Chapter 7
Cut Sets and Cut Vertices

In this chapter, we find a type of subgraph of a graph G where removal from G separates some vertices from others in G. This type of subgraph is known as cut set of G. Cut set has a great application in communication and transportation networks.

7.1 Cut Sets and Fundamental Cut Sets

7.1.1 Cut Sets

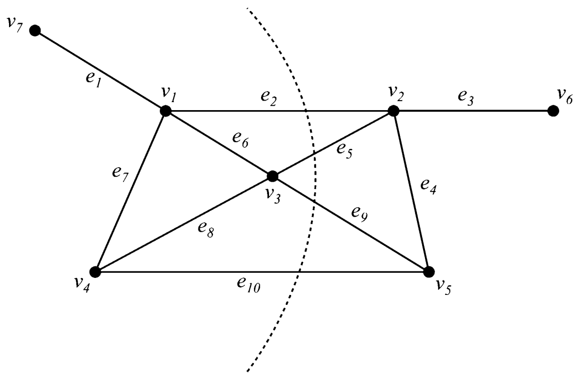

Fig. 7.1 A cut set $\{e_2, e_5, e_9, e_{10}\}$ of the graph

In a connected graph G, the set of edges is said to be a *cut set* of G if removal of the set from G leaves G disconnected but no proper subsets of this set does not do so.

In the graph shown in Fig. 7.1, the set of edges $\{e_2, e_5, e_9, e_{10}\}$ is a cut set of the graph. In Fig. 7.1, it is represented by a dotted curve. It can be noted that the edge set $\{e_2, e_5, e_4\}$ is also a cut set of the graph. $\{e_1\}$ is a cut set containing only one edge. Removal of the set $\{e_2, e_5, e_4, e_9, e_{10}\}$ disconnects the graph but it is not cut set because its proper subset $\{e_2, e_5, e_4\}$ is a cut set.

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Corollary

(1) *Every edge of a tree is a cut set of the tree.*

(2) *A cut set is a subgraph of G.*

7.1.2 Fundamental Cut Set (or Basic Cut Set)

Let T be a spanning tree of a connected graph G. A cut set S of G containing exactly one branch of T is called a *Fundamental cut set* of G with regard to T.

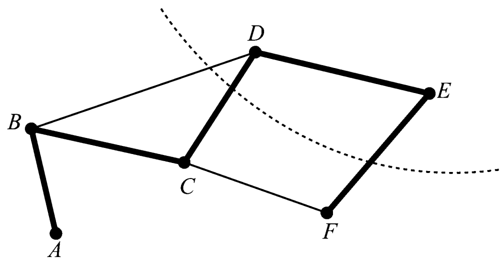


Fig. 7.2 $\{BD, CD, EF\}$ is a cut set but not a fundamental cut set

In Fig. 7.2, $T = \{AB, BC, CD, DE, EF\}$ is a spanning tree (shown by bold lines). The set of edges $\{BD, BC\}$ is a cut set containing one branch BC of T. So, $\{BD, BC\}$ is a Fundamental cut set of G w.r.t T. In the same graph the set $\{BD, CD, EF\}$ (as shown by dotted curve in Fig. 7.2) is a cut set but not a Fundamental Cut set with regard to T because it contains two edges of T.

7.2 Cut Vertices

A vertex v of a connected graph G is said to be a *cut vertex* if its deletion from G (together with the edges incident to it) disconnects the graph.

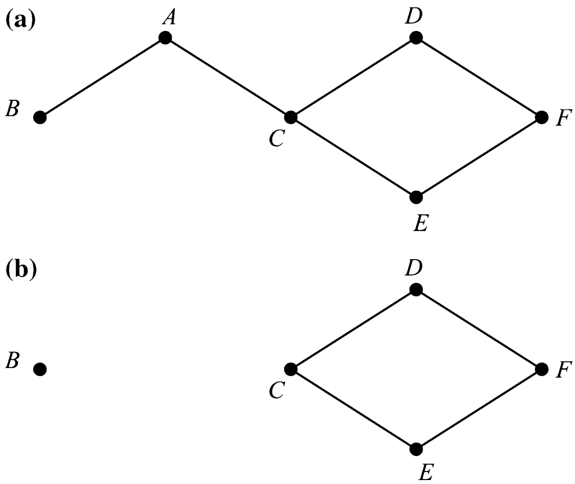


Fig. 7.3 **a** Vertex A is a cut vertex of the graph, **b** Disconnected graph after the removal of vertex A

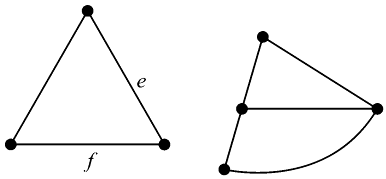


Fig. 7.4 A disconnected graph with two connected components

In the graph G shown in Fig. 7.3a removal of the vertex A (together with the edges incident to it) leaves a disconnected graph shown in Fig. 7.3b. So, A is a cut vertex, we note that C is also a cut vertex. But F is not a cut vertex of this graph.

Corollary

(1) *Every vertex* (*with degree greater than one*) *of a tree is cut vertex.*

(2) *A graph may have no cut vertex at all. For example, the graph in* Fig. 7.4 *has no cut vertex. Another examples are* K_2, K_3, K_4, *etc. They have no cut vertex.*

7.2.1 Cut Set with respect to a Pair of Vertices

If a cut set puts two vertices v_1 and v_2 into two different components. Then, it is called a cut set with regard to v_1 and v_2.

In the graph shown in Fig. 7.3a, $\{AC\}$ is a cut set with regard to the vertices B and E. This is not a cut set with regard to A and B.

7.3 Separable Graph and its Block

7.3.1 Separable Graph

A connected graph (or a connected component of a graph) is said to be *separable* if it has a cut vertex.

On the other hand, a connected graph (or a connected component of a graph) which is not separable is called *non-separable* graph.

The graph in Fig. 7.3a is separable. On the other hand, each of the two connected components of the graph in Fig. 7.4 is non-separable. Again each of the two components in Fig. 7.3b is nonseparable.

7.3.2 Block

A separable graph consists of two or more non-separable subgraphs. Each of these non-separable subgraphs is called a *Block*. The graph in Fig. 7.3a has the following Blocks shown in Fig. 7.5.

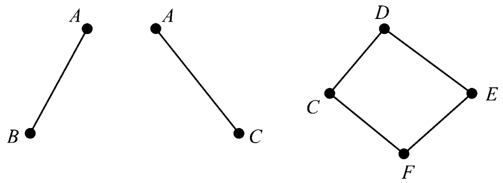


Fig. 7.5 Three Blocks of the graph in Fig. 7.3a

It can be noted that each of these blocks has no cut vertex.

But between the following two subgraphs of Fig. 7.3a shown in Fig. 7.6, first one is not a block because it is further separable as A is cut vertex of it.

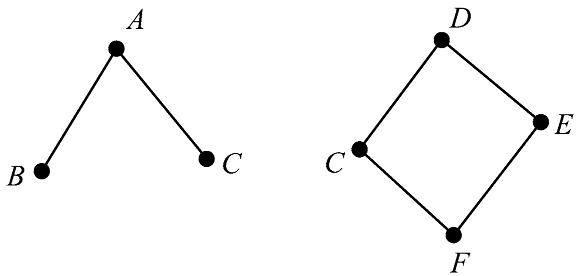


Fig. 7.6 The first subgraph of Fig.7.3a is not a Block whereas the second subgraph is a Block

7.4 Edge Connectivity and Vertex Connectivity

7.4.1 Edge Connectivity of a Graph

Let G be a graph (may be disconnected) having k components. The minimum number of edges whose deletion from G increases the number of components of G is called *edge connectivity* of G. It is denoted by $\lambda(G)$.

In Fig. 7.4, a graph having two components is shown. We see that if one edge is deleted from the graph, the number of its components still remains 2. But if two particular edges, say e and f are deleted then number of components becomes 3. So, the edge connectivity of the graph is 2.

Corollary

(1) *The number of edges in the smallest cut set of a graph is its edge connectivity.*
(2) *The edge connectivity of a tree is* 1.

7.4.2 Vertex Connectivity of a Graph

Let G be a graph (may be disconnected). The minimum number of vertices (together with the edges incident to it) whose deletion from G increase the number of components of G is called *vertex connectivity* of G. It is denoted by $\kappa(G)$.

k-connected and k-edge connected: A graph G is *k-connected* if $\kappa(G) = k$, and G is *k-edge connected* if $\lambda(G) = k$

For Example, the vertex connectivity of the graph in Fig. 7.7 is 2.

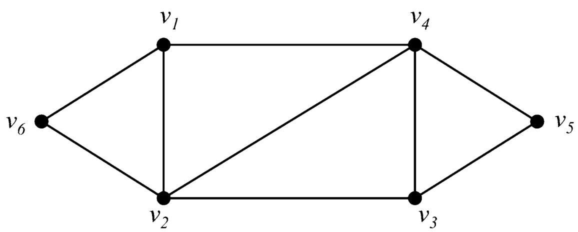


Fig. 7.7 The vertex connectivity of the graph is 2

Corollary

(1) *The vertex connectivity of tree* is 1.
(2) *The vertex connectivity of a connected separable graph is* 1.

Theorem 7.1 *Every cut set in a connected graph contains at least one branch of every spanning tree of the graph.*

Proof Let, S be a cut set of G. Let T be a spanning tree of G. Suppose that, S does not contain any branch of T. Then all edges of T are present in $G - S$. It means that $G - S$ is connected graph. It implies that S is not a cut set. Hence a cut set must contain at least one branch of a spanning tree of G. □

For example, in Fig. 7.8, $\{a, c\}$ is not a cut set, so it should contain one branch of T to become a cut set.

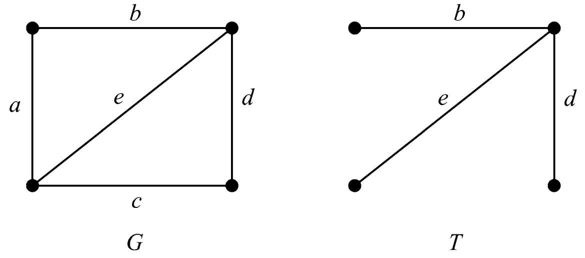


Fig. 7.8 A tree G and its spanning tree T

Theorem 7.2 *A vertex v in a connected graph G is a cut vertex if and only if there exists two vertices a and b distinct from v in G such that every path connecting a and b passes through v.*

Proof If v is a cut vertex of G, $G - v$ is a disconnected graph. Let us select two vertices a and b in two different components of $G - v$. Then there exists no path from a to b in $G - v$. Since, G is connected graph there exists a path P from a to b in G. If the path does not contain the vertex v, then removal of v from G will not disconnect the vertices a and b, which is a contradiction to the fact that a and b lies in two different components of $G - v$. Hence every path between a and b passes through v.

Conversely, if every path from a to b contains the vertex v then removal of v from G disconnects a and b. Hence, a and b lies in different components of G which implies that $G - v$ is disconnected graph. Therefore, v is cut vertex of G. □

Theorem 7.3 *The edge connectivity of a graph $\leq$ the smallest degree of all vertices of the graph.*

Proof Let v_k be the vertex with smallest degree in G. Let $d(v_k)$ be the degree of v_k. Vertex v_k can be separated from G by removing the $d(v_k)$ edges incident on vertex v_k. Therefore, removal of $d(v_k)$ edges disconnects the graph. Hence, the edge connectivity of a graph cannot exceed the smallest degree of all vertices of the graph. □

Theorem 7.4 *In any graph, the vertex connectivity $\leq$ the edge connectivity.*

Proof Let λ denote the edge connectivity of G. Therefore, there exists a cut set S in G containing λ edges.

Then, S partitions the vertices of G into two subsets V_1 and V_2 such that every edge in S joins a vertex in V_1 to a vertex in V_2. By removing at most λ vertices from V_1 or V_2 on which the edges of S are incident, we will be able to remove S (together with all other edges incident on these vertices) from G. Thus, removal of at most λ vertices from G will disconnect the graph. Hence, the vertex connectivity is less than or equal to λ. □

Corollary *Every cut set in a non-separable graph with more than two vertices contains at least two edges.*

Proof A graph is nonseparable if its vertex connectivity is at least two. In view of Theorem 7.4, edge connectivity $\geq$ vertex connectivity. Hence, edge connectivity of a non-separable graph is at least two which is possible if the graph has at least two edges. □

Theorem 7.5 *The maximum vertex connectivity of a connected graph with n vertices and e edges $(e \geq n-1)$ is the integral part of the number $2e/n$,* i.e., $\lfloor 2e/n \rfloor$. (*The floor function* ***floor*(*x*)** = $\lfloor x \rfloor$ *is the largest integer not greater than x*)

Proof We know that every edge in G contributes two degrees. Thus the sum of degrees of all the vertices is $2e$. Since, this sum $2e$ is divided among n vertices, therefore, there must be at least one vertex in G whose degree is less than or equal to the number $2e/n$.

Therefore, using Theorem 7.3, the edge connectivity of $G \leq 2e/n$.

Consequently, it follows from Theorem 7.4, vertex connectivity $\leq$ edge connectivity $\leq 2e/n$.

Hence, maximum vertex connectivity possible is $\lfloor 2e/n \rfloor$. □

Theorem 7.6 (Whitney's Inequality) *For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$ i.e. vertex connectivity $\leq$ the edge connectivity $\leq$ the minimum degree of the graph G*

Proof We shall first prove $\lambda(G) \leq \delta(G)$.

If G has no edges, then $\lambda = 0$ and $\delta = 0$. If G has edges, then we get a disconnected graph, when all edges incident with a vertex of minimum degree are removed. Thus, in either case, $\lambda(G) \leq \delta(G)$.

Now, from Theorem 7.4, it follows that $\kappa(G) \leq \lambda(G)$. Hence, it is proved. □

Example 7.1 Find the edge connectivity, vertex connectivity and minimum degree of the following graph in Fig. 7.9

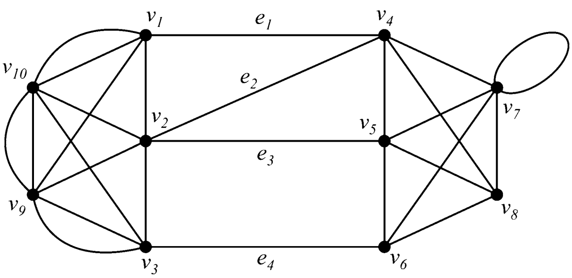


Fig. 7.9

Solution:
The vertex connectivity of the given graph is three because removal of v_1, v_2, v_3 or v_4, v_5, v_6 disconnects the graph.

The edge connectivity of this graph is four. $S = \{e_1, e_2, e_3, e_4\}$ is one such cut set. It can be observed that the degree of each vertex is at least four.

Therefore, $\kappa(G) = 3,\ \lambda(G) = 4$ *and* $\delta(G) = 4$.

Example 7.2 Find the edge connectivity, vertex connectivity and minimum degree of the following graph in Fig. 7.10.

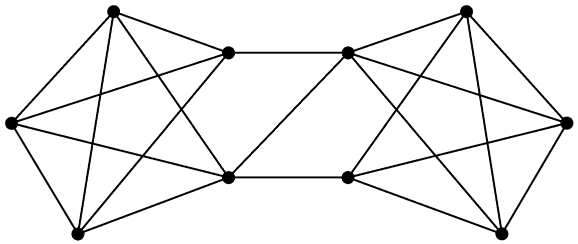


Fig. 7.10

Solution:
The vertex connectivity of the given graph is two because removal of at least two vertices are required to disconnect the graph.

The edge connectivity of this graph is three because removal of at least three edges are required to disconnect the graph. It can be observed that the degree of each vertex is at least four.

Therefore, $\kappa(G) = 2,\ \lambda(G) = 3$ *and* $\delta(G) = 4$.

Example 7.3 Show that, the edge connectivity $\lambda(G)$, vertex connectivity $\kappa(G)$, and minimum degree $\delta(G)$of the following graph in Fig. 7.11 are equal. Is the given graph separable?

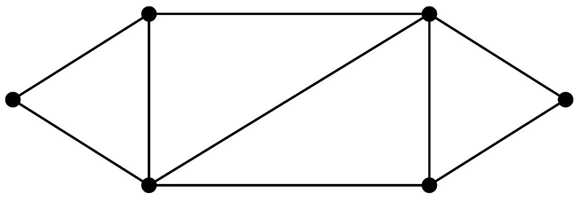


Fig. 7.11

Solution:
The vertex connectivity of the given graph is two because removal of at least two vertices are required to disconnect the graph. So, $\kappa(G) = 2$

The edge connectivity of this graph is two because removal of at least two edges are required to disconnect the graph. Therefore, $\lambda(G) = 2$

It can be observed that the degree of each vertex is at least two.

Therefore, $\kappa(G) = \lambda(G) = \delta(G) = 2$.

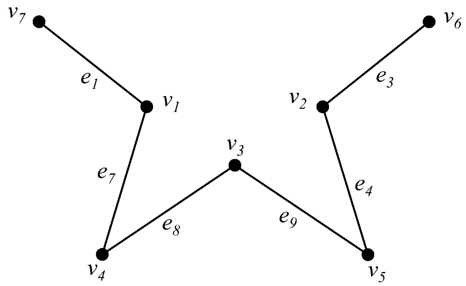
Moreover, the given graph in Fig. 7.11 is nonseparable, since it has no cut vertex.

Example 7.4 Find the fundamental cut sets of the graph in Fig. 7.1

Solution:

The Fig. 7.12 shows the spanning tree obtained by DFS.

Fig. 7.12 A spanning tree obtained by DFS



From the simple graph Fig. 7.1, we see that there are $n - 1 = 7 - 1 = 6$ fundamental cut sets with regard to the branches e_1, e_3, e_4, e_7, e_8 and e_9 of the spanning tree in Fig. 7.12.

Fundamental cut sets	Corresponding branch
$\{e_2, e_6, e_7\}$	e_7
$\{e_2, e_5, e_4\}$	e_4
$\{e_2, e_6, e_8, e_{10}\}$	e_8
$\{e_2, e_5, e_9, e_{10}\}$	e_9
$\{e_1\}$	e_1
$\{e_3\}$	e_3

Exercises:

1. Prove that a vertex v of a tree T is a cut vertex if and only if $d(v) > 1$.
2. Let T be a tree with at least three vertices. Prove that there is a cut vertex v of T such that every vertex adjacent to v, except for possibly one, has degree 1.
3. Let v be a cut vertex of the simple connected graph G. Prove that v is not a cut vertex of its complement $\overline{G}$.
4. Let G be a simple connected graph with at least two vertices and let v be a vertex in G of smallest possible degree, say k.
   (a) Prove that $\kappa(G) \leq k$ where $\kappa(G)$ is called vertex connectivity of G. It is the smallest number of vertices in G whose deletion from G leaves either a disconnected graph or K_1.
   (b) Prove that $\kappa(G) \leq 2e/n$, where e is the number of edges and n is the number of vertices in G.

5. Let G be a Hamiltonian graph. Show that G does not have a cut vertex.
6. Find the edge connectivity and vertex connectivity of the following graph in Fig. 7.13.

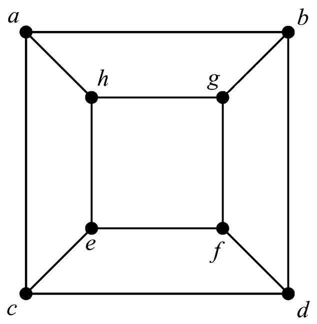


Fig. 7.13