

Calculus I

Exercise 2

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1 Prove that

$$a \leq b \iff \forall \epsilon > 0 \quad a < b + \epsilon$$

1.1 $a \leq b \Rightarrow \forall \epsilon > 0 \quad a < b + \epsilon$:

It is given that:

$$0 < \epsilon \qquad a \leq b$$

Therefore, because according to axiom *O3* an ordered field is adhering to addition:

$$a + 0 < b + \epsilon$$

It is worth mentioning that due to the uniqueness of zero, $b + \epsilon$ must be *greater* than zero, rather than greater than or *equal* to zero.

1.2 $a \leq b \Leftarrow \forall \epsilon > 0 \quad a < b + \epsilon$:

We'll rephrase using contraposition:

$$a > b \Rightarrow \exists \epsilon > 0 \quad a \geq b + \epsilon$$

We need to find an $\epsilon > 0$ so that $a \geq b + \epsilon$.

Let $\epsilon = a - b$, now, we can see that:

$$\begin{aligned} b + \epsilon &= b + (a - b) = a \\ a &\geq b + \epsilon \end{aligned}$$

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2 Prove that

$$\forall m, n \in \mathbb{N} \quad mn \in \mathbb{N}$$

Let's assume that m is some arbitrary natural number.

We'll prove using induction, starting with $n = 1$: According to axiom *M3*:

$$m \cdot 1 = m$$

It is given that $m \in \mathbb{N}$, therefore this case is valid. Now, let's assume that it is true for a general n , i.e. $n = n$, and:

$$mn \in \mathbb{N}$$

Now, we'll check $n = n + 1$: According to axiom *D*:

$$m(n + 1) = mn + m$$

We've assumed that $mn \in \mathbb{N}$, and it is given that $m \in \mathbb{N}$.

In addition, we've shown in exercise *2a* that the natural numbers adhere to addition.

Therefore:

$$mn \in \mathbb{N}, m \in \mathbb{N} \implies (mn + m) \in \mathbb{N}$$

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3 Prove the following:

3.1 Prove:

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

We'll prove this by induction.

3.1.1 $n = 1$

we can see that the statement is true for $n = 1$:

$$\sum_{k=1}^1 k^2 \stackrel{(def)}{=} 1^2 = \frac{1(1+1)(2+1)}{6}$$

3.1.2 $n = n$

Now, we'll assume it is true for $n = n$, i.e.:

$$\sum_{k=1}^n k^2 \stackrel{(def)}{=} 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3.1.3 $n = n + 1$

And now, we'll check $n = n + 1$:

$$\sum_{k=1}^{n+1} k^2 \stackrel{(def)}{=} 1^2 + 2^2 + \dots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

Now, according to the definition:

$$\sum_{k=1}^n k^2 + (n+1)^2 = \sum_{k=1}^{n+1} k^2$$

Therefore, we need to check if:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

And after solving each side, we receive:

$$0 = 0$$

Therefore, we've proved that the statement is true for $n = n + 1$.

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3.2 Prove:

$$\sum_{i=0}^{n-1} x^i = \frac{1_{\mathbb{F}} - x^n}{1_{\mathbb{F}} - x}$$

Note: I'll use $1_{\mathbb{F}}$ and 1 interchangeably while proving this.
We'll prove this by induction.

3.2.1 $n = 1$

we can see that the statement is true for $n = 1$, because by definition, $a^0 = 1$:

$$\sum_{i=0}^0 x^i \stackrel{(def)}{=} x^0 = \frac{1_{\mathbb{F}} - x^1}{1_{\mathbb{F}} - x} = \frac{1_{\mathbb{F}} - x}{1_{\mathbb{F}} - x} = 1$$

3.2.2 $n = k$

Now, let's assume that the statement is true for $n = k$, i.e.:

$$\sum_{i=0}^{k-1} x^i \stackrel{(def)}{=} 1 + x + x^2 \dots + x^{k-1} = \frac{1_{\mathbb{F}} - x^k}{1_{\mathbb{F}} - x}$$

3.2.3 $n = k + 1$

$$\sum_{i=0}^k x^i \stackrel{(def)}{=} 1 + x + x^2 \dots + x^{k-1} + x^k = \frac{1_{\mathbb{F}} - x^{k+1}}{1_{\mathbb{F}} - x}$$

According to the definition:

$$\sum_{i=0}^{k-1} x^i + x^k = \sum_{i=0}^k x^i$$

Therefore, we need to prove that:

$$\frac{1_{\mathbb{F}} - x^k}{1_{\mathbb{F}} - x} + x^k = \frac{1_{\mathbb{F}} - x^{k+1}}{1_{\mathbb{F}} - x}$$

$$\frac{1_{\mathbb{F}} - x^k}{1_{\mathbb{F}} - x} + x^k = \frac{1_{\mathbb{F}} - x^k + (1_{\mathbb{F}} - x)x^k}{1_{\mathbb{F}} - x} = \frac{1_{\mathbb{F}} - x^k + x^k - x^{k+1}}{1_{\mathbb{F}} - x} = \frac{1_{\mathbb{F}} - x^{k+1}}{1_{\mathbb{F}} - x}$$

Thus, we've proved that:

$$\sum_{i=0}^{k-1} x^i + x^k = \sum_{i=0}^k x^i$$

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4

$$m, n, s, t \in \mathbb{F} \quad n, t > 0$$

Prove:

$$\frac{m}{n} < \frac{s}{t} \implies \frac{m}{n} < \frac{m+s}{n+t} < \frac{s}{t}$$

Lemma 1. *We'll show that if $b, d > 0$:*

$$\frac{a}{b} > \frac{c}{d} \implies \frac{a}{b} > \frac{a+c}{b+d}$$

It is given that:

$$ab^{-1} > cd^{-1}$$

We'll multiply both sides by bd and then add ab :

$$\begin{aligned} ad &> bc \\ ab + ad &> ab + bc \end{aligned}$$

We'll pack using distributivity and then multiply by $b^{-1}(b+d)^{-1}$:

$$\begin{aligned} a(b+d) &> b(a+c) \\ ab^{-1} &> (a+c)(b+d)^{-1} \\ \frac{a}{b} &> \frac{a+c}{b+d} \end{aligned}$$

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Using *Lemma 1*, and the fact that $n, t > 0$:

$$\frac{m+s}{n+t} < \frac{s}{t}$$

Similarly, *Lemma 1* can be used in its mirrored form to show that:

$$\frac{m}{n} < \frac{m+s}{n+t}$$

Therefore, according to transitivity:

$$\frac{m}{n} < \frac{m+s}{n+t} < \frac{s}{t}$$

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8

$$x, y \geq 0, \quad n \in \mathbb{N}$$

Prove:

$$x < y \iff x^n < y^n$$

8.1 $x < y \implies x^n < y^n$

Lemma 2.

$$\begin{aligned} a, b, c, d &> 0 \\ a > b, \quad c > d \end{aligned}$$

Because $a > b$, we can multiply both by c , which is positive:

$$ac > bc$$

Similarly:

$$bc > bd$$

Therefore, due to transitivity:

$$bd < bc < ac$$

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We'll prove this by induction.

8.1.1 $n = 1$

By definition, $a^1 = a$, therefore we can see the statement is true for $n=1$:

$$x^1 = x < y = y^1$$

8.1.2 $n = k$

Let's assume it is true for $n = k$, that is:

$$x < y \implies x^k < y^k$$

8.1.3 $n = k + 1$

$$\begin{aligned}x &< y \\x^{k+1} &= x^k \cdot x \\y^{k+1} &= y^k \cdot y\end{aligned}$$

According to our $n = k$ assumption:

$$x^k < y^k$$

Therefore, because it is given that $x, y \geq 0$, and *Lemma 2*, we can show that:

$$x^{k+1} = x^k x < y^k y = y^{k+1}$$

We've shown:

$$x < y \implies x^n < y^n$$

8.2 $x < y \iff x^n < y^n$

The contrapositive form of this assertion is:

$$x \geq y \implies x^n \geq y^n$$

Similarly to how we proved the \implies part, we'll prove it by induction:

8.2.1 $n = 1$

We can easily see that this is correct, as $a^1 = a$:

$$x^1 = x \geq y = y^1$$

8.2.2 $n = k$

Let's assume it is true for $n = k$, that is:

$$x \geq y \implies x^k \geq y^k$$

8.2.3 $n = k + 1$

$$\begin{aligned}x &\geq y \\x^{k+1} &= x^k \cdot x \\y^{k+1} &= y^k \cdot y\end{aligned}$$

According to our $n = k$ assumption:

$$x^k \geq y^k$$

Therefore, because it is given that $x, y \geq 0$, and *Lemma 2*, we can show that:

$$x^{k+1} = x^k x \geq y^k y = y^{k+1}$$

We've shown:

$$x \geq y \implies x^n \geq y^n$$

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