EECE\CS 253 Image Processing

Lecture Notes: The 1&2-Dimensional Fourier Transforms

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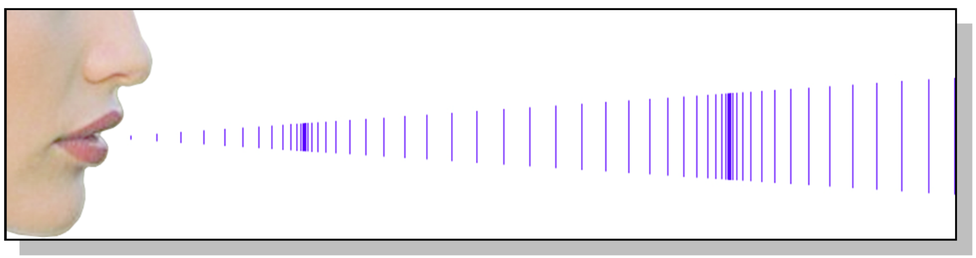
Signal:

A measurable phenomenon that changes over time or throughout space.

sound



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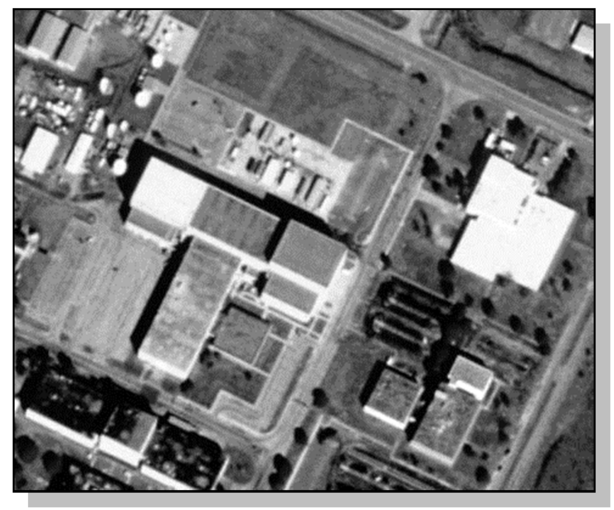


Signals: Space-Time vs. Frequency-Domain Representation

Space/time representation: a graph of the measurements with respect to a point in time and/or positions in space.

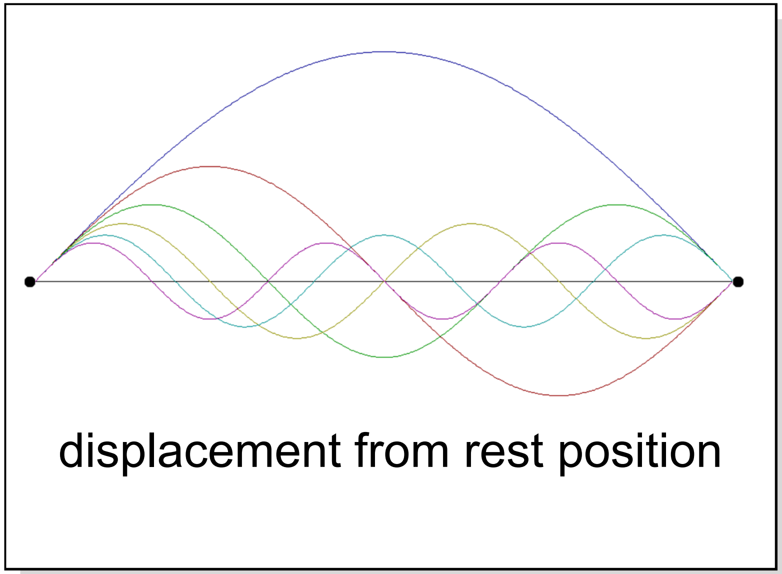
Fact: signals undulate (otherwise they'd contain no information).

Frequency-domain representation: an exact description of a signal *in terms of* its undulations.

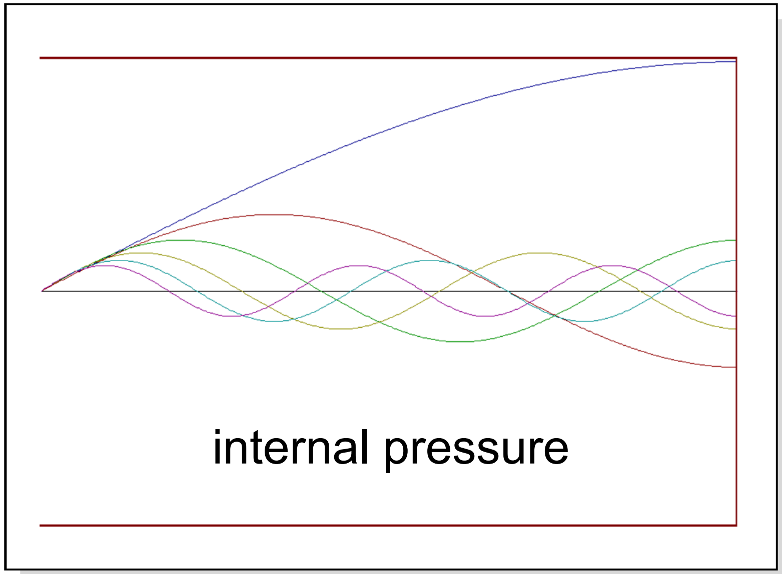
Origin of Sounds

- The mechanical vibrations of an object in an atmosphere.
- Vibrations: internal elastic motions of the material.
- The surface of the object undulates causing compressions and rarefactions in the air which propagate through the air away from the surface.
- An object vibrates with different *modes*.
- A mode is a vibratory pattern with a distinctive shape — part of the object surface moves out while another part moves in — a *standing wave*.

Vibratory Modes / Standing Waves: Examples



string modes



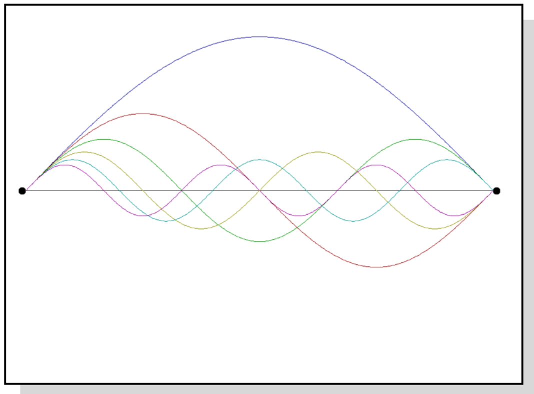
pipe modes

Note that the modes are all sinusoids.

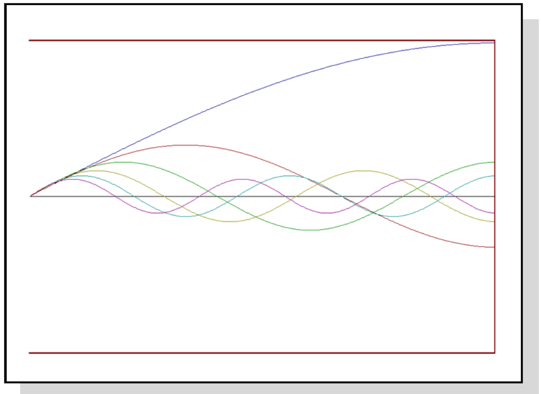
Note that the negatives of these also will occur

Sound Waves:

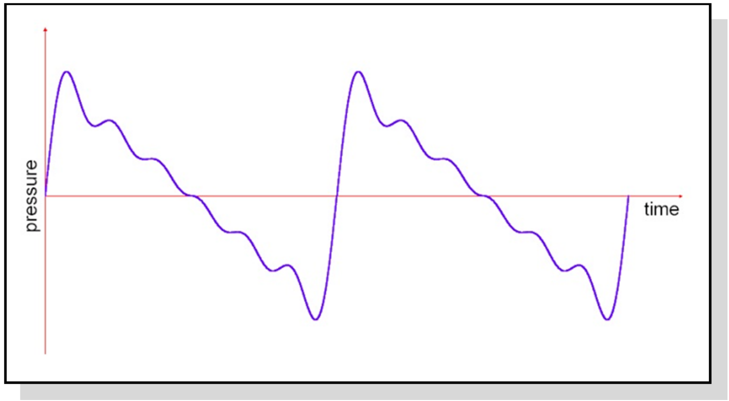
Emerge from the superposition of the modes.

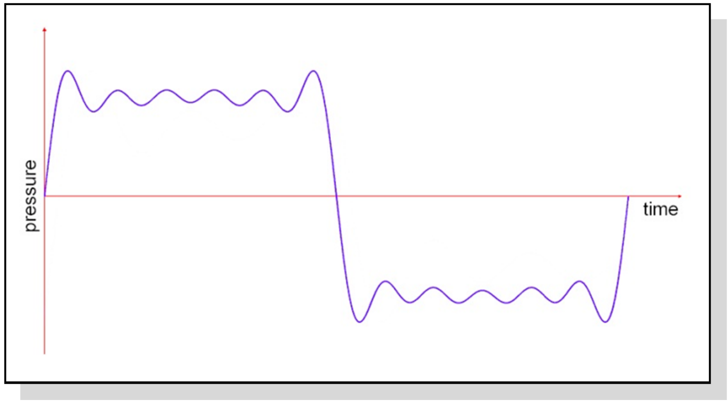


string sound ↓



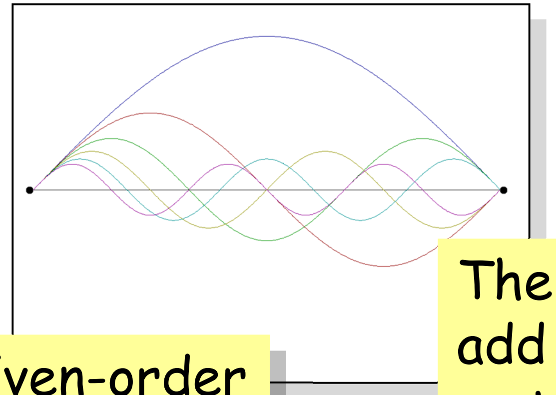
pipe sound ↓

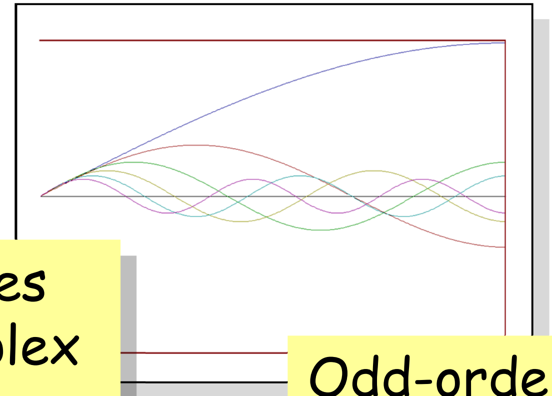




Sound Waves:

Emerge from the superposition of the modes.

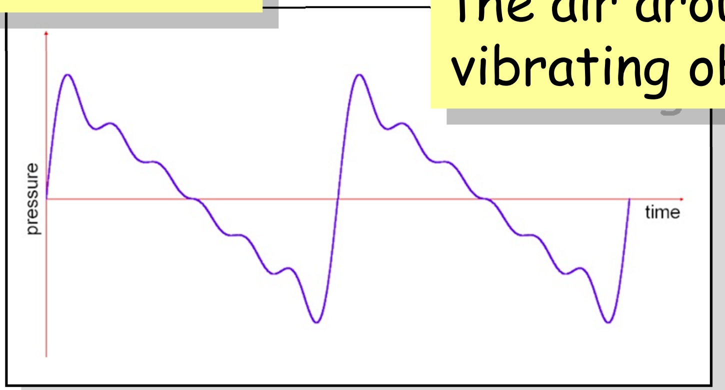


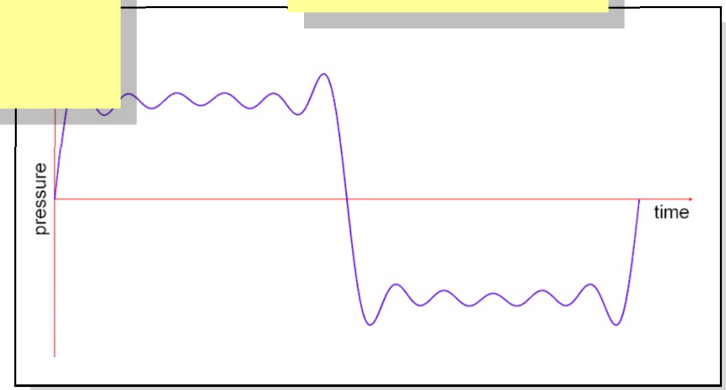


Even-order harmonics

The vibratory modes add up to one complex motion that pushes the air around the vibrating object

Odd-order harmonics





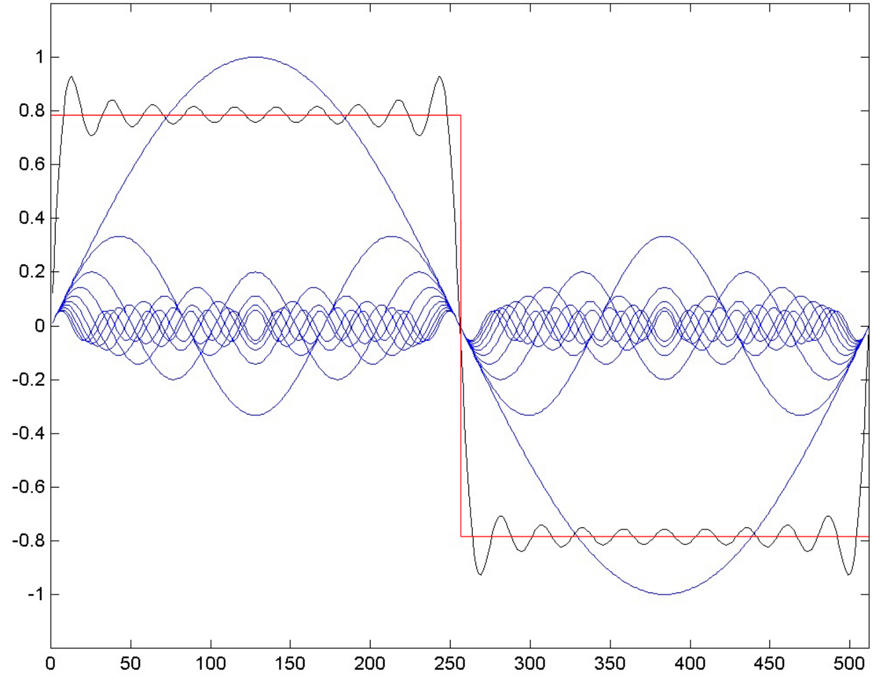
Fact: Any Real Signal has a Frequency-Domain Representation

Odd-order harmonics

$$\mathrm{sq}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \sin\left[\frac{2\pi}{\lambda}(2n+1)\,t\right]$$

The modes shown (blue) sum to the rippling square wave (black).

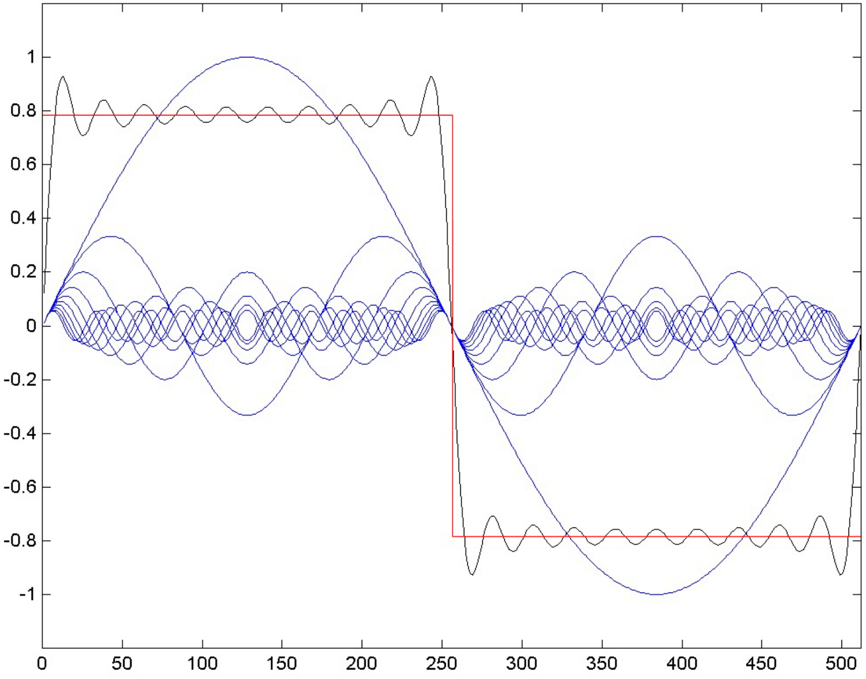
As the number of modes in the sum becomes large, it approaches a square wave (red).



Frequency-Domain Representation

Any periodic signal can be described by a sum of sinusoids.

$$\mathrm{sq}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \sin\left[\frac{2\pi}{\lambda}(2n+1)\,t\right]$$



The sinusoids are called "basis functions".

The multipliers are called "Fourier coefficients".

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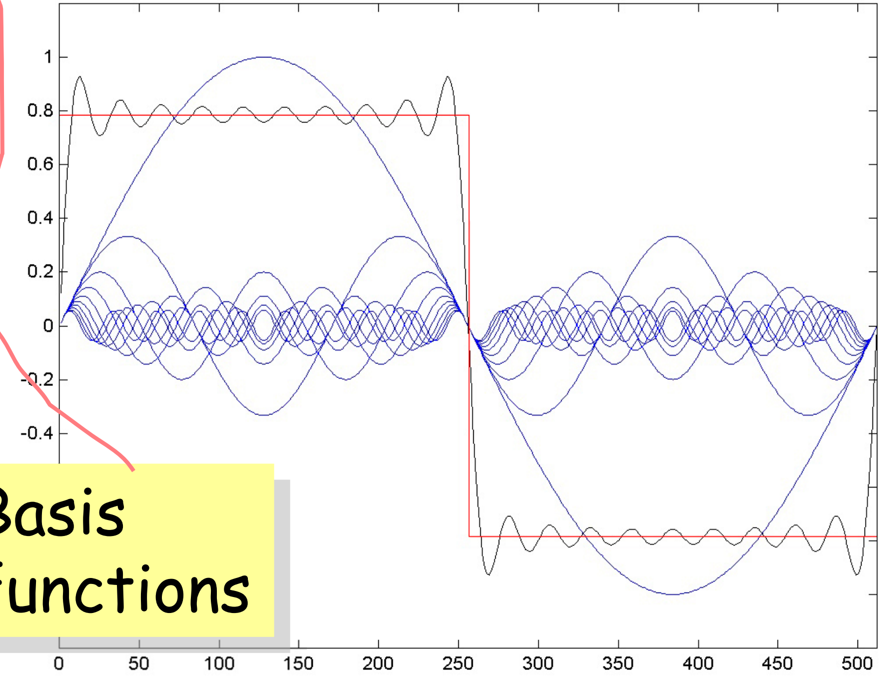
Basis functions



Frequency-Domain Representation

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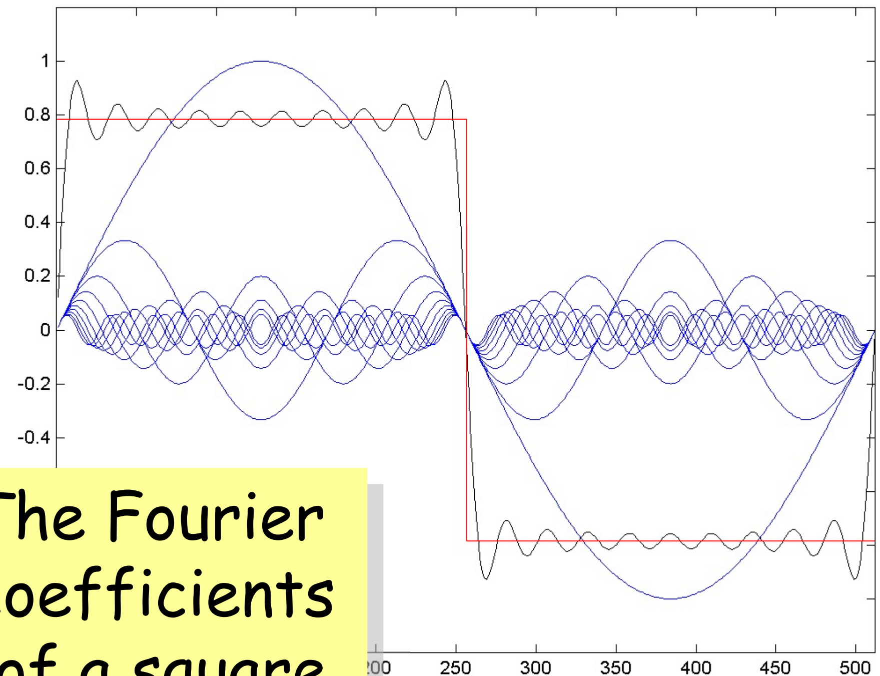
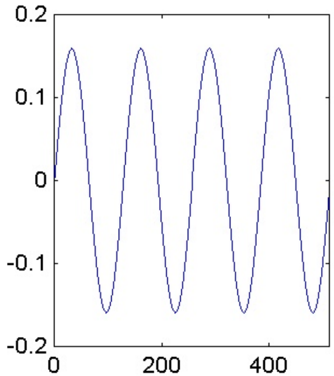


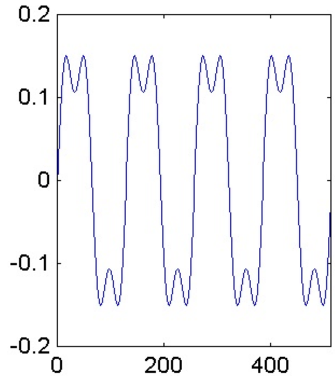
The Fourier coefficients (of a square wave).

Example: Partial Sums of a Square Wave

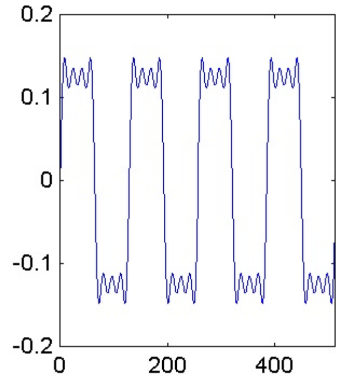
The limit of the given sequence of partial sums[1] is exactly a square wave



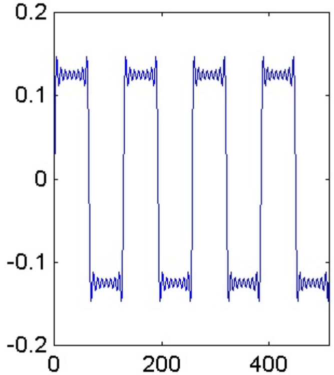
1 sine



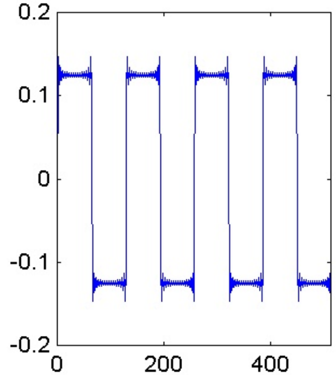
2 sines



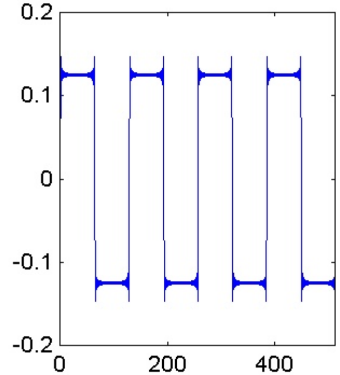
4 sines



8 sines



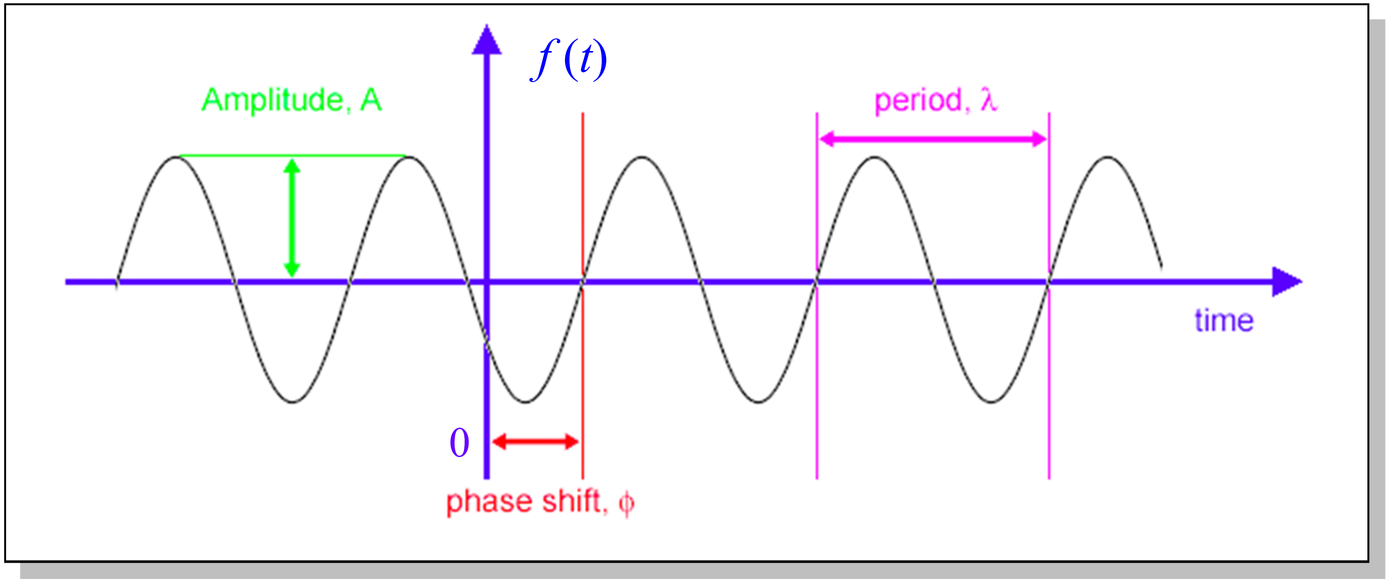
16 sines



32 sines

[1] the limit as n approaches infinity of the sum of n sines.

Anatomy of a Sinusoid



$$f(t) = A\sin\left(\frac{2\pi}{\lambda}t - \phi\right)$$

$1/\lambda$ is the frequency of the sinusoid (Hz).
$2\pi/\lambda$ is the angular frequency (radians/s).

The Inner Product: a Measure of Similarity

The similarity between functions f and g on the interval $(-\lambda/2, \lambda/2)$ can be defined by

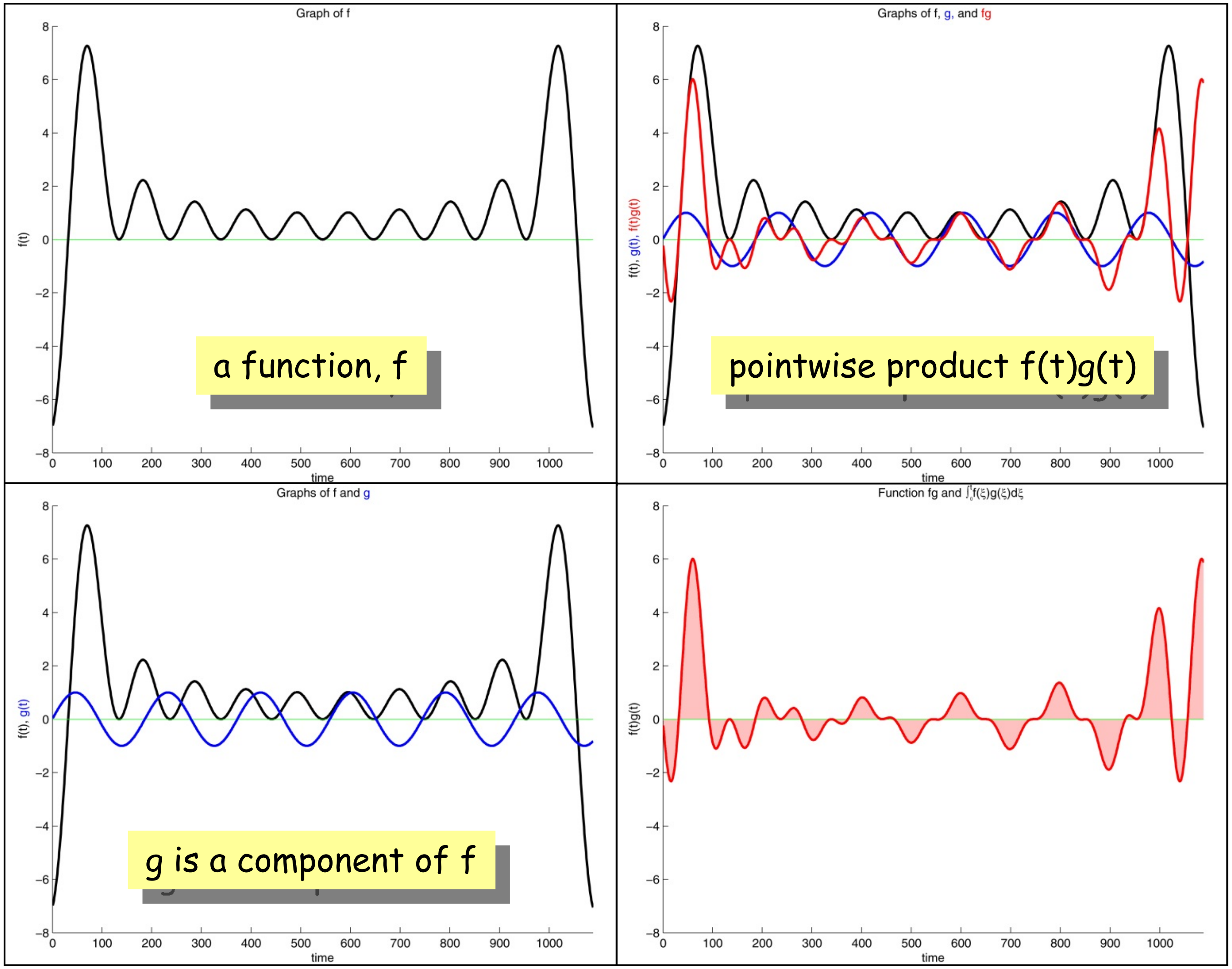
$$\langle f, g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\, g^*(t)\, dt$$

where $g^*(t)$ is the complex conjugate of $g(t)$.

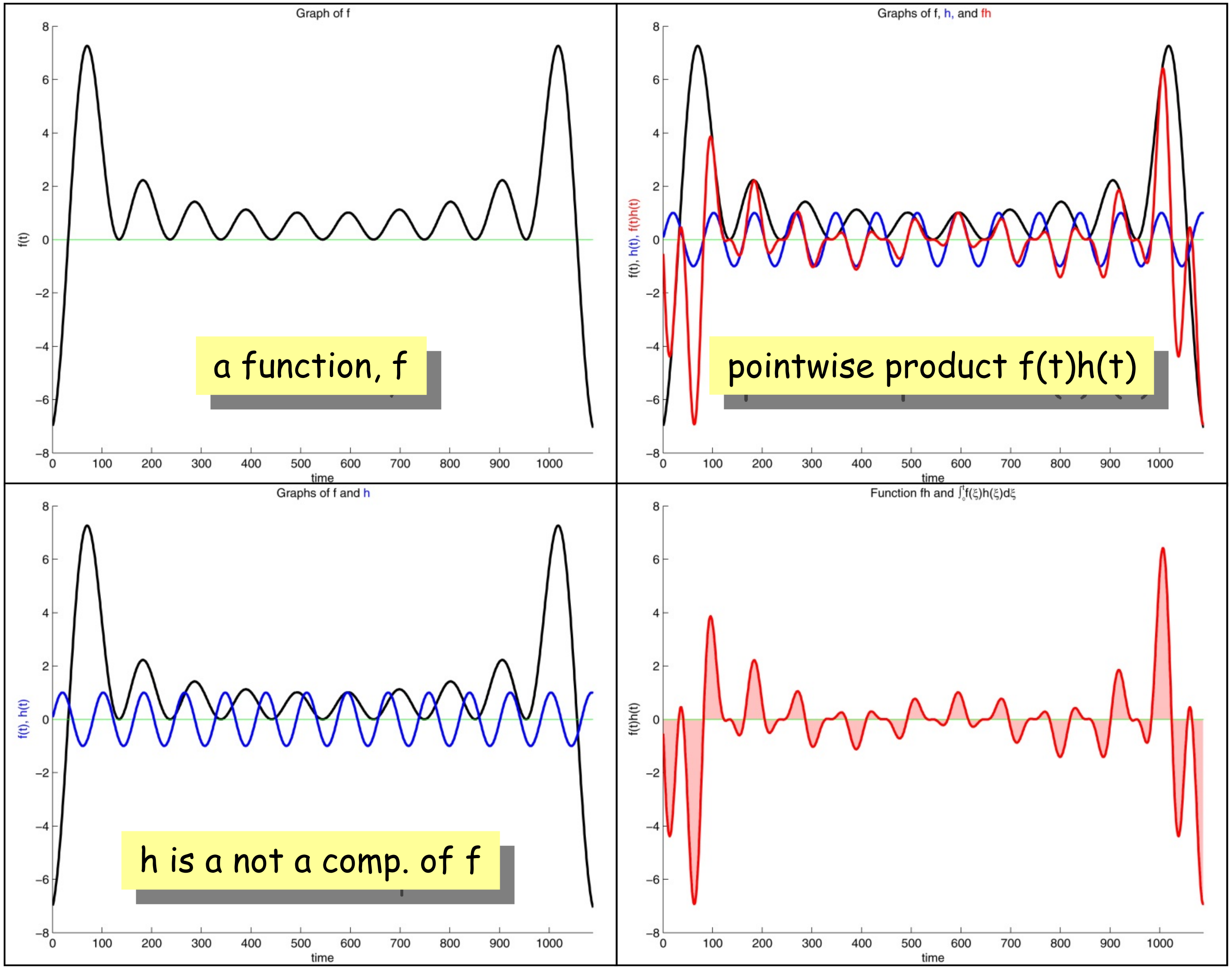
This number, called the *inner product of* f and g, can also be thought of as the amount of g in f or as the projection of f onto g.

If f and g have the same energy, then their inner product is maximal if $f = g$. On the other hand if $\langle f, g\rangle = 0$, then f and g have nothing in common.

Inner Products



Inner Products



Inner Product of a Periodic Function and a Sinusoid

$$\langle f,g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\sin\left(\tfrac{2\pi}{\lambda}t\right)dt$$

$$\langle f,g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\cos\left(\tfrac{2\pi}{\lambda}t\right)dt$$

$$\langle f,g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\left[\cos\left(\tfrac{2\pi}{\lambda}t\right)-i\sin\left(\tfrac{2\pi}{\lambda}t\right)\right]dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t)e^{-i\frac{2\pi}{\lambda}t}\,dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t)e^{-i\omega t}\,dt$$

3 different representations

$$e^{-i\frac{2\pi}{\lambda}t} = \cos\left(\tfrac{2\pi}{\lambda}t\right)-i\sin\left(\tfrac{2\pi}{\lambda}t\right)$$

$$\omega = \frac{2\pi}{\lambda}$$

Inner Product of a Periodic Function and a Sinusoid

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \sin\left(\tfrac{2\pi}{\lambda} t\right) dt$$

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \cos\left(\tfrac{2\pi}{\lambda} t\right) dt$$

real number results yield the amplitude of that sinusoid in the function.

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(\text{[illegible]} t)\Big] dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\frac{2\pi}{\lambda}t} dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\omega t} dt$$

Inner Product of a Periodic Function and a Sinusoid

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \sin\left(\tfrac{2\pi}{\lambda} t\right) dt$$

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \cos\left(\tfrac{2\pi}{\lambda} t\right) dt$$

$$\begin{aligned} \langle f, g \rangle &= \int_{-\lambda/2}^{\lambda/2} f(t) \left[\cos\left(\tfrac{2\pi}{\lambda} t\right) - i \sin\left(\tfrac{2\pi}{\lambda} t\right)\right] dt \\ &= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\frac{2\pi}{\lambda}t} dt \\ &= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\omega t} dt \end{aligned}$$

Complex number result yields the amplitude and phase of that sinusoid in the function.

The Fourier Series

is the decomposition of a λ-periodic signal into a sum of sinusoids.

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n}{\lambda} t\right) + B_n \sin\left(\frac{2\pi n}{\lambda} t\right)$$

The representation of a function by its Fourier Series is the sum of sinusoidal "basis functions" multiplied by coefficients.

$$\text{periodic}: \quad \exists \lambda \in \mathbb{R} \text{ such that } f(t \pm n\lambda) = f(t)$$

$$A_n = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t) \left[\cos\left(\frac{2\pi n}{\lambda} t - \varphi_n\right)\right] dt \text{ for } n \geq 0$$

$$B_n = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t) \left[\sin\left(\frac{2\pi n}{\lambda} t - \varphi_n\right)\right] dt \text{ for } n \geq 0$$

Fourier coefficients are generated by taking the inner product of the function with the basis.

The basis functions correspond to modes of vibration.

The Fourier Series

can also be written in terms of complex exponentials

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{+i\frac{2\pi n}{\lambda}t} = \sum_{n=-\infty}^{\infty} |C_n| e^{+i\left(\frac{2\pi n}{\lambda}t + \phi_n\right)}$$

$$= \sum_{n=-\infty}^{\infty} |C_n| \cos\left(\frac{2\pi n}{\lambda}t + \phi_n\right) + |C_n| \sin\left(\frac{2\pi n}{\lambda}t + \phi_n\right)$$

$$i = \sqrt{-1}$$

$$C_n = |C_n| e^{+i\phi_n}$$

$$C_n = |C_n| e^{+i\phi_n} = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\frac{2\pi n}{\lambda}t}\, dt$$

$$= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(t) \left[\cos\left(\frac{2\pi n}{\lambda}t + \phi_n\right) + |C_n| \sin\left(\frac{2\pi n}{\lambda}t + \phi_n\right)\right] dt$$

$$e^{\pm ix} = \cos x + i \sin x$$

$$f(t + n\lambda) = f(t)$$

for all integers n

The Fourier Series

Cont'd. on next page.

Relationship between the real and the complex Fourier Series

$$f(t) = \sum_{n=0}^{\infty} [A_n \cos \omega_n t + B_n \sin \omega_n t], \quad \text{where } \omega_n = \frac{2\pi n}{\lambda}$$

$$= \frac{2}{\lambda} \sum_{n=0}^{\infty} \left[\int_{-\lambda/2}^{\lambda/2} f(\eta) \cos \omega_n \eta \, d\eta \cos \omega_n t + \int_{-\lambda/2}^{\lambda/2} f(\eta) \sin \omega_n \eta \, d\eta \sin \omega_n t \right]$$

$$= \frac{2}{\lambda} \sum_{n=0}^{\infty} \int_{-\lambda/2}^{\lambda/2} f(\eta) [\cos \omega_n \eta \cos \omega_n t + f(\eta) \sin \omega_n \eta \sin \omega_n t] \, d\eta$$

$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} \int_{-\lambda/2}^{\lambda/2} f(\eta) \cos(\omega_n \eta - \omega_n t) \, d\eta$$

The sine-plus-cosine form results from the projection of f onto a cosine that is in phase with the current time.

Relationship between the real and the complex Fourier Series (cont'd.)

Cont'd. on next page.

Claim: $$0 = \sum_{n=-\infty}^{\infty} \sin(\omega_n \eta - \omega_n t).$$

Therefore: $$\int_{-\lambda/2}^{\lambda/2} \left[f(\eta) \sum_{n=-\infty}^{\infty} \sin(\omega_n \eta - \omega_n t) \right] d\eta = 0.$$

Thus: $$-i \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \left[\int_{-\lambda/2}^{\lambda/2} f(\eta) \sin(\omega_n \eta - \omega_n t)\, d\eta \right] = 0.$$

Then add zero to the equation at the end of the previous page:

$$f(t) = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \left[\int_{-\lambda/2}^{\lambda/2} f(\eta) \cos(\omega_n \eta - \omega_n t)\, d\eta \right] - i \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \left[\int_{-\lambda/2}^{\lambda/2} f(\eta) \sin(\omega_n \eta - \omega_n t)\, d\eta \right].$$

Relationship between the real and the complex Fourier Series (cont'd.)

$$f(t)=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\cos(\omega_n\eta-\omega_n t)\,d\eta\right]-i\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\sin(\omega_n\eta-\omega_n t)\,d\eta\right]$$

$$=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)\left[\cos\omega_n(\eta-t)-i\sin\omega_n(\eta-t)\right]d\eta$$

$$=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)e^{-i\omega_n(\eta-t)}d\eta$$

$$=\sum_{n=-\infty}^{\infty}\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(\eta)e^{-i\frac{2\pi n}{\lambda}\eta}d\eta\, e^{+i\frac{2\pi n}{\lambda}t}$$

$$=\sum_{n=-\infty}^{\infty}C_n e^{+i\frac{2\pi n}{\lambda}t}=\sum_{n=-\infty}^{\infty}|C_n|e^{i\phi_n}e^{+i\frac{2\pi n}{\lambda}t}=\sum_{n=-\infty}^{\infty}|C_n|e^{+i\left(\frac{2\pi n}{\lambda}t+\phi_n\right)}$$

Then some algebraic manipulations lead to the result.

Relationship between the real and the complex Fourier Series (cont'd.)

$$f(t)=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\boxed{\cos(\omega_n\eta-\omega_n t)}\,d\eta\right]-i\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\boxed{\sin(\omega_n\eta-\omega_n t)}\,d\eta\right]$$

$$=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)\boxed{[\cos\omega_n(\eta-t)-i\sin\omega_n(\eta-t)]\,d\eta}$$

$$=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)\boxed{e^{-i\omega_n(\eta-t)}}\,d\eta$$

$$=\sum_{n=-\infty}^{\infty}\boxed{\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(\eta)\,e^{-i\frac{2\pi n}{\lambda}\eta}\,d\eta}\,e^{+i\frac{2\pi n}{\lambda}t}$$

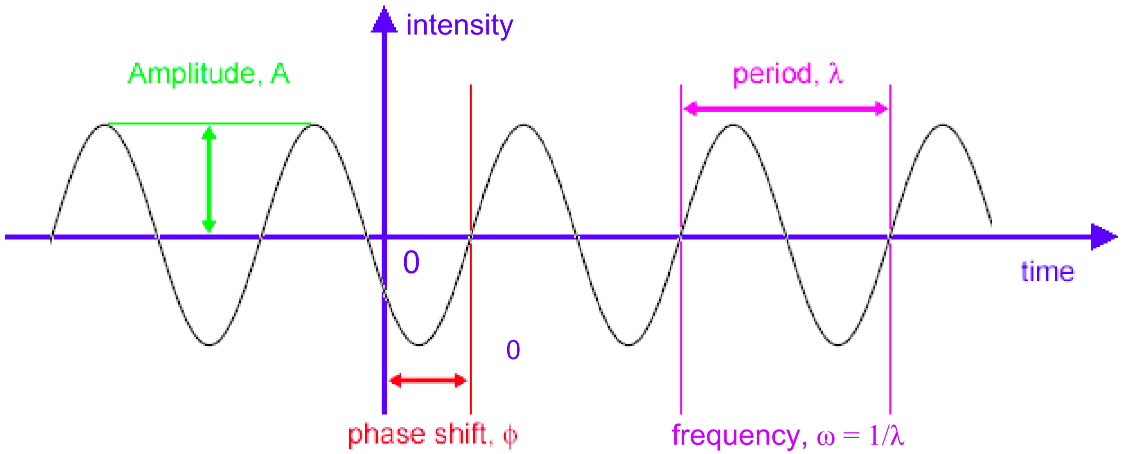
$$=\sum_{n=-\infty}^{\infty}\boxed{C_n}\,e^{+i\frac{2\pi n}{\lambda}t}=\sum_{n=-\infty}^{\infty}|C_n|\,e^{i\phi_n}e^{+i\frac{2\pi n}{\lambda}t}=\sum_{n=-\infty}^{\infty}|C_n|\,e^{+i\left(\frac{2\pi n}{\lambda}t+\phi_n\right)}$$

Then some algebraic manipulations lead to the result.

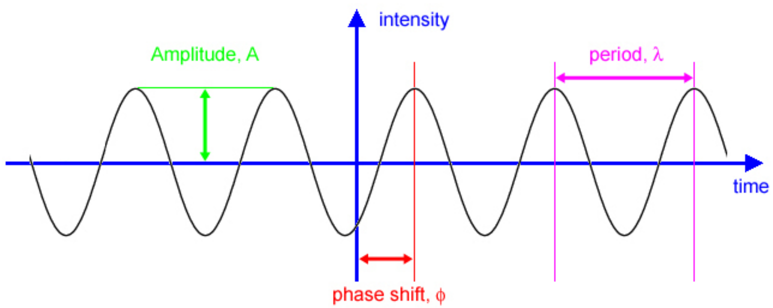
Why are Fourier Coefficients Complex Numbers?

$$f(t)=\sum_{n=-\infty}^{\infty} C_n e^{+i\frac{2\pi n}{\lambda}t} \quad \text{where} \quad C_n = |C_n|\, e^{+i\phi_n}.$$

C_n represents the amplitude, $A=|C_n|$, and relative phase, ϕ, of that part of the original signal, $f(t)$, that is a sinusoid of frequency $\omega_n = n/\lambda$.



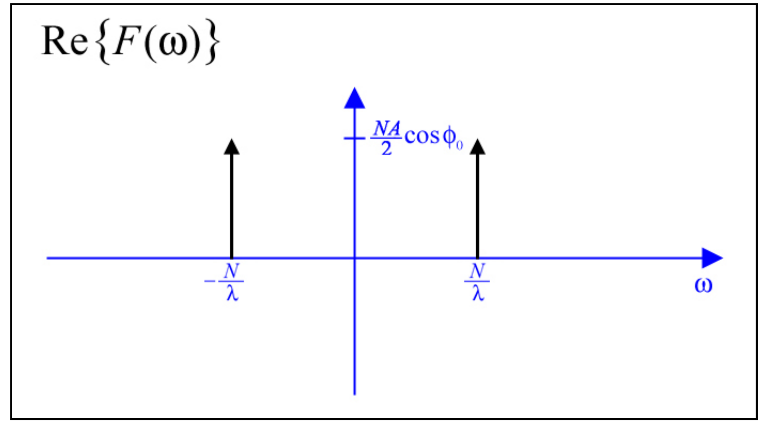
What about real + imaginary?



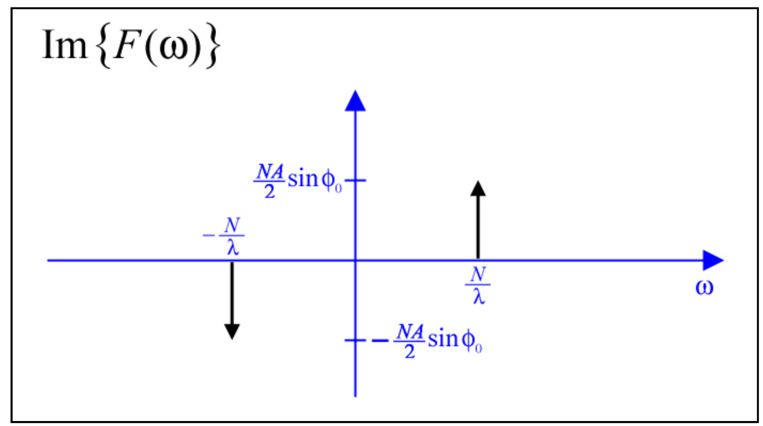
$\Im$

$$f(n) = A\cos[(2\pi n/\lambda) - \phi_0], \text{ for } n = 0, \ldots, N\text{-}1.$$

The FS of a cosine is a pair of impulses with complex amplitudes

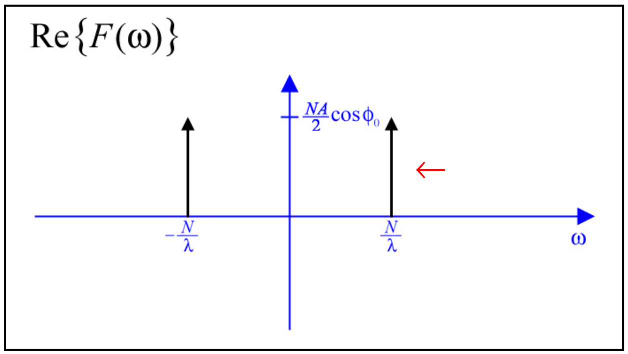


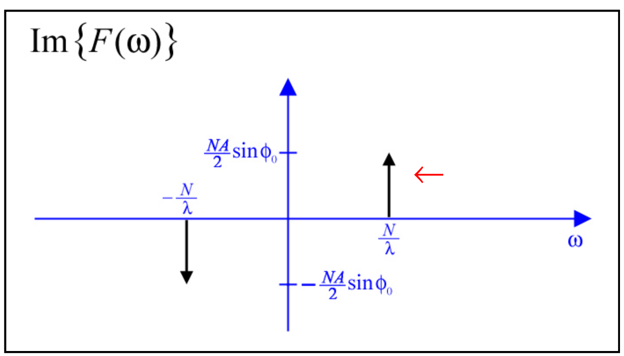
$$F(\omega) = \left(\tfrac{NA}{2}\cos\varphi\right)[\delta(\omega + N/\lambda) + \delta(\omega - N/\lambda)]$$
$$+\, i\left(\tfrac{NA}{2}\sin\varphi\right)[-\delta(\omega + N/\lambda) + \delta(\omega - N/\lambda)]$$



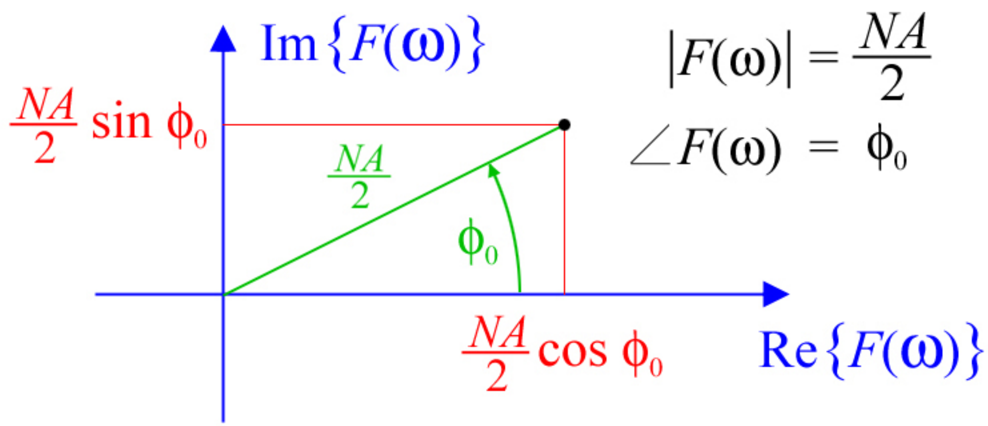
Real + Imaginary to Magnitude & Phase

The real and imaginary parts at the positive frequency, N/λ ...





Complex Value at $\omega = +N/\lambda$

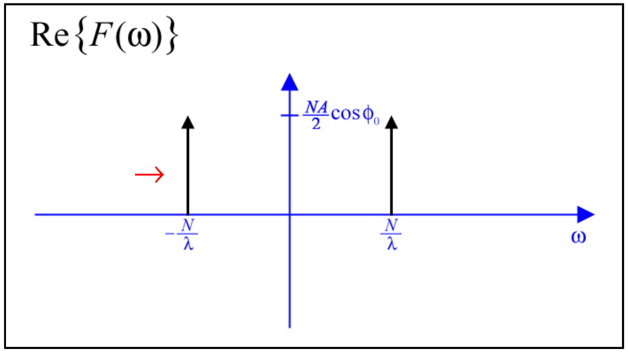


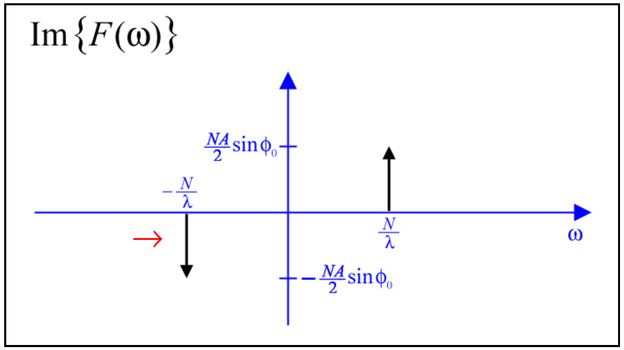
$$F(\omega = +N/\lambda) = \frac{NA}{2}\cos\phi_0 + i\,\frac{NA}{2}\sin\phi_0$$

... form a magnitude, NA/2, and a phase, ϕ_0.

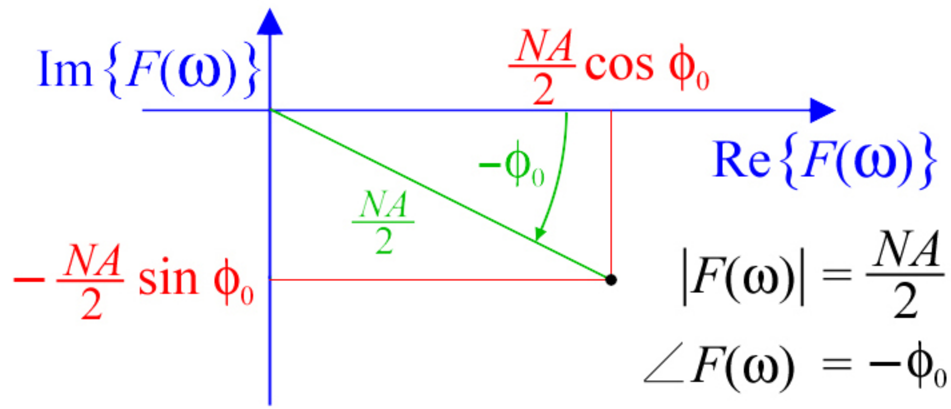
Real + Imaginary to Magnitude & Phase

The real and imaginary parts at the negative frequency, -N/λ ...





Complex Value at $\omega = -N/\lambda$



$$|F(\omega)| = \frac{NA}{2}$$

$$\angle F(\omega) = -\phi_0$$

$$F(\omega = -N/\lambda) = \frac{NA}{2}\cos\phi_0 - i\,\frac{NA}{2}\sin\phi_0$$

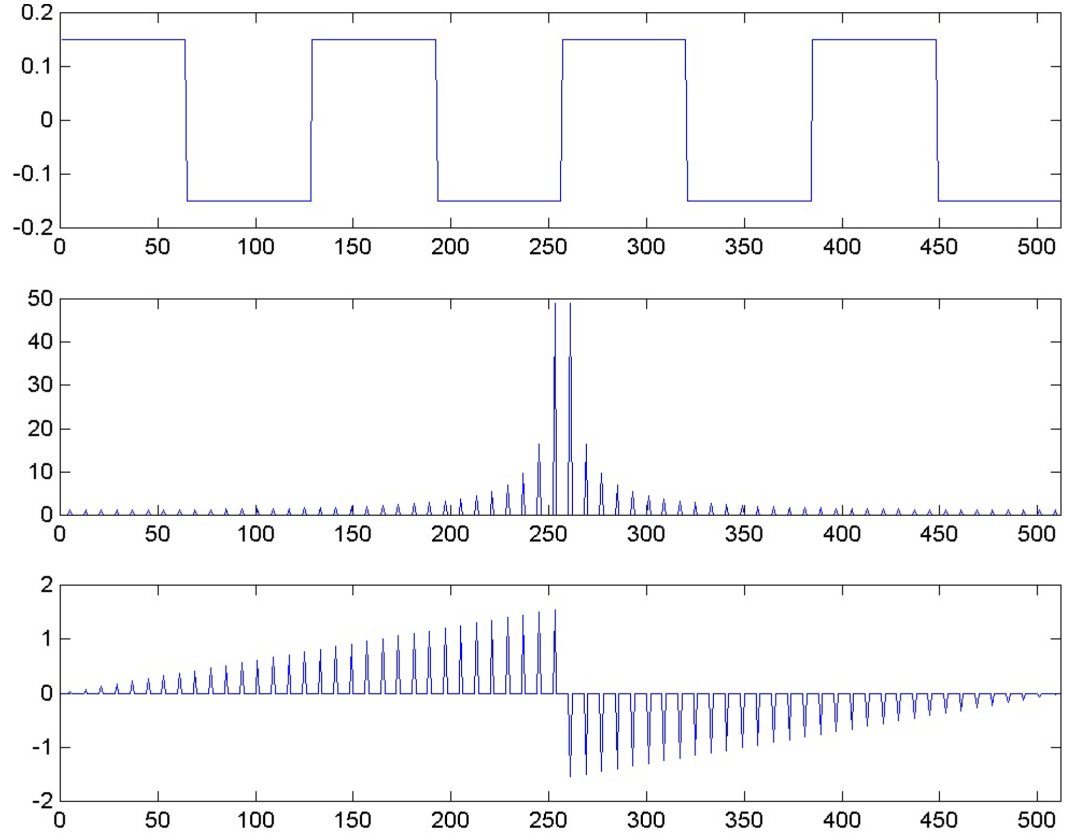
... form a magnitude, NA/2, and a phase, $-\phi_0$.

Fourier Series of a Square Wave

Time-domain signal

Fourier magnitude

Fourier phase



The Fourier Transform

is the decomposition of a *nonperiodic* signal into a continuous sum* of sinusoids.

$$F(\omega) = |F(\omega)| e^{i\Phi(\omega)} = \int_{-\infty}^{\infty} f(t)\, e^{i2\pi\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)[\cos(2\pi\omega t) + i\sin(2\pi\omega t)]\, dt$$

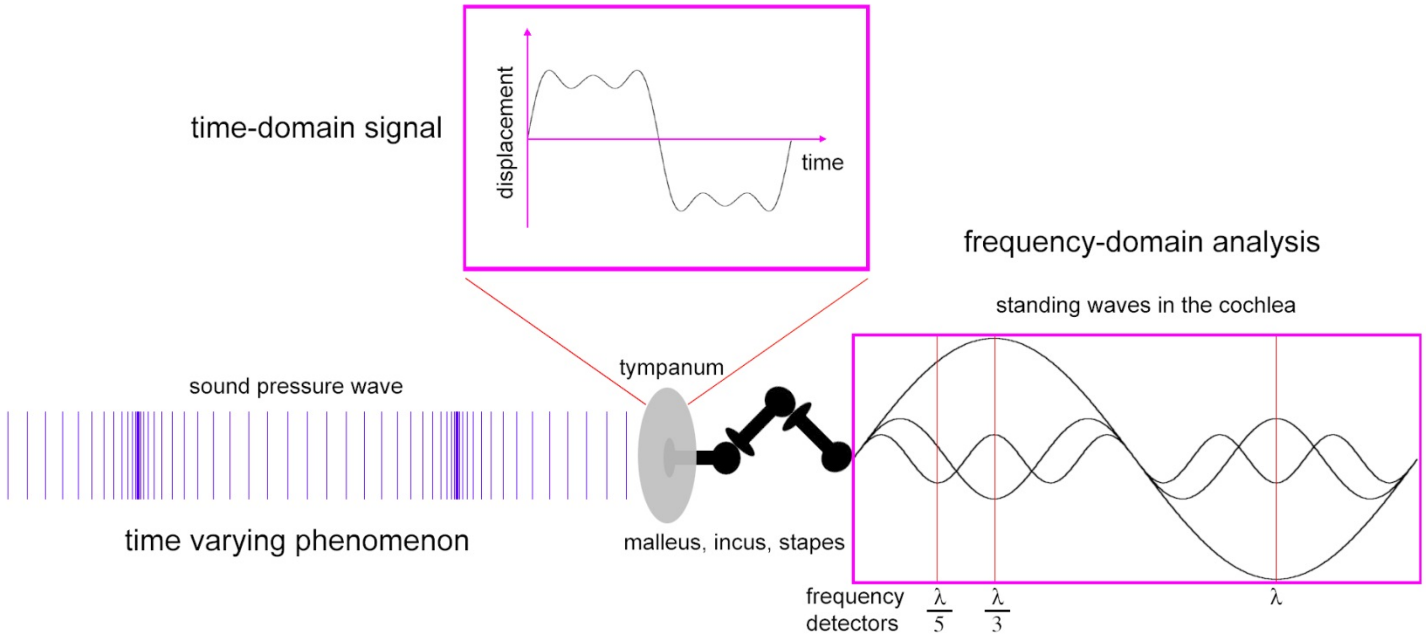
$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i2\pi\omega t}\, d\omega = \int_{-\infty}^{\infty} |F(\omega)|\, e^{-i(2\pi\omega t + \Phi(\omega))}\, d\omega$$

$$= \int_{-\infty}^{\infty} F(\omega)[\cos(2\pi\omega t) - i\sin(2\pi\omega t)]\, d\omega$$

$$= \int_{-\infty}^{\infty} |F(\omega)|[\cos(2\pi\omega t + \Phi(\omega)) - i\sin(2\pi\omega t + \Phi(\omega))]\, d\omega$$

* *i.e.*, an integral.

Mammals Use the FT in Hearing



The Discrete Fourier Transform

A discrete signal, $\{h_k \mid k = 0, 1, 2, \ldots, N-1\}$, of finite length N can be represented as a weighted sum of N sinusoids, $\{e^{-i2\pi kn/N} \mid n = 0, 1, 2, \ldots, N-1\}$ through

$$h_k = \sum_{n=0}^{N-1} H_n e^{-i2\pi kn/N}$$

where the set, $\{H_n \mid n = 0, 1, 2, \ldots, N-1\}$, are the Fourier coefficients defined as the projection of the original signal onto sinusoid, n, given by:

$$H_n = \frac{1}{N} \sum_{k=0}^{N-1} h_k e^{+i2\pi kn/N}$$

The Two-Dimensional Fourier Transform

Primary Uses of the FT in Image Processing:

- Explains why down-sampling can add distortion to an image and shows how to avoid it.
- Useful for certain types of noise reduction, deblurring, and other types of image restoration.
- For feature detection and enhancement, especially edge detection.

The Fourier Transform: Discussion

The expressions

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\omega t}\, dt = \langle f(t),\ e^{+i2\pi\omega t} \rangle$$

continuous signals defined over all real numbers

and

$$H_n = \frac{1}{N} \sum_{n=0}^{N-1} h_k e^{-i2\pi kn/N} = \langle h_k,\ e^{+i2\pi kn/N} \rangle$$

discrete signals with N terms or samples.

for the Fourier coefficients are "inner products" which can be thought of as measures of the similarity between the functions $f(t)$ and $e^{+i2\pi\omega t}$ for $t \in (-\infty, \infty)$ or between the sequences $\{h_k\}_{k=0}^{N-1}$ and $\{e^{+i2\pi kn/N}\}_{k=0}^{N-1}$.

In the context of inner products, the complex exponentials

$$\left\{ e^{-i2\pi\omega t} \,\middle|\, \omega\in\Re \text{ and } \omega\in(-\infty,\infty)\right\} \text{ and } \left\{ e^{-i2\pi kn/N} \,\middle|\, \ldots,-2,-1,0,1,2,\ldots \right\}$$

are called "orthogonal sets" since they have the property:

$$\left\langle e^{-i2\pi\omega_1 t},\ e^{-i2\pi\omega_2 t}\right\rangle = \int_{-\infty}^{\infty} e^{-i2\pi\omega_1 t}\cdot e^{+i2\pi\omega_2 t}\,dt = \begin{cases} \infty, & \text{if } \omega_1=\omega_2 \\ 0, & \text{if } \omega_1\neq\omega_2 \end{cases}$$

$$\left\langle e^{-i2\pi jn/N},\ e^{-i2\pi kn/N}\right\rangle = \sum_{n=0}^{N-1} e^{-i2\pi jn/N}\cdot e^{+i2\pi kn/N} = \begin{cases} c, & \text{if } j=k \\ 0, & \text{if } j\neq k \end{cases},$$

The function sets are called "orthogonal basis sets"

They are called "basis sets" since for any function[1], $f(t)$, of a real variable there exists a complex-valued function F(w), and for any sequence[1], h_k, there exist complex numbers, H_n, such that

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i2\pi\omega t}\, d\omega \quad \text{and} \quad h_k = \sum_{n=0}^{N-1} H_n e^{-i2\pi kn/N}.$$

[1] with finite energy.

Consider the 2-dimensional functions

$$\left\{ e^{-i2\pi(ux+vy)} \,\middle|\, u,v,x,y \in \Re \right\} \text{ and } \left\{ e^{-i2\pi(\frac{jm}{M}+\frac{kn}{N})} \,\middle|\, j,m \in 0,\ldots,M-1,\;\; k,n \in 0,\ldots,N-1 \right\}$$

These are, likewise, orthogonal:

$$\left\langle e^{-i2\pi(u_1x+v_1y)},\, e^{-i2\pi(u_2x+v_2y)} \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi(u_1x+v_1y)} \cdot e^{+i2\pi(u_2x+v_2y)}\,dxdy$$

$$= \begin{cases} \infty, & \text{if } u_1=u_2 \text{ and } v_1=v_2 \\ 0, & \text{otherwise} \end{cases},$$

$$\left\langle e^{-i2\pi\left(\frac{j_1m}{M}+\frac{k_1n}{N}\right)},\, e^{-i2\pi\left(\frac{j_2m}{M}+\frac{k_2n}{N}\right)} \right\rangle = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-i2\pi\left(\frac{j_1m}{M}+\frac{k_1n}{N}\right)} \cdot e^{+i2\pi\left(\frac{j_2m}{M}+\frac{k_2n}{N}\right)}$$

$$= \begin{cases} c, & \text{if } j_1=j_2 \text{ and } k_1=k_2 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore

$$\left\{ e^{-i2\pi(ux+vy)} \,\middle|\, u, v, x, y \in \mathbb{R} \right\} \quad \text{and} \quad \left\{ e^{-i2\pi\left(\frac{jm}{M}+\frac{kn}{N}\right)} \,\middle|\, j, k, m, n, M \in \mathbb{Z} \right\}$$

are orthogonal basis sets. This suggests that function $f(x,y)$ defined on the real plane, and sequence $\{\{ h_{mn} \}\}$ for integers m and n have analogous Fourier representations,

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{+i2\pi(ux+vy)}\, du dv \quad \text{and} \quad h_{mn} = \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} H_{jk} e^{+i2\pi\left(\frac{jm}{M}+\frac{kn}{N}\right)}.$$

where the Fourier coefficients are given by

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx dy \quad \text{and} \quad H_{jk} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h_{mn} e^{-i2\pi\left(\frac{jm}{M}+\frac{kn}{N}\right)}.$$

(True for finite energy functions $f(x,y)$ and $\{\{ h_{mn} \}\}$.)

Continuous Fourier Transform

$$\mathbf{I}(r,c) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathfrak{I}(v,u)\, e^{+i2\pi(vr+uc)}\, du\,dv$$

$$\mathfrak{I}(v,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{I}(r,c)\, e^{-i2\pi(vr+uc)}\, dc\,dr$$

The continuous Fourier transform assumes a continuous image exists in a finite region of an infinite plane.

Discrete Fourier Transform

The discrete Fourier transform assumes a digital image exists on a closed surface, a torus.

$$\mathbf{I}(r,c) = \sum_{v=0}^{R-1}\sum_{u=0}^{C-1} \mathfrak{I}(v,u)\, e^{+i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

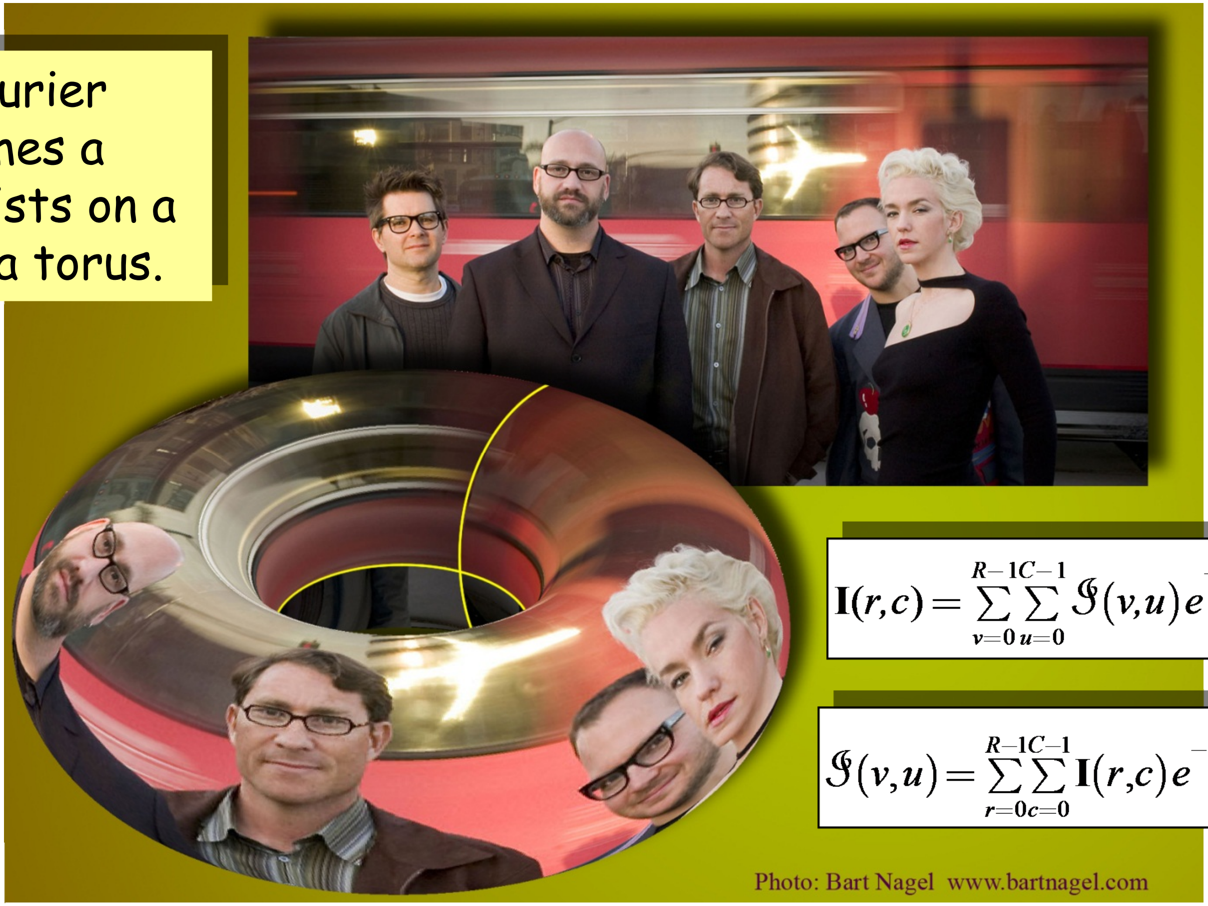
$$\mathfrak{I}(v,u) = \sum_{r=0}^{R-1}\sum_{c=0}^{C-1} \mathbf{I}(r,c)\, e^{-i2\pi\left(\frac{rv}{R}+\frac{cu}{C}\right)}$$

The BoingBoing Bloggers

Photo: Bart Nagel www.bartnagel.com

Discrete Fourier Transform

The discrete Fourier transform assumes a digital image exists on a closed surface, a torus.



$$\mathbf{I}(r,c) = \sum_{v=0}^{R-1}\sum_{u=0}^{C-1} \mathfrak{I}(v,u)\, e^{+i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

$$\mathfrak{I}(v,u) = \sum_{r=0}^{R-1}\sum_{c=0}^{C-1} \mathbf{I}(r,c)\, e^{-i2\pi\left(\frac{rv}{R}+\frac{cu}{C}\right)}$$

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The 2D Fourier Transform of a Digital Image

Let $\mathbf{I}(r,c)$ be a single-band (intensity) digital image with R rows and C columns. Then, $\mathbf{I}(r,c)$ has Fourier representation

$$\mathbf{I}(r,c) = \sum_{u=0}^{R-1} \sum_{v=0}^{C-1} \mathfrak{I}(v,u)\, e^{+i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)},$$

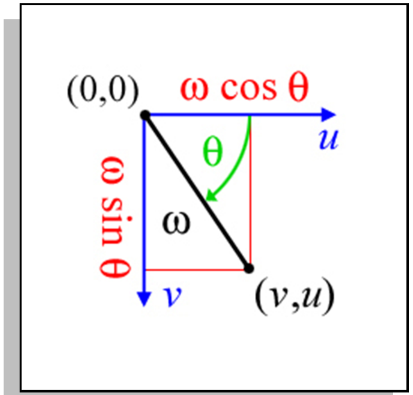
these complex exponentials are 2D sinusoids.

where

$$\mathfrak{I}(v,u) = \frac{1}{RC} \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} \mathbf{I}(r,c)\; e^{-i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

are the R x C Fourier coefficients.

What are 2D sinusoids?



To simplify the situation assume $R = C = N$. Then

$$e^{\pm i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)} = e^{\pm i\frac{2\pi}{N}(vr+uc)} = e^{\pm i\frac{2\pi\omega}{N}(r\sin\theta + c\cos\theta)},$$

where

$$v = \omega\sin\theta, \quad u = \omega\cos\theta, \quad \omega = \sqrt{v^2+u^2}, \quad \text{and} \quad \theta = \tan^{-1}\left(\tfrac{v}{u}\right).$$

Write

$$\lambda = \frac{N}{\omega},$$

Then by Euler's relation,

$$e^{\pm i2\pi\frac{1}{\lambda}(r\sin\theta + c\cos\theta)} = \cos\left[\tfrac{2\pi}{\lambda}(r\sin\theta + c\cos\theta)\right] \pm i\sin\left[\tfrac{2\pi}{\lambda}(r\sin\theta + c\cos\theta)\right].$$

Note: since images are indexed by row & col with r down and c to the right, θ is positive in the clockwise direction.

Cont'd. on next page.

What are 2D sinusoids? (cont'd.)

Both the real part of this,

$$\mathrm{Re}\left\{ e^{\pm i 2\pi \frac{1}{\lambda}(r\sin\theta + c\cos\theta)} \right\} = +\cos\left[\tfrac{2\pi}{\lambda}(r\sin\theta + c\cos\theta)\right]$$

and the imaginary part,

$$\mathrm{Im}\left\{ e^{\pm i 2\pi \frac{1}{\lambda}(r\sin\theta + c\cos\theta)} \right\} = \pm\sin\left[\tfrac{2\pi}{\lambda}(r\sin\theta + c\cos\theta)\right]$$

are sinusoidal "gratings" of unit amplitude, period λ and direction θ.

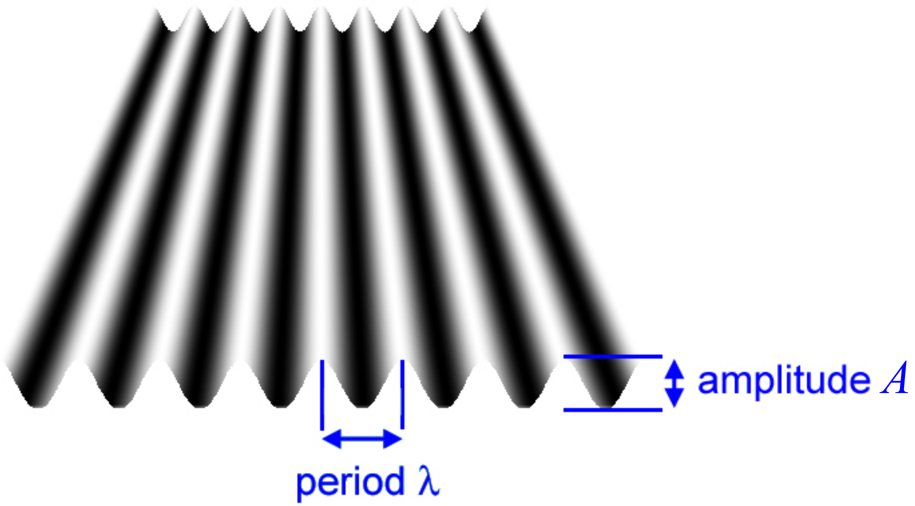
Then $\dfrac{2\pi\omega}{N}$ is the radian frequency, and $\dfrac{\omega}{N}$ the frequency, of the wavefront

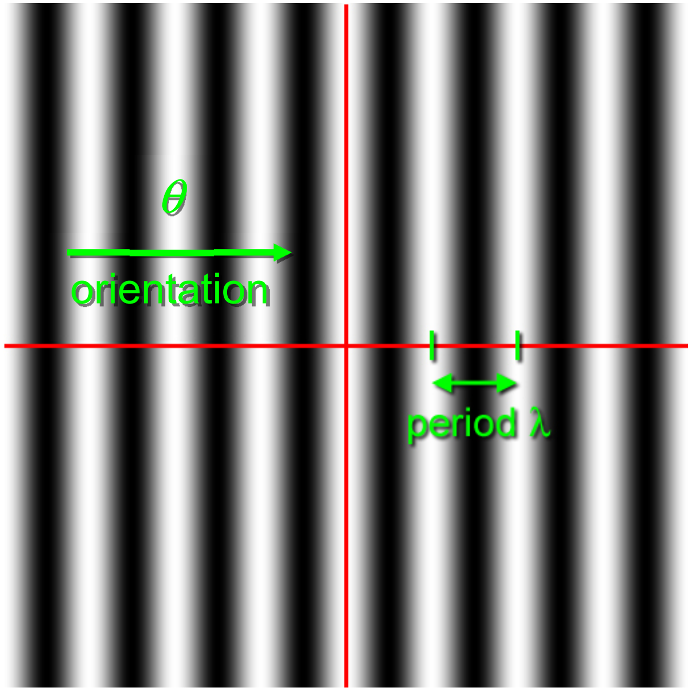
and $\lambda = \dfrac{N}{\omega}$ is the wavelength in pixels in the wavefront direction.

2D Sinusoids:

$$\mathbf{I}(r,c)=\frac{A}{2}\left\{\cos\left[\frac{2\pi}{\lambda}(r\cdot\sin\theta+c\cdot\cos\theta)+\varphi\right]+1\right\}$$

... are plane waves with grayscale amplitudes, periods in terms of lengths, ...

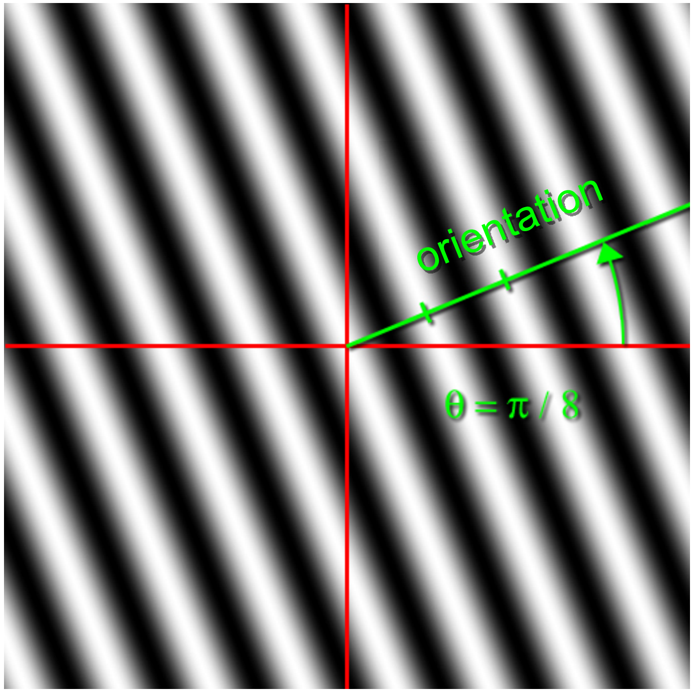


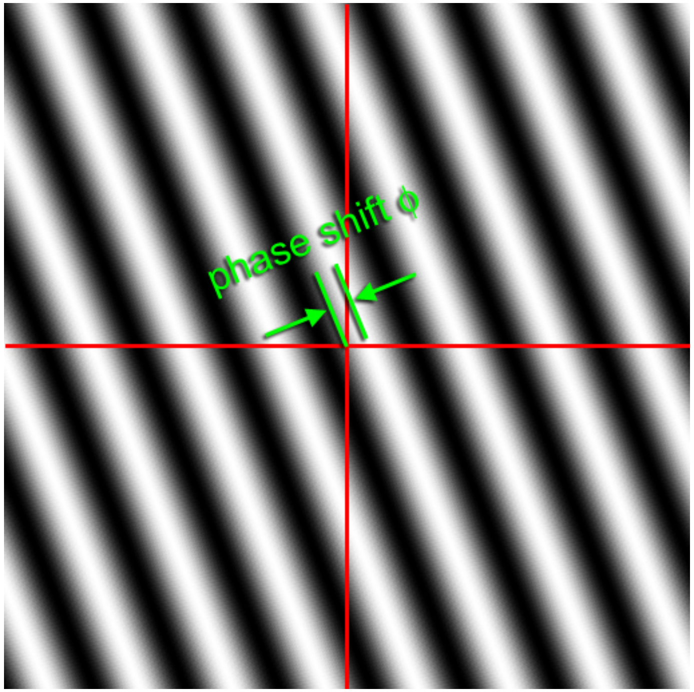


ϕ = phase shift

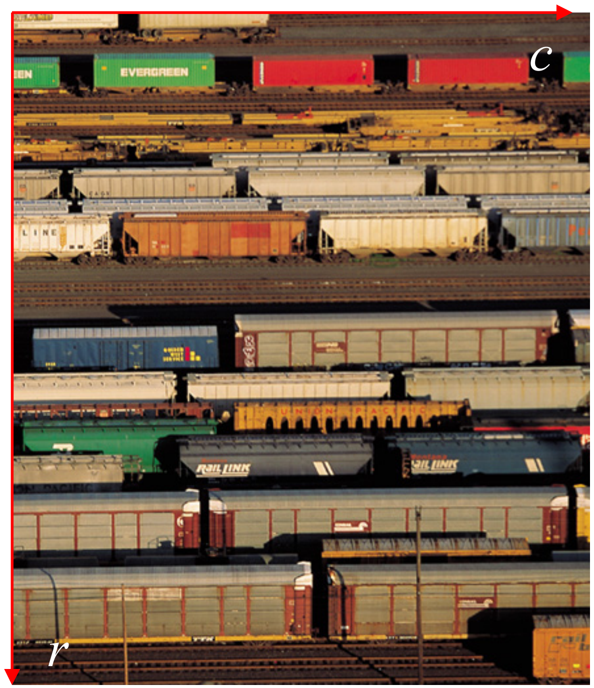
2D Sinusoids:

... specific orientations, and phase shifts.

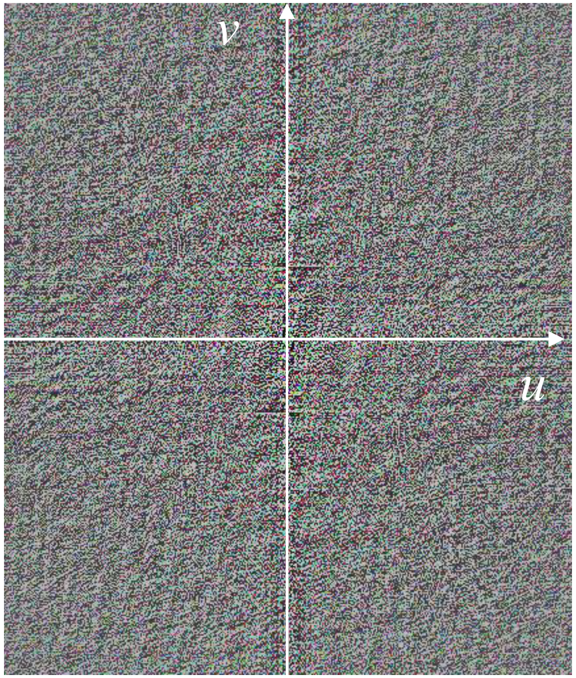




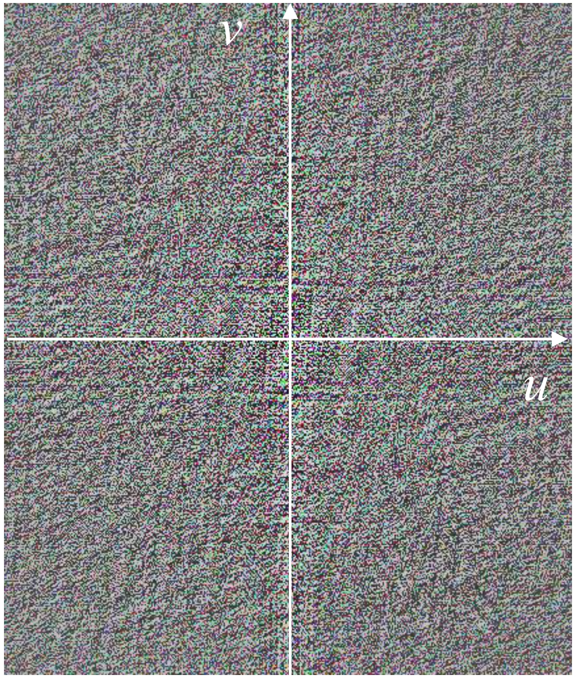
The Fourier Transform of an Image



$\mathbf{I}$



$\mathrm{Re}[\mathcal{F}\{\mathbf{I}\}]$

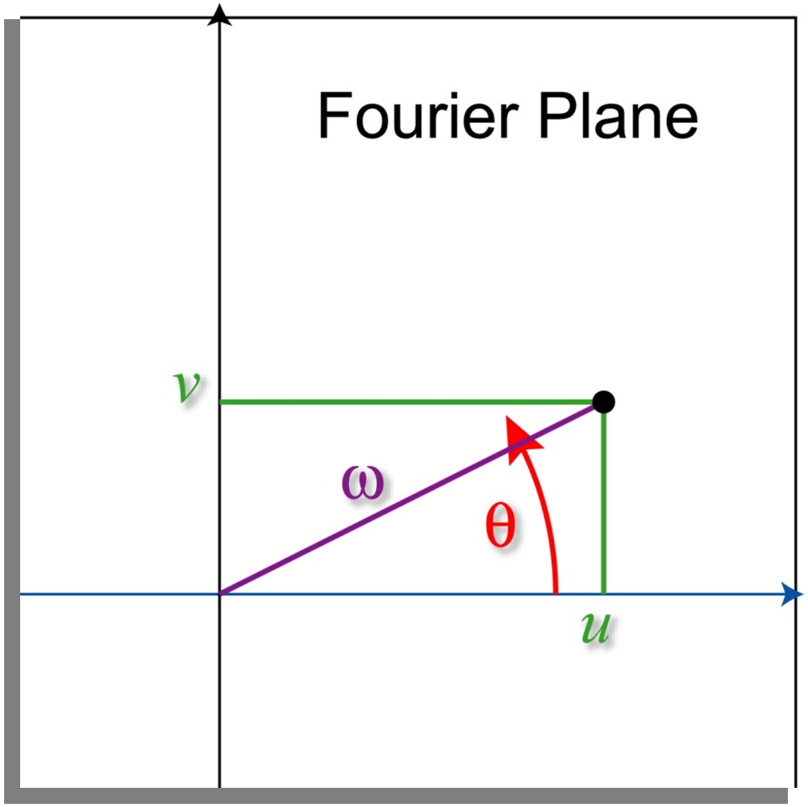


$\mathrm{Im}[\mathcal{F}\{\mathbf{I}\}]$

Points on the Fourier Plane

If R=C=N the point at column freq. u and row freq. v represents a sinusoid with freq. ω and orientation θ.

If R ≠ C then $\omega = 1/\lambda$ where λ is the length of vector (C/u, R/v) and the wavefront orientation is $\theta = \tan^{-1}[(v/R)/(u/C)]$.



Points on the Fourier Plane (of a Digital Image)

More about this later (pp. 66-86).

In the Fourier transform of an $R \times C$ digital image, positions u and v indicate the number of repetitions of the sinusoid in those directions. Therefore the wavelengths along the column and row axes are

$$\lambda_u = \frac{C}{u} \quad \text{and} \quad \lambda_v = \frac{R}{v} \quad \text{pixels,}$$

and the wavelength in the wavefront direction is

$$\lambda_{\mathrm{wf}} = \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2}.$$

The frequency is the fraction of the sinusoid traversed over one pixel,

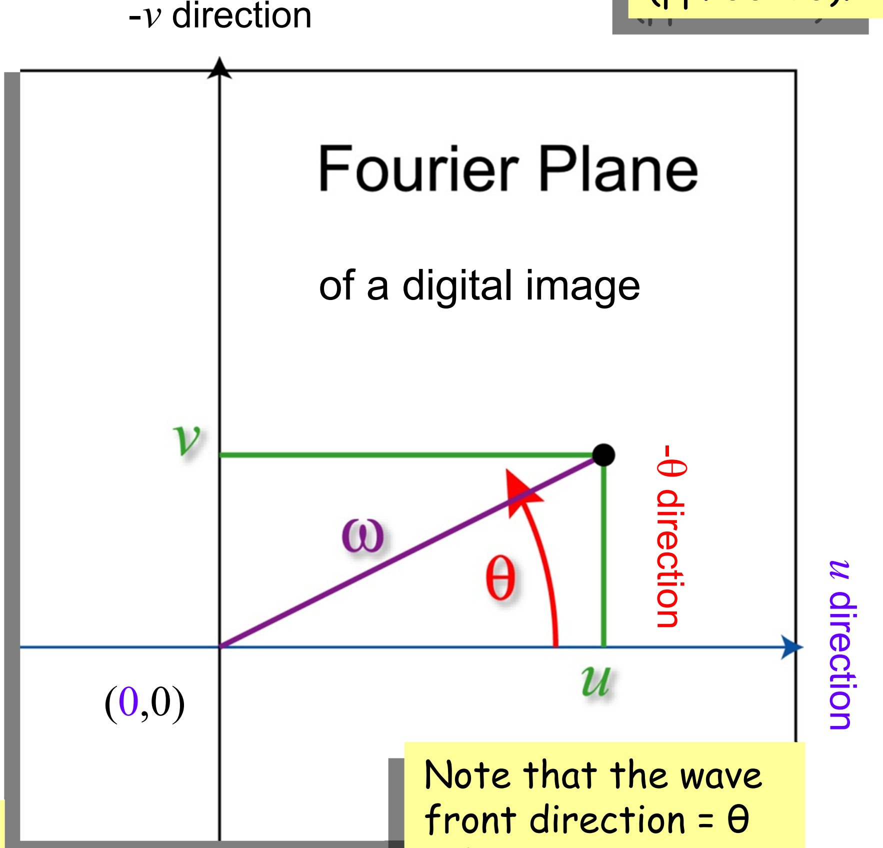
$$\omega_u = \frac{u}{C}, \quad \omega_v = \frac{v}{R}, \text{ and}$$

$$\omega_{\mathrm{wf}} = 1 \Big/ \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2} \quad \text{cycles.}$$

The wavefront direction is given by

$$\theta_{\mathrm{wf}} = \tan^{-1}\left(\frac{\omega_v}{\omega_u}\right) = \tan^{-1}\left(\frac{v\,C}{u\,R}\right).$$

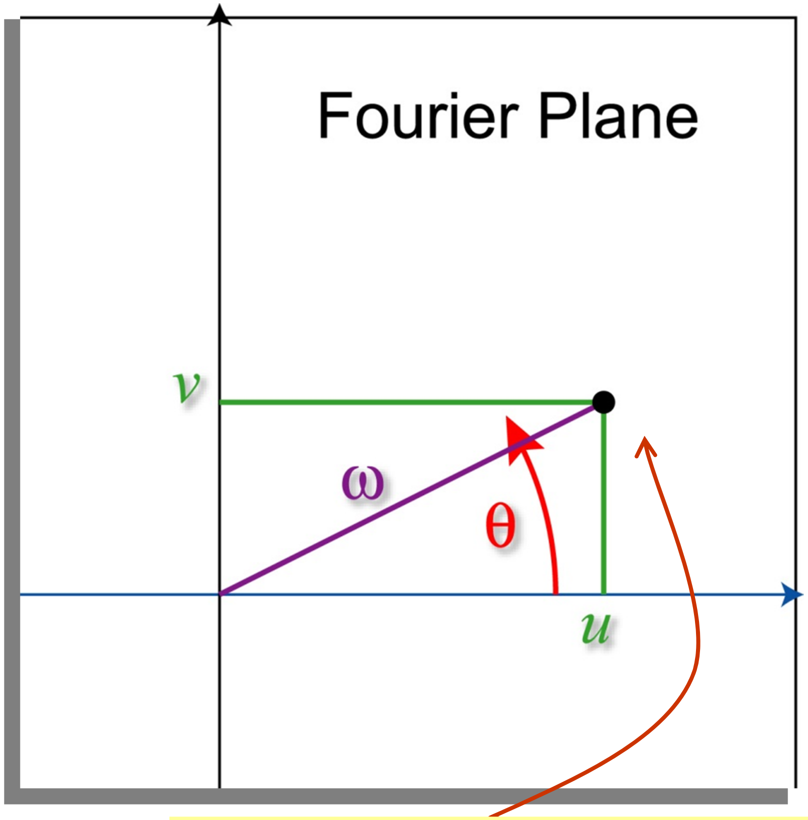
$\dfrac{\text{row freq.}}{\text{column freq.}}$

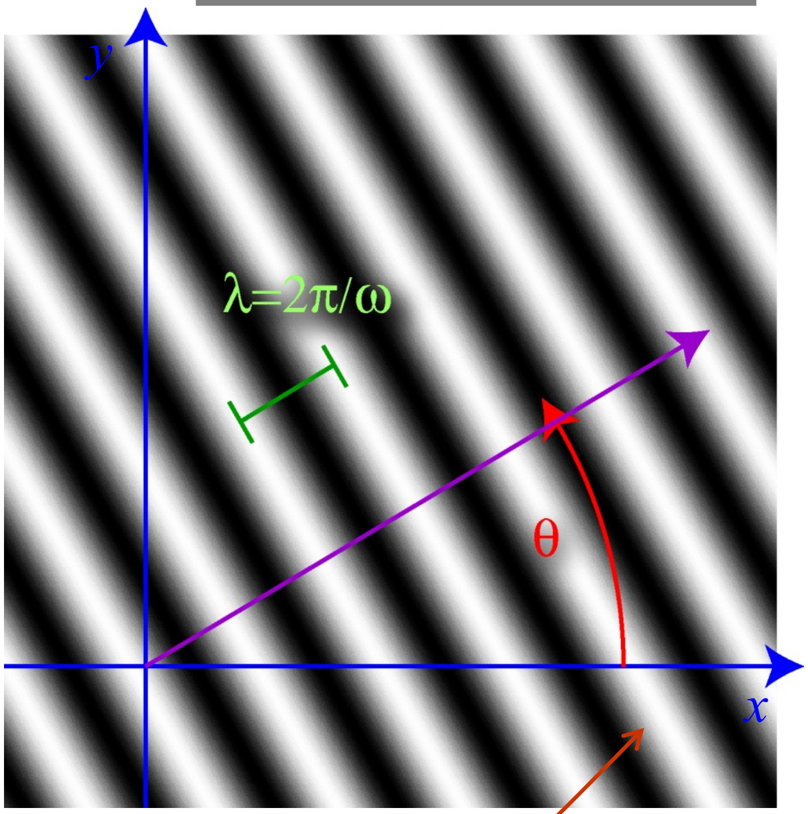


Note that the wave front direction = θ only if R=C.

Points on the Fourier Plane

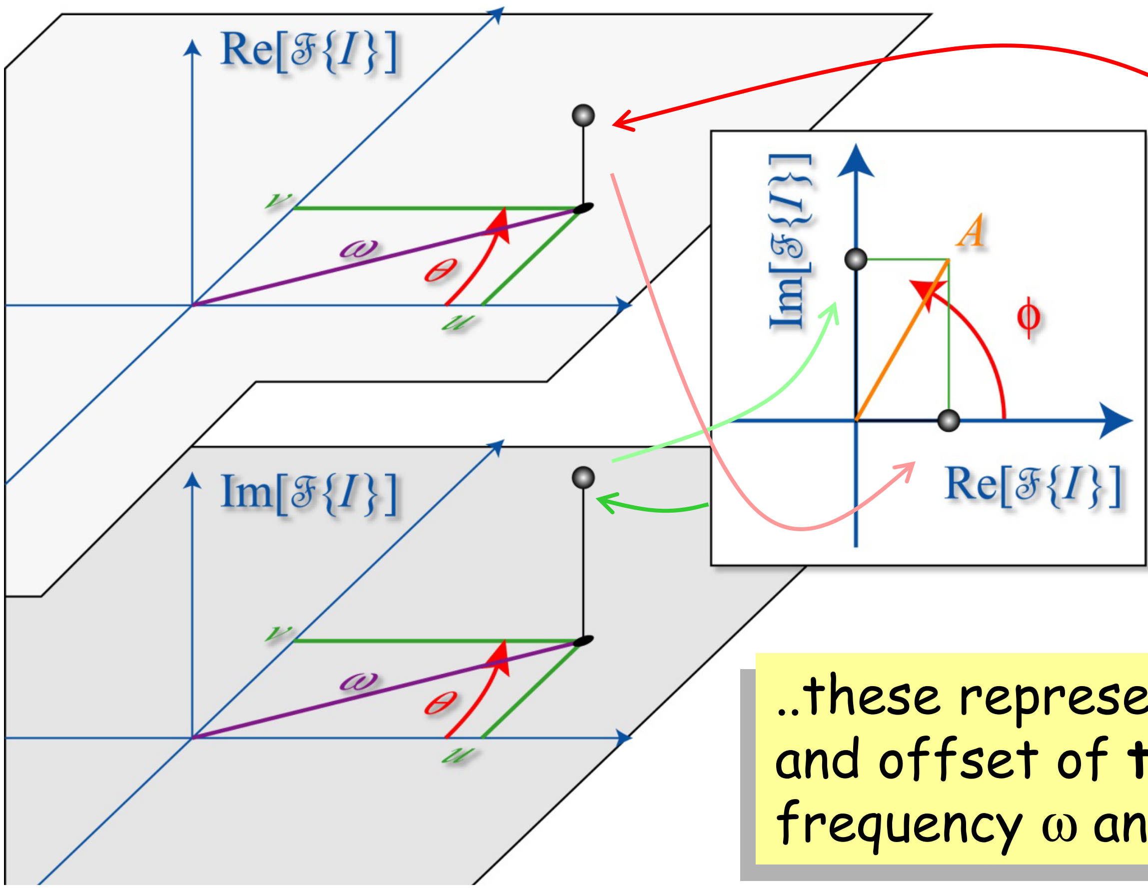
Note that θ is the wavefront direction only if R=C.





This point represents this particular sinusoidal grating

The Value of a Fourier Coefficient …



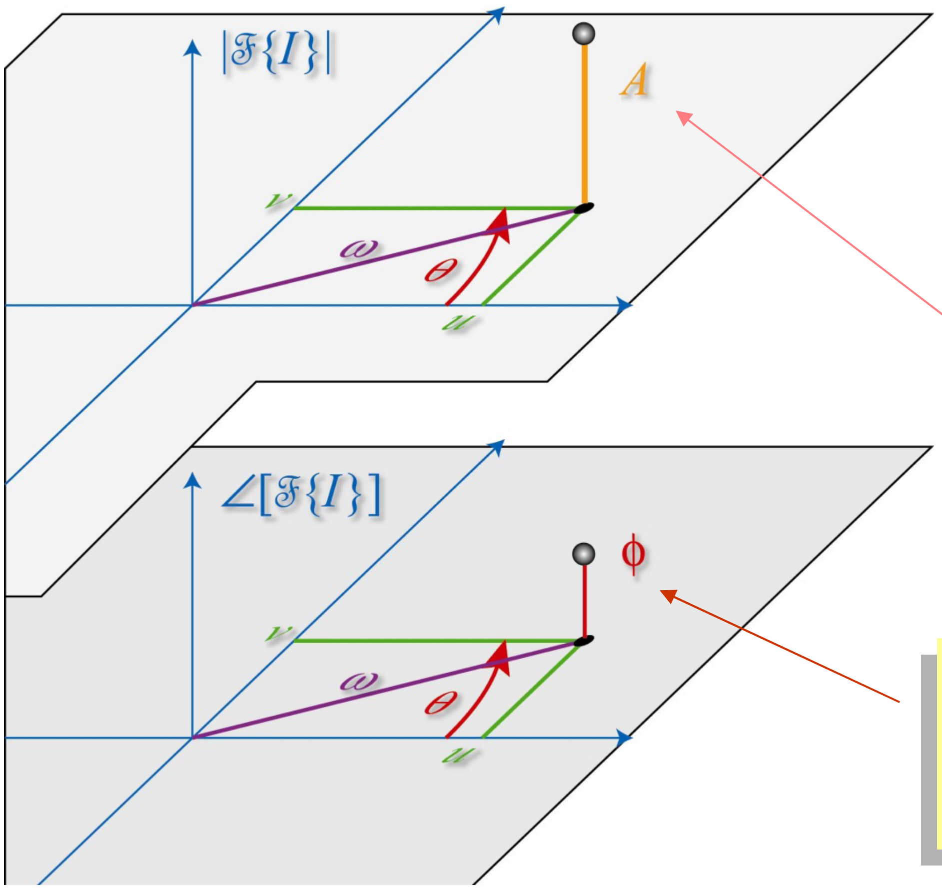
… is a complex number with a real part and an imaginary part.

If you represent that number as a magnitude, A, and a phase, ϕ, …

..these represent the amplitude and offset of **the** sinusoid with frequency ω and direction θ.*

*See p. 49.

The Value of a Fourier Coefficient

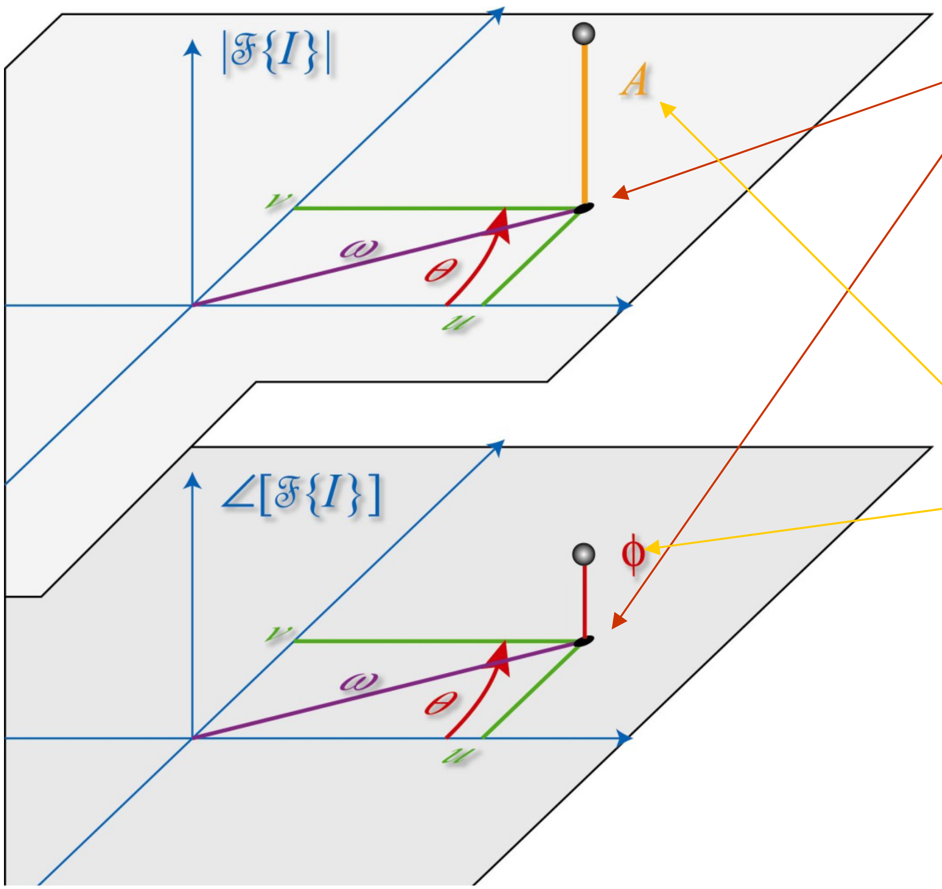


The magnitude and phase representation makes more sense physically…

…since the Fourier magnitude, $A\ (\omega,\theta)$, at point (ω,θ) represents the amplitude of the sinusoid…

and the phase, $\phi(\omega,\theta)$, represents the offset of the sinusoid relative to origin.

The Fourier Coefficient at (u,v)



So, the point (u,v) on the Fourier plane…

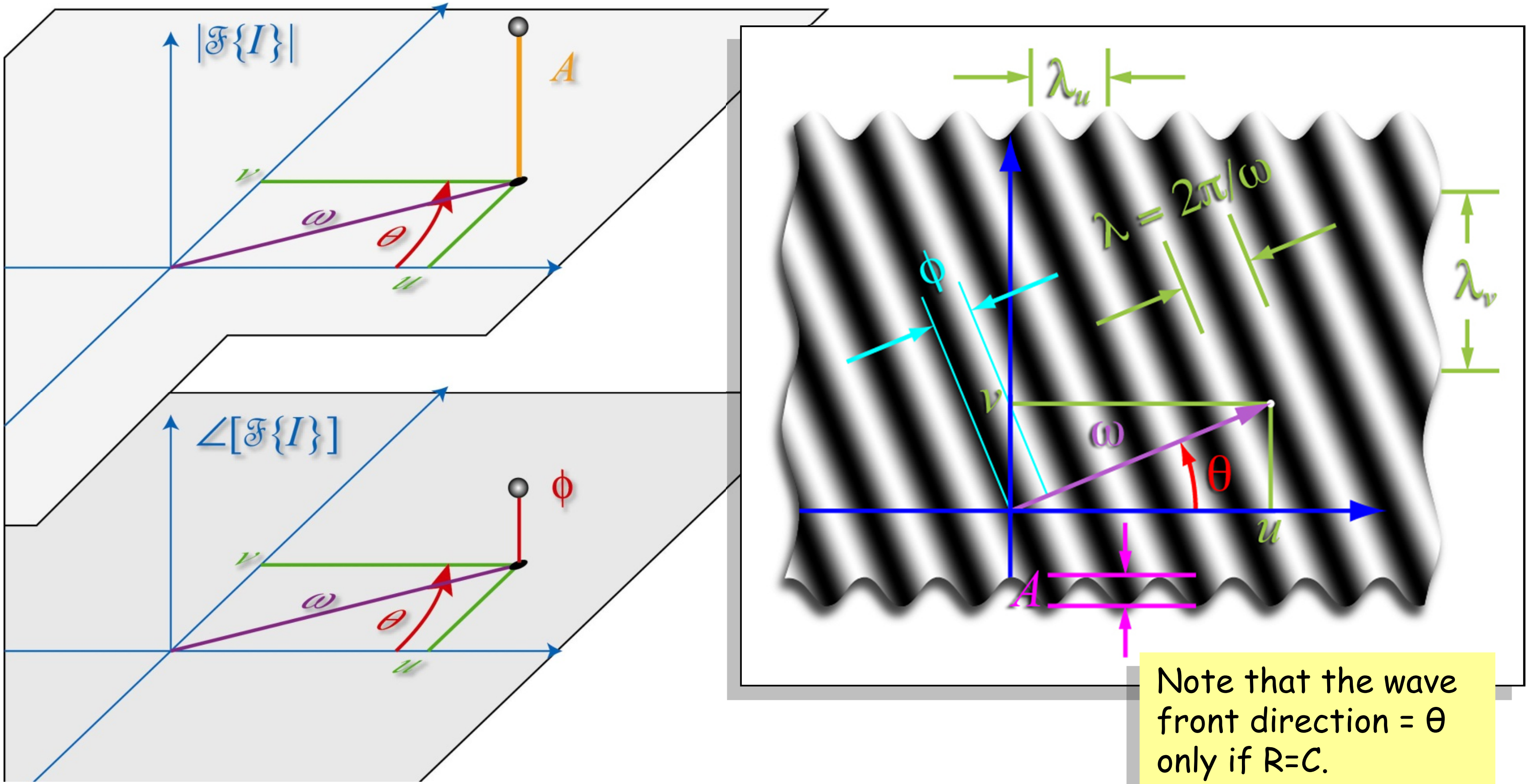
…represents a sinusoidal grating of frequency ω and orientation θ.*

The complex value, $F(u,v)$, of the FT at point (u,v)…

…represents the amplitude, A, and the phase offset, ϕ, of the sinusoid.

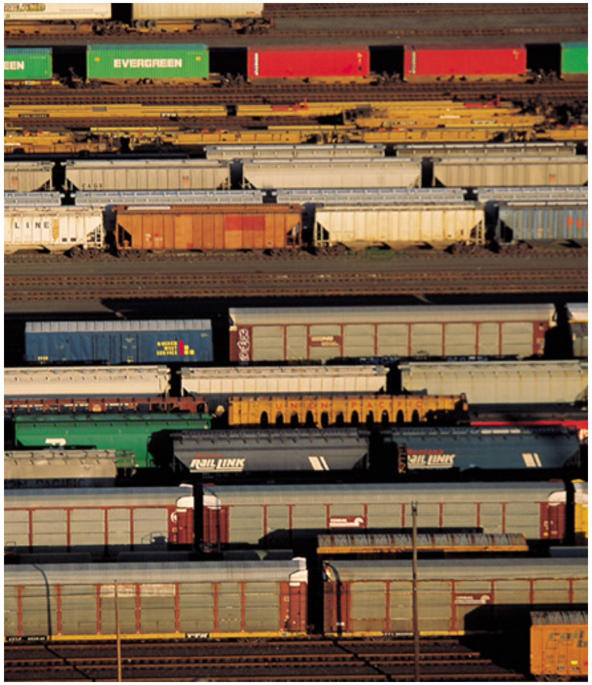
*See p. 49.

The Sinusoid from the Fourier Coeff. at (*u*,*v*)

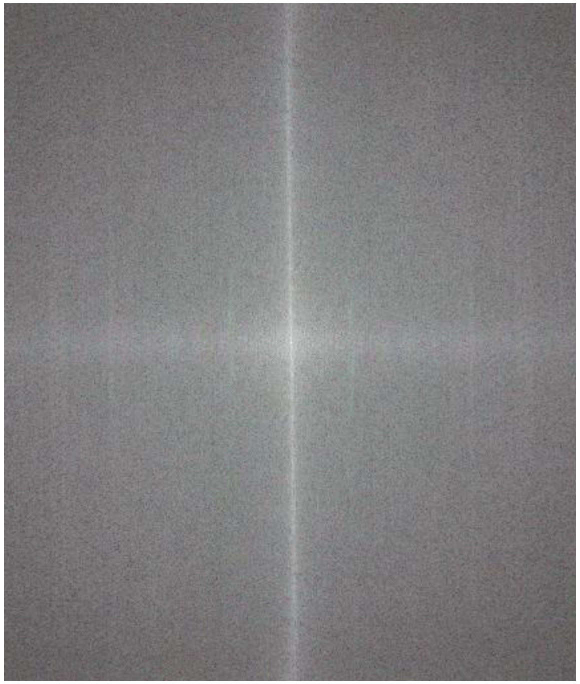


Note that the wave front direction = θ only if R=C.

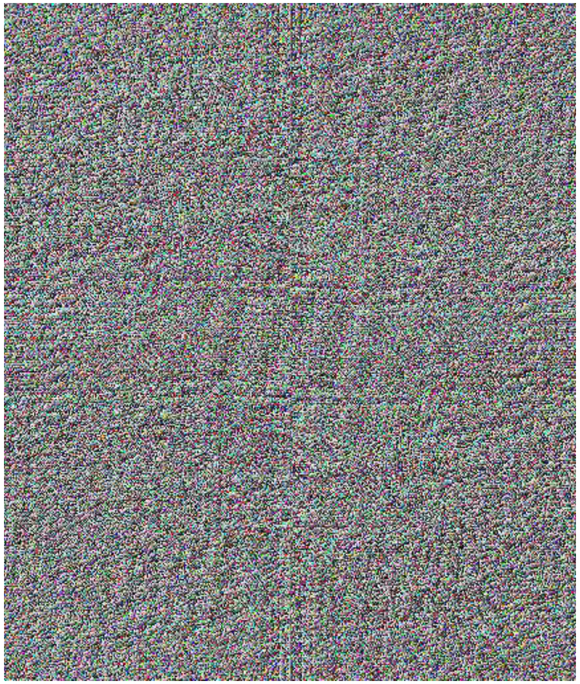
FT of an Image (Magnitude + Phase)



$\mathbf{I}$

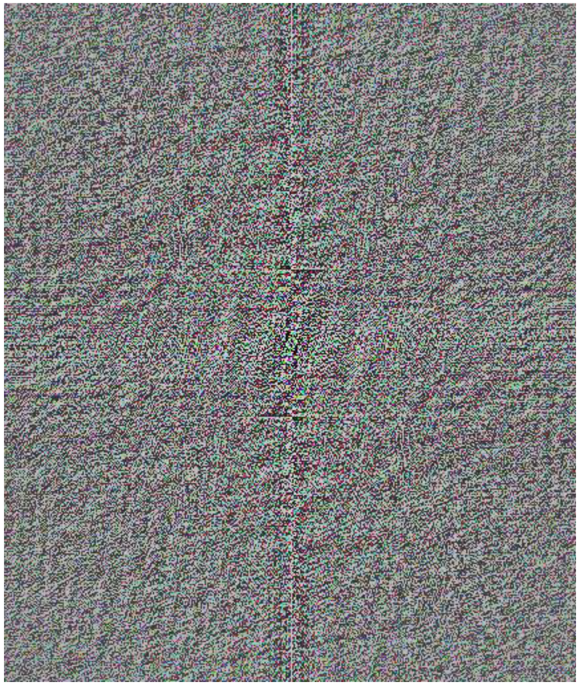


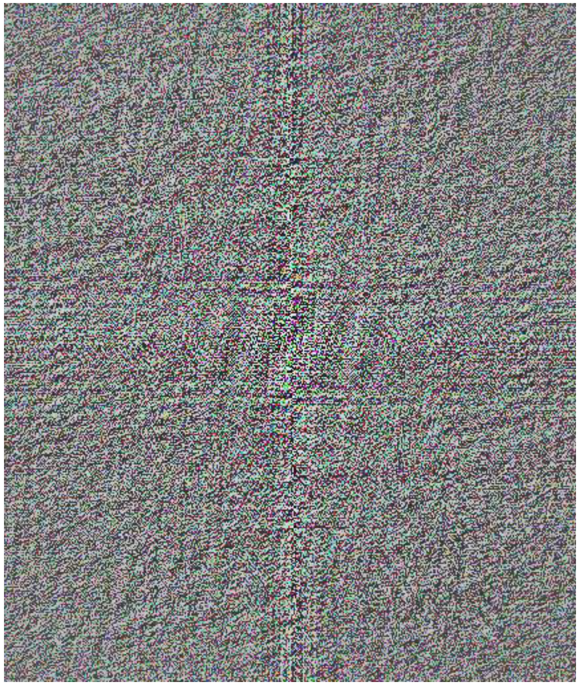
$\log\{|\mathcal{F}\{\mathbf{I}\}|^2+1\}$



$\angle[\mathcal{F}\{\mathbf{I}\}]$

FT of an Image (Real + Imaginary)





$\mathbf{I}$ $\quad$ $\mathrm{Re}[\mathscr{F}\{\mathbf{I}\}]$ $\quad$ $\mathrm{Im}[\mathscr{F}\{\mathbf{I}\}]$

The Power Spectrum

For display, the log of the power spectrum is often used.

The power spectrum of a signal is the square of the magnitude of its Fourier Transform.

$$\begin{aligned}\left|\mathfrak{F}(u,v)\right|^2 &= \mathfrak{F}(u,v)\,\mathfrak{F}^*(u,v)\\ &= \left[\mathrm{Re}\,\mathfrak{F}(u,v) + i\,\mathrm{Im}\,\mathfrak{F}(u,v)\right]\left[\mathrm{Re}\,\mathfrak{F}(u,v) - i\,\mathrm{Im}\,\mathfrak{F}(u,v)\right]\\ &= \left[\mathrm{Re}\,\mathfrak{F}(u,v)\right]^2 + \left[\mathrm{Im}\,\mathfrak{F}(u,v)\right]^2.\end{aligned}$$

At each location (u,v) it indicates the squared intensity of the frequency component with period $\lambda = 1\big/\sqrt{u^2+v^2}$ and orientation $\theta = \tan^{-1}(v/u)$.

```
For display in Matlab:
PS = fftshift(2*log(abs(fft2(I))+1));
```

On the Computation of the Power Spectrum

The power spectrum (PS) is defined by $\mathrm{PS}(\mathrm{I}) = \left|\mathfrak{F}\{\mathrm{I}(u,v)\}\right|^2$.

We take the base-e logarithm of the PS in order to view it. Otherwise its dynamic range could be too large to see everything at once. We add 1 to it first so that the minimum value of the result is 0 rather than –infinity, which it would be if there were any zeros in the PS. Recall that

$\log(f^2) = 2\log(f)$.

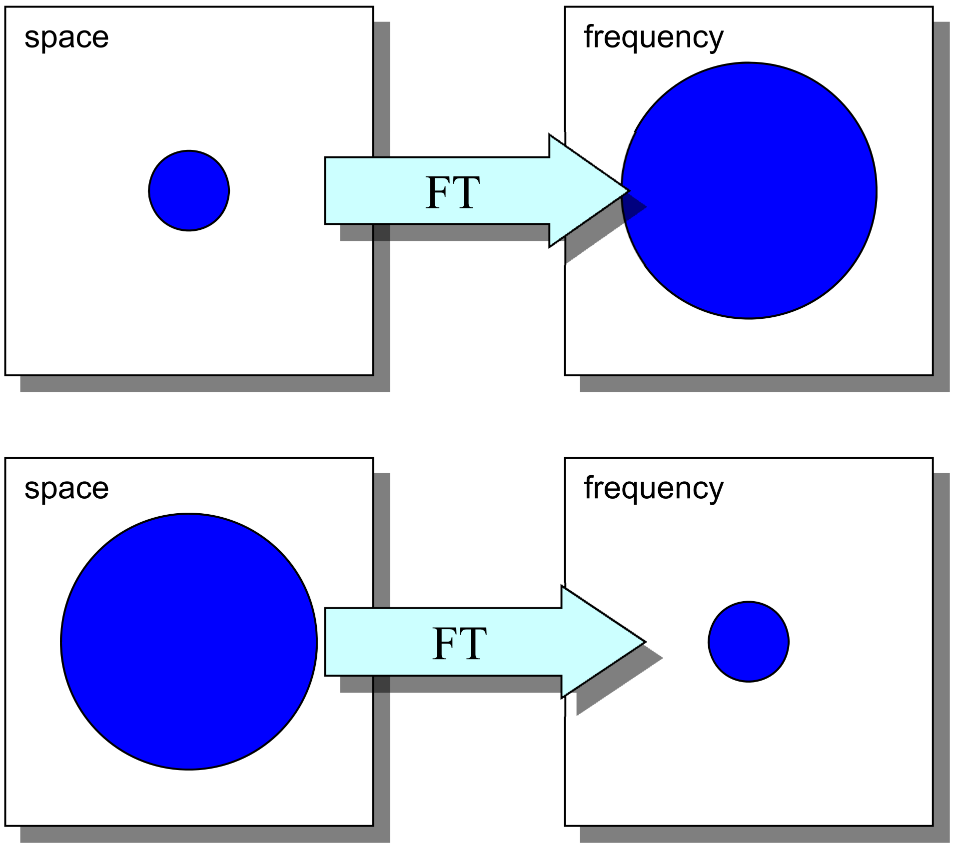
Multiplying by 2 is not necessary if you are generating a PS for viewing, since you'll probably have to scale it into the range 0-255 anyway. It is much easier to see the structures in a Fourier plane if the origin is in the center. Therefore we usually perform an fftshift on the PS before it is displayed.

```
>> PS = fftshift(log(abs(fft2(I))+1));
>> M  = max(PS(:));
>> image(uint8(255*(PS/M)));
```

If the PS is being calculated for later computational use -- for example the autocorrelation of a function is the inverse FT of the PS of the function -- it should be calculated by

```
>> PS = abs(fft2(I)).^2;
```

The Uncertainty Relation

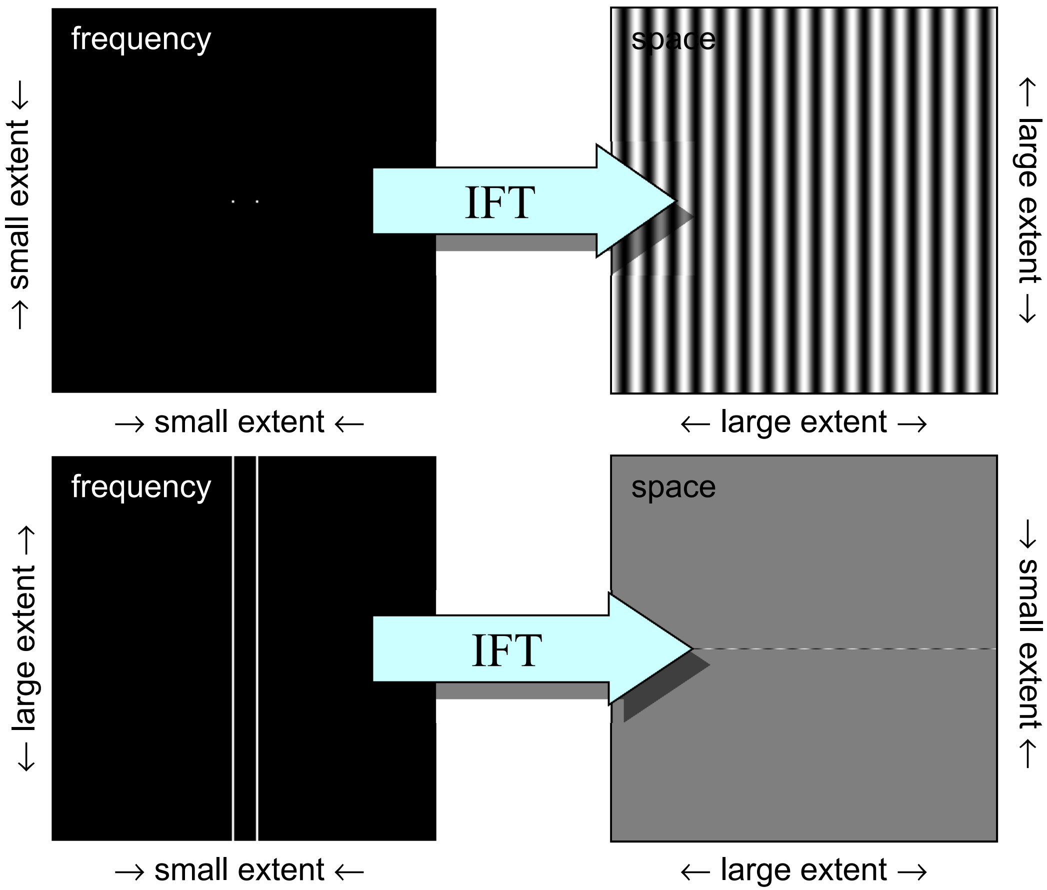


If $\Delta x \, \Delta y$ is the extent of the object in space and if $\Delta u \, \Delta v$ is its extent in frequency then,

$$\Delta x \, \Delta y \cdot \Delta u \, \Delta v \geq \frac{1}{16\pi^2}$$

A small object in space has a large frequency extent and vice-versa.

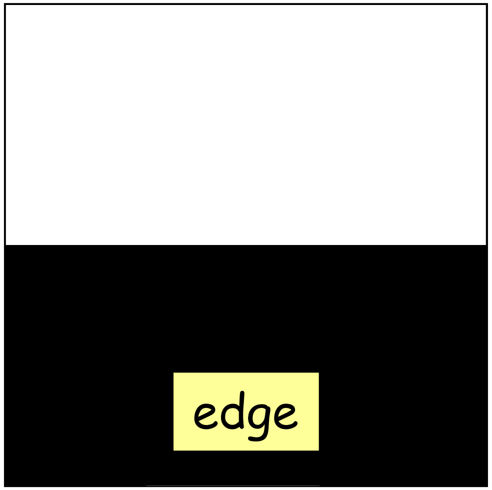
The Uncertainty Relation

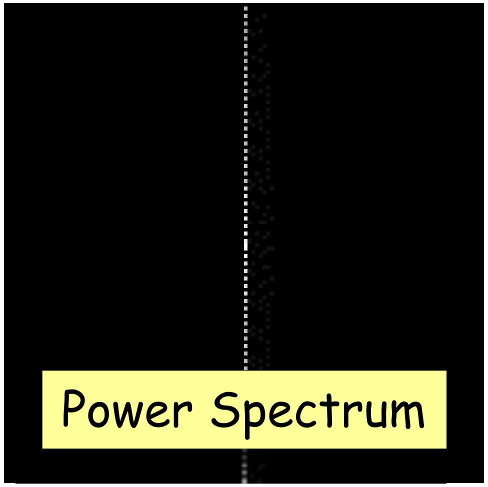


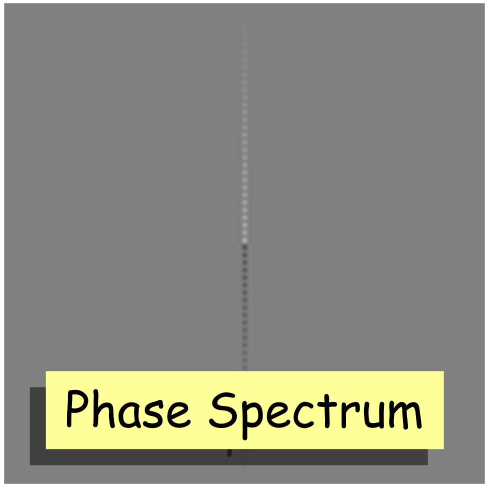
Recall: a symmetric pair of impulses in the frequency domain becomes a sinusoid in the spatial domain.

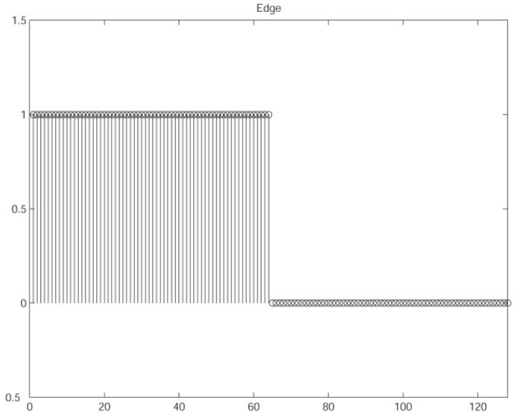
A symmetric pair of lines in the frequency domain becomes a sinusoidal line in the spatial domain.

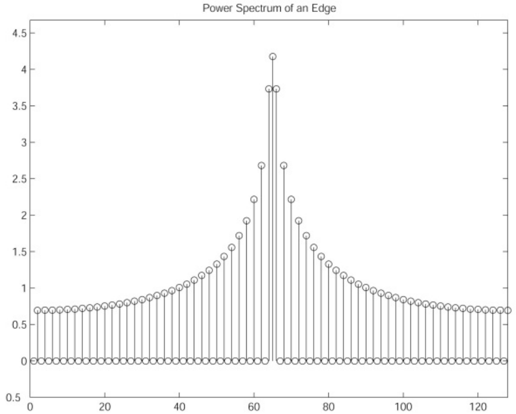
The Fourier Transform of an Edge

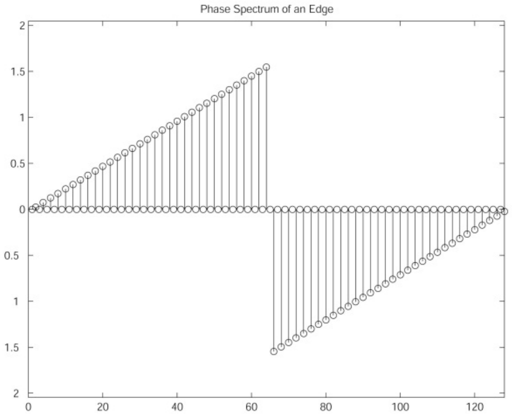




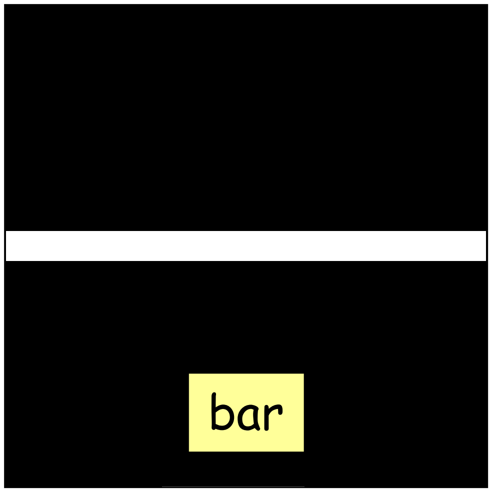


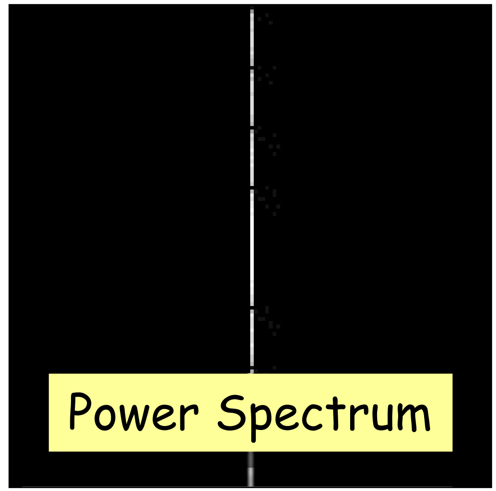


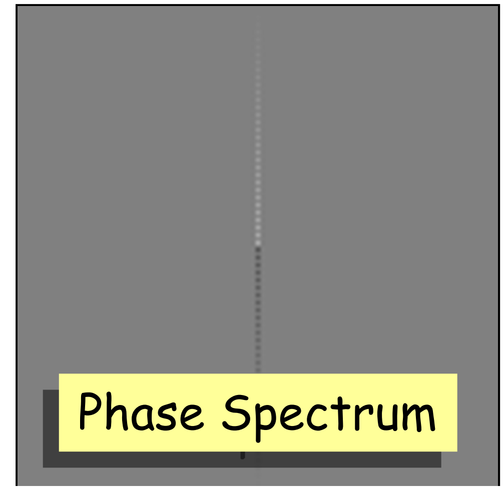


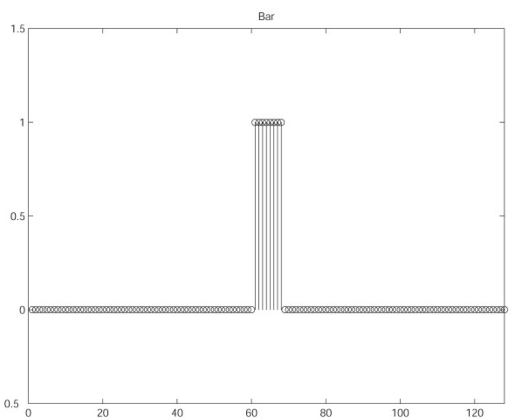


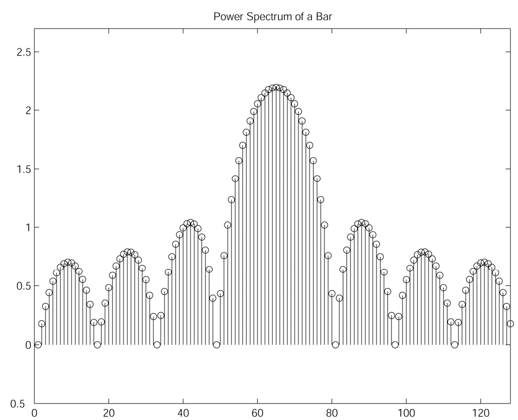
The Fourier Transform of a Bar

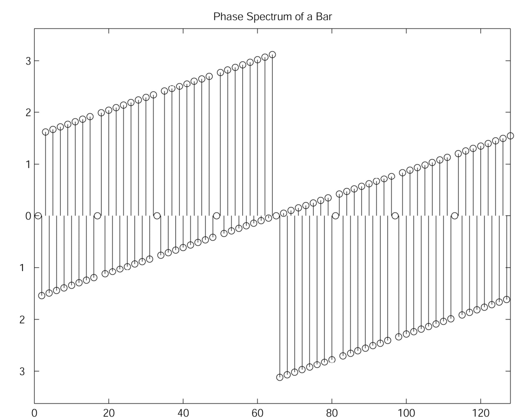












Coordinate Origin of the FFT

Center = (floor(R/2)+1, floor(C/2)+1)

Even

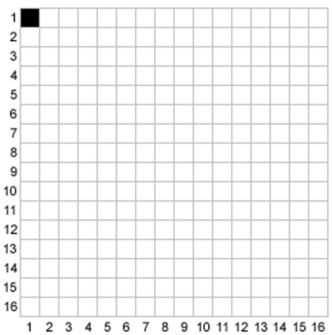


Image Origin

Odd

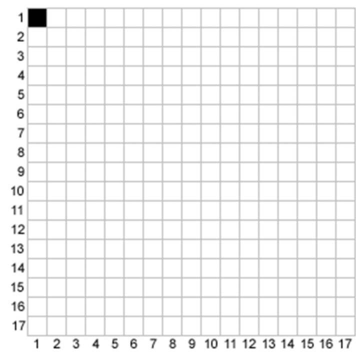
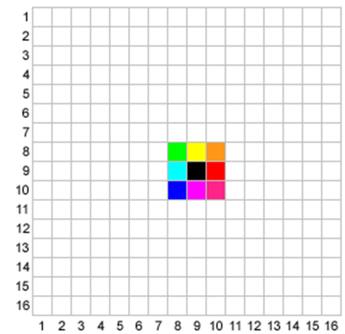


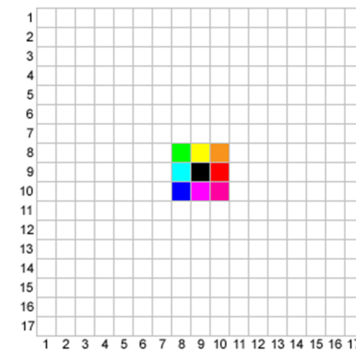
Image Origin

Even

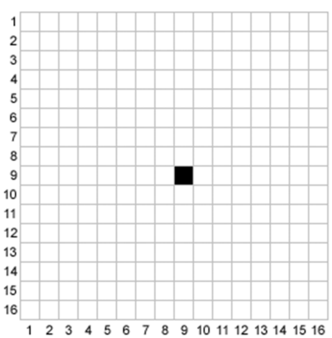


Weight Matrix Origin

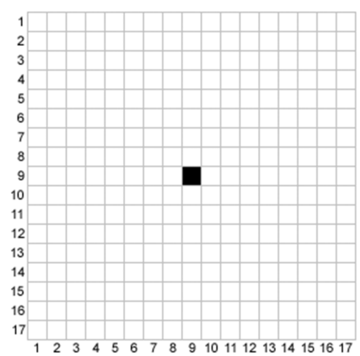
Odd



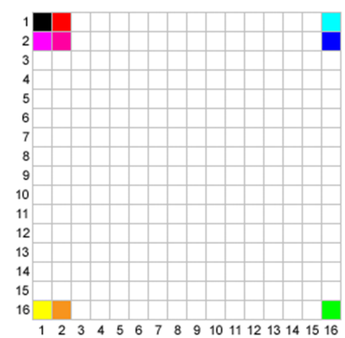
Weight Matrix Origin



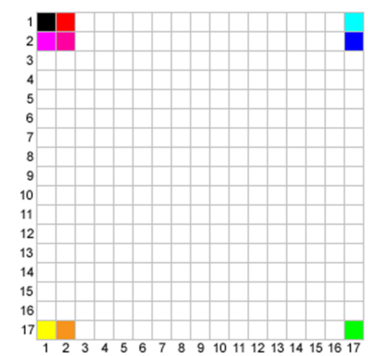
After FFT shift



After FFT shift



After IFFT shift



After IFFT shift

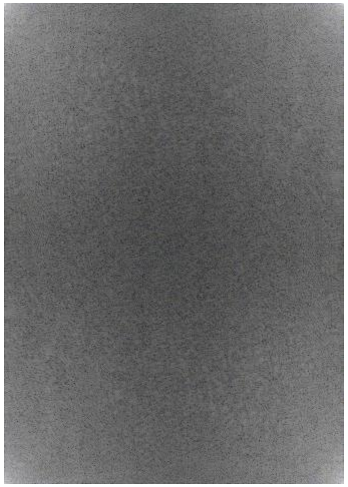
Matlab's `fftshift` and `ifftshift`

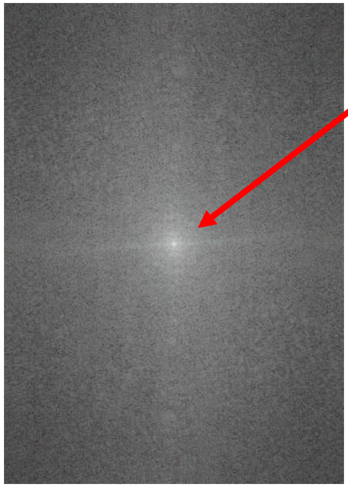
`I = ifftshift(J):`

`J = fftshift(I):`

origin

from FFT2
or ifftshift





origin

after fftshift

$\mathbf{J}\left(\lfloor R/2 \rfloor + 1, \lfloor C/2 \rfloor + 1\right) \rightarrow \mathbf{I}(1,1)$

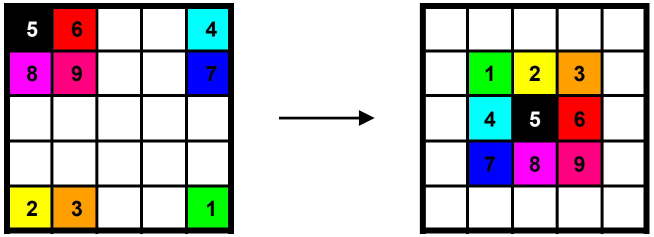
$\mathbf{I}(1,1) \rightarrow \mathbf{J}\left(\lfloor R/2 \rfloor + 1, \lfloor C/2 \rfloor + 1\right)$

where $\lfloor x \rfloor$ = `floor(x)` = the largest integer smaller than x.

Matlab's `fftshift` and `ifftshift`

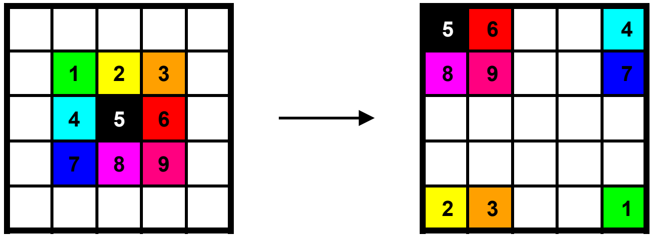
`J = fftshift(I):`

$$\mathbf{I}(1,1) \rightarrow \mathbf{J}(\ R/2\ +1,\ \ C/2\ +1)$$



`I = ifftshift(J):`

$$\mathbf{J}(\ R/2\ +1,\ \ C/2\ +1) \rightarrow \mathbf{I}(1,1)$$



where x = `floor(x)` = the largest integer smaller than x.

Points on the Fourier Plane (of a Digital Image)

In the Fourier transform of an $R \times C$ digital image, positions u and v indicate the number of repetitions of the sinusoid in those directions. Therefore the wavelengths along the column and row axes are

$$\lambda_u = \frac{C}{u} \quad \text{and} \quad \lambda_v = \frac{R}{v} \quad \text{pixels,}$$

and the wavelength in the wavefront direction is

$$\lambda_{\mathrm{wf}} = \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2}.$$

The frequency is the fraction of the sinusoid traversed over one pixel,

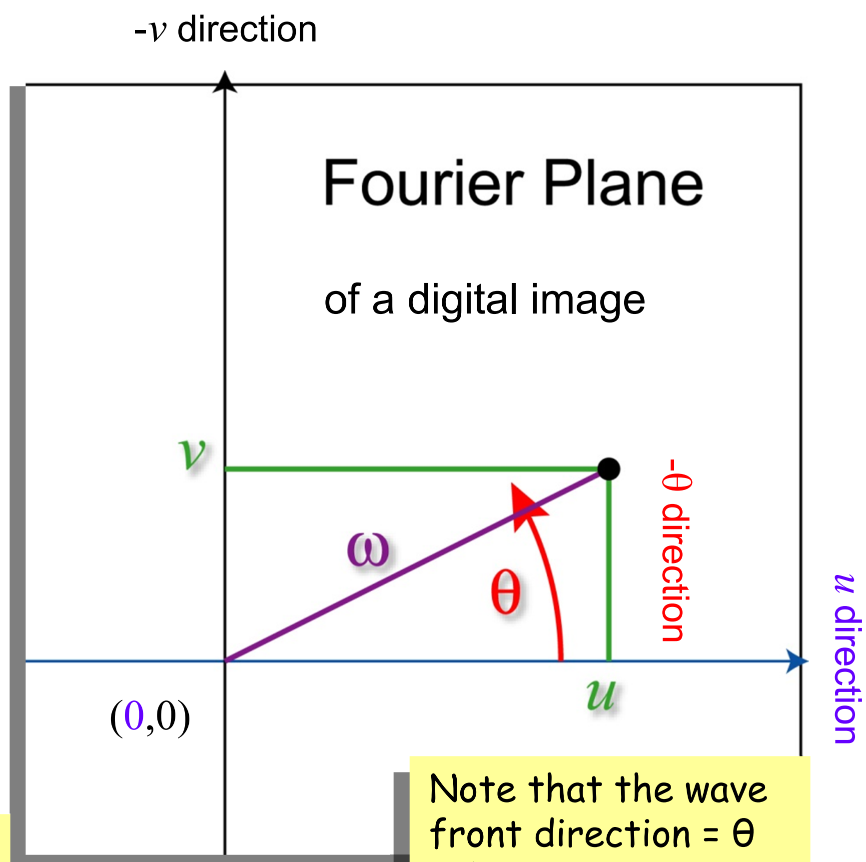
$$\omega_u = \frac{u}{C}, \; \omega_v = \frac{v}{R}, \text{ and}$$

$$\omega_{\mathrm{wf}} = 1 \Big/ \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2} \quad \text{cycles.}$$

The wavefront direction is given by

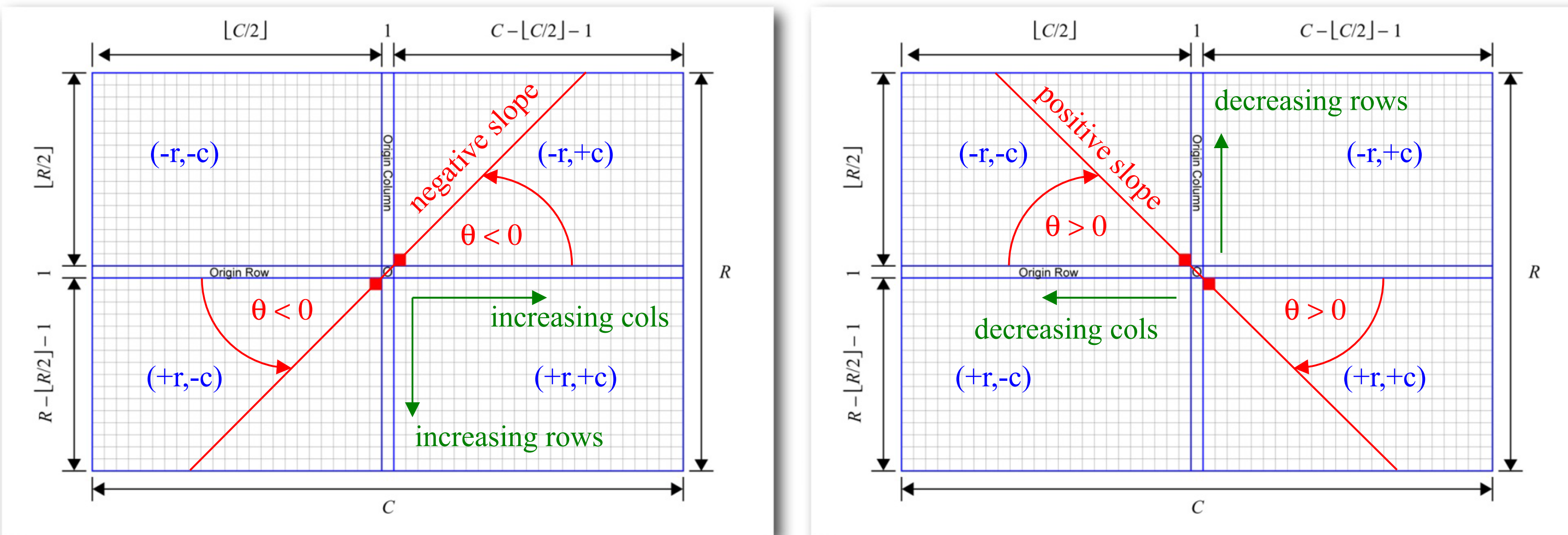
$$\theta_{\mathrm{wf}} = \tan^{-1}\left(\frac{\omega_v}{\omega_u}\right) = \tan^{-1}\left(\frac{vC}{uR}\right).$$

$\frac{\text{row freq.}}{\text{column freq.}}$



Note that the wave front direction = θ only if R=C.

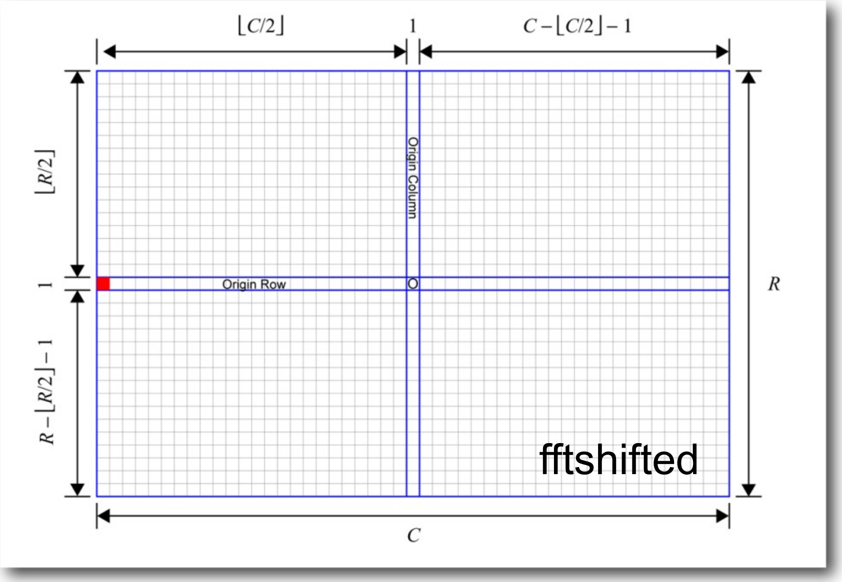
Coordinates and Directions in the Fourier Plane

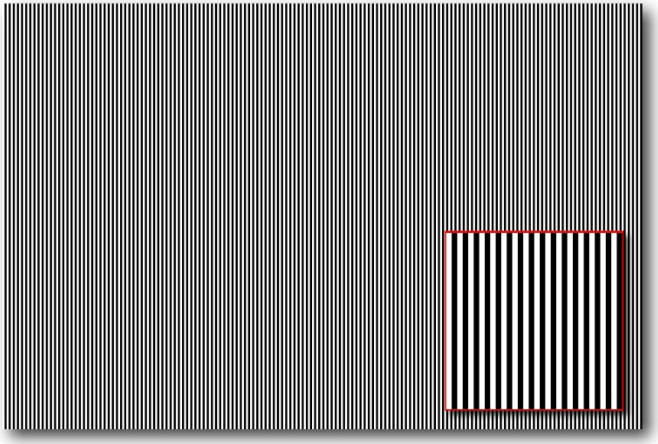


Since rows increase down and columns to the right, slopes and angles are opposite those of a right-handed coordinate system.

Inverse FFTs of Impulses

"horizontal" is the wavefront direction.

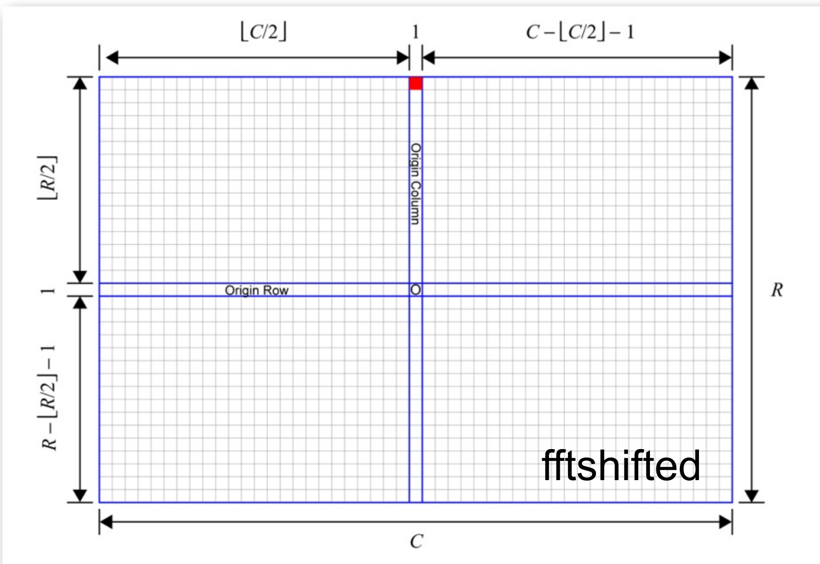


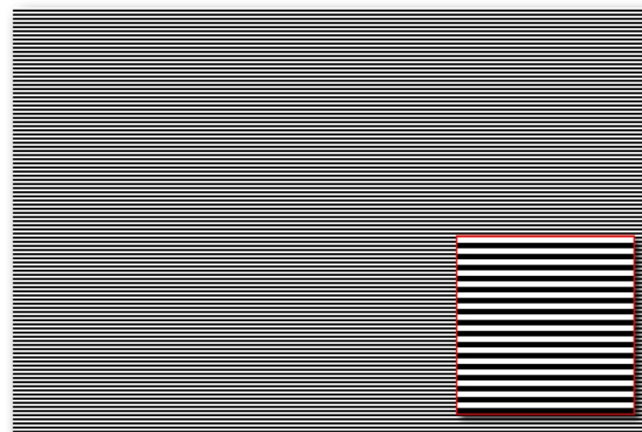


highest-possible-frequency horizontal sinusoid (C is even)

Inverse FFTs of Impulses

"vertical" is the wavefront direction.

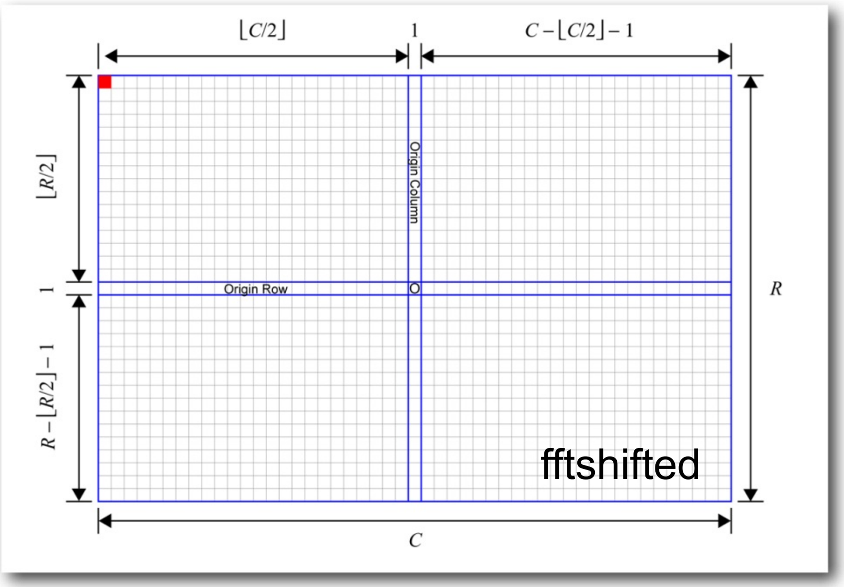


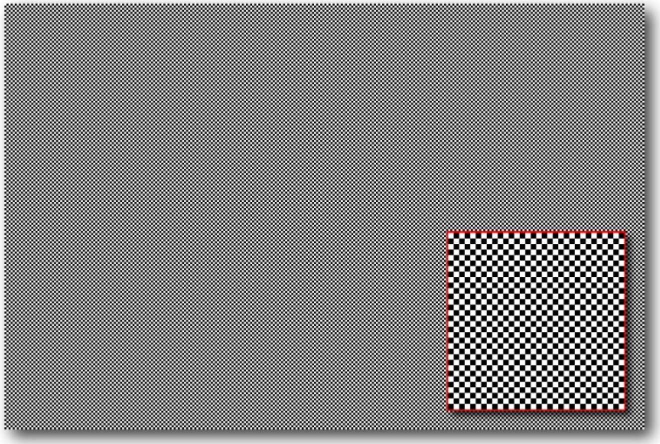


highest-possible-frequency vertical sinusoid (*R* is even)

Inverse FFTs of Impulses

a checker-board pattern.

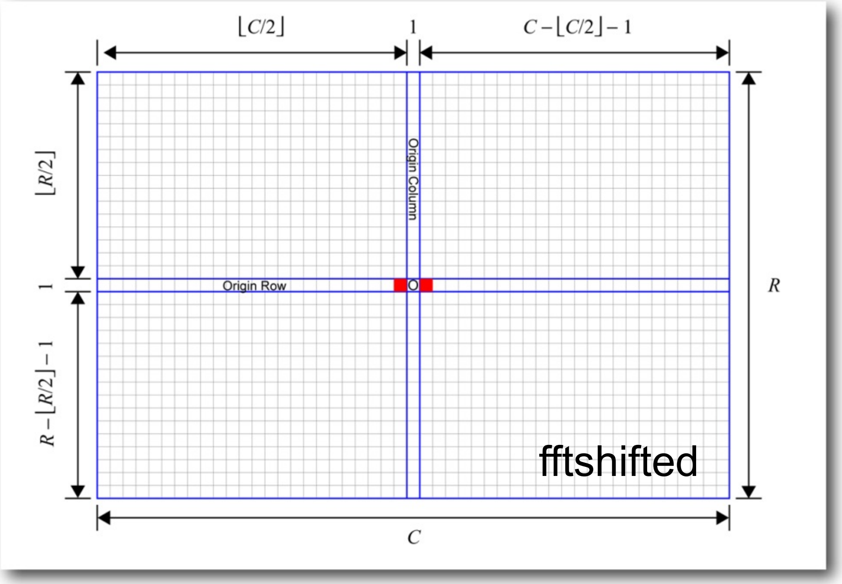


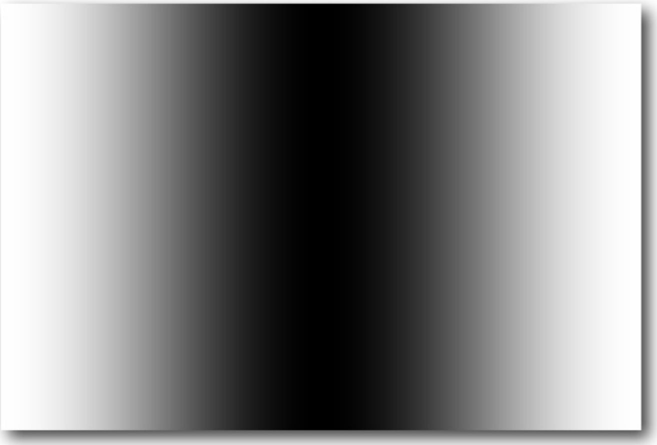


highest-possible-freq horizontal+vertical sinusoid (R & C even)

Inverse FFTs of Impulses

"horizontal" is the wavefront direction.

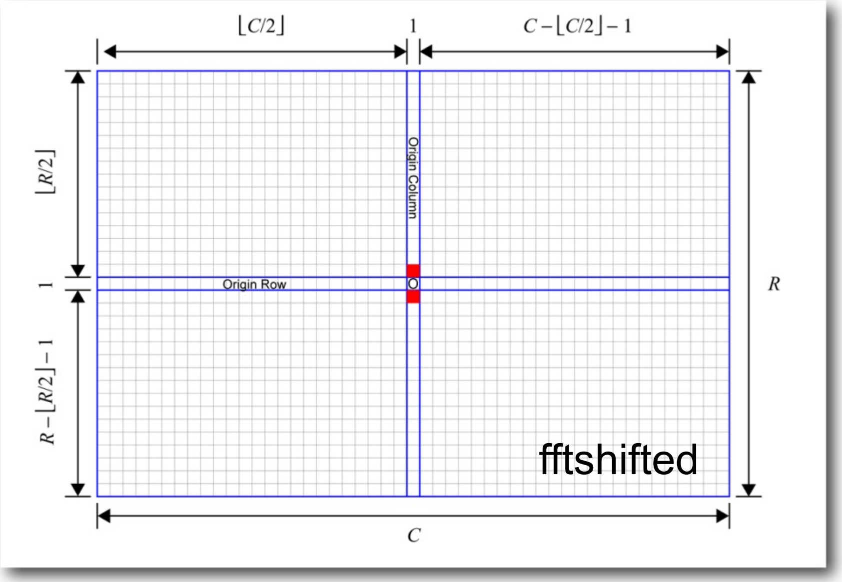


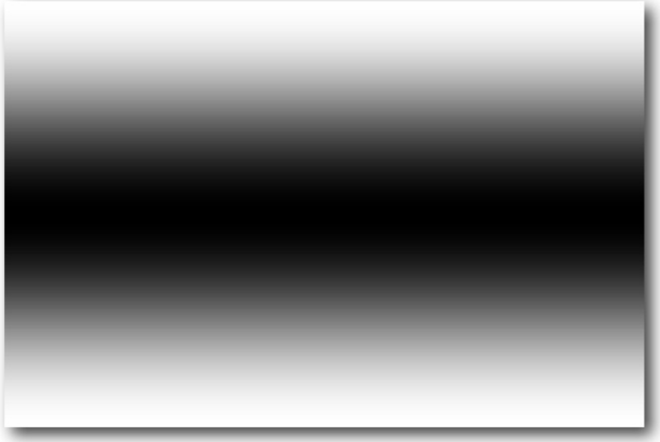


lowest-possible-frequency horizontal sinusoid

Inverse FFTs of Impulses

"vertical" is the wavefront direction.

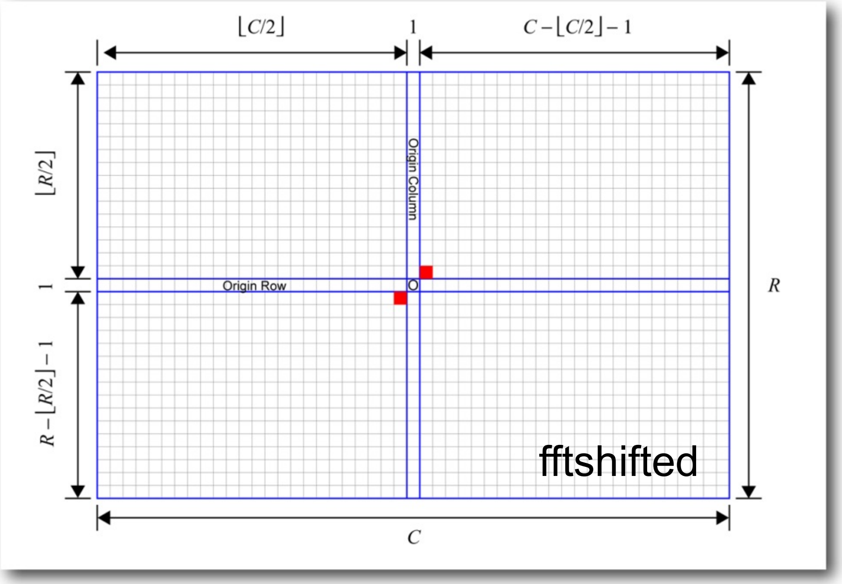


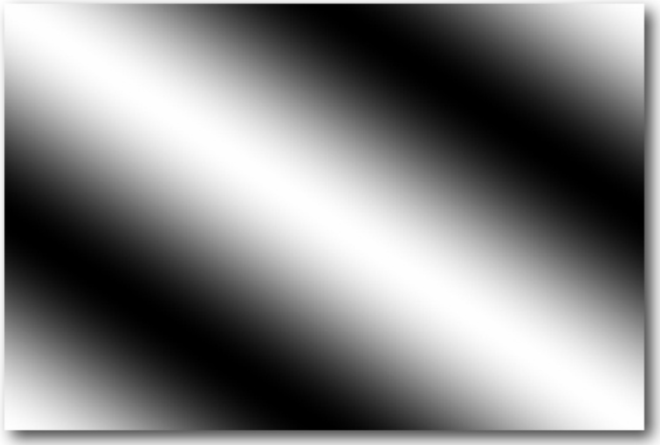


lowest-possible-frequency vertical sinusoid

Inverse FFTs of Impulses

"negative diagonal" is the wavefront direction.

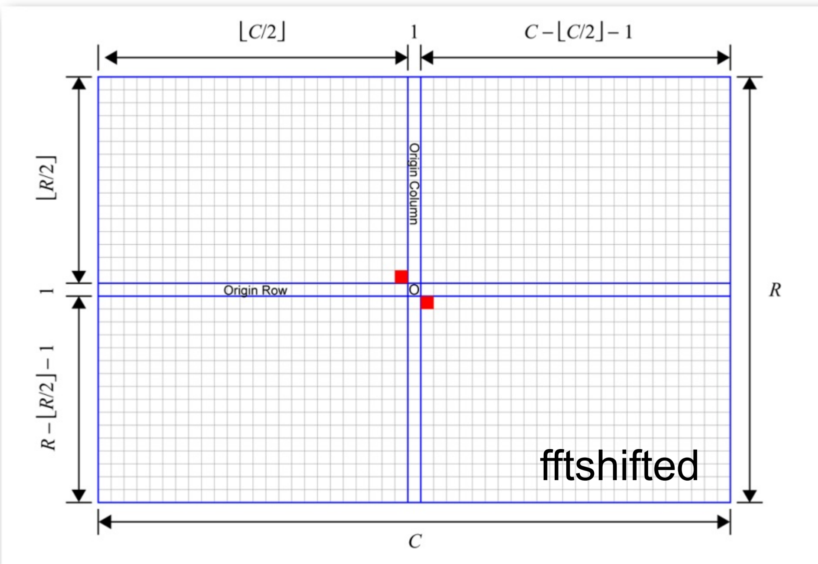


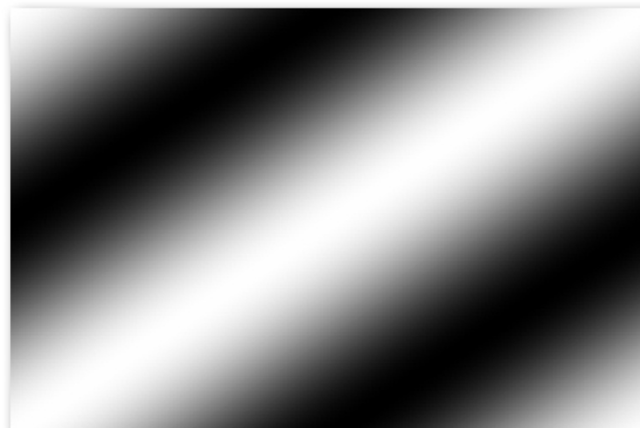


lowest-possible-frequency negative diagonal sinusoid

Inverse FFTs of Impulses

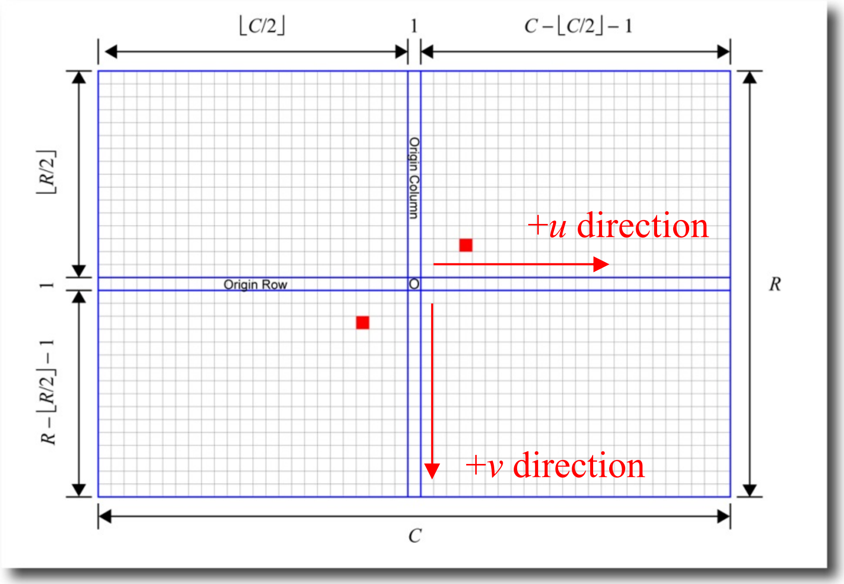
"positive diagonal" is the wavefront direction.

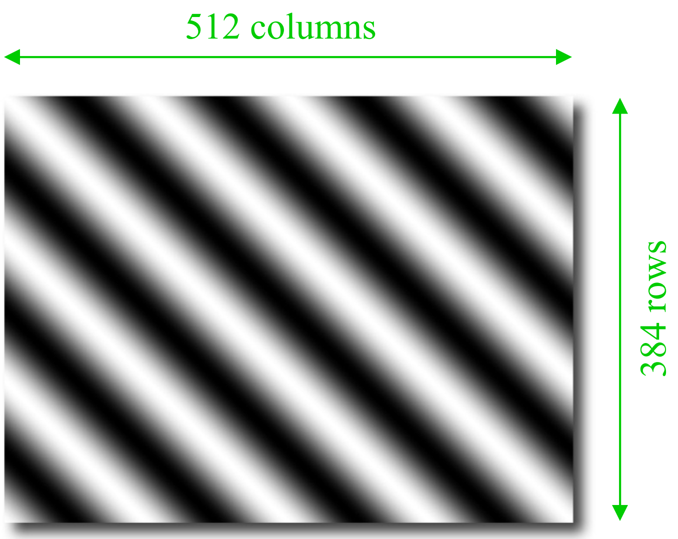




lowest-possible-frequency positive diagonal sinusoid

Frequencies and Wavelengths in the Fourier Plane





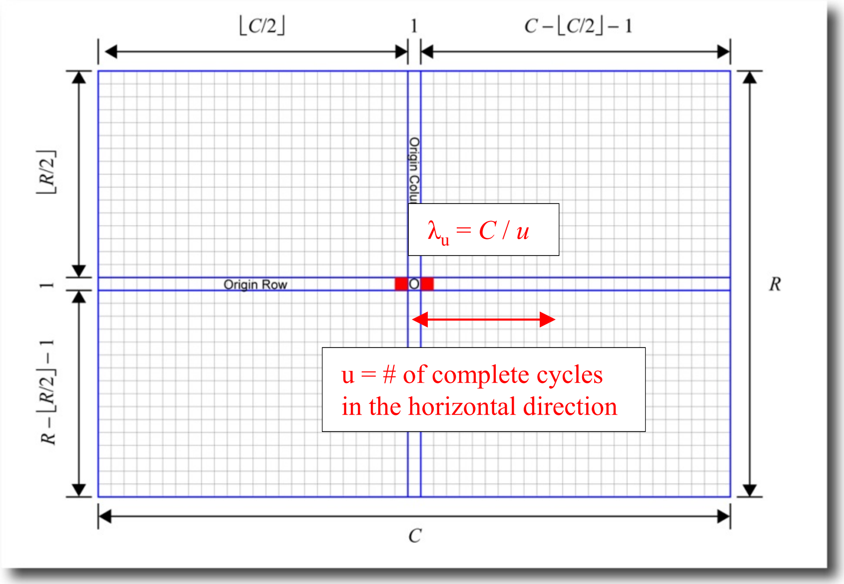
Note this …

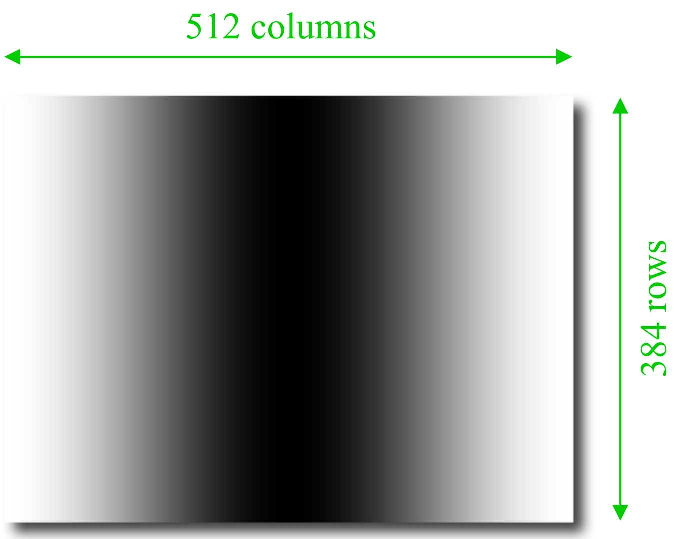
… and this.

frequencies: $(u,v) = (4,3)$; wavelengths: $(\lambda_u, \lambda_v) = (128,128)$

How can that be?

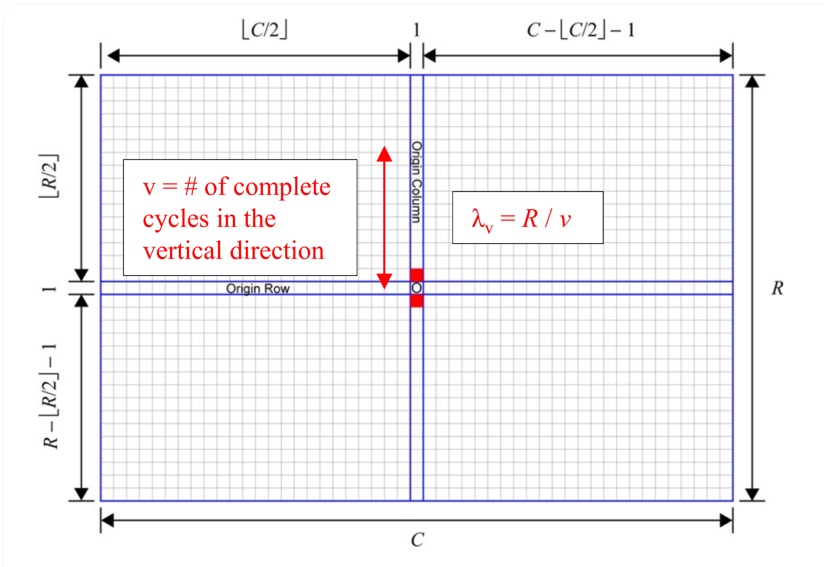
Frequencies and Wavelengths in the Fourier Plane

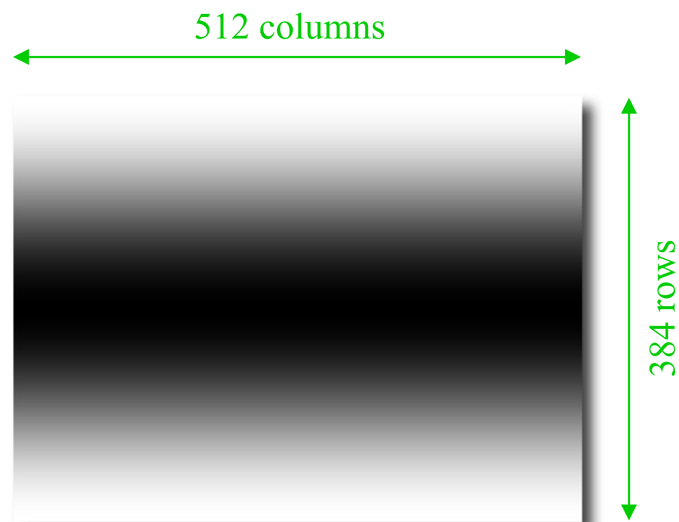




frequencies: $(u,v) = (1,0)$; wavelength: $\lambda_u = 512$

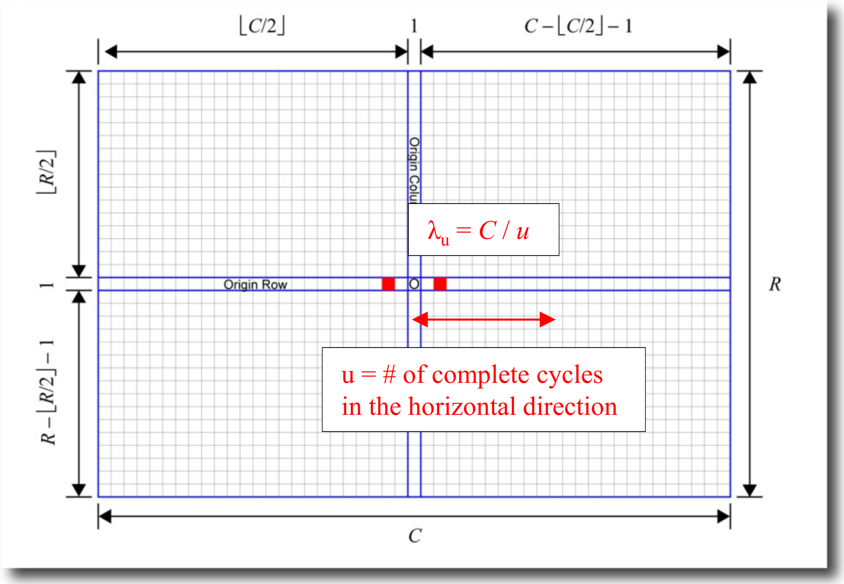
Frequencies and Wavelengths in the Fourier Plane

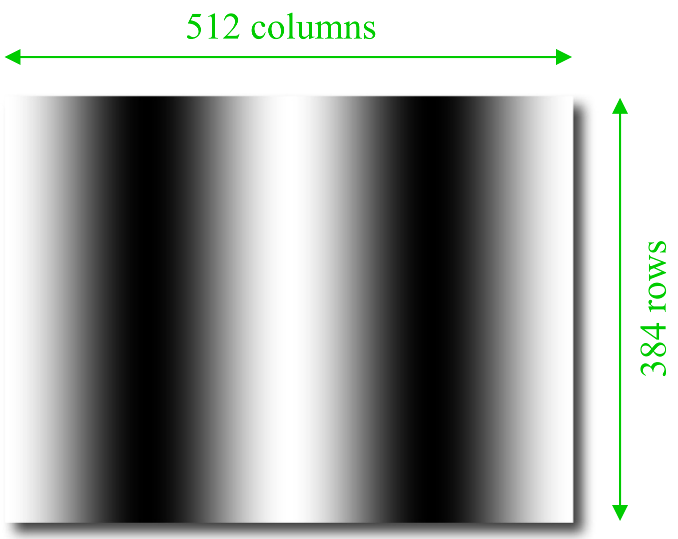




frequencies: $(u,v) = (0,1)$; wavelength: $\lambda_v = 384$

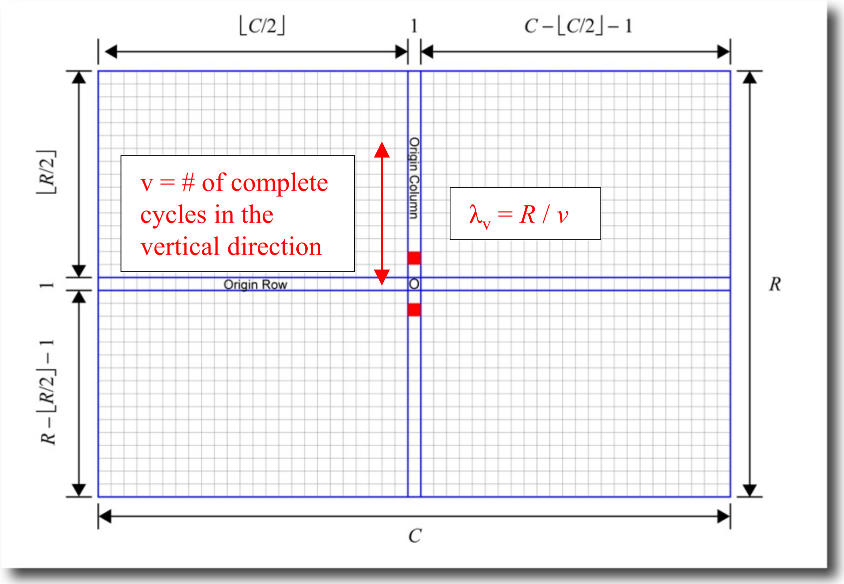
Frequencies and Wavelengths in the Fourier Plane

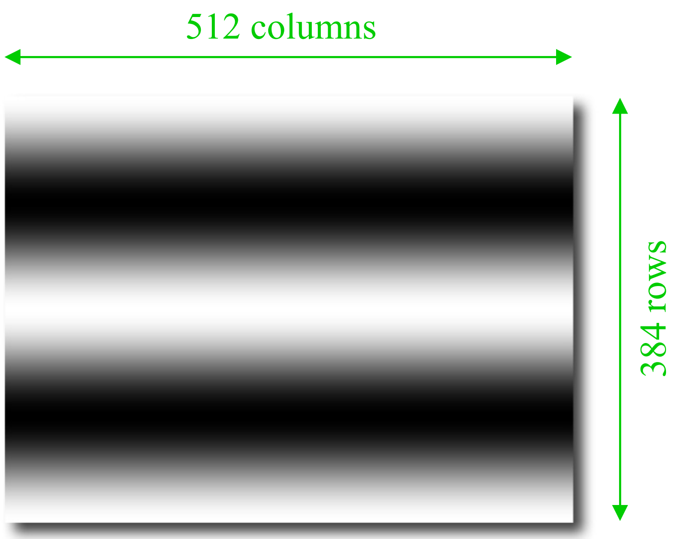




frequencies: $(u,v) = (2,0)$; wavelength: $\lambda_u = 256$

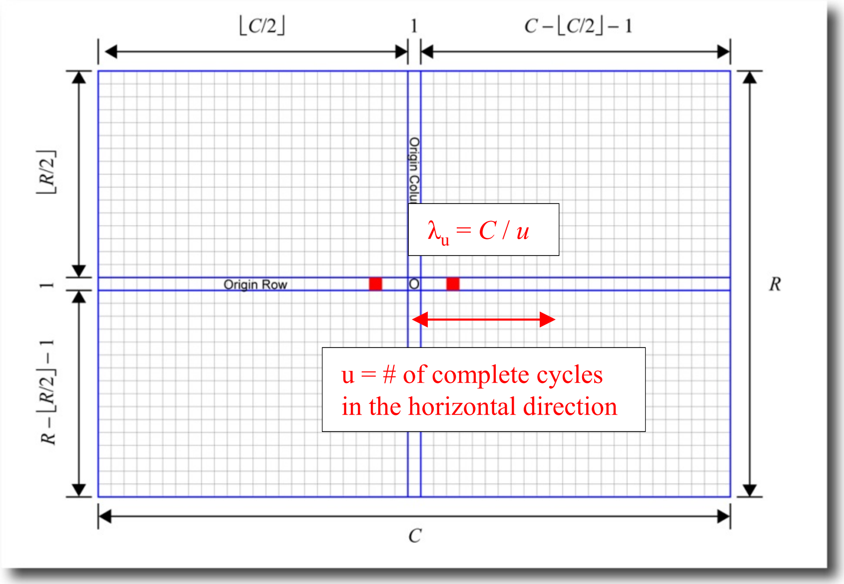
Frequencies and Wavelengths in the Fourier Plane

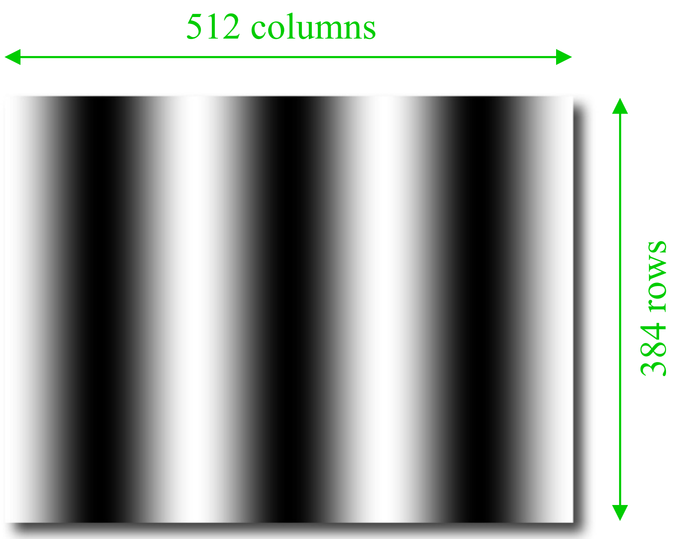




frequencies: $(u,v) = (0,2)$; wavelength: $\lambda_v = 192$

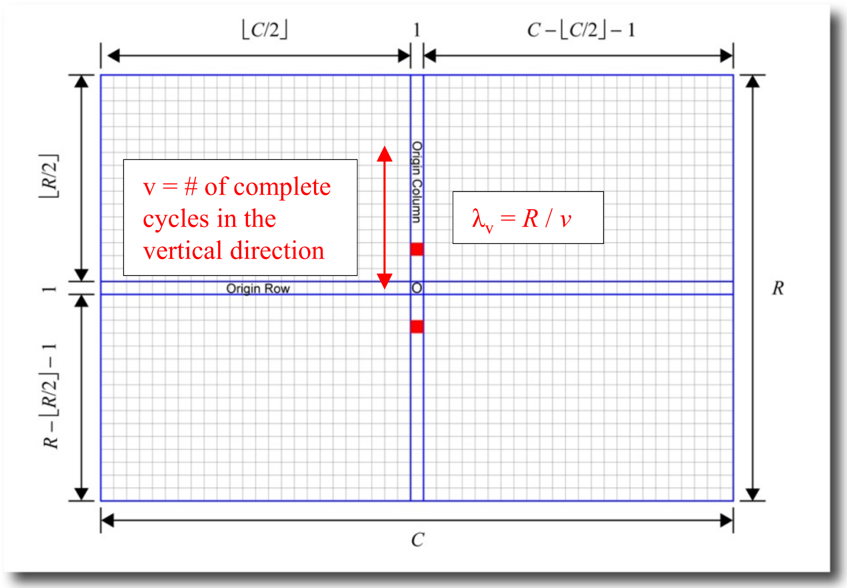
Frequencies and Wavelengths in the Fourier Plane

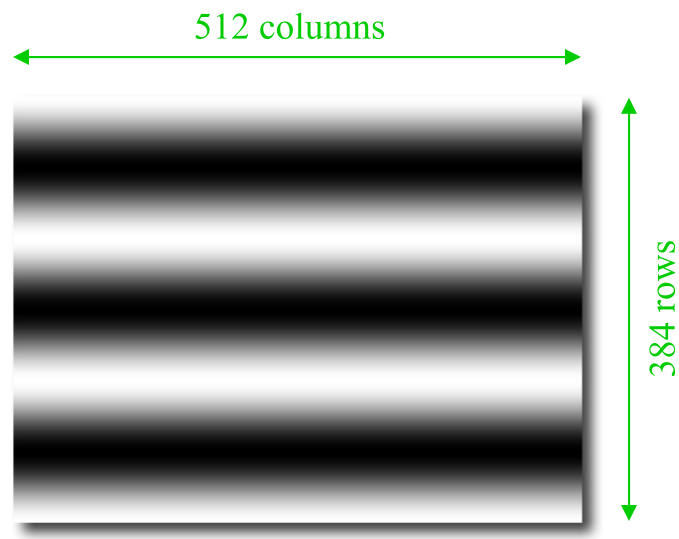




frequencies: $(u,v) = (3,0)$; wavelength: $\lambda_u = 170\tfrac{2}{3}$

Frequencies and Wavelengths in the Fourier Plane

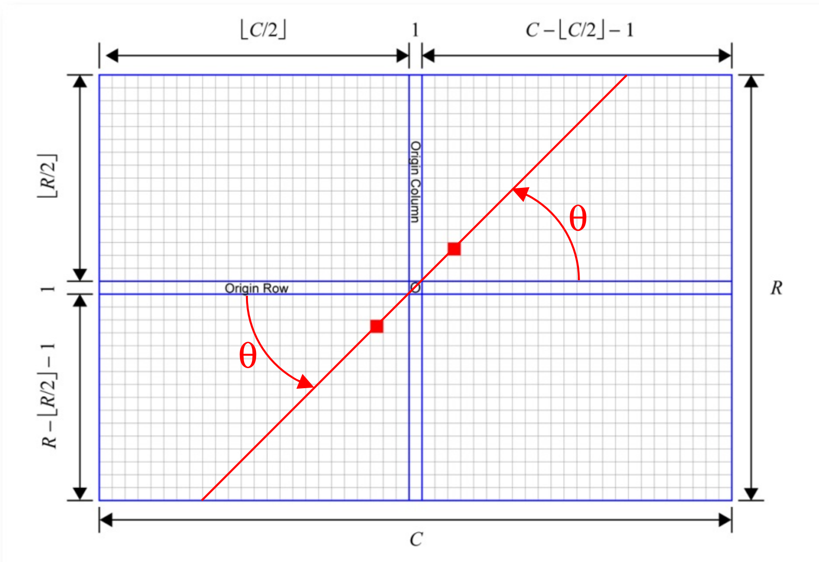


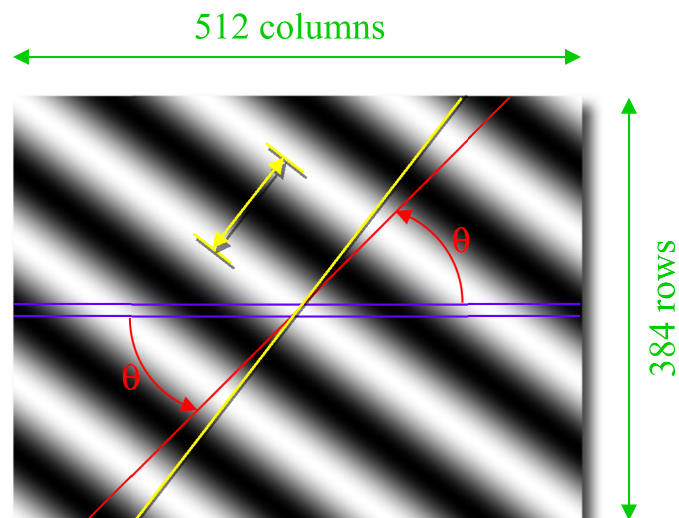


frequencies: $(u,v) = (0,3)$; wavelength: $\lambda_v = 128$

In the Fourier plane of a **square image**, the orientation of the line through the point pair = the orientation of the wave front in the image. **Not so for a non-square image.**

In the F plane the angle is -45° in this image it's about -53° (yellow line). That's because the fraction of R covered by one pixel is 4/3 the fraction of C covered by one pixel.





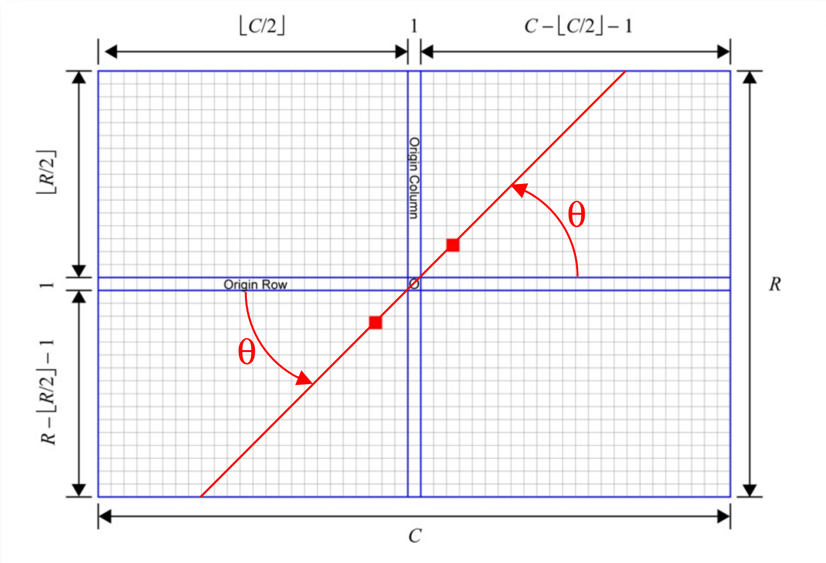
Also as a result, the wavelength is 213⅓.

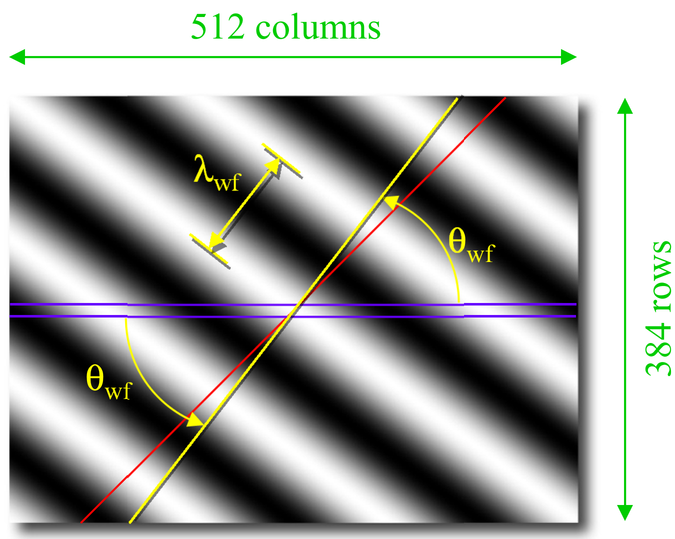
frequencies: $(u,v) = (3,3)$; wavelengths: $(\lambda_u, \lambda_v) = (170\tfrac{2}{3}, 128)$

In general the slope of the wavefront direction in the image is given by (v/R) / (u/C). Therefore its angle is

$$\theta_{\mathrm{wf}} = \tan^{-1}\left(\frac{vC}{uR}\right),$$

Fr… engths in the Fourier Plane



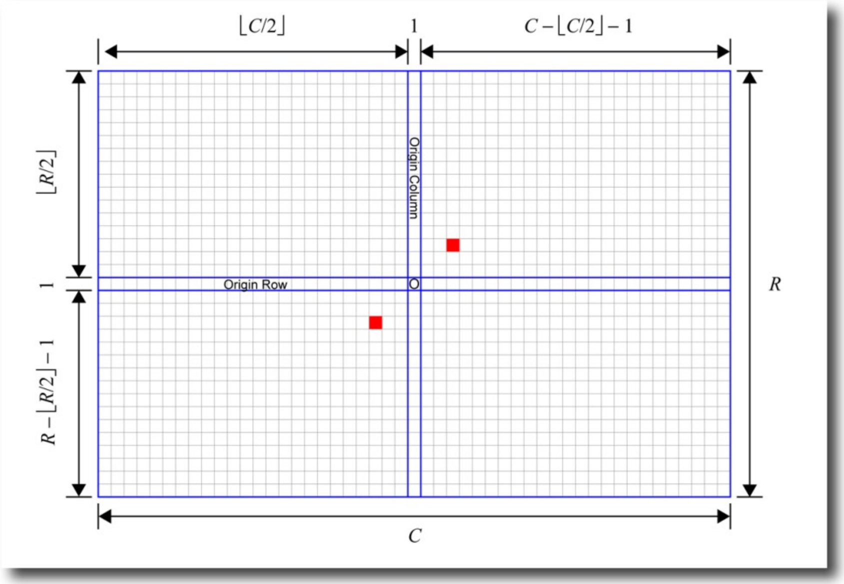


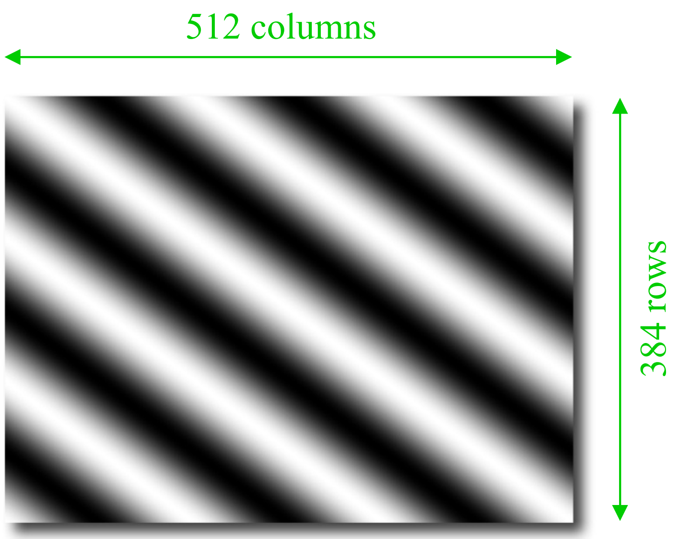
frequencies: $(u,v) = (3,3)$; wavelengt…

and the wavelength is:

$$\lambda_{\mathrm{wf}} = \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2},$$

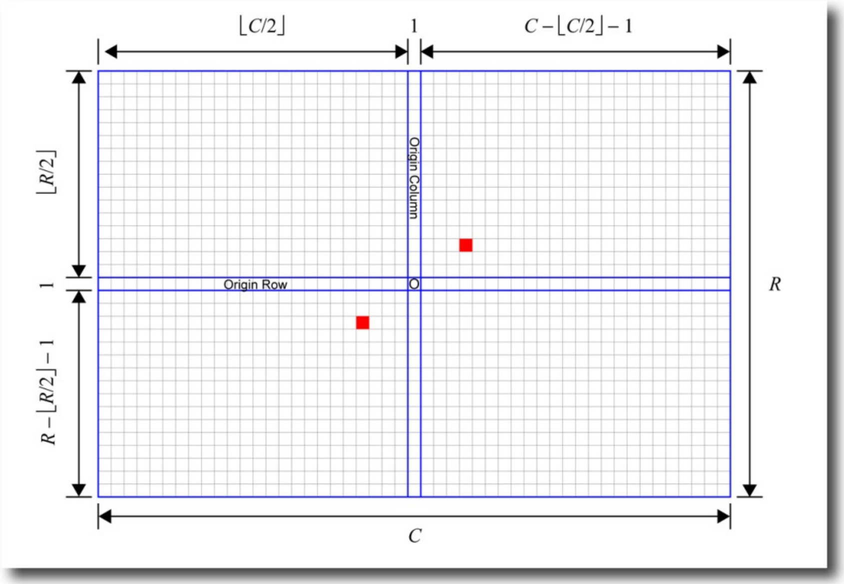
Frequencies and Wavelengths in the Fourier Plane

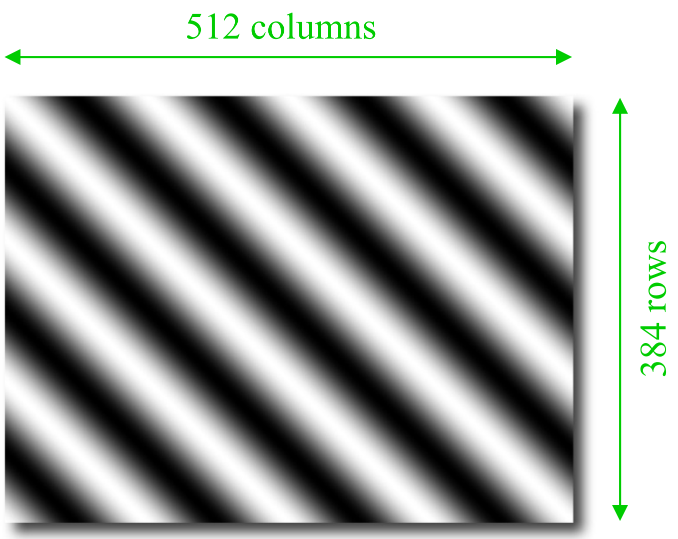




frequencies: $(u,v) = (3,3)$; wavelengths: $(\lambda_u, \lambda_v) = (170\frac{2}{3}, 128)$

Frequencies and Wavelengths in the Fourier Plane



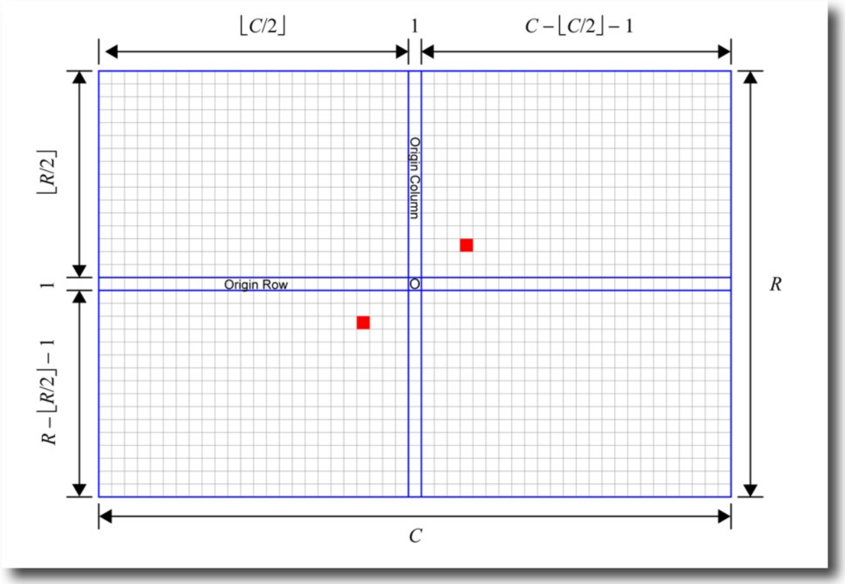


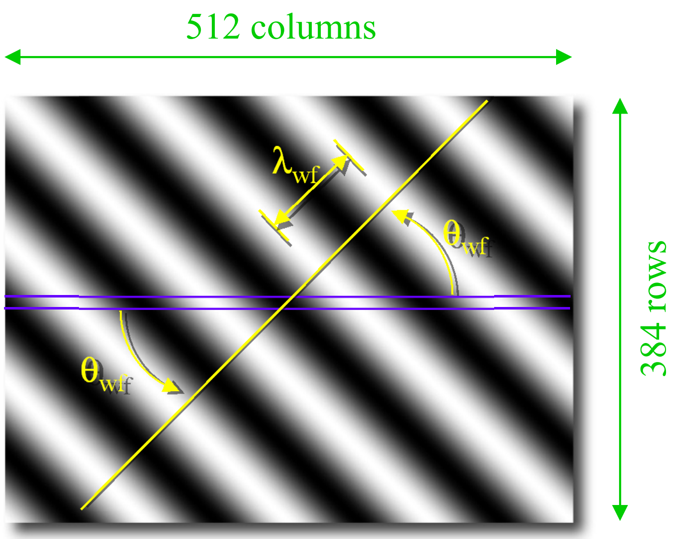
frequencies: $(u,v) = (4,3)$; wavelengths: $(\lambda_u, \lambda_v) = (128,128)$

The ratio R/C = ¾ in this image. Therefore at frequency (4,3) the wave front angle is

$$\theta_{wf} = \tan^{-1}\left(\frac{3\cdot 512}{4\cdot 384}\right) = \tan^{-1}\left(\frac{3\cdot 4}{4\cdot 3}\right) = \tan^{-1}(1) = 45^\circ,$$

Fr… Fourier Plane



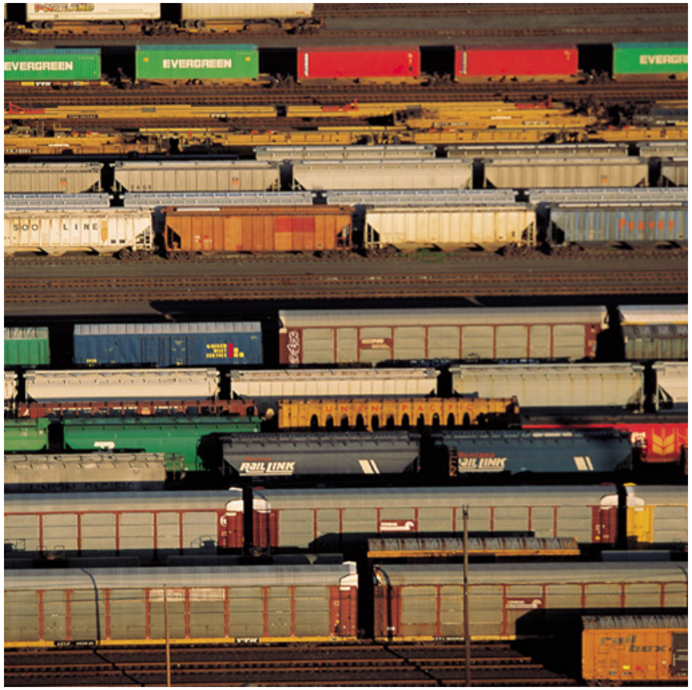


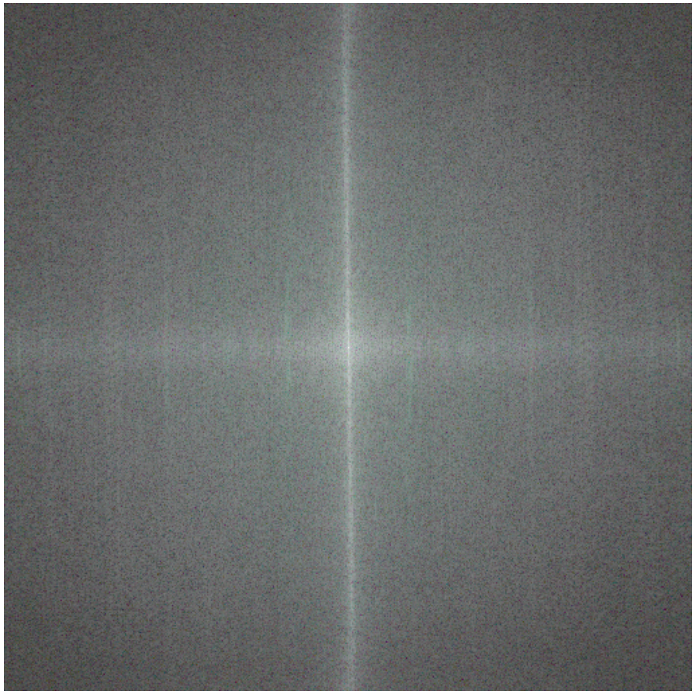
frequencies: $(u,v) = (4,$…

and the wavelength is

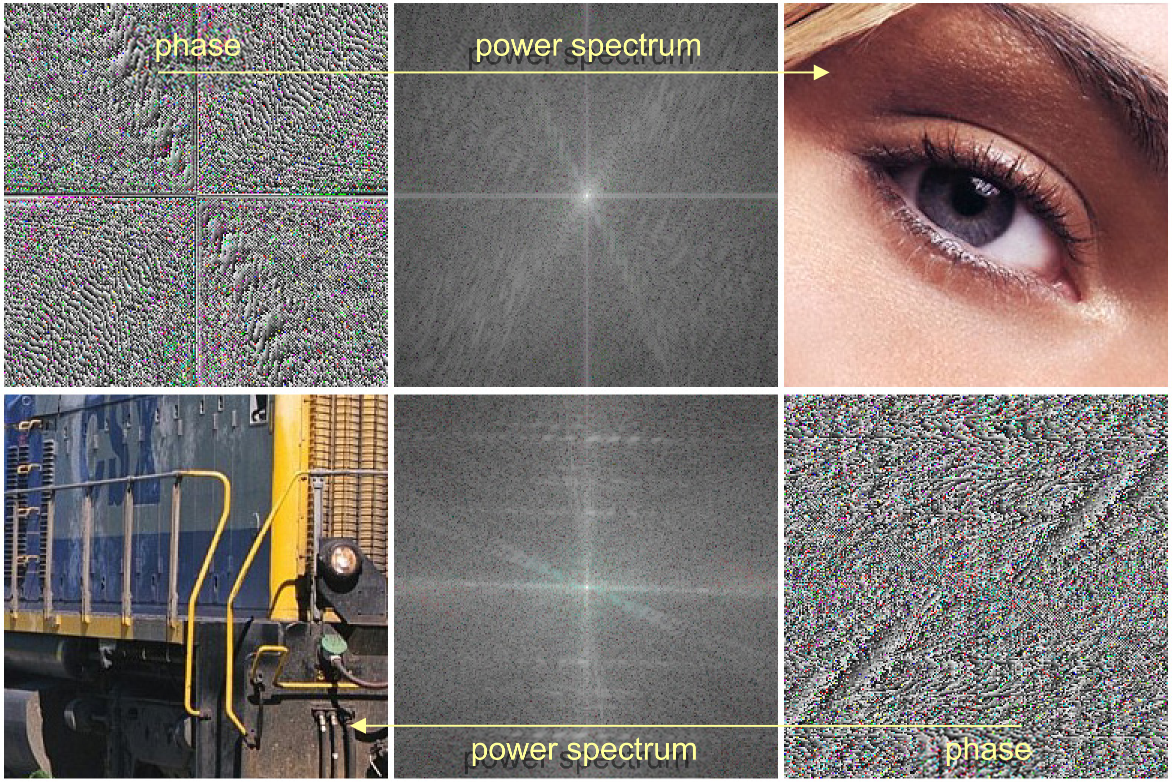
$$\lambda_{wf} = \sqrt{\left(\frac{512}{4}\right)^2 + \left(\frac{384}{3}\right)^2} = \sqrt{2\cdot 128^2} = 128\sqrt{2},$$

Power Spectrum of an Image



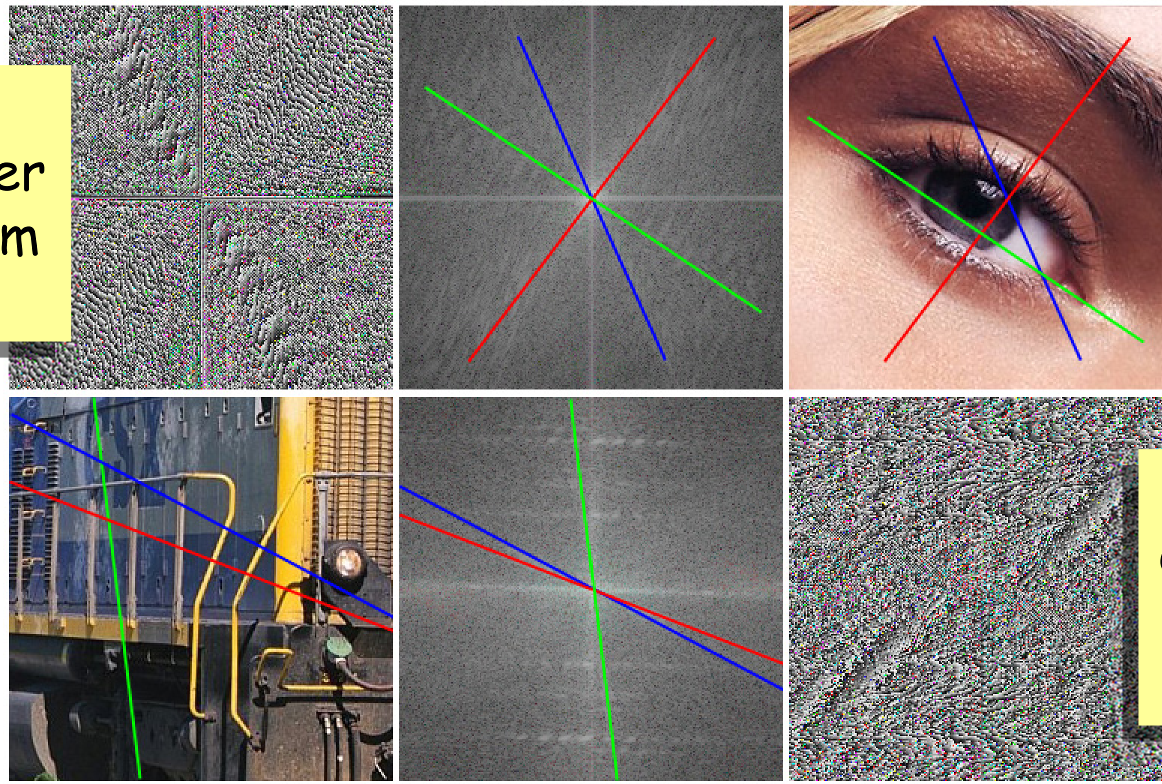


Relationship between Image and FT



Features in the FT and in the Image

Lines in the Power Spectrum are …



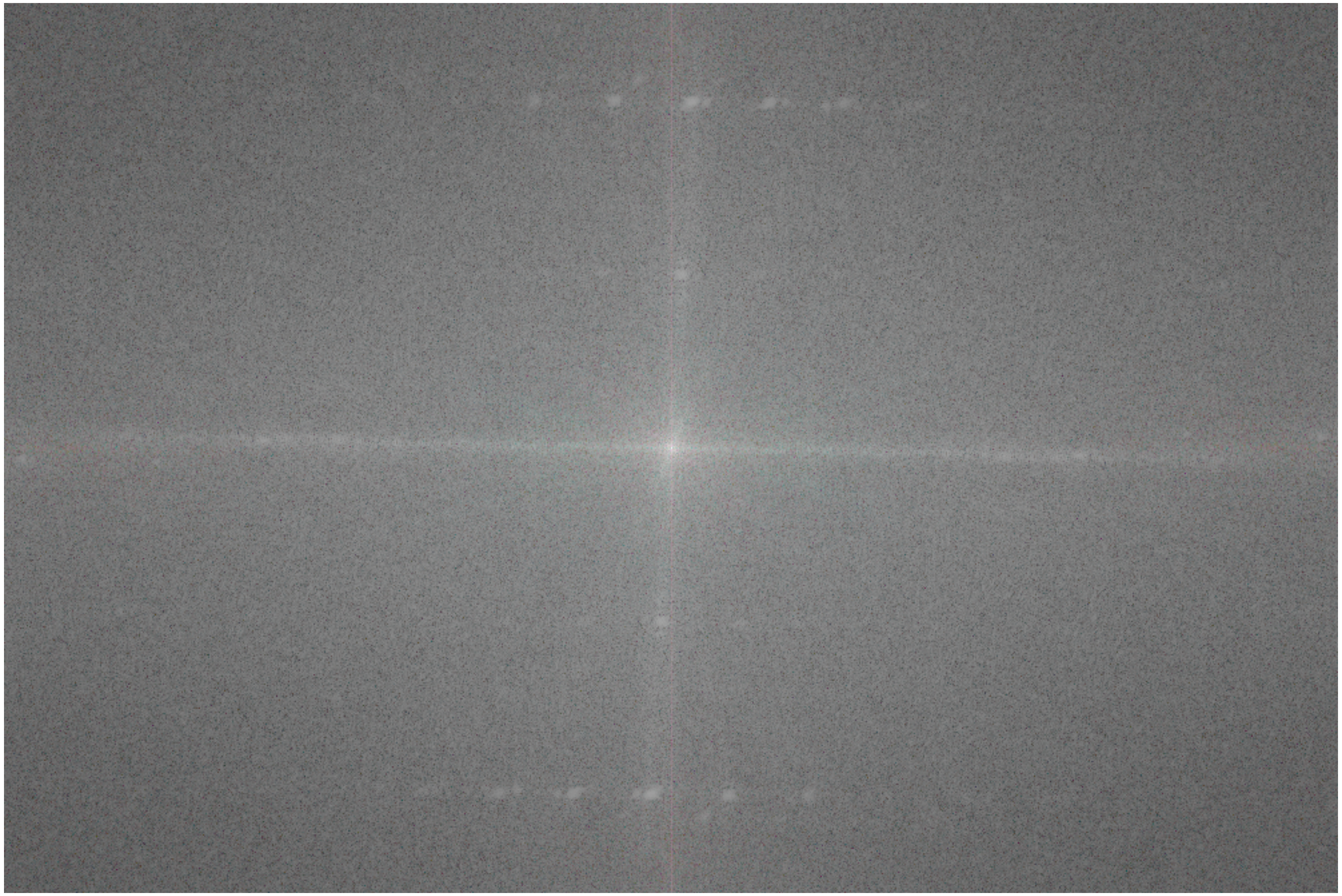
… perpen-dicular to lines in the image.

Fourier Magnitude and Phase

I

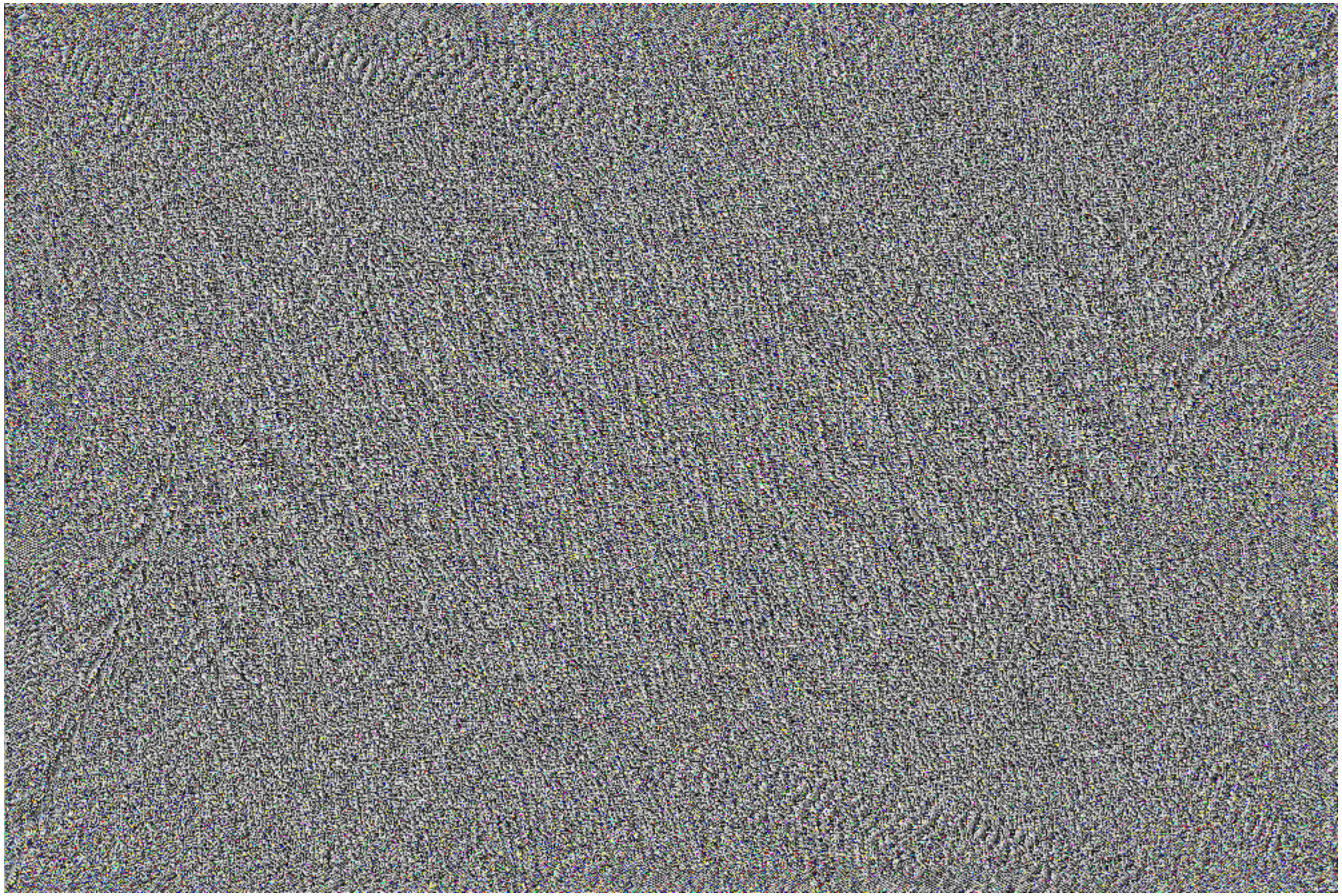
Fourier Magnitude

$\log\left|\mathfrak{F}\{\mathbf{I}\}\right|$



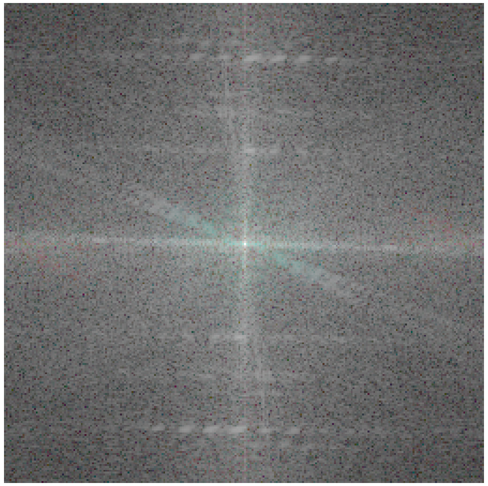
Fourier Phase

$\angle \mathfrak{F}\{\mathbf{I}\}$

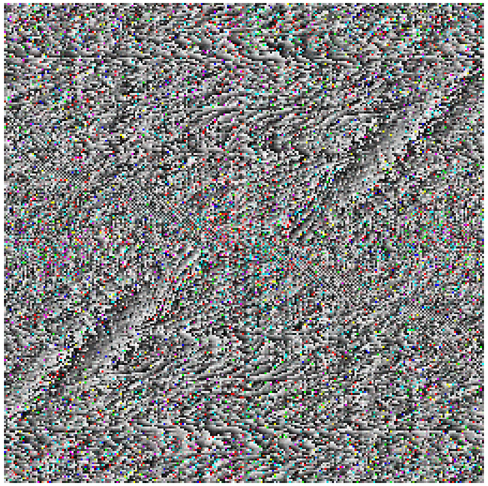


Q: Which contains more visually relevant information; magnitude or phase?

original image

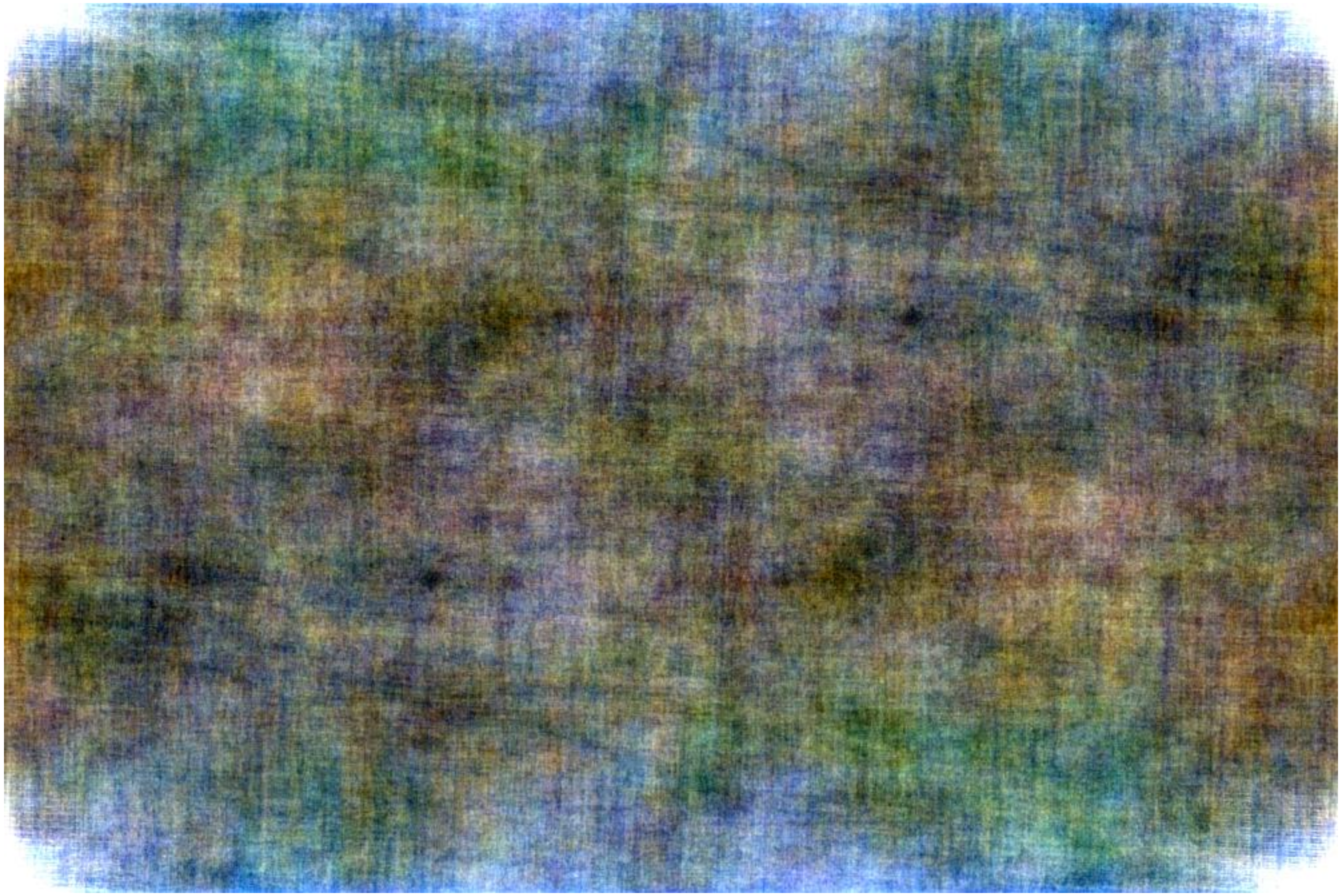


Fourier log magnitude



Fourier phase

Magnitude Only Reconstruction



Phase Only Reconstruction