3 Calculate AC, BC, ABC and BAC

$$AC = \begin{bmatrix} c_1^1 & c_2^1 & c_3^1 \\ c_1^3 & c_2^3 & c_3^3 \\ c_1^2 & c_2^2 & c_3^2 \end{bmatrix}$$
$$BC = \begin{bmatrix} c_1^3 & c_2^3 & c_3^3 \\ c_1^1 & c_2^1 & c_3^1 \\ c_1^2 & c_2^2 & c_3^2 \end{bmatrix}$$
$$A(BC) = \begin{bmatrix} c_1^3 & c_2^3 & c_3^3 \\ c_1^2 & c_2^2 & c_3^2 \\ c_1^1 & c_2^1 & c_3^1 \end{bmatrix}$$
$$B(AC) = \begin{bmatrix} c_1^2 & c_2^2 & c_3^2 \\ c_1^1 & c_2^1 & c_3^1 \\ c_1^3 & c_2^3 & c_3^3 \end{bmatrix}$$

6 Prove that

$$A(\lambda B) = B(\lambda A) = \lambda(AB)$$

We'll prove this by showing:

$$[A(\lambda B)]_j^i = [B(A\lambda)]_j^i = [\lambda(AB)]_j^i$$

Therefore, if for all of the matrices, all of the cells are identical, the matrices are the same.

$$[A(\lambda B)]_j^i = \sum_{k=1}^{n} a_k^i(b_j^k\lambda)$$
$$[B(A\lambda)]_j^i = \sum_{k=1}^{n} b_j^k(a_k^i\lambda)$$
$$[\lambda(AB)]_j^i = \lambda(\sum_{k=1}^{n} a_k^i b_j^k)$$

As all of the operations are happening inside $\mathbb{F}$, we can factor the λ, and therefore:

$$[A(\lambda B)]_j^i = \sum_{k=1}^{n} a_k^i(b_j^k\lambda) = \lambda(\sum_{k=1}^{n} a_k^i b_j^k)$$
$$[B(A\lambda)]_j^i = \sum_{k=1}^{n} b_j^k(a_k^i\lambda) = \lambda(\sum_{k=1}^{n} a_k^i b_j^k)$$

Therefore, we've shown that:

$$[A(\lambda B)]_j^i = [B(A\lambda)]_j^i = [\lambda(AB)]_j^i$$

And therefore:

$$A(\lambda B) = B(\lambda A) = \lambda(AB)$$

$\square$

7 Prove that

$$A(\lambda C + D) = O$$

According to question **5**:

$$A(\lambda C + D) = A\lambda C + AD$$

According to question **6**:

$$A\lambda C + AD = \lambda \cdot AC + AD$$

It is given that $AC = AD = O$, therefore:

$$\lambda \cdot AC + AD = \lambda \cdot O + O$$

According to the properties of matrix scalar multiplication:

$$a \cdot O = O$$

Therefore:

$$\lambda \cdot O + O = O + O = O$$

□

10

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

11 Calculate A^{2020}

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We've proved in question *16A* $\forall a, b \in \mathbb{F}$:

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$$

Therefore, we can see that:

$$A^n = \begin{bmatrix} 1 & 2^{n-1} \\ 0 & 1 \end{bmatrix}$$
$$A^{2020} = \begin{bmatrix} 1 & 2^{2019} \\ 0 & 1 \end{bmatrix}$$

14

We'll prove this using induction over the sequence's length as ***s***.

14.0.1 $\underline{s = 1}$**:**

If the sequence's length is 1, it is trivial that the statement is true, as $A_1 \in GL_n(\mathbb{F})$. That is because $A_1 = A_1$, and $(A_1)^{-1} = (A_1)^{-1}$.

14.0.2 $\underline{s = s - 1}$**:**

We'll assume the statement is true for any sequence of an arbitrary length ***s-1***, and we'll mark it as ***B***, that is:

$$B = A_1 \cdot ... \cdot A_{s-1} \in GL_n(\mathbb{F})$$
$$B^{-1} = (A_1 \cdot ... \cdot A_{s-1})^{-1} = A_{s-1}^{-1} \cdot ... \cdot A_1^{-1}$$

14.0.3 $\underline{s = s}$**:**

Now, we'll prove the statement is true for a sequence with length ***s***.
Let A_s equal to the identity matrix of size n:

$$A_s = \mathbb{I}_n$$

Now, we can prove $(A_1 \cdot ... \cdot A_s) \in GL_n(\mathbb{F})$:

$$\begin{aligned} &A_1 \cdot ... \cdot A_s = B \cdot A_s \\ &= B \cdot \mathbb{I}_n \\ &= B \\ &B \in GL_n(\mathbb{F}) \\ &A_1 \cdot ... \cdot A_s \in GL_n(\mathbb{F}) \end{aligned}$$

Let's show that $(A_1 \cdot ... \cdot A_s)^{-1} = B^{-1}$:

$$\begin{aligned} &(A_1 \cdot ... \cdot A_s)^{-1} = (A_1 \cdot ... \cdot A_{s-1} \cdot A_s)^{-1} \\ &= (B \cdot A_s)^{-1} \\ &= (B \cdot \mathbb{I}_n)^{-1} \\ &= B^{-1} \end{aligned}$$

Now, using the fact that $A_s = \mathbb{I}_n = A_s^{-1}$, we'll show that $A_s^{-1} \cdot ... \cdot A_1^{-1} = B^{-1}$:

$$\begin{aligned} &A_s^{-1} \cdot ... \cdot A_1^{-1} = A_s^{-1} \cdot A_{s-1}^{-1} \cdot ... \cdot A_1^{-1} \\ &= A_s^{-1} \cdot B^{-1} \\ &= \mathbb{I}_n \cdot B^{-1} \\ &= B^{-1} \end{aligned}$$

Now, we can conclude:

$$(A_1 \cdot ... \cdot A_s)^{-1} = B^{-1} = A_s^{-1} \cdot ... \cdot A_1^{-1}$$

□

17

17.1 $\underline{\implies:}$

It is given that C is invertible, therefore $det(C) \neq 0$, that is:

$$\begin{aligned} det(C) &= A \cdot B - O \cdot O \\ &= AB \neq O \end{aligned}$$

17.2 $\underline{\impliedby:}$

19 Prove A is reversible

$$A^3 - 2A + I = O$$

We'll start by deducting I and multiplying the equation by $(-I)$:

$$\begin{aligned} 2A - A^3 &= I \\ A(2I - A^2) &= I \\ A(\sqrt{2}I + A)(\sqrt{2}I - A) &= I \end{aligned}$$

A is reversible if and only if there exists A^{-1} such that $AA^{-1} = I$, therefore:

$$A^{-1} = (\sqrt{2}I + A)(\sqrt{2}I - A)$$

$\square$