QBS 120 - Problem Set 2

Rob Frost

1. (Based on Rice 3.18) Let X and Y have the joint density function $f(x, y) = kxy, 0 \leq y \leq x \leq 2$ or 0 elsewhere.

   A Describe the region over which the density if positive and use it in determining limits of integration to answer the following questions.

   B Find k.

   C Find the marginal densities of X and Y.

   D Find the conditional densities of Y given X and X given Y.

2. (Based on Rice 3.20) If X_1 is uniform on [0,1], and, conditional on X_1, X_2 is uniform on $[X_1, 2]$, find the joint and marginal distributions of X_1 and X_2.

3. (Optional*) Rice 3.53: Consider forming a random rectangle in two ways. Let U_1, U_2, U_3 be independent standard normal variables. One rectangle has sides U_1 and U_2 and the other is square with side U_3. Find the probability that the area of the square is greater than the area of the other rectangle. Hint: to find the distribution of the rectangle's area, adapt the approach on page 98 for the quotient Z=Y/X to the product Z=XY; in this case, the density of Z is given by:

$$f_Z(z) = \int_{-\infty}^{\infty} 1/|x| f_{X,Y}(x, z/x) dx$$

   To find the distribution of the square's area, start with the CDF and differentiate to get the density. If you get stuck on the integration by parts details, feel free to wait for the official solutions.

4. (Based on Rice 3.71) Let $X_1, ..., X_n$ be independent RVs all with the same density f. Find an expression for the probability that the interval $[X_{(1)}, \infty)$ encompasses at least 100v% of the probability mass of density f. Note: remember that the notation $X_{(1)}$ refers to the first order statistic.

5. (Based on Rice 4.31) Let X be uniformly distributed on the interval [1,4]. Find E[1/X]. Is E[1/X] = 1/E[X]? Note: find E[X] using the definition of expectation, don't just plug in the expectation of a U(a,b) RV.

6. (Based on Rice 4.49) Two independent measurements, X and Y, are taken of a quantity μ. E[X] = E[Y] = μ, but σ_x and σ_y are unequal. The two measurements are combined by means of a weighted average to give:

$$Z = \alpha X + (1 - \alpha) Y$$

   where α is a scalar and $0 \leq \alpha \leq 1$.

A Show that $E[Z] = \mu$.

B If X and Y are not independent, what is $E[Z]$?

C What is Var(Z)? Does this result hold if X and Y are not independent?

D Find α in terms of σ_X and σ_Y to minimize Var(Z).

E Under what circumstances is it better (i.e., minimizing variance) to use the average $(X + Y)/2$ than either X or Y alone?

7. (Based on Rice 4.57) If X and Y are independent random variables, find E[XY] and Var(XY) in terms of the means and variances of X and Y. Hints: 4.1.1 Corollary A, realize that $E[X^2] = Var(X) + E[X]^2$