Discrete Math

Exercise 2

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1

1.1 $f(C_1 \backslash C_2) = f(C_1) \backslash f(C_2)$

The statement is incorrect.
Let:

$$
\begin{aligned}
&A = \{1, 2, 3\} \\
&B = \{4, 5, 6\} \\
&f(1) = 4 \\
&f(2) = 5 \\
&f(3) = 4 \\
&C_1 = \{1, 2\} \\
&C_2 = \{2, 3\}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&f(C_1 \backslash C_2) = f(1) = 4 \\
&f(C_1) \backslash f(C_2) = \{4, 5\} \backslash \{4, 5\} = \varnothing
\end{aligned}
$$

Therefore, we can see that:

$$f(C_1 \backslash C_2) \neq f(C_1) \backslash f(C_2)$$

1.2 $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$

In order to prove this identity, we'll need to show the following:

$$
\begin{aligned}
&f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2) \\
&f(C_1) \cup f(C_2) \subseteq f(C_1 \cup C_2)
\end{aligned}
$$

1.2.1 $f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2)$

Let $y \in f(C_1 \cup C_2)$.
Therefore, there exists $x \in C_1 \cup C_2$ such that $f(x) = y$.
Hence, there are two possibilities, $x \in C_1$ or $x \in C_2$.
If $x \in C_1$, then $y \in f(C_1)$, and similarly, if $x \in C_2$, then $y \in f(C_2)$.
Therefore, $y \in f(C_1)$ or $y \in f(C_2)$, or formally:

$$y \in f(C_1) \cup f(C_2)$$

1.2.2 $f(C_1) \cup f(C_2) \subseteq f(C_1 \cup C_2)$

Let $y \in f(C_1) \cup f(C_2)$.
If $y \in f(C_1)$, then there exists an $x \in C_1$ such that $f(x) = y$.
If $x \in C_1$, then $x \in C_1 \cup C_2$, and therefore $f(x) = y$ still stands.
Therefore:

$$y \in f(C_1 \cup C_2)$$

$\square$

2

2.1

The statement is true. First, let's look at: $f(C) \cap D$
Let:

$$y \in f(C) \cap D$$

Therefore:

$$y \in f(c) \tag{1}$$
$$y \in D \tag{2}$$

Now, let's look at: $f(C \cap f^{-1}(D))$
Let:

$$x \in C \cap f^{-1}(D)$$

Therefore:

$$x \in C \tag{3}$$
$$x \in f^{-1}(D) \tag{4}$$

If we apply f on 3 and 4, we'll return to 1 and 2, accordingly.

2.2

The statement is incorrect, here's a counter example:

$$\begin{aligned}
&A = \{1\} \\
&B = \{2, 3\} \\
&C = A = \{1\} \\
&D = \{2\} \\
&f(1) = 3
\end{aligned}$$

Now, we can see:

$$\begin{aligned}
&f(C) \cup D = \{2, 3\} \\
&f(C \cup f^{-1}(D)) = f(\{1\} \cup \{\}) = f(\{1\}) = f(1) = 3 \\
&\{2, 3\} \neq \{3\}
\end{aligned}$$

$\square$

3

It is given that $g \circ f$ is surjective, therefore:

$$(\forall y \in C)\ (\exists x \in A)\ g(f(x)) = y$$

We want to show that g is surjective, that is:

$$(\forall y \in C)\ (\exists x \in B)\ g(x) = y$$

Because f is defined as:

$$f : A \longrightarrow B$$

We can read $g \circ f$ surjection as: "f outputs g's input, and then every output of g is surjective".
Therefore, we can conclude that if $g \circ f$ is surjective, g **must** be surjective as well.

4

4.1

The statement is incorrect.
Let:

$$\begin{aligned} &A = \{1, 2\} \\ &B = \{3, 4\} \\ &C = \{5\} \\ &f(1) = 3 \\ &f(2) = 3 \\ &g(1) = 4 \\ &g(2) = 4 \\ &h(3) = 5 \\ &h(4) = 5 \end{aligned}$$

Therefore, we can see that h is **surjective**, $h \circ f = h \circ g$, but $g \neq f$. $\square$

4.2

The statement is correct, let's show it.
it is given that $h \circ f = h \circ g$, therefore:

$$\forall x \in A \;\; h(f(x)) = h(g(x))$$

However, because it is given that h is **injective**, we can conclude that:

$$g(x) = f(x)$$

And therefore:

$$g = f$$

$\square$

5

5.1 Compute $g \circ f$

$$h_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 6 & 4 & 1 & 7 & 5 \end{pmatrix}$$

5.2 Compute $f \circ g$

$$h_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 7 & 1 & 5 & 4 & 6 \end{pmatrix}$$

5.3 Find the order of h_1 and h_2

First, we'll factor h_1 and h_2:

$$h_1 = (13675)(2)(4)$$
$$h_2 = (13764)(2)(5)$$

Therefore, the order of h_1 and h_2 is 5.

5.4 Find the permutation h_1^{5781} and h_2^{5782}

5.4.1 h_1^{5781}

We can see that:

$$5781 \ mod \ 5 = 1$$

Therefore:

$$h_1^{5781} = h_1^1 = h_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 6 & 4 & 1 & 7 & 5 \end{pmatrix}$$

5.4.2 h_2^{5782}

We can see that:

$$5782 \ mod \ 5 = 2$$

Therefore:

$$h_2^{5782} = h_2^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 6 & 3 & 5 & 1 & 4 \end{pmatrix}$$