Thesis for the degree of Master of Science in Engineering Physics

Dynamics of inertial particles in random flows

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Abstract

Acknowledgments

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Introduction

Most fluid systems in nature contain more than one species of particles, and it is therefore important to understand the behavior and dynamics of these kinds of particle systems. They are described by the Navier-Stokes equations commonly used for single-component fluids, but with moving boundary conditions. This would be hard to solve explicitly and it would furthermore become problematic to analyse the properties of the system.

In order to analyse the dynamics of inertial particles the equation of motion should be be expressed as an ordinary differential because then the tools of dynamical systems theory are accessible. The Maxey-Riley equation is one of those equations.

This thesis concerns the investigation of small finite-size particles in a fluid, where the density of the particles differs from that of the fluid.

The forming of rain droplets in clouds is not fully understood, and more sophisticated models are needed in order to take the great size of the rain droplets into account.

1.1 Outline

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Nature of Flows

compressible vs incompr. viscuous reynolds number laminar vs turbulent flow newtonian vs nonNewtonian

2.1 Dynamics of Fluids

tracers (active tracer, passive tracer). define $\boldsymbol{u}, \boldsymbol{r}, t$

2.1.1 Material Derivative

To measure changes of an arbitrary material property $\alpha(\boldsymbol{r}, t)$ of a fluid which depends on time t and position $\boldsymbol{r}(t) \equiv (x(t), y(t), z(t))$, it is possible to measure α locally at a fixed point in space. That constitutes the *Eulerian derivative* $\partial\alpha/\partial t$. It is also possible to follow the flow of the fluid and measure the property changes along fluid trajectories. It would then be necessary to take the derivative with respect to all variables. A small change $\mathrm{d}\alpha$ during time $\mathrm{d}t$ will then be given by

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\partial\alpha}{\partial t} + \frac{\partial\alpha}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\alpha}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\alpha}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}. \tag{2.1}$$

Since the velocity of the fluid is $\boldsymbol{u} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$, Eq. (2.1) can be written as

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\partial\alpha}{\partial t} + (\boldsymbol{u} \cdot \nabla)\alpha. \tag{2.2}$$

This is the *material derivative* with respect to α. α could denote any property of the fluid, e.g. pressure, temperature, density, momentum, and the list goes on. Notice that the first term on the right-hand side is the Eulerian derivative, and the second term accounts for spatial variations of α. It is in the field of fluid dynamics common to denote the material derivative by $\mathrm{D}/\mathrm{D}t$.

A very useful property is the acceleration of a fluid element at position $\boldsymbol{r}$, which is obtained by setting $\alpha \equiv \boldsymbol{u}$ in Eq. (2.2):

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}. \tag{2.3}$$

We could also insert $\alpha \equiv \rho_f$ in Eq. (2.2), where ρ_f is the density of the fluid, to obtain

$$\frac{\mathrm{d}\rho_f}{\mathrm{d}t} = \frac{\partial \rho_f}{\partial t} + \boldsymbol{u} \cdot \nabla \rho_f. \tag{2.4}$$

Using the *mass continuity equation*

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \boldsymbol{u}) = \frac{\partial \rho_f}{\partial t} + \nabla \rho_f \cdot \boldsymbol{u} + \rho_f \nabla \cdot \boldsymbol{u} = 0, \tag{2.5}$$

Eq. (2.4) transforms into

$$\frac{\mathrm{d}\rho_f}{\mathrm{d}t} = -\nabla \rho_f \cdot \boldsymbol{u} - \rho_f \nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \rho_f = -\rho_f \nabla \cdot \boldsymbol{u}. \tag{2.6}$$

An *incompressible flow* is defined as having constant density, which implies that $\mathrm{d}\rho_f/\mathrm{d}t = 0$ and consequently

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2.7}$$

This condition applies to all incompressible flows.

2.1.2 The Navier-Stokes Equations

The Navier-Stokes equations are very useful and have many applications in many different areas.

The motion of fluids is governed by the Navier-Stokes equations. These equations are derived using Newton's second law as well as the conservation laws for momentum, mass and energy. Furthermore, it is assumed that the fluid is a continuum. Denote $\boldsymbol{u}(\boldsymbol{r}, t)$ the velocity of the fluid at position $\boldsymbol{r}(t) \equiv (x(t), y(t), z(t))$ and time t. The general form of the equations is

$$\rho_f \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f}, \tag{2.8}$$

where $\boldsymbol{f}$ is the sum of all body forces, and $\boldsymbol{\sigma}$ is the stress tensor given by

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{2.9}$$

2.2 Turbulence

Turbulent systems are chaotic and irregular, motion of fluids.

2.3 Simulating flows

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Colloidal Systems

Most fluid systems in nature contain more than one species of particles, and it is therefore important to understand the behavior and dynamics of these kinds of particle systems. They are described by the Navier-Stokes equations commonly used for single-component fluids, but with moving boundary conditions. This would be hard to solve explicitly and it would furthermore become problematic to analyse the properties of the system.

In order to analyse the dynamics of inertial particles the equation of motion should be be expressed as an ordinary differential because then the tools of dynamical systems theory are accessible. The Maxey-Riley equation is one of those equations.

This thesis concerns the investigation of small finite-size particles in a fluid, where the density of the particles differs from that of the fluid.

The forming of rain droplets in clouds is not fully understood, and more sophisticated models are needed in order to take the great size of the rain droplets into account.

3.1 Equations of Motion

In order to describe the dynamics of particles in a turbulent fluid it is important to construct a model with appropriate approximations. The Navier-Stokes equations with moving boundary conditions could in principle be used to describe the motion of the particles. These equations are, however, very difficult to solve for turbulent systems, both analytically and numerically. It is necessary to construct more simple models that are easier to analyse but sufficiently accurate to describe the relevant phenomena. These models may be very illuminating as new interesting properties can be discovered of the system.

3.1.1 Advective Model

Let us first consider point particles with no inertia. This approximation is valid when the particle mass and size are negligible. The equation of motion is in this model given by

$$\dot{\boldsymbol{r}}(t) = \boldsymbol{u}(\boldsymbol{r}(t), t), \tag{3.1}$$

where $\boldsymbol{r}(t)$ is the particle position at time t and $\boldsymbol{u}$ is the fluid velocity. The dots over variables denote time derivatives. As evident from this equation, the particle will in this model follow the flow completely and at every point take on the velocity of the fluid. This phenomenon is called *advection*.

This model is valid in numerous cases and has been used extensively in applications, especially in the early days. However, it does not account for many interesting properties of ....

3.1.2 Stokes' Law

To improve upon the simple advective model, let us consider finite-size particles. These particles have inertia, and the result is that the particles do not only follow the flow of the surrounding fluid. To account for particle inerta, consider a spherical particle with radius a and mass m_p. Now, as depicted in Fig 2.1, the velocity of the flow close to the particle surface differs from the flow velocity a little distance away. Let us neglect this effect and instead consider the relative velocity between the particle and the fluid as $\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{r}, t) - \dot{\boldsymbol{r}}$. The particle Reynolds number is then defined to be $\mathrm{Re}_p \equiv L_0|\boldsymbol{w}|/\nu$.

The fricitonal force, commonly referred to as Stokes' drag, is

$$\boldsymbol{F} = 6\pi a\nu\rho_f\boldsymbol{w}, \tag{3.2}$$

and, assuming that this is the dominant force determining the motion of the particle, the equation of motion becomes

$$\ddot{\boldsymbol{r}} = \gamma[\boldsymbol{u}(\boldsymbol{r}, t) - \dot{\boldsymbol{r}}], \tag{3.3}$$

where $\gamma = 6\pi a\nu\rho_f/m$ is the damping rate. This equation is hard to evaluate because the fluid velocity $\boldsymbol{u}$ depends on $\boldsymbol{r}$.

This model assumes that the major forces are due to viscuous drag, and that the particle size is small so that the velocity field changes negligibly across the particle. Moreover, the interaction between particles is not taken into account, and effects of the inertia of the displaced fluid parcels are neglected. We will next consider corrections to the Stokes' law (3.3) using the Maxey-Riley equation.

3.1.3 The Maxey-Riley Equation

The Maxey-Riley equation is valid for small, spherical and rigid particles advected by a smooth flow, i.e. flows that are predominantly laminar. The particles should have small Reynolds numbers Re_p, and the velocity difference across the particle must be small.

We will consider one term at a time.

Buoyancy Force

Previously we have neglected the gravitational force and assumed that the dynamics of the particles are mainly influenced by the fluid, but how would rain droplets fall if not for gravity? The bouyancy force is the correction of the differences of density between the particles and the surrounding fluid. It has the simple form

$$\boldsymbol{f}_b = (\rho_p - \rho_f)\boldsymbol{g}, \tag{3.4}$$

where $\boldsymbol{g}$ is the gravitational acceleration and $\boldsymbol{f}_b$ is the force per unit volume. If the particle density is less than that of the fluid they are referred to as *bubbles*. [lite om bubbles].

There are also systems with *neutrally buoyant particles*, e.g. plankton in the oceans, where $\rho_p = \rho_f$. [lite om fysiken i detta fall]]However, this thesis will mainly focus on systems where $\rho_p > \rho_f$.

Force of the Undisturbed Flow

The effects of the undisturbed fluid is evaluated in position $\boldsymbol{r}$, at the center of the particle sphere. The force is the same as the flow would exert on a fluid element of the same size as the particle, and it is applied in the direction of the trajectory of the *fluid element*, not the particle trajectory. Using Newton's second law and the fluid acceleration given in Eq. (2.3) the force becomes

$$\boldsymbol{f}_f = \rho_f \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}. \tag{3.5}$$

Added Inertia

An accelerating particle moves an amount of fluid as it travels along its trajectory, because it cannot be in the same position as fluid particles simultaneously. An exception would be if the particle is inertialess and completely follows the flow. This gives rise to a drag force which causes the particle to lose kinetic energy, and the result is that the particle appears to have additional inertia. This phenomenon is commonly referred to as the *added-mass effect*. The added-mass term has the form

$$\boldsymbol{f}_m = -\frac{\rho_f}{2}\left\{\ddot{\boldsymbol{r}} - \frac{\mathrm{D}}{\mathrm{D}t}\left[\boldsymbol{u} + \frac{1}{10}a^2\nabla^2\boldsymbol{u}\right]\right\}. \tag{3.6}$$

The minus sign accounts for the frictional nature of the force. The term $a^2\nabla^2\boldsymbol{u}/10$ is the *Faxen correction* and is due to spatial variations of the velocity field. This term has mostly been neglected mainly for practical reasons, but also because the radius a of the particle is usually small so that the velocity field of the fluid does not change significantly across the particle. It is easy to verify using Eq. (3.6) that if the particle is inertialess and follows the flow the particle and the fluid at position $\boldsymbol{r}$ would accelerate towards the same direction and $\boldsymbol{f}_m = 0$, i.e. the particle displaces no fluid.

Stokes' Drag

Stokes' drag is the frictional force due to the viscosity of the fluid. This was discussed in Subsec. 3.1.2, but now a Faxen correction is included to take the flow variations into account. The form of Stokes' drag is then

$$\boldsymbol{f}_d = -\frac{9\rho_f\nu}{2a^2}\boldsymbol{Q}, \tag{3.7}$$

where

$$\boldsymbol{Q} = \dot{\boldsymbol{r}} - \boldsymbol{u} - \frac{1}{6}a^2\nabla^2\boldsymbol{u}. \tag{3.8}$$

Basset-Boussinesq History Term

The integral is the *Basset-Boussinesq history force*, which accounts for the fact that the vorticity diffuses away from the particle due to viscosity. wiki: This force is usually neglected, but it can be large if the particle accelerate at a high rate.

The Maxey-Riley Equation

The final form of the Maxey-Riley equation is obtained by summing all the forces so far. The equation of motion becomes

$$\begin{aligned}\rho_p \ddot{\boldsymbol{r}} =& (\rho_p - \rho_f)\boldsymbol{g} + \rho_f \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - \frac{\rho_f}{2}\left\{\ddot{\boldsymbol{r}} - \frac{\mathrm{D}}{\mathrm{D}t}\left[\boldsymbol{u} + \frac{1}{10}a^2\nabla^2\boldsymbol{u}\right]\right\} \\ & - \frac{9\rho_f \nu}{2a^2}\left\{\boldsymbol{Q} + a\int_0^t \mathrm{d}\tau \frac{\dot{\boldsymbol{Q}}(\tau)}{\sqrt{\pi\nu(t-\tau)}}\right\}.\end{aligned} \tag{3.9}$$

*The conditions where MR holds. (small particles) *I will consider corrections for larger particles (faxen corrections will change)

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