Linear Algebra I

Exercise 3

Aviv Vaknin
316017128

4 Find the *RREF* form for the given matrices:

4.1

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Working on the first column, we'll apply $A_2 - 2A_1$, and $A_3 - 4A_1$:

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & 5 & 6 \\ 0 & 0 & 13 & 16 \end{bmatrix}$$

For the second column, we'll apply $A_2 \cdot (-1)$, and then $A_1 - 2A_2$:

$$\begin{bmatrix} 1 & 0 & 10 & 11 \\ 0 & 1 & -5 & -6 \\ 0 & 0 & 13 & 16 \end{bmatrix}$$

For the third column, we'll apply $A_3 \cdot \frac{1}{13}$, $A_1 - 10A_3$ and $A_2 + 5A_3$, and therefore:

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & -\frac{17}{13} \\ 0 & 1 & 0 & \frac{2}{13} \\ 0 & 0 & 1 & \frac{16}{13} \end{bmatrix}$$

4.2

$$B = \begin{bmatrix} 0 & 0 & 1 & 3 & -2 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 2 & 3 & 8 & 2 \\ 0 & 1 & 1 & 3 & 3 \end{bmatrix}$$

We'll start with the second column, and replace B_1 with B_2.
Then we'll use B_1 to zero out the rest of the column, namely $B_3 - 2B_1$ and $B_4 - B_1$:

$$\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & -1 & -4 & 2 \\ 0 & 0 & -1 & -3 & 3 \end{bmatrix}$$

Now, for the third column, we'll apply $B_1 - 2B_2$, $B_3 + B_2$ and $B_4 + B_2$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the fourth column, we'll add $3B_3$ to $B2$, and then multiply B_3 by (-1).

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the last column, we'll apply $B_1 - 4B_4$ and $B_2 + 2B_4$, and therefore:

$$rref(B) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4.3

$$C = \begin{bmatrix} 0 & 3 & 1 \\ 5 & -4 & 2 \\ 2 & 2 & 7 \\ 1 & -1 & 0 \\ 0 & 5 & 3 \end{bmatrix}$$

We'll start with the first column, and replace C_4 with C_1, and then apply $C_2 - 5C_1$ and $C_3 - 2C_1$:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 5 & 3 \end{bmatrix}$$

Now, for the second column, we'll apply $C_1 + C_2$, $C4 - 3C_2$ and $C5 - 5C_2$:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & -7 \end{bmatrix}$$

For the third and last column, we'll replace C_3 and C_4, multiply $C3$ by $-\frac{1}{5}$ and then apply $C_1 - 2C_3$, $C_2 - 2C_3$ and $C5 + 7C_3$, and get:

$$rref(C) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

8 Find all of the solutions, $\mathbb{F} = Q$

First, let's convert the equations system into a matrix:

$$A = \begin{bmatrix} \frac{1}{3} & 2 & -6 & \mid 0 \\ -4 & 0 & 5 & \mid 0 \\ -3 & 6 & -13 & \mid 0 \\ -\frac{7}{3} & 2 & -\frac{8}{3} & \mid 0 \end{bmatrix}$$

And then find its *rref*:

$$rref(A) = \begin{bmatrix} 1 & 0 & -\frac{5}{4} & \mid 0 \\ 0 & 1 & -\frac{67}{24} & \mid 0 \\ 0 & 0 & 0 & \mid 0 \\ 0 & 0 & 0 & \mid 0 \end{bmatrix}$$

Therefore, we've received the following system:

$$\begin{Bmatrix} X_1 - \frac{5}{4}X_3 = 0 \\ X_2 - \frac{67}{24}X_3 = 0 \end{Bmatrix}$$

We'll mark X_3 as t_1, therefore:

$$X_1 = \frac{5}{4}t_1$$
$$X_2 = \frac{67}{24}t_1$$

Therefore, the solution set is:

$$\mathbb{S} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ \begin{bmatrix} \frac{5}{4}t_1 \\ \frac{67}{24}t_1 \\ t_1 \end{bmatrix} : t_1 \in Q \right\}$$

10 Find all of the solutions, if there are any, $\mathbb{F} = Q$

First, let's convert the equations system into a matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 & \mid 1 \\ 2 & 0 & 2 & \mid 1 \\ 1 & -3 & 4 & \mid 2 \end{bmatrix}$$

And get it to its *rref* form:

$$\begin{bmatrix} 1 & 0 & 1 & \mid \frac{1}{2} \\ 0 & 1 & -1 & \mid -\frac{1}{2} \\ 0 & 0 & 0 & \mid 0 \end{bmatrix}$$

Therefore, we've received the following system:

$$\begin{Bmatrix} X_1 + X_3 = \frac{1}{2} \\ X_2 - X_3 = -\frac{1}{2} \end{Bmatrix}$$

We'll mark X_3 as t_1, therefore:

$$X_1 = \frac{1}{2} - t_1$$
$$X_2 = -\frac{1}{2} + t_1$$

Therefore, the solution set is:

$$\mathbb{S} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ \begin{bmatrix} \frac{1}{2} - t_1 \\ -\frac{1}{2} + t_1 \\ t_1 \end{bmatrix} : t_1 \in Q \right\}$$

13 Show that the equations system has a single solution iff $ad - bc \neq 0$

Last week, we've shown that if $ad - bc \neq 0$, its matrix will be of the form: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Therefore, the matrix will represent the *identity matrix*.
In the lecture, we've learned that the identity matrix only has a single solution.
In this exercise specifically, that is because $X_1 = B^1$, and $X_2 = B^2$,
as there aren't any free variables.

21 Show that matrix addition is associative

Let A, B and C are $M_{m\times n}(\mathbb{F})$ matrices.
We need to show that:

$$(A+B)+C = A+(B+C)$$

According to the definition of matrix addition:

$$((A+B)+C)^i_j = (a^i_j + b^i_j) + c^i_j$$

$$(A+(B+C))^i_j = a^i_j + (b^i_j + c^i_j)$$

Since $a^i_j,\ b^i_j,\ c^i_j \in \mathbb{F}$, according to the additive associativity axiom:

$$(a^i_j + b^i_j) + c^i_j = a^i_j + (b^i_j + c^i_j)$$

$$(A+B)+C = A+(B+C)$$

$$\blacksquare$$