Linear Algebra I

Exercise 1

Aviv Vaknin
316017128

2 Prove: $((a+b)+c)+d = (a+b)+(c+d) = a+(b+(c+d))$

2.1 $((a+b)+c)+d = (a+b)+(c+d)$:

First of all, we'll take a look at the left side of the equation.
We'll mark:

$$h = (a+b)$$

Therefore:

$$(h+c)+d = ((a+b)+c)+d$$

$$(h+c)+d \overset{(A1)}{=} h+(c+d)$$

We'll subtitue h with $(a+b)$:

$$h+(c+d) = (a+b)+(c+d)$$

$$\Downarrow$$

$$((a+b)+c)+d = (a+b)+(c+d)$$

2.2 $a+(b+(c+d)) = (a+b)+(c+d)$:

We'll mark:

$$h = (c+d)$$

Therefore:

$$a+(b+h) = a+(b+(c+d))$$

$$a+(b+h) \overset{(A1)}{=} (a+b)+h$$

We'll subtitue h with $(c+d)$:

$$(a+b)+h = (a+b)+(c+d)$$

$$\Downarrow$$

$$a+(b+(c+d)) = (a+b)+(c+d)$$

$$\blacksquare$$

8 Is B a field?

If B fulfills all 9 field's axioms, B is a field, let's check them one by one.

8.1 A1 - Additive Associativity

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \end{pmatrix}\right] \oplus \begin{pmatrix} e \\ f \end{pmatrix} \overset{?}{=} \begin{pmatrix} a \\ b \end{pmatrix} \oplus \left[\begin{pmatrix} c \\ d \end{pmatrix} \oplus \begin{pmatrix} e \\ f \end{pmatrix}\right]$$

$$\begin{pmatrix} a+c \\ b+d \end{pmatrix} \oplus \begin{pmatrix} e \\ f \end{pmatrix} \overset{?}{=} \begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c+e \\ d+f \end{pmatrix}$$

$$\begin{pmatrix} a+c+e \\ b+d+f \end{pmatrix} = \begin{pmatrix} a+c+e \\ b+d+f \end{pmatrix}$$

8.2 M1 - Multiplicative Associativity

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \odot \begin{pmatrix} c \\ d \end{pmatrix}\right] \odot \begin{pmatrix} e \\ f \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} a \\ b \end{pmatrix} \odot \left[\begin{pmatrix} c \\ d \end{pmatrix} \odot \begin{pmatrix} e \\ f \end{pmatrix}\right]$$

$$\begin{pmatrix} ac \\ bd \end{pmatrix} \odot \begin{pmatrix} e \\ f \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} a \\ b \end{pmatrix} \odot \begin{pmatrix} ce \\ df \end{pmatrix}$$

$$\begin{pmatrix} ace \\ bdf \end{pmatrix} = \begin{pmatrix} ace \\ bdf \end{pmatrix}$$

8.3 A2 - Additive Commutativity

$$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} c \\ d \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a+c \\ b+d \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} c+a \\ d+b \end{pmatrix}$$

As $a, b \in \mathbb{R}$, we can use axiom A2:

$$\begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} c+a \\ d+b \end{pmatrix}$$

8.4 M2 - Multiplicative Commutativity

$$\begin{pmatrix} a \\ b \end{pmatrix} \odot \begin{pmatrix} c \\ d \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} c \\ d \end{pmatrix} \odot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} ac \\ bd \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} ca \\ db \end{pmatrix}$$

As $a, b, c, d \in \mathbb{R}$, we can use axiom M2:

$$\begin{pmatrix} ac \\ bd \end{pmatrix} = \begin{pmatrix} ca \\ db \end{pmatrix}$$

8.5 A3 - Additive Identity

Now, we'll look for 0_B:

$$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a+0 \\ b+0 \end{pmatrix}$$

As $a, b \in \mathbb{R}$, we can use axiom A3:

$$\begin{pmatrix} a+0 \\ b+0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0_B$$

8.6 M3 - Multiplicative Identity

Now, we'll look for 1_B:

$$\begin{pmatrix} a \\ b \end{pmatrix} \odot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1a \\ 1b \end{pmatrix}$$

As $a, b \in \mathbb{R}$, we can use axiom M3:

$$\begin{pmatrix} 1a \\ 1b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1_B$$

This answers (ii).

8.7 A4 - Additive Inverse

Now, let us find if exists a member, $\begin{pmatrix} c \\ d \end{pmatrix} \in B$, for any other member $\begin{pmatrix} a \\ b \end{pmatrix} \in B$, so that:

$$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \end{pmatrix} = 0_B$$

We'll start by adding the inverse of every number to itself:

$$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} a + (-a) \\ b + (-b) \end{pmatrix}$$

As $a, b \in \mathbb{R}$, we can use axiom A4 to find the sum's value:

$$\begin{pmatrix} a + (-a) \\ b + (-b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0_B$$

This answers (i).

8.8 M4 - Multiplicative Inverse

Now, let us find if exists a member, $\begin{pmatrix} c \\ d \end{pmatrix} \in B$, for any other member $\begin{pmatrix} a \\ b \end{pmatrix} \in B$, so that:

$$\begin{pmatrix} a \\ b \end{pmatrix} \odot \begin{pmatrix} c \\ d \end{pmatrix} = 1_B$$

Let's multiply every number by its multiplicitive inverse:

$$\begin{pmatrix} a \\ b \end{pmatrix} \odot \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \end{pmatrix} = \begin{pmatrix} a \cdot \frac{1}{a} \\ b \cdot \frac{1}{b} \end{pmatrix}$$

As $a, b \in \mathbb{R}$, we can use axiom M4:

$$\begin{pmatrix} a \cdot \frac{1}{a} \\ b \cdot \frac{1}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1_B$$

8.9 D - Distributivity

In order to display distributivity, we'll need to show the B equivalent of:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

i.e., we need to prove:

$$\begin{pmatrix} a \\ b \end{pmatrix} \odot \left[\begin{pmatrix} c \\ d \end{pmatrix} \oplus \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} ac \\ bd \end{pmatrix} \oplus \begin{pmatrix} ae \\ bf \end{pmatrix}$$

First, let's solve:

$$\left[\begin{pmatrix} c \\ d \end{pmatrix} \oplus \begin{pmatrix} e \\ f \end{pmatrix} \right]$$

We use B's binary additive operator:

$$\begin{pmatrix} c \\ d \end{pmatrix} \oplus \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} c + e \\ d + f \end{pmatrix}$$

We use B's binary multiplicative operator:

$$\begin{pmatrix} a \\ b \end{pmatrix} \odot \begin{pmatrix} c + e \\ d + f \end{pmatrix} = \begin{pmatrix} a(c + e) \\ b(d + f) \end{pmatrix}$$

As $a, b, c, d, e, f \in \mathbb{R}$, we can use the distributivity axiom to unpack:

$$\begin{pmatrix} a(c + e) \\ b(d + f) \end{pmatrix} = \begin{pmatrix} ac + ae \\ bd + bf \end{pmatrix} = \begin{pmatrix} ac \\ bd \end{pmatrix} \oplus \begin{pmatrix} ae \\ bf \end{pmatrix}$$

This answers (iii).

$$\blacksquare$$

23 Prove $-(a+b) = (-a) + (-b) = -a - b$:

According to axiom *A4*:

$$-(a+b) + (a+b) = 0$$

Now, let's add $-a-b$ to both sides of the equation:

$$-(a+b) + (a+b) - a - b = -a - b$$

And use axioms *A1* and *A2*:

$$-(a+b) + (a + (-a)) + (b + (-b)) = -a - b$$

According to axiom *A4*:

$$-(a+b) + (0) + (0) = -a - b$$

And according to axiom *A3* and the definition of deduction:

$$-(a+b) = -a - b = (-a) + (-b)$$

■

26 Prove $(-a)(-b) = ab$:

According to multiplicative identity axiom:

$$(-a)(-b) = ((-1) \cdot a)((-1) \cdot b)$$

According to multiplicative commutativity axiom:

$$((-1) \cdot a)((-1) \cdot b) = (-1) \cdot (-1) \cdot a \cdot b$$

And multiplicative identity axiom again:

$$(-1) \cdot (-1) \cdot a \cdot b = 1 \cdot a \cdot b = ab$$

■

30 Prove $a \neq 0 \implies 0/a = 0$:

30.1 Lemma: $a \cdot 0 = 0$

According to the distributivity and multiplicative identity axioms:

$$a \cdot 0 + a = a(0+1) = a \cdot 1 = a$$

Therefore:

$$a \cdot 0 + a = a$$

We'll subtract a from both sides of the equation:

$$a \cdot 0 + a - a = a - a$$

$$a \cdot 0 = 0$$

$$\blacksquare$$

30.2 $a \neq 0 \implies 0/a = 0$

According to the definition of division:

$$0/a = 0 \cdot a^{-1}$$

According to the lemma:

$$0 \cdot a^{-1} = 0$$

$$\blacksquare$$

37 Solve in $\mathbb{R}$:

$$\sum_{i=1}^{53} (-1)^i$$

We can notice that 53 is an odd number, and therefore (-1) is multiplied by itself an odd number of times.
Therefore, we can say that the following two are identical:

$$\sum_{i=1}^{53} (-1)^i = \sum_{i=1}^{1} (-1)^i = -1$$

$$\blacksquare$$

38 Solve in $\mathbb{R}$:

$$\sum_{k=3}^{20} [k \cdot k - (k-1) \cdot (k-1)]$$

First, let's understand this by solving:

$$\sum_{k=3}^{4} [k \cdot k - (k-1) \cdot (k-1)]$$

$$\sum_{k=3}^{4} [k \cdot k - (k-1) \cdot (k-1)] = 4^2 - 3^2 + 3^2 - 2^2 = 4^2 - 2^2$$

Therefore, we can extrapolate:

$$\sum_{k=3}^{20} [k \cdot k - (k-1) \cdot (k-1)] = 20^2 - 2^2$$

$$\blacksquare$$

49

$$a_1X_1 + a_2X_2 + a_3X_3 = b$$

49.1 $\underline{(1, 1, 2, 1)}$:

First, we'll apply the coefficients:

$$X_1 + X_2 + 2X_3 = 1$$

Change the equation to a single variable equation:

$$X_1 = 1 - t_2 - 2t_3$$

Therefore:

$$\mathbb{S} = \left\{ \begin{pmatrix} 1 - t_2 - 2t_3 \\ t_2 \\ t_3 \end{pmatrix} : t_2, t_3 \in \mathbb{R} \right\}$$

49.2 $\underline{(0, 1, 6, 3)}$:

First, we'll apply the coefficients:

$$0X_1 + X_2 + 6X_3 = 3$$

Change the equation to a single variable equation:

$$X_2 = 3 - 0t_1 - 6t_3 = 3 - 6t_3$$

Therefore:

$$\mathbb{S} = \left\{ \begin{pmatrix} t_1 \\ 3 - 6t_3 \\ t_3 \end{pmatrix} : t_1, t_3 \in \mathbb{R} \right\}$$

49.3 $\underline{(0,3,6,3)}$:

First, we'll apply the coefficients:

$$0X_1 + 3X_2 + 6X_3 = 3$$

We'll divide both sides of the equation by 3:

$$0X_1 + X_2 + 2X_3 = 1$$

Change the equation to a single variable equation:

$$X_2 = 1 - 0t_1 - 2t_3 = 1 - 2t_3$$

Therefore:

$$\mathbb{S} = \left\{ \begin{pmatrix} t_1 \\ 1 - 2t_3 \\ t_3 \end{pmatrix} : t_1, t_3 \in \mathbb{R} \right\}$$

$\blacksquare$