

1

1.1

While i and ii are statements, iii isn't a statement, because we haven't received any information about x 's value.

1.2

$i)$

Statement:

$$\forall n \in \mathbb{F} \quad \exists m \in \mathbb{F} \mid n = m + m$$

Negated Statement:

$$\exists n \in \mathbb{F} \quad \forall m \in \mathbb{F} \mid n \neq m + m$$

$ii)$

Statement:

$$\forall m, n \in \mathbb{F} \quad n = m + m \rightarrow -n = -m - m$$

Negated Statement:

$$\exists m, n \in \mathbb{F} \quad n = m + m \wedge -n \neq -m - m$$

1.3

$i)$

Statement:

$$\forall n \in \mathbb{F} \quad \exists m \in \mathbb{F} \mid n = m + m$$

Negated Statement:

$$\exists n \in \mathbb{F} \quad \forall m \in \mathbb{F} \mid n \neq m + m$$

2

i is the formal representation of a field's additive inverse axiom, i.e. A4.

On the other hand, ii states that in the field \mathbb{F} , there's a certain number, x , that if we'll add it to **any** other number in \mathbb{F} , we'll receive $0_{\mathbb{F}}$.

The two statements are **not** logically equal.

3

3.1 Prove $\forall a, b \in \mathbb{F} \quad -(a - b) = (b - a)$

First, let's find $(a - b)$'s inverse:

$$(a - b) + x = 0$$

We'll add $(b - a)$ to both sides of the equation:

$$(a - b) + (b - a) + x = (b - a)$$

And find the inverse:

$$x = (b - a)$$

Now, we can easily see that $(a - b)$ and $(b - a)$ are the inverses of each other. And due to the additive inverse axiom (A4) :

$$-(a - b) = x = (b - a)$$

■

3.2 Prove the 'uniqueness of multiplicative inverse' property

It is given that $ab, ac = 1_{\mathbb{F}}$, and we need to prove that $b = c = a^{-1}$.

3.2.1 $ab = 1_{\mathbb{F}}$

According to the multiplicative inverse property (M4), we can deduct:

$$b = a^{-1}$$

3.2.2 $ac = 1_{\mathbb{F}}$

Exactly as above (M4), we can deduct:

$$c = a^{-1}$$

Therefore, we can conclude:

$$b = c = a^{-1}$$

■

- 4 H is a set that satisfies all of the field axioms,
 $H \neq \emptyset$, $1_H = 0_H$
Prove that H contains only a single member.

Adding two 0_H should result in a 0_H , due to axiom A3:

$$0_H + 0_H = 0_H$$

However, because $1_H = 0_H$, it also means that:

$$1_H + 1_H = 0_H$$

Because of that, we can conclude that no other members exist in H , except $1_H = 0_H$

■

5 \mathbb{F} is an ordered field, prove the following:

$$\text{5.1 } \underline{\forall x, y \in \mathbb{F} \ 0_{\mathbb{F}} < x < y \iff 0_{\mathbb{F}} < y^{-1} < x^{-1}}$$

$$\text{5.1.1 } \underline{0_{\mathbb{F}} < x < y \implies 0_{\mathbb{F}} < y^{-1} < x^{-1}}:$$

It is given that:

$$x < y$$

We'll multiple both sides of the inequality by 1, using axiom M_4 :

$$xyy^{-1} < yxx^{-1}$$

It is given that $x, y > 0$ therefore we can divide the equation by xy :

$$y^{-1} < x^{-1}$$

$$\text{5.1.2 } \underline{0_{\mathbb{F}} < x < y \iff 0_{\mathbb{F}} < y^{-1} < x^{-1}}:$$

It is given that:

$$y^{-1} < x^{-1}$$

We'll multiple both sides of the inequality by 1, using axiom M_4 :

$$y^{-1}xx^{-1} < x^{-1}yy^{-1}$$

It is given that $x^{-1}, y^{-1} > 0$ therefore we can divide the equation by $x^{-1}y^{-1}$:

$$x < y$$

■

$$\mathbf{5.2} \quad \underline{x, y, z, w \in \mathbb{F} \mid x < y, z \leq w \implies x + z < y + w}$$

It is given that:

$$x < y$$

We'll add $(z + w)$ to both sides, according to axiom $O3$:

$$x + (z + w) < y + (z + w)$$

According to axiom $A1$, we'll rearrange the inequality:

$$(x + z) + w < (y + w) + z$$

It is given that $z \leq w$, therefore if we'll remove w from the left side, and z from the right side, the inequality should remain correct:

$$x + z < y + w$$

■

$$\mathbf{5.3} \quad \underline{\forall x, y \in \mathbb{F} (0_{\mathbb{F}} < xy) \iff ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y))}$$

$$\mathbf{5.3.1} \quad \underline{0_{\mathbb{F}} < xy \implies ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y)) :}$$