I DIFFERENTIAL EQUATIONS.

Def^n:- An equation which involves derivatives or differentials or differential coefficients, is called a differential eq^n.

Ex.
$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5 = e^x.$$

$$\left[1 + \frac{dy}{dx}\right]^{3/2} + \frac{54}{\left(\frac{dy}{dx}\right)} = xy.$$

differential eq^n

- ordinary diff. eq^n
- partial diff eq^n.

Formation of diff eq^n. ← $\boxed{\text{ordinary diff eq}^n}$

↓

First order first degree diff. eq^n.

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- variable separable method
- exact diff. eq^{n}s
- reducible to exact diff. eq^n
  - rule I
  - rule II
  - rule III
  - rule IV
  - rule V
- linear diff. eq^{n}s
- reducible to linear diff. eq^{n}s.

ordinary diff. eqⁿ:

A diff. eqⁿ in which all the differential coefficients of the eqⁿ are ordinary or all diff. coefficients invove only one independent variable is ordinary diff eqⁿ.

Ex. $\frac{dy}{dx} + 5 = \sin x$

Order of a diff. eqⁿ:

The order of a diff. eqⁿ is the order of the heighest derivative appearing in it.

Ex. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4 = \tan x$. $\longrightarrow$ order $= \underline{\underline{2}}$

$\therefore$ diff. eqⁿ is of order 2 or second order eqⁿ.

$\frac{d^3y}{dx^2} + \frac{dy}{dx} + 4e^x = 0 \longrightarrow$ order $= \underline{\underline{3}}$

$\therefore$ third order diff. eqⁿ.

Degree of a diff. eqⁿ.

The degree or power or index of the highest ordered derivative in the differential eqⁿ, when it is free from radicals and fractions, is known as the degree of a differential eqⁿ.

Ex. i) $x\frac{dy}{dx} + y^2 = \cos x$, order $= 1$ degree $= 1$

~~iii~~ ans. $\longrightarrow$ $\because$ $x\left(\frac{dy}{dx}\right)^1 + y^2 = \cos x$

ii) $(x+y)dx + (3x-2y)dy = 0$

ans. $\longrightarrow$ $(x+y)dx = -(3x-2y)\,dy$

$$\therefore \frac{-(x+y)}{(3x-2y)} = \frac{dy}{dx}$$

$$\therefore \frac{-(x+y)}{(3x-2y)} = \left(\frac{dy}{dx}\right)^1 \quad \therefore \text{order} = 1, \text{degree} = 1$$

iii) $x^2 \frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^3 = \sin x$.

$$\rightarrow x^2\left(\frac{d^2y}{dx^2}\right)^1 + x\left(\frac{dy}{dx}\right)^3 = \sin x \quad \therefore \text{order} = 2, \text{degree} = 1$$

iv) $\frac{dx}{dy} + \frac{d}{\left(\frac{dy}{dx}\right)^2} = e^x$

$$\rightarrow \frac{1}{\frac{dy}{dx}} + \frac{d}{\left(\frac{dy}{dx}\right)^2} = e^x \Rightarrow \frac{\frac{dy}{dx}}{\left(\frac{dy}{dx}\right)^2} + \frac{d}{\left(\frac{dy}{dx}\right)^2} = e^x$$

$$\frac{\frac{dy}{dx} + d}{\left(\frac{dy}{dx}\right)^2} = e^x$$

$$\therefore \frac{dy}{dx} + d = e^x\left(\frac{dy}{dx}\right)^2 \quad \therefore \text{order} = 1 \quad \text{degree} = 2$$

$$\rightarrow \left(\frac{dy}{dx}\right)^1 + d = e^x\left(\frac{dy}{dx}\right)^2$$

v) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = 5\frac{d^2y}{dx^2}$

squaring both sides,

$$\left\{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}\right\}^2 = \left\{5\frac{d^2y}{dx^2}\right\}^2$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{\cancel{2}}\times\cancel{2}} = 25\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 25\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow 1^3+3(1)^2\left(\frac{dy}{dx}\right)^2+3(1)\left\{\left(\frac{dy}{dx}\right)^2\right\}^2+\left\{\left(\frac{dy}{dx}\right)^2\right\}^3 = 25\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow 1+3\left(\frac{dy}{dx}\right)^2+3\left(\frac{dy}{dx}\right)^4+\left(\frac{dy}{dx}\right)^6 = 25\left(\frac{d^2y}{dx^2}\right)^2$$

$\therefore$ order = 2 and degree = 2

Formation Of differential Equation:

Ordinary diff. eqn can be formed by eliminating certain arbitrary constants from its general soln.

If there are n arbitrary constants in the general soln then it is to be differentiated n times and eliminating those n arbitrary constants from these $(n+1)$ eqns, we will get the reqd n^{th} ordered diff. eqn.

Ex. Find D.E. whose general soln is, $y = c_1x + c_2e^x$.

→ given, $y = c_1x + c_2e^x$ —— (1)

here two constants are there c_1 and c_2.

$\therefore$ diff. (1) two times,

$$\frac{dy}{dx} = \frac{d}{dx}(c_1x + c_2e^x)$$

$$\frac{dy}{dx} = c_1 + c_2e^x \quad \text{—— (ii)}$$

again diff.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(c_1 + c_2e^x) = 0 + c_2e^x = c_2e^x \quad \text{—— (iii)}$$

put, $c_2e^x = \frac{d^2y}{dx^2}$ in (ii)

$$\frac{dy}{dx} = c_1 + \frac{d^2y}{dx^2}$$

$$c_1 = \frac{dy}{dx} - \frac{d^2y}{dx^2}$$

∴ from (i)

$$y = \left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\right)x + \frac{d^2y}{dx^2}$$

$$y = x\frac{dy}{dx} - x\frac{d^2y}{dx^2} + \frac{d^2y}{dx^2}$$

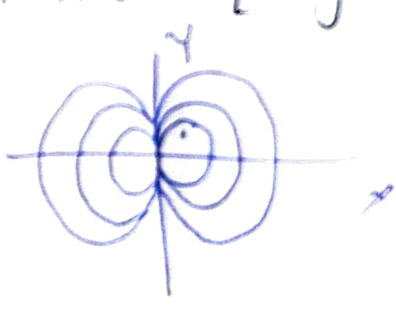
$$y - x\frac{dy}{dx} + x\frac{d^2y}{dx^2} - \frac{d^2y}{dx^2} = 0$$

$$y - x\frac{dy}{dx} + (x-1)\frac{d^2y}{dx^2} = 0$$

this is req^d second order diff eq^n

Q8) Find the D.E. of all circles touching y-axis at the origin and centres on x-axis. ... [May 08, Dec 03]

→ eq^n of such circle is,



$$(x-a)^2 + (y-0)^2 = a^2$$

$$x^2 - 2ax + \cancel{a^2} + y^2 = \cancel{a^2}$$

$$\therefore x^2 + y^2 = 2ax \quad \text{—— (i)}$$

only one constant diff 1 times, eq^n(i)

$$2x + 2y\frac{dy}{dx} = 2a$$

$$\cancel{2}\left(x + y\frac{dy}{dx}\right) = \cancel{2}a$$

$$\therefore a = x + y\frac{dy}{dx}$$

put in(i)

$$x^2 + y^2 = 2\left(x + y\frac{dy}{dx}\right)x$$

$$\therefore x^2 + y^2 = 2x^2 + 2xy\frac{dy}{dx}$$

$$\therefore 2x^2 + 2xy\frac{dy}{dx} - x^2 - y^2 = 0$$

$$\therefore x^2 + 2xy\frac{dy}{dx} - y^2 = 0$$

this is diff. eq^n.

Q3) from D. E. whose general sol^n is, $y = ae^{4x} + be^{3x}$ where a and b are arbitrary constants. [May 03, 11]

→ given G.S. is, $y = ae^{4x} + be^{3x}$ ——(i) $\Rightarrow ae^{4x} + be^{3x} - y = 0$ —— (i)

eliminating arbitrary constants, ∴ diff. 2 times eq^n(i) as their are two arbitrary constants.

$$\cancel{\therefore \frac{dy}{dx} = ae^{4x}\cdot 4 + be^{3x}\cdot 3}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(ae^{4x} + be^{3x}\right)$$

$$= ae^{4x}\cdot 4 + be^{3x}\cdot 3$$

$$\frac{dy}{dx} = 4ae^{4x} + 3be^{3x} \text{ —— (ii)}$$

$$\therefore 4ae^{4x} + 3be^{3x} - \frac{dy}{dx} = 0 \text{ —— (ii)}$$

first degree order is,

$$\frac{dy}{dx} = f(x,y) \quad \text{or,} \quad M + N\frac{dy}{dx} = 0 \quad \text{or,} \quad M\,dx + N\,dy = 0$$

here M and N are constants or functions of x and y

There are 8 methods of solving eqns

1) Variable separable form [VSF]
2) reducible to VSF
3) Homogeneous diff. eqn
4) Non Homogeneous diff. eqn
5) Exact diff eqn.
6) Reducible to exact D.E.
7) Linear diff. eqn. (LDE)
8) Reducible to LDE

steps:
separate $\frac{dy}{dx}$
Take all like terms together like terms with dy

3) Exact Differential Equation:

Definition: A diff. eq^n of the form $M(x,y)dx + N(x,y)dy$ is said to be diff. eq^n if there exists a fu^n $u(x,y)$ such that, $du = Mdx + Ndy = 0$

$\therefore$ its sol^n is $u(x,y) = $ constant.

Theorem I: The necessary and sufficient cond^n for the diff. eq^n $Mdx + Ndy = 0$ to be exact is,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The general sol^n of Exact DE may be obtained by,

$$\int\limits_{x \text{ constants}} N\,dy + \int [\text{Terms of } M \text{ not containing } y]\,dx = c$$

or $\displaystyle\int\limits_{x \text{ cont}} N\,dy + \int\limits_{\text{free } y} M\,dx = c$ or $\displaystyle\int\limits_{y \text{ con}} M\,dx + \int \left[\text{Terms of } N \text{ not containing } x\right] dy = c$

Q1) Solve $(1+xy^2)\,dx + (1+x^2y)\,dy = 0$

consider the form, $Mdx + Ndy = 0$

Here, $M = 1 + xy^2$

$N = 1 + x^2y$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[1 + xy^2\right] = 0 + x \cdot 2y = 2xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[1 + x^2y\right] = \left[0 + y \cdot 2x\right] = 2xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

this is exact DE.

$\therefore$ its G.S. is,

$$\int\limits_{y \text{ con}} M\,dx + \int \left[\text{terms of } N \text{ not containing } x\right] dy = c$$

$\therefore \int_{y\,con} (1+xy^2)\,dx + \int (1)\,dy = c$ [$\because N = 1+x^2y$ and x^2y contains x $\therefore$ neglect it]

$\therefore$ ~~xy~~.

$\therefore \int_{y\,con} 1\,dx + \int_{y\,con} xy^2\,dx + y = c$

$\therefore x + y^2 \int x\,dx + y = c$

$\therefore x + y^2 \times \frac{x^2}{2} + y = c$

$\therefore x + \frac{x^2y^2}{2} + y = c$

This is req^d G.S.

Q2) solve $\frac{dy}{dx} = \frac{x+y-2}{y-x-4}$ — [Dec-14]

$\therefore (y-x-4)\,dy = (x+y-2)\,dx$

$\therefore (x+y-2)\,dx - (y-x-4)\,dy = 0$

$\therefore M\,dx + N\,dy = 0$

here, $M = x+y-2$

$N = -(y-x-4) = -y+x+4$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[x+y-2] = 0+1-0 = \underline{\underline{1}}$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[-y+x+4] = 0+1+0 = \underline{\underline{1}}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ given is exact.

and its sol^n is,

$\int_{y\,con} M\,dx + \int$ [Those terms of N does not cont. x] $dy = c$

$$\Rightarrow \int_{y\,\text{con.}} (x+y-2)\,dx + \int (-y+4)\,dy = c$$

$$\Rightarrow \left[\int_{y\,\text{con}} x\,dx + \int_{y\,\text{con}} y\,dx - \int_{y\,\text{con}} 2\,dx\right] + \int -y\,dy + \int 4\,dy = c$$

$$\Rightarrow \frac{x^2}{2} + y\int dx - 2x - \frac{y^2}{2} + 4y = c$$

$$\Rightarrow \frac{x^2}{2} + xy - 2x - \frac{y^2}{2} + 4y = c$$

This is req^d diff. eq^n.

Q3) solve $\frac{dy}{dx} = \frac{1+y^2+3x^2y}{1-2xy-x^3}$ [May-17, M-4]

$$\rightarrow (1-2xy-x^3)\,dy = (1+y^2+3x^2y)\,dx$$

$$\therefore (1+y^2+3x^2y)\,dx - (1-2xy-x^3)\,dy = 0$$

$$\therefore M\,dx + N\,dy = 0$$

Here, $M = 1+y^2+3x^2y$ &

$N = -(1-2xy-x^3) = -1+2xy+x^3$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[1+y^2+3x^2y] = 0+2y+3x^2 = 3x^2+2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[-1+2xy+x^3] = 0+2y+3x^2 = 3x^2+2y$$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ eq^n is exact.

$\therefore$ sol^n is,

$$\int_{y\,\text{con}} M\,dx + \int [\text{those terms of } N \text{ not containing } x]\,dy = c$$

$$\Rightarrow \int_{y\,\text{con}} (1+y^2+3x^2y)\,dx + \int [-1]\,dy = c$$

$$\therefore \left[\int_{y\,\text{con}} 1\,dx + \int_{y\,\text{con}} y^2\,dx + \int_{y\,\text{con}} 3x^2y\,dx\right] + \int -1\,dy = c$$

$$\therefore x + y^2\int dx + 3y\int x^2\,dx - y = c$$

$$\therefore x + y^2 \cdot x + \cancel{3}y \cdot \frac{x^3}{\cancel{3}} - y = c$$

$$\therefore x + xy^2 + x^3y - y = c$$

Q4) Solve, $\left[\log(x^2+y^2) + \dfrac{2x^2}{x^2+y^2}\right]dx + \dfrac{2xy}{x^2+y^2}\,dy = 0$

[Dec-07, May-08, 10, 13]

→ Here, $M = \log(x^2+y^2) + \dfrac{2x^2}{x^2+y^2}$

$$N = \frac{2xy}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[\log(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right]$$

$$= \frac{1}{x^2+y^2}(0+2y) + 2x^2 \times \frac{\partial}{\partial y}\frac{1}{x^2+y^2}$$

$$= \frac{2y}{x^2+y^2} + 2x^2 \times \frac{-1}{(x^2+y^2)^2} \times (0+2y)$$

$$= \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2}$$

$$= \frac{2y(x^2+y^2)}{(x^2+y^2)^2} - \frac{4x^2y}{(x^2+y^2)^2}$$

$$= \frac{2x^2y + 2y^3 - 4x^2y}{(x^2+y^2)^2} = \frac{2y^3 - 2x^2y}{(x^2+y^2)^2} \quad \text{——— (1)}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{2xy}{x^2+y^2}\right]$$

$$= 2y\frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right]$$

$$= 2y\left\{\frac{(x^2+y^2)\times\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}\right\}$$

$$= 2y\left\{\frac{(x^2+y^2) - x(2x+0)}{(x^2+y^2)^2}\right\}$$

$$= 2y\left\{\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right\}$$

$$= 2y\left\{\frac{y^2-x^2}{(x^2+y^2)^2}\right\} = \frac{2y^3-2x^2y}{(x^2+y^2)^2} \quad \text{—— (11)}$$

thus from (1) and (11) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ eq^n is exact.

$\therefore$ its sol^n is,

$$\int_{y\,con} M\,dx + \int\left[\text{those terms of } N \text{ does not contain } x\right]dy = c$$

$$\Rightarrow \int_{y\,con}\left[\log(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right]dx + \int 0\,dy = c$$

$$\Rightarrow \int_{y\,con}\log(x^2+y^2)\,dx + \int_{y\,con}\frac{2x^2}{x^2+y^2}\,dx = c$$

$$\Rightarrow \int_{y\,con}\underset{u}{[\log(x^2+y^2)]}\times\underset{v}{(1)}\,dx + \int_{y\,con}\frac{2x^2}{x^2+y^2}\,dx = c \quad \text{—— (1)}$$

now we find, by integration by parts ~~$\int u v dx = u \int v dx -$~~

~~$\int \log(x^2+y^2) \times (1)$~~ $\int u v dx = u \int v dx - \int \left[\frac{du}{dx} \times \int v dx\right] dx$

$\therefore \int \log(x^2+y^2) \times (1)\, dx$

y con u v

$= \log(x^2+y^2) \int 1 dx - \int \left[\frac{d}{dx} \log(x^2+y^2) \times \int 1 dx\right] dx$

$= \log(x^2+y^2) \times x - \int \frac{1}{x^2+y^2} \times (2x+0) \times x\, dx$

$= x \log(x^2+y^2) - \int \frac{2x^2}{x^2+y^2} dx$

put it in (1)

$\therefore x \log(x^2+y^2) - \cancel{\int \frac{2x^2}{x^2+y^2} dx} + \cancel{\int \frac{2x^2}{x^2+y^2} dx} = c$

$\Rightarrow x \log(x^2+y^2) = c$. which is reqd G.S.

H.W. $(y^2 e^{xy^2} + 4x^3) dx + (2xy e^{xy^2} - 3y^2) dy = 0$

ans. → $e^{xy^2} + x^4 - y^3 = c$

Q5) solve $\frac{dy}{dx} = \frac{y+1}{(y+2)e^y - x}$

$\rightarrow [(y+2)e^y - x] dy = (y+1) dx$

$\therefore (y+1) dx - [(y+2)e^y - x] dy = 0$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $M = (y+1)$, $N = -[(y+2)e^y - x]$

$= -(y+2)e^y + x$

$\therefore$ solⁿ is,

$\int (y+1) dx + \int -(y+2)e^y dy = c$

y con u v

$$\Rightarrow (y+1)\int dx - \int (y+2)e^y\,dy = c$$

$$\Rightarrow (y+1)x - \left\{(y+2)\int e^y\,dy - \int\left[\frac{d}{dy}(y+2)\times\int e^y\,dy\right]dy\right\} = c$$

$$\Rightarrow (y+1)x - \left\{(y+2)e^y - \int[(1+0)\times e^y]\,dy\right\} = c$$

$$\Rightarrow x(y+1) - (y+2)e^y + \int e^y\,dy = c$$

$$\Rightarrow x(y+1) - (y+2)e^y + e^y = c$$

$$\Rightarrow x(y+1) + e^y(-(y+2)+1) = c$$

$$\Rightarrow x(y+1) + e^y(-y-2+1) = c$$

$$\Rightarrow x(y+1) + e^y(-y-1) = c$$

$$\Rightarrow x(y+1) + e^y[-(y+1)] = c$$

$$\Rightarrow x(y+1) - e^y(y+1) = c$$

$$\Rightarrow (y+1)(x-e^y) = c$$

Solve, $\left[\frac{y^2}{(y-x)^2} - \frac{1}{x}\right]dx + \left[\frac{1}{y} - \frac{x^2}{(x-y)^2}\right]dy = 0$

exact.

$\therefore$ sol^n is,

$$\int_{y\,con}\left(\frac{y^2}{(y-x)^2} - \frac{1}{x}\right)dx + \int \frac{1}{y}\,dy = c$$

$$\left[\int_{y\,con}\frac{y^2}{(y-x)^2}\,dx - \int\frac{1}{x}\,dx\right] + \log y = c$$

$$\left[y^2\int\frac{1}{(y-x)^2}\,dx - \log x\right] + \log y = c$$

$$y^2\times\left(\frac{1}{y-x}\right) - \log x + \log y = c \Rightarrow \frac{y^2}{y-x} + \log y - \log x = c$$

$$\Rightarrow \frac{y^2}{y-x} + \log\left(\frac{y}{x}\right) = c$$

this is req^d G.S.

Q Solve, $\left(\frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}\right)dx - \frac{x}{(x-y)^2}dy = 0$ [May-09]

$$M\,dx + N\,dy = 0$$

$\longrightarrow$ Given eq^n is exact.

$\therefore$ its G.S. is.

$$\int_{y\ \text{con}} \left(\frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}\right)dx = c$$

$$\int_{y\ \text{con}} \frac{y}{(x-y)^2}\,dx - \int \frac{1}{2\sqrt{1-x^2}}\,dx$$

$$\Rightarrow y\int_{y\ \text{con}} \frac{1}{(x-y)^2}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx = c$$

$$\Rightarrow y \times \frac{-1}{(x-y)} - \frac{1}{2}\sin^{-1}x = c$$

$$\Rightarrow \frac{-y}{(x-y)} - \frac{1}{2}\sin^{-1}x = c$$

Reducible to exact diff. eqⁿ

Solve, * $x(x-y)\frac{dy}{dx} = y(x+y)$

[May-18, M4]

Theory :→ A function $f(x,y)$ is said to be integrating factor for the eqⁿ $M\,dx + N\,dy = 0$

If I.F. $(M\,dx + N\,dy) = 0$ is an exact diff. eqⁿ.

In other words, an I.F. is a multiplying factor by which the non-exact DE. can be made exact.

If $M\,dx + N\,dy = 0$ is not an exact D.E. then there are six rules for finding integrating factor.

Rule - I → If $M\,dx + N\,dy = 0$ is homogeneous diff. eqⁿ in x and y and $xM + yN \neq 0$ then the I.F.

$$\star \boxed{I.F. = \frac{1}{xM + yN}}$$

Rule - II → If the form of diff. eqⁿ $M\,dx + N\,dy = 0$ is, $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$ and $xM - yN \neq 0$

then $\boxed{I.F. = \frac{1}{xM - yN}}$

Here, $f_1(xy)$ and $f_2(xy)$ are functions of (xy)

for example, $f_1(xy) = x^2y^2 + xy + 1$

$f_2(xy) = xy - 3$

Rule - III → Let $M\,dx + N\,dy = 0$ be non exact diff. eqⁿ.

If $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = f(x)$ (function of x only)

then I.F. is $e^{\int f(x)dx}$

$$\boxed{I.F.\ e^{\int f(x)dx}}$$

Rule 4 → Let $M\,dx + N\,dy = 0$ be nonexact diff. eqⁿ.

If $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y) \quad [\text{fuⁿ of } y \text{ only}]$$

then, $\boxed{I.F. = e^{\int f(y)\,dy}}$

Rule 5 → If the eqⁿ $M\,dx + N\,dy = 0$ can be written as

$$x^a y^b (my\,dx + nx\,dy) + x^{a_1} y^{b_1} (m_1 y\,dx + n_1 x\,dy) = 0$$

where, $a, b, m, n, a_1, b_1, m_1, n_1$ are all constants. m, n, m_1, n_1 are nonzero and $mn_1 - nm_1 \neq 0$.

Then I.F. will be $x^h y^k$ where h and k are constants such that after multiplying the eqⁿ by I.F., the condⁿ of exactness is satisfied.

$$I.F. = x^h y^k.$$

Rule 6 → Integrating factors found by Inspection:

In a number of cases, the I.F. can be found after recognizing each group as being a part of an exact differential. In this connection the following integrable combⁿ provide quite useful.

Examples:-)

Rule I → Solve, $x(x-y)\frac{dy}{dx} = y(x+y)$ — [May-18, M-4]

$\rightarrow x(x-y)\,dy = y(x+y)\,dx$

$\Rightarrow y(x+y)\,dx - x(x-y)\,dy = 0$ —— (1)

$M\,dx + N\,dy = 0$

$M = y(x+y)$, $N = -x(x-y)$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

because, $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[y(x+y)] = \frac{\partial}{\partial y}[xy + y^2]$

$= \boxed{x + 2y}$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[-x(x-y)] = \frac{\partial}{\partial x}[-x^2+xy]$

$= \frac{\partial}{\partial x}(-x^2) + \frac{\partial}{\partial x}(xy)$

$= -2x+y.$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

and $xM+yN = x[y(x+y)] + y[-x(x-y)]$

$= xy(x+y) - xy(x-y)$

$= \cancel{x^2y} + xy^2 - \cancel{x^2y} + xy^2 = 2xy^2 \neq 0$

$\therefore$ I.F. is, $\frac{1}{2xy^2}$

multiplying with eq^n ①

$\therefore \frac{1}{2xy^2}\left[y(x+y)\,dx - x(x-y)\,dy\right] = 0$

$\therefore \frac{\cancel{y}(x+y)}{2xy^{\cancel{2}}}dx - \frac{\cancel{x}(x-y)}{2\cancel{x}y^2}dy = 0$

$\therefore \frac{x+y}{2xy}dx - \frac{x-y}{2y^2}dy = 0$

$\therefore \left(\frac{\cancel{x}}{2\cancel{x}y} + \frac{\cancel{y}}{2x\cancel{y}}\right)dx + \left(\frac{-x}{2y^2} + \frac{\cancel{y}}{2y^{\cancel{2}}}\right)dy = 0$

$\therefore \left(\frac{1}{2y} + \frac{1}{2x}\right)dx + \left(\frac{-x}{2y^2} + \frac{1}{2y}\right)dy = 0$

$\left[M = \frac{1}{2y} + \frac{1}{2x}\right]$ and $N = \frac{-x}{2y^2} + \frac{1}{2y}$

here, $\frac{\partial M}{\partial y} = \frac{1}{2} \times \frac{-1}{y^2} + 0$, $\frac{\partial N}{\partial x} = \frac{-1}{2y^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

and eq^n becomes exact after multiplying it with I.F.

$\therefore$ G.S. is, $\int\limits_{y\,con} M\,dx + \int\limits_{free\,x} N\,dy = c$

$$\Rightarrow \int\limits_{y\,con}\left(\frac{1}{2y}+\frac{1}{2x}\right)dx + \int \frac{1}{2y}\,dy = c$$

$$\Rightarrow \int\limits_{y\,con}\frac{1}{2y}\,dx + \int\limits_{y\,con}\frac{1}{2x}\,dx + \int\frac{1}{2y}\,dy = c$$

$$\Rightarrow \frac{x}{2y} + \frac{1}{2}\log x + \frac{1}{2}\log y = c$$

$$\Rightarrow \frac{x}{2y} + \frac{1}{2}(\log x + \log y) = c$$

$$\Rightarrow \frac{x}{2y} + \frac{1}{2}\log(xy) = c$$

Q 2) Solve, $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$

$\rightarrow$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact.

$\therefore$ Find I.F. $\rightarrow$ $xM + yN = x(xy - 2y^2) + y[-(x^2 - 3xy)]$

$$= \cancel{x^2y} - 2xy^2 - \cancel{x^2y} + 3xy^2$$

$$= xy^2 \neq 0$$

$\therefore$ I.F. is $\frac{1}{xy^2}$

Q 3) Solve, $x^2y\,dx - (x^3 + y^3)\,dy = 0$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

and Find I.F. $\rightarrow$ $xM + yN = x(x^2y) + y(-(x^3 + y^3))$

$$= \cancel{x^3y} - \cancel{x^3y} - y^4 = -y^4 \neq 0$$

$\therefore$ I.F. is, $\frac{-1}{y^4}$.

Q4) solve, $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eq^n not exact, $M=(x^2y-2xy^2)$, $N=-(x^3-3x^2y)$

$\therefore xM+yN = x(x^2y-2xy^2)+y[-(x^3-3x^2y)]$

$= \cancel{x^3y}-2x^2y^2-\cancel{x^3y}+3x^2y^2 = x^2y^2 \neq 0$

$\therefore$ I.F. is, $\frac{1}{xM+yN} = \frac{1}{x^2y^2}$.

Q5) $(x^2-3xy+2y^2)dx+(3x^2-2xy)dy=0$. [Dec-05]

$M\,dx+N\,dy=0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eq^n is not exact d.e.

$\therefore xM+yN$ is I.F.

as $xM+yN = x(x^2-3xy+2y^2)+y(3x^2-2xy)$

$= x^3-\cancel{3x^2y}+\cancel{2xy^2}+\cancel{3x^2y}-\cancel{2xy^2}$

$\Rightarrow x^3 \neq 0$

$\therefore$ I.F. is $\frac{1}{xM+yN} = \frac{1}{x^3}$

Rule 2) Examples,

Solve, $(1+xy)y\,dx+(1-xy)x\,dy=0$ — (1) [Dec-17, M-4]

$M\,dx+N\,dy=0$

$M=y(1+xy)$

$N=x(1-xy)$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$\therefore$ given D.E. is not exact.

$xM-yN = xy(1+xy)-yx(1-xy)$

$= \cancel{xy}+x^2y^2-\cancel{xy}+x^2y^2$

$= 2x^2y^2 \neq 0$

∴ I.F. is, $\frac{1}{xM - yN} = \frac{1}{2x^2y^2}$ is I.F.

∴ after multiplying $\frac{1}{2x^2y^2}$ with eqⁿ ①, it will become exact.

i.e. $\frac{1}{2x^2y^2}\left[(1+xy)y\,dx + (1-xy)x\,dy\right]$ is exact D.E.

i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $M = \frac{1}{2x^2y^2}(1+xy)y$ and $N = \frac{x(1-xy)}{2x^2y^2}$

$= \frac{1}{2xy^2} - \frac{xy}{2x^2y^2}$

$= \frac{1}{2xy^2} - \frac{1}{2y}$

and its solⁿ is,

$$\int_{y\,\text{con.}} M\,dx + \int_{\text{free } x} N\,dy = c$$

$$\therefore \int_{y\,\text{con}} \frac{1+xy}{2x^2y}\,dx + \int 0\,dy = \cancel{c}$$

$$\therefore \int \frac{1}{2y}\left(\frac{1+xy}{x^2}\right)dy = c$$

$$\therefore \frac{1}{2y}\int\left(\frac{1}{x^2} + \frac{xy}{x^2}\right)dy = c$$

$$\therefore \frac{1}{2y}\int\left(\frac{1}{x^2} + \frac{y}{x}\right)dy = c$$

$$\therefore \frac{1}{2y}\left[-\frac{1}{x} + y\log x\right] = c$$

$$\Rightarrow \int_{y\,\text{con}} \frac{1+xy}{2x^2y}\,dx + \int -\frac{1}{2y}\,dy = c$$

$$\Rightarrow \frac{1}{2y}\int \frac{1+xy}{x^2}\,dx - \int \frac{1}{2y}\,dy = c$$

$$\therefore \frac{1}{2y}\int\left(\frac{1}{x^2} + \frac{xy}{x^2}\right)dx - \int \frac{1}{2y}\,dy = c$$

$$\therefore \cancel{\frac{1}{2y}\left[-\frac{y}{x} + y\log x\right] - \frac{y}{2}\log y}$$

$$\therefore \frac{1}{2y}\left[\int \frac{1}{x^2}\,dx + \int \frac{y}{x}\,dx\right] - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

$$\therefore \frac{1}{2y}\left[-\frac{1}{x} + y\int \frac{1}{x}\,dx\right] - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

$$\therefore \frac{1}{2y}\left[-\frac{1}{x} + y\log x\right] - \frac{1}{2}\log y = c$$

$$\therefore -\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\therefore -\frac{1}{2}\left(\frac{1}{xy} - \log x + \log y\right) = c$$

$\therefore \frac{1}{xy} + \log y - \log x = c$

$\therefore \frac{1}{xy} + \log\left(\frac{y}{x}\right) = c$

Solve, $y(1+xy)dx + x(1+xy+x^2y^2)dy = 0$ —— ①

—[Dec-08, 11]

$\longrightarrow$ $M\,dx + N\,dy = 0$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Here, $Mx - Ny = y(1+xy)x - x(1+xy+x^2y^2)y$

$= xy(1+xy) - xy(1+xy+x^2y^2)$

$= \cancel{xy} + \cancel{x^2y^2} - \cancel{xy} - \cancel{x^2y^2} - x^3y^3 \neq 0$

$\therefore$ I.F. is $\frac{1}{Mx-Ny} = \frac{1}{-x^3y^3} = \frac{-1}{x^3y^3}$

$\therefore$ by multiplying eqn ① with $\frac{-1}{x^3y^3}$ it will become exact.

$\therefore$ i.e. the eqn

$\frac{-1}{x^3y^3}\left[y(1+xy)dx + x(1+xy+x^2y^2)dy\right] = 0$ will become exact.

$\therefore \frac{-1}{x^3y^2}(1+xy)dx - \frac{1}{x^2y^3}(1+xy+x^2y^2)dy = 0$

$\therefore \left(\frac{-1}{x^3y^2} - \frac{1}{x^2y}\right)dx - \left(\frac{1}{x^2y^3} + \frac{1}{xy^2} + \frac{1}{y}\right)dy = 0$

$\therefore$ G.S. is,

$\int\limits_{y\,con} M\,dx + \int\limits_{free\,x} N\,dy = c$

$\Rightarrow \int\limits_{y\,con}\left(\frac{-1}{x^3y^2} - \frac{1}{x^2y}\right)dx + \int\limits_{free\,x} \frac{-1}{y}\,dy = c$

$$\Rightarrow \int_{y\,\text{const}} \frac{-1}{x^3y^2}\,dx + \int_{y\,\text{const}} \frac{-1}{x^2y}\,dx - \int \frac{1}{y}\,dy = c$$

$$\Rightarrow \frac{-1}{y^2}\int \frac{1}{x^3}\,dx - \frac{1}{y}\int \frac{1}{x^2}\,dx - \int \frac{1}{y}\,dy = c$$

$$\Rightarrow \cancel{\frac{-1}{y^2} \times \frac{-1}{x} + \frac{1}{y}\log x - \log y = c}$$

$$\Rightarrow \frac{-1}{y^2} \times \frac{-1}{2} \times \int \frac{-2}{x^3}\,dx - \frac{1}{y} \times -1 \times \int \frac{-1}{x^2}\,dx - \int \frac{1}{y}\,dy = c$$

$$\Rightarrow \frac{-1}{y^2} \times \frac{-1}{2} \times \frac{1}{x^2} - \frac{1}{y} \times -1 \times \frac{1}{x} - \log y = c$$

$$\Rightarrow \frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = c \quad \left[\text{by using, } \int f'(x)\,dx = f(x)\right]$$

is the req^d D.E.

Q Solve, $(x^2y^2+2)y\,dx + (2-2x^2y^2)x\,dy = 0$ —— (1)

$M = y(x^2y^2+2)$

$N = (2-2x^2y^2)x$

Now, $Mx - Ny = xy(x^2y^2+2) - (2-2x^2y^2)xy$

$= xy[(x^2y^2 + \cancel{2} - \cancel{2} + 2x^2y^2)]$

$= xy[3x^2y^2] = 3x^3y^3$

$\therefore$ I.F. is, $\dfrac{1}{Mx-Ny} = \dfrac{1}{3x^3y^3}$

now after multiplying by I.F. eq^n (1) becomes exact,

$$\therefore \frac{1}{3x^3y^3}\left[(x^2y^2+2)y\,dx + (2-2x^2y^2)x\,dy\right] = 0$$

$$\therefore \frac{x^2y^2+2}{3x^3y^2}\,dx + \frac{2-2x^2y^2}{3x^2y^3}\,dy = 0$$

$\therefore M = \frac{x^2y^2+2}{3x^3y^2}$; $N = \frac{2-2x^2y^2}{3x^2y^3}$

$= \frac{\cancel{x^2}\cancel{y^2}}{3x^{\cancel{3}}\cancel{y^2}} + \frac{2}{3x^3y^2}$ $\quad \therefore N = \frac{2}{3x^2y^3} - \frac{2\cancel{x^2}\cancel{y^2}}{3\cancel{x^2}y^{\cancel{3}}}$

$M = \frac{1}{3x} + \frac{2}{3x^3y^2}$ $\quad N = \frac{2}{3x^2y^3} - \frac{2}{3y}$

$\therefore$ Its G.S. is,

$$\underset{y\,con}{\int} M\,dx + \underset{free\,x}{\int} N\,dy = c$$

$$\Rightarrow \underset{y\,con.}{\int}\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int \frac{-2}{3y}\,dy = c$$

$$\Rightarrow \left[\int \frac{1}{3x}dx + \int \frac{2}{3x^3y^2}dx\right] - \frac{2}{3}\int \frac{1}{y}dy = c$$

$$\Rightarrow \left[\frac{1}{3}\int \frac{1}{x}dx + \frac{2}{3y^2}\int \frac{1}{x^3}dx\right] - \frac{2}{3}\int \frac{1}{y}dy = c$$

$$\Rightarrow \left[\frac{1}{3}\int \frac{1}{x}dx + \frac{2}{3y^2} \times \frac{-1}{2} \times \int \frac{-2}{x^3}dx\right] - \frac{2}{3}\int \frac{1}{y}dy = c$$

$$\Rightarrow \frac{1}{3}\log x + \left(\frac{-1}{3}\right) \times \frac{1}{y^2} \times \frac{1}{x^2} - \frac{2}{3}\log y = c$$

$$\Rightarrow \frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$$

$$\Rightarrow \frac{1}{3}\left(\log x - \frac{1}{x^2y^2} - 2\log y\right) = c$$

$$\Rightarrow \log x - \frac{1}{x^2y^2} - 2\log y = 3c = c_1$$

$$\Rightarrow \log x - \frac{1}{x^2y^2} - 2\log y = c_1$$

is reqd D.E.

Solve,

Rule: only $f(x,y)$ must be seen Purely of $f(x,y)$ But Non-homogenous.

$$y(xy+2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0 \quad \text{--- (1)}$$

[Dec-03, May-09]

$$M\,dx + N\,dy = 0$$

as, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eq^n is not exact.

and $Mx - Ny = y(xy+2x^2y^2)x - x(xy - x^2y^2)y$

$$= \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3$$

$$= 3x^3y^3$$

$\therefore$ I.F. is $\frac{1}{3x^3y^3}$

$\therefore$ after multiplying it with eq^n (1) it will become exact.

$$\therefore \frac{1}{3x^3y^3}\left[y(xy+2x^2y^2)\,dx + x(xy - x^2y^2)\,dy\right] = 0$$

$$\therefore \frac{1}{3x^3y^2}(xy+2x^2y^2)\,dx + \frac{1}{3x^2y^3}(xy - x^2y^2)\,dy = 0$$

$$\therefore \left(\frac{xy}{3x^3y^2} + \frac{2x^2y^2}{3x^3y^2}\right)dx + \left(\frac{xy}{3x^2y^3} - \frac{x^2y^2}{3x^2y^3}\right)dy = 0$$

$$\therefore \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

now this is exact.

$\therefore$ its G.S. is,

$$\int_{y\ \text{con}} M\,dx + \int_{\text{free } x} N\,dy = c$$

$$\Rightarrow \int_{y\ \text{con}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c$$

$$\Rightarrow \int \frac{1}{3x^2y}\,dx + \int \frac{2}{3x}\,dx - \int \frac{1}{3y}\,dy = c$$

$$\Rightarrow \frac{1}{3y}\int \frac{1}{x^2}\,dx + \frac{2}{3}\int \frac{1}{x}\,dx - \frac{1}{3}\int \frac{1}{y}\,dy = c$$

$$\Rightarrow \frac{1}{3y} \times -1 \times \int \frac{-1}{x^2}\,dx + \frac{2}{3}\int \frac{1}{x}\,dx - \frac{1}{3}\int \frac{1}{y}\,dy = c$$

$$\Rightarrow \frac{-1}{3y} \times \frac{1}{x} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

$$\Rightarrow \frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

Rule 3 :-

Solve, $(x^4 e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0$ — ①

$\rightarrow$ $M = \underline{x^4 e^x} - 2mxy^2$

$N = 2mx^2y$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eq^n is not exact diff. eq^n.

but, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{[0-(2mx)2y] - [(2my)(2x)]}{2mx^2y}$

$= \frac{-4mxy - 4mxy}{2mx^2y}$

$= \frac{-8\cancel{mxy}}{2\cancel{m}x^{\cancel{2}}\cancel{y}} = \frac{-4}{x}$ function of x only

$= f(x)$

$\therefore$ I.F. $= e^{\int f(x)dx}$

$= e^{\int \frac{-4}{x}dx}$

$= e^{-4\int \frac{1}{x}dx}$

$= e^{-4\log x}$

$= e^{\log x^{-4}}$

$= x^{-4} = \frac{1}{x^4}$.

after multiplying (ii) with I.F. the eq^n becomes exact.

$\therefore$ eq^n (i) becomes exact.

$$\frac{1}{x^4}\left\{[x^4e^x - 2mxy^2]dx + 2mx^2y\,dy\right\} = 0$$

$$\Rightarrow \frac{1}{x^4}(x^4e^x - 2mxy^2)dx + \frac{1}{x^4}\times 2mx^2y\,dy = 0$$

$$\Rightarrow \frac{x^4e^x - 2mxy^2}{x^4}dx + \frac{2mx^2y}{x^4}dy = 0$$

$$\Rightarrow \left(e^x - \frac{2my^2}{x^3}\right)dx + \frac{2my}{x^2}dy = 0$$

now, this is exact diff. eq^n and its G.S. is,

$$\int_{y\,con} M\,dx + \int_{free\,x} N\,dy = c$$

$$\Rightarrow \int_{y\,con}\left(e^x - \frac{2my^2}{x^3}\right)dx + \int 0\,dy = c$$

$$\Rightarrow \int_{y\,con} e^x\,dx - \int_{y\,con}\frac{2my^2}{x^3}dx = c$$

$$\Rightarrow \int e^x\,dx - 2my^2\int\frac{1}{x^3}dx = c$$

$$\Rightarrow e^x - 2my^2\times\frac{-1}{2}\times\int\frac{-2}{x^3}dx = c$$

$$\Rightarrow e^x - \cancel{2}my^2\times\frac{-1}{\cancel{2}}\times\frac{1}{x^2} = c$$

$$\Rightarrow e^x + \frac{my^2}{x^2} = c$$

Solve $\frac{dx}{dy} - e^{x-y} = 4x^3e^{-y}$ [Dec-16, M-4]

$\rightarrow \frac{dx}{dy} = 4x^3e^{-y} + e^{x-y}$

$\Rightarrow dx = (4x^3e^{-y} + e^{x-y})dy$

$\Rightarrow dx - (4x^3e^{-y} + e^{x-y})dy = 0$

$M = 1$, $N = -(4x^3e^{-y} + e^{x-y})$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eq^n is not exact.

now, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{0 \cancel{-}\left[\frac{\partial}{\partial x}(-(4x^3e^{-y} + e^{x-y}))\right]}{\cancel{-}(4x^3e^{-y} + e^{x-y})}$$

$$= -\frac{\frac{\partial}{\partial x}[(4x^3e^{-y} + e^x \cdot e^{-y})]}{(4x^3e^{-y} + e^x \cdot e^{-y})}$$

$$= -\frac{\frac{\partial}{\partial x}[(4x^3 + e^x)e^{-y}]}{(4x^3 + e^x)e^{-y}}$$

$$= -\frac{\cancel{e^{-y}}\frac{\partial}{\partial x}(4x^3 + e^x)}{(4x^3 + e^x)\cancel{e^{-y}}}$$

$$= -\frac{(12x^2 + e^x)}{(4x^3 + e^x)} = f(x)$$ function of x only

$\therefore$ its I.F. is, $e^{\int f(x)dx} = e^{f}$

$$= e^{\int \frac{-(12x^2 + e^x)}{(4x^3 + e^x)}dx} = e^{-\int \frac{12x^2 + e^x}{4x^3 + e^x}dx}$$

using, $\int \frac{f'(x)}{f(x)} dx = \log f(x)$

$$= e^{-\log(4x^3+e^x)} = e^{\log(4x^3+e^x)^{-1}} = (4x^3+e^x)^{-1} = \frac{1}{4x^3+e^x}$$

Q Solve, $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$ [May 14, 15, M-4]

$\rightarrow 2xy\,dy = (x^2+y^2+1)dx$

$\Rightarrow (x^2+y^2+1)dx - 2xy\,dy = 0$ —— ①

$M\,dx + N\,dy = 0$

$M = x^2+y^2+1$ and $N = -2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eqn is not exact.

now,
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-(-2y)}{-2xy} = \frac{2y+2y}{-2xy} = \frac{4\cancel{y}}{-2x\cancel{y}}$$
$$= \frac{-2}{x} = f(x)$$

fun of x only.

$$\therefore \text{I.F. is, } e^{\int f(x)dx} = e^{\int \frac{-2}{x}dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x}$$
$$= e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

I.F. is $\frac{1}{x^2}$, after multiplying it with eqn ① it will become exact.

$$\therefore \frac{1}{x^2}\left[(x^2+y^2+1)dx - 2xy\,dy\right] = 0$$

$$\therefore \frac{x^2+y^2+1}{x^2}dx - \frac{2\cancel{x}y}{x^{\cancel{2}}}dy = 0$$

~~this~~ $\therefore \frac{x^2+y^2+1}{x^2}dx - \frac{2y}{x}dy = 0$

now this is exact and its solⁿ is,

$$\int_{y\,con.} M\,dx + \int_{free\,x} N\,dy = c$$

$$\Rightarrow \int_{y\,con.} \frac{x^2+y^2+1}{x^2}\,dx + \int 0\,dy = c$$

$$\Rightarrow \int_{y\,con.} \left(\frac{x^2}{x^2} + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx = c$$

$$\Rightarrow \int_{y\,con} 1 + \frac{y^2}{x^2} + \frac{1}{x^2}\,dx = c$$

$$\Rightarrow \int_{y\,con} 1\,dx + \int_{y\,con} \frac{y^2}{x^2}\,dx + \int_{y\,con} \frac{1}{x^2}\,dx = c$$

$$\Rightarrow x + y^2 \int \frac{1}{x^2}\,dx + \int \frac{1}{x^2}\,dx = c$$

$$\Rightarrow x + y^2 \times (-1) \times \int \frac{-1}{x^2}\,dx + (-1) \times \int \frac{-1}{x^2}\,dx = c$$

$$\Rightarrow x - y^2 \times \frac{1}{x} + (-1) \times \frac{1}{x} = c$$

$$\Rightarrow x - \frac{y^2}{x} - \frac{1}{x} = c$$

Q solve, $(x^2+y^2+x)\,dx + xy\,dy = 0$ —— (1) DEC-13, M-4.

$$M\,dx + N\,dy = 0$$

$M = x^2+y^2+x$ and $N = xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ eqⁿ is not exact.

∴ we find its int. factor,

∴ its I.F. is, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(0+2y+0)-(y)}{xy} = \frac{2y-y}{xy}$$

$= \frac{y}{xy} = \frac{1}{x} = f(x)$ which is funⁿ of x alone.

and its I.F. is $e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$\therefore$ after multiplying eqⁿ ① with I.F. it will become exact.

$\therefore x[(x^2+y^2+x)dx + xy\,dy] = 0$

$\Rightarrow (x^3+xy^2+x^2)dx + x^2y\,dy = 0$

now this is exact.

and its G.S. is,

$$\int_{y\,con.} M\,dx + \int_{free\ x} N\,dy = c$$

$$\Rightarrow \int_{y\,con} (x^3+xy^2+x^2)dx + \int 0\,dy = c$$

$$\Rightarrow \int_{y\,con} x^3dx + \int_{y\,con} xy^2dx + \int_{y\,con} x^2dx = c$$

$$\Rightarrow \frac{x^4}{4} + y^2\int_{y\,con} x\,dx + \int_{y\,con} x^2dx = c$$

$$\Rightarrow \frac{x^4}{4} + y^2 \times \frac{x^2}{2} + \frac{x^3}{3} = c$$

$$\Rightarrow \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$$

Q solve, $(x\sec^2 y - x^2\cos y)dy = (\tan y - 3x^4)dx$ ~~(x)~~

$(\tan y - 3x^4)dx - (x\sec^2 y - x^2\cos y)dy = 0$ —— ①

$M\,dx + N\,dy = 0$

$M = (\tan y - 3x^4)$, $N = -(x\sec^2 y - x^2\cos y)$

here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eqⁿ is not exact

Rule 4) → Examples :-

Q Solve, $y(2x^2y+e^x)dx = (e^x+y^3)dy$ —①

$\Rightarrow$ ~~$(e^x+y^3)dy$~~

$\Rightarrow y(2x^2y+e^x)dx - (e^x+y^3)dy = 0$

$M = y(2x^2y+e^x) = 2x^2y^2 + ye^x$

$N = -(e^x+y^3)$

here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eq^n is not exact.

$\therefore$ we find I.F.

$$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-(e^x+0) - (2x^2 \times 2y + e^x)}{y(2x^2y+e^x)}$$

$$= \frac{-e^x - 4x^2y - e^x}{y(2x^2y+e^x)} = \frac{-2e^x - 4x^2y}{y(2x^2y+e^x)}$$

$$= \frac{-2\cancel{(e^x+2x^2y)}}{y\cancel{(2x^2y+e^x)}} = \frac{-2}{y} = f(y)$$

fun^n of y only.

$$\therefore \text{I.F. is, } e^{\int f(y)dy} = e^{\int \frac{-2}{y}dy} = e^{-2\int \frac{1}{y}dy}$$

$$= e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

$\therefore$ after multiplying by I.F. it will become exact.

$$\therefore \frac{1}{y^2}\left\{ y(2x^2y+e^x)dx - (e^x+y^3)dy \right\} = 0$$

$\therefore \frac{y}{y^2}(2x^2y+e^x)dx - \frac{e^x+y^3}{y^2}dy=0$

$\Rightarrow \frac{2x^2y+e^x}{y}dx - \left(\frac{e^x}{y^2}+\frac{y^3}{y^2}\right)dy=0$

$\Rightarrow \left(2x^2+\frac{e^x}{y}\right)dx - \left(\frac{e^x}{y^2}+y\right)dy=0$ it is now exact

$\therefore$ its G.S. is,

$\int\limits_{y\,con} M\,dx + \int\limits_{free\,x} N\,dy = c$

$\Rightarrow \int\limits_{y\,con}\left(2x^2+\frac{e^x}{y}\right)dx + \int -y\,dy = c$

$\Rightarrow \frac{2x^3}{3}+\frac{1}{y}\cdot e^x - \frac{y^2}{2}=c$

$\Rightarrow \frac{2x^3}{3}+\frac{e^x}{y}-\frac{y^2}{2}=c$

Solve, $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$ —— [Dec 09, 11, M-4]

$M\,dx+N\,dy=0$

$M=y^4+2y$ and

$N=xy^3+2y^4-4x$

here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eq^n is not exact.

$\therefore$ we find I.F.

$\therefore \frac{\cancel{\frac{\partial M}{\partial y}}\,\cancel{-}\,\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{\cancel{N}\; M} = \frac{\cancel{y^3+0-4-(4y^3+2)}}{\cancel{xy^3+2y^4-4x}}$

$\therefore \frac{y}{y^2}(2x^2y+e^x)dx - \frac{e^x+y^3}{y^2}dy=0$

$\Rightarrow \frac{2x^2y+e^x}{y}dx - \left(\frac{e^x}{y^2}+\frac{y^3}{y^2}\right)dy=0$

$\Rightarrow \left(2x^2+\frac{e^x}{y}\right)dx - \left(\frac{e^x}{y^2}+y\right)dy=0$ it is now exact

$\therefore$ its G.S. is,

$\int\limits_{y\,\text{con}} M\,dx + \int\limits_{\text{free }x} N\,dy = c$

$\Rightarrow \int\limits_{y\,\text{con}} \left(2x^2+\frac{e^x}{y}\right)dx + \int -y\,dy = c$

$\Rightarrow \frac{2x^3}{3}+\frac{1}{y}\cdot e^x - \frac{y^2}{2}=c$

$\Rightarrow \frac{2x^3}{3}+\frac{e^x}{y} - \frac{y^2}{2}=c$

Solve, $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0$ — [Decog, 11, M-4]

$M\,dx+N\,dy=0$

$M=y^4+2y$ and

$N=xy^3+2y^4-4x$

here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ eqn is not exact.

$\therefore$ we find I.F.

$\therefore \frac{\cancel{\frac{\partial M}{\partial y}} - \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \cancel{\frac{y^3+0-4-(4y^3+2)}{xy^3+2y^4-4x}}$

$$\Rightarrow \frac{(y^3+0-4)-(4y^3+2)}{y^4+2y} = \frac{y^3-4-4y^3-2}{y^4+2y} = \frac{-3y^3-6}{y(y^3+2)}$$

$$= \frac{-3\cancel{(y^3+2)}}{y\cancel{(y^3+2)}} = \frac{-3}{y} = f(y) \text{ it is fun}^n \text{ of } y \text{ only.}$$

$\therefore$ I.F. is,

$$e^{\int f(y)dy} = e^{\int \frac{-3}{y}dy} = e^{-3\int \frac{1}{y}dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

after multiplying ① with I.F. it will become exact.

$$\therefore \frac{1}{y^3}\left\{(y^4+2y)dx + (xy^3+2y^4-4x)dy\right\} = 0$$

$$\therefore \frac{y^4+2y}{y^3}dx + \frac{xy^3+2y^4-4x}{y^3}dy = 0$$

$$\Rightarrow \left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0$$

now this is exact and its G.S. is,

$$\int_{y\,con.} M dx + \int_{free\ x} N dy = c$$

$$\Rightarrow \int_{y\,con} \left(y+\frac{2}{y^2}\right)dx + \int 2y\,dy = c$$

$$\Rightarrow \left(y+\frac{2}{y^2}\right)\times x + \cancel{2}\frac{y^2}{\cancel{2}} = c$$

$$\Rightarrow x\left(y+\frac{2}{y^2}\right) + y^2 = c$$

Solve, $y\log y\,dx + (x-\log y)\,dy = 0$ — Dec-10.

$\rightarrow M = y\log y$

$N = x - \log y$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ eqn is not exact.

$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1-[y.\frac{1}{y}+\log y]}{y\log y} = \frac{\cancel{1}-\cancel{1}-\cancel{\log y}}{y\,\cancel{\log y}}$

$= \frac{-1}{y} = f(y)$

$\therefore$ I.F. is, $e^{\int f(y)dy} = e^{\int \frac{-1}{y}dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y}$

$= e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$

now,

$\frac{1}{y}[y\log y\,dx + (x-\log y)\,dy] = 0$

is exact,

$\frac{1}{\cancel{y}} \times \cancel{y}\log y\,dx + \frac{1}{y}(x-\log y)\,dy = 0$

$\Rightarrow \log y\,dx + \frac{1}{y}(x-\log y)\,dy = 0$

$\Rightarrow \log y\,dx + \left(\frac{x}{y} - \frac{\log y}{y}\right)dy = 0$

$\therefore$ its soln is,

$\int_{y\,\text{con}} M\,dx + \int_{x\,\text{free}} N\,dy = c$

$\Rightarrow \int_{y\,\text{con}} \log y\,dx + \int \frac{-\log y}{y}\,dy = c$

$\Rightarrow \log y \int_{y\,\text{con}} 1\,dx - \int \frac{\log y}{y}\,dy = c$ — (1)

Solve, $y\log y\,dx + (x-\log y)\,dy = 0$ —— Dec-10.

$\rightarrow M = y\log y$

$N = x - \log y$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ eq^n is not exact.

$$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1-[y\cdot\frac{1}{y}+\log y]}{y\log y} = \frac{\cancel{1}-\cancel{1}-\cancel{\log y}}{y\cancel{\log y}}$$

$$= \frac{-1}{y} = f(y)$$

$\therefore$ I.F. is, $e^{\int f(y)dy} = e^{\int \frac{-1}{y}dy} = e^{-\int\frac{1}{y}dy} = e^{-\log y}$

$= e^{\log y^{-1}}$

$= y^{-1} = \frac{1}{y}$

now,

$\frac{1}{y}[y\log y\,dx + (x-\log y)\,dy] = 0$

is exact,

$\frac{1}{\cancel{y}}\times\cancel{y}\log y\,dx + \frac{1}{y}(x-\log y)\,dy = 0$

$\Rightarrow \log y\,dx + \frac{1}{y}(x-\log y)\,dy = 0$

$\Rightarrow \log y\,dx + \left(\frac{x}{y} - \frac{\log y}{y}\right)dy = 0$

$\therefore$ its sol^n is,

$$\int_{y\,con} M\,dx + \int_{x\,free} N\,dy = c$$

$$\Rightarrow \int_{y\,con} \log y\,dx + \int \frac{-\log y}{y}\,dy = c$$

$$\Rightarrow \log y\int_{y\,con} 1\,dx - \int\frac{\log y}{y}\,dy = c \quad \text{—— (1)}$$

using $\int [f(x)]^n f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}$

$\therefore \int \dfrac{\log y}{y}\,dx = \int \underbrace{[\log y]^1}_{[f(x)]^1} \times \underbrace{\dfrac{1}{y}}_{f'(x)}\,dy = \dfrac{(\log y)^{1+1}}{1+1} = \dfrac{(\log y)^2}{2}$

put it in ①

$\therefore (\log y)(x) - \dfrac{(\log y)^2}{2} = c$

Solve, $(2x + e^x \log y) y\,dx + e^x\,dy = 0$ ——①

$M = (2x + e^x \log y) y = 2xy + e^x y \log y$

$N = e^x$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

$\therefore \dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \dfrac{\cancel{e^x - (0 + e^x \times \frac{1}{y})}}{\cancel{2x + e^x \log y}}$

$= \dfrac{e^x - [2x \cdot 1 + e^x (\cancel{y} \times \frac{1}{\cancel{y}} + \log y)]}{(2x + e^x \log y) y}$

$= \dfrac{e^x - [2x + e^x (1 + \log y)]}{y(2x + e^x \log y)}$

$= \dfrac{\cancel{e^x} - 2x - \cancel{e^x} - e^x \log y}{y(2x + e^x \log y)}$

$= \dfrac{-\cancel{(2x + e^x \log y)}}{y\cancel{(2x + e^x \log y)}} = \underline{\underline{\dfrac{-1}{y}}} = f(y)$ which is the fun of y only.

$\therefore$ its I.F. is, $e^{\int f(y)\,dy} = e^{\int \frac{-1}{y}\,dy} = e^{-\int \frac{1}{y}\,dy} = e^{-\log y}$

$= e^{\log y^{-1}} = y^{-1} = \dfrac{1}{y}$

after multiplying by it eqn (1) becomes exact.

$\therefore \frac{1}{y}\left\{(2x+e^x \log y)\, y\, dx + e^x\, dy\right\} = 0$ is exact.

$\therefore$ ~~it's d.e. is~~,

$\Rightarrow \frac{(2x+e^x \log y)\cancel{y}}{\cancel{y}}\, dx + \frac{e^x}{y}\, dy = 0$ is exact

$\therefore$ its soln is,

$$\int_{y\,con} M\, dx + \int_{x\,free} N\, dy = c$$

$$\Rightarrow \int_{y\,con} (2x+e^x \log y)\, dx + \int 0\, dy = c$$

$$\Rightarrow \int_{y\,con} 2x\, dx + \int_{y\,con} e^x \log y\, dx = c$$

$$= \frac{\cancel{2}x^2}{\cancel{2}} + \log y \int e^x\, dx = c$$

$$\Rightarrow x^2 + (\log y)\, e^x = c$$

(L.D.E)

• Linear Differential eqⁿ:-

Defⁿ:- i) The eqⁿ of the form, $\boxed{\frac{dy}{dx} + p(x)y = \varphi(x)}$ — ① [eqⁿ linear in y]

where, $p(x)$ and $\varphi(x)$ are functions of x only or constants and y is a dependent variable.

~~Theorem I~~: The general solⁿ of eqⁿ ① is,

$$y e^{\int p\,dx} = \int \varphi \cdot e^{\int p\,dx}\, dx + c$$

i.e. $y.(I.F.) = \int \varphi . (I.F)\, dx + c$

here I.F. is, $e^{\int p(x)\,dx}$

ii) linear in x

$$\frac{dx}{dy} + p(y)x = \varphi(y)$$

$$I.F. = e^{\int p(y)\,dy}$$

now its G.S. is,

$$x.(I.F) = \int \varphi(y)(I.F.)\,dy + c$$

i.e. $x \cdot e^{\int P(y) dy} = \int \phi(y) \cdot e^{\int P(y) dy} dy + c$

working rule

To solve LDE of the first order, we consider the following steps,

Step I: Standard form:

Arrange the given DE in standard form as,

$$\frac{dy}{dx} + P(x) y = \phi(x) \text{ or,}$$

$$\frac{dx}{dy} + P(y) x = \phi(y)$$

Step II Integrating factor:

Step III General solⁿ.

Examples:

Solve, $\frac{dy}{dx} + y \cot x = \sin 2x$ ——— Dec-18, Q-4

Step I: standard form,

$\frac{dy}{dx} + [\cot x] y = \sin 2x$ comparing with, $\frac{dy}{dx} + p(x) y = \phi(x)$

where, $p(x) = \cot x$, $\phi(x) = \sin 2x$

$\therefore$ D.E. (1) is linear in y

$\therefore$ I.F. is, $e^{\int p(x) dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

$\therefore$ G.S. is,

$$y (I.F.) = \int \phi(x) . (I.F.) dx + c$$

$$\therefore y \times \sin x = \int \sin 2x \times \sin x \, dx + c$$

$$= \int 2 \sin x . \cos x . \sin x \, dx + c$$

$= \int 2\sin^2 x\,(\cos x)\,dx + c$

$= 2\int \sin^2 x\,(\cos x)\,dx + c$

$= 2 \times \frac{1}{3} \times \int 3\sin^2 x \times \cos x\,dx + c$

$= \frac{2}{3} \times \int d\left(\sin^3 x\right) + c$ using, $\int f'(x)dx = f(x)+c$

$= \frac{2}{3} \times \sin^3 x + c$

$y\sin x = \frac{2\sin^3 x}{3} + c$

$y = \frac{2\sin^3 x + c}{3\sin x} = \frac{2\sin^3 x}{3\sin x} + \frac{c}{3\sin x}$

$= \frac{2}{3}\sin^2 x + c_1 \operatorname{cosec} x$

This is req^d g.s.

Solve, $\cos x \frac{dy}{dx} + y = \sin x$. [May-18, M-4]

$\longrightarrow \frac{dy}{dx} + \frac{y}{\cos x} = \frac{\sin x}{\cos x}$

$\Rightarrow \frac{dy}{dx} + \sec x\, y = \tan x$

comp. with, $\frac{dy}{dx} + p(x)y = Q(x)$

$p(x) = \sec x$, $Q(x) = \tan x$.

$\therefore$ I.F. is $e^{\int p(x)dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)}$

$= (\sec x + \tan x)$

$\therefore$ Q.S, is.

$y\ I.F. = \int Q(x) \times I.F\, dx + c$

$\therefore y(\sec x + \tan x) = \int \tan x (\sec x + \tan x)\, dx + c$

$= \int (\tan x \times \sec x + \tan^2 x)\, dx + c$

$= \int \tan x \sec x\, dx + \int \tan^2 x\, dx + c$

$= \int d(\sec x)\cancel{d} + \int (\cancel{1+} \sec^2 x - 1)\, dx + c$

$= \sec x + \int \sec^2 x\, dx - \int 1\, dx + c$

$= \sec x + \int d(\tan x) - x + c$

$= \sec x + \tan x - x + c.$

Solve, $(1+y^2) + (x - e^{-\tan^{-1} y}) \dfrac{dy}{dx} = 0$

$\rightarrow \cancel{(1+y^2) + x \dfrac{dy}{dx} + e^{-\tan^{-1} y} \dfrac{dy}{dx} = 0}$

$(1+y^2) = -(x - e^{-\tan^{-1} y}) \dfrac{dy}{dx}$

$\therefore (1+y^2) \dfrac{dx}{dy} = -(x - e^{-\tan^{-1} y})$

$\therefore (1+y^2) \dfrac{dx}{dy} = -x + e^{-\tan^{-1} y}$

$\therefore (1+y^2) \dfrac{dx}{dy} + x = e^{-\tan^{-1} y}$

$\therefore \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{e^{-\tan^{-1} y}}{1+y^2}.$

$\therefore \dfrac{dx}{dy} + \left(\dfrac{1}{1+y^2}\right) x = \dfrac{e^{-\tan^{-1} y}}{1+y^2}$

$\therefore \frac{dx}{dy} + P(y)\,x = Q'(y)$

$P(y) = \frac{1}{1+y^2}$, $Q(y) = \frac{e^{-\tan^{-1}y}}{1+y^2}$

$\therefore$ I.F. is, $e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

$\therefore$ G.S. is,

$x \cdot \text{I.F.} = \int Q(y)\,\text{I.F.}\,dy + c$

$x \cdot e^{\tan^{-1}y} = \int \frac{e^{-\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y}\,dy + c$

$= \int \frac{e^{-\tan^{-1}y} \times e^{\tan^{-1}y}}{1+y^2}\,dy + c$

$= \int \frac{dy}{1+y^2} + c$

$\therefore x \cdot e^{\tan^{-1}y} = \tan^{-1}y + c$ is the G.S.

Q solve, $x\cos x \frac{dy}{dx} + (\cos x - x\sin x)y = 1$.

standard form,

$\frac{dy}{dx} + \frac{(\cos x - x\sin x)}{x\cos x}\,y = \frac{1}{x\cos x}$

where, $P(x) = \frac{\cos x - x\sin x}{x\cos x}$ and $Q(x) = \frac{1}{x\cos x}$

I.F. is, $e^{\int P(x)dx} = e^{\int \frac{\cos x - x\sin x}{x\cos x}dx} = e^{\int \frac{d(x\cos x)}{x\cos x}}$

$= e^{\log|x\cos x|} = x\cos$

using $\int \frac{f'(x)}{f(x)} dx = \log f(x)$

$\therefore$ G.S. is,

$y.\text{I.F.} = \int Q(x) \times \text{I.F.}\, dx + c$

$\therefore y \times (x \cos x) = \int \frac{1}{\cancel{x \cos x}} \times \cancel{x \cos x}\, dx + c$

$\therefore x y \cos x = \int dx + c$

$\therefore x y \cos x = x + c$

Solve, $y \log y\, dx + (x - \log y) dy = 0$ [May-16 M-4]

here,

$y \log y \frac{dx}{dy} + (x - \log y) = 0$

$\therefore y \log y \frac{dx}{dy} + x = \log y$

$\therefore \frac{dx}{dy} + \left(\frac{1}{y \log y}\right) x = \frac{\log y}{y \log y}$

$\frac{dx}{dy} + p(y)\, x = \cancel{\log e}\; q(y)$

where, $p(y) = \frac{1}{y \log y}$, $q(y) = \frac{\cancel{\log y}}{y \cancel{\log y}} = \frac{1}{y}$

$\therefore$ I.F. is $e^{\int p(y) dy} = e^{\int \frac{1}{y \log y} dy} = e^{\int \frac{\frac{1}{y}}{\log y} dy}$

$\therefore e^{\int \frac{d(\log y)}{\log y}} = e^{\log(\log y)} = \underline{\underline{\log y}}$

$\therefore$ G.S. is,

$x.\text{I.F.} = \int Q(y).\text{I.F.}\, dy + c$

$\Rightarrow x.\log y = \int \frac{1}{y} \cancel{\log y} \times \log y\, dy + c$

$\Rightarrow x\log y = \int \frac{\log y}{y} dy + c$

$= \int (\log y)\left(\frac{1}{y}\right) + c$

$x\log y = \frac{(\log y)^2}{2} + c$ using, $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$

here, $f(x) = \log y$.

solve, $(1-x^2)\frac{dy}{dx} = 1 + xy$ — [Dec-14, M-4]

$\rightarrow \frac{dy}{dx} = \frac{1}{1-x^2} + \frac{xy}{1-x^2}$

$\Rightarrow \frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2}$

$\Rightarrow \cancel{\frac{dy}{dx} + \frac{xy}{}}$

$\Rightarrow \frac{dy}{dx} + \left(\frac{-x}{1-x^2}\right)y = \frac{1}{1-x^2}$

$\frac{dy}{dx} + P(x)y = Q(x)$

$\therefore$ I.F. is $e^{\int P(x)dx} = e^{\int \frac{-x}{1-x^2}dx} = e^{\frac{1}{2}\int \frac{-2x}{1-x^2}dx}$

$= e^{\frac{1}{2}\int \frac{d(1-x^2)}{1-x^2}} = e^{\frac{1}{2}\times\log(1-x^2)}$

$= e^{\log(1-x^2)^{1/2}} = (1-x^2)^{1/2}$

$\therefore$ its G.S. is,

$y.\text{I.F.} = \int Q(x).\times \text{I.F.}dx + c$

$\therefore y\sqrt{1-x^2} = \int \frac{1}{1-x^2} \times \sqrt{1-x^2}\,dx + c$

$= \int \frac{1}{\sqrt{1-x^2}} dx + c \Rightarrow y\sqrt{1-x^2} = \sin^{-1}x + c$

Q Solve, $\frac{dy}{dx} + \frac{4x}{1+x^2} y = \frac{1}{(x^2+1)^3}$ [May-11 M-4]

$\rightarrow$ $\frac{dy}{dx} + P(x) y = Q(x)$

here, $P(x) = \frac{4x}{1+x^2}$ and $Q(x) = \frac{1}{(x^2+1)^3}$

Standard form:

Given D.E. is L.D.E. in y.

Its integrating factor is,

$$I.F. = e^{\int P(x) dx}$$

$$= e^{\int \frac{4x}{1+x^2} dx}$$

$$= e^{2\int \frac{2x}{1+x^2} dx}$$

$$= e^{2\log(1+x^2)} = e^{\log(1+x^2)^2} = (1+x^2)^2$$

$\therefore$ its general solⁿ is,

$$y \cdot I.F. = \int Q(x) \cdot I.F. \, dx + c$$

$$\therefore y \cdot (1+x^2)^2 = \int \frac{1}{(x^2+1)^3} \times (1+x^2)^2 dx + c$$

$$\therefore y \cdot (1+x^2)^2 = \int \frac{1}{x^2+1} dx + c$$

$$\therefore y \cdot (1+x^2)^2 = \tan^{-1} x + c$$

which is reqᵈ G.S.

Reducible to the Linear form:- (Bernoulli's Equation

1. Bernaulli's diff. eqⁿ ing.

A differential equation of the form

$$\frac{dy}{dx} + p(x)y = \varphi(x)y^{n} \quad \text{——} \quad (i)$$

is called Bernoulli's DE in y.

Method of solⁿ :-

1. Dividing by y^n on both sides of eqⁿ(1) we get,

$$\frac{\frac{dy}{dx} + p(x)y}{y^n} = \frac{\varphi(x)\cancel{y^n}}{\cancel{y^n}}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = \varphi(x)$$

Substitute difficult term.

put, $y^{1-n} = u$

$$\therefore (1-n)y^{\cancel{1}-n-\cancel{1}}\frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

$$\Rightarrow \frac{1}{1-n}\frac{du}{dx} + p(x)u = \varphi(x)$$

$$\Rightarrow \frac{du}{dx} + (1-n)p(x)u = (1-n)\varphi(x)$$

This is linear in u and can be solved by the method of LDE.

Bernoulli's Differential equation in x:

General form,

$$\cancel{\frac{dx}{dy} + p(x)x}\ \frac{dx}{dy} + p(y)x = \varphi(y)x^n.$$

$$\therefore x^{-n}\frac{dx}{dy} + p(y)x^{1-n} = \varphi(y)$$

put, $x^{1-n} = u$ and proceed as above.

3. If the form of differential eq^n is,

$$f'(y)\frac{dy}{dx} + p(x)f(y) = \varphi(x)$$

put, $f(y) = u \Rightarrow f'(y)\frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow \frac{du}{dx} + p(x)u = \varphi(x)$ This is linear in u.

4. Similarly for $f'(x)\frac{dx}{dy} + p(y)f(x) = \varphi(y)$

put, $f(x) = u$

$\therefore f'(x)\frac{dx}{dy} = \frac{du}{dy}$

$\therefore f'(x)\frac{dx}{dy} + p(y)f(x) = \varphi(y)$

$\Rightarrow \frac{du}{dy} + p(y) \cdot u = \varphi(y)$ This is linear in u.

Solve, $xy - \frac{dy}{dx} = y^3 e^{-x^2}$. [May-2019, M-4]

reduce to LDE.

we have,

$$-\frac{dy}{dx} + xy = y^3 e^{-x^2}$$

$\Rightarrow \frac{dy}{dx} - xy = -y^3 e^{-x^2} \Rightarrow \frac{dy}{dx} + p(x)y = \varphi(x)y^n$ form.

$\Rightarrow$ ~~d~~ devide by y^3 on both sides,

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} - \frac{xy}{y^3} = -\frac{\cancel{y^3}e^{-x^2}}{\cancel{y^3}}$$

$$\Rightarrow y^{-3}\frac{dy}{dx} - \frac{x}{y^2} = -e^{-x^2}$$

$$\Rightarrow y^{-3}\frac{dy}{dx} - xy^{-2} = -e^{-x^2} \quad \text{— (1)}$$

put, $y^{-2} = u$

$\Rightarrow -2y^{-2-1} \times \frac{dy}{dx} = \frac{du}{dx} \Rightarrow -2y^{-3} \times \frac{dy}{dx} = \frac{du}{dx} \Rightarrow y^{-3}\frac{dy}{dx} = \frac{-1}{2}\frac{du}{dx}$

put in ①

$\therefore \frac{-1}{2}\frac{du}{dx} - xu = -e^{-x^2}$.

$\therefore$ multiply by -2 on both sides,

$\Rightarrow \frac{du}{dx} - (-2)xu = -(-2)e^{-x^2}$

$\Rightarrow \frac{du}{dx} + 2xu = 2e^{-x^2}$.

form, $\frac{du}{dx} + p(x)u = Q(x)$ here, $p(x) = 2x$ and $Q(x) = 2e^{-x^2}$.

$\therefore$ I.F. is, $e^{\int p(x)dx} = e^{\int 2x\,dx} = e^{2x^2/2} = e^{x^2}$.

$\therefore$ its G.S. is,

$u \cdot I.F. = \int Q(x) \times I.F.\, dx + c$

$= \int 2e^{-x^2} \times e^{x^2} dx + c$

$= \int 2\,dx + c$

$\Rightarrow u \cdot e^{x^2} = 2x + c$

$\Rightarrow y^{-2} \times e^{x^2} = 2x + c$

$\Rightarrow \frac{e^{x^2}}{y^2} = 2x + c$.

This is req^d general sol^n.

Solve $x\frac{dy}{dx} + y = y^2 \log x$. [May-15]

Reduce to LDE.

we have, $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2 \log x}{x}$

form, $\frac{dy}{dx} + p(x)y = q(x)y^n$

Here, $p(x) = \frac{1}{x}$ and $q(x) = \frac{\log x}{x}$ & $n = 2$

devide by y^2 on both sides,

$$\therefore \frac{\frac{dy}{dx} + \frac{1}{x} y}{y^2} = \frac{\frac{\log x}{x} \cancel{y^2}}{\cancel{y^2}}$$

$$\therefore y^{-2}\frac{dy}{dx} + y^{1-2}\frac{1}{x} = \frac{\log x}{x}$$

$$\therefore y^{-2}\frac{dy}{dx} + y^{-1}\frac{1}{x} = \frac{\log x}{x} \quad \text{---} (1)$$

put $y^{-1} = u$

diff. w.r.t. x

$$\therefore \frac{d}{dx}(y^{-1}) = \frac{du}{dx}$$

$$\therefore -1 y^{-1-1} \frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore y^{-2}\frac{dy}{dx} = \frac{-du}{dx}$$

put in (1)

$$\therefore \frac{-du}{dx} + u \times \frac{1}{x} = \frac{\log x}{x}$$

$$\therefore \frac{du}{dx} - \frac{1}{x}u = \frac{\log x}{x}$$

form, $\frac{du}{dx} + p(x)u = Q(x)$

$p(x) = \frac{-1}{x}$ and $Q(x) = \frac{\log x}{x}$

and its I.F. is, $e^{\int p(x)dx} = e^{\int \frac{-1}{x}dx} = e^{-\int \frac{1}{x}dx}$

$$= e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

and its G.S. is,

$$u \cdot I.F. = \int Q(x) \cdot I.F. \times dx + c$$

$$u \cdot \frac{1}{x} = \int \frac{\log x}{x} \times \frac{1}{x} dx + c \quad \text{---} (1)$$

put $\log x = v$ ... $x = e^v$

diff w.r.t x

$$\frac{d}{dx}(\log x) = \frac{dv}{dx}$$

$$\frac{1}{x} = \frac{dv}{dx}$$

$$\frac{1}{x}dx = dv$$

from ①,

$$y^{-1}\frac{1}{x} = \int \frac{v}{e^v}dv + c$$

$$\Rightarrow \frac{1}{yx} = \int ve^{-v}dv + c$$

$$= v\int e^{-v}dv - \int\left[\frac{dv}{dv} \times \int e^{-v}dv\right]dv + c$$

$$= \frac{v.e^{-v}}{-1} - \int \frac{e^{-v}}{-1}dv + c$$

$$= -ve^{-v} + \int e^{-v}dv + c$$

$$= -ve^{-v} + \frac{e^{-v}}{-1} + c$$

$$= -ve^{-v} - e^{-v} + c$$

$$= -e^{-v}(v+1) + c$$

$$= -\frac{1}{e^v}(v+1) + c$$

$$\Rightarrow \frac{1}{yx} = -\frac{1}{x}(1 + \log x) + c$$

Solve, $(x^3y^3 + xy)\frac{dy}{dx} = 1$ [May-05, Dec 09]

→ Reduce to LDE.

Let, $(x^3y^3 + xy)\frac{dy}{dx} = 1$

$$\cancel{\frac{dy}{dx}}\ x^3y^3 + xy = \frac{dx}{dy}$$

$$\frac{dx}{dy} - yx = x^3y^3 \quad \text{①}$$

$$= \cancel{\frac{1}{2}}\left\{ t\int e^t dt - \int\left(\frac{dt}{dt} \times \int e^t dt\right) dt \right\} + c$$

$$= \cancel{\frac{1}{2}}\left\{ te^t - \int 1 \times e^t dt \right\} + c$$

$$= \cancel{\frac{1}{2}}\left\{ te^t - e^t \right\} + c$$

$$\therefore ue^{y^2} = \cancel{\frac{1}{2}}\left\{ y^2 e^{y^2} - e^{y^2} \right\} + c$$

$$\therefore -x^{-2} e^{y^2} = \cancel{\frac{1}{2}} \times (y^2 - 1) e^{y^2} + c$$

$$\therefore -x^{-2} = \cancel{\frac{1}{2}} \times (y^2 - 1) \frac{\cancel{e^{y^2}}}{\cancel{e^{y^2}}} + \frac{c}{e^{y^2}}$$

$$= \cancel{\frac{y^2 - 1}{2}} \cancel{+ ce^{-y^2}}$$

$$\therefore -\frac{1}{x^2} = (y^2 - 1) + ce^{-y^2}$$

Solve: $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$ — May-12

→ Reduce to LDE:

Dividing by $\sec y$, we have,

$$\frac{1}{\sec y}\left[\frac{dy}{dx} - \frac{\tan y}{1+x}\right] = \frac{1}{\cancel{\sec y}}(1+x) e^x \cancel{\sec y}.$$

$$\Rightarrow \cos y \frac{dy}{dx} - \cos y \times \frac{\tan y}{1+x} = (1+x) e^x$$

$$\Rightarrow \cos y \frac{dy}{dx} - \cancel{\cos y} \times \frac{\frac{\sin y}{\cancel{\cos y}}}{1+x} = (1+x) e^x$$

$$\Rightarrow \cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x) e^x \quad \text{—— (1)}$$

Let, $\sin y = u$

$\Rightarrow \cos y \frac{dy}{dx} = \frac{du}{dx}$

put in (1)

$\therefore \frac{du}{dx} - \frac{1}{1+x} u = (1+x)e^x$

Linear eqn, $\frac{du}{dx} + P(x)u = Q(x)$

$\therefore$ I.F. is, $e^{\int P(x)dx} = e^{\int -\frac{1}{1+x}dx} = e^{-\int \frac{1}{1+x}dx} = e^{-\log|1+x|}$

$= e^{\log|1+x|^{-1}} = (1+x)^{-1} = \frac{1}{1+x}$

$\therefore$ G.S. is,

$u \cdot I.F. = \int Q(x) \times I.F.\, dx + c$

$\Rightarrow (\sin y)\left(\frac{1}{1+x}\right) = \int \cancel{(1+x)}e^x \times \frac{1}{\cancel{1+x}} dx + c$

$= \int e^x dx + c$

$\Rightarrow \frac{\sin y}{1+x} = e^x + c$