1 Prove that

$$a \leq b \iff \forall \epsilon > 0 \quad a < b + \epsilon$$

1.1 $\underline{a \leq b \Rightarrow \forall \epsilon > 0 \quad a < b + \epsilon}$:

It is given that:

$$0 < \epsilon \qquad\qquad\qquad\qquad a \leq b$$

Therefore, because according to axiom $O3$ an ordered field is adhering to addition:

$$a + 0 < b + \epsilon$$

It is worth mentioning that due to the uniqueness of zero, $b + \epsilon$ must be *greater* than zero, rather than greater than or *equal* to zero.

1.2 $\underline{a \leq b \Leftarrow \forall \epsilon > 0 \quad a < b + \epsilon}$:

We'll rephrase using contraposition:

$$a > b \Rightarrow \exists \epsilon > 0 \quad a \geq b + \epsilon$$

We need to find an $\epsilon > 0$ so that $a \geq b + \epsilon$.
Let $\epsilon = a - b$, now, we can see that:

$$b + \epsilon = b + (a - b) = a$$
$$a \geq b + \epsilon$$

$$\blacksquare$$

2 Prove that

$$\forall m, n \in \mathbb{F} \quad mn \in \mathbb{F}$$

Let's assume that m is some arbitrary natural number.
We'll prove using induction, starting with $n = 1$: According to axiom *M3*:

$$m \cdot 1 = m$$

It is given that $m \in \mathbb{F}$, therefore this case is valid. Now, let's assume that it is true for a general n, i.e. $n = n$, and:

$$mn \in \mathbb{F}$$

Now, we'll check $n = n + 1$: According to axiom D:

$$m(n + 1) = mn + m$$

We've assumed that $mn \in \mathbb{F}$, and it is given that $m \in \mathbb{F}$.
In addition, we've shown in exercise *2a* that the natural numbers adhere to addition.
Therefore:

$$mn \in \mathbb{F},\ m \in \mathbb{F} \implies (mn + m) \in \mathbb{F}$$

$$\blacksquare$$

3 Prove

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

We'll prove this by induction: