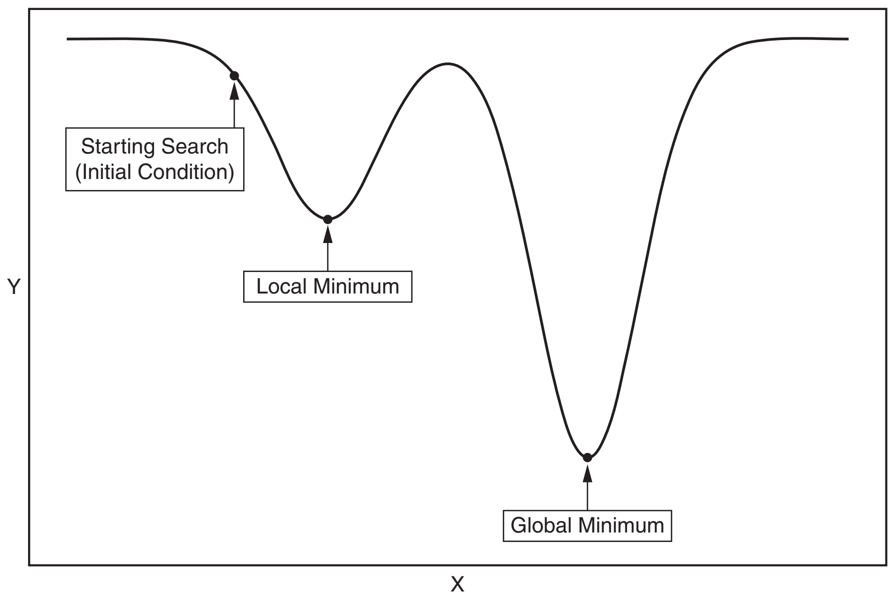
CONVEX OPTIMIZATION PROBLEM

One type of mathematical programming problem is a *convex optimization problem*. To appreciate what a convex optimization problem is, let's look at the types of solutions that can be produced by an optimization model: a global solution and a local solution. Exhibit 34–1 provides an illustration for understanding the two types of solutions.

EXHIBIT 34–1



(a) Global mimimum solution



(b) Local and global minimum solutions

Panel a of Exhibit 34–1 shows an unconstrained minimization problem. The curve in the exhibit is said to be *convex*. Every point on the curve is a solution to the objective function, and these points are referred to as *feasible solutions*. Suppose that the search for the minimum value begins at the point on the left-hand side of the graph (referred to as the *initial condition*). Clearly, with the convex shape shown, there is a minimum value that is referred to as a *global solution*.

Now look at panel b. What the algorithms used for solving a mathematical programming problem do is search for optimal solutions. If the algorithm starts at the initial condition and moves right, the algorithm will identify a minimum value, but it is clearly not the global solution. Instead, the search would produce a *local solution*. The mathematical function shown in panel b is a nonconvex function. There can be more than one local solution to a mathematical programming problem when the mathematical function is nonconvex. In contrast, with a convex function, as shown in panel a of the exhibit, there can only be one global solution.

Exhibit 34–1 provides an illustration of an unconstrained minimization problem. Suppose that the problem is an unconstrained maximization problem, as shown in Exhibit 34–2. Panel a shows a mathematical function that is *concave*, and as a result, there is only one global solution. Panel b shows a nonconcave mathematical function and, as can be seen, has two local solutions.

Thus, in a minimization optimization problem, a convex function will allow for only a global solution, whereas there could be one or more local solutions if the function is nonconvex. Similarly, when there is a maximization optimization problem, there is only a global solution when the function is concave, but one or more local solutions when it is nonconcave.

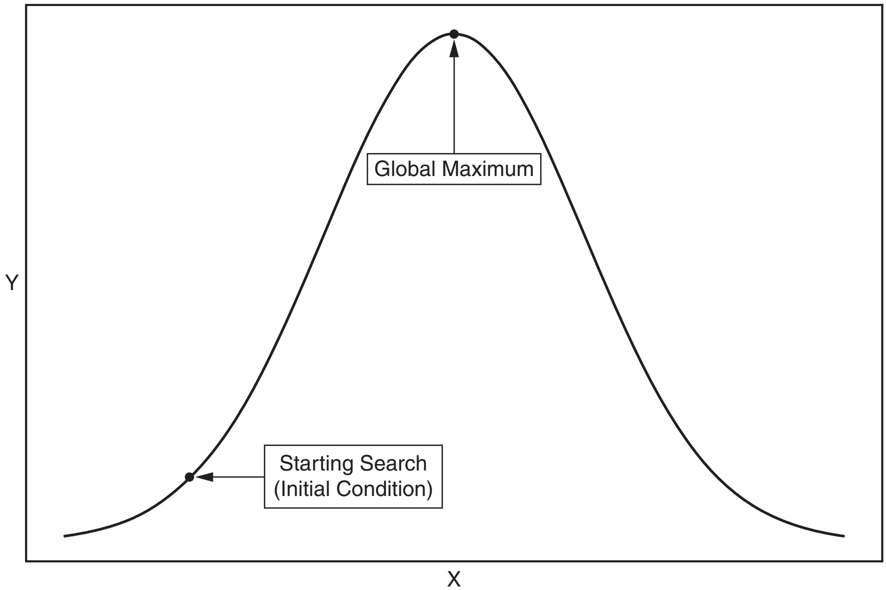
The name given to the class of optimization problems where there is only a global solution is a *convex optimization problem* (despite the fact that in the case of a maximization optimization problem that function is concave). Convex problems allow for more general types of constraints. The hierarchy of convex optimization problems used most frequently in bond portfolio management is[1]

- Second-order cone problem;
- Quadratic problem;
- Linear problem.

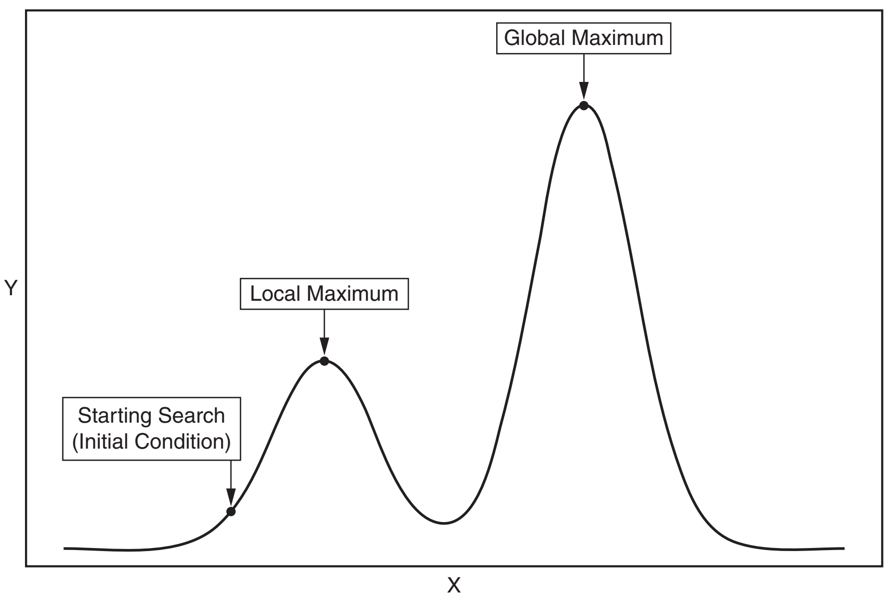
Solving a second-order cone problem (SOCP) is not as simple as solving linear or quadratic problems. The first reason is that a SOCP is relatively more difficult to set up. The second reason is that a SOCP is much more computationally intensive. Finally, even though there is commercial software that can be used to solve any convex problem, there is also specialized software that is more efficient if the objective function has certain additional properties. In contrast to linear and quadratic optimization problems described next, modern SOCP software can handle only hundreds of decision variables.

1. The actual hierarchy is conic problem, semidefinite problem, second-order cone problem, quadratic problem, and linear problem.

EXHIBIT 34–2



(a) Global maximum solution



(b) Local and global maximum solutions

We describe each of these special types of convex optimization problems below starting from the bottom of the hierarchy.

Linear Program

The objective function and all the constraints in a linear optimization model are linear. For example, a structured portfolio strategy may call for construction of a portfolio to satisfy a liability or set of liabilities at a minimum cost. Suppose that the acceptable universe of bonds is 2,500 issues made up of Treasuries, agencies, and all investment-grade corporate bonds. The decision variables are the amounts to be invested in an issue. There are then 2,500 decision variables, one representing each of the acceptable issues in the universe.

Let

$w_i =$ par amount to purchase of a particular issue ($i = 1, 2, \ldots, 2{,}500$);

$p_i =$ price of issue i (as a percent of par value P, e.g., 0.90, 1.00, 1.05).

Then the total cost of the portfolio is

$$w_1\, p_1 + w_2\, p_2 + \cdots + w_{2500}\, p_{2500}$$

This expression represents the objective function. The objective function is linear, and the portfolio manager would seek to minimize the objective function subject to the constraints imposed.

Two examples of a linear constraint would be that a minimum of \$10 million of the portfolio must be invested in Treasury issues and that no more than \$35 million may be invested in BBB-rated corporate bonds. Mathematically, assuming that issues 1, 2, …, 150 are the Treasury issues, the first constraint can be expressed as

$$w_1\, p_1 + w_2\, p_2 + \cdots + w_{150}\, p_{150} \geq \$10{,}000{,}000.$$

Assuming that issues 425–500 are BBB-rated corporate bonds, the second constraint can be expressed mathematically as

$$w_{425}\, p_{425} + w_{426}\, p_{426} + \cdots + w_{500}\, p_{500} \leq \$35{,}000{,}000.$$

One common linear constraint in structured portfolio strategies is a restriction on portfolio duration. For example, a dedicated portfolio strategy (discussed later in this chapter) would constrain portfolio duration to match the duration of the liability. Assuming no short selling, all the decision variables are then constrained to be greater than or equal to zero.

Quadratic Program

The restriction imposed on employing linear programming is that both the objective function and the constraints be linear. It is common in structured portfolio strategies and asset-allocation models, however, to have a quadratic objective function. For example, a common objective in constructing an indexed portfolio is to minimize tracking error. As explained in Chapter 32, tracking error is the standard deviation of the active returns. The square of the tracking error is used in portfolio construction. Solving this problem requires the use of a mathematical programming model that permits a quadratic objective function. In the case of asset-allocation models, the objective is to allocate funds among asset classes to minimize portfolio risk. Portfolio risk is measured by the variance of the portfolio, which is expressed mathematically as a quadratic expression.

Linear programming is inappropriate in these applications. *Quadratic programming* does permit a quadratic objective function and permits the constraints to be a nonlinear expression.

Second-Order Cone Program

A SOCP is a type of convex optimization in which a linear function is minimized subject to linear constraints and the intersection of second-order cones. By using second-order (quadratic) cones, a bond portfolio manager can incorporate curvature information into the model to solve more complicated problems.

Two alternatives to the standard deviation as a measure of portfolio risk are value at risk (VaR) and conditional value at risk (CVaR). They can be used in optimization problems for asset allocation, as explained later in this chapter. CVaR has better properties than VaR as a risk measure but also is convex. As a result of this nonlinear function for CVaR, it can be used in asset-allocation problems as a risk measure. It can be demonstrated that by using CVaR, it is possible to combine several risk factors and produce an optimization problem that is a SOCP.[2] SOCP formulations arise mostly in robust optimization applications (described later).

MIXED-INTEGER PROGRAMMING

When setting up an optimization problem, it is assumed that the decision variables are divisible and result in whole numbers. Suppose that a bond portfolio manager wants to solve a convex program problem to determine the optimal number of bonds to purchase to accomplish some investment objective (i.e., some objective function). Suppose that the solution is to buy a par value equal to $1,145,375 of

2. See Georg C. Pflug, "Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk," in Stanislav Uryasev (ed.), *Probabilistic Constrained Optimization* (Boston: Springer, 2000), pp. 272–281; Anna G. Quaranta and Alberto Zaffaroni, "Robust Optimization of Conditional Value at Risk and Portfolio Selection," *Journal of Banking & Finance*, Vol. 32 (2008), pp. 2046–2056; R. Tyrell Rockafellar and Stanislav Uryasev, "Optimization of Conditional Value-at-Risk," *Journal of Risk*, Vol. 2 (2000), pp. 21-42; and R. Tyrell Rockafellar and Stanislav Uryasev, "Conditional Value-at-Risk for General Loss Distributions," *Journal of Banking & Finance*, Vol. 26 (2002), pp. 1443–1471.

one bond and $934,311 of another. These two purchases would be challenging in the bond market because these quantities of each bond would be an odd lot. So, is purchasing $1,145,375 of one bond and $934,311 of another the optimal solution? In fact, (1) the modified (rounded) optimal solution may no longer satisfy all the constraints or (2) there may be another solution that is better than the modified optimal solution. To handle this type of problem, *mixed-integer programming* is used. This type of optimization problem restricts some of the decision variables to take on integer values, whereas other decision variables may be continuous variables. Mixed-integer programming is used when transaction sizes in round lots are sought or where one of two issues but not both may be part of the optimal solution.

OPTIMIZATION UNDER UNCERTAINTY

In the mathematical programming models described thus far, the parameters of the model are known. There are many problems faced by bond portfolio managers where the parameters in the constraints or objective function are not known with certainty because either (1) the data are measured with error or (2) some of the data represent information about the future and therefore are stochastic. Mathematical programming where the parameters are not known with certainty is referred to as *optimization under uncertainty*.

The three general approaches for incorporating uncertainty into optimization problems are

- Dynamic programming;
- Stochastic programming;
- Robust optimization.

Although the fields of dynamic programming, stochastic programming, and robust optimization have some overlap, historically they have evolved independently of each other.

Dynamic Programming

Dynamic programming is used to solve a large multistage optimization problem sequentially, starting at the last stage and proceeding backward. By doing so, the optimization process keeps track only of the optimal paths from any given time period onward. There is an underlying dynamic system and an objective function (called a *reward* or a *cost* function depending on whether the problem is a maximization or a minimization) that is additive over time.

The dynamic system at any point in time t is described by a set of state variables. In the case of a problem where a maximization of the objective function (i.e., the total reward) is sought, at every state the goal is to find the value for the decision variables such that the current stage forward is maximized.

Stochastic Programming[3]

In *stochastic programming*, random variables with specified probability distributions are used to characterize the uncertainty associated with the parameters and optimize the expected value of the objective function. There are discrete time periods for decisions to be made. A key concept in stochastic programming is *recourse*, which is simply the ability of the decision maker to take some corrective action after a random event occurs. In implementing a stochastic programming model, it is necessary to generate scenarios for the random variables.

In a two-stage stochastic programming model, the decision variables are of two types. The first-stage decision variables must be made prior to the actual realization of the parameters that become known.[4] After random events have occurred and the random parameters become known, the second-stage decision variables (also called *recourse variables*) are made, thereby improving the model. There is a cost associated with each stage owing to uncertainty of the parameters. The objective is to select the first-stage decision variables such that the total of the first-stage costs and the expected value of the random second-stage costs is minimized.

In multistage stochastic programming, there are recourse and corrective decisions at successive stages. There are improvements to the model because information about the parameters becomes known at each stage.

Robust Optimization[5]

There is a major problem with dynamic and stochastic programming formulations. It is that in practice it is often difficult to obtain detailed information about the probability distributions of the uncertainties in the model. At the same time, depending on the number of scenarios involved in the formulation, dynamic and stochastic programming methods can be prohibitively costly computationally. Unlike stochastic programming, *robust optimization* replaces probability distributions with *uncertainty sets*. An uncertainty set is the set of values that the model's parameters can take. The inputs are not traditional forecasts such as expected

3. For an introduction to stochastic programming, see John R. Birge and Francois Louveaux, *Introduction to Stochastic Programming* (New York: Springer-Verlag, 1997). For an introduction with application to finance, see William T. Ziemba and Raymond G. Vickson, *Stochastic Optimization Models in Finance* (New York: Academic Press, 1978); Koray Simsek, "Introduction to Stochastic Programming and Its Applications to Finance," in Frank J. Fabozzi (ed.), *Encyclopedia on Financial Models*, Vol. III (Hoboken, NJ: John Wiley & Sons, 2013), pp. 123–135; and Li-Yong Yu, Xiao-Dong Ji, and Shou-Yang Wang, "Stochastic Programming Models in Financial Optimization: Survey," *Advanced Modeling and Optimization*, Vol. 5, No. 1 (2003), pp. 1–26.
4. For this reason, first-stage decision variables are also referred to as *here-and now variables*.
5. For a more detailed explanation of robust optimization, see Frank J. Fabozzi, Petter N. Kolm, Dessislava A. Pachamanova, and Sergio F. Focardi, *Robust Portfolio Optimization and Management* (Hoboken, NJ: John Wiley & Sons, 2007).

returns and risk but rather uncertainty sets containing point estimates (e.g., confidence intervals around the forecasts).

In practice, selection of the uncertainty set for a particular application is typically based on statistical estimates and defined in smart ways that do not lead to overly conservative or computationally challenging formulations. The objective in robust optimization is to select the decision variables that perform best should the most adverse parameter value from within the uncertainty sets occur. That is, the decision maker considers the worst possible outcome and selects the decision variables based on that. While robust optimization shares some features with stochastic programming, classic robust optimization focuses on the worst case, whereas classic stochastic programming focuses on the average over scenarios.

Some practitioners have questioned whether using robust optimization is worthwhile. When inaccuracy is assumed in the expected return estimates, some tests with simulated and real market data suggest that robust optimization outperforms classic mean-variance optimization (discussed below) in terms of total excess return a large percentage (70%–80%) of the time.[6] This finding, however, has been challenged by others.[7] The reason for much of the difference is in how the uncertainty of the parameters is modeled. Therefore, finding a suitable degree of robustness and appropriate definitions of uncertainty sets can have a significant impact on portfolio performance.

Independent tests by practitioners and academics using both simulated and actual market data appear to confirm that robust optimization generally results in more stable portfolio weights; that is, it eliminates the extreme corner solutions resulting from traditional mean-variance optimization. This fact has implications for portfolio rebalancing in the presence of transaction costs and taxes, because transaction costs and taxes can add substantial expenses when the portfolio is rebalanced. Depending on the particular robust formulations employed, robust mean variance optimization also appears to improve worst-case portfolio performance and results in smoother and more consistent portfolio returns. Finally, by preventing large swings in positions, robust optimization typically makes better use of the turnover budget and risk constraints.

Robust optimization, however, is not a panacea. By using robust portfolio optimization formulations, portfolio managers are likely to trade off the optimality of their portfolio allocation in cases in which nature behaves as they predicted for protection against the risk of inaccurate estimation. Therefore, portfolio managers using this technique should not expect to do better than classic portfolio optimization when estimation errors have little impact or when typical scenarios occur. However, portfolio managers should expect insurance in scenarios in which their

6. See, e.g., Sebastián Ceria and Robert A. Stubbs, "Incorporating Estimation Errors into Portfolio Selection: Robust Portfolio Construction," *Journal of Asset Management*, Vol. 7, No. 2 (2006), pp. 109–127.

7. See, e.g., Jyh-Huei Lee, Dan Stefek, and Alexander Zhelenyak, "Robust Portfolio Optimization: A Closer Look," *MSCI Barra Research Insights Report* (June 2006).

estimates deviate from the actual realized values by up to the amount they have prespecified in the modeling process.

Applications to Bond Portfolio Management

There are a good number of models for managing bond portfolios using stochastic programming with recourse. The first such model was proposed in 1972 by Bradley and Crane,[8] who proposed a bond portfolio-management model that allows for interest-rate and price changes as well as portfolio rebalancing.

Building on discrete-time fixed-income options theory, Hiller and Eckstein[9] propose a stochastic programming model for managing asset/liability portfolios in the face of interest-rate-contingent claims. They refer to their model as *stochastic dedication*. Golub et al.[10] develop a two-stage stochastic program with recourse to deal with the dynamics of interest rates, cash-flow uncertainty, and liquidity, as well as defaults in managing a fixed-income portfolio. The optimization model is then integrated with simulation analysis. The performance of the model is compared with an immunization (a strategy discussed below). Dupačová and Bertocchi[11]explain how a bond portfolio can be formulated as a multiperiod stochastic programming problem with random recourse and apply it to a real-life problem from the Italian bond market.

Because of the uncertainty of the cash flows due to prepayments for mortgage-backed securities and callable bonds, stochastic programming is well suited to optimization modeling. A stochastic programming model for this problem within an asset/liability framework was proposed by McKendall, Zenios, and Holmer[12]; Vassiadou-Zeniou and Zenios[13] proposed robust optimization for managing a portfolio of callable bonds.

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8. Stephen P. Bradley and Dwight B. Crane, "A Dynamic Model for Bond Portfolio Management," *Management Science*, Vol. 19, No. 2 (1972), pp. 139–151.
9. Randall S. Hiller and Jonathan Eckstein, "Stochastic Dedication: Designing Fixed Income Portfolios Using Massively Parallel Benders Decomposition," *Management Science*, Vol. 39, No. 11 (1993), pp. 1422–1438.
10. Bennett Golub, Martin Holmer, Raymond McKendall, Lawrence Pohlman, and Stavros A. Zenios, "A Stochastic Programming Model for Money Management," *European Journal of Operational Research*, Vol. 85, No. 2 (1995), pp. 282–296.
11. Jitka Dupačová and Marida Bertocchi, "Management of Bond Portfolios Via Stochastic Programming: Postoptimality and Sensitivity Analysis," in Jaroslav Doležal and Jiří Fidler *J.* (eds), *System Modelling and Optimization. IFIP — The International Federation for Information Processing* (Boston, MA: Springer, 1996), pp. 574–581.
12. Raymond McKendall, Stavros A. Zenios, and Martin Holmer, "Stochastic-Programming Models for Portfolio Optimization with Mortgage-Backed Securities: Comprehensive Research Guide," in Rita L. D'Ecclesia and Stavros A. Zenios (eds.), *Operations Research Models in Quantitative Finance*: *Contributions to Management Science* (Heidelberg: Physica-Verlag, 1994). See also Stavros A. Zenios, Martin R. Holmer, Raymond McKendall, and Christiana Vassiadou-Zeniou, "Dynamic Models for Fixed-Income Portfolio Management Under Uncertainty," *Journal of Economic Dynamics and Control*, Vol. 22 (1998), pp. 1517–1541.
13. Christiana Vassiadou-Zeniou and Stavros A. Zenios, "Robust Optimization Models for Managing Bond Portfolios," *European Journal of Operational Research*, Vol. 91 (1996), pp. 264–273.

OPTIMIZATION WITH MULTIPLE OBJECTIVE FUNCTIONS

A bond portfolio may encounter a problem where there are multiple objective functions for which an optimal solution is sought. Typically, the way this is tackled is to reformulate the optimization problem as one with a single objective. There are several approaches for doing so. One way is to assign weights to each of the objective functions and to then make the model's objective function the optimization of the weighted sum of the objectives. A second approach is to optimize what the portfolio manager feels is the most important objective function and reformulate the other objective functions as constraints while assigning to each of the other objective functions a bound of the value that the portfolio manager is willing to tolerate.

BOND PORTFOLIO MANAGEMENT APPLICATIONS

Cash-Flow Matching

Consider an asset manager who is managing funds for the corporate sponsor of a defined-benefit pension plan that needs to ensure a particular stream of semiannual cash payments over the next 4 years for retiring plan participants. For example, the pension plan may have semiannual obligations representing annuity payments.

Let the cash obligations (i.e., the amount that must be paid out to the beneficiaries) for the eight payment dates over the next 4 years be represented by m_t (m_1, . . . , m_8). The assumed obligations for each of the eight semiannual obligations are given in Exhibit 34–3. The asset manager on behalf of its client, the pension plan, is considering investing in five different Treasury securities. Over the next eight payment dates (i.e., semiannually), bond i pays out coupons c_i (c_{i1}, . . . , c_{i8}). If the bond matures on date t, the corresponding c_{it} equals the coupon payment plus the principal. The bonds currently trade at ask prices $p_i = (p_1, \ldots, p_5)$. The relevant data are provided in Exhibit 34–4. The asset manager would like to ensure that the coupon payments from the bonds cover the pension plan's obligations.

The objective is to be able to create a minimum-cost portfolio of the five Treasury securities to satisfy the eight payments that must be paid to the beneficiaries. The decision variables in this illustration are the amount of each bond to be purchased. If we let x_i be the amount invested in Treasury security i, then there are five decision variables ($x_1, \ldots, x_5$). Because the objective is create a minimum-cost portfolio comprised of the five bonds, then $p_i x_i$ is the cost of purchasing Treasury security x_i; the cost for the entire portfolio is

$$p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 + p_5x_5.$$

Then the objective is to minimize the above expression. Therefore, the objective function is

$$\text{Min } (p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 + p_5x_5).$$

There are 13 constraints. There is a constraint for each of the eight time periods that specify that the cash flow generated from all five Treasury securities

EXHIBIT 34–3

Cash Obligations for the Cash-Flow Matching Illustration

Time Period Due	Amount of Obligation ($)
1	1,000,000
2	2,000,000
3	1,000,000
4	2,000,000
5	8,000,000
6	12,000,000
7	4,000,000
8	10,000,000

EXHIBIT 34-4

Data for the Five Treasury Securities for the Cash-Flow Matching Illustration

Treasury Security	Coupon Rate (%)	Years to Maturity	Price per $100 par Value ($)
1	2.5	2.5	102.36
2	5.0	3.0	110.83
3	3.0	3.5	96.94
4	4.0	4.0	114.65
5	3.5	4.0	96.63

must be greater than or equal to the amount of the obligation. For each time period ($t = 1, \ldots, 8$),

$$c_{1t}x_1 + c_{2t}x_2 + c_{3t}x_3 + c_{4t}x_4 + c_{5t}x_5 \geq m_t.$$

The final constraint is that the amount purchased must be greater than or equal to zero. That is, short selling (i.e., a value for x_i less than zero) is not permitted. Thus a constraint for each of the five decision variables is that $x_\mathrm{i} > 0$.

In practice, the amounts for the bond investments may need to be presented as round lots. This can be achieved by using integer variables. New integer variables z_i ($z_1, \ldots, z_5$) are introduced that correspond to the number of lots to buy of each bond. Suppose that a lot for bond i is l_i. To obtain the optimal number of lots of each bond to purchase, we rewrite the problem formulation and add the following 10 more constraints:

$$x_\mathrm{i} = z_i l_i, \qquad i = 1, \ldots, 5;$$

$$z_i \text{ integer}, \qquad i = 1, \ldots, 5.$$

The cash-flow matching problem is then

$$\min\ (p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 + p_5x_5)$$

$$\text{subject to}\quad c_{1t}x_1 + c_{2t}x_2 + c_{3t}x_3 + c_{4t}x_4 + c_{5t}x_5 \geq m_t, \qquad t = 1, \ldots, 8$$

$$x_{\mathrm{i}} \geq 0, \qquad I = 1, \ldots, 5$$

$$x_{\mathrm{i}} = z_i l_i, \qquad i = 1, \ldots, 5$$

$$z_i \text{ integer}, \qquad i = 1, \ldots, 5$$

Exhibit 34–5 shows the optimization problem for our cash-flow matching problem. The question is what type of optimization problem this is. Because the objective function and the constraints are linear, it seems that this is a linear programming problem. However, there are decision variables that can take on any positive value, and there are decision variables that can only take on integer values. Therefore, this is a mixed-integer linear programming problem.

Asset Allocation

The objective of an asset-allocation model is to divide funds among major asset classes so as either (1) to maximize expected return subject to a certain level of risk or (2) to minimize risk subject to a certain level of expected return. The two are equivalent statements of the asset-allocation problem. Asset allocation is used in multiasset strategies where the allocation is to be made among major asset classes, or it can be used to allocate funds among only the sectors of the bond class. Here we will illustrate the allocation among bond sectors.

The basic inputs for an asset-allocation model are expected returns of the major asset classes, risk estimates, and correlations (or covariances) for each major asset class. The portfolio's expected return is simply the weighted average of the expected value of each asset class, where the weight is the proportion of the portfolio devoted to the asset class. That is, if x_i denotes the proportion of the portfolio allocated to asset class i and r_i denotes the expected return for asset class i, the portfolio's expected return, assuming that there are m major asset classes, can be expressed as

$$E(r_p) = x_1r_1 + x_2r_2 + \cdots + x_mr_m,$$

where $E(r_p)$ is the expected return for the portfolio.

The expected risk of a portfolio is measured by the variance of the portfolio. Unlike the expected portfolio return, which is a weighted average of the expected return of each asset class, the portfolio variance also depends on the covariance (or correlation) between the asset classes. For example, if there are only two asset classes, the portfolio variance is calculated as

$$\mathrm{var}(r_p) = x_1^2\ \mathrm{var}(r_1) + x_2^2\ \mathrm{var}(r_2) + 2x_1\, x_2\, \mathrm{cov}(r_1, r_2),$$

EXHIBIT 34–5

Optimization Problem for the Cash-Flow Matching Illustration

$$\min: \$102.36x_1 + \$110.83x_2 + \$96.84x_3 + \$114.65x_4 + \$96.63x_5$$

subject to: constraints to satisfy the amount needed to meet semiannual payments

$$\$2.50x_1 + \$5.00x_2 + \$3.00x_3 + \$4.00x_4 + \$3.50x_5 \geq \$1{,}000{,}000$$
$$\$2.50x_1 + \$5.00x_2 + \$3.00x_3 + \$4.00x_4 + \$3.50x_5 \geq \$2{,}000{,}000$$
$$\$2.50x_1 + \$5.00x_2 + \$3.00x_3 + \$4.00x_4 + \$3.50x_5 \geq \$1{,}000{,}000$$
$$\$2.50x_1 + \$5.00x_2 + \$3.00x_3 + \$4.00x_4 + \$3.50x_5 \geq \$2{,}000{,}000$$
$$\$102.50x_1 + \$5.00x_2 + \$3.00x_3 + \$4.00x_4 + \$3.50x_5 \geq \$8{,}000{,}000$$
$$\$105.00x_2 + \$3.00x_3 + \$4.00x_4 + \$3.50x_5 \geq \$12{,}000{,}000$$
$$\$103.00x_3 + \$4.00x_4 + \$3.50x_5 \geq \$4{,}000{,}000$$
$$\$104.00x_4 + \$103.50x_5 \geq \$10{,}000{,}000$$

Nonshorting constraints:

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_3 \geq 0$$
$$x_4 \geq 0$$
$$x_5 \geq 0$$

Round-lot constraints:

$$x_1 = z_1 l_1 \qquad z_1 \text{ integer}$$
$$x_2 = z_2 l_2 \qquad z_2 \text{ integer}$$
$$x_3 = z_3 l_3 \qquad z_3 \text{ integer}$$
$$x_4 = z_4 l_4 \qquad z_4 \text{ integer}$$
$$x_5 = z_5 l_5 \qquad z_5 \text{ integer}$$

where

var (r_p) = portfolio return variance;

var (r_i) = variance of asset class i returns;

cov(r_1, r_2) = covariance between the returns for asset classes 1 and 2.

Alternatively, the portfolio variance can be expressed as

$$\text{var}(r_p) = x_1^2 \text{ var}(r_1) + x_2^2 \text{ var}(r_2) + 2x_1\, x_2 \text{ std}(r_1) \text{ std}(r_2) \text{ corr}(r_1, r_2),$$

where

std(r_i) = standard deviation of the returns of asset class i;

corr(r_1, r_2) = correlation between the returns for asset classes 1 and 2.

A constraint that is imposed is that the sum of the weights must equal 1. That is, if there are m asset classes,

$$x_1 + x_2 + \cdots + x_m = 1.$$

If no short selling of an asset class is permitted, then there are m constraints imposed requiring that $x_i > 0$ ($i = 1, \ldots, m$).

Because the portfolio variance is a quadratic expression (the decision variables—the x_i's—are squared), this is a quadratic programming problem. Given the portfolio expected return and variance, quadratic programming will generate a set of *efficient portfolios*. These are portfolios that will give the minimum portfolio variance for a given level of expected portfolio return.

The asset-allocation model can be extended further. Beyond developing a model to minimize portfolio risk, the decision maker can use concepts such as value at risk and conditional value at risk. Another important application of optimization is to the area of risk budgeting. This problem involves the optimal allocation of risk among bond portfolio managers. The problem can be cast in a framework similar to that of the asset-allocation decision, where the objective is to minimize the expected active risk (tracking error) for a given level of expected alpha.

Illustration of Asset Allocation Using Mean-Variance Analysis

Let's now illustrate how to apply mean-variance analysis to select fixed-income asset classes using Portfolio Visualizer.[14] We will assume the following five asset classes:

Global bonds (U.S. dollar hedged);

Long-term U.S. Treasuries;

High-yield corporate bonds;

Short-term corporate bonds;

Long-term corporate bonds.

The data that will be used to obtain the return inputs are monthly returns from January 1, 2009, to December 31, 2020. For the five asset classes, the expected return (i.e., average historical return) and standard deviation of return are shown in Exhibit 34–6, whereas Exhibit 34–7 shows the correlation of returns between each asset class. For the optimization model in this illustration, several constraints

14. https://www.portfoliovisualizer.com/optimize-portfolio.

EXHIBIT 34–6

Expected Return and Standard Deviation for Asset-Allocation Illustration

Asset Class	Expected Return	Standard Deviation
Global bonds (USD hedged)	6.20%	3.32%
Long-term Treasury	6.38%	11.88%
High-yield corporate bonds	9.56%	6.93%
Short-term investment grade bonds	3.80%	2.17%
Long-term corporate bonds	9.04%	8.70%

EXHIBIT 34–7

Return Correlations for the Asset-Allocation Illustration

Asset Class	Global Bonds (USD Hedged)	Long-Term Treasuries	High-Yield corporate Bonds	Short-Term Investment Grade Bonds	Long-Term Corporate Bonds
Global bonds (USD hedged)	1.000	0.410	0.570	0.740	0.750
Long-term Treasuries	0.410	1.000	0.230	0.100	0.730
High-yield corporate bonds	0.570	0.230	1.000	0.750	0.360
Short-term investment grade bonds	0.740	0.100	0.750	1.000	0.570
Long-term corporate bonds	0.750	0.730	0.360	0.570	1.000

were imposed on the amount that can be allocated to some of the asset classes. Specifically,

Asset Class	Minimum Weight	Maximum Weight
Global bonds (USD hedged)	20%	40%
Long-term Treasuries	10%	20%
High-yield corporate bonds	20%	40%
Short-term investment grade bonds	5%	—
Long-term corporate bonds	10%	—

The mean-variance asset-allocation model will generate a portfolio that provides the highest expected portfolio return for a given standard deviation. Each portfolio created for a given standard deviation is referred to as an *efficient portfolio*. The set of all efficient portfolios is referred to as the *efficient set*, and if plotted on a diagram with the vertical axis being the expected return and the horizonal axis the portfolio standard deviation, it is called the *efficient frontier*. Exhibit 34–8 shows 50 efficient portfolios for this optimization problem for different standard deviations for the portfolio.

The optimal efficient portfolio must be selected from among all the efficient portfolios. However, in order to do so, the bond portfolio manager must decide on the criterion that will be used to select the best from among all the efficient portfolios. That efficient portfolio is the *optimal portfolio*.

A commonly used criterion is maximization of the Sharpe ratio. For each of the 50 efficient portfolios shown in Exhibit 34–8, the last column shows the Sharpe ratio. An examination of the last column shows that the maximum Sharpe ratio is 1.562, which is found for efficient portfolios 11–24. The Sharpe ratios are calculated to three decimal places. If more decimal places are shown, it would be seen that efficient portfolio 21 is the optimal allocation. Exhibit 34–9 shows the optimal allocation, the corresponding portfolio expected return, and the portfolio standard deviation.

Extension of Asset-Allocation Optimization

While we have illustrated asset allocation using variance (standard deviation) as a measure of risk, it is not the only portfolio risk measure that can be used. Other risk measures and other criterion can be used to select the optimal portfolio. These other risk measures include the conditional value at risk (CVaR) and drawdown. For a mean-CVaR optimization problem, the objective function is to minimize the expected tail loss. In the case where drawdowns are viewed as the risk, the objective function is the minimization of the maximum drawdown. The optimal portfolios for both risk measures are shown in Exhibit 34–9. When a benchmark is used, the risk measure that is used is tracking error, which we discussed in Chapter 32.

In our illustration, we used maximization of the Sharpe ratio. We explained other risk-adjusted returns (reward/risk ratios) that can be used, such as the Sortino ratio. Recall that this measure requires the specification of a minimum acceptable return. Exhibit 34–10 shows the optimal allocation for three minimum acceptable returns.

Indexing

Indexing involves constructing a portfolio to match the performance of a specified index. There are three popular methodologies for designing a portfolio to replicate an index: (1) the cell approach, (2) the optimization approach, and (3) the variance-minimization approach.[15]

15. For a model that uses both optimization and simulation to track an index, see K. J. Worzel, C. Vassiadou-Zeniou, and Stavros A. Zenios, "Integrated Simulation and Optimization Models for Tracking Fixed-Income Indices," *Operations Research*, Vol. 42 (1994), pp. 223–233.

EXHIBIT 34–8

Efficient Portfolios

Efficient Portfolio	Global Bonds (USD Hedged) (%)	Long-Term Treasury (%)	High-Yield Corporate Bonds (%)	Short-Term Investment-Grade Bonds (%)	Long-Term Corporate Bonds (%)	Portfolio Expected Return (%)	Standard Deviation (%)	Sharpe Ratio
1	40.00	10.00	27.52	12.48	10.00	7.14	4.11	1.559
2	40.00	10.00	27.86	12.14	10.00	7.16	4.12	1.559
3	40.00	10.00	28.20	11.80	10.00	7.18	4.13	1.559
4	40.00	10.00	28.54	11.46	10.00	7.20	4.14	1.560
5	40.00	10.00	28.88	11.12	10.00	7.22	4.15	1.560
6	40.00	10.00	29.22	10.78	10.00	7.24	4.16	1.560
7	40.00	10.00	29.56	10.44	10.00	7.26	4.17	1.561
8	40.00	10.00	29.90	10.10	10.00	7.28	4.18	1.561
9	40.00	10.00	30.24	9.76	10.00	7.30	4.20	1.561
10	40.00	10.00	30.58	9.42	10.00	7.32	4.21	1.561
11	40.00	10.00	30.92	9.08	10.00	7.34	4.22	1.562
12	40.00	10.00	31.26	8.74	10.00	7.36	4.23	1.562
13	40.00	10.00	31.60	8.40	10.00	7.38	4.24	1.562
14	40.00	10.00	31.94	8.06	10.00	7.40	4.25	1.562
15	40.00	10.00	32.28	7.72	10.00	7.42	4.26	1.562
16	40.00	10.00	32.62	7.38	10.00	7.44	4.28	1.562
17	40.00	10.00	32.96	7.04	10.00	7.46	4.29	1.562

(*Continued*)

EXHIBIT 34–8

Efficient Portfolios (*Continued*)

Efficient Portfolio	Global Bonds (USD Hedged) (%)	Long-Term Treasury (%)	High-Yield Corporate Bonds (%)	Short-Term Investment-Grade Bonds (%)	Long-Term Corporate Bonds (%)	Portfolio Expected Return (%)	Standard Deviation (%)	Sharpe Ratio
18	40.00	10.00	33.30	6.70	10.00	7.48	4.30	1.562
19	40.00	10.00	33.64	6.36	10.00	7.50	4.31	1.562
20	40.00	10.00	33.98	6.02	10.00	7.52	4.32	1.562
21	40.00	10.00	34.33	5.67	10.00	7.54	4.33	1.562
22	40.00	10.00	34.67	5.33	10.00	7.56	4.35	1.562
23	39.99	10.00	35.01	5.00	10.00	7.58	4.36	1.562
24	39.40	10.00	35.60	5.00	10.00	7.60	4.37	1.562
25	7,497.81	10.00	36.19	5.00	10.00	7.61	4.39	1.561
26	38.22	10.00	36.78	5.00	10.00	7.63	4.40	1.560
27	37.63	10.00	37.37	5.00	10.00	7.65	4.41	1.560
28	37.04	10.00	37.96	5.00	10.00	7.67	4.43	1.559
29	36.45	10.00	38.55	5.00	10.00	7.69	4.44	1.558
30	35.86	10.00	39.14	5.00	10.00	7.71	4.46	1.557
31	35.27	10.00	39.73	5.00	10.00	7.73	4.47	1.556
32	34.62	10.00	40.00	5.00	10.38	7.75	4.49	1.552
33	33.92	10.00	40.00	5.00	11.08	7.77	4.52	1.546
34	33.23	10.00	40.00	5.00	11.77	7.79	4.55	1.540

EXHIBIT 34–8

Efficient Portfolios (*Continued*)

Efficient Portfolio	Global Bonds (USD Hedged) (%)	Long-Term Treasury (%)	High-Yield Corporate Bonds (%)	Short-Term Investment-Grade Bonds (%)	Long-Term Corporate Bonds (%)	Portfolio Expected Return (%)	Standard Deviation (%)	Sharpe Ratio
35	32.53	10.00	40.00	5.00	12.47	7.81	4.58	1.534
36	31.83	10.00	40.00	5.00	13.17	7.83	4.61	1.528
37	31.14	10.00	40.00	5.00	13.86	7.85	4.64	1.522
38	30.44	10.00	40.00	5.00	14.56	7.87	4.68	1.516
39	29.75	10.00	40.00	5.00	15.25	7.89	4.71	1.509
40	29.05	10.00	40.00	5.00	15.95	7.91	4.74	1.503
41	28.35	10.00	40.00	5.00	16.65	7.93	4.77	1.497
42	27.66	10.00	40.00	5.00	17.34	7.95	4.80	1.491
43	26.96	10.00	40.00	5.00	18.04	7.97	4.83	1.485
44	26.27	10.00	40.00	5.00	18.73	7.99	4.87	1.479
45	25.57	10.00	40.00	5.00	19.43	8.01	4.90	1.473
46	24.87	10.00	40.00	5.00	20.13	8.03	4.93	1.467
47	24.18	10.00	40.00	5.00	20.82	8.05	4.96	1.461
48	23.48	10.00	40.00	5.00	21.52	8.07	5.00	1.455
49	22.78	10.00	40.00	5.00	22.22	8.09	5.03	1.449
50	22.09	10.00	40.00	5.00	22.91	8.11	5.06	1.443

Note: The expected portfolio return is an annual return based on average monthly return. In the calculation of the Sharpe ratio, the 3-month Treasury bill return is used as the risk-free rate.

EXHIBIT 34–9

Optimal Asset Allocation and Corresponding Portfolio Expected Return and Standard Deviation Using Different Portfolio Risk Measures

Asset Allocation	Mean Variance	Mean CVaR	Mean Maximum Drawdown
Global bonds (USD hedged)	40.00%	20.00%	21.89%
Long-term Treasuries	10.00%	10.00%	14.08%
High-yield corporate bonds	34.33%	20.00%	40.00%
Short-term investment-grade bonds	5.67%	40.00%	14.02%
Long-term corporate bonds	10.00%	10.00%	10.00%
Expected portfolio return	7.54%	6.22%	7.54%
Portfolio standard deviation	4.33%	3.83%	4.46%
Maximum drawdown	–5.16%	–4.15%	–5.04%

EXHIBIT 34–10

Asset Allocation Maximizing the Sortino Ratio

	Minimum Acceptable Return		
Asset Allocation	**0%**	**3%**	**5%**
Global bonds (USD hedged)	25.88%	35.00%	20.00%
Long-term Treasuries	10.00%	10.00%	10.00%
High-yield corporate bonds	20.00%	40.00%	40.00%
Short-term investment-grade bonds	34.12%	5.00%	5.00%
Long-term corporate bonds	10.00%	10.00%	25.00%
Expected portfolio return	6.36%	7.74%	8.17%
Portfolio standard deviation	3.79%	4.4%	5.16%
Maximum drawdown	–4.34%	–5.51%	–5.88%

The cell approach divides the index into cells, each representing a different characteristic of the index. The most common characteristics used to distinguish a bond index are (1) duration, (2) coupon, (3) maturity, (4) market sector (Treasury, corporate, mortgage backed), (5) credit rating, and (6) optionality. The objective is to select from all the issues in the index one or more issues in each cell that can be used to represent the entire cell. No optimization is involved with this methodology.

The optimization approach requires the portfolio manager to construct a portfolio that not only will match a cell breakdown and satisfy other constraints but also will optimize some objective. An objective could be to maximize the yield to maturity or some other yield measure, to maximize convexity, or to maximize total return.[16] Constraints other than matching the cell breakdown might include not investing more than a specified amount in one issuer or group of issuers and overweighting certain sectors for enhanced indexing. Depending on the mathematical characterization of the objective function and constraints, either linear or quadratic programming is used to solve for the optimal portfolio.

With the variance-minimization methodology, the bond portfolio manager seeks to minimize the tracking error. This approach requires historical data to estimate forward-looking tracking error. Quadratic programming is used because the objective function is the tracking error.

Optimized Portfolio Rebalancing

Once the optimal solution or optimal portfolio is found in any particular application, the optimization process does not end. In most strategies, an initially optimal portfolio must be rebalanced periodically as the characteristics of the portfolio change. For example, in an immunized portfolio, the duration for the individual issues changes over time as the issues approach maturity and as yields in the market change. Therefore, the portfolio's duration will change or wander from its target.

Rebalancing requires the purchase of issues (either new issues not in the original optimal portfolio or additional amounts of issues included in the original portfolio) and the sale of some or all of the issues in the original optimal portfolio. An optimization model can be used to minimize the transaction costs associated with rebalancing a portfolio.

Immunization

Immunization is a structured portfolio strategy designed to protect against interest-rate changes. A single-period immunization addresses one future liability when the objective is to construct a portfolio to minimize the risk that changes in interest rates will result in not satisfying the objective. Life insurance companies use single-period immunization in managing portfolios to satisfy their obligations in that the duration of the portfolio must be equal to the maturity of the liability. However, as explained in Chapter 13, duration is limited to a parallel shift in the yield curve. Other researchers have presented methodologies for immunizing under more general shifts in the yield curve.[17]

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16. For a mathematical presentation of this methodology, see Christina Seix and Ravi Akoury, "Bond Indexation: The Optimal Quantitative Approach," *Journal of Portfolio Management*, Vol. 12, No. 3 (1986), pp. 50–53.

17. See Andrea J. Heuson, Thomas F. Gosnell, Jr., and W. Brian Barrett, "Yield Curve Shifts and the Selection of Immunization Strategies," *Journal of Fixed Income*, Vol. 5, No. 2 (1995), pp. 53–64; and Joel R. Barber and Mark L. Cooper "Immunization Using Principal Component Analysis," *Journal of Portfolio Management*, Vol. 23, No. 1 (1996), pp. 99–105.

Multiperiod immunization is used in the case of more than one future liability. Examples of multiple liabilities are liabilities of a defined-benefit pension fund, liabilities of annuity policies issued by life insurance companies, and liabilities resulting from a state-sponsored lottery. Once again, the objective is to construct a minimum-cost portfolio designed to satisfy all the liabilities regardless of how interest rates change. Two requirements of multiperiod immunization are that (1) the duration of the portfolio must be equal to the duration of the liabilities and (2) the convexity of the portfolio must be greater than or equal to the convexity of the liabilities. Zenios and Kang[18] apply portfolio immunization strategy in a multiperiod stochastic optimization framework.

18. Stavros A. Zenios and Pan Kang, "Mean-Absolute Deviation and Portfolio Optimization for Mortgage-Backed Securities," *Annals of Operations Research*, Vol. 45 (1993), pp. 433–450.

CHAPTER
35

MACHINE LEARNING

Historically, financial econometric tools, particularly linear regression, have been the workhorse of bond portfolio managers to identify patterns in data. *Machine learning* (ML) is changing this by providing bond portfolio managers with the opportunity to use modern nonlinear and highly dimensional techniques. Bond portfolio managers build predictive models using information that contains complex patterns that when properly analyzed can be used in formulating investment strategies so as to enhance portfolio returns. To accomplish this, members of the portfolio management team must be capable of analyzing both structured and unstructured data. ML, which is a discipline within the field of data science, provides the analytical tools for extracting insights from a wide range of data sets.

The ML approach to bond analysis and portfolio management is in principle a consequence of the diffusion of low-cost, high-performance computers based on using a family of very flexible models that can approximate sample data with unlimited precision. Basically, ML is the branch of data science that studies how algorithms learn automatically from experience. The application of ML allows members of a portfolio-management team to accomplish tasks that in the past could only have been performed by humans.

ML does not depend on any specific financial theory. Consequently, when applying ML to different tasks in the investment-management process, ML models do not rely on finance theory but instead rely on purely statistical analysis of financial phenomena. ML algorithms impose constraints on model complexity for the purpose of retaining some forecasting capability when applied to out-of-sample data.

The ability of ML to identify patterns that can be used to enhance returns without relying on some financial theory is achieved by the development of a sequence of actions that is automatically optimized based on experience without being preprogrammed to do so. For solving the problem at hand, it is the sequence of actions that allows the computer to improve performance as experience increases. During the learning process, there is either no intervention or very limited human intervention. Typically, the sequence of actions is performed by the computer (i.e., machine, hence *machine* learning) to solve the problem by a set of rules that the computer must follow. These rules are referred to as *algorithms*. The data inputs (variables and data sets) are searched for patterns using the ML algorithm.

For the purpose of formulating the rules (i.e., algorithm), the entire data set is not used. Instead, only a certain portion of the data set is used in deriving the rules, and that portion is referred to as the *training data* or *training set*. In testing the performance of the rules, the balance of the data set is used. This portion of the data set, referred to as the *testing data* and *out-of-sample data*, allows the testing of the algorithm on unseen data.

ML is particularly useful for analyzing large and complex data. In the absence of ML algorithms, it is less likely (some argue nearly impossible) that the bond portfolio-management team would be capable of identifying a potentially profitable pattern. Moreover, because ML is able to learn patterns from a data set, this makes ML suitable for analyzing the phenomena involved in complex systems such as those faced by bond portfolio teams operating in the bond market. In contrast to ML, traditional statistical tools used in financial econometrics are not able to identify patterns from the data set, leading to misspecified models that result in bond portfolio strategies that are in fact false discoveries (i.e., Type II error, as discussed in Chapter 30).

In ML, there are four categories of computer learning processes that differ based on the information provided to the machine during the training period:

- Supervised learning;
- Unsupervised learning;
- Semisupervised learning;
- Reinforcement learning.

The key to understanding the difference between learning styles is that they depend on the feedback provided to the computer during the training phase. As a result, each learning style produces different answers to different types of questions and can be used for different applications in bond portfolio management.

The purpose of this chapter is to briefly describe the four types of ML and to review how they have been applied in the management of bond portfolios. Before doing so, it is important to understand the different types of data.

TYPES OF DATA

There are three ways to classify data:

- Quantitative and qualitative data;
- Structured, unstructured, and semistructured data;
- Labeled and unlabeled data.

Quantitative Data Versus Qualitative Data

The value for *quantitative data* is numerical. Generally, all real numbers are possible.[1] In some analyses, quantitative data are discrete values only, such as integers (i.e., 1, 2, 3, …). Time is expressed this way. Typically, most quantitative data are

1. In mathematics, a *real number* is defined as a quantity that can be expressed as an infinite decimal expansion.

decimal values. The monthly return on a portfolio, fund, or investment strategy is measured using decimals, and financial metrics are used in the analysis of a bond's issuer (e.g., return on assets, return on equity, leverage ratios, current ratio, debt-service coverage, and turnover ratios), and the recovery rate on a defaulted debt instrument. One can perform transformations and computations with quantitative data.

Qualitative data are information about qualities. With qualitative data, certain attributes of an item can only be assigned to categories. For example, the sectors of the U.S. fixed-income market are items that can be categorized as belonging to the specific sectors "U.S. Treasury," "agency," "corporate," "municipal," "mortgage," and "asset backed." Another example would be the credit ratings assigned to debt obligations by commercial rating agencies such as Standard & Poor's, Moody's, and Fitch Ratings. Some qualitative data have only two classifications. Examples are investment-grade bonds versus non-investment-grade bonds, defaulted versus nondefaulted issues, and callable versus noncallable bonds. Typically, a numerical code is assigned to indicate the different categories. For example, in the case of sectors of the fixed-income market, a 1 can be assigned to a bond issue that is a U.S. Treasury security, 2 can be assigned to a bond that is an agency security, 3 can be assigned to a corporate bond, and so on. Unlike quantitative data, one cannot perform any computation with qualitative data because they are simply names of the underlying attribute.

In some applications, quantitative data can be converted into nonquantitative data. For example, the maturity of a bond is quantitative data. However, in some analyses, the data can be transformed into qualitative data based on the amount of time remaining to maturity. A bond analyst may decide to classify holdings in a portfolio based on short maturities, intermediate maturities, and long maturities. The specific length for the classification is determined by the portfolio manager. For example, some portfolio managers view a short-term maturity as debt issues with a maturity of 1 year or less, while another may view it as less than 2 years.

Structured Data Versus Unstructured Data Versus Semistructured Data

Data are also classified as structured data, unstructured data, and semistructured data. In general, *structured data* consist of information that is objectively represented. The information can be quantitative data or nonquantitative data. Data that possess the opposite attributes to those of structured data are referred to as *unstructured data*. They can be textual or nontextual (e.g., images). Such data do not have a structure that is recognizable, and for this reason, the information is difficult to categorize. Until they are processed for use, unstructured data are stored in their native format. Unstructured data that might be used by bond analysts would include government filings by a company, a bond prospectus, and reports by other analysts about a bond issue or bond sector. These forms of unstructured data are referred to as *textual data*. Unstructured data also include photos and videos. For example, if a bond analyst is interested in looking at the activity of a major retailer's sales on an ongoing basis, computer satellite images of that retailer's parking lot might be used.

Semistructured data fall between structured and unstructured data. Such data do not have a strict data model structure to be treated as structured data. Certain data within semistructured data will have a label (discussed next), but parts of the data will be unstructured. Emails are an example of semistructured data. Although emails are similar to structured data because they have the name of the sender, the name of the recipient, and the time the email was sent, what makes emails unstructured is the content of the email. Moreover, when an email has an attachment, that is also unstructured data.

There is increased use of unstructured and semistructured data in bond analysis because of the development of the field of data science. With respect to asset management, this field involves applying advanced analytical techniques to extract information from data for the purpose of making a wide range of investment decisions. The techniques, or algorithms, developed in data science can be used to extract information from unstructured and semistructured data (as well as improve the way structured data are analyzed).

Labeled Versus Unlabeled Data

Data that have not been tagged with a label that identifies their characteristics, classifications, or properties are referred to as *unlabeled data*. Examples of unlabeled data that relate to finance would be audio recordings of webcasts by security analysts or company representatives, printed articles about a company by the media or by sell-side security analysts, tweets, and Security and Exchange Commission (SEC) filings by a public company.

For a database with unlabeled data, some of that data can be enhanced by an expert assigning a meaningful or informative label to them. The task of labeling data that are unlabeled is sometimes referred to as *classifying* or *tagging* the data, and the resulting data are referred to as *labeled data*. For example, consider the use of tweets from social media platforms such as Twitter that are used for quantifying market sentiment on a real-time basis. The objective is to provide a label representative of the overall sentiment of the tweet.

The task of labeling data can be done manually by humans or automatically by another ML algorithm that has been trained to perform that task. The task is critical because any mistake or inaccuracy in labeling adversely impacts the informativeness of the data. As a result, this will adversely affect the objective of what is being sought by using an ML algorithm.

The importance of the labeling task also means that ML algorithms will not eliminate the need for human judgment in the investment process. For ML to work effectively, experts are needed to label unlabeled data. The expert may be a member of the investment-management team or an external consultant third-party vendor retained for this purpose.

SUPERVISED LEARNING ALGORITHMS

In ML, *supervised learning algorithms* allow the computer to learn from training data that already contain the correct labels for each input. That is, labeled data are used. In supervised learning, the computer identifies patterns that link the input variables to the dependent variable. (In the terminology of ML, the dependent variable is referred to as a *response variable* and the explanatory variables as *features*.) The label *supervised learning* is given because the data set will include the features as well as the correct labels. Because the correct answers to the question at hand are provided as part of the data set, the computer can search for patterns in the feature space that can distinguish between categories of the response variable.

Regression ML and classification ML algorithms are the two types of supervised learning algorithms. The purpose of a *regression ML algorithm* is to predict a numerical value as the output. That is, regression algorithms are used to predict continuous variables. A simple example of a regression algorithm is a bond portfolio manager trying to predict the resulting price change of an intermediate-term corporate bond to changes in the 10-year Treasury rate (i.e., estimate the empirical duration). This is the same as the regression analysis described in Chapter 31. There is no difference between ML as used by practitioners of ML and ML used by econometricians.

As the name suggests, the second type of supervised learning, the *supervised classification algorithm*, involves developing a set of rules to determine which group a training example should belong to. This means that the labels for each training example will correspond to a category or group so that the algorithm can learn to identify patterns that allow the machine to categorize features that distinguish between groups or labels. Categorizing the members of a high-yield corporate bond index as to whether an issue will be upgraded, downgraded, or remain the same is an example of an application of this type of supervised classification algorithm. In this application, there are three responses that the algorithm can produce: upgrade, downgrade, and unchanged rating. Unlike a regression ML, which generates a numerical value associated with the input values (i.e., features), the goal of this supervised classification algorithm is to identify which of the three categories each high-yield corporate bond is most likely to fall into.

However, it is important to note that there is considerable overlap between regression and classification ML algorithms. Many of the common ML algorithms can be used in both classification and regression settings depending on the response variable fed to the algorithm and the way it is trained. This is similar to the connection between a linear regression and a logistic regression, where the linear regression predicts a continuous variable and the logistic regression predicts binary variables.

Commonly Used Supervised Learning Algorithms

The following are popular supervised learning algorithms used in asset management in addition to linear regression:

- *Decision Tree.* For this algorithm, the process begins at the root of the tree (i.e., the initial node) and moves on as questions at each node are answered, and based on the answers, moves along the branch of the decision tree that corresponds to that particular condition.
- *Naive Bayes.* This algorithm is used for classification based on Bayes' theorem. For a given set of predictors in the data set, it is assumed that they are independent, meaning that the features are uncorrelated with one another.
- *Random Forest.* This algorithm is used for both regression and classification problems. It is a collection of decision trees. It is a commonly used ensemble method (see below).
- *Support Vector Machine* (SVM). This algorithm is used in solving both regression and classification problems. SVMs are useful when there are many features in the data set. The goal of this algorithm is to synthesize these features into a hyperplane that can help categorize the data or predict a numerical output.
- *Linear Regularization Models.* This class of models uses regularization to shrink the coefficients for each of the model coefficients. For this reason, they are particularly useful when working with a large set of features. The three most common regularization models are: *least absolute shrinkage and selection operator* (LASSO), *ridge regression*, and *elastic net*. LASSO regressions allow coefficients to shrink to zero, which means that certain features will not influence model predictions; thus LASSO provides some form of feature selection as well. Ridge regressions do not shrink coefficients to zero but will result in less important features having a smaller impact on predictions. Lastly, elastic net is a combination of the LASSO and ridge models because it simultaneously shrinks coefficients and eliminates uninformative features.
- *Neural Networks.* There are many types of neural networks used in ML. In general, neural networks aim to replicate how the human brain learns. Neural networks work by using an *input layer* (which is the set of features of variables used), one or more *hidden layers*, and an *output layer*. Each of the layers has a set of nodes (similar to neurons in the brain) that connect to each of the nodes in the subsequent layer based on what is referred to as an *activation function*. Based on the accuracy of the model, which is determined by the loss function (typically mean squared error for regression or cross-entropy for classification), the model will use gradient descent to adjust model parameters for better

predictions. For regression problems, the output layer will consist of a single node with the numerical value of the label. This is also the case for binary classification problems, except the value will either be 0 or 1. When more than three categories or classifications are used, the output layer will have the same number of nodes as the number of categories.

Ensemble Methods in Supervising Learning Models

Ensemble methods combine several ML algorithms into one predictive model with the goal of either (1) decreasing the variance of an estimate, (2) reducing the bias, or (3) improving the predictive power. *Bootstrap aggregation*, commonly referred to as *bagging*, is used to decrease the variance of an estimate by averaging multiple estimates. For example, suppose that a parameter of interest is estimated using subsets of a data set that are selected randomly with replacement and estimated using decision trees. Suppose further that N subsets are estimated so that there are N predictions for the given observation. Bagging simply involves computing the average of the N estimates for the parameter.

Boosting is a method to reduce biases. To understand this ensemble method, it is necessary to understand the concept of a *weak learner* and a *strong learnecr*. Weak learners (also referred to as *base learners*) are models that produce predictions that are just slightly better than just guessing. A strong learner, in contrast, is a good or near-optimal model measured by its predictive ability. The objective in boosting is to convert weak learners into strong learners. In other words, boosting seeks to create a stronger classification algorithm from a number of weak classification algorithms. The most commonly used boosting algorithm is AdaBoost.

UNSUPERVISED LEARNING

While it is not difficult to define supervised learning, defining unsupervised learning is not easy. In contrast to supervised learning, there are no labels for the inputs in the case of *unsupervised learning algorithms*. Determining the structure of the data is the objective in unsupervised learning, and that is based on the statistical properties of the unlabeled data. Using a set of training data that are unlabeled, the ML algorithm then seeks to discover groupings of the data that have similar statistical characteristics.

In principle, the method of unsupervised learning applies to all available data. One can apply unsupervised learning to a sample and then generalize to the entire population. For example, a financial application performs clustering of price time series on sample data and then applies the same clustering to new data. Unsupervised learning is often used for exploratory analyses to identify hidden patterns, particularly when there is a lack of labeled training data.

Three examples of unsupervised learning are principal component analysis, clustering analysis, and neural networks. We described principal component analysis (PCA) in Chapter 31. As with ML regression, there is no difference between PCA when applied by an ML practitioner and an econometrician. Below we describe the other two types of unsupervised learning algorithms.

Clustering Analysis

To understand clustering analysis, consider the problem of marking-to-market debt instruments that are illiquid and difficult to price. An unsupervised ML algorithm can be written to identify debt instruments that have characteristics similar to those of an illiquid debt instrument for which a price is sought. If an unsupervised learning algorithm can identify an appropriate cluster for the illiquid debt instrument, the characteristics of the cluster can be used to mark-to-market the illiquid one and potentially other illiquid bonds in the portfolio that are difficult to price. The two commonly used clustering algorithms are hierarchical cluster analysis and k-means.

Hierarchical Cluster Analysis

In *hierarchical cluster analysis* the objective is to build a hierarchy of clustering. There are two techniques that have been used: agglomerative hierarchical cluster analysis and divisive hierarchical cluster analysis. The more commonly used method is the *agglomerative hierarchical cluster analysis* (also referred to as the *bottom-up approach* to hierarchical clustering). The process begins by treating each data point as an individual cluster. With each iteration, clusters that are similar are combined until a specified number of clusters is obtained. In contrast, when the process starts with all data points being a cluster and then at each iteration there is a splitting to create smaller clusters, we have the case of *divisive hierarchical cluster analysis* (also referred to as the *top-down approach* to hierarchical clustering). Hierarchical cluster analysis is visualized using *dendrograms*, which are tree diagrams that show the sequences of cluster mergers (in the case of agglomerative hierarchical cluster analysis) or splitting of clusters (in the case of divisive hierarchical cluster analysis).

k-Means

Similar to the above, the objective of *k-means clustering* is to partition observations into groups or clusters. This algorithm requires that the user prespecify the number of clusters (k) to find. The algorithm then chooses *k centroids*, which are randomly selected observations in the data set. Each data point thereafter is grouped with the centroid to which it is closest based on some measure of distance (typically Euclidean or Manhattan distance). This is an iterative process where the algorithm will adjust the cluster centroids in order to minimize the *within-cluster sum of squares* (WCSS), which is, as the name suggests, the sum of squared distances of each observation to the centroid of its respective cluster. A useful tool for determining the number of clusters in the data set is the *elbow method*, which plots the number of clusters on the horizontal axis and the WCSS on the vertical axis. The idea is to choose the number of clusters where the WCSS starts to plateau (stops decreasing as rapidly). Thus the plot will form an elbow shape, where one selects the cluster where the elbow begins.

SEMISUPERVISED LEARNING ALGORITHMS

When the training data include both unlabeled and labeled data, we have the case of *semisupervised learning algorithms*. The goal of semisupervised learning is to evaluate how combining unlabeled and labeled data may alter the learning behavior and then create algorithms that benefit from using both data types. The appeal of semisupervised learning is that it can use unlabeled data to improve supervised learning algorithms when there are limited labeled data or when it is too costly to obtain additional labeled data. Another potential advantage of semisupervised learning algorithms is that they can start with unlabeled and labeled data but eventually predict labels for new unlabeled data.

REINFORCEMENT LEARNING

A fundamental principle of learning theory is that learning is interactive. *Reinforcement learning* (RL) is slightly different from the other forms of ML discussed previously. The goal of a reinforcement learning algorithm is to determine the optimal action given a certain state of the world. Because subsequent states of the world depend on the actions previously taken, RL algorithms are designed to explore different states of the world to gain new information for model updates. Take, for example, the game of chess; the optimal next move will depend on the move made now and the opponent's next move. That is, there is not a single correct action to take because it depends on the previous actions taken as well as how that affects the current state (i.e., the positions of each piece in the chess example).

RL is characterized by the set of actions the algorithm can take, the environment, and the state. The environment dictates the rules of the game or the dynamics of the world being explored. The state, as previously mentioned, is the current landscape or the specific instance of the current environment. The goal of the RL algorithm is to choose the actions that maximize the cumulative reward, which is some user-defined function that determines how well the algorithm is performing. Based on the action taken in each state, the algorithm is provided feedback via the reward function in order to update the model to make the best decisions moving forward.

In thinking about the difference between RL and the other ML methods previously mentioned in this chapter, it is important to understand that the algorithm is trying to understand a dynamic system, like that of chess. This makes implementing RL models much more difficult than standard supervised and unsupervised models because of the fact that there is not a single label being applied to the data set. Instead, RL attempts to link the state of the world to optimal actions that help to maximize some reward. In addition, the algorithm is also tasked with selecting possible suboptimal actions in order to explore whether those actions could lead to better outcomes.

ML APPLICATIONS TO BOND PORTFOLIO MANAGEMENT

In this section we describe several applications of ML to bond portfolio management.

Predicting Default Rates on Commercial Mortgage Loans

Finding relative value in tranches of a commercial mortgage-backed securities (CMBS) deal over another deal is crucial for enhancing returns of portfolio managers who invest in this sector of the structured finance market. Much of the work in finding relative value comes from investigating the underlying collateral and loan details for each individual commercial property within a deal. The collateral must be analyzed for its potential to default on future payments in order to assess the potential performance of the properties in the loan pool. By investigating default potential among the different properties within a loan pool, a portfolio manager can identify CMBS tranches that contain properties that have lower projected default risk than other tranches. If portfolio managers find that the default risks involved have been incorrectly embedded into the CMBS tranche's price, there is an opportunity to enhance portfolio returns. In order to predict defaults and future performance of property loans, a model has to be developed that incorporates property characteristics of the individual properties. There have been a good number of studies that propose models to predict future default probabilities of residential and commercial properties.

The logistic regression discussed in Chapter 23 has been the statistical model traditionally used for default prediction for loans in general. In a 1998 study, Episcopos, Pericli, and Hu[2] compared logistic regression and neural networks to predict commercial mortgage defaults using seven independent variables from property types and regions. They concluded that the neural network performs better when predicting loan default than the logistic regression benchmark.

More recently, Cowden, Fabozzi, and Nazemi[3] applied various ML algorithms to predict commercial real estate loan defaults for loans in CMBS. They assessed the performance of the following ML classification techniques: support vector

2. Andreas Episcopos, Athanasios Pericli, and Jianxun Hu, "Commercial Mortgage Default: A Comparison of Logit with Radial Basis Function Networks," *Journal of Real Estate Finance and Economics*, Vol. 17, No. 2 (1998), pp. 163–178.
3. Chad Cowden, Frank J. Fabozzi, and Abdolreza Nazemi, "Default Prediction of Commercial Real Estate Properties Using Machine Learning," *Journal of Portfolio Management*, Vol. 45, No. 7 (2019), pp. 55–67.

machine (SVM), random forest, boosting, and classification trees, comparing the performance to the typical statistical technique. The principal finding of their study was that the SVM technique for predicting defaults on commercial property loans significantly outperformed other methods. This ML model enables an optimization program to attribute significance to many individual property metrics that impact defaults, the two most popular metrics being loan-to-value ratio and debt-service coverage ratio, in order to find the best classifier between defaulting and nondefaulting commercial loans. SVM allows the testing of a number of different property metrics instead of focusing attention on only one or two individual components of a property's financial attributes. Moreover, the boosting technique identified the ratio of the capitalization-rate spread to the average capitalization-rate spread of property type as the most important driver of defaults in commercial real estate loans.

Predicting Corporate Bond Recovery Rates

Several studies have employed ML for predicting recovery rates for defaulted corporate bonds. Bastos[4] illustrated how an ensemble of 251 models derived from the same regression method yielded more accurate forecasts of recovery rates than a single model. More specifically, using bootstrap aggregation (bagging) to build an ensemble of regression trees, he showed that his results are valid for 255 corporate bonds and loans both during out-of-sample estimation and during cross-validation. Other studies report that ML approaches such as regression trees and support-vector regressions outperform traditional statistical methods for predicting recovery rates of corporate bonds out of sample.

Prepayment Modeling for Agency Residential MBS

Key to the valuation of agency RMBS is the modeling of voluntary prepayment and default behaviors of the underlying borrowers in the mortgage pool. As Zhang et al.[5] state, prepayment modeling is "among the most complex areas of financial

4. See Abdolreza Nazemi, Farnoosh Fatemi Pour, Konstantin Heidenreich, and Frank J. Fabozzi, "Fuzzy Decision Fusion Approach for Loss-Given-Default Modeling," *European Journal of Operational Research*, Vol. 262, No. 2 (2017), pp. 780–791; Abdolreza Nazemi, Konstantin Heidenreich, and Frank J. Fabozzi, "Improving Corporate Bond Recovery Rate Prediction Using Multi-Factor Support Vector Regressions," *European Journal of Operational Research* Vol. 271, No. 2 (2018), pp. 664–675; Abdolreza Nazemi, Friedrich Baumann, and Frank J. Fabozzi, "Intertemporal Defaulted Bond Recoveries Prediction Via Machine Learning," *European Journal of Operational Research* Vol. 297, No. 3 (2022), pp. 1162–1177; Min Qi and Xinlei Zhao, "Comparison of Modeling Methods for Loss Given Default," *Journal of Banking & Finance*, Vol. 35, No. 11 (2011), pp. 2842–2855; and Xiao Yao, Jonathan Crook, and Galina Andreeva, "Support Vector Regression for Loss Given Default Modelling," *European Journal of Operational Research*, Vol. 240, No. 2 (2015), pp. 528–538; and; Egon A. Kalotay and Edward I. Altman, "Intertemporal Forecasts of Defaulted Bond Recoveries and Portfolio Losses," *Review of Finance*, Vol. 21, No. 1 (2017), pp. 433–463.
5. Jiawei Zhang, Xiaojian Zhao, Joy Zhang, Fei Teng, Siyu Lin, and Hongyuan, "Agency MBS Prepayment Model Using Neural Networks," *Journal of Structured Finance*, Vol. 24, No. 4 (2019), pp. 17–33.

modeling." The complexities are due to a large data set, a large set of risk factors, and difficulties in model specification and estimation. The proliferation of both pool and loan-level data coupled with access to ML algorithms have opened the door to the application of ML to mortgage prepayment modeling. The modular prepayment model, one that relies on defined functions to predict mortgage prepayment, has dominated the MBS market nearly since its inception. However, ML models are beginning to make inroads and, in some cases, replace traditional modular prepayment models and might aptly be described as *second generation* mortgage prepayment models.

Zhang et al.[6] showed how neural network models can be used for prepayment modeling of 30-year agency MBS pools. They report that the models (1) provided very good in-time, out-of-sample error tracking and are able to accurately replicate the overall speeds with training using only 10% of the pool sample data and (2) have good out-of-sample error tracking for the overall speeds. Using only 10% of the pool sample data from years prior to 2016 for training the model, the neural network model they proposed was able to accurately forecast the overall speeds between 2016 and 2018.

Schultz and Fabozzi[7] proposed an ML mortgage prepayment model using a boosted gradient classifier trained at the loan level and generalized to the pool level. (A gradient boosted classifier is a tree-based model using an ensemble of weak learners to create a strong committee for prediction.) The authors applied the model to the Freddie Mac loan-level data set because it represents the more comprehensive release of loan-level data of the two government-sponsored enterprises (Freddie Mac and Fannie Mae). They used, in their model, features that are well known to impact mortgage prepayments: (1) external time-dependent variables (housing turnover rate, seasonal factors, updated borrower loan-to-value ratio), (2) internal time-dependent variables (loan age, borrower incentive), and (3) continuous variables (debt-to-income ratio, credit score). In addition to these sets of three features, categorical variables are also included. Typically, categorical variables relate to borrower or loan characteristics such as loan purpose, property type, property location, and servicer. The classifier is trained on 101,202,381 transaction records covering the period January 2006 to December 2020. Each feature of the model is scored according to its contribution to the model's predictive accuracy. The purpose of feature importance in ML is to assist in the determination of the relationship between the target variable and the features. The dominate features of the model are borrower incentive, loan age, lagged mortgage curve, and home appreciation. A backtest of the test data for origination years between 2011 and 2020 indicated that overall the model performed well, capturing differences in the seasoning ramp

6. The description of the modeling in the article was an artificial intelligence collaboration between the Ernst & Young's Quantitative Advisory Services and the Securitized Products Research group at MSCI.
7. Glenn M. Schultz and Frank J. Fabozzi, "Rise of the Machines: Application of Machine Learning to Mortgage Prepayment Modeling," *Journal of Fixed Income* Vol. 31, Issue 3 (2022), pp. 6–19. February 2021.

of each cohort. The notable exceptions were 2011–2012. Nonetheless, overall the model appeared to perform well across the vintage cohorts examined.

Measuring Bond Liquidity

In Chapter 24 we discussed measuring bond liquidity. ML is being used by several major asset-management firms to account for the numerous factors that impact a bond's liquidity risk. The information provided from ML applications to measure liquidity risk can be used to comply with the SEC's liquidity rules (discussed in Chapter 24) and incorporated into various bond portfolio-management and trading activities. According to BlackRock's head of liquidity research, Stefano Pasquali, ML has been known since the 1980s to be able to improve liquidity modeling.[8]

8. Cited in Faye Kiburn, "Firms Eye Machine Learning for Liquidity Risk Models," *Waterstechnology* (September 22, 2017). Available at: https://www.waterstechnology.com/market-data-data-analytics/data-display-analytics/3424646/firms-eye-machine-learning-for-liquidity-risk-models.

INDEX

CHAPTER
THIRTY-SIX

VALUATION OF BONDS WITH EMBEDDED OPTIONS

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The complication in building a model to value bonds with embedded options and option-type derivatives is that cash flows will depend on interest rates in the future. Academicians and practitioners have attempted to capture this interest rate uncertainty through various models, often designed as single or multi-factor models. These models attempt to capture the stochastic behavior of rates.

In practice, these elegant mathematical models must be converted to numeric applications. Here we focus on one such model—a single-factor model that assumes a stationary variance or, as it is more often called, volatility. We demonstrate how to move from the yield curve to a valuation lattice. Effectively, the lattice is a representation of the model, capturing the distribution of rates over time. In our illustration we will present the lattice as a binomial tree, the most simple lattice form.

The lattice holds all the information required to perform the valuation of certain option-like interest rate products. First, the lattice is used to generate cash flows over the life of the security. Next, the interest rates on the lattice are used to compute the present value of those cash flows.

There are several interest rate models that have been used in practice to construct an interest rate lattice. These are described in other chapters. In each case, interest rates can realize one of several possible rates when we move from one period to the next. A lattice model where it is assumed that only two rates are possible in the next period given the current rate is called a *binomial model*. A lattice model where it is assumed that interest rates can take on three possible rates in the next period is called a *trinomial model*. There are even more complex models that assume more than three possible rates in the next period can be realized.

Regardless of the underlying assumptions, each model shares a common restriction. The interest rate tree generated must produce a value for an on-the-run optionless issue that is consistent with the current par yield curve. In effect, the value estimated by the model must be equal to the observed market price for the optionless instrument. Under these conditions, the model is said to be "arbitrage free." A lattice that produces an arbitrage-free valuation is said to be "fair." The lattice is used for valuation only when it has been calibrated to be fair. More on calibration below.

In this chapter we show how to value bonds with embedded options using the lattice methodology. We begin by demonstrating how an interest rate lattice is constructed. Then we use the model to value bonds with an embedded option. The lattice methodology also can be used to value floating-rate securities with option-type derivatives, options on bonds, caps, floors, swaptions, and forward-start swaps.[1]

THE INTEREST RATE LATTICE

Exhibit 36-1 provides an example of a binomial interest rate tree, which consists of a number of "nodes" and "legs." Each leg represents a one-year interval over time. A simplifying assumption of one-year intervals is made to illustrate the key principles. The methodology is the same for smaller time periods. In fact, in practice, the selection of the length of the time period is critical, but we need not be concerned with this nuance here.

The distribution of future interest rates is represented on the tree by the nodes at each point in time. Each node is labeled as N and has a subscript, a combination of L's and H's. The subscripts indicate whether the node is lower or higher on the tree, respectively, relative to the other nodes. Thus node N_{HH} is reached when the one-year rate realized in the first year is the higher of the two rates for that period, then the highest of the rates in the second year.

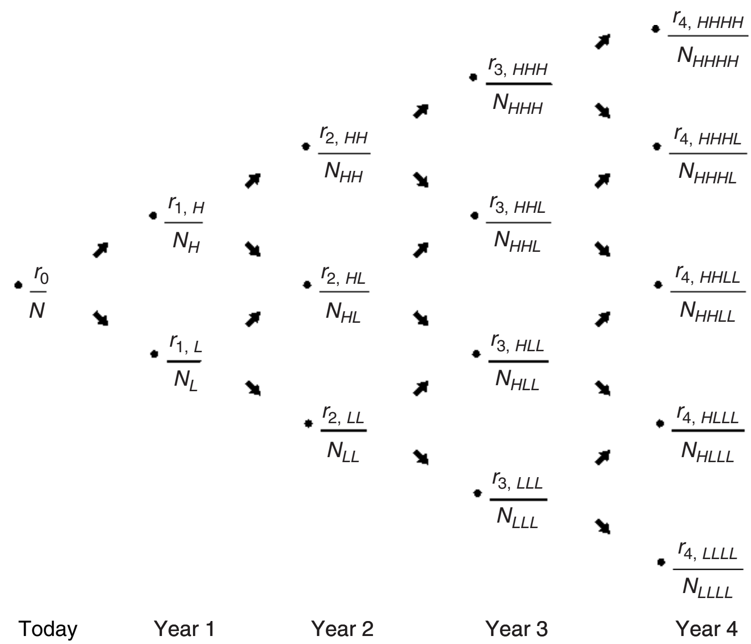
The root of the tree is N, the only point in time at which we know the interest rate with certainty. The one-year rate today (i.e., at N) is the current one-year spot rate, which we denote by r_0.

We must make an assumption concerning the probability of reaching one rate at a point in time. For ease of illustration, we have assumed that rates at any point in time have the same probability of occurring; in other words, the probability is 50% on each leg.

1. These applications of the lattice methodology are presented in Frank J. Fabozzi, Andrew Kalotay, and Michael Dorigan, "Yield Curves and Valuation Lattices" and "Using the Lattice Model to Value Bonds with Embedded Options, Floaters, Options, and Caps/Floors," Chapters 13 and 14 in Frank J. Fabozzi (ed.), *Interest Rate, Term Structure, and Valuation Modeling* (Hoboken, NJ: Wiley, 2002).

EXHIBIT 36-1

Four-Year Binomial Interest Rate Tree



The interest rate model we will use to construct the binomial tree assumes that the one-year rate evolves over time based on a log-normal random walk with a known (stationary) volatility. Technically, the tree represents a one-factor model. Under the distributional assumption, the relationship between any two adjacent rates at a point in time is calculated via the following equation:

$$r_{1,H} = r_{1,L}e^{2\sigma\sqrt{t}}$$

where σ is the assumed volatility of the one-year rate, t is time in years, and e is the base of the natural logarithm. Since we assume a one-year interval, that is, $t = 1$, we can disregard the calculation of the square root of t in the exponent.

For example, suppose that $r_{1,L}$ is 4.4448% and σ is 10% per year, then

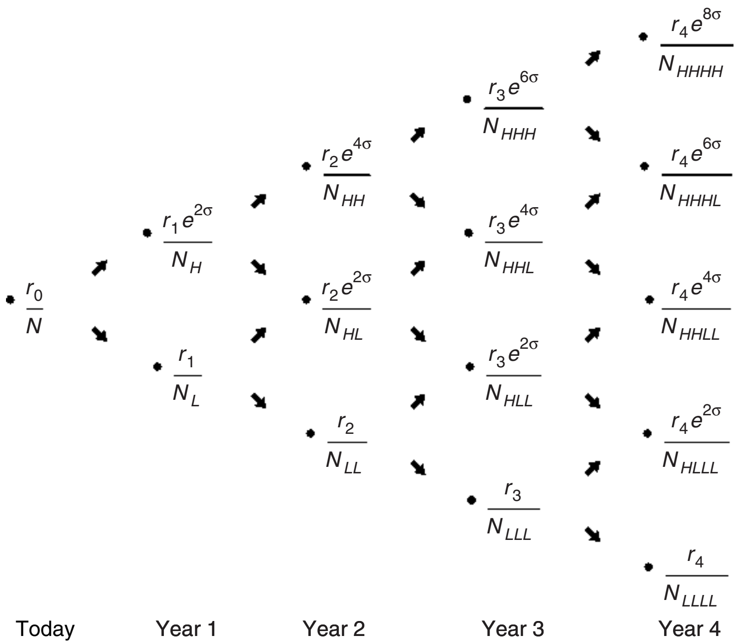
$$r_{1,H} = 4.4448\%(e^{2\times0.10}) = 5.4289\%$$

In the second year, there are three possible values for the one-year rate. The relationship between $r_{2,LL}$ and the other two one-year rates is as follows:

$$r_{2,HH} = r_{2,LL}\,(e^{4\sigma}) \qquad \text{and} \qquad r_{2,HL} = r_{2,LL}(e^{2\sigma})$$

EXHIBIT 36-2

Four-Year Binomial Interest Rate Tree with One-Year Rates*



*r_t is the lowest one-year rate at each point in time.

Thus, for example, if $r_{2,LL}$ is 4.6958%, and assuming once again that σ is 10%, then

$$r_{2,HH} = 4.6958\%(e^{4\times 0.10}) = 7.0053\%$$

and

$$r_{2,HL} = 4.6958\%(e^{2\times 0.10}) = 5.7354\%$$

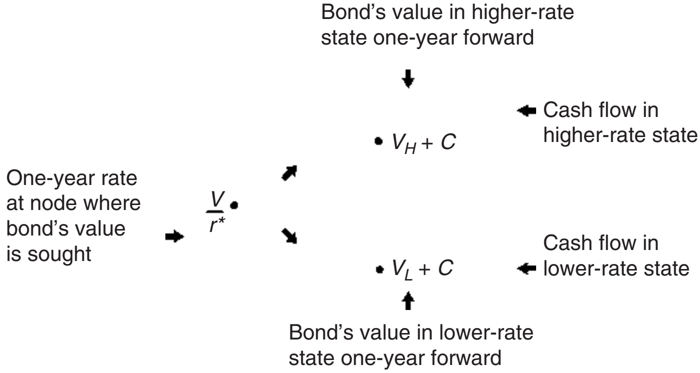
This relationship between rates holds for each point in time. Exhibit 36-2 shows the interest rate tree using this new notation.

Determining the Value at a Node

In general, to get a security's value at a node, we follow the fundamental rule for valuation: The value is the present value of the expected cash flows. The appropriate discount rate to use for cash flows one-year forward is the one-year rate at the node where we are computing the value. Now there are two present values in this case: the present value of the cash flows in the state where the one-year rate is the higher rate and one where it is the lower-rate state. We have assumed that the probability of

EXHIBIT 36-3

Calculating a Value at a Node



both outcomes is equal. Exhibit 36-3 provides an illustration for a node assuming that the one-year rate is r^* at the node where the valuation is sought and letting

V_H = the bond's value for the higher one-year rate state
V_L = the bond's value for the lower one-year rate state
C = coupon payment

From where do the future values come? Effectively, the value at any node depends on the future cash flows. The future cash flows include (1) the coupon payment one year from now and (2) the bond's value one year from now, both of which may be uncertain. Starting the process from the last year in the tree and working backward to get the final valuation resolves the uncertainty. At maturity, the instrument's value is known with certainty—par. The final coupon payment can be determined from the coupon rate or from prevailing rates to which it is indexed. Working back through the tree, we realize that the value at each node is calculated quickly. This process of working backward is often referred to as *recursive valuation*.

Using our notation, the cash flow at a node is either

$V_H + C$ for the higher one-year rate
$V_L + C$ for the lower one-year rate

The present value of these two cash flows using the one-year rate at the node, r^*, is

$$\frac{V_H + C}{(1 + r^*)} = \text{present value for the higher one-year rate}$$

$$\frac{V_L + C}{(1 + r^*)} = \text{present value for the lower one-year rate}$$

Then the value of the bond at the node is found as follows:

$$\text{Value at a node} = \frac{1}{2}\left[\frac{V_H + C}{(1+r^*)} + \frac{V_L + C}{(1+r^*)}\right]$$

CALIBRATING THE LATTICE

We noted earlier the importance of the no-arbitrage condition that governs the construction of the lattice. To ensure that this condition holds, the lattice must be calibrated to the current par yield curve, a process we demonstrate here. Ultimately, the lattice must price optionless par bonds at par.

Assume the on-the-run par yield curve for a hypothetical issuer as it appears in Exhibit 36-4. The current one-year rate is known, 3.50%. Hence the next step is to find the appropriate one-year rates one-year forward. As before, we assume that volatility σ is 10% and construct a two-year tree using the two-year bond with a coupon rate of 4.2%, the par rate for a two-year security.

Exhibit 36-5 shows a more detailed binomial tree with the cash flow shown at each node. The root rate for the tree r_0 is simply the current one-year rate, 3.5%. At the beginning of year 2, there are two possible one-year rates, the higher rate and the lower rate. We already know the relationship between the two. A rate of 4.75% at N_L has been chosen arbitrarily as a starting point. An iterative process determines the proper rate (i.e., trial-and-error). The steps are described and illustrated below. Again, the goal is a rate that, when applied in the tree, provides a value of par for the two-year 4.2% bond.

Step 1. Select a value for r_1. Recall that r_1 is the lower one-year rate. In this first trial, we arbitrarily selected a value of 4.75%.

Step 2. Determine the corresponding value for the higher one-year rate. As explained earlier, this rate is related to the lower one-year rate as follows: $r_1 e^{2\sigma}$. Since r_1 is 4.75%, the higher one-year rate is 5.8017% (= 4.75% $e^{2\times 0.10}$). This value is reported in Exhibit 36-5 at node N_H.

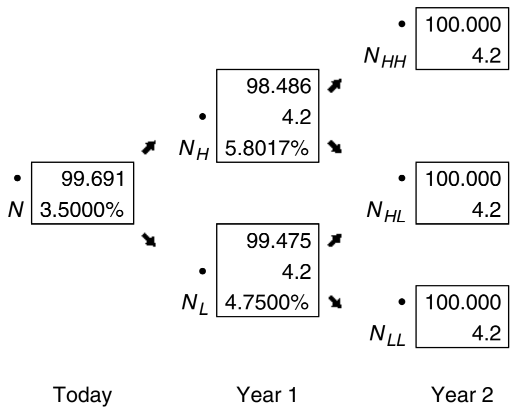
EXHIBIT 36-4

Issuer Par Yield Curve

Maturity	Par Rate	Market Price
1 year	3.50%	100
2 years	4.20%	100
3 years	4.70%	100
4 years	5.20%	100

EXHIBIT 36-5

The One-Year Rates for Year 1 Using the Two-Year 4.2% On-the-Run Issue: First Trial



Step 3. Compute the bond's value one year from now. This value is determined as follows:

a. Determine the bond's value two years from now. In our example, this is simple. Since we are using a two-year bond, the bond's value is its maturity value ($100) plus its final coupon payment ($4.2). Thus it is $104.2.

b. Calculate V_H. Cash flows are known. The appropriate discount rate is the higher one-year rate, 5.8017% in our example. The present value is $98.486 (= $104.2/1.058017).

c. Calculate V_L. Again, cash flows are known—the same as those in step 3b. The discount rate assumed for the lower one-year rate is 4.75%. The present value is $99.475 (= $104.2/1.0475).

Step 4. Calculate *V*.

a. Add the coupon to both V_H and V_L to get the cash flow at N_H and N_L, respectively. In our example we have $102.686 for the higher rate and $103.675 for the lower rate.

b. Calculate *V*. The one-year rate is 3.50%. (*Note:* At this point in the valuation, r^* is the root rate, 3.50%.) Therefore, $99.691 = 1/2($99.214 + $100.169)

Step 5. Compare the value in step 4 to the bond's market value. If the two values are the same, then the r_1 used in this trial is the one we seek. If, instead, the value found in step 4 is not equal to the market value of the bond, this means that the value r_1 in this trial is not the one-year rate that is consistent with the current yield curve. In this case, the five steps are repeated with a different value for r_1.

When r_1 is 4.75%, a value of \$99.691 results in step 4, which is less than the observed market price of \$100. Therefore, 4.75% is too large, and the five steps must be repeated trying a lower rate for r_1.

Let's jump right to the correct rate for r_1 in this example and rework steps 1 through 5. This occurs when r_1 is 4.4448%. The corresponding binomial tree is shown in Exhibit 36-6. The value at the root is equal to the market value of the two-year issue (par).

We can "grow" this tree for one more year by determining r_2. Now we will use the three-year on-the-run issue, the 4.7% coupon bond, to get r_2. The same five steps are used in an iterative process to find the one-year rates in the tree two years from now. Our objective is now to find the value of r_2 that will produce a bond value of \$100. Note that the two rates one year from now of 4.4448% (the lower rate) and 5.4289% (the higher rate) do not change. These are the fair rates for the tree one-year forward.

The problem is illustrated in Exhibit 36-7. The cash flows from the three-year 4.7% bond are in place. All we need to perform a valuation are the rates at the start of year 3. In effect, we need to find r_2 such that the bond prices at par. Again, an arbitrary starting point is selected, and an iterative process produces the correct rate.

The completed version of Exhibit 36-7 is found in Exhibit 36-8. The value of r_2, or equivalently $r_{2,LL}$, that will produce the desired result is 4.6958%. The corresponding rates $r_{2,HL}$ and $r_{2,HH}$ would be 5.7354% and 7.0053%, respectively.

EXHIBIT 36-6

The One-Year Rates for Year 1 Using the Two-Year 4.2% On-the-Run Issue

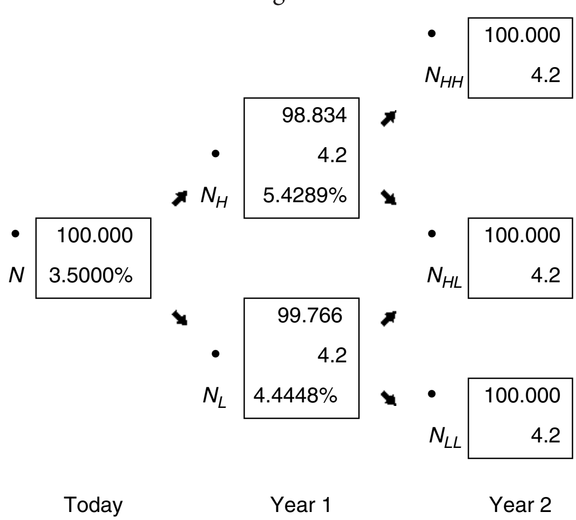


EXHIBIT 36-7

Information for Deriving the One-Year Rates for Year 2 Using the Three-Year 4.7% On-the-Run Issue

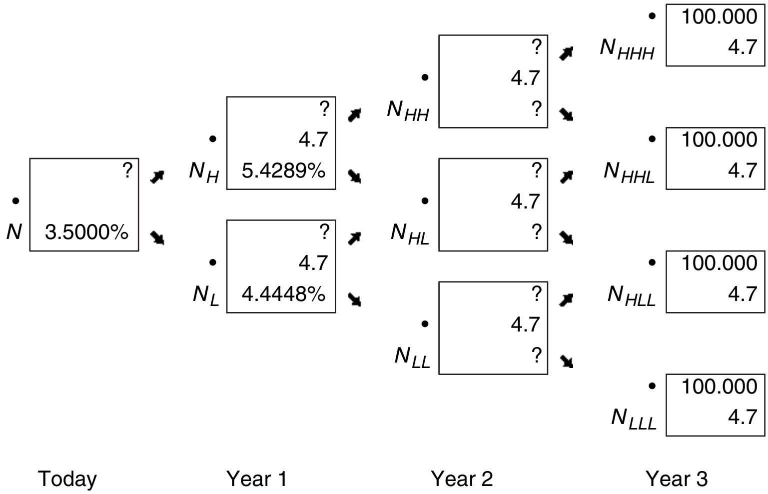
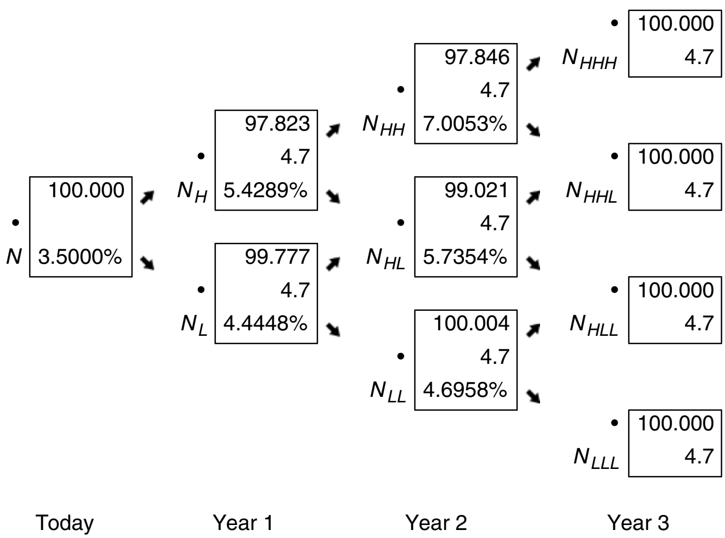


EXHIBIT 36-8

The One-Year Rates for Year 2 Using the Three-Year 4.7% On-the-Run Issue



To verify that these are the correct one-year rates two years from now, work backward from the four nodes at the right of the tree in Exhibit 36-8. For example, the value in the box at N_{HH} is found by taking the value of \$104.7 at the two nodes to its right and discounting at 7.0053%. The value is \$97.846. Similarly, the value in the box at N_{HL} is found by discounting \$104.70 by 5.7354% and at N_{LL} by discounting at 4.6958%.

USING THE LATTICE FOR VALUATION

To illustrate how to use the lattice for valuation purposes, consider a 6.5% option-free bond with four years remaining to maturity. Since this bond is option-free, it is not necessary to use the lattice model to value it. All that is necessary to obtain an arbitrage-free value for this bond is to discount the cash flows using the spot rates obtained from bootstrapping the yield curve shown in Exhibit 36-4. The spot rates are as follows:

1 year	3.5000%
2 years	4.2147%
3 years	4.7345%
4 years	5.2707%

Discounting the 6.5% four-year option-free bond with a par value of \$100 at the above spot rates would give a bond value of \$104.643.

Exhibit 36-9 contains the fair tree for a four-year valuation. Exhibit 36-10 shows the various values in the discounting process using the lattice in Exhibit 36-9. The root of the tree shows the bond value of \$104.643, the same value found by discounting at the spot rate. This demonstrates that the lattice model is consistent with the valuation of an option-free bond when using spot rates.

FIXED-COUPON BONDS WITH EMBEDDED OPTIONS

The valuation of bonds with embedded options proceeds in the same fashion as in the case of an option-free bond. However, the added complexity of an embedded option requires an adjustment to the cash flows on the tree depending on the structure of the option. A decision on whether to call or put must be made at nodes on the tree where the option is eligible for exercise. Examples for both callable and putable bonds follow.

Valuing a Callable Bond

In the case of a call option, the call will be made when the present value (PV) of the future cash flows is greater than the call price at the node where the decision to exercise is being made. Effectively, the following calculation is made:

$$V_t = \min[\text{call price, PV(future cash flows)}]$$

EXHIBIT 36-9

Binomial Interest Rate Tree for Valuing Up to a Four-Year Bond for Issuer (10% Volatility Assumed)

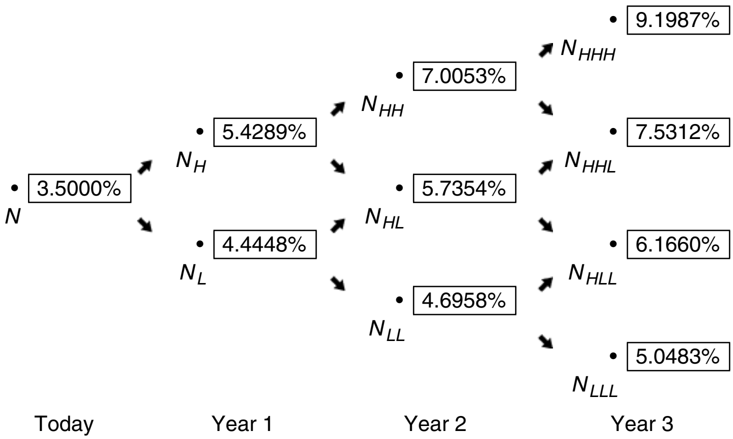
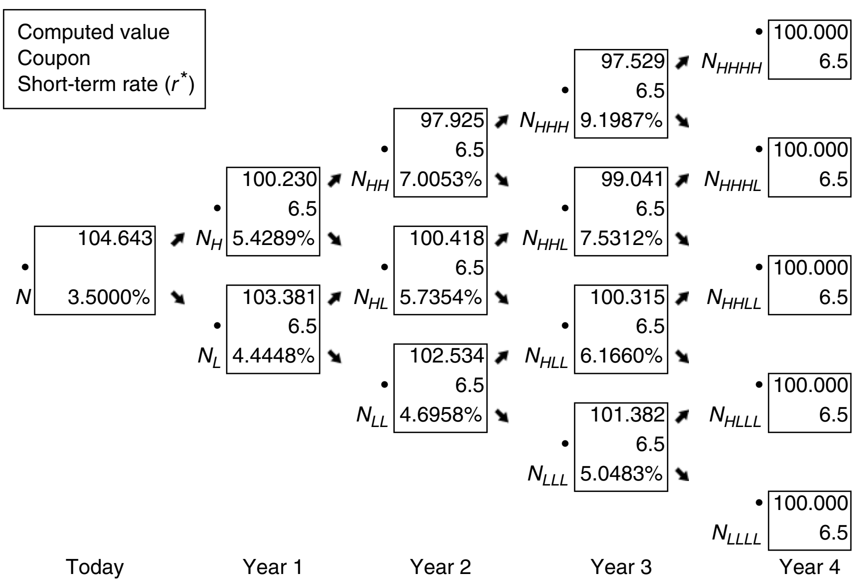


EXHIBIT 36-10

Valuing an Option-Free Bond with Four Years to Maturity and a Coupon Rate of 6.5% (10% Volatility Assumed)



where V_t represents the PV of future cash flows at the node. This operation is performed at each node where the bond is eligible for call.

For example, consider a 6.5% bond with four years remaining to maturity that is callable in one year at $100. We will value this bond, as well as the other instruments in this chapter, using a binomial tree. Exhibit 36-9 is the binomial interest rate tree that was derived earlier in this chapter and then used to value an option-free bond. In constructing the binomial tree in Exhibit 36-9, it is assumed that interest rate volatility is 10%. This binomial tree will be used throughout this chapter.

Exhibit 36-11 shows that two values are now present at each node of the binomial tree. The discounting process explained earlier is used to calculate the first of the two values at each node. The second value is the value based on whether the issue will be called. Again, the issuer calls the issue if the PV of future cash flows exceeds the call price. This second value is incorporated into the subsequent calculations.

In Exhibits 36-12 and 36-13, certain nodes from Exhibit 36-11 are featured. Exhibit 36-12 shows nodes where the issue is not called (based on the simple call rule used in the illustration) in years 2 and 3.[2] The values reported in this case are the same as in the valuation of an option-free bond. Exhibit 36-13 shows some nodes where the issue is called in years 2 and 3. Notice how the methodology changes the cash flows. In year 3, for example, at node N_{HLL} the recursive valuation process produces a PV of 100.315. However, given the call rule, this issue would be called. Therefore, 100 is shown as the second value at the node, and it is this value that is then used as the valuation process continues. Taking the process to its end, the value for this callable bond is 102.899.

The value of the call option is computed as the difference between the value of an optionless bond and the value of a callable bond. In our illustration, the value of the option-free bond is 104.643 (calculated earlier in this chapter). The value of the callable bond is 102.899. Hence the value of the call option is 1.744 (=104.634 − 102.899).

Valuing a Putable Bond

A putable bond is one in which the bondholder has the right to force the issuer to pay off the bond prior to the maturity date. The analysis of the putable bond follows closely that of the callable bond. In the case of the putable, we must establish the rule by which the decision to put is made. The reasoning is similar to that for the callable bond. If the PV of the future cash flows is less than the put price (i.e., par), then the bond will be put. In equation form,

$$V_t = \max[\text{put price, PV(future cash flows)}]$$

Exhibit 36-14 is analogous to Exhibit 36-3. It shows the binomial tree with the values based on whether or not the investor exercises the put option at each node.

2. We assume cash flows occur at the end of the year.

EXHIBIT 36-11

Valuing a Callable Bond with Four Years to Maturity, a Coupon Rate of 6.5%, and Callable After the First Year at 100 (10% Volatility Assumed)

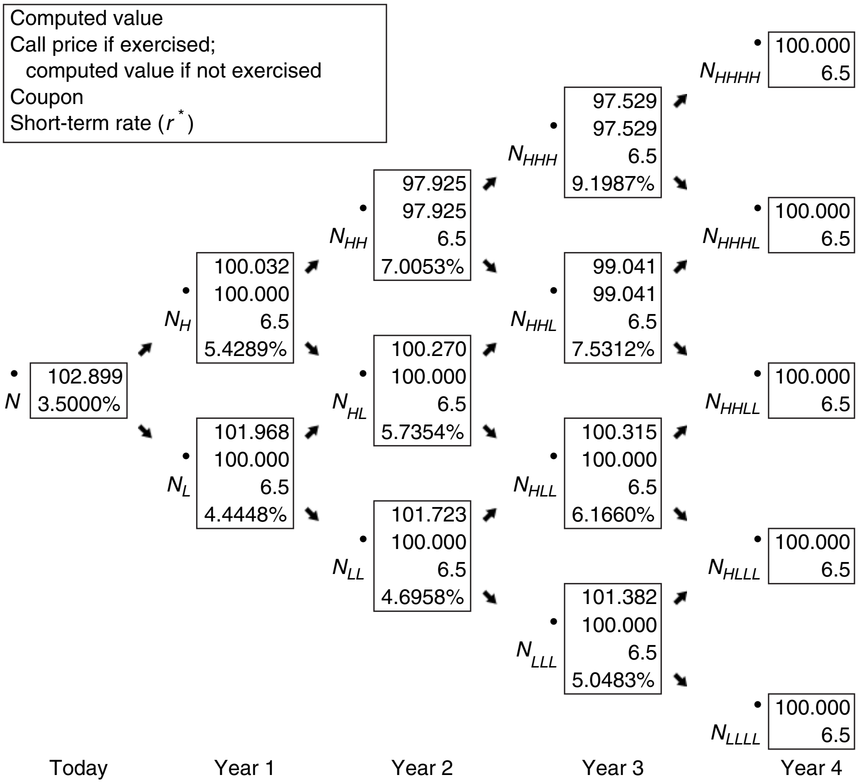


EXHIBIT 36-12

Featured Nodes in Years 2 and 3 for a Callable Bond: Nodes Where Call Option Is Not Exercised

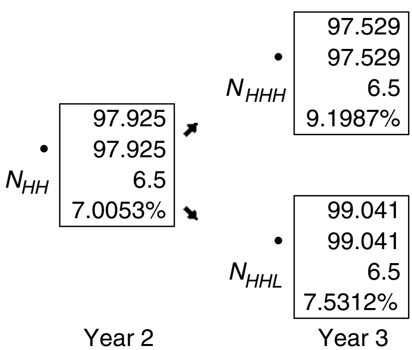


EXHIBIT 36-13

Featured Nodes in Years 2 and 3 for a Callable Bond: Selected Nodes Where the Call Option Is Exercised

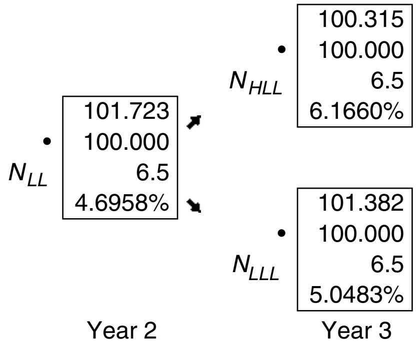
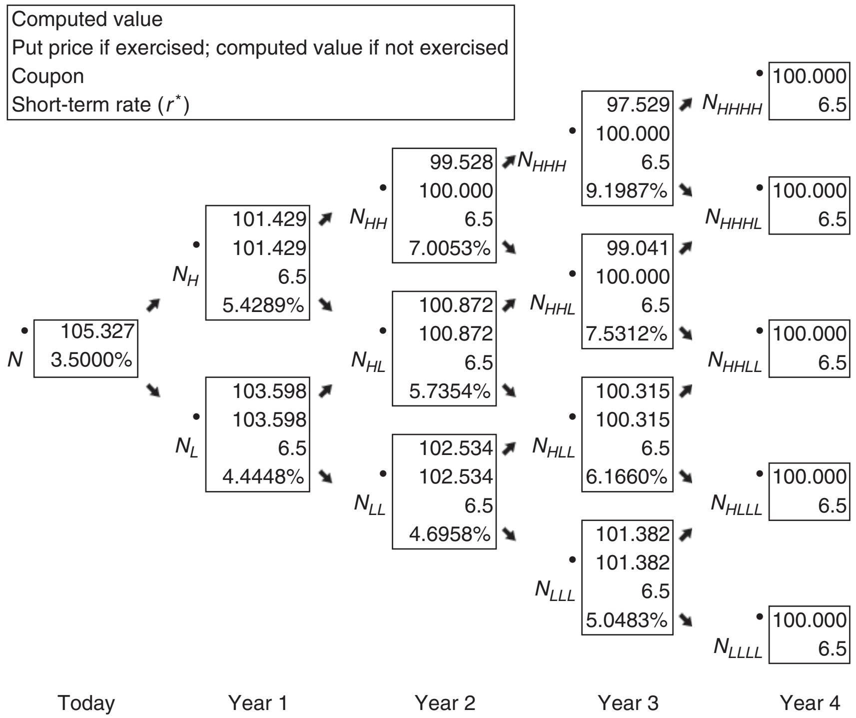


EXHIBIT 36-14

Valuing a Putable Bond with Four Years to Maturity, a Coupon Rate of 6.5%, and Putable after the First Year at 100 (10% Volatility Assumed)



The bond is putable any time after the first year at par. The value of the bond is 105.327. Note that the value is greater than the value of the corresponding option-free bond.

With the two values in hand, we can calculate the value of the put option. Since the value of the putable bond is 105.327 and the value of the corresponding option-free bond is 104.643, the value of the embedded put option purchased by the investor is effectively 0.684.

Suppose that a bond is both putable and callable. The procedure for valuing such a structure is to adjust the value at each node to reflect whether the issue would be put or called. Specifically, at each node there are two decisions about the exercising of an option that must be made. If it is called, the value at the node is replaced by the call price. The valuation procedure then continues using the call price at that node. If the call option is not exercised at a node, it must be determined whether or not the put option will be exercised. If it is exercised, then the put price is substituted at that node and is used in subsequent calculations.

VALUATION OF TWO MORE EXOTIC STRUCTURES

The lattice-based recursive valuation methodology is robust. To further support this claim, we address the valuation of two more exotic structures—the step-up callable note and the range floater.

Valuing a Step-Up Callable Note

Step-up callable notes are callable instruments whose coupon rate is increased (i.e., "stepped up") at designated times. When the coupon rate is increased only once over the security's life, it is said to be a *single step-up callable note*. A *multiple step-up callable note* is a step-up callable note whose coupon is increased more than one time over the life of the security. Valuation using the lattice model is similar to that for valuing a callable bond described earlier except that the cash flows are altered at each node to reflect the coupon characteristics of a step-up note.

Suppose that a four-year step-up callable note pays 4.25% for two years and then 7.5% for two more years. Assume that this note is callable at par at the end of year 2 and year 3. We will use the binomial tree given in Exhibit 36-9 to value this note.

Exhibit 36-15 shows the value of the note if it were not callable. The valuation procedure is the now familiar recursive valuation from Exhibits 36-12 and 36-13. The coupon in the box at each node reflects the step-up terms. The value is 102.082. Exhibit 36-16 shows that the value of the single step-up callable note is 100.031. The value of the embedded call option is equal to the difference in the optionless step-up note value and the step-up callable note value, 2.051.

EXHIBIT 36-15

Valuing a Single Step-Up Noncallable Note with Four Years to Maturity (10% Volatility Assumed)

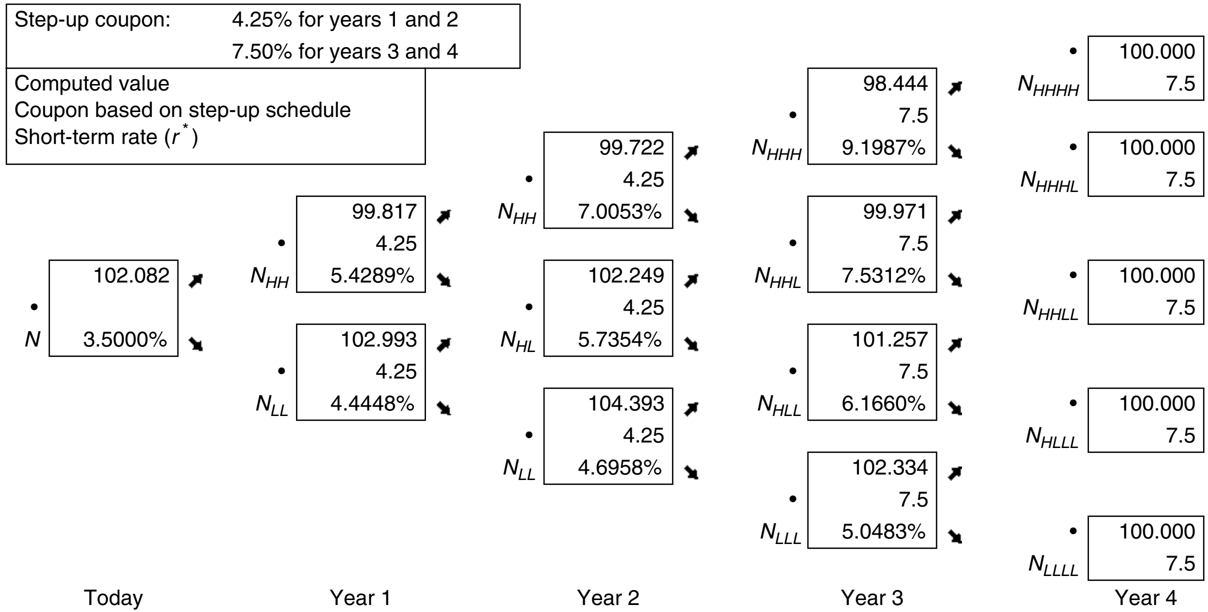


EXHIBIT 36-16

Valuing a Single Step-Up Callable Note with Four Years to Maturity, Callable in Two Years at 100 (10% Volatility Assumed)

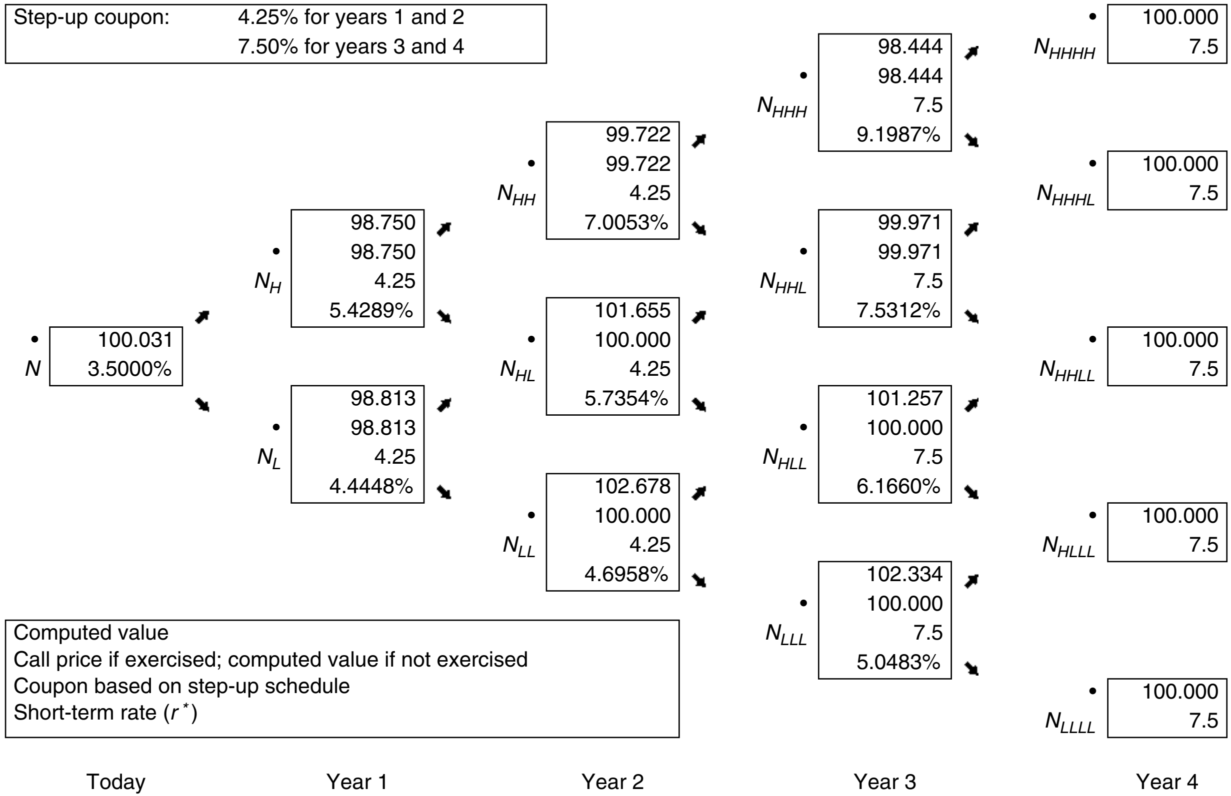


EXHIBIT 36-17

Coupon Schedule (Bands) for a Range Note

	Year 1	Year 2	Year 3
Lower limit	3.00%	4.00%	5.00%
Upper limit	5.00%	6.25%	8.00%

Now we move to another structure where the coupon floats with a reference rate but is restricted. In this next case, a range is set in which the bond pays the reference rate when the rate falls within a specified range, but outside the range no coupon is paid.

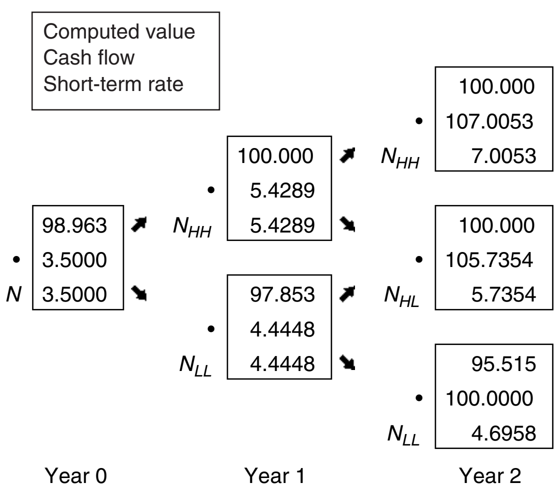
Valuing a Range Note

A *range note* is a security that pays the reference rate only if the rate falls within a band. If the reference rate falls outside the band, whether the lower or upper boundary, no coupon is paid. Typically, the band increases over time.

To illustrate, suppose that the reference rate is, again, the one-year rate and the note has three years to maturity. Suppose further that the band (or coupon schedule) is defined as in Exhibit 36-17. Exhibit 36-18 holds our tree and the cash

EXHIBIT 36-18

Valuation of a Two-Year Range Floater



flows expected at the end of each year. Either the one-year reference rate is paid, or nothing. In the case of this three-year note, there is only one state in which no coupon is paid. Using our recursive valuation method, we can work back through the tree to the current value, 98.963.

EXTENSIONS

We next demonstrate how to compute the option-adjusted spread, effective duration, and the convexity for a fixed income instrument with an embedded option.

Option-Adjusted Spread

We have concerned ourselves with valuation to this point. However, financial market transactions determine the actual price for a fixed income instrument, not a series of calculations on an interest rate lattice. If markets are able to provide a meaningful price (usually a function of the liquidity of the market in which the instrument trades), this price can be translated into an alternative measure of value, the option-adjusted spread (OAS).

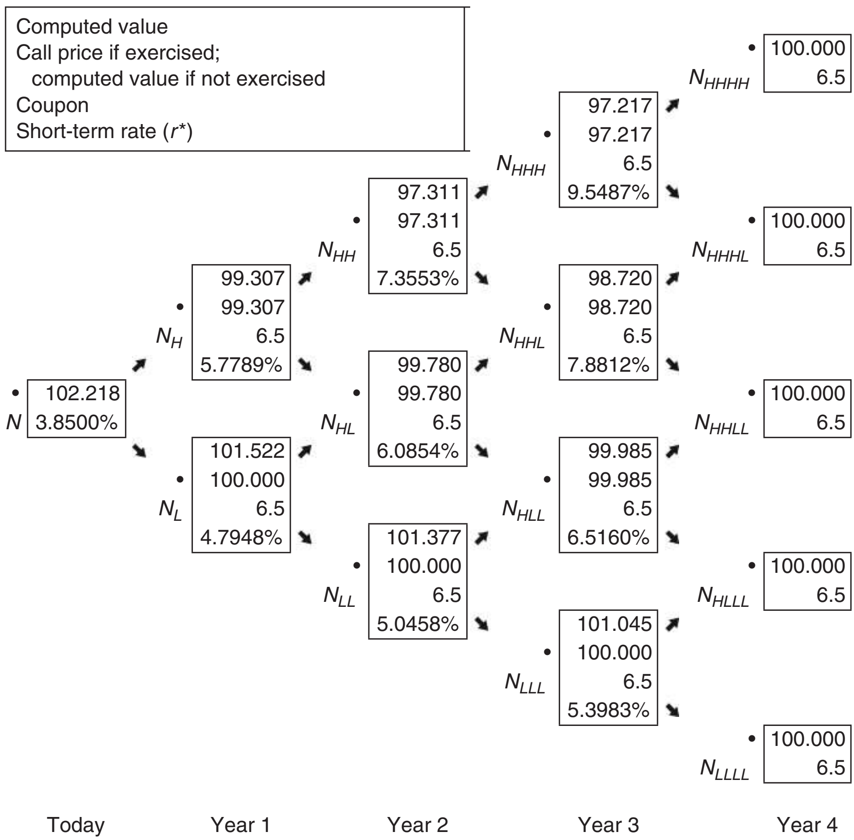
The OAS for a security is the fixed spread (usually measured in basis points) over the benchmark rates that equates the output from the valuation process with the actual market price of the security. For an optionless security, the calculation of OAS is a relatively simple, iterative process. The process is much more analytically challenging with the added complexity of optionality. And just as the value of the option is volatility-dependent, the OAS for a fixed income security with embedded options or an option-like interest-rate product is volatility-dependent.

Recall our illustration in Exhibit 36-11, where the value of a callable bond was calculated as 102.899. Suppose that we had information from the market that the price is actually 102.218. We need the OAS that equates the value from the lattice with the market price. Since the market price is lower than the valuation, the OAS is a positive spread to the rates in the exhibit, rates that we assume to be benchmark rates.

The solution in this case is 35 basis points, which is incorporated into Exhibit 36-19 that shows the value of the callable bond after adding 35 basis points to each rate. The simple binomial tree provides evidence of the complex calculation required to determine the OAS for a callable bond. In Exhibit 36-11, the bond is called at N_{HLL}. However, once the tree is shifted 35 basis points in Exhibit 36-19, the PV of future cash flows at N_{HLL} falls below the call price to 99.985, so the bond is not called at this node. Hence, as the lattice structure grows in size and complexity, the need for computer analytics becomes obvious.

EXHIBIT 36-19

Demonstration That the Option-Adjusted Spread Is 35 Basis Points for a 6.5% Callable Bond Selling at 102.218 (Assuming 10% Volatility)*



*Each one-year rate is 35 basis points greater than in Exhibit 36-11.

Effective Duration and Effective Convexity

Duration and convexity provide a measure of the interest rate risk inherent in a fixed income security. We rely on the lattice model to calculate the effective duration and effective convexity of a bond with an embedded option and other option-like securities. The formulas for these two risk measures are given below:

$$\text{Effective duration} = \frac{V_{-} - V_{+}}{2V_0(\Delta r)}$$

$$\text{Effective convexity} = \frac{V_{+} + V_{-} - 2V_0}{2V_0(\Delta r)^2}$$

where V_- and V_+ are the values derived following a parallel shift in the yield curve down and up, respectively, by a fixed spread. The model adjusts for the changes in the value of the embedded call option that result from the shift in the curve in the calculation of V_- and V_+.

Note that the calculations must account for the OAS of the security. Below we provide the steps for the proper calculation of V_+. The calculation for V_- is analogous.

Step 1. Given the market price of the issue, calculate its OAS.

Step 2. Shift the on-the-run yield curve up by a small number of basis points (Δr).

Step 3. Construct a binomial interest-rate tree based on the new yield curve from step 2.

Step 4. Shift the binomial interest-rate tree by the OAS to obtain an "adjusted tree." That is, the calculation of the effective duration and convexity assumes a constant OAS.

Step 5. Use the adjusted tree in step 4 to determine the value of the bond, V_+.

We can perform this calculation for our four-year callable bond with a coupon rate of 6.5%, callable at par selling at 102.218. We computed the OAS for this issue as 35 basis points. Exhibit 36-20 holds the adjusted tree following a shift in the yield curve up by 25 basis points and then adding 35 basis points (the OAS) across the tree. The adjusted tree is then used to value the bond. The resulting value V_+ is 101.621.

To determine the value of V_-, the same five steps are followed except that in step 2, the on-the-run yield curve is shifted down by a small number of basis points (Δr). It can be demonstrated that for our callable bond, the value for V_- is 102.765.

The results are summarized below:

$$\Delta r = 0.0025$$
$$V_+ = 101.621$$
$$V_- = 102.765$$
$$V_0 = 102.218$$

Therefore,

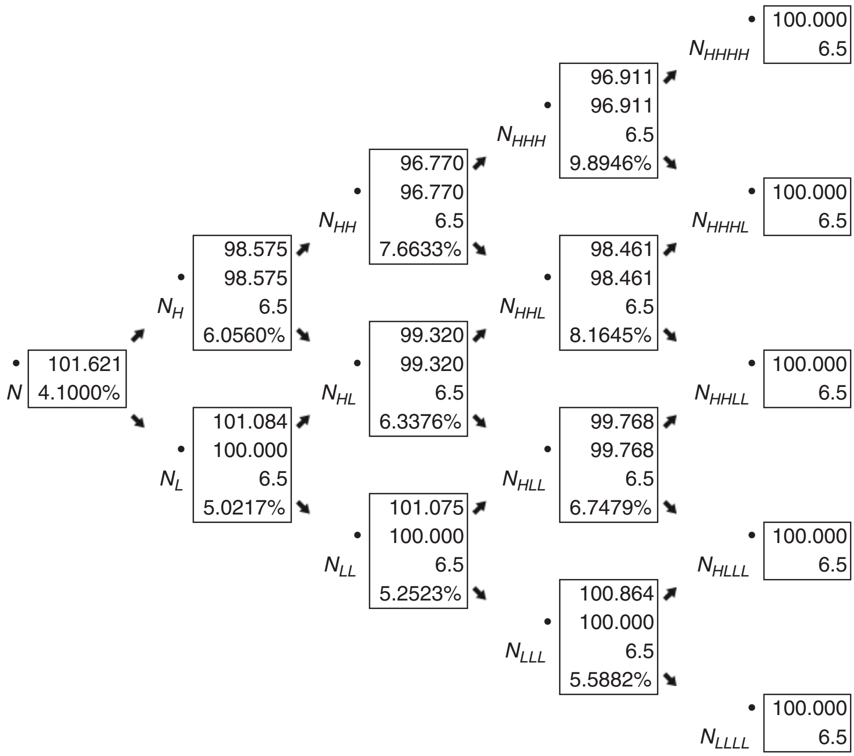
$$\text{Effective duration} = \frac{102.765 - 101.621}{2(102.218)(0.0025)} = 2.24$$

$$\text{Effective convexity} = \frac{101.621 + 102.765 - 2(102.218)}{2(102.218)(0.0025)^2} = -39.1321$$

Notice that this callable bond exhibits negative convexity.

EXHIBIT 36-20

Determination of V_+ for Calculating Effective Duration and Convexity*



*+25 basis point shift in on-the-run yield curve.

KEY POINTS

- For bonds with embedded options, the expected cash flow will depend on future interest-rate levels, which in turn depend on expected interest-rate volatility.
- An interest rate lattice provides a robust means for the valuation of a number of fixed income securities and derivatives.
- Given the market price of a bond, a lattice model can be used to obtain the option-adjusted spread to a benchmark yield curve based on an assumed interest rate volatility.

- A bond's OAS is the fixed spread (usually measured in basis points) over the benchmark rates that equates the output from the valuation process with the actual market price of the security.
- Effective duration and convexity can be computed for a bond by changing the yield-curve up and down by a given number of basis points and calculating what the new prices would be on the revised interest rate tree. These new prices are then used in the standard duration and convexity formula.