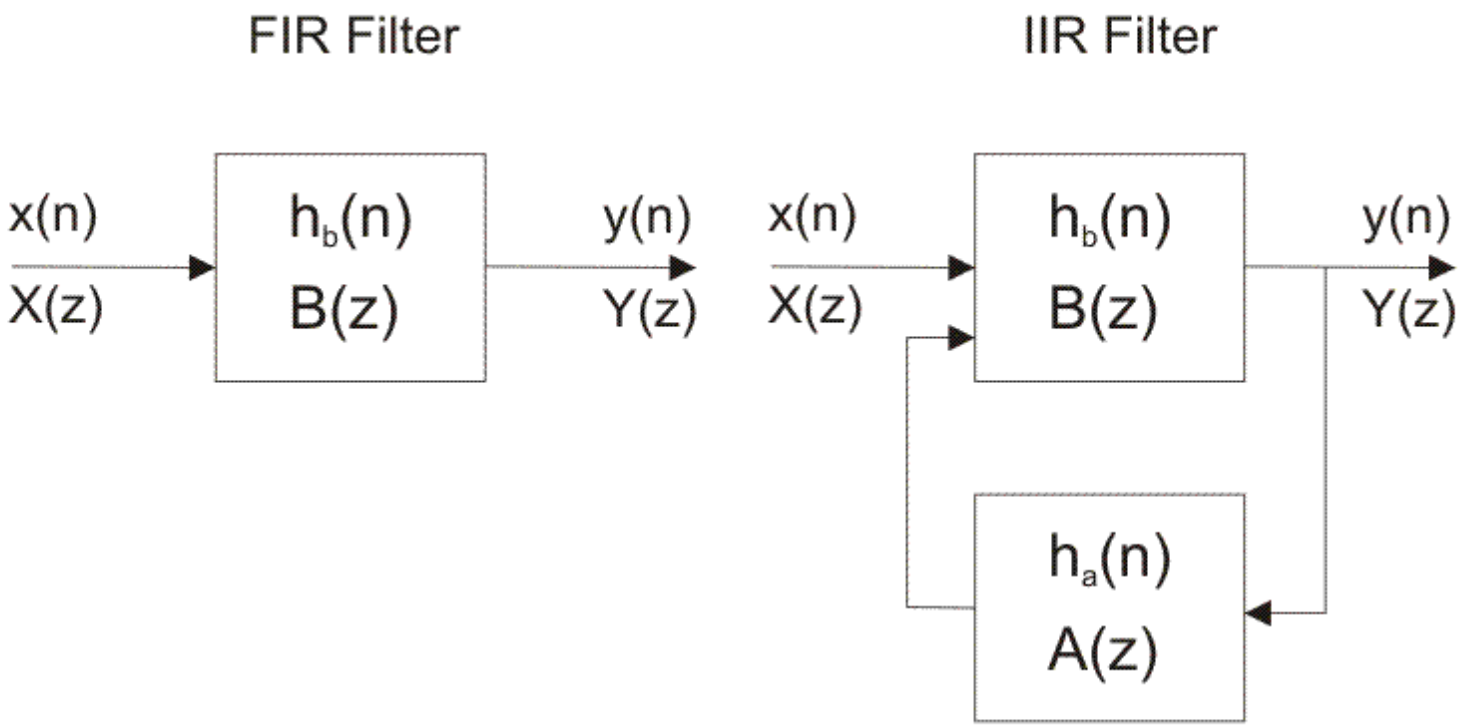
FIR & IIR



IIR

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}.$$

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

$$\begin{aligned} h[n] = T_d h_c(nT_d) &= \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] \\ &= \sum_{k=1}^{N} T_d A_k (e^{s_k T_d})^n u[n]. \end{aligned}$$

$$\boxed{H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}.}$$

IIR Filters

Form the ratio $Y(z)/X(z) = H(z)$

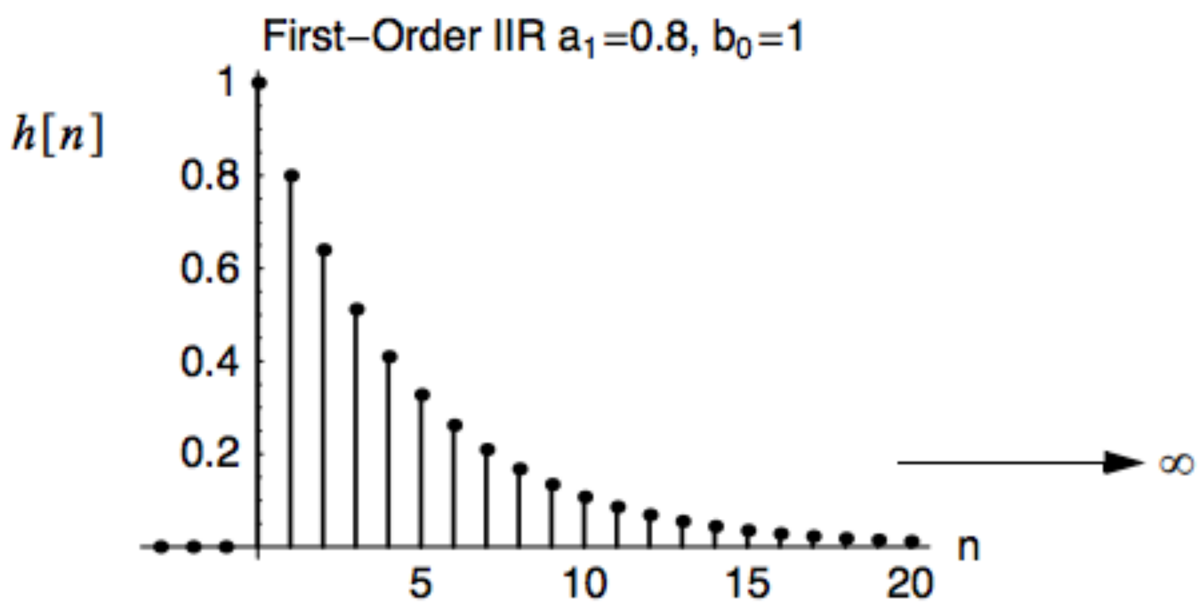
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 - a_1 z^{-1} - \dots - a_N z^{-N}}$$

IIR Filters

Example: First-Order IIR with $b_0 = 1, a_1 = 0.8$

- The impulse response is

$$h[n] = 0.8^n u[n]$$



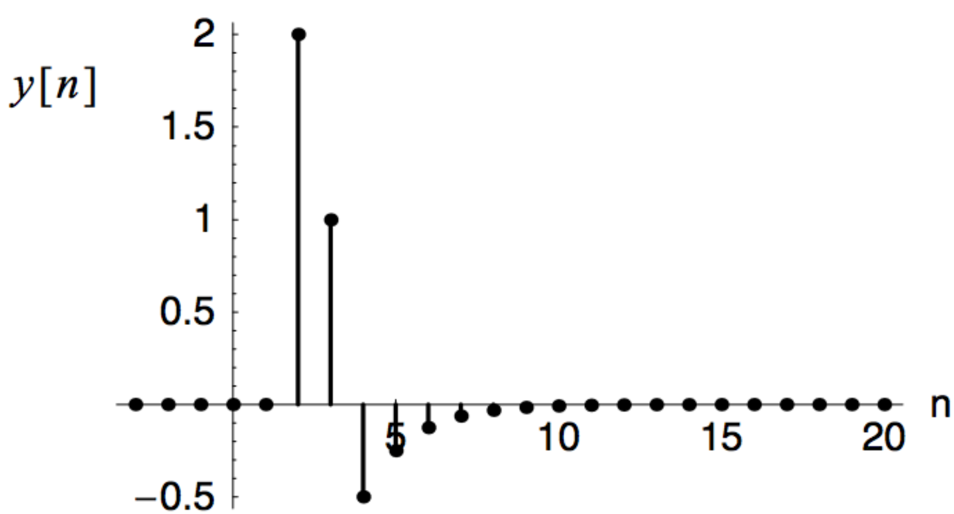
IIR Filters

Example: $x[n] = 2\delta[n-2] - \delta[n-4]$, $a_1 = 0.5$ and $b_0 = 1$

- Using the above result, it follows that

$$y[n] = 2(0.5)^{n-2}u[n-2] - (0.5)^{n-4}u[n-4]$$

- Plotting this function results in



IIR Filters

The coefficients of the numerator polynomial, denoted $B(z)$, correspond to the feed-forward terms of the difference equation

The coefficients of the denominator polynomial, denoted $A(z)$, for $z^{-l}, l > 0$ correspond to the feedback terms of the difference equation

We have used various MATLAB functions that take as input `b` and `a` coefficient vectors, e.g., `filter(b,a,...)`, `freqz(b,a,...)`, and `zplane(b,a)`

In terms of the general IIR system we now identify those vectors as

$$\mathtt{b} = [b_0, b_1, \ldots, b_M]$$

$$\mathtt{a} = [1, -a_1, -a_2, \ldots, -a_N]$$

IIR Filters

Example: Impulse Response Using MATLAB

- Suppose that $a_1 = 0.5$, $b_0 = -3$, and $b_1 = 2$

```
>> n = 0:20;
>> x = [1 zeros(1,20)]; % impulse sequence input
>> y = filter([-3,2],[1 -0.5],x);
>> stem(n,y,'filled')
>> axis([0 10 -3.1 .6])
>> grid
>> ylabel('Impulse Response h[n]')
>> xlabel('Time Index (n)')
```

IIR Filters

<u>Example</u>: `y = filter([1 1],[1 -0.8],x)`

- We wish to find the system function, impulse response, and difference equation that corresponds to the given `filter()` expression
- By inspection

$$H(z) = \frac{1+z^{-1}}{1-0.8z^{-1}}$$

- The impulse response using page 8–5, eqns (8.7)—(8.9)

$$h[n] = \delta[n] + (1+0.8^{-1})(0.8)^n u[n-1]$$

- The difference equation is

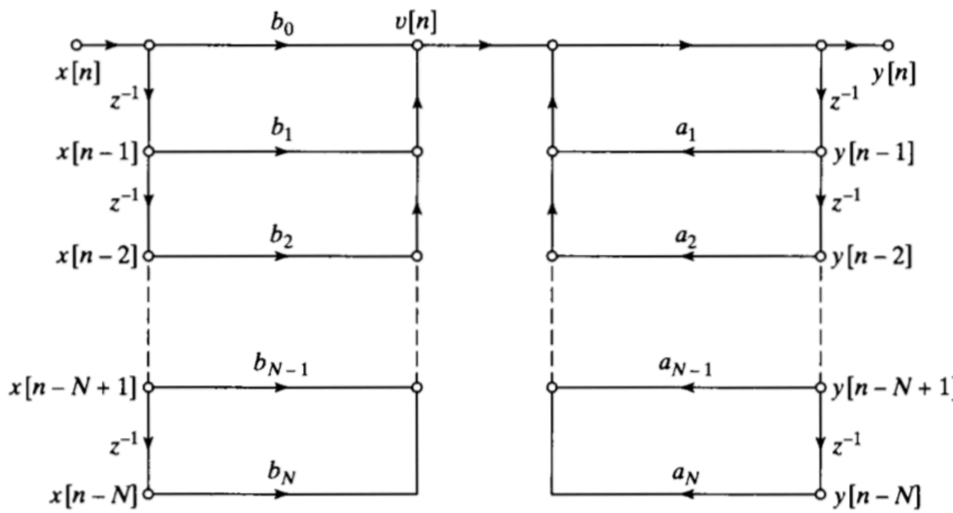
$$y[n] = 0.8y[n-1] + x[n] + x[n-1]$$

IIR Filters

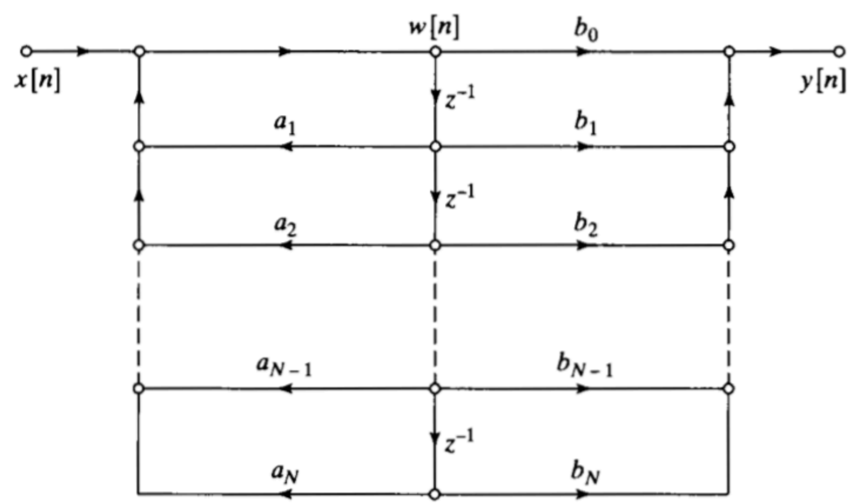
$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k],$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}.$$

Direct Form I structure



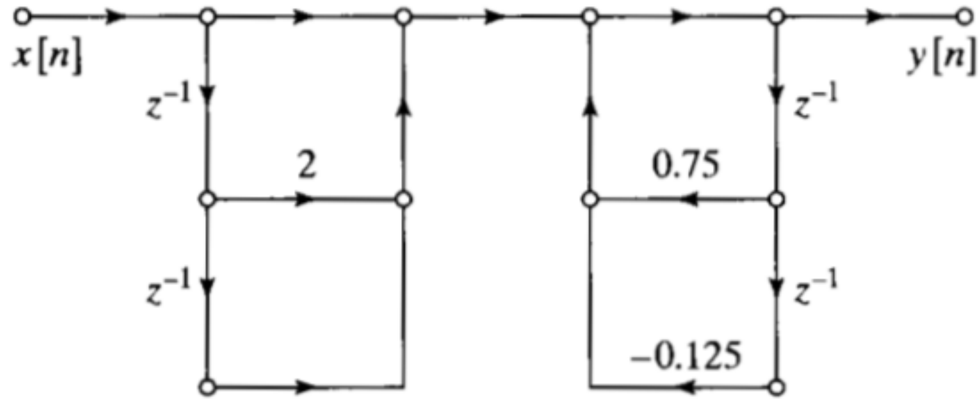
Direct Form II structure



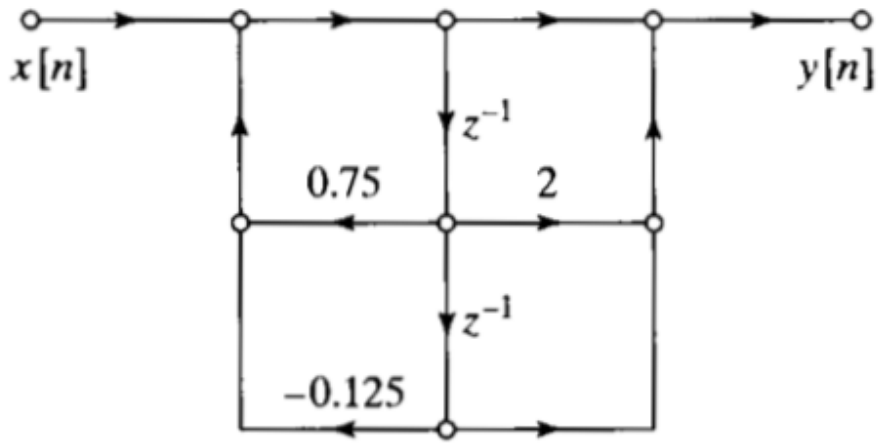
$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}.$$

IIR Filters

Direct Form I structure



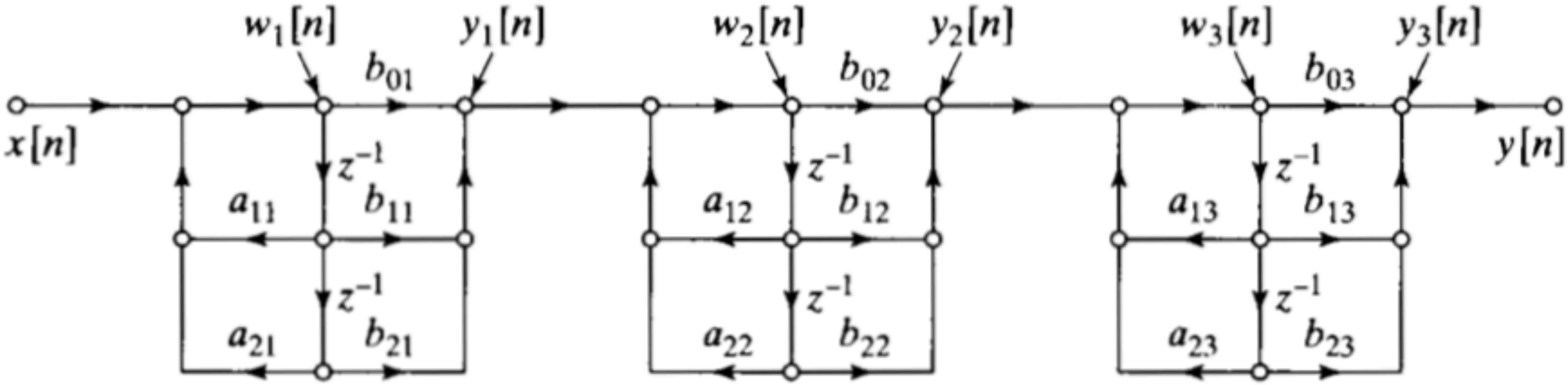
Direct Form II structure



IIR Filters

$$H(z) = A\frac{\prod_{k=1}^{M_1}(1 - f_k z^{-1})\prod_{k=1}^{M_2}(1 - g_k z^{-1})(1 - g_k^* z^{-1})}{\prod_{k=1}^{N_1}(1 - c_k z^{-1})\prod_{k=1}^{N_2}(1 - d_k z^{-1})(1 - d_k^* z^{-1})},$$

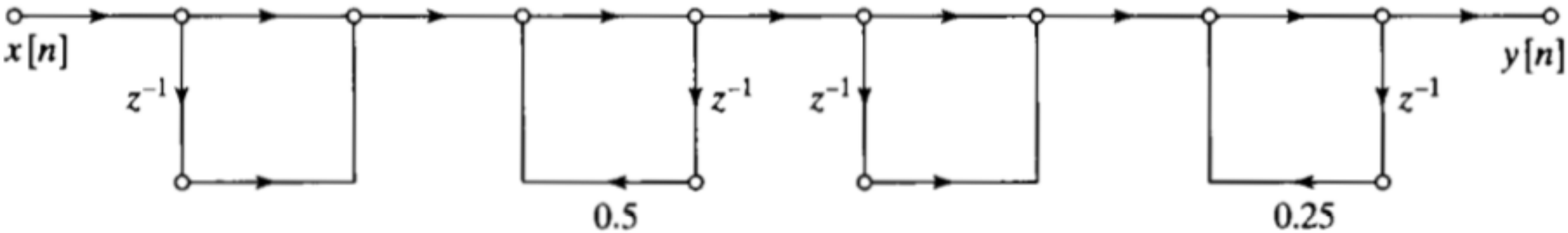
Cascade structure

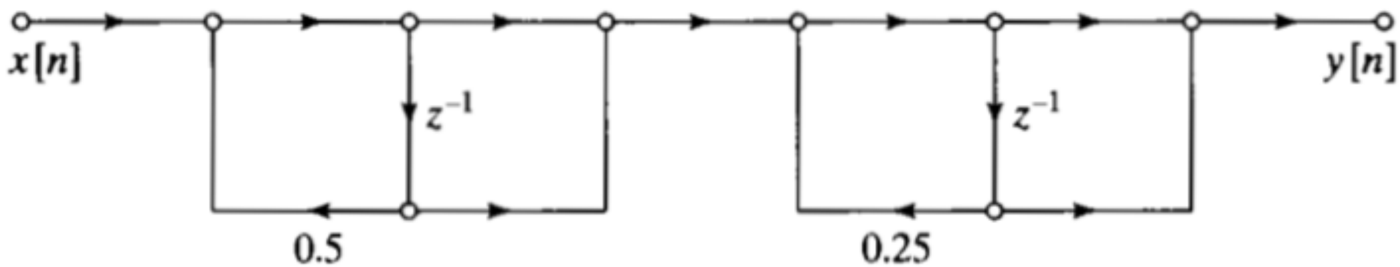


IIR Filters

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})}.$$

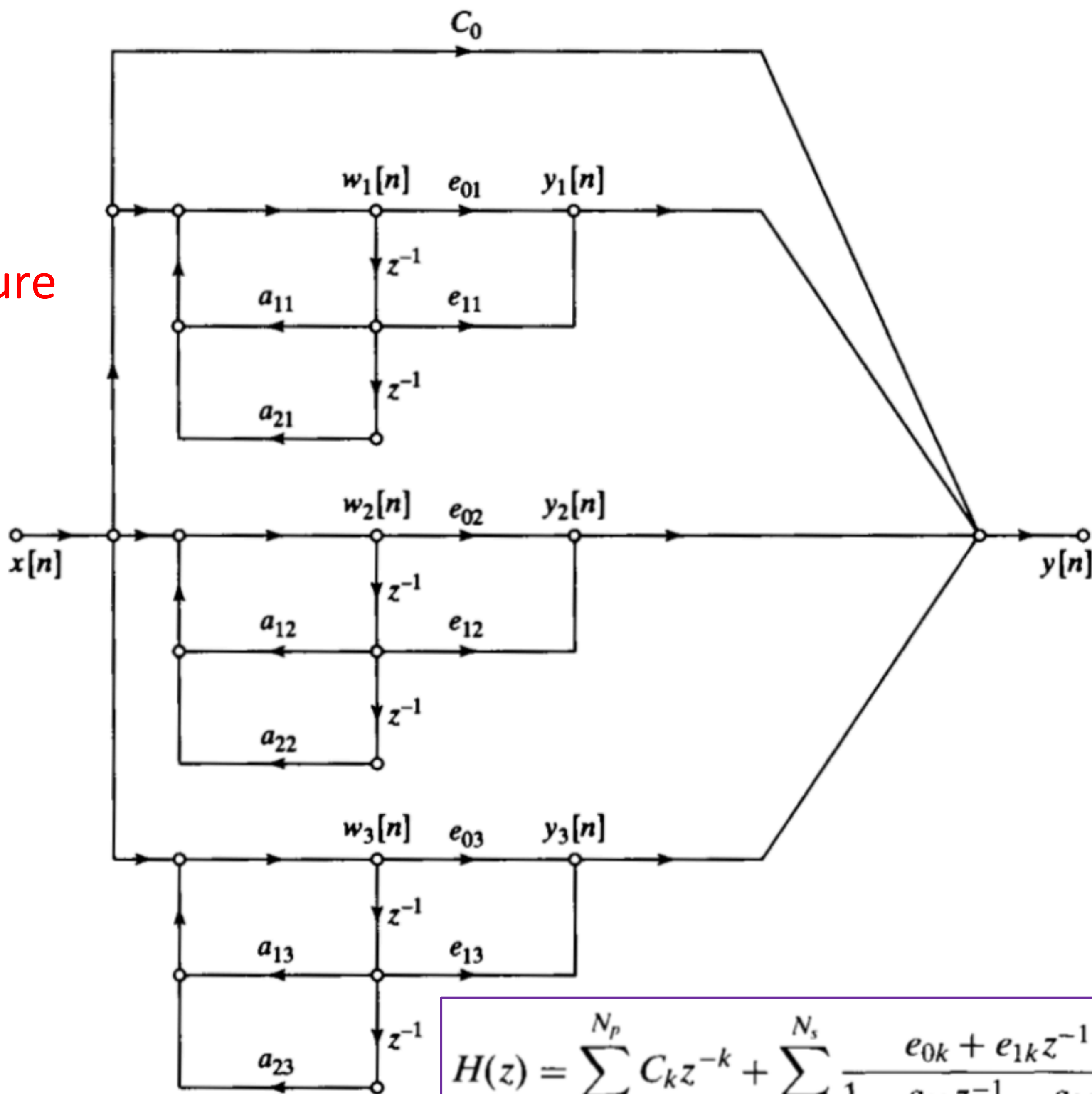
Cascade structure





IIR Filters

Parallel structure



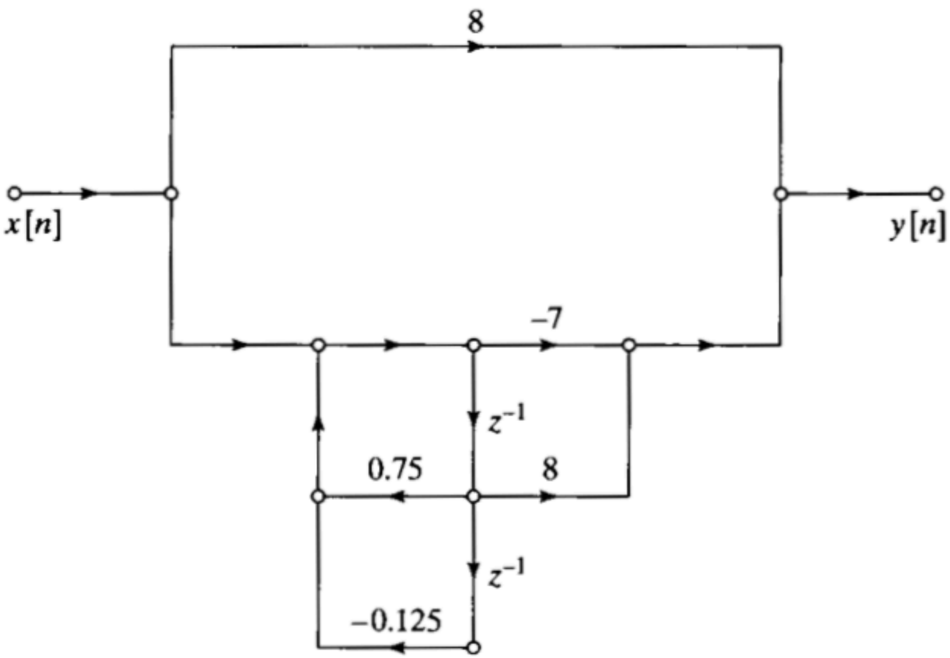
$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_s} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}},$$

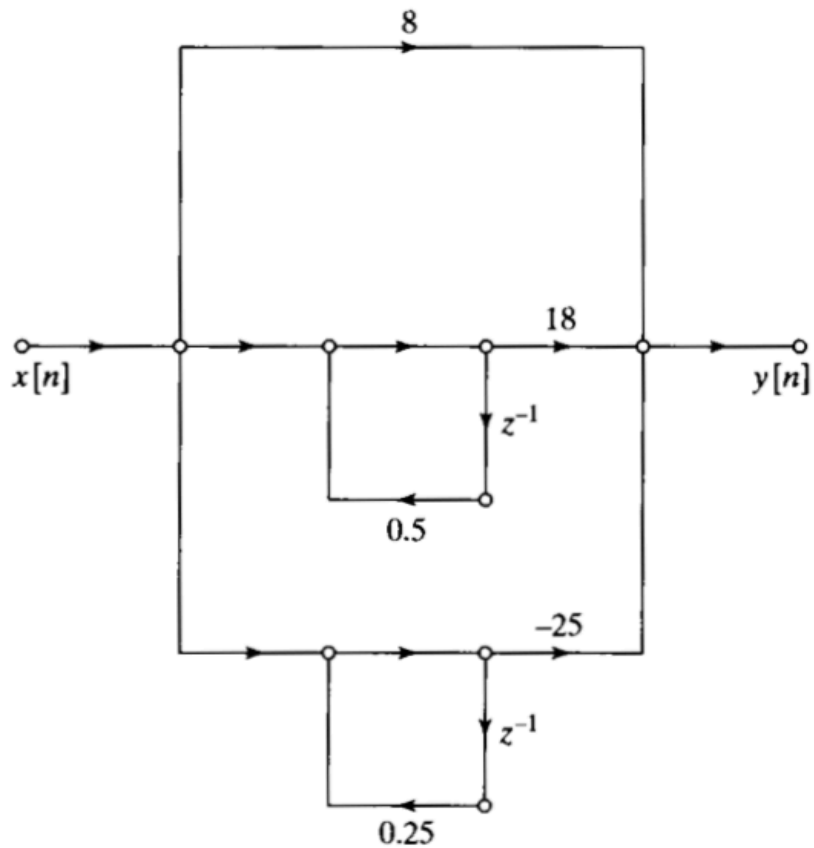
IIR Filters

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}}.$$

$$H(z) = 8 + \frac{18}{1 - 0.5z^{-1}} - \frac{25}{1 - 0.25z^{-1}}.$$

Parallel structure





IIR Filters

Pole Locations and Stability

- We know that

$$h[n] = a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}} = H(z) \tag{8.16}$$

- We note that this system has a pole at $z = a$ and a zero at $z = 0$
- The impulse response decays to zero so long as $|a| < 1$, which is equivalent to requiring that the pole lies inside the unit circle
- **System Stability**: Causal LTI IIR systems, initially at rest, are stable if all of the poles of the system function lie inside the unit circle

IIR Filters

Example: Second-order $H(z)$

- Suppose that

$$H(z) = \frac{1 + 0.2z^{-1}}{1 - 1.4z^{-1} + 0.81z^{-2}} = \frac{z(z+0.2)}{z^2 - 1.4z + 0.81}$$

$$= \frac{z(z+0.2)}{(z-(0.7+j0.4\sqrt{2}))(z-(0.7-j0.4\sqrt{2}))}$$

- In polar form the poles are $p_{1,2} = 0.9e^{\pm j0.680}$, so the poles are inside the unit circle and the system is stable
- We can check stability using `zplane()` to plot the poles and zeros for us

```
>>> zplane([1 0.2],[1 -1.4 0.81])
```

IIR Filters - BP

<u>Example</u>: $H(z) = (1 + 0.2z^{-1})/(1 - 1.4z^{-1} + 0.81z^{-2})$

- Making the substitution $z = e^{j\hat{\omega}}$ we have

$$H(e^{j\hat{\omega}}) = \frac{1 + 0.2e^{-j\hat{\omega}}}{1 - 1.4e^{-j\hat{\omega}} + 0.81e^{-j2\hat{\omega}}}$$

- We can use `freqz()` to plot the magnitude and phase

```
>> w = -pi:(pi/500):pi;
>> H = freqz([1 .2],[1 -1.4 0.81],w);
```

IIR Filters - LP

<u>Example</u>: $H(z) = 1/(1 - 0.8z^{-1})$

- Here we have

$$H(e^{j\hat{\omega}}) = \frac{1}{1 - 0.8e^{-j\hat{\omega}}}$$

Plotting using `freqz()` we have

```
>> w = -pi:(pi/500):pi;
>> H = freqz(1,[1 -0.8],w);
```

IIR Filters - HP

<u>Example</u>: $H(z) = 1/(1 + 0.8z^{-1})$

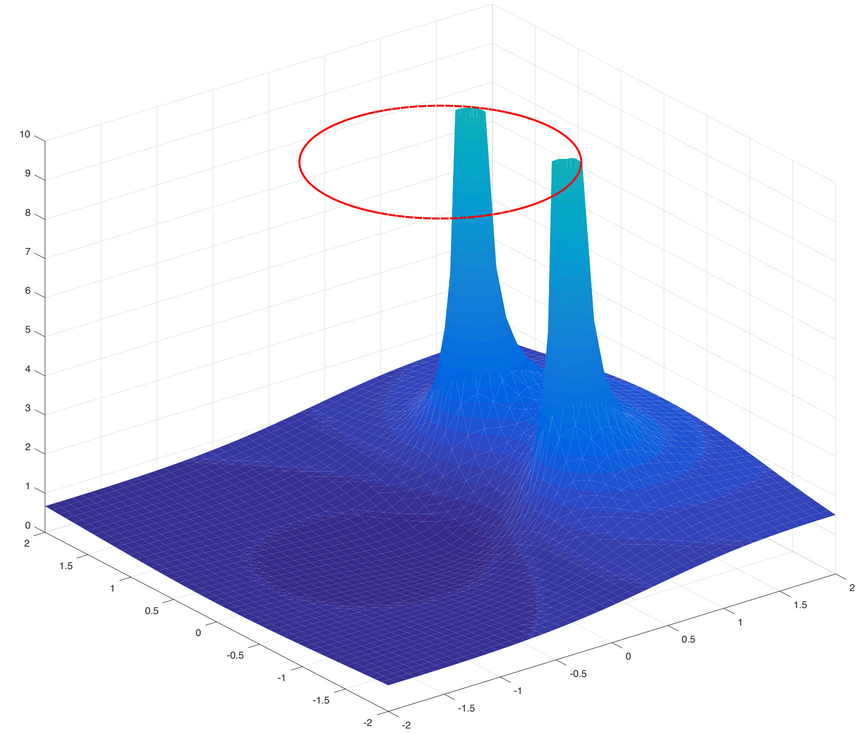
- Here we have

$$H(e^{j\hat{\omega}}) = \frac{1}{1 + 0.8e^{-j\hat{\omega}}}$$

Plotting using `freqz()` we have

```
>> w = -pi:(pi/500):pi;
>> H = freqz(1,[1 0.8],w);
```

IIR Filters - HP



- Consider the second-order system

$$H(z) = \frac{1 + 0.2z^{-1}}{1 - 1.4z^{-1} + 0.81z^{-2}}$$

- A 3D surface plot of $|H(z)|$ can help clarify how the frequency response is obtained by evaluating $H(z)$ around the unit circle

Ideal Frequency Selective Filters

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases}$$

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty.$$

$$H_{\mathrm{hp}}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & \omega_c < |\omega| \le \pi, \end{cases}$$

$$h_{\mathrm{hp}}[n] = \delta[n] - h_{\mathrm{lp}}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}.$$

Ideal Delay System

$$h_{\mathrm{id}}[n] = \delta[n - n_d], \qquad H_{\mathrm{id}}(e^{j\omega}) = e^{-j\omega n_d},$$

$$\boxed{\begin{aligned} &|H_{\mathrm{id}}(e^{j\omega})| = 1, \\ &\sphericalangle H_{\mathrm{id}}(e^{j\omega}) = -\omega n_d, \qquad |\omega| < \pi, \end{aligned}}$$

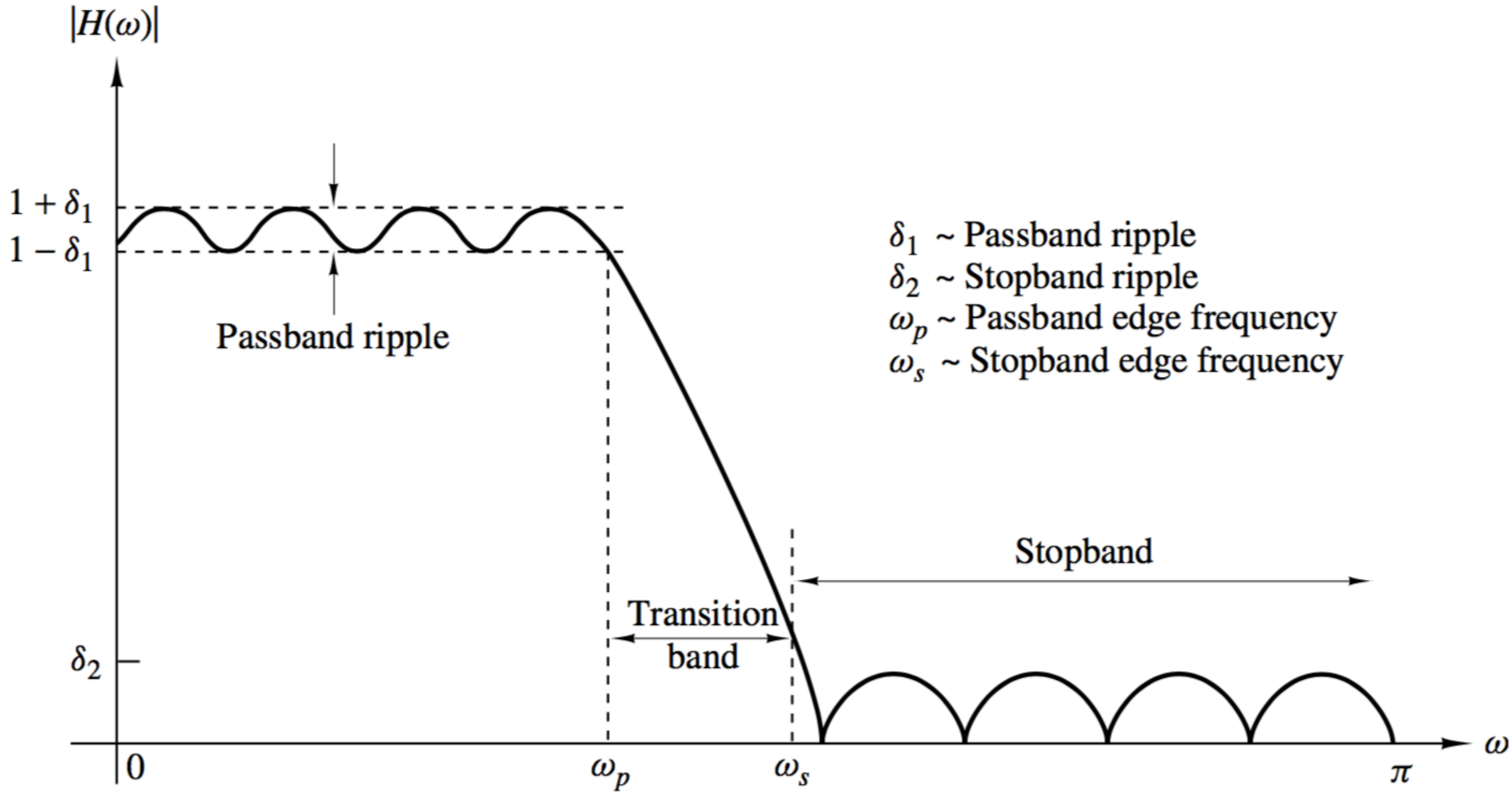
Ideal Delay System

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi. \end{cases}$$

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \qquad -\infty < n < \infty.$$

$$\boxed{\tau(\omega) = \mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}.}$$

Practical Freq-Selective Filters



Symmetric & Antisymmetric FIR Filters

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_{M-1} x(n-M+1)$$

$$= \sum_{k=0}^{M-1} b_k x(n-k)$$

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)$$

$$\boxed{H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}}$$

$$\boxed{z^{-(M-1)} H(z^{-1}) = \pm H(z)}$$

Symmetric & Antisymmetric FIR Filters

An FIR filter has linear phase if its unit sample response satisfies the condition

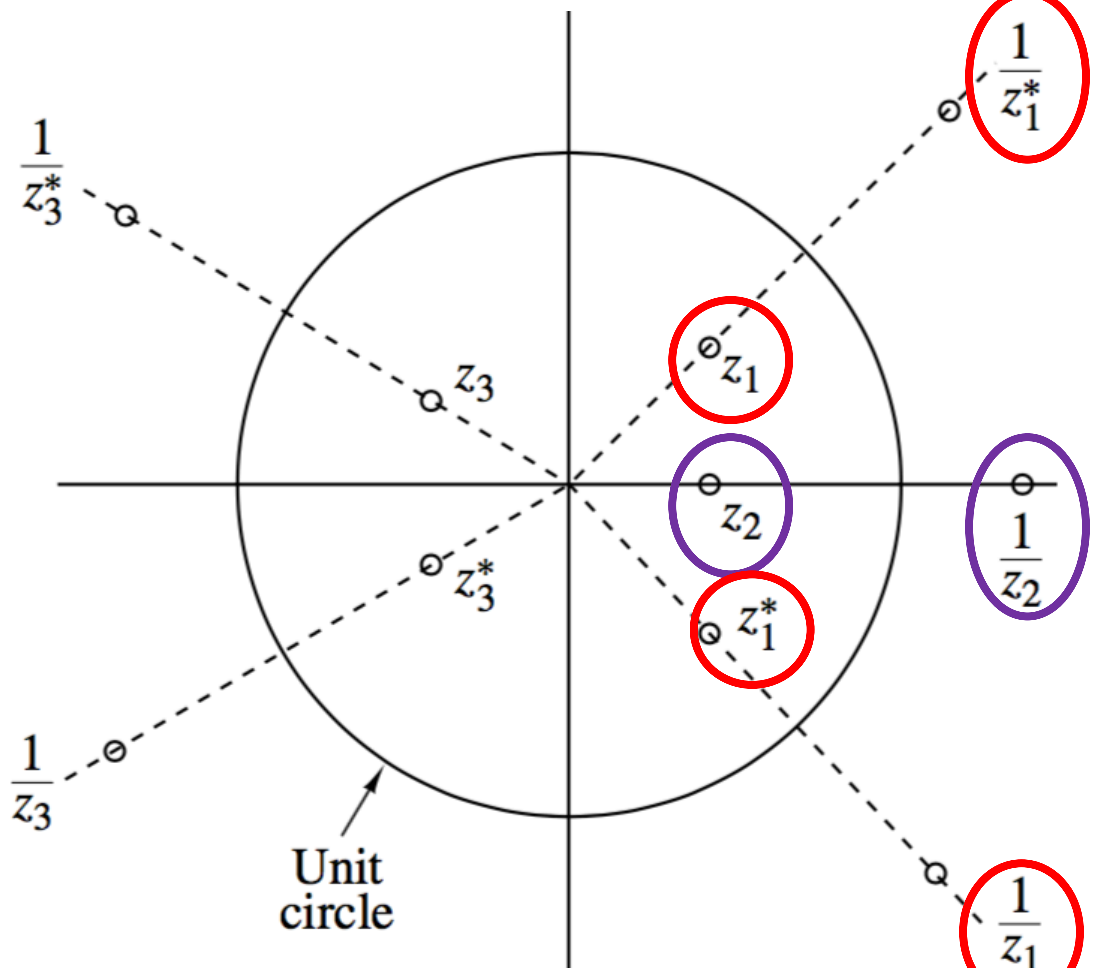
$$h(n) = \pm h(M - 1 - n), \qquad n = 0, 1, \ldots, M - 1$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$

$$\boxed{= z^{-(M-1)/2} \left\{ h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{(M-3)/2} h(n) \left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2} \right] \right\}, \quad M \text{ odd}}$$

$$\boxed{= z^{-(M-1)/2} \sum_{n=0}^{(M/2)-1} h(n)[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}], \qquad M \text{ even}}$$

This result implies that the roots of the polynomial $H(z)$ are identical to the roots of the polynomial $H(z^{-1})$. Consequently, the roots of $H(z)$ must occur in reciprocal pairs. In other words, if z_1 is a root or a zero of $H(z)$, then $1/z_1$ is also a root. Furthermore, if the unit sample response $h(n)$ of the filter is real, complex-valued roots must occur in complex-conjugate pairs. Hence, if z_1 is a complex-valued root, z_1^* is also a root. As a consequence of (2.6), $H(z)$ also has a zero at $1/z_1^*$. Figure 2.1 illustrates the symmetry that exists in the location of the zeros of a linear-phase FIR filter.



$$\boxed{z^{-(M-1)} H(z^{-1}) = \pm H(z)}$$

Symmetric & Antisymmetric FIR Filters

The frequency response characteristics of linear-phase FIR filters are obtained by evaluating $H(z)$ on the unit circle.

$$\boxed{H(\omega) = H_r(\omega) e^{-j\omega(M-1)/2}}$$

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\omega\left(\frac{M-1}{2} - n\right), \quad M \text{ odd}$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\cos\omega\left(\frac{M-1}{2} - n\right), \quad M \text{ even}$$

$$\boxed{\Theta(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi, & \text{if } H_r(\omega) < 0 \end{cases}}$$

Linear FIR Filters using Windows

$$\boxed{H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

In general, the unit sample response $h_d(n)$ obtained from $H_d(\omega)$ is infinite in duration and must be truncated at some point, say at $n = M - 1$, to yield an FIR filter of length M. Truncation of $h_d(n)$ to a length $M - 1$ is equivalent to multiplying $h_d(n)$ by a "rectangular window," defined as

$$w(n) = \begin{cases} 1, & n = 0, 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{\begin{aligned} h(n) &= h_d(n) w(n) \\ &= \begin{cases} h_d(n), & n = 0, 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}}$$

Linear FIR Filters using Windows

It is instructive to consider the effect of the window function on the desired frequency response $H_d(\omega)$. Recall that multiplication of the window function $w(n)$ with $h_d(n)$ is equivalent to convolution of $H_d(\omega)$ with $W(\omega)$, where $W(\omega)$ is the frequency-domain representation (Fourier transform) of the window function, that is,

$$W(\omega) = \sum_{n=0}^{M-1} w(n)e^{-j\omega n} = \sum_{n=0}^{M-1} e^{-j\omega n}$$

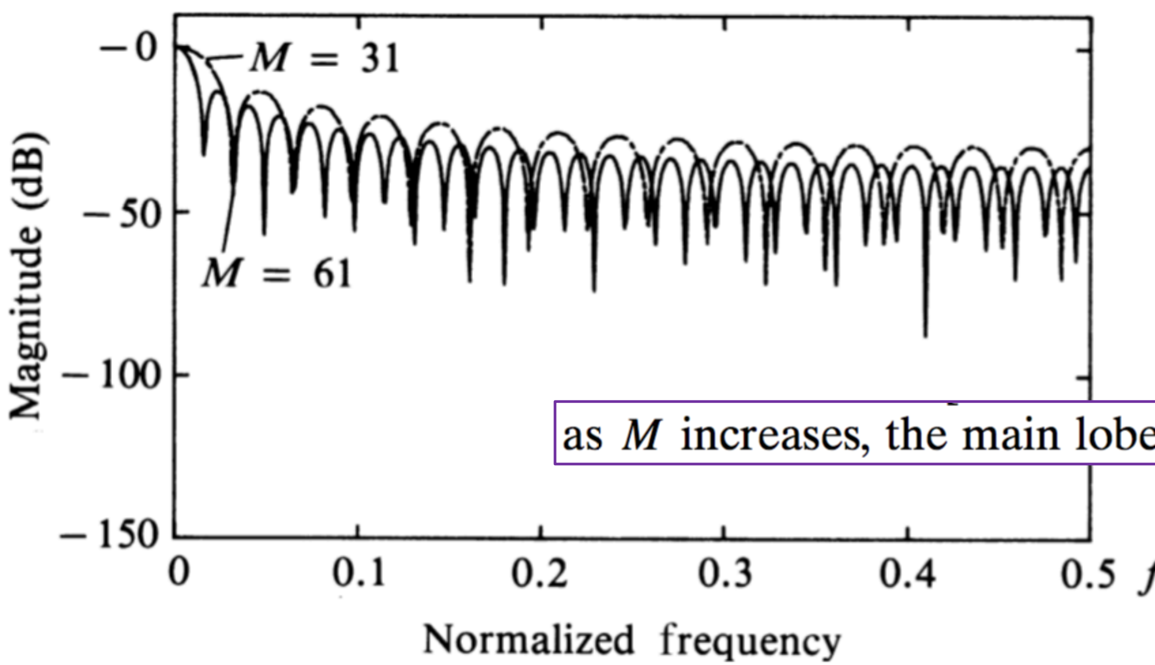
$$= \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = e^{-j\omega(M-1)/2} \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

This window function has a magnitude response

$$|W(\omega)| = \frac{|\sin(\omega M/2)|}{|\sin(\omega/2)|}, \qquad \pi \le \omega \le \pi$$

and a piecewise linear phase

$$\Theta(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right), & \text{when } \sin(\omega M/2) \ge 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi, & \text{when } \sin(\omega M/2) < 0 \end{cases}$$

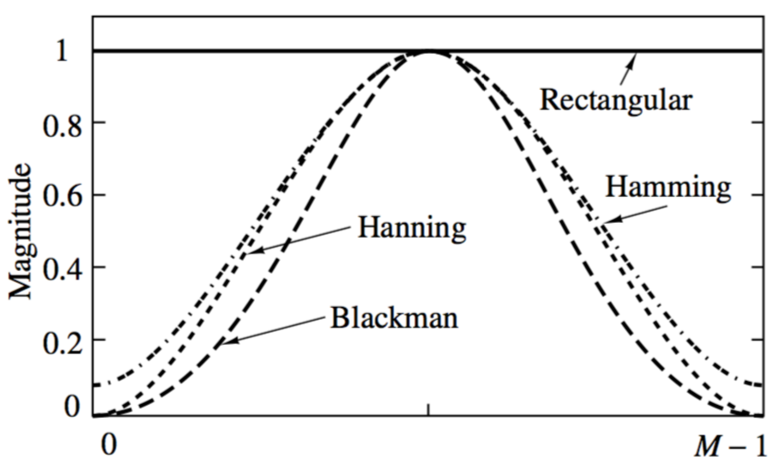


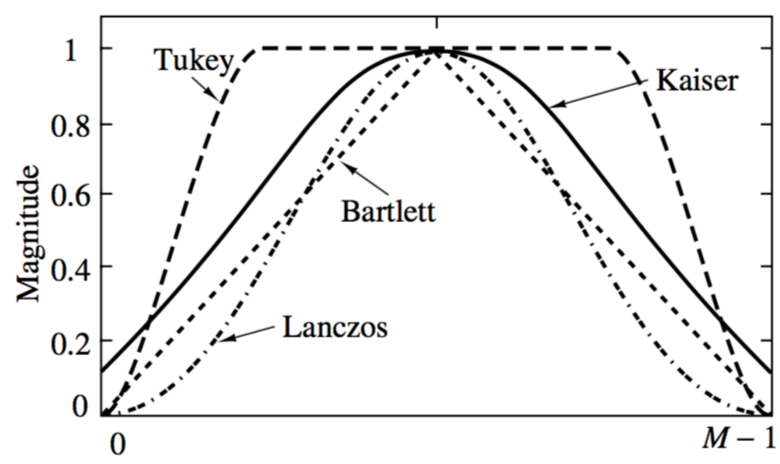
as M increases, the main lobe becomes narrower.

Linear FIR Filters using Windows

Thus the convolution of $H_d(\omega)$ with $W(\omega)$ yields the frequency response of the (truncated) FIR filter. That is,

$$\boxed{H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\nu)W(\omega - \nu)d\nu}$$





Linear FIR Filters using Windows

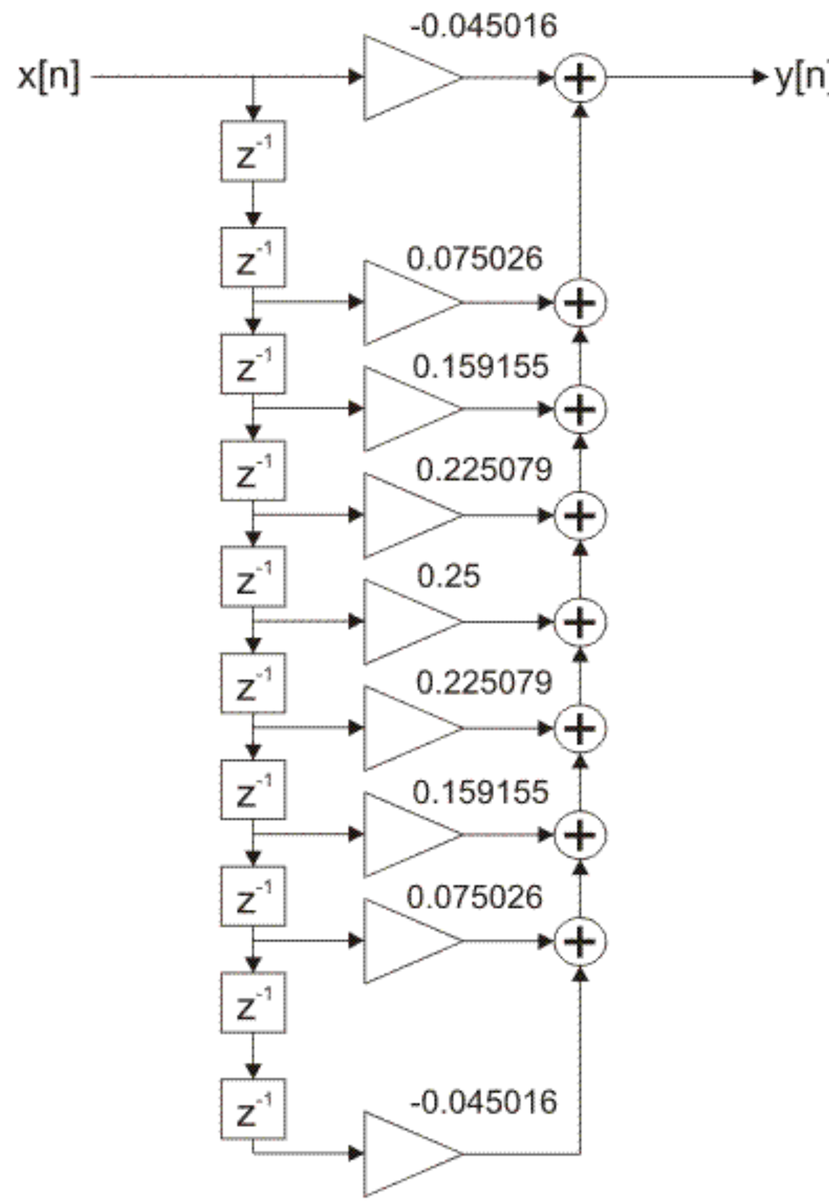
TABLE 1 Window Functions for FIR Filter Design

Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \dfrac{M-1}{2} \right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{M-1}{2}\right)^2 - \left(n - \dfrac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\dfrac{M-1}{2}\right)\right]}$

Example 1

- Filter order – N=10
- Sampling frequency – fs=20KHz
- Passband cut-off frequency – fc=2.5K

Method – filter design using rectangular window



Linear FIR Filters using Windows

`bartlett`	Bartlett window
`barthannwin`	Modified Bartlett-Hanning window
`blackman`	Blackman window
`blackmanharris`	Minimum 4-term Blackman-Harris window
`bohmanwin`	Bohman window
`chebwin`	Chebyshev window
`flattopwin`	Flat Top window
`gausswin`	Gaussian window
`hamming`	Hamming window
`hann`	Hann window
`kaiser`	Kaiser window
`nuttallwin`	Nuttall defined minimum 4-term Blackman-Harris window
`parzenwin`	Parzen (de la Valle-Poussin) window
`rectwin`	Rectangular window
`triang`	Triangular window
`tukeywin`	Tukey window

Example 2

Consider the ideal, or "brick wall", digital low-pass filter with a cutoff frequency of ω_0 rad/s. This filter has magnitude 1 at all frequencies less than ω_0, and magnitude 0 at frequencies between ω_0 and π. Its impulse response sequence $h(n)$ is

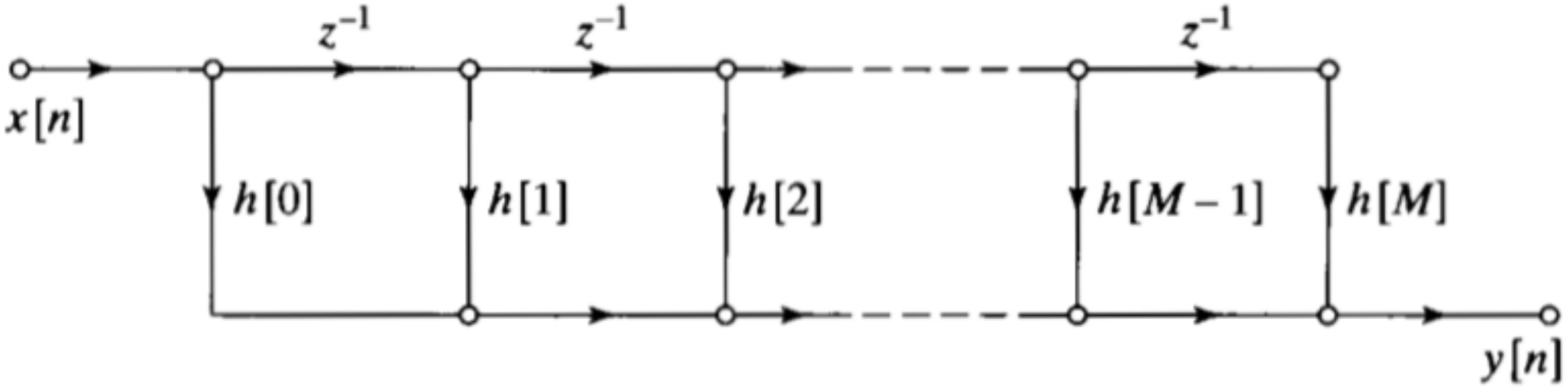
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}d\omega = \frac{\omega_0}{\pi}sinc\left(\frac{\omega_0}{\pi}n\right)$$

This filter is not implementable since its impulse response is infinite and noncausal. To create a finite-duration impulse response, truncate it by applying a window. Retain the central section of impulse response in the truncation to obtain a linear phase FIR filter. For example, a length 51 filter with a lowpass cutoff frequency ω_0 of 0.4π rad/s has the impulse response,

FIR Filters

$$\boxed{y[n] = \sum_{k=0}^{M} b_k x[n-k].}$$

Direct Form



FIR Filters

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} = \prod_{k=1}^{M_s} (b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}),$$

Cascade Form