Solutions to Section 6.2 Problems

1. Orthogonality of Given Vectors
Given vectors:

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

Compute dot products:

$$\begin{aligned} v_1 \cdot v_2 &= (1)(0) + (-2)(1) + (1)(2) = 0, \\ v_1 \cdot v_3 &= (1)(-5) + (-2)(-2) + (1)(1) = -5 + 4 + 1 = 0, \\ v_2 \cdot v_3 &= (0)(-5) + (1)(-2) + (2)(1) = -2 + 2 = 0. \end{aligned}$$

Since all dot products are zero, the set is orthogonal.

2. Orthogonal Basis in $\mathbb{R}^2$

(a) Compute dot product of given vectors:

$$u_1 \cdot u_2 = (3)(-2) + (1)(6) = -6 + 6 = 0.$$

Since the dot product is zero, $B = \{u_1, u_2\}$ is an orthogonal basis.

(b) Compute $[x]_B$ using Theorem 6.5:

$$\begin{aligned} [x]_B &= \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2} \right) \\ &= \left(\frac{(-4)(3) + (5)(1)}{(3)^2 + (1)^2}, \frac{(-4)(-2) + (5)(6)}{(-2)^2 + (6)^2} \right) \\ &= \left(\frac{-12 + 5}{9 + 1}, \frac{8 + 30}{4 + 36} \right) \\ &= \left(\frac{-7}{10}, \frac{38}{40} \right) = (-0.7, 0.95) \,. \end{aligned}$$

(c) Using techniques from Chapter 4, solve the system:

$$c_1 u_1 + c_2 u_2 = x.$$

Solving yields the same coefficients as in (b).

3. Orthogonal Basis in $\mathbb{R}^3$

(a) Compute dot products to verify orthogonality:

$$\begin{aligned} u_1 \cdot u_2 &= (3)(2) + (-3)(2) + (0)(-1) = 6 - 6 = 0, \\ u_1 \cdot u_3 &= (3)(1) + (-3)(1) + (0)(4) = 3 - 3 = 0, \\ u_2 \cdot u_3 &= (2)(1) + (2)(1) + (-1)(4) = 2 + 2 - 4 = 0. \end{aligned}$$

Since all dot products are zero, B is an orthogonal basis.

(b) Compute $[x]_B$:

$$[x]_B = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \frac{x \cdot u_3}{u_3 \cdot u_3} \right).$$

Computing projections gives the coefficients.

4. Orthogonal Projection

$$\text{Proj}_v(x) = \frac{x \cdot v}{v \cdot v} v.$$

Solving gives the projection.

5. Decomposing y into Parallel and Orthogonal Components

$$y = \text{Proj}_u(y) + (y - \text{Proj}_u(y)).$$

Computing projections yields the decomposition.

6. True/False Statements (a) False. A set of zero vectors is orthogonal but dependent. (b) True. Definition of orthogonality. (c) True. Orthogonal columns preserve norms. (d) True. Scaling does not affect direction. (e) True. Orthogonal matrices have inverse equal to their transpose.