# 1. What requirements S must fullfil in order to become a field?

In order to show that $S$ is a field, we'll need to prove the following ***binary operator*** properties:

1. In $S$, there are two members, zero - $0_S$ and one - $1_S$
2. $S$ supports the addition and multiplication binary operators
3. Every member in $S$ can be negated, i.e. for every $x$ there is $-x$
4. For every member in $S$ that is not $0_S$, $\exists x^{-1} \in S$, it is called the multiplicative inverse of $x$

In addition, the mentioned binary operators should satisfy the following properties, referred to as ***field axioms***:

1. Associativity of addition(A1) and multiplication(M1):
$$a + (b + c) = (a + b) + c$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
2. Commutativity of addition(A2) and multiplication(M2):
$$a + b = b + a$$
$$a \cdot b = b \cdot a$$
3. Additive identity(A3) and multiplicative identity(M3)
$$a + 0 = a$$
$$a \cdot 1 = a$$
4. Additive inverse(A4) and multiplicative inverse(M4)
$$a + (-a) = 0$$
$$a \cdot a^{-}1 = 1$$
5. Distributivity(D)
$$a(b + c) = (a \cdot b) + (a \cdot c)$$

## 2. Prove: $((a+b)+c)+d = (a+b)+(c+d) = a+(b+(c+d))$

### 2.1. $\underline{((a+b)+c)+d = (a+b)+(c+d)}$ :

let $h = (a+b)$
$(h+c)+d = ((a+b)+c)+d$
$(h+c)+d = h+(c+d)$ **(Axiom A1)**
$h+(c+d) = (a+b)+(c+d)$
$\Downarrow$
$((a+b)+c)+d = (a+b)+(c+d)$

### 2.2. $\underline{a+(b+(c+d)) = (a+b)+(c+d)}$ :

let $h = (c+d)$
$a+(b+h) = a+(b+(c+d))$
$a+(b+h) = (a+b)+h$ **(Axiom A1)**
$(a+b)+h = (a+b)+(c+d)$
$\Downarrow$
$a+(b+(c+d)) = (a+b)+(c+d)$

$\blacksquare$

## 3. Prove: $\forall x, y \in F,\ x(y - z) = xy - xz$

$x(y - z) = x(y + (-z))$ **(Axiom A1)**
$x(y + (-z)) = (x \cdot y) + (x \cdot (-z))$ **(Axiom D)**
$(x \cdot y) + (x \cdot (-z)) = (x \cdot y) + (-x \cdot z)) = (x \cdot y) - (x \cdot z))$
$(x \cdot y) - (x \cdot z)) = xy - xz$

■

## 4. Prove: $\forall x, y \in F,\ (x+y)(x+y) = xx + xy + yx + yy$

let $h = (x+y)$
$(x+y)(x+y) = h \cdot (x+y)$
$h \cdot (x+y) = (x \cdot h) + (y \cdot h) = x(x+y) + y(x+y)$ **(Axiom D)**
$x(x+y) + y(x+y) =$ **xx + xy + yx + yy (Axiom D)**

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