Linear Algebra I

Exercise 5

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2

2.1 Calculate A^2, B^2, AB and BA

$$A^2 = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

2.2

$$A^k = \begin{bmatrix} 1 & k & (A^{k-1} + (k-1)) & 0 & 0 \\ 0 & 1 & k & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & k \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^k = \begin{bmatrix} 1 & 0 & k & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & w \end{bmatrix}$$

When x, y, z and w alternate between $-1, 0$ and 1.

4.3 Find P, R and show P as a multiplication of elementary matrices

$$R = \begin{bmatrix} 1 & 0 & 15 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5 Decide whether the matrix is invertible or not. If it is invertible, find its inverse.

5.1

U_1 is invertible.

$$U_1^{-1} = \begin{bmatrix} \frac{9}{14} & -\frac{1}{14} & -\frac{3}{14} \\ \frac{10}{7} & -\frac{5}{7} & -\frac{1}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \end{bmatrix}$$

5.2

U_2 is not invertible, because its rref is:

$$rref(U_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$$

5.3

U_3 is not invertible, because its rref is:

$$rref(U_3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$$

13 Solve $AX = b$

13.1

$$b = b_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$AX = b$ and therefore $X = A^{-1}b$.

$$A^{-1}b_1 = \begin{bmatrix} -15 \\ \frac{25}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

13.2

$$A^{-1}b_2 = \begin{bmatrix} -\frac{23}{2} \\ \frac{11}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

13.3

$$b_1 + b_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

$$A^{-1}(b_1 + b_2) = \begin{bmatrix} -32 \\ 18 \\ 2 \\ 3 \end{bmatrix}$$

14

We're given that $(v_1, ..., v_s)$ is the set of solutions for the homogenous system $AX = 0$. That is:

$$\forall v \in (v_1, ..., v_s) \quad A \cdot v = 0$$

Because of that, we can conclude that any linear combination of $(v_1, ..., v_s)$ is also a solution for the homogenous system.
It is true for both solution addition (i.e. $v_1 + v_2$ or scalar multiplication (i.e. $v_1 \cdot \lambda$), for example:

$$\begin{aligned} A \cdot v_1 \cdot \lambda = & \\ &= (A \cdot (v_1)) \cdot \lambda \\ &= (0) \cdot \lambda \\ &= 0 \end{aligned}$$

$$\begin{aligned} A \cdot (v_1 + v_2) = & \\ &= (A \cdot v_1) + (A \cdot v_2) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

□

15

It is given that:

$$\begin{aligned} S: &\quad \text{solution set of } AX = b \\ T: &\quad \text{solution set of } CAX = Cb \end{aligned}$$

We need to show that $S \subseteq T$, that is, we need to show:

$$\forall x \in \mathbb{F}_c \quad Ax = b \implies \quad CAx = Cb$$

Therefore:

$$CAx = C(Ax) = C(b) = Cb$$

□

17 $AB = I_m$

It is given that $AB = I_m$, therefore, by the definition of the invertible matrix we can conclude that:

$$B = A^{-1}$$
$$A = B^{-1}$$

Therefore, as both A and B are invertible, they are square matrices by definition, and therefore $m = n$.

17.1 Prove $BX = 0$ has a single solution

We've shown that B is invertible.
By definition, if a homogenous linear system is made out of an invertible matrix, then the system has exactly one solution - which is the **trivial** solution.

17.2 Prove $m \leq n$

We've shown that $m = n$, therefore $m \leq n$.

17.3 $BC = I_n$

As $BC = I_n$, we can conclude:

$$C = B^{-1}$$

However, we've already shown that:

$$A = B^{-1}$$

Due to the uniquness of a matrix' inverse, we can conclude that $A = C$. $\square$