1 $A \subseteq \mathbb{F}$, prove $m \in \mathbb{F}$ is the infimum of A <u>***if and only if***</u>:

1. m is the lower bound of A
2. $(\forall \epsilon > 0)\ (\exists a \in A)\ (a < m + \epsilon)$

1.1 <u>$\implies$:</u>

We'll start with assuming $m \in \mathbb{F}$ is the infimum of A.
By definition, the infimum is a lower bound **(1)**.
Therefore:

$$\forall a \in A \ \ a \geq m$$

and, suppose m' is another lower bound for A:

$$m \geq m'$$

First, by definition, an infimum is a lower bound.
Regarding the second item, let's suppose by contradiction:

$$\exists \epsilon > 0 \ \ \forall a \in A \ \ a \geq m + \epsilon$$
$$m + \epsilon > m$$

Therefore, we can see that $\underline{m + \epsilon}$ is a lower bound for A, bigger than the infimum.
We've reached a contradiction, as we've assumed m is the infimum of A **(2)**.

1.2 <u>$\impliedby$:</u>

Let's suppose that m is not the infimum.
Therefore, exists another lower-bound - m' so that $m' > m$.
Let $\epsilon > 0$:

$$\epsilon = m' - m$$
$$m' = \epsilon + m$$

Therefore:

$$\forall \epsilon > 0 \ \ \exists a \in A \ \ a \geq \epsilon + m$$

Which is in contradiction with our initial assumption:

$$(\forall \epsilon > 0)\ (\exists a \in A)\ (a < m + \epsilon)$$

$\square$

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$$A \subseteq \mathbb{F}$$
$$s \in \mathbb{F}$$

2.1 Prove $s = \max(A) \iff (s = \sup(A)) \wedge (s \in A)$

2.1.1 $\implies$:

Let's start by supposing $s = \max(A)$.
By definition, the maximum is always part of the set, therefore **(1)**:

$$s \in A$$

Let's show that $\max(A) = \sup(A)$:
We know from definition, that s is an upper-bound of A.
Let's take any arbitrary $\epsilon > 0$, therefore because $s \in A$:

$$s > s - \epsilon$$

Therefore, by the definition of the supremum(proposition 5.7 in class), s is A's supremum.**(2)**

2.1.2 $\impliedby$:

Let's suppose that $s = \sup(A)$ and $s \in A$.
According to the definition of the maximum, if s is the maximum of A, it must be:

1. a member of A
2. a supremum of A

Both are given. □

2.2 Prove that if A has a maximum, there's only one.

By definition, if a set has a maximum, it fulfills the following properties:

1. it is a member of the set
2. it is a supremum of the set

Suppose m is the maximum of A.
By definition, it must be part of A, satisfying **(1)**. Because of **(2)**, it must be a supremum.

2.2.1 n's smaller than m

According to the definition of the upper-bound:

$$\forall \epsilon > 0 \ \ \exists a \in A \ \ a > m - \epsilon$$

Therefore, $m - \epsilon$ is smaller than m, and is not an upper-bound, therefore not a maxmium.

2.2.2 n's bigger than m

Likewise, $m + \epsilon$ is bigger than m, and is by definition an upper-bound.
However, we've stated that m is the upper-bound of A, and according to the upper-bound definition,
Every number that is bigger than the upper-bound of a set, is not a member of the set.
Therfore, we've shown that there can only exist a single maximum, if any. $\square$

3 $\varnothing \neq A, B \subseteq \mathbb{F}$ Prove or disprove:

3.1

Let $A = \mathbb{R}$, and $B = \mathbb{N}$.
Therefore, we can see that A is not lower-bounded, while B is lower-bounded by 1. Disproved. $\square$

3.2

If the claim is incorrect, then B mustn't have an upper-bound.
Since B is a subset of A, B must be bounded from above as well, that is due to the definition of an upper bound.
In formal notation, as $B \subseteq A$:

$$(\forall b \in B)\ (\exists a \in A)\ (a \geq b)$$

However, since A is bounded from above:

$$(\forall a \in A)\ (\exists m)\ (m \geq a)$$

Therefore, due to transitivity:

$$m \geq b$$

$\square$

4 $\varnothing \neq A, B \subseteq \mathbb{F} \quad \exists \sup(A), \exists \sup(B)$ Prove or disprove:

4.1 $\underline{B \subseteq A \implies \sup(B) \leq \sup(A)}$

Let $m = \sup(A) = \max(A)$.
Therefore, according to the definition of the *supremum*:

$$\forall a \in A \; m \geq a$$

Now, let's take a look at the maximum of B:

$$\forall b \in B \; m' \geq b$$

Since $B \subseteq A$, the maximum value of B can, at the most be equal to m, that is:

$$m' \leq m$$

$\square$

4.2 $\underline{(B \subseteq A) \;\wedge\; (B \neq A) \implies \sup(B) < \sup(A)}$

Let:

$$A = \{1, 2, 3\}$$
$$B = A \backslash \{1\}$$

We can clearly see that $(B \subseteq A) \;\wedge\; (B \neq A)$.
However, $\sup(B) = \sup(A) = 3$, thus the claim is incorrect. $\square$

4.3 $\underline{-A = \{-a \mid a \in A\}}$

It is given that A necessarily has a supremum.
Therefore, -A is necessarily bounded from below by that same supremum, as a negative.
Let m be A's supremum, and -m be -A's infimum.
According to the definition of the *supremum*:

$$\forall a \in A \; m \geq a$$

We'll multiply by *(-1)*:

$$-m \leq -a$$

Note that this is exactly the definition of the infimum. Thus, we can conclude that -A is bounded from below, $\exists \inf(-A)$ and that $\inf(-A) = -\sup(A)$. $\square$

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