Homework 4: The PCA and MUSIC algorithms

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***Abstract*— In this document, we present the solution for homework 4. Trying to follow the guidelines for writing papers, we develop each the requirements of the assignment on Sections II, III, and IV.**

I. INTRODUCTION

II. THEORY

We want to explain an observed variable x_i with a variable z_i through a Gaussian distribution. This time, we include a rotation, dilation and translation in the form

$$x_i, \leftarrow \boldsymbol{W} z_i + \boldsymbol{\mu}.$$

At the same time, with the matrix $\boldsymbol{W}$ we can discard the dimensions of a particular signal which do not pose a significant contribution to the composition of the signal. That is Principal Component Analysis: determine which dimensions (components) are truly important in the representation of a signal. There are two approach.

A. Basic PCA

The MUSIC power spectrum is given by:

$$S^{MUSIC}(k) = \frac{1}{\boldsymbol{e}_k^H R^{-1} \boldsymbol{e}_k},$$

which after discarding the signal eigenvectors to achieve perpendicularity and at the same time a singularity on the expression, we have

$$S^{MUSIC}(k) = \frac{1}{\boldsymbol{e}_k^H \boldsymbol{V}_n \boldsymbol{V}_n^H \boldsymbol{e}_k},$$

where $\boldsymbol{e}_k = [1 \exp{(j\omega_k)}, \ldots, \exp{(j(L-1)\omega_k)}]$. We embed the observed variable in $\boldsymbol{e}_k$.

To obtain an equivalent MUSIC power spectrum using the signal eigenvectors, and based on the descomposition of matrix $\boldsymbol{V} = [\boldsymbol{V}_s \boldsymbol{V}_n]$we start with the expression:

$$\boldsymbol{V}\boldsymbol{V}^H = \boldsymbol{V}_s \boldsymbol{V}_s^H + V_n \boldsymbol{V}_n^H = \boldsymbol{I},$$

where the dimensions of all the product matrices are the same. Solving for $V_n \boldsymbol{V}_n^H = \boldsymbol{I} - \boldsymbol{V}_s \boldsymbol{V}_s^H$, we obtain

$$S^{MUSIC\prime}(k) = \frac{1}{\boldsymbol{e}_k^H (\boldsymbol{I} - \boldsymbol{V}_s \boldsymbol{V}_s^H) \boldsymbol{e}_k},$$

The depiction of the two expression is presented in Fig. 1.

B. Probabilistic PCA

In contrast to the basic PCA, there is an iterative process based on the Expectation Maximization. The algorithm is

1) Initialize W.
2) $Z = (W^H W)^- 1 W^H X$.
3) $W_p = X Z^H (Z Z^H)^{-1}$.
4) $W_{pp} = W_p W_p^H R W_p$.
5) $W = \text{gramschmidt}(W_{pp})$.
6) $\Lambda = W^H R W$.
7) Go to 1).

where the function gramschmidt($*$) transform a nonsingular matrix (request also well-conditioned matrix) into an orthonormal form.

III. EXPERIMENTS

In a set of D antennas, a snapshot $x[n]$ is received. Our purpose is to find the angle of the source of such a signal. The code for the generation of the artificially generated data is given on the assignment. We need to generate 100 samples of the data where we embed the source angles $\theta_1 = \pi/2$, $\theta_2 = \pi$, and $\theta_3 = 3\pi/2$. The amplitude of the signals were chosen arbitrarily to be $A_1 = 5.2$, $A_2 = 3.5$, and $A_3 = 2.7$.

A. Basic PCA

There is a sequence of steps to implement the standard PCA algorithm and test it into the MUSIC algorithm: After generating the data as explained above, we compute the autocorrelation matrix of the data, all the eigenvectors and we compute the inverse of the power spectrum of the noise eigenvectors, which is depicted in Fig. 1(a).

B. Probabilistic PCA

We initialize $\boldsymbol{W}$ as an identity matrix for the algorithm described in Section II-B.

$$\lambda_{PPCA} = \begin{bmatrix} 4696.6967 + 0j \\ 3938.0821 + 3.4106e-13j \\ 3898.0878 + 2.2737e-13j \\ 1033.6755 + 1.4210e-14j \\ 1071.7745 + 1.4210e-14j \\ 296.8125 - 4.2632e-14j \\ 291.1196 + 7.10546e-14j \\ 4053.5474 - 1.1368e-13j \\ 1281.6756 + 4.2632e-14j \\ 510.0582 + 8.5265e-14j \end{bmatrix}$$

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In contrast, the eigenvalues reported from the basic PCA are

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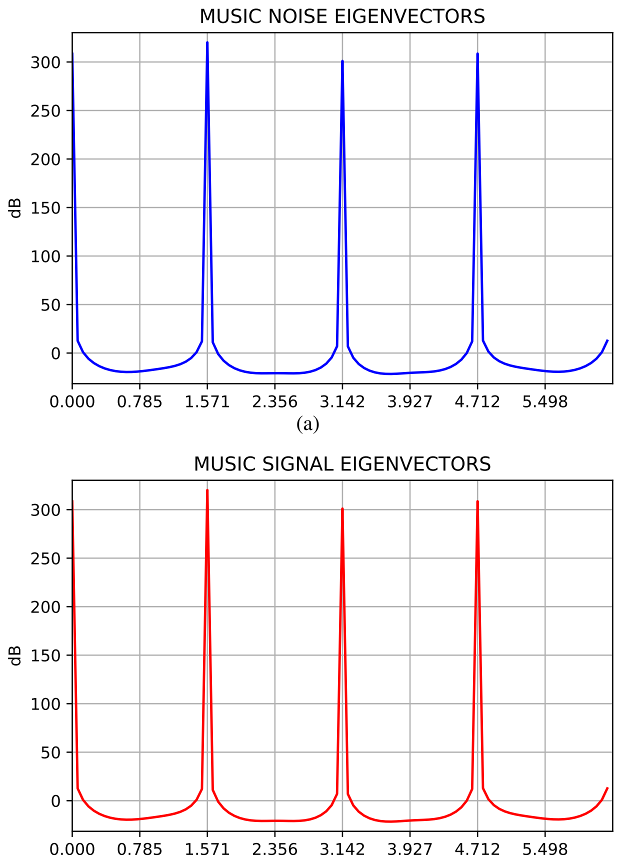

(a)

(b)

Fig. 1. MUSIC Power Spectrum using (a) noise eigenvectors, (b) signal eigenvectors.

$$\lambda_{signal} = \begin{bmatrix} 4709.4496 - 5.2921e-15j \\ 510.5888 - 2.16221e-14j \\ 211.3332 - 4.6848e-14j \\ 148.5890 + 8.6841e-14j \end{bmatrix}.$$

$$\lambda_{noise} = \begin{bmatrix} -2.9908e-13 + 1.4491e-14j \\ 9.3037e-14 - 5.2206e-14j \\ 7.3982e-14 + 3.2462e-14j \\ -7.4499e-14 - 1.9217e-14j \\ 2.5000e-14 - 1.9694e-15j \\ -3.5562e-14 + 2.38656e-14j \end{bmatrix}.$$

where evidently there is not a visible relationship between the eigenvalues reported from the basic PCA and the probabilistic approach.

IV. Conclusion

In this final assignment, we took the problem of detecting the origin of a signal using an array of antennas. The goal is to confront the two approaches of PCA: the basic PCA and the probabilistic PCA.

The results obtained on the probabilistic PCA are dubious. We are concerned with line 6) of the algorithm presented on Section II-B

$$\begin{aligned} \mathbf{\Lambda} &= \boldsymbol{W}^H \boldsymbol{R} \boldsymbol{W}, \\ &= \mathbf{\Lambda}^H \boldsymbol{V}^H \boldsymbol{V} \mathbf{\Lambda} \boldsymbol{V}^H \boldsymbol{V} \mathbf{\Lambda}, \\ &= \mathbf{\Lambda}\mathbf{\Lambda}\mathbf{\Lambda}?, \end{aligned}$$