lecture 3 Cauchy Riemann equations

Recall $\Omega \subseteq \mathbb{C}$ open set

$f : \Omega \to \mathbb{C}$, $z_0 \in \Omega$

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

$(h \in \mathbb{C})$

$f(z) = u(x,y) + i\,v(x,y)$

$z = x + iy$.

Assume $f'(z_0)$ exists.

$$f'(z_0) = \lim_{h \to 0} \frac{\left[u(x_0+h_1, y_0+h_2) + i\,v(x_0+h_1, y_0+h_2) - u(x_0,y_0) - i\,v(x_0,y_0) \right]}{h_1 + i h_2}$$

$z_0 = x_0 + i y_0$

$h = h_1 + i h_2$

Let $h \to 0$ along x-axis.

$f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0)$

Let $h \to 0$ along y-axis

$f'(z_0) = \frac{1}{i}\left(u_y(x_0, y) + i\,v_y(x_0, y_0) \right)$

Comparing

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \text{CR- equations}$$

Theorem $f = u+iv$ is complex diff at $z_0 \in \Omega$

$\Rightarrow$ u, v satisfies CR- equations

$$f'(z) = u_x + i v_x$$

Corollary

(1) $f \in \mathcal{H}(\Omega)$ + $f'(z) = 0 \ \forall z \in \Omega$
+ Ω domain

$\Rightarrow$ f is constant.

$f = u + iv$

$u : \Omega \to \mathbb{R}$

$f' = u_x + i v_x$

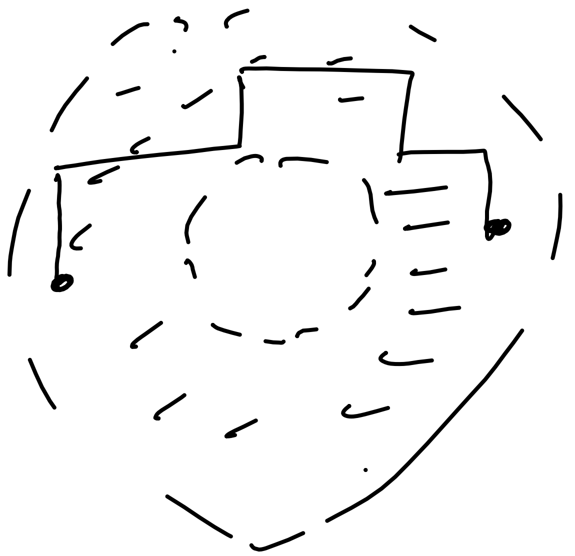
$u_x = 0 = v_x$

$u_y = -v_x = 0$

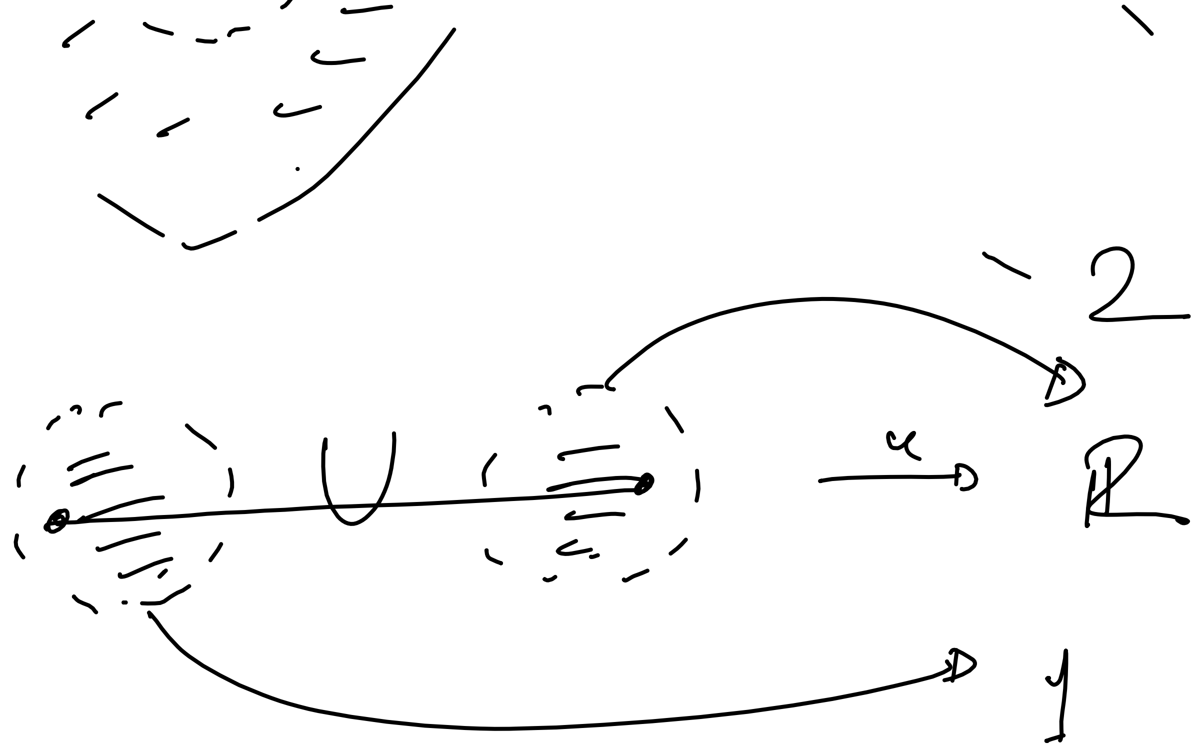
$\Rightarrow$ u is const, v is const

$\Rightarrow$ f = const.

domain / region
= open connected set



- connected



$u_x = 0 = u_y$

② $f(z) = \bar{z}$

$u = x \qquad v = -y$

$u_x = 1 \neq v_y = -1$

$\Rightarrow$ f is not complex diff at any point

③ $f : \Omega \to \mathbb{C}$ is real valued function. Ω - domain

Can it be holomorphic?

$v = 0$

$$\left.\begin{aligned} u_x &= v_y = 0 \\ u_y &= -v_x = 0 \end{aligned}\right\} \Rightarrow u = \text{Const}$$

$f = \text{const}.$

Example (CR-equations are not sufficient)

$$f(z) = \frac{\bar{z}^2}{z} \qquad z \neq 0$$

$$= 0 \qquad z = 0$$

$(\Omega = \mathbb{C})$

Claim f is not complex diff at 0, but CR equations are satisfied at $z_0 = 0$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\bar{h}^2}{h \cdot h}$$

$$= \lim_{h \to 0} \frac{\bar{h}^2}{h^2} \quad \text{— does not exist.}$$

$h \to 0$ along x-axis then the limit is $= 1$

$h \to 0$ along y-axis.

$$\lim_{h \to 0} \frac{(-iy)^2}{(iy)^2} = 1$$

$h \to 0$ along $x = y$ line

$$\lim_{x \to 0} \frac{(x - ix)^2}{(x + ix)^2} = \left(\frac{1-i}{1+i}\right)^2 = -1$$

$$f(z) = \bar{z}^2/z \qquad z \neq 0$$
$$= 0 \qquad z = 0$$

$$u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$v(x,y) = \frac{y^3 - 3x^2y}{x^2+y^2} \qquad (x,y) \neq (0,0)$$
$$(x,y) = (0,0)$$

$$u_x(0,0) = \lim_{h_1 \to 0} \frac{u(h_1,0) - u(0,0)}{h_1} \qquad h_1 \in \mathbb{R}$$

$$\| \quad = 1$$

$$v_y(0,0) = \lim_{h_2 \to 0} \frac{v(0,h_2) - v(0,0)}{h_2} = 1$$

<u>Theom</u>. $f = u + iv$

Ω - open.

Assme u, v satisfies CR equations

+ u_x, u_y, v_x, v_y are continuous on Ω ($\subseteq \mathbb{C} = \mathbb{R}$)

$\Longrightarrow$ f is holomorphic on Ω

<u>Example</u>

① $f(z) = e^x(\cos y + i\sin y)$

$u(x,y) = e^x \cos y \qquad v = e^x \sin y$.

$u_x = e^x \cos y = v_y = e^x \cos y$

$u_y = -e^x \sin y \qquad v_x = e^x \sin y$

Thus f is holomorphic on $\mathbb{C}$.

$$e^z := e^x(\cos y + i \sin y) \in \mathcal{H}(\mathbb{C})$$

- $$\frac{d}{dz}(e^z) = u_x + i v_x = e^z$$

- $$e^z \cdot e^{z'} = e^{z+z'} \quad z, z' \in \mathbb{C}$$

- $$|e^z| = e^x \neq 0$$
so $e^z \neq 0 \quad \forall\, z \in \mathbb{C}$.

(2) $$f = \underbrace{x^3 - 3xy^2}_{u} + i\,(\underbrace{3x^2y - y^3}_{v})$$

$$u_x = v_y$$
$$u_y = -v_x$$

holomorphic on $\mathbb{C}$.

Exercise (CR in polar form)

$x = r\cos\theta \quad y = r\sin\theta$

$u = u(x, y) \quad v = v(x,y)$

$$\frac{\partial u}{\partial r} = u_r = u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r}$$
$$= u_x \cos\theta + u_y \sin\theta$$

$$\boxed{\text{CR - equations in polar form.} \quad u_r = \frac{1}{r} v_\theta \quad v_r = -\frac{1}{r} u_\theta}$$

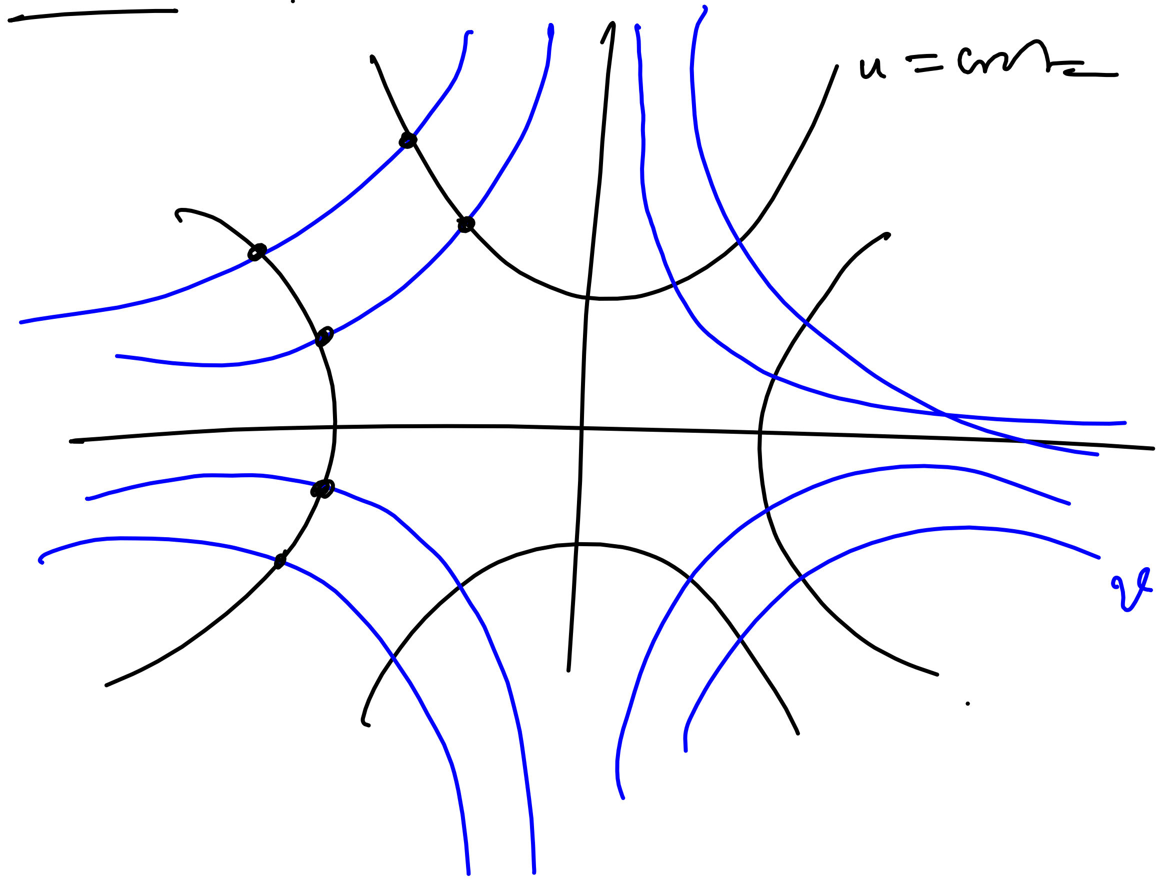
Orthogonal family

$f = u + iv$

$f(z) = z^2 = (x^2 - y^2) + 2ixy$

$u = x^2 - y^2 \qquad v = 2xy$

Plot $u = \text{const} \qquad v = \text{const}$



Th. & Suppose $f'(z_0) \neq 0$.

& the level curves $u(x,y) = c_1$, $v(x,y) = c_2$

intersect at z_0

$\Rightarrow$ u & v intersect orthogonally at z_0.

$f(z) = \frac{1}{z} \in H(\mathbb{C} \setminus (0))$

$= \frac{x - iy}{x^2+y^2}$

$u = \frac{x}{x^2+y^2}$

$v = -\frac{y}{x^2+y^2}$

& $u = 1$ $\quad x^2+y^2 = x$