

# Calculus I

## Exercise 1

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# 1

## 1.1

While  $i$  and  $ii$  are statements,  $iii$  isn't a statement, because we haven't received any information about  $x$ 's value.

## 1.2

$i)$

Statement:

$$\forall n \in \mathbb{F} \quad \exists m \in \mathbb{F} \mid n = m + m$$

Negated Statement:

$$\exists n \in \mathbb{F} \quad \forall m \in \mathbb{F} \mid n \neq m + m$$

$ii)$

Statement:

$$\forall m, n \in \mathbb{F} \quad n = m + m \rightarrow -n = -m - m$$

Negated Statement:

$$\exists m, n \in \mathbb{F} \quad n = m + m \wedge -n \neq -m - m$$

## 1.3

$i)$

Statement:

$$\forall n \in \mathbb{F} \quad \exists m \in \mathbb{F} \mid n = m + m$$

Negated Statement:

$$\exists n \in \mathbb{F} \quad \forall m \in \mathbb{F} \mid n \neq m + m$$

# 2

$i$  is the formal representation of a field's additive inverse axiom, i.e. A4.

On the other hand,  $ii$  states that in the field  $\mathbb{F}$ , there's a certain number,  $x$ , that if we'll add it to **any** other number in  $\mathbb{F}$ , we'll receive  $0_{\mathbb{F}}$ .

The two statements are **not** logically equal.

### 3

#### 3.1 Prove $\forall a, b \in \mathbb{F} \quad -(a - b) = (b - a)$

First, let's find  $(a - b)$ 's inverse:

$$(a - b) + x = 0$$

We'll add  $(b - a)$  to both sides of the equation:

$$(a - b) + (b - a) + x = (b - a)$$

And find the inverse:

$$x = (b - a)$$

Now, we can easily see that  $(a - b)$  and  $(b - a)$  are the inverses of each other. And due to the additive inverse axiom (A4) :

$$-(a - b) = x = (b - a)$$

■

#### 3.2 Prove the 'uniqueness of multiplicative inverse' property

It is given that  $ab, ac = 1_{\mathbb{F}}$ , and we need to prove that  $b = c = a^{-1}$ .

##### 3.2.1 $ab = 1_{\mathbb{F}}$

According to the multiplicative inverse property (M4), we can deduct:

$$b = a^{-1}$$

##### 3.2.2 $ac = 1_{\mathbb{F}}$

Exactly as above (M4), we can deduct:

$$c = a^{-1}$$

Therefore, we can conclude:

$$b = c = a^{-1}$$

■

- 4  $H$  is a set that satisfies all of the field axioms,  
 $H \neq \emptyset$ ,  $1_H = 0_H$   
**Prove that  $H$  contains only a single member.**

Adding two  $0_H$  should result in a  $0_H$ , due to axiom A3:

$$0_H + 0_H = 0_H$$

However, because  $1_H = 0_H$ , it also means that:

$$1_H + 1_H = 0_H$$

Because of that, we can conclude that no other members exist in  $H$ , except  $1_H = 0_H$

■

## 5 $\mathbb{F}$ is an ordered field, prove the following:

$$\text{5.1 } \underline{\forall x, y \in \mathbb{F} \ 0_{\mathbb{F}} < x < y \iff 0_{\mathbb{F}} < y^{-1} < x^{-1}}$$

$$\text{5.1.1 } \underline{0_{\mathbb{F}} < x < y \implies 0_{\mathbb{F}} < y^{-1} < x^{-1}}:$$

It is given that:

$$x < y$$

We'll multiple both sides of the inequality by 1, using axiom  $M_4$ :

$$xyy^{-1} < yxx^{-1}$$

It is given that  $x, y > 0$  therefore we can divide the equation by  $xy$ :

$$y^{-1} < x^{-1}$$

$$\text{5.1.2 } \underline{0_{\mathbb{F}} < x < y \iff 0_{\mathbb{F}} < y^{-1} < x^{-1}}:$$

It is given that:

$$y^{-1} < x^{-1}$$

We'll multiple both sides of the inequality by 1, using axiom  $M_4$ :

$$y^{-1}xx^{-1} < x^{-1}yy^{-1}$$

It is given that  $x^{-1}, y^{-1} > 0$  therefore we can divide the equation by  $x^{-1}y^{-1}$ :

$$x < y$$

■

$$\mathbf{5.2} \quad \underline{x, y, z, w \in \mathbb{F} \mid x < y, \ z \leq w \implies x + z < y + w}$$

It is given that:

$$x < y$$

We'll add  $(z + w)$  to both sides, according to axiom  $O3$ :

$$x + (z + w) < y + (z + w)$$

According to axiom  $A1$ , we'll rearrange the inequality:

$$(x + z) + w < (y + w) + z$$

It is given that  $z \leq w$ , therefore if we'll remove  $w$  from the left side, and  $z$  from the right side, the inequality should remain correct:

$$x + z < y + w$$

■

$$\mathbf{5.3} \quad \underline{\forall x, y \in \mathbb{F} \ (0_{\mathbb{F}} < xy) \iff ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y))}$$

$$\mathbf{5.3.1} \quad \underline{0_{\mathbb{F}} < xy \implies ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y)) :}$$

Due to the ordered field's trichotomy axiom,  $x, y$  must be  $> 0$  or  $< 0$ , it is known that  $xy > 0$  and therefore  $x, y \neq 0$  (as proven before).

If  $x > 0$ :

$$xy > 0$$

Let's divide by  $x$ :

$$y > 0$$

Else, if  $x < 0$ :

$$xy > 0$$

If we divide by  $x$ , the  $>$  will change to a  $<$ , as proven previously in exercise 2.5:

$$y < 0$$

Therefore, we can see that if  $xy > 0$ ,  $x, y > 0$  or  $x, y < 0$  must be true.

$$\mathbf{5.3.2} \quad \underline{0_{\mathbb{F}} < xy \iff ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y)) :}$$

First, let's assume that  $0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y$ :

$$x > 0$$

According to ordered field's axiom 4, we can multiply both sides of the inequality by  $y$ :

$$xy > 0$$

Now, let's assume that  $0_{\mathbb{F}} > x \wedge 0_{\mathbb{F}} > y$ :

$$x < 0$$

If we multiply both sides of the equation by  $y$ (which is negative), the inequality will change signs:

$$xy > 0$$

■

## 5.4 Prove:

$$0 < b \in \mathbb{F} \quad \forall a \in \mathbb{F} \quad a^2 < b^2 \implies -b < a < b$$

Due to the ordered field's trichotomy axiom,  $a$  is one of the following:

- $a < 0$
- $a = 0$
- $a > 0$

Therefore, we'll need to show that the statement is true for all three.

### 5.4.1 $a = 0$ :

$$b > 0$$

Using the ordered field's  $O3$  axiom, we'll subtract  $b$  from both sides:

$$b - b > -b$$

Using  $A3$ :

$$-b < 0$$

Now, according to transitivity:

$$-b < 0 = a < b$$

### 5.4.2 $a > 0$ :

First, we'll need a lemma to help us demonstrate an idea.

#### **Lemma 1.**

*We want to show that*

$$a, b > 0 \quad a^2 > b^2 \implies a > b$$

*We'll prove that by contraposition:*

$$a \leq b \implies a^2 \leq b^2$$

*We'll multiply the left side by  $b$ , and by  $a$ :*

$$a \leq b$$

$$a^2 \leq a \cdot b$$

$$a \cdot b \leq b^2$$

*Therefore, according to the transitivity axiom:*

$$a^2 \leq b^2$$

As shown in the lemma, we know that  $b > a$ .  
 In addition, it was proven in class that if  $b > 0$ ,  $-b < 0$ .  
 Therefore, according to transitivity:

$$-b < 0 < a < b$$

### 5.4.3 $a < 0$ :

One side of the inequality is easy to show:

$$b > 0 > a$$

Now, we'll try to show that  $-b < a$ :

$$b^2 > a^2$$

We'll deduct  $a^2$  from both sides, according to O3:

$$b^2 - a^2 > 0$$

Now, using distributivity:

$$(b + a)(b - a) > 0$$

We can see that there are two possible scenarios where this inequality is true:

$$(b + a) > 0 \wedge (b - a) > 0$$

$$(b + a) < 0 \wedge (b - a) < 0$$

However, because  $b > a$ , we can tell that the second scenario cannot happen, therefore:

$$b - a > 0$$

$$b + a > 0$$

Therefore we can show that:

$$a > -b$$

■

## 6 Prove or disprove the following, for $a, b, x, y \in \mathbb{F}$ :

### 6.1 $ab < a + b$

Let  $a, b = 0_{\mathbb{F}}$ :

$$0_{\mathbb{F}} \cdot 0_{\mathbb{F}} < 0_{\mathbb{F}} + 0_{\mathbb{F}}$$

According to Axioms A3 and M3 it yields the **false** statement:

$$0_{\mathbb{F}} < 0_{\mathbb{F}}$$



## 6.2 $x^2 < y^2 \implies x < y$

In order to disprove this statement, we need to find  $x, y \in \mathbb{F}$  such that:

$$x^2 < y^2 \wedge x \geq y$$

As an example, we can take:

$$x = 1_{\mathbb{F}}$$

$$y = -(1_{\mathbb{F}} + 1_{\mathbb{F}})$$

We can see that  $x \geq y$ .

$x^2$ :

$$x^2 = 1_{\mathbb{F}} \cdot 1_{\mathbb{F}} = 1_{\mathbb{F}}$$

$y^2$ :

$$y^2 = -(1_{\mathbb{F}} + 1_{\mathbb{F}}) \cdot -(1_{\mathbb{F}} + 1_{\mathbb{F}}) = 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}}$$

We can see that  $y^2 > x^2$ , thus, the statement is **false**.

### 6.3 $x < y \implies x^2 < y^2$

In order to disprove this statement, we need to find  $x, y \in \mathbb{F}$  such that:

$$x < y \wedge x^2 \geq y^2$$

We can use the same examples from 6.2, only swapping the  $x$  and the  $y$ :

$$y = 1_{\mathbb{F}}$$

$$x = -(1_{\mathbb{F}} + 1_{\mathbb{F}})$$

We can see that  $y \geq x$ .

$y^2$ :

$$y^2 = 1_{\mathbb{F}} \cdot 1_{\mathbb{F}} = 1_{\mathbb{F}}$$

$x^2$ :

$$x^2 = -(1_{\mathbb{F}} + 1_{\mathbb{F}}) \cdot -(1_{\mathbb{F}} + 1_{\mathbb{F}}) = 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}}$$

We can see that  $x^2 > y^2$ , thus, the statement is **false**.

## 7 **Prove** $\forall a \in \mathbb{F} \mid -a| = |a|$ :

According to the definition of absolute value, there are two different scenarios for  $a$ :

### 7.0.1 $a \geq 0$ :

Since  $a \geq 0$ , using the abs definition we'll get:

$$|a| = a$$

According to the ordered field axioms  $-a < 0$ , and therefore:

$$|(-a)| = -(-a) = a$$

We can see that:

$$a \geq 0 \implies |-a| = |a| = a$$

### 7.0.2 $a < 0$ :

Since  $a < 0$ , using the abs definition will yield:

$$|a| = -a$$

We've previously proved that the negative of a negative is positive, i.e.  $-a > 0$ , therefore:

$$|-a| = -a$$

We can see that:

$$a < 0 \implies |-a| = |a| = -a$$

■

## 8 Prove the following for $a, b \in \mathbb{F}$ :

### 8.1 $\max(a, b) = \frac{1}{2}(a + b + |a - b|)$

This splits into three different cases:

- $a < b$
- $a = b$
- $a > b$

#### 8.1.1 $a < b$ :

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|)$$

Since  $a < b$ , the term  $|a - b|$  is equal to  $b - a$ , therefore:

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + b - a) = \frac{1}{2}(2b) = b$$

#### 8.1.2 $a = b$ :

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|)$$

Since  $a = b$ , the term  $|a - b|$  is equal to 0, therefore:

$$\max(a, b) = \frac{1}{2}(a + b + 0) = \frac{1}{2}(a + b) = \frac{a + b}{2} = a = b$$

#### 8.1.3 $a > b$ :

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|)$$

Since  $a > b$ , the term  $|a - b|$  is equal to  $a - b$ , therefore:

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + a - b) = \frac{1}{2}(2a) = a$$

## 8.2 $\min(a, b) = \frac{1}{2}(a + b - |a - b|)$

This splits into three different cases:

- $a < b$
- $a = b$
- $a > b$

### 8.2.1 $a < b$ :

$$\min(a, b) = \frac{1}{2}(a + b - |a - b|)$$

Since  $a < b$ , the term  $|a - b|$  is equal to  $b - a$ , therefore:

$$\min(a, b) = \frac{1}{2}(a + b - |a - b|) = \frac{1}{2}(a + b - (b - a)) = a$$

### 8.2.2 $a = b$ :

$$\min(a, b) = \frac{1}{2}(a + b - |a - b|)$$

Since  $a = b$ , the term  $|a - b|$  is equal to 0, therefore:

$$\min(a, b) = \frac{1}{2}(a + b - 0) = \frac{1}{2}(a + b) = \frac{a + b}{2} = a = b$$

### 8.2.3 $a > b$ :

$$\min(a, b) = \frac{1}{2}(a + b - |a - b|)$$

Since  $a > b$ , the term  $|a - b|$  is equal to  $a - b$ , therefore:

$$\min(a, b) = \frac{1}{2}(a + b - |a - b|) = \frac{1}{2}(a + b - (a - b)) = \frac{1}{2}(2b) = b$$

■