Calculus I

Exercise 5

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1 Decide whether the following series converge or diverge.

1.2

$$b_n = \frac{5 + (-1)^n n}{4n^2 + 1}$$

b_n converges to 0.
Therefore, for every $\epsilon > 0$ we need to find an $N \in \mathbb{N}$ such that for every $n > N$:

$$\left| \frac{5 + (-1)^n n}{4n^2 + 1} - 0 \right| < \epsilon$$

Let $\epsilon > 0$.
We'll choose $N \in \mathbb{N}$ such that:

$$N > \frac{2}{\epsilon}$$

For every natural $n > N$:

$$\frac{2}{n} < \frac{2}{N}$$

Therefore, for every $n > N$ we'll get:

$$\left| \frac{5 + (-1)^n n}{4n^2 + 1} - 0 \right| < \frac{8n}{4n^2} = \frac{2}{n} < \frac{2}{N} < \frac{2}{\frac{2}{\epsilon}} = \epsilon$$

Thus, by definition:

$$lim(b_n) = 0$$

$\square$

1.3

$$\begin{aligned} c_n &= (n+1)^2 - n^2 \\ &= n^2 + 2n + 1 - n^2 \\ &= 2n + 1 \\ &> n \end{aligned}$$

c_n is not bounded because $c_n > n$.
Because the series is not bounded, it diverges. $\square$

1.4

$$d_n = \frac{\sqrt{16n^2+3}}{n}$$

Let $\epsilon > 0$.
We'll choose $N \in \mathbb{N}$ such that:

$$N > \frac{1}{\epsilon}$$

And that for every natural $n > N$:

$$\frac{1}{n} < \frac{1}{N}$$

Therefore, for every $n > N$ we'll get:

$$\left| \frac{\sqrt{16n^2+3}}{n} - 4 \right| < \frac{1}{n} < \frac{1}{N} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

Thus, by definition:

$$lim(d_n) = 4$$

□

2

2.1 Write formally: $(an)_{n=1}^{\infty}$ converges

$$\exists L \in \mathbb{R} \quad \forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |a_n - L| < \epsilon$$

2.2 Write formally: $(an)_{n=1}^{\infty}$ not converges

$$\forall L \in \mathbb{R} \quad \exists \epsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n > N \quad |a_n - L| \geq \epsilon$$

2.3 Prove a_n diverges

$$a_n = \begin{cases} 1 + \frac{1}{n} & n \ is \ even \\ 2 & n \ is \ odd \end{cases}$$

Although a_n is bounded, it diverges.
We can see that because for even n's, the values will keep getting smaller and smaller, but never smaller than 1.
Therefore, $\lim(1 + \frac{1}{n}) = 1$.
In addition, while n is odd, we'll get the permanent series 2, whose limit is $lim(2) = 2$.
However, as we're constantly "moving" between two different limits, we can see that the definition of the limit won't hold for any of the limits, i.e. 1 *or* 2.
Therefore, neither of them is the limit of the series, and the series diverges. □

3 Prove or disprove

3.2

The statement is incorrect.
As an example, let a_n be:

$$a_n = 1^n$$

Therefore, a_n diverges as it doesn't converge into a single value.
In other words, for any $\epsilon > 0$ there doesn't exist $N \in \mathbb{N}$ such that $\forall n > N \quad |a_n - L| < \epsilon$.
On the other hand, $|a_n|$ converges to 1, and therefore:

$$lim(|a_n|) = 1$$

$\square$

3.4

The statement is correct.
The regular definition of the limit is: "$\forall \epsilon > 0 \quad \exists N \in \mathbb{N}$", which states that for any possible ϵ - as small or as big as we might find, we'll always be able to find some N.
However, this statement is **more strict**.
According to the statement, an $N \in \mathbb{N}$ exists that satisfies the definition for **every** ϵ.
Therefore, it doesn't contradict the traditional definition of the limit, and we can assume that:

$$lim(a_n) = L$$

$\square$

4 a_n converges to $L \in \mathbb{R}$, prove the following:

4.1 $\underline{L \geq 0}$

Let's assume by contradiction that $L < 0$.
Let $\lambda \in \mathbb{R}$ such that:

$$\lambda = \frac{L}{2}$$
$$L < \lambda < 0$$

According to (10.9) from lecture, we can assume that from a certain point, $a_n < \lambda$.
However, we've reached a contradiction:

$$0 \leq a_n < \lambda < 0$$

Therefore, we've proved that $L \geq 0$. $\square$

4.2 $\underline{L = 0 \implies lim(\sqrt{a_n}) = 0}$

It is given that (a_n) converges to 0 as $L = 0$.
Therefore by definition, for any arbitrary $\epsilon > 0$, we'll be able to find some $N \in \mathbb{N}$ such that $\forall n > N$:

$$|a_n| < \epsilon$$

$(\sqrt{a_n})$ is the same series, but every member in the series is being rooted.
Therefore, for **that same** N, every $n > N$ will converge as well, to $\sqrt{0} = 0$:

$$lim(\sqrt{a_n}) = \sqrt{0} = 0$$

$\square$

4.3 $\underline{L > 0 \implies lim(\sqrt{a_n}) = \sqrt{L}}$

Similarly to 4.2, we know that $(\sqrt{a_n})$ is the same series, but every member in the series is being rooted.
Therefore, for **that same** N, every $n > N$ will converge as well.
This time however, we know that $L > 0$, and therefore (a_n) converges to a number that is bigger than 0, which is L.
Therefore, $(\sqrt{a_n})$ converges to the square root of a number that is bigger than 0 - which is L, that is:

$$lim(\sqrt{a_n}) = \sqrt{lim(a_n)} = \sqrt{L}$$

$\square$

5

5.2

We learn from 5.1, that once we take any sequence of the form $a_n = x^n$, as long as $x > 0$, at "very large" numbers, a_n will be "very large".
Thus, a_n diverges, as it keeps increasing - exponentially.
Therefore, once we try to calculate the limit of $\frac{1}{x^n}$:

$$a_n = \frac{1}{x^n}$$
$$lim(a_n) = 0$$

In addition, using the fact that x^n is exponentially bigger than n, and the law of limits division:

$$lim(n) = n$$
$$lim(\frac{n}{x^n}) = 0$$

5.3

We need to prove the following:

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n-1}}$$

Let:

$$x = \sqrt{\frac{2}{n-1}} > 0$$

According to (5.1):

$$\begin{aligned}(1+x)^n &> \frac{n(n-1)}{2} \cdot x^2 \\ &= \frac{n(n-1)}{2} \cdot \frac{2}{n-1} \\ &= n\end{aligned}$$

Notice that we were able to divide by $(n-1)$ as it is given that $2 \leq n \in \mathbb{N}$.
We'll take the **n-th root** of both sides of the equation,
By utilizing what we've proved in exercise 2 question 8: $a > b \implies a^n > b^n$

$$\begin{aligned}(1+x)^n &> n \\ 1+x &> \sqrt[n]{n} \\ 1+\sqrt{\frac{2}{n-1}} &> \sqrt[n]{n}\end{aligned}$$

$\square$

5.4

We need to prove:

$$\lim(\sqrt[n]{n}) = 1$$

Therefore, we need to show that $\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n > N$:

$$\left| \sqrt[n]{n} - 1 \right| < \epsilon$$

in (5.3) we've shown that:

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n-1}} \tag{1}$$

Using (1) we can see:

$$\sqrt[n]{n} - 1 < \sqrt{\frac{2}{n-1}} \tag{2}$$

Now, we can use (2) to bound the term from above:

$$\begin{aligned} \left| \sqrt[n]{n} - 1 \right| &= \sqrt[n]{n} - 1 \\ &< \sqrt{\frac{2}{n-1}} \\ &< \frac{2}{n-1} \end{aligned}$$

Therefore, for every $N > \frac{2}{\epsilon} + 1$:

$$\begin{aligned} \left| \sqrt[n]{n} - 1 \right| &< \frac{2}{n-1} \\ &< \frac{2}{N-1} \\ &< \frac{2}{\frac{2}{\epsilon} + 1 - 1} \\ &= \epsilon \end{aligned}$$

Therefore, by the limit's definition:

$$\lim(\sqrt[n]{n}) = 1$$

□

6

6.1

The statement is **true**.
It is given that (a_n) and $(a_n + b_n)$ converge, therefore, both of them have limits.
By definition, the limit of the sum of sequences, is the sum of the limits, therefore:

$$\lim(a_n + b_n) = \lim(a_n) + \lim(b_n)$$
$$\lim(b_n) = \lim(a_n + b_n) - \lim(a_n)$$

Therefore, we can conclude that b_n converges. □

6.2

The statement is **false**, here's an example:

$$a_n = \frac{1}{n^2}$$
$$b_n = n$$
$$a_n \cdot b_n = \frac{n}{n^2} = \frac{1}{n}$$

We can clearly see that a_n and $a_n \cdot b_n$ converge to 0, yet b_n diverges. □

6.3

The statement is **false**.
As an example, let's define a_n and b_n:

$$a_n = \begin{cases} n & \text{n is odd} \\ \frac{1}{n^2} & \text{n is even} \end{cases}$$
$$b_n = \begin{cases} \frac{1}{n^2} & \text{n is odd} \\ n & \text{n is even} \end{cases}$$

Now, we can clearly see that both of these sequences do not converge and therefore - do not have limits.
However, we can see that $a_n b_n$ **does** converge, and it converges to 0. □

7

7.1

Let:

$$G = \sqrt[n]{y_1 \cdot ... \cdot y_n}$$
$$S = y_1 + ... + y_n$$

We need to prove:

$$G \leq \frac{S}{n}$$

First, we'll show:

$$\prod_{i=1}^{n} \frac{y_i}{G} = 1$$

$$\begin{aligned}
\prod_{i=1}^{n} \frac{y_i}{G} &= \frac{y_1 \cdot ... \cdot y_n}{G^n} \\
&= \frac{y_1 \cdot ... \cdot y_n}{(\sqrt[n]{y_1 \cdot ... \cdot y_n})^n} \\
&= \frac{y_1 \cdot ... \cdot y_n}{y_1 \cdot ... \cdot y_n} \\
&= 1
\end{aligned}$$

This will let us use the assumption from question 7 on exercise 4:

$$\prod_{i=1}^{n} \frac{y_i}{G} = 1 \implies \sum_{i=1}^{n} \frac{y_i}{G} \geq n$$

Therefore:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{y_i}{G} &= \frac{y_1 + ... + y_n}{G} \\
&= \frac{S}{G} \geq n
\end{aligned}$$

Now we can apply some inequality algebra:

$$\frac{G}{S} \leq \frac{1}{n}$$
$$G \leq \frac{S}{n}$$

$\square$