1. What requirements S must fullfil in order to become a field?

In order to show that S is a field, we'll need to prove the following ***binary operator*** properties:

1. In S, there are two members, zero - 0_S and one - 1_S
2. S supports the addition and multiplication binary operators
3. Every member in S can be negated, i.e. for every x there is $-x$
4. For every member in S that is not 0_S, $\exists x^{-1} \in S$, it is called the multiplicative inverse of x

In addition, the mentioned binary operators should satisfy the following properties, referred to as ***field axioms***:

1. Associativity of addition(A1) and multiplication(M1):

$$a + (b + c) = (a + b) + c$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

2. Commutativity of addition(A2) and multiplication(M2):

$$a + b = b + a$$

$$a \cdot b = b \cdot a$$

3. Additive identity(A3) and multiplicative identity(M3)

$$a + 0 = a$$

$$a \cdot 1 = a$$

4. Additive inverse(A4) and multiplicative inverse(M4)

$$a + (-a) = 0$$

$$a \cdot a^{-}1 = 1$$

5. Distributivity(D)

$$a(b + c) = (a \cdot b) + (a \cdot c)$$

2. Prove: $((a+b)+c)+d = (a+b)+(c+d) = a+(b+(c+d))$

2.1. $\underline{((a+b)+c)+d = (a+b)+(c+d)}$:

let $h = (a+b)$

$(h+c)+d = ((a+b)+c)+d$

$(h+c)+d = h+(c+d)$ **(A1)**

$h+(c+d) = (a+b)+(c+d)$

$\Downarrow$

$((a+b)+c)+d = (a+b)+(c+d)$

2.2. $\underline{a+(b+(c+d)) = (a+b)+(c+d)}$:

let $h = (c+d)$

$a+(b+h) = a+(b+(c+d))$

$a+(b+h) = (a+b)+h$ **(A1)**

$(a+b)+h = (a+b)+(c+d)$

$\Downarrow$

$a+(b+(c+d)) = (a+b)+(c+d)$

$\blacksquare$

3. Prove: $\forall x, y \in F,\ x(y-z) = xy - xz$

$x(y-z) = x(y+(-z))$ **(A1)**

$x(y+(-z)) = (x \cdot y) + (x \cdot (-z))$ **(D)**

$(x \cdot y) + (x \cdot (-z)) = (x \cdot y) + (-x \cdot z)) = (x \cdot y) - (x \cdot z))$

$(x \cdot y) - (x \cdot z)) = xy - xz$

$\blacksquare$

4. **Prove:** $\forall x, y \in F,\ (x+y)(x+y) = xx + xy + yx + yy$

let $h = (x+y)$

$(x+y)(x+y) = h \cdot (x+y)$

$h \cdot (x+y) = (x \cdot h) + (y \cdot h) = x(x+y) + y(x+y)$ **(D)**

$x(x+y) + y(x+y) = xx + xy + yx + yy$ **(D)**

$\blacksquare$

5. **Prove:** $\forall x, y \in F,\ (x+y)(x-y) = xx - yy$

$(x+y)(x-y) = xx - xy + yx - yy$ **(ex. 4)**

$xx - xy + yx - yy = xx + (-xy + yx) - yy$ **(A1)**

$(-xy + yx) = (-xy + xy)$ **(A2)**

$(-xy + xy) = 0$ **(A4)**

$\Downarrow$

$xx + (-xy + yx) - yy = xx + 0 - yy = xx - yy$ **(A3)**

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6. Prove: $(a = b) \wedge (c = d) \Rightarrow (a + c = b + d) \wedge (ac = bd)$

6.1. $\underline{(a = b) \wedge (c = d) \Rightarrow a + c = b + d}$:

$c = d = x$

$a = b$

$a + x = b + x$ **(Consistency with addition)**

$a + x = a + c$ **(x=c)**

$b + x = b + d$ **(x=d)**

$\Downarrow$

$a + c = b + d$

6.2. $\underline{(a = b) \wedge (c = d) \Rightarrow ac = bd}$:

$c = d = x$

$a = b$

$ax = bx$ **(Consistency with multiplication)**

$ax = ac$ **(x=c)**

$bx = bd$ **(x=d)**

$\Downarrow$

$ac = bd$

■

7. $A = \left\{ \begin{pmatrix} 1 \\ a \end{pmatrix} \middle| a \in \mathbb{R} \right\}$

7.1. Does A have a neutral additive member?

$\begin{pmatrix} 1 \\ a \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ a+0 \end{pmatrix}$
$\begin{pmatrix} 1 \\ a+0 \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}$ **(A3)**
$\Downarrow$
$0_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

7.2. Does A have a neutral multiplicative member?

$\begin{pmatrix} 1 \\ a \end{pmatrix} \odot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ a \cdot 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ a \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}$ **(M3)**
$\Downarrow$
$1_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

7.3. Is A a field?