Problem Set 6

October 27, 2020

0.1 Problem 1: (Based on Rice 9.1) A coin is thrown independently 10 times to test the hypothesis that the probability of a heads is 1/2 vs the alternative that the probability is not 1/2. The test rejects if the number of observed heads is 0 or 10.

(a) What is the significance level of the test?

(b) If the probability of heads is 0.1, what is the power of the test?

(c) If the test instead rejects if the number of observed heads is 1 or 9, what is the significance level?

a)

$$\alpha = P(x = 0|H_o) + P(x = 10|H_o)\alpha = \binom{10}{0} * (.5)^0 * (.5)^{10} + \binom{10}{10} * (.5)^{10} * (.5)^0 = .001953 \quad (1)$$

b)

$$\alpha = \binom{10}{0} * (.1)^0 * (.9)^{10} + \binom{10}{10} * (.1)^{10} * (.9)^0 = .348678 \quad (2)$$

c)

$$\alpha = P(x \leqslant 1|H_o) + P(x \geqslant 9|H_o)\alpha = P(x = 0|H_o) + P(x = 1|H_o) + P(x = 9|H_o) + P(x = 10|H_o)\alpha = \binom{10}{0} * (.5) \quad (3)$$

0.2 Problem 2: Which of the following hypotheses are simple, and which are composite? Justify your answers.

(a) X follows a uniform distribution on [0,1].

(b) A die is unbiased.

(c) X follows a normal distribution with mean 0 and variance > 10.

(d) X follows a normal distribution with mean = 0.

a) This hypotheses is simple because we know our parameters and the range of values that the random variable x can take on is known.

b) Since the dice is a standard 6-sided dice, we know p=1/6 and therefore the distribution is specified and is thus a simple hypothesis.

c) The parameters are not fully specified completely. Sigma^2 can take any value greater than 10 and therefore the distribution is not fully known and thus it is a composite hypothesis.

d) Once again, we do not know our full parameters. Sigma^2 can take any value and therefore we don't know the full distribution and thus conclude it is a composite hypothesis.

0.3 Problem 3: True or false and state why:

(a) The significance level of a statistical test is equal to the probability that the null hypothesis is true.

(b) If the significance level of a test is decreased, the power would be expected to increase.

(c) If a test is rejected at the significance level , the probability that the null hypothesis is true equals alpha.

(d) The probability that the null hypothesis is falsely rejected is equal to the power of the test.

(e) A type I error occurs when the test statistic falls in the rejection region of the test.

(f) A type II error is more serious than a type I error.

(g) The power of a test is determined by the null distribution of the test statistic.

(h) The likelihood ratio is a random variable.

a) False. The significance level is the conditional probability of rejecting the null hypothesis when the null hypothesis is true.

b) False. Power should decrease since the test rejects less often if it has a lower significance level.

c) False, going off our logic for part a, we are unable to evaluate the probability that our null hyp. being true.

d) False. power of the test is the probability that the null hypothesis is correctly rejected.

e) False. When this happens it may lead to either a correct decision of the null being fals, or to the type I error, it could be either one.

f) False. It depends on the context. Most of the time Type I is worse, however life isn't black and white and thus sometimes depending on the modeling of the question a Type II error could be worse.

g) Flase. The power is determined by the conditional prob of correctly rejecting the null hypothesis when Ha is true or 1 - Type II error.

h) True! The likelihood ratio is a function of the data which results in a random variable and also is a ratio of the conditional pdf/pmf of data (X) under our null and alternative hypothesis.

0.4 Problem 5: Suppose that the null hypothesis is true, that the distribution of the test statistic, T say, is continuous with cdf F and that the test rejects for large values of T. Let V denote the p-value of the test.

(a) Show that V = 1−F(T).

(b) Conclude that the null distribution of V is uniform.

(c) If the null hypothesis is true, what is the probability that the p-value is greater than .1?

(d) Show that the test that rejects if V < alpha has significance level alpha.

a)

V = P(T>t) = 1-P(T<=t) = 1-F(T) , because P(T<=t) = F(T)

b)

As F(T) [] U(0,1) -> X [] U(0,1. If 1-X follows a uniform distribution then 1-F(T) also follows a uniform distribution. Thus, V follows a uniform distribution.

c)

P(T>t)>.1

1-P(T<=t)>.1

1-F(T)>.1

F(T)<1-.9

-> F(T) < .90

d)

P(T>t) < alpha

1-P(T<=t) < alpha

1-F(t) < alpha

F(t) > 1 - alpha. When F(T) >1-alpha we can reject our test. We reject the test if V < alpha has significance alpha.

0.5 Problem 7: Nylon bars were tested for brittleness. Each of the 280 bars was molded under similar conditions and was tested in five places. Assuming that each bar has uniform composition, the number of breaks on a given bar should be binomially distributed with five trials and an unknown probability of p of failure. If the bars are all of the same uniform strength, p should be the same for all of them; if they are of different strengths, p should vary from bar to bar. Thus, the H0 is that the p's are all equal. The following table summarizes the outcome of the experiment:

(a) Under the given assumption, the data in the table consist of 280 observations of inde- pendent binomial random variables. Find the mle of p.

(b) Pooling the last three cells, test the agreement of the observed frequency distribution with the binomial distribution using Pearson's chi-square test.

a)

$$\hat{p} = \frac{\sum_{i=0}^{5} iN_i}{5(\sum_{i=0}^{5} N_i)} = \frac{\sum_{k=1}^{280} X_k}{280} * \frac{1}{5} = \frac{1}{5}\bar{X}\hat{p} = \frac{0*157+1*69+2*35+3*17+4*1+5*1}{280} * \frac{1}{5} = .711 \tag{4}$$

b)

$$X^2 = \sum_{i=0}^{5} \frac{(O_i - E_i)^2}{E_i} = \frac{(157-130.1)^2}{130.1} + \frac{(69-107.8)^2}{107.8} + \frac{(35-35.7)^2}{35.7} + \frac{(19-6.4)^2}{6.4} = 44.347 \tag{5}$$

0.6 Problem 8: In this problem you will develop and apply a test for normality based on the coefficient of skewness as defined in Section 9.9 of Rice.

(a) Write an R function to compute the coefficient of skewness, b1. 2

(b) Use simulation to approximate the sampling distribution of b1 when the data is modeled by 100 independent N (0, 1) RVs.

(c) Test 1000 data sets each containing 100 independent N (0, 1) RVs for normality. For this test H0 is that the data are iid N(0,1) and HA is that the data are not iid N(0,1). Use the sampling distribution of b1 to compute an approximate p-value for each test. Generate a Q-Q plot of these p-values relative to a theoretical U (0, 1) distribution. Does the shape of this plot match your expectations? Explain.

(d) Test 1000 data sets each containing 100 independent Poisson RVs with = 1 for nor- mality. Compute an approximate p-value for each test. Generate a Q-Q plot of these p-values relative to a theoretical U(0,1) distribution. Does the shape of this plot match your expectations? Explain.

[3]:
```
if (!requireNamespace("moments", quietly = TRUE))
    install.packages("moments")
```

```
Installing package into '/Users/valazeinali/Library/R/3.6/library'
(as 'lib' is unspecified)



The downloaded binary packages are in
/var/folders/bx/dp3pc2gx453dz73c35_y9_8h0000gn/T//RtmpjSux50/downloaded_packages
```

a) the equation for skewness below:

$$b_1 = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^3}{s^3} \tag{6}$$

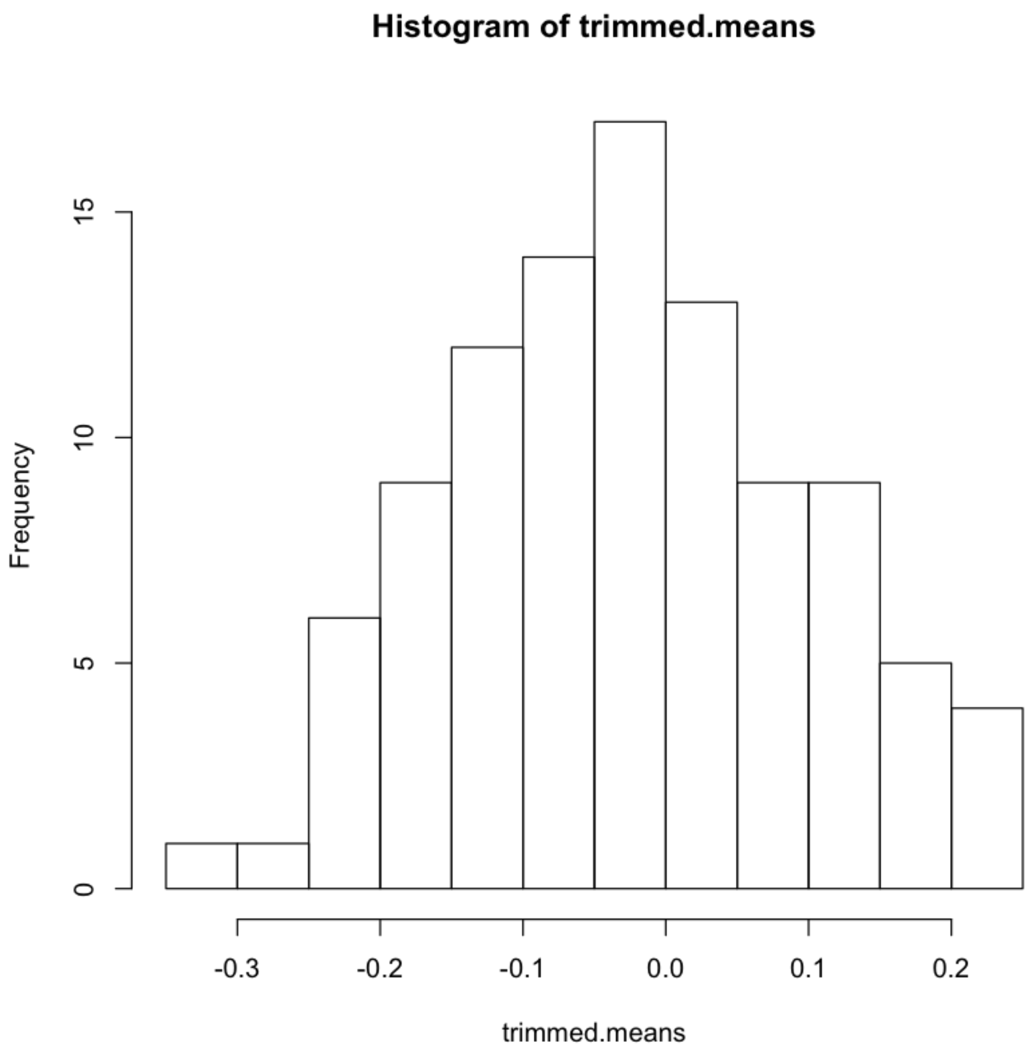
[4]:
```
#a) using prebuilt function that uses formula shown above
library(moments)

x <- rnorm(100)
skewness(x)
```

-0.107035705150691

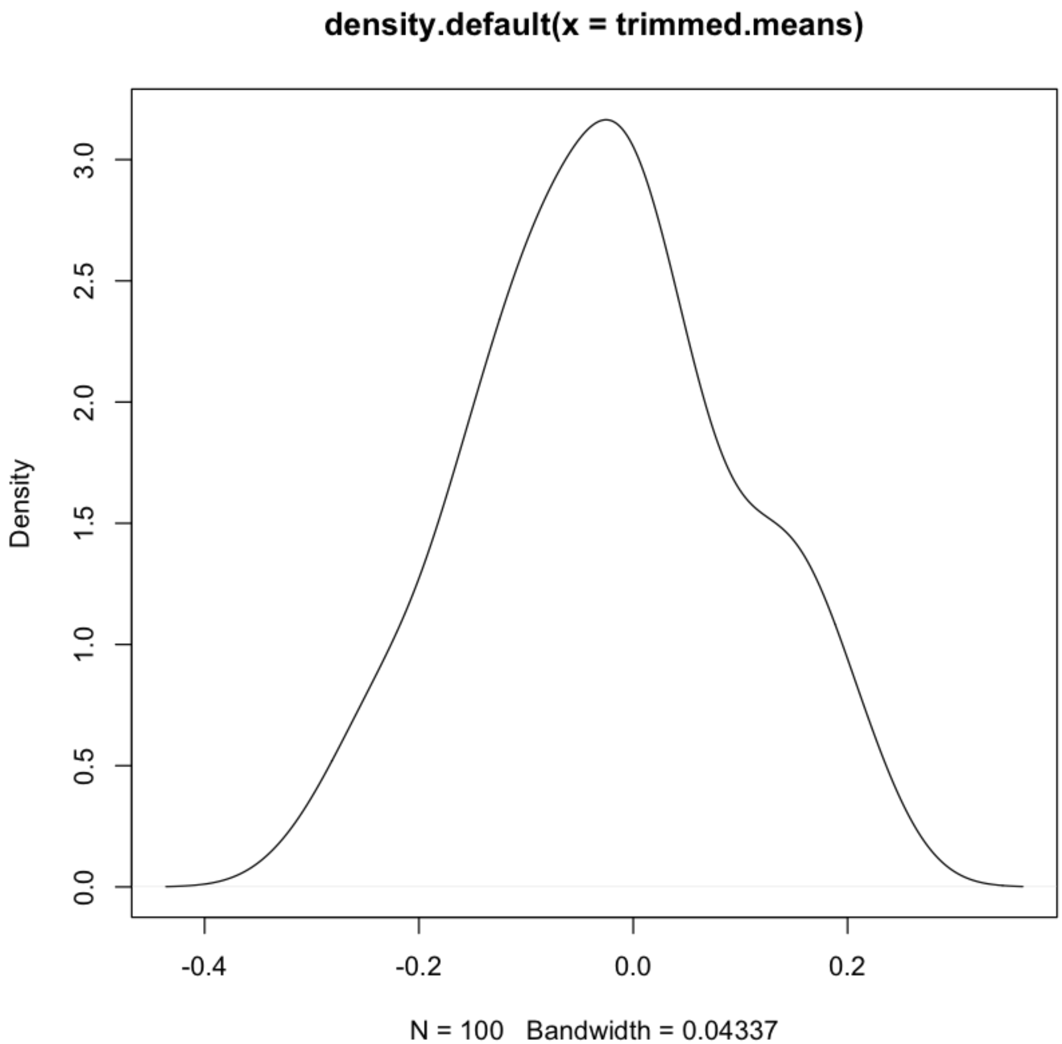
[5]:
```
#b)
B = 100
n= length(x)
```

```
bootstrap.data = matrix(sample(x, B*n, replace=T),
nrow=B, ncol=n)
trimmed.means = apply(bootstrap.data, 1, function(x) {
mean(x, trim=0.2)
})
```

[6]:
```
hist(trimmed.means)
```



[8]:
```
plot(density.default(x=trimmed.means))
```



[ ]:
```
#c & d unfortunately couldn't figure out. I will go to office hours to ask
 ↪approach.
```