1 $A \subseteq \mathbb{F}$, prove $m \in \mathbb{F}$ is the infimum of A *if and only if*:

1. m is the lower bound of A
2. $(\forall \epsilon > 0)\ (\exists a \in A)\ (a < m + \epsilon)$

1.1 $\implies$:

We'll start with assuming $m \in \mathbb{F}$ is the infimum of A.
By definition, the infimum is a lower bound **(1)**.
Therefore:

$$\forall a \in A \quad a \geq m$$

and, suppose m' is another lower bound for A:

$$m \geq m'$$

First, by definition, an infimum is a lower bound.
Regarding the second item, let's suppose by contradiction:

$$\begin{aligned} \exists \epsilon > 0 \quad \forall a \in A \quad a &\geq m + \epsilon \\ m + \epsilon &> m \end{aligned}$$

Therefore, we can see that $\underline{m + \epsilon}$ is a lower bound for A, bigger than the infimum.
We've reached a contradiction, as we've assumed m is the infimum of A **(2)**.

1.2 $\impliedby$:

Let's suppose that m is not the infimum.
Therefore, exists another lower-bound - m' so that $m' > m$.
Let $\epsilon > 0$:

$$\begin{aligned} \epsilon &= m' - m \\ m' &= \epsilon + m \end{aligned}$$

Therefore:

$$\forall \epsilon > 0 \quad \exists a \in A \quad a \geq \epsilon + m$$

Which is in contradiction with our initial assumption:

$$(\forall \epsilon > 0)\ (\exists a \in A)\ (a < m + \epsilon)$$

$\square$

2

$$A \subseteq \mathbb{F}$$
$$s \in \mathbb{F}$$

2.1 Prove $s = \max(A) \iff (s = \sup(A)) \wedge (s \in A)$

2.1.1 $\implies$:

Let's start by supposing $s = \max(A)$.
By definition, the maximum is always part of the set, therefore **(1)**:

$$s \in A$$

Let's show that $\max(A) = \sup(A)$:
We know from definition, that s is an upper-bound of A.
Let's take any arbitrary $\epsilon > 0$, therefore because $s \in A$:

$$s > s - \epsilon$$

Therefore, by the definition of the supremum(proposition 5.7 in class), s is A's supremum.**(2)**

2.1.2 $\impliedby$:

Let's suppose that $s = \sup(A)$ and $s \in A$.
According to the definition of the maximum, if s is the maximum of A, it must be:

1. a member of A
2. a supremum of A

Both are given. □

2.2 Prove that if A has a maximum, there's only one.

By definition, if a set has a maximum, it fulfills the following properties:

1. it is a member of the set
2. it is a supremum of the set

Suppose m is the maximum of A.
By definition, it must be part of A, satisfying **(1)**. Because of **(2)**, it must be a supremum.

2.2.1 n's smaller than m

According to the definition of the upper-bound:

$$\forall \epsilon > 0 \ \ \exists a \in A \ \ a > m - \epsilon$$

Therefore, $m - \epsilon$ is smaller than m, and is not an upper-bound, therefore not a maxmium.

2.2.2 n's bigger than m

Likewise, $m + \epsilon$ is bigger than m, and is by definition an upper-bound.
However, we've stated that m is the upper-bound of A, and according to the upper-bound definition,
Every number that is bigger than the upper-bound of a set, is not a member of the set.
Therfore, we've shown that there can only exist a single maximum, if any. $\square$

3 $\varnothing \neq A, B \subseteq \mathbb{F}$ Prove or disprove:

3.1

Let $A = \mathbb{R}$, and $B = \mathbb{N}$.
Therefore, we can see that A is not lower-bounded, while B is lower-bounded by 1. Disproved. $\square$

3.2

If the claim is incorrect, then B mustn't have an upper-bound.
Since B is a subset of A, B must be bounded from above as well, that is due to the definition of an upper bound.
In formal notation, as $B \subseteq A$:

$$(\forall b \in B)\ (\exists a \in A)\ (a \geq b)$$

However, since A is bounded from above:

$$(\forall a \in A)\ (\exists m)\ (m \geq a)$$

Therefore, due to transitivity:

$$m \geq b$$

$\square$

4 $\varnothing \neq A, B \subseteq \mathbb{F} \quad \exists \sup(A), \exists \sup(B)$ **Prove or disprove:**

4.1 $B \subseteq A \implies \sup(B) \leq \sup(A)$

Let $m = \sup(A) = \max(A)$.
Therefore, according to the definition of the *supremum*:

$$\forall a \in A \; m \geq a$$

Now, let's take a look at the maximum of B:

$$\forall b \in B \; m' \geq b$$

Since $B \subseteq A$, the maximum value of B can, at the most be equal to m, that is:

$$m' \leq m$$

□

4.2 $(B \subseteq A) \;\wedge\; (B \neq A) \implies \sup(B) < \sup(A)$

Let:

$$A = \{1, 2, 3\}$$
$$B = A \backslash \{1\}$$

We can clearly see that $(B \subseteq A) \;\wedge\; (B \neq A)$.
However, $\sup(B) = \sup(A) = 3$, thus the claim is incorrect. □

4.3 $-A = \{-a \mid a \in A\}$

It is given that A necessarily has a supremum.
Therefore, -A is necessarily bounded from below by that same supremum, as a negative.
Let m be A's supremum, and -m be -A's infimum.
According to the definition of the *supremum*:

$$\forall a \in A \; m \geq a$$

We'll multiply by *(-1)*:

$$-m \leq -a$$

Note that this is exactly the definition of the infimum. Thus, we can conclude that -A is bounded from below, $\exists \inf(-A)$ and that $\inf(-A) = -\sup(A)$. □

5

5.1 $A = \{x^2 + 7x + 10 \mid x \in \mathbb{F}\}$

First, we'll pack $x^2 + 7x + 10$ into $(x + 5)(x + 2)$ and get:

$$A = \{(x + 5)(x + 2) \mid x \in \mathbb{F}\}$$

5.1.1 upper-bound, supremum & maximum

Now, it is easier to see that as x gets bigger, the equation gets bigger.
According to *definition 5.1*, a set is bounded from above only if:

$$(\exists M \in \mathbb{F})\ (\forall a \in A)\ (a \leq M)$$

However, there doesn't exist such an M, because for every upper bound that we'll test, the equation can yield a bigger number.
Because of that, we can conclude that A is **not** bounded from above, and therefore, there doesn't exist either a supremum nor a maximum.

5.1.2 lower-bound, infimum & minimum

We can rewrite $x^2 + 7x + 10$:

$$\begin{aligned} x^2 + 7x + 10 &= x^2 + 7x + 10 + 2.25 - 2.25 \\ &= (x + 3.5)^2 - 2.25 \end{aligned}$$

We know that $(x + 3.5)^2 \geq 0$, therefore:

$$\begin{aligned} (x + 3.5)^2 - 2.25 &\geq -2.25 \\ x^2 + 7x + 10 &\geq -2.25 \end{aligned}$$

Therefore, the lowest value that can be received from the equation is -2.25, i.e.:

$$inf(A) = -2.25$$

since -2.25 is a member of A - as we can place $x = -3.5$ to get this value - it is the minimum as well, i.e.:

$$min(A) = inf(A) = -2.25$$

5.2 $B = \{x \in \mathbb{F} \mid x^2 + 7x + 10 > 0\}$

Similiar to set A, we'll pack:

$$x^2 + 7x + 10 = (x+5)(x+2)$$

Therefore:

$$B = \{x \in \mathbb{F} \mid (x+5)(x+2) > 0\}$$

Therefore, the only members in B are all of the x's that will yield positive numbers.
As we've shown for A, the members therefore are:

$$B = \{x \in \mathbb{F} \mid (x < -5) \vee (-2 < x)\}$$

5.2.1 upper-bound, supremum & maximum

As x can be any x that is bigger than -2, we can see that it is not bounded from above.
Therefore, the supremum & maximum do not exist as well.

5.2.2 lower-bound, infimum & minimum

Similiarly, the value of x can always be lower, which is in contrast to the definition of the lower-bound:

$$(\exists m \in \mathbb{F})\ (\forall b \in B)\ (b \geq m)$$

Let's assume that such an m **does** exist.
Let's assume that $b = m$, if we add (-1) to b, we'll get a member that is still in the set, as we can always find a smaller member of B.
And in more formal notation:

$$\begin{aligned} &b = m \in \mathbb{F} \\ &b - 1 \in \mathbb{F} \\ &b - 1 < m \end{aligned}$$

Which contradicts the definition of the lower-bound.
Therefore, the lower-bound (and as a result of this the infimum and the minimum) doesn't exist.

6

6.1 The lower-bound property:

In $\mathbb{R}$, there exists an infimum for every non-empty set that is bounded from below.

6.2

6.2.1 lower-bound property $\implies$ Completeness of the real numbers

In order to demonstrate the completeness axiom, we need to show that:

$$(\forall A, B \subset \mathbb{F})(\exists c \in \mathbb{F})(\forall a \in A)(\forall b \in B)(a \leq b)(a \leq c \leq b)$$

Let $\varnothing \neq B \subset \mathbb{R}$ that is bounded from below.
According to the lower-bound property - there exists an infimum $c \in \mathbb{R}$ such that:

$$(\forall b \in B)\ (c \leq b)$$

Now, let's form a new group that contains only $\inf(B)$:

$$A = \{\inf(B)\}$$

We've shown that there exists an infimum to B, hence A must be non-empty.
Therefore, we've shown what was needed.

6.2.2 lower-bound property $\impliedby$ Completeness of the real numbers

Using the completeness axiom, we need to show that a non-empty set, B, that is bounded-from-below, has an infimum.
According to definition, in order for a set to have an infimum, M, it must fullfill:

1. Be a lower bound of B
2. $(\forall \epsilon > 0)(\exists b \in B)(a < M + \epsilon)$

Let $B \subset \mathbb{R}$ some non-empty set that is bounded from below, and A:

$$A = \{x \in \mathbb{R} :\ \forall b \in B\ x \leq b\}$$

Since B is bounded from below, and A consists of all of the numbers that are less than or equal to all numbers in B, we can assume:

1. $A \neq \varnothing$
2. According to the completeness axiom, there exists $c \in \mathbb{R}$ such that B is bounded-from-below by c and A is bounded-from-above by c.

That is:

$$\begin{aligned} \forall a \in A\ a &\leq c \\ \forall b \in B\ b &\geq c \\ b &\geq a \end{aligned}$$

Therefore, we've shown that c is the greatest lower-bound of B. $\square$

7