CSE 321 - Introduction into Algorithm Design

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Homework 1:

(1) $T_1(n) = 3n^4 + 3n^3 + 1$

$T_2(n) = 3^n$

$T_3(n) = (n-2)!$

$T_4(n) = \ln^2 n$

$T_5(n) = 2^{2n}$

$T_6(n) = \sqrt[3]{n}$

$T_4(n) < T_6(n) < T_1(n) < T_2(n) < T_5(n) < T_3(n)$

$$\lim_{n\to\infty} \frac{T_4(n)}{T_6(n)} = \lim_{n\to\infty} \frac{\ln^2 n}{n^{1/3}} = \left(\cancel{\lim_{n\to\infty} \frac{2/n \ln n}{1/3 \; 1/n^{2/3}}} = \cancel{\lim_{n\to\infty} 6 \frac{\ln n}{n^{1/3}}} = \cancel{\lim_{n\to\infty} 6 \frac{1/n}{1/3 \; 1/n^{2/3}}} = \right.$$

$$\left. = \cancel{\lim_{n\to\infty} \frac{18}{n^{1/3}} = 0} \right) = \lim_{n\to\infty} \frac{\ln \ln n}{n^{1/3}} = \lim_{n\to\infty} 3 \frac{1}{n^{1/3} \ln(n)} = 0$$

$T_4(n) \in o(T_6(n)) \Rightarrow T_4(n) \in O(T_6(n))$ ✓

$$\lim_{n\to\infty} \frac{T_4(n)}{T_1(n)} = \lim_{n\to\infty} \frac{\ln^2(n)}{3n^4 + 3n^3 + 1} = \lim_{n\to\infty} \frac{2\ln(n)\, 1/n}{12n^3 + 9n^2} = \lim_{n\to\infty} \frac{2 \ln n}{12n^4 + 9n^3} =$$

$$= \lim_{n\to\infty} \frac{2/n}{48n^3 + 27n^2} = \lim_{n\to\infty} \frac{2}{48n^4 + 27n^3} = 0$$ ✓

$T_4(n) \in O(T_1(n))$

$$\lim_{n\to\infty} \frac{T_1(n)}{T_2(n)} = \frac{3n^4 + 3n^3 + 1}{3^n} = \lim_{n\to\infty} \frac{12n^3 + 9n^2}{3^n \ln(3)} = \lim_{n\to\infty} \frac{36n^2 + 18n}{\ln^2(3)\, 3^n} =$$

$$= \lim_{n\to\infty} \frac{72n + 18}{\ln^2(3)\, 3^n} = \lim_{n\to\infty} \frac{72}{\ln^3(3)\, 3^n} = 0$$

$T_1(n) \in O(T_2(n))$ ✓

$$\lim_{n\to\infty} \frac{T_2(n)}{T_5(n)} = \lim_{n\to\infty} \frac{3^n}{2^{2n}} = \left(\frac{3}{4}\right)^n = 0$$

$T_2(n) \in O(T_5(n))$ ✓

$$\lim_{n\to\infty} \frac{T_5(n)}{T_3(n)} = \lim_{n\to\infty} \frac{4^n}{(n-2)!} \approx \lim_{n\to\infty} \frac{4^n}{\sqrt{2\pi(n-2)}\left(\frac{n-2}{e}\right)^n \left(\frac{e}{n-2}\right)^2} \Rightarrow$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$(n-2)! = \sqrt{2\pi(n-2)}\left(\frac{n-2}{e}\right)^{n-2} = \sqrt{2\pi(n-2)}\left(\frac{n-2}{e}\right)^n \left(\frac{e}{n-2}\right)^2$$

$$\Rightarrow \lim_{n\to\infty} \underbrace{\left(\frac{4e}{n-2}\right)^n}_{=0} \cdot \frac{(n-2)^2}{\sqrt{2\pi(n-2)}\cdot e} = 0$$

$T_5(n) \in O(T_3(n))$ ✓

2 a) This algorithm finds median value of the given array.

plum - min value of the array
watermelon - max value of the given array
orange - median value of the given array
orangeTime has no particular role, procedure of finding min and max values could have been done without this unnecessary complication.

b) i) Worst case occures when min = L[n] :

$W(n) = 2n+1+n = 3n+1 \in \Theta(n)$

j) Best case occures when min=L[1] :

$B(n) = n+1+n = 2n+1 \in \Theta(n)$

k) $A(n) = \Theta(n)$ Because $W(n) = B(n) = \Theta(n)$

③ a) $\sum_{i=0}^{n-1} (i^2+1)^2 = f(n)$

$$\int_0^{n-1} (x^2+1)^2 dx \le f(n) \le \int_0^{n} (x^2+1)^2 dx$$

$$\int_0^{n-1} (x^2+1)^2 dx = \int_0^{n-1} (x^4 + 2x^2 + 1)\, dx = \left(\frac{x^5}{5} + \frac{2}{3}x^3 + x\right)\Bigg|_0^{n-1} =$$

$$= \frac{(n-1)^5}{5} + \frac{2(n-1)^3}{3} + (n-1)$$

$$\int_0^{n-} (x^2+1)^2 dx = \frac{1}{5} n^5 + \frac{2}{3} n^3 + n$$

$$\frac{1}{5}(n-1)^5 + \frac{2}{3}(n-1)^3 + (n-1) \le f(n) \le \frac{1}{5} n^5 + \frac{2}{3} n^3 + n$$

$f(n) \in \Theta(n^5)$

b) $\sum_{i=2}^{n-1} \log(i)^2 = f(x)$

$$\int_1^{n-1} \log(x^2) dx \le f(x) \le \int_2^{n} \log(x^2) dx$$

$$\int_1^{n-1} \log(x^2) dx = 2\int_1^{n-1} \log(x) dx = 2 \log e \int_1^{n-1} \ln(x) dx = 2 \log e (x \ln x - x)\Bigg|_1^{n-1}$$

$$= 2\log(e)\left((n-1)\ln(n-1) - (n-1) - (1 \ln 1 - 1)\right) = 2\log(e)\left((n-1)\ln(n-1) - n + 2\right)$$

$$= 2\log(e)(n-1)\ln(n-1) - 2\log(e)n + 4\log(e)$$

$\wedge$

$f(n)$

$$\int_2^{n} \log(x^2) dx = 2\log(e)(x \ln x - x)\Bigg|_2^{n} = 2\log(e)(n \ln(n) - n - 2\ln 2 + 2)$$

$$2\log(e)\left(\underline{(n-1)\ln(n-1)} - n + 2\right) \le f(n) \le 2\log(e)(n \ln(n) - n - 2\ln 2 + 2)$$

$f(n) \in \Theta(n \log n)$

c) $\sum_{i=1}^{n}(i+1)2^{i-1} = f(n)$

$$\int_0^n (x+1)2^{x-1}dx \le f(n) \le \int_1^{n+1}(x+1)2^{x-1}dx$$

$$\int_0^n (x+1)2^{x-1}dx = \int_0^n (x2^{x-1}+2^{x-1})dx = \int_0^n x2^{x-1}dx + \int_0^n 2^{x-1}dx =$$

$=$

$$\int_0^n x2^{x-1}dx = \frac{1}{2}\int_0^n x2^x dx = \frac{1}{2}\left(\frac{x2^x}{\ln 2}\Big|_0^n - \frac{1}{\ln 2}\int_0^n 2^x dx\right) = \frac{1}{2}\left(\left(\frac{x2^x}{\ln 2}\right)\Big|_0^n -\right.$$

$\int u\,dv = uv - \int v\,du$ $\quad$ $x=u$, $dv = 2^x dx$, $dx = du$, $v = \frac{2^x}{\ln 2}$

$$\left.-\frac{1}{\ln 2}\left(\frac{2^x}{\ln 2}\right)\Big|_0^n\right) = \frac{1}{2}\left(\frac{n2^n}{\ln 2} - \frac{1}{\ln 2}\left(\frac{2^n}{\ln 2} - \frac{1}{\ln 2}\right)\right) = \frac{1}{2}\left(\frac{n2^n}{\ln 2} - \frac{2^n}{(\ln 2)^2} +\right.$$

$$\left.+\frac{1}{(\ln 2)^2}\right) = \frac{1}{2}\left(\frac{n2^n}{\ln 2} - \frac{2^n+1}{(\ln 2)^2}\right)$$

$$\int_1^{n+1}(x+1)2^{x-1}dx = \frac{1}{2}\left(\left(\frac{x2^x}{\ln 2}\right)\Big|_1^{n+1} - \frac{1}{\ln 2}\left(\frac{2^x}{\ln 2}\right)\Big|_1^{n+1}\right) = \frac{1}{2}\left(\frac{(n+1)2^{n+1}}{\ln 2} - \frac{2}{\ln 2} -\right.$$

$$\left.-\frac{2^{n+1}}{(\ln 2)^2} + \frac{2}{(\ln 2)^2}\right) = \frac{1}{2}\left(\frac{(n+1)2^{n+1}-2}{\ln 2} - \frac{2^{n+1}+2}{(\ln 2)^2}\right)$$

$$\frac{1}{2}\left(\frac{n2^n}{\ln 2} - \frac{2^n+1}{(\ln 2)^2}\right) \le f(n) \le \frac{1}{2}\left(\frac{(n+1)2^{n+1}-2}{\ln 2} - \frac{2^{n+1}+2}{(\ln 2)^2}\right)$$

$$f(n) \in \Theta(n2^n)$$

(d) $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i+j) = f(n)$

$$\int_0^{n-1}\int_0^{x-1} (x+y)\,dy\,dx \le f(n) \le \int_0^{n}\int_0^{x} (x+y)\,dy\,dx$$

$$\int_0^{n-1}\int_0^{x-1} (x+y)\,dy\,dx = \int_0^{n-1} \left(xy + \frac{y^2}{2}\right)\Big|_0^{x-1} dx = \int_0^{n-1} \left(\frac{3}{2}x^2 - 2x + \frac{1}{2}\right) dx =$$

$$= \left(\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right)\Big|_0^{n-1} = \frac{1}{2}(n-1)^3 - (n-1)^2 + \frac{1}{2}(n-1)$$

$$\int_0^{n}\int_0^{x-1} (x+y)\,dy\,dx = \int_0^{n} \left(xy + \frac{y^2}{2}\right)\Big|_0^{x-1} = \left(\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right)\Big|_0^{n} =$$

$$= \frac{1}{2}n^3 - n^2 + \frac{1}{2}n$$

$$\frac{1}{2}(n-1)^3 - (n-1)^2 + \frac{1}{2}(n-1) \le f(n) \le \frac{1}{2}n^3 - n^2 + \frac{1}{2}n$$

$$f(n) \in \Theta(n^3)$$

a)
```
int sum=0;
int i=0,j=0;
for(i=0; i<=(n-1); ++i) {
    for(j=0;j<pow(pow(i,2)+1,2); ++j;
  sum+= j;
}
```

d)
```
int i=0, j=0, k=0, sum=0;
for( i=0; i<= (n-1); ++i) {
    for( j=0; j<=(i-1) ;++j)
        for( k=0; k<(i+j); ++k);
    sum+=k;
}
```

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(4) $n = 2^K - 1$

$\log_2(n+1) = K$

$$\sum_{i=0}^{K} 2^i = 2^{K+1} - 1 = f(x)$$

$$\int_0^{\log_2(n+1)} 2^x \, dx \leq f(x) \leq \int_0^{\log_2(n+2)} 2^x \, dx$$

$$\int_0^{\log_2(n+1)} 2^x \, dx = \left(\frac{2^x}{\ln 2}\right)\Bigg|_0^{\log_2(n+1)} = \frac{2^{\log_2(n+1)}}{\ln 2} - \frac{1}{\ln 2} = \frac{n+1}{\ln 2} - \frac{1}{\ln 2} =$$

$$= \frac{n}{\ln 2} = \frac{1}{\ln 2} \cdot n$$

$$\int_0^{\log(n+2)} 2^x \, dx = \frac{n+2}{\ln 2} - \frac{1}{\ln 2} = \frac{1}{\ln 2} \cdot (n+1) = \frac{n}{\ln 2} + \frac{1}{\ln 2}$$

$f(x) \in \Theta(n)$

5 a) $n^3 \in O(3^{2n})$

$$\lim_{n\to\infty} \frac{n^3}{3^{2n}} = \lim_{n\to\infty} \frac{n^3}{9^n} = \lim_{n\to\infty} \frac{3n^2}{9^n \ln 9} = \lim_{n\to\infty} \frac{6n}{9^n (\ln 9)^2} =$$

$$= \lim_{n\to\infty} \frac{6}{9^n (\ln 9)^3} = 0$$

$n^3 \in o(3^{2n}) \to n^3 \in O(3^{2n})$ ✓

b) $n \in o(\log \log n)$ ✗

$$\lim_{n\to\infty} \frac{n}{\log\log n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n \log n (\log 10)^2}} = \lim_{n\to\infty} n \log(n) (\log(10))^2 = \infty$$

$n \in \omega(\log \log n)$

d) $$\lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n} = \lim_{n\to\infty} \frac{\sqrt{n^2(10+\frac{7}{n}+\frac{3}{n^2})}}{n} = \lim_{n\to\infty} \frac{\cancel{n}\sqrt{10+\frac{7}{n}+\frac{3}{n^2}}}{\cancel{n}}$$

$$= \lim_{n\to\infty} \sqrt{10+\frac{7}{n}+\frac{3}{n^2}} = \sqrt{10}$$

$\sqrt{10n^2+7n+3} \in \Theta(n)$