1 Prove that

$$a \leq b \iff \forall \epsilon > 0 \quad a < b + \epsilon$$

1.1 $\underline{a \leq b \Rightarrow \forall \epsilon > 0 \quad a < b + \epsilon}$:

It is given that:

$$0 < \epsilon \qquad\qquad\qquad\qquad a \leq b$$

Therefore, because according to axiom $O3$ an ordered field is adhering to addition:

$$a + 0 < b + \epsilon$$

It is worth mentioning that due to the uniqueness of zero, $b + \epsilon$ must be *greater* than zero, rather than greater than or *equal* to zero.

1.2 $\underline{a \leq b \Leftarrow \forall \epsilon > 0 \quad a < b + \epsilon}$:

We'll rephrase using contraposition:

$$a > b \Rightarrow \exists \epsilon > 0 \quad a \geq b + \epsilon$$

We need to find an $\epsilon > 0$ so that $a \geq b + \epsilon$.
Let $\epsilon = a - b$, now, we can see that:

$$b + \epsilon = b + (a - b) = a$$
$$a \geq b + \epsilon$$

$$\blacksquare$$

2 Prove that

$$\forall m, n \in \mathbb{N} \quad mn \in \mathbb{N}$$

Let's assume that m is some arbitrary natural number.
We'll prove using induction, starting with $n = 1$: According to axiom *M3*:

$$m \cdot 1 = m$$

It is given that $m \in \mathbb{N}$, therefore this case is valid. Now, let's assume that it is true for a general n, i.e. $n = n$, and:

$$mn \in \mathbb{N}$$

Now, we'll check $n = n + 1$: According to axiom D:

$$m(n + 1) = mn + m$$

We've assumed that $mn \in \mathbb{N}$, and it is given that $m \in \mathbb{N}$.
In addition, we've shown in exercise *2a* that the natural numbers adhere to addition.
Therefore:

$$mn \in \mathbb{N},\ m \in \mathbb{N} \implies (mn + m) \in \mathbb{N}$$

$$\blacksquare$$

3 Prove the following:

3.1 Prove:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

We'll prove this by induction.

3.1.1 $n = 1$

we can see that the statement is true for $n = 1$:

$$\sum_{k=1}^{1} k^2 \stackrel{(def)}{=} 1^2 = \frac{1(1+1)(2+1)}{6}$$

3.1.2 $n = n$

Now, we'll assume it is true for $n = n$, i.e.:

$$\sum_{k=1}^{n} k^2 \stackrel{(def)}{=} 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3.1.3 $n = n + 1$

And now, we'll check $n = n + 1$:

$$\sum_{k=1}^{n+1} k^2 \stackrel{(def)}{=} 1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

Now, according to the definition:

$$\sum_{k=1}^{n} k^2 + (n+1)^2 = \sum_{k=1}^{n+1} k^2$$

Therefore, we need to check if:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

And after solving each side, we receive:

$$0 = 0$$

Therefore, we've proved that the statement is true for $n = n + 1$.

$$\blacksquare$$

3.2 Prove:

$$\sum_{i=0}^{n-1} x^i = \frac{1_{\mathbb{F}} - x^n}{1_{\mathbb{F}} - x}$$

Note: I'll use $1_{\mathbb{F}}$ and 1 interchangeably while proving this.
We'll prove this by induction.

3.2.1 $n = 1$

we can see that the statement is true for $n = 1$, because by definition, $a^0 = 1$:

$$\sum_{i=0}^{0} x^i \overset{(def)}{=} x^0 = \frac{1_{\mathbb{F}} - x^1}{1_{\mathbb{F}} - x} = \frac{1_{\mathbb{F}} - x}{1_{\mathbb{F}} - x} = 1$$

3.2.2 $n = k$

Now, let's assume that the statement is true for $n = k$, i.e.:

$$\sum_{i=0}^{k-1} x^i \overset{(def)}{=} 1 + x + x^2 ... + x^{k-1} = \frac{1_{\mathbb{F}} - x^k}{1_{\mathbb{F}} - x}$$

3.2.3 $n = k + 1$

$$\sum_{i=0}^{k} x^i \overset{(def)}{=} 1 + x + x^2 ... + x^{k-1} + x^k = \frac{1_{\mathbb{F}} - x^{k+1}}{1_{\mathbb{F}} - x}$$

According to the definition:

$$\sum_{i=0}^{k-1} x^i + x^k = \sum_{i=0}^{k} x^i$$

Therefore, we need to prove that:

$$\frac{1_{\mathbb{F}} - x^k}{1_{\mathbb{F}} - x} + x^k = \frac{1_{\mathbb{F}} - x^{k+1}}{1_{\mathbb{F}} - x}$$

$$\frac{1_{\mathbb{F}} - x^k}{1_{\mathbb{F}} - x} + x^k = \frac{1_{\mathbb{F}} - x^k + (1_{\mathbb{F}} - x)x^k}{1_{\mathbb{F}} - x} = \frac{1_{\mathbb{F}} - x^k + x^k - x^{k+1}}{1_{\mathbb{F}} - x} = \frac{1_{\mathbb{F}} - x^{k+1}}{1_{\mathbb{F}} - x}$$

Thus, we've proved that:

$$\sum_{i=0}^{k-1} x^i + x^k = \sum_{i=0}^{k} x^i$$

$$\blacksquare$$

4

$$m, n, s, t \in \mathbb{F} \quad n, t > 0$$

Prove:

$$\frac{m}{n} < \frac{s}{t} \implies \frac{m}{n} < \frac{m+s}{n+t} < \frac{s}{t}$$

?

8

$$x, y \geq 0, \;\; n \in \mathbb{N}$$

Prove:

$$x < y \iff x^n < y^n$$

8.1 $\underline{x < y \implies x^n < y^n}$

Lemma 1.

$$a, b, c, d > 0$$
$$a > b, \;\; c > d$$

Because $a > b$, we can multiply both by c, which is positive:

$$ac > bc$$

Similarly:

$$bc > bd$$

Therefore, due to transitivity:

$$bd < bc < ac$$

$$\blacksquare$$

We'll prove this by induction.

8.1.1 $\underline{n = 1}$

By definition, $a^1 = a$, therefore we can see the statement is true for n=1:

$$x^1 = x < y = y^1$$

8.1.2 $\underline{n = k}$

Let's assume it is true for $n = k$, that is:

$$x < y \implies x^k < y^k$$

8.1.3 $n = k + 1$

$$x < y$$
$$x^{k+1} = x^k \cdot x$$
$$y^{k+1} = y^k \cdot y$$

According to our $n = k$ assumption:

$$x^k < y^k$$

Therefore, because it is given that $x, y \geq 0$, and *Lemma 1*, we can show that:

$$x^{k+1} = x^k x < y^k y = y^{k+1}$$

We've shown:

$$x < y \implies x^n < y^n$$

8.2 $x < y \impliedby x^n < y^n$

The contrapositive form of this assertion is:

$$x \geq y \implies x^n \geq y^n$$

Similarly to how we proved the $\implies$ part, we'll prove it by induction:

8.2.1 $n = 1$

We can easily see that this is correct, as $a^1 = a$:

$$x^1 = x \geq y = y^1$$

8.2.2 $n = k$

Let's assume it is true for $n = k$, that is:

$$x \geq y \implies x^k \geq y^k$$

8.2.3 $n = k + 1$

$$x \geq y$$
$$x^{k+1} = x^k \cdot x$$
$$y^{k+1} = y^k \cdot y$$

According to our $n = k$ assumption:

$$x^k \geq y^k$$

Therefore, because it is given that $x, y \geq 0$, and *Lemma 1*, we can show that:

$$x^{k+1} = x^k x \geq y^k y = y^{k+1}$$

We've shown:

$$x \geq y \implies x^n \geq y^n$$

$$\blacksquare$$