7 Find the solution set for:

$$X + Y - Z = -1$$

$$X - Y - Z = -1$$

First, let's isolate X:

$$X = Z - Y - 1$$

$$X = Z + Y - 1$$

From the above, we can conclude that $Y = 0$. If we change one of the original equations, we'll get:

$$X = Z - 1$$

Therefore, the solution set for X, Y, Z is:

$$S = \left\{ \begin{pmatrix} z-1 \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\}$$

$$\blacksquare$$

8 Show that the inverse of the inheritance rule is not true.

The equations system:

$$x^1 + x^2 = 1$$

$$x^2 - x^3 = 3$$

We can generate the equation L, which is a linear form of the mentioned two equations, by multiplying the first equation by 3, and the second one by (-1).

$$L : 3x^1 + 2x^2 + x^3 = 0$$

As we can easily see, the following solution to L is not a solution to the equations system:

$$x^1 = 0$$

$$x^2 = 0$$

$$x^3 = 0$$

$$\blacksquare$$

10 Are the two equation systems equivalent? if yes, write every equation as a linear form of the other system.

10.1 Showing the systems are equivalent

10.1.1 Right-hand side system

First, let's find the solutions set of the right-hand side system.
We can easily see from the first equation that $X^1 = X^3$, in addition, let's declare $t^1 = X^3$.
Now, from the second equation, we get:

$$X^2 = -3t^1$$

This leads us to the following solutions set for the right-hand side system:

$$S = \left\{ \begin{pmatrix} t^1 \\ -3t^1 \\ t^1 \end{pmatrix} : t^1 \in \mathbb{R} \right\}$$

10.1.2 Left-hand side system

Let's find the linear form of $L_2 - 2L_3$:

$$X^2 + 3x^3 = 0$$

Which leads us to:

$$X^2 = -3x^3$$

In addition, we'll mark X^3 as t^1, i.e.:

$$X^2 = -3t^1$$

Now, we'll find the linear form of $L_2 - 2L_1$:

$$3x^1 - 3t^1 = 0$$

Which leads to:

$$3x^1 = 3t^1$$
$$x^1 = t^1$$

This leads us to the following solutions set for the left-hand side system:

$$S = \left\{ \begin{pmatrix} t^1 \\ -3t^1 \\ t^1 \end{pmatrix} : t^1 \in \mathbb{R} \right\}$$

As we can see, the two systems have identical solution sets, thus, they're equivalent.

10.2 Write as a linear form of the other system

10.2.1 Left-hand side system as a linear form of the other system

$$-X^1 + X^2 + 4X^3 = 0$$

If we multiply the first and second equations by (-1) and (1), respectively, we'll receive the intended equation.

$$X^1 + 3X^2 + 8X^3 = 0$$

If we multiply the first and second equations by (1) and (3), respectively, we'll receive the intended equation.

$$\frac{1}{2}X^1 + X^2 + \frac{5}{2}X^3 = 0$$

If we multiply the first and second equations by $(\frac{1}{2})$ and (1), respectively, we'll receive the intended equation.

10.2.2 Right-hand side system as a linear form of the other system

$$X^1 - X^3 = 0$$

If we multiply the first and second equations by (-1) and (1), respectively, we'll receive the intended equation.

13 Does a homogenous system of m linear equations with n variables with a single solution exist?

13.1 $\underline{n = 3,\ m = 4}$

It exists, here's an example of such a system:

$$x + y + z = 0$$

$$2x + 2y + 2z = 0$$
$$3x + 3y + 3z = 0$$
$$4x + 4y + 4z = 0$$

13.2 $\underline{n = 4,\ m = 3}$

It exists, here's an example of such a system:

$$w + x + y + z = 0$$

$$2w + 2x + 2y + 2z = 0$$
$$3w + 3x + 3y + 3z = 0$$

16 Write the equation system that the given matrix, is its coefficients matrix

$$0x^1 + 0x^2 + x^3 + 4x^4 + 3x^5 = 0$$
$$2x^1 + 4x^2 + 2x^3 + 6x^4 + 7x^5 = 0$$
$$3x^1 + 6x^2 + 2x^3 + 5x^4 + 8x^5 = 0$$

18 Can A and B be row-equivalent? explain why.

There aren't any elementary row operations that can be applied to A or B, so that they'll be row-equivalent.
That is because their columns and rows count are different.
In other words, there does not exist a sequence $e_1, e_2 ... e_s$ such that:

$$A = e_s(e_{s-1}(...e_1(B)))$$

19

19.1 What are the inverse operations to $e_1^{-1}, e_2^{-1}, e_3^{-1}$?

19.1.1 e_1^{-1}:

The inverse is simply:

$$e_1^{-1} = A^1 \cdot (-2)^{-1}$$

Such that:

$$e_1^{-1}(e_1(A^1)) = A^1$$

19.1.2 e_2^{-1}:

The inverse is simply itself:

$$e_2^{-1} = e_2$$

Such that:

$$e_2(e_2(A)) = A$$

19.1.3 e_3^{-1}:

Because e_3 is:

$$e_3 = A^3 + 3 \cdot A^1$$

The inverse is:

$$e_3^{-1} = A^3 - 3 \cdot A^1$$

19.2 Execute $e_1^{-1}, e_2^{-1}, e_3^{-1}$ on the matrix

$$e_1(A) = \begin{bmatrix} 4 & 2 & -2 & 6 \\ 1 & -2 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$e_2(e_1(A)) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 4 & 2 & -2 & 6 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$e_3(e_2(e_1(A))) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 4 & 2 & -2 & 6 \\ 3 & -6 & 2 & 4 \end{bmatrix}$$

23 B is row-equivalent to A, and C is row-equivalent to B. Prove C is row-equivalent to A.

First, let's prove that every row is row-equivalent to itself.
Let's suppose that e is an elementary operation that multiplies the first row by 1.
Because of that, we can see that A is row-equivalent to A:

$$A = e(A) \tag{1}$$

That proves that every matrix is row-equivalent to itself.
It is given that B is row-equivalent to A, therefore by definition, there exists a series of elementary operations $e_1, e_2...e_j$ such that:

$$B = e_j(e_{j-1}(...e_1(A))) \tag{2}$$

Since every operation e has an inverse operation e^{-1}, using (2) we can see that:

$$e_j^{-1}(B) = e_j^{-1}(e_j(e_{j-1}(...e_1(A)))) = e_{j-1}(...e_1(A))$$

We can now use induction to see that if we keep performing these operations, we'll eventually arrive at A.
Therefore, since we've placed operations on B and from that arrived to A,
we've shown that if g is row-equivalent to h, h is row-equivalent to g.
Therefore, that implies that both A and C are row-equivalent to B, and therefore, they're row-equivalent as well.

■

24 Prove:

$$ad - bc \neq 0 \implies \begin{bmatrix} a & b \\ c & d \end{bmatrix} \iff \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$