Calculus I

Exercise 1

Aviv Vaknin
316017128

1

1.1

While i and ii are statements, iii isn't a statement,
because we haven't received any information about x's value.

1.2

$i)$
Statement:

$$\forall n \in \mathbb{F} \ \ \exists m \in \mathbb{F} \ \Big| \ n = m + m$$

Negated Statement:

$$\exists n \in \mathbb{F} \ \ \forall m \in \mathbb{F} \ \Big| \ n \neq m + m$$

$ii)$
Statement:

$$\forall m, n \in \mathbb{F} \ \ n = m + m \to -n = -m - m$$

Negated Statement:

$$\exists m, n \in \mathbb{F} \ \ n = m + m \ \wedge \ -n \neq -m - m$$

1.3

$i)$
Statement:

$$\forall n \in \mathbb{F} \ \ \exists m \in \mathbb{F} \ \Big| \ n = m + m$$

Negated Statement:

$$\exists n \in \mathbb{F} \ \ \forall m \in \mathbb{F} \ \Big| \ n \neq m + m$$

2

i is the formal representation of a field's additive inverse axiom, i.e. A4.
On the other hand, ii states that in the field $\mathbb{F}$, there's a certain number, x, that if we'll add it to **any** other number in $\mathbb{F}$, we'll receive $0_{\mathbb{F}}$.
The two statements are **not** logically equal.

3

3.1 Prove $\forall\ a, b \in \mathbb{F}\ \ -(a-b) = (b-a)$

First, let's find $(a-b)$'s inverse:

$$(a-b) + x = 0$$

We'll add $(b-a)$ to both sides of the equation:

$$(a-b) + (b-a) + x = (b-a)$$

And find the inverse:

$$x = (b-a)$$

Now, we can easily see that $(a-b)$ and $(b-a)$ are the inverses of each other.
And due to the additive inverse axiom (A4) :

$$-(a-b) = x = (b-a)$$

$$\blacksquare$$

3.2 Prove the 'uniquness of multiplicitive inverse' property

It is given that $ab, ac = 1_{\mathbb{F}}$, and we need to prove that $b = c = a^{-1}$.

3.2.1 $ab = 1_{\mathbb{F}}$

According to the multiplicitive inverse property (M4), we can deduct:

$$b = a^{-1}$$

3.2.2 $ac = 1_{\mathbb{F}}$

Exactly as above (M4), we can deduct:

$$c = a^{-1}$$

Therefore, we can conclude:

$$b = c = a^{-1}$$

$$\blacksquare$$

4 H is a set that satisfies all of the field axioms, $H \neq \emptyset$, $1_H = 0_H$ Prove that H contains only a single member.

Adding two 0_H should result in a 0_H, due to axiom A3:

$$0_H + 0_H = 0_H$$

However, because $1_H = 0_H$, it also means that:

$$1_H + 1_H = 0_H$$

Because of that, we can conculde that no other members exist in H, except $1_H = 0_H$

$$\blacksquare$$

5 $\mathbb{F}$ is an ordered field, prove the following:

5.1 $\underline{\forall x, y \in \mathbb{F}\ 0_{\mathbb{F}} < x < y \iff 0_{\mathbb{F}} < y^{-1} < x^{-1}}$

5.1.1 $\underline{0_{\mathbb{F}} < x < y \implies 0_{\mathbb{F}} < y^{-1} < x^{-1}}$:

It is given that:

$$x < y$$

We'll multiple both sides of the inequality by 1, using axiom *M4*:

$$xyy^{-1} < yxx^{-1}$$

It is given that $x, y > 0$ therefore we can divide the equation by xy:

$$y^-1 < x^-1$$

5.1.2 $\underline{0_{\mathbb{F}} < x < y \impliedby 0_{\mathbb{F}} < y^{-1} < x^{-1}}$:

It is given that:

$$y^{-1} < x^{-1}$$

We'll multiple both sides of the inequality by 1, using axiom *M4*:

$$y^{-1}xx^{-1} < x^{-1}yy^{-1}$$

It is given that $x^-1, y^-1 > 0$ therefore we can divide the equation by x^-1y^-1:

$$x < y$$

$$\blacksquare$$

5.2 $\underline{x, y, z, w \in \mathbb{F} \big| \; x < y, \; z \leq w \implies x + z < y + w}$

It is given that:

$$x < y$$

We'll add $(z + w)$ to both sides, according to axiom *O3*:

$$x + (z + w) < y + (z + w)$$

According to axiom *A1*, we'll rearrange the inequality:

$$(x + z) + w < (y + w) + z$$

It is given that $z \leq w$, therefore if we'll remove w from the left side, and z from the right side, the inequality should remain correct:

$$x + z < y + w$$

$$\blacksquare$$

5.3 $\forall x, y \in \mathbb{F}\ (0_{\mathbb{F}} < xy) \iff ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y))$

5.3.1 $0_{\mathbb{F}} < xy \Longrightarrow ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y))$:

Due to the ordered field's trichotomy axiom, x, y must be > 0 or < 0, it is known that $xy > 0$ and therefore $x, y \neq 0$ (as proven before).
If $x > 0$:

$$xy > 0$$

Let's divide by x:

$$y > 0$$

Else, if $x < 0$:

$$xy > 0$$

If we divide by x, the $>$ will change to a $<$, as proven previously in exercise 2.5:

$$y < 0$$

Therefore, we can see that if $xy > 0$, $x, y > 0$ or $x, y < 0$ must be true.

5.3.2 $0_{\mathbb{F}} < xy \Longleftarrow ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y))$:

First, let's assume that $0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y$:

$$x > 0$$

According to ordered field's axiom 4, we can multiply both sides of the inequality by y:

$$xy > 0$$

Now, let's assume that $0_{\mathbb{F}} > x \wedge 0_{\mathbb{F}} > y$:

$$x < 0$$

If we multiply both sides of the equation by y(which is negative), the inequality will change signs:

$$xy > 0$$

$$\blacksquare$$

5.4 Prove:

$$0 < b \in \mathbb{F} \ \ \forall a \in \mathbb{F} \ \ a^2 < b^2 \Longrightarrow -b < a < b$$

Due to the ordered field's trichotomy axiom, a is one of the following:

- $a < 0$
- $a = 0$
- $a > 0$

Therefore, we'll need to show that the statement is true for all three.

5.4.1 $a = 0$:

$$b > 0$$

Using the ordered field's $O3$ axiom, we'll subtract b from both sides:

$$b - b > -b$$

Using $A3$:

$$-b < 0$$

Now, according to transitivity:

$$-b < 0 = a < b$$

5.4.2 $a > 0$:

First, we'll need a lemma to help us demonstrate an idea.

Lemma 1.
We want to show that

$$a, b > 0 \ \ a^2 > b^2 \Longrightarrow \ a > b$$

We'll prove that by contraposition:

$$a \leq b \Longrightarrow \ a^2 \leq b^2$$

We'll multiply the left side by b, and by a:

$$a \leq b$$

$$a^2 \leq a \cdot b$$

$$a \cdot b \leq b^2$$

Therefore, according to the transitivity axiom:

$$a^2 \leq b^2$$

As shown in the lemma, we know that $b > a$.
In addition, it was proven in class that if $b > 0$, $-b < 0$.
Therefore, according to transitivity:

$$-b < 0 < a < b$$

5.4.3 $a < 0$:

One side of the inequality is easy to show:

$$b > 0 > a$$

Now, we'll try to show that $-b < a$:

$$b^2 > a^2$$

We'll deduct a^2 from both sides, according to O3:

$$b^2 - a^2 > 0$$

Now, using distributivity:

$$(b + a)(b - a) > 0$$

We can see that there are two possible scenarios where this inequality is true:

$$(b + a) > 0 \wedge (b - a) > 0$$

$$(b + a) < 0 \wedge (b - a) < 0$$

However, because $b > a$, we can tell that the second scenario cannot happen, therefore:

$$b - a > 0$$

$$b + a > 0$$

Therefore we can show that:

$$a > -b$$

$$\blacksquare$$

6 Prove or disprove the following, for $a, b, x, y \in \mathbb{F}$:

6.1 $\underline{ab < a + b}$

Let $a, b = 0_{\mathbb{F}}$:

$$0_{\mathbb{F}} \cdot 0_{\mathbb{F}} < 0_{\mathbb{F}} + 0_{\mathbb{F}}$$

According to Axioms $A3$ and $M3$ it yields the **false** statement:

$$0_{\mathbb{F}} < 0_{\mathbb{F}}$$

6.2 $\underline{x^2 < y^2 \Longrightarrow x < y}$

In order to disprove this statement, we need to find $x, y \in \mathbb{F}$ such that:

$$x^2 < y^2 \ \wedge \ x \geq y$$

As an example, we can take:

$$x = 1_{\mathbb{F}}$$

$$y = -(1_{\mathbb{F}} + 1_{\mathbb{F}})$$

We can see that $x \geq y$.
x^2:

$$x^2 = 1_{\mathbb{F}} \cdot 1_{\mathbb{F}} = 1_{\mathbb{F}}$$

y^2:

$$y^2 = -(1_{\mathbb{F}} + 1_{\mathbb{F}}) \cdot (-(1_{\mathbb{F}} + 1_{\mathbb{F}})) = 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}}$$

We can see that $y^2 > x^2$, thus, the statement is **false**.

6.3 $\underline{x < y \Longrightarrow x^2 < y^2}$

In order to disprove this statement, we need to find $x, y \in \mathbb{F}$ such that:

$$x < y \ \wedge \ x^2 \geq y^2$$

We can use the same examples from 6.2, only swapping the x and the y:

$$y = 1_{\mathbb{F}}$$

$$x = -(1_{\mathbb{F}} + 1_{\mathbb{F}})$$

We can see that $y \geq x$.
y^2:

$$y^2 = 1_{\mathbb{F}} \cdot 1_{\mathbb{F}} = 1_{\mathbb{F}}$$

x^2:

$$x^2 = -(1_{\mathbb{F}} + 1_{\mathbb{F}}) \cdot (-(1_{\mathbb{F}} + 1_{\mathbb{F}})) = 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}}$$

We can see that $x^2 > y^2$, thus, the statement is **false**.

7 Prove $\forall a \in \mathbb{F} \ |-a| = |a|$:

According to the definition of absolute value, there are two different scenarios for a:

7.0.1 $\underline{a \geq 0}$:

Since $a \geq 0$, using the abs definition we'll get:

$$|a| = a$$

According to the ordered field axioms $-a < 0$, and therefore:

$$|(-a)| = -(-a) = a$$

We can see that:

$$a \geq 0 \Longrightarrow |-a| = |a| = a$$

7.0.2 $\underline{a < 0}$:

Since $a < 0$, using the abs definition will yield:

$$|a| = -a$$

We've previously proved that the negative of a negative is positive, i.e. $-a > 0$, therefore:

$$|-a| = -a$$

We can see that:

$$a < 0 \Longrightarrow |-a| = |a| = -a$$

$$\blacksquare$$

8 Prove the following for $a, b \in \mathbb{F}$:

8.1 $\underline{\max(a,b) = \frac{1}{2}(a + b + |a - b|)}$

This splits into three different cases:

- $a < b$
- $a = b$
- $a > b$

8.1.1 $\underline{a < b}$:

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|)$$

Since $a < b$, the term $|a - b|$ is equal to $b - a$, therefore:

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + b - a) = \frac{1}{2}(2b) = b$$

8.1.2 $\underline{a = b}$:

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|)$$

Since $a = b$, the term $|a - b|$ is equal to 0, therefore:

$$\max(a,b) = \frac{1}{2}(a + b + 0) = \frac{1}{2}(a + b) = \frac{a + b}{2} = a = b$$

8.1.3 $\underline{a > b}$:

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|)$$

Since $a > b$, the term $|a - b|$ is equal to $a - b$, therefore:

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + a - b) = \frac{1}{2}(2a) = a$$

8.2 $\min(a,b) = \frac{1}{2}(a+b-|a-b|)$

This splits into three different cases:

- $a < b$
- $a = b$
- $a > b$

8.2.1 $a < b$:

$$\min(a,b) = \frac{1}{2}(a+b-|a-b|)$$

Since $a < b$, the term $|a-b|$ is equal to $b-a$, therefore:

$$\min(a,b) = \frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-(b-a)) = a$$

8.2.2 $a = b$:

$$\min(a,b) = \frac{1}{2}(a+b-|a-b|)$$

Since $a = b$, the term $|a-b|$ is equal to 0, therefore:

$$\min(a,b) = \frac{1}{2}(a+b-0) = \frac{1}{2}(a+b) = \frac{a+b}{2} = a = b$$

8.2.3 $a > b$:

$$\min(a,b) = \frac{1}{2}(a+b-|a-b|)$$

Since $a > b$, the term $|a-b|$ is equal to $a-b$, therefore:

$$\min(a,b) = \frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-(a-b)) = \frac{1}{2}(2b) = b$$

$$\blacksquare$$