1

1.1

While i and ii are statements, iii isn't a statement,
because we haven't received any information about x's value.

1.2

$i)$
Statement:

$$\forall n \in \mathbb{F} \ \ \exists m \in \mathbb{F} \ \Big| \ n = m + m$$

Negated Statement:

$$\exists n \in \mathbb{F} \ \ \forall m \in \mathbb{F} \ \Big| \ n \neq m + m$$

$ii)$
Statement:

$$\forall m, n \in \mathbb{F} \ \ n = m + m \rightarrow -n = -m - m$$

Negated Statement:

$$\exists m, n \in \mathbb{F} \ \ n = m + m \ \wedge \ -n \neq -m - m$$

1.3

$i)$
Statement:

$$\forall n \in \mathbb{F} \ \ \exists m \in \mathbb{F} \ \Big| \ n = m + m$$

Negated Statement:

$$\exists n \in \mathbb{F} \ \ \forall m \in \mathbb{F} \ \Big| \ n \neq m + m$$

2

i is the formal representation of a field's additive inverse axiom, i.e. A4.
On the other hand, ii states that in the field $\mathbb{F}$, there's a certain number, x, that if we'll add it to **any** other number in $\mathbb{F}$, we'll receive $0_{\mathbb{F}}$.
The two statements are **not** logically equal.

3

3.1 Prove $\forall\ a, b \in \mathbb{F}\ \ -(a-b) = (b-a)$

First, let's find $(a-b)$'s inverse:

$$(a-b) + x = 0$$

We'll add $(b-a)$ to both sides of the equation:

$$(a-b) + (b-a) + x = (b-a)$$

And find the inverse:

$$x = (b-a)$$

Now, we can easily see that $(a-b)$ and $(b-a)$ are the inverses of each other.
And due to the additive inverse axiom (A4) :

$$-(a-b) = x = (b-a)$$

$\blacksquare$

3.2 Prove the 'uniquness of multiplicitive inverse' property

It is given that $ab, ac = 1_{\mathbb{F}}$, and we need to prove that $b = c = a^{-1}$.

3.2.1 $ab = 1_{\mathbb{F}}$

According to the multiplicitive inverse property (M4), we can deduct:

$$b = a^{-1}$$

3.2.2 $ac = 1_{\mathbb{F}}$

Exactly as above (M4), we can deduct:

$$c = a^{-1}$$

Therefore, we can conclude:

$$b = c = a^{-1}$$

$\blacksquare$

4 H is a set that satisfies all of the field axioms, $H \neq \emptyset$, $1_H = 0_H$ Prove that H contains only a single member.

Adding two 0_H should result in a 0_H, due to axiom A3:

$$0_H + 0_H = 0_H$$

However, because $1_H = 0_H$, it also means that:

$$1_H + 1_H = 0_H$$

Because of that, we can conculde that no other members exist in H, except $1_H = 0_H$

$$\blacksquare$$

5 $\mathbb{F}$ is an ordered field, prove the following:

5.1 $\underline{\forall x, y \in \mathbb{F}\ 0_{\mathbb{F}} < x < y \iff 0_{\mathbb{F}} < y^{-1} < x^{-1}}$

5.1.1 $\underline{0_{\mathbb{F}} < x < y \implies 0_{\mathbb{F}} < y^{-1} < x^{-1}}$:

It is given that:

$$x < y$$

We'll multiple both sides of the inequality by 1, using axiom *M4*:

$$xyy^{-1} < yxx^{-1}$$

It is given that $x, y > 0$ therefore we can divide the equation by xy:

$$y^-1 < x^-1$$

5.1.2 $\underline{0_{\mathbb{F}} < x < y \impliedby 0_{\mathbb{F}} < y^{-1} < x^{-1}}$:

It is given that:

$$y^{-1} < x^{-1}$$

We'll multiple both sides of the inequality by 1, using axiom *M4*:

$$y^{-1}xx^{-1} < x^{-1}yy^{-1}$$

It is given that $x^-1, y^-1 > 0$ therefore we can divide the equation by x^-1y^-1:

$$x < y$$

$$\blacksquare$$

5.2 $x, y, z, w \in \mathbb{F} \big| \ x < y, \ z \leq w \implies x + z < y + w$

It is given that:

$$x < y$$

We'll add $(z + w)$ to both sides, according to axiom *O3*:

$$x + (z + w) < y + (z + w)$$

According to axiom *A1*, we'll rearrange the inequality:

$$(x + z) + w < (y + w) + z$$

It is given that $z \leq w$, therefore if we'll remove w from the left side, and z from the right side, the inequality should remain correct:

$$x + z < y + w$$

$$\blacksquare$$

5.3 $\forall x, y \in \mathbb{F}\ (0_{\mathbb{F}} < xy) \iff ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y))$

5.3.1 $0_{\mathbb{F}} < xy \implies ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y))$:

Due to the ordered field's trichotomy axiom, x, y must be > 0 or < 0, it is known that $xy > 0$ and therefore $x, y \neq 0$ (as proven before).
If $x > 0$:

$$xy > 0$$

Let's divide by x:

$$y > 0$$

Else, if $x < 0$:

$$xy > 0$$

If we divide by x, the $>$ will change to a $<$, as proven previously in exercise 2.5:

$$y < 0$$

Therefore, we can see that if $xy > 0$, $x, y > 0$ or $x, y < 0$ must be true.

5.3.2 $0_{\mathbb{F}} < xy \impliedby ((x < 0_{\mathbb{F}} \wedge y < 0_{\mathbb{F}}) \vee (0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y))$:

First, let's assume that $0_{\mathbb{F}} < x \wedge 0_{\mathbb{F}} < y$:

$$x > 0$$

According to ordered field's axiom 4, we can multiply both sides of the inequality by y:

$$xy > 0$$

Now, let's assume that $0_{\mathbb{F}} > x \wedge 0_{\mathbb{F}} > y$:

$$x < 0$$

If we multiply both sides of the equation by y(which is negative), the inequality will change signs:

$$xy > 0$$

$$\blacksquare$$

5.4 <u>Prove:</u>

$$0 < b \in \mathbb{F} \ \ \forall a \in \mathbb{F} \ \ a^2 < b^2 \Longrightarrow -b < a < b$$

Due to the ordered field's trichotomy axiom, a is one of the following:

- $a < 0$
- $a = 0$
- $a > 0$

5.4.1 <u>$a = 0$:</u>

$$b > 0$$

Using the ordered field's O3 axiom:

$$b - b > -b$$

And the additive inverse identity axiom:

$$0 > -b$$

In addition:

$$a^2 < b^2 \Longrightarrow a < b$$

Now, according to transitivity:

$$-b < 0 = a < b$$

5.4.2 <u>$a > 0$:</u>

$$a^2 < b^2 \Longrightarrow a < b$$

We'll multiply the result by (-1), and the inequality signs will change accordingly:

$$-a > -b$$