UNIVERSITY OF COPENHAGEN
DEPARTMENT OF MATHEMATICAL SCIENCES

MSc Project Outside Course Scope

Regularity results for some fully non-linear elliptic PDE

Date: September 26, 2023

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Title and subtitle:	[Regularity results for some fully nonlinear elliptic PDE]
Description:	The purpose of this report is to establish Hölder regularity results for some fully nonlinear elliptic PDE, of second order. For this purpose, the viscosity solution is an essential construction, and we briefly introduce the appropriate function spaces that arise in this setting. We show the Harnack inequality and the Evans-Krylov theorem in this setting, and then we move on to solving the Dirichlet problem for concave equations with constant coefficients.
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Date:	01-05-2023

Table of content

1 Introduction and Viscosity Solutions

We shall consider equations of the form

$$F(D^2u, x) = f(x),$$

where u and f are defined on a bounded domain $\Omega \subseteq \mathbb{R}^n$, and $F : \mathcal{S} \times \Omega \to \mathbb{R}$ is a uniformly elliptic operator, meaning that there exists positive constants $\lambda \leq \Lambda$ such that

$$\lambda\|N\| \leq F(M+N, x) - F(M, x) \leq \Lambda\|N\| \qquad \forall M \in \mathcal{S} \quad \forall N \geq 0 \quad \forall x \in \Omega.$$

Here $\mathcal{S}$ is the space of symmetric $n \times n$ matrices, $N \geq 0$ means that $N \in \mathcal{S}$ is non-negative symmetric, and $\|M\| = \sup_{|z|\leq 1} |Mz|$ is the usual operator norm on matrices. Throughout this report we shall only consider F and f to be at least continuous in their domains. The simplest class of examples of uniformly elliptic operators are affine operators $Lu = a_{ij}u_{ij} + c$. The ellipticity condition is guaranteed when the matrices $(a_{ij}(x))$ have a uniform lower and upper bound on the eigenvalues. Another useful class of equations are given by Pucci's extremal operators $\mathcal{M}^+$ and $\mathcal{M}^-$, which are defined as:

$$\mathcal{M}^+(M) = \mathcal{M}^+(M, \lambda, \Lambda) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

and

$$\mathcal{M}^-(M) = \mathcal{M}^-(M, \lambda, \Lambda) := \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i.$$

Here, $e_i = e_i(M)$ are the eigenvalues of M. A simple diagonalization argument (see p.15 in [1]) yields that $\mathcal{M}^+(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} L_A M$ and $\mathcal{M}^-(M) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} L_A M$, where $\mathcal{A}_{\lambda,\Lambda} \subset \mathcal{S}$ is the space of symmetric matrices with eigenvalues in $[\lambda, \Lambda]$, and $L_A M = \operatorname{tr}(AM)$ is the corresponding linear functional. We now define the so-called viscosity solutions, which turn out to be the correct notion of weak solution for fully nonlinear elliptic equations:

Definition 1. $u \in C(\Omega)$ is a viscosity subsolution [resp. supersolution] of $F(D^2u, x) = f(x)$ in Ω if for all $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ and $u - \varphi$ has a local maximum [resp. minimum] at x_0 then

$$F(D^2\varphi(x_0), x_0) \geq f(x_0) \qquad\qquad [\text{resp. } F(D^2\varphi(x_0, x_0) \leq f(x_0)].$$

Additionally, u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

The basic idea for this definition is to put the derivatives on the test function φ via the maximum principle. Indeed, using sup and inf convolutions (see p. 58 in [2]) one can demonstrate that a comparison principle (and thus also a weak maximum principle) is built into the above definition. We skip these details. It is easy to see that u is a viscosity subsolution if and only if $F(D^2\varphi(x_0), x_0) \geq 0$ for all x_0 and $\varphi \in C^2(\Omega)$ that touches u by above at x_0 (and similarly for supersolutions). Furthermore, a simple approximation argument (see Proposition 2.4 in [1]) shows that $\varphi \in C^2(\Omega)$ may always be assumed to be a paraboloid (if convenient). Now, in order to construct a regularity theory for fully nonlinear equations we need an appropriate notion of function spaces to work in. This is done using Pucci's extremal operators that are defined above.

Definition 2. We define the class of subsolutions $\underline{S}(f) = \underline{S}(\lambda, \Lambda, f)$ as the continuous functions such that $\mathcal{M}^+(D^2u, \lambda, \Lambda) \geq f(x)$ in the viscosity sense in Ω. Similarly, we define the class of supersolutions $\overline{S}(f) = \overline{S}(f, \lambda, \Lambda)$ as the continuous functions such that $\mathcal{M}^-(D^2u, \lambda, \Lambda) \leq f(x)$ in Ω. Finally we define $S(f) = \underline{S}(f) \cap \overline{S}(f)$ and $S^*(f) = \underline{S}(-|f|) \cap \overline{S}(|f|)$.

We also use the shorthand $S(0) = S$ and similarly for sub- and supersolutions. Working specifically with these function spaces turns out to be very advantageous, as they have many useful properties. We shall skip the details here (see p. 15-16 in [1]), but one of the main benefits is that both $\underline{S}$, $\overline{S}$, and S are closed under uniform limits in compact sets. Now, to illustrate why S may be of use when studying regularity, suppose that we have $u \in C^2(\Omega) \cap S$. By picking $\varphi = u$ as the test function and evaluating at each point, we get from the definition of Pucci's extremal operators that there exists $A = (a_{ij}(x))$ with eigenvalues in $[\lambda, \Lambda]$ such that $a_{ij}(x)u_{ij} = 0$ (The entries in A might not be continuous in x). This means that sufficiently regular elements of S are in fact classical solutions to some linear uniformly elliptic operator in nondivergence form. Furthermore, we have the following result which implies that S also contains all viscosity solutions to fully nonlinear equations:

Proposition 1.1. *let u satisfy $F(D^2u, x) \geq f(x)$ in the viscosity sense in Ω. Then for any $\varphi \in C^2(\Omega)$ we have*

$$u - \phi \in \underline{S}(\lambda/n, \Lambda, f(x) - F(D^2\varphi(x), x)).$$

A similar result holds for supersolutions.

Proof. Omitted, see Proposition 2.13 in [1]. □

In particular, taking $\varphi = 0$ we obtain a suitable class S to place viscosity solutions of some general equation. This means that it suffices to obtain regularity results S (in particular we avoid having to linearize the equation $F(D^2u, x) = 0$). We state without proof a general uniqueness result for viscosity solutions (this is Theorem 3.1 in [3])

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and $u, v \in C(\overline{\Omega})$ be viscosity solutions of*

$$F(D^2u, x) = 0 \quad in\ \Omega,$$

where F is uniformly elliptic in the space of symmetric matrices $\mathcal{S}$. If $u = v$ on $\partial\Omega$ then $u = v$ on Ω.

Actually the result in [3] is more general, since it shows existence for an even more general class of equations (which are not considered in this report). Now, we also state the following result which can be proved using Proposition 1.1 and the fact that S is closed under uniform limits in compact sets (see p. 49 in [1]). Recall that $F : \mathcal{S} \times \Omega \to \mathbb{R}$ is concave in the space $\mathcal{S}$ of symmetric matrices if for all $M, N \in \mathcal{S}$ and $x \in \Omega$, and any $\alpha \in [0, 1]$ we have that:

$$F(\alpha M + (1-\alpha)N, x) \geq \alpha F(M, x) + (1-\alpha)F(N, x).$$

Proposition 1.3. *Assume that $F(D^2u) = 0$ where $u \in C^2(\Omega)$ and F is concave as above. Then for any $e \in \partial B_1$ we have that $u_{ee} \in \underline{S}(\lambda/n, \Lambda, 0)$ in Ω.*

2 Krylov-Safonov Harnack Inequality

With the above expository overview in place, we can start to do some analysis, and here present the Harnack inequality for the class of viscosity solutions S. Such an inequality is important for studying regularity of second order PDE, and in this case where we consider a very general class of equations it is very difficult to prove. In brief, using the ABP-estimate on the class S (see Theorem 3.2 in [**?**]) one can obtain an unnormalized Harnack inequality in cubical domains. Recall that $Q_r(x) := B_r(x)_{\|\cdot\|_\infty} = \{y \in \mathbb{R}^n \mid \|x-y\|_\infty < r\}$, where we shall omit specifying the centerpoint if $x = 0$.

Lemma 2.1. *Suppose that $u \in S^*(f) \cap C(\overline{Q}_{4\sqrt{n}})$ satisfies $u \geq 0$ in $Q_{4\sqrt{n}}$ and $\inf_{Q_{1/4}} u \leq 1$. Then there exists universal constants $\varepsilon_0 > 0$ and $C > 0$ such that $\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0$ implies $\sup_{Q_{1/4}} u \leq C$.*

Proof. Omitted, see ch. 4 in [1]. □

We can then obtain something more akin to the standard Harnack inequality by a rescaling argument:

Theorem 2.2. *Suppose that $u \in S^*(f)$ satisfies $u \geq 0$ in Q_1. Then there exists a universal constant $C > 0$ such that*

$$\sup_{Q_{1/2}} u \leq C(\inf_{Q_{1/2}} u + \|f\|_{L^n(Q_1)})$$

Proof. First suppose that $u \in S^*(f) \cap C(\overline{Q}_{4\sqrt{n}})$ and $u \geq 0$ in $Q_{4\sqrt{n}}$. For any $\delta > 0$, the modified function $u_\delta := u/(\inf_{Q_{1/4}} u + \delta + \varepsilon_0^{-1}\|f\|_{L^n(Q_{4\sqrt{n}})}) \in S^*(f_\delta)$, where f_δ is defined by the same rescaling, satisfy all the conditions in Lemma 2.1, so after taking the limit $\delta \to 0^+$ we get:

$$\sup_{Q_{1/4}} u \leq C(\inf_{Q_{1/4}} u + \|f\|_{L^n(Q_{4\sqrt{n}})}).$$

The final estimate will now follow from a standard covering procedure: Let $H \subseteq \widetilde{H} \subseteq Q_1$ be open sets, such that we have N cubes $Q_\rho^1, \ldots, Q_\rho^N$ of fixed radius (and with center-points $x_1, \ldots, x_N$, which are omitted for brevity) satisfying $H \subseteq Q_\rho^1 \cup \cdots \cup Q_\rho^N$ and $Q_{R\rho}^1 \cup \cdots \cup Q_{R\rho}^N \subseteq \widetilde{H}$, with $R > 0$ being some constant. Suppose furthermore that no two cubes are disjoint. Rescaling the above inequality we get that

$$\sup_{Q_\rho^i} u \leq C(\inf_{Q_\rho^i} u + \rho\|f\|_{L^n(Q_{R\rho})}),$$

for all $1 \leq i \leq N$, with $\rho > 0$ is sufficiently small, and $R = 16\sqrt{n}$ in this particular case. Additionally we have that

$$\inf_{Q_\rho^j} u \leq \inf_{Q_\rho^j \cap Q_\rho^{j+1}} u \leq \sup_{Q_\rho^j \cap Q_\rho^{j+1}} u \leq \sup_{Q_\rho^{j+1}} u,$$

for all $1 \leq j \leq N-1$. Finally, after a possible relabelling there exists $i_1 \leq i_2$ such that $\sup_H u = \sup_{Q_\rho^{i_1}}$ and $\inf_{Q_\rho^{i_2}} u = \inf_H u$. Using these inequalities repeatedly we get that

$$\sup_H u \leq C(\inf_H u + \rho\|f\|_{L^n(\widetilde{H})}),$$

where the constant C now also depends on N (i.e. on the covering of H). Now, selecting $H = Q_{1/2}$ and $\widetilde{H} = Q_1$ we get the desired result. □

The covering argument used in this proof is rather standard, and will from now on be applied in further proofs without much elaboration. Note that it immediately allows us to formulate the Harnack inequality in terms of balls $B_{1/2}$ and B_1 (properly scaled), as well as most other reasonable sets $H \subseteq \widetilde{H} \subseteq \Omega$. A slight modification of this can also be applied to general L^p-norms, for instance in the weak Harnack inequality:

Theorem 2.3. *(1) Let $u \in \overline{S}(f)$ in Q_1 satisfy $u \geq 0$ in Q_1, where f is continuous and bounded. Then there exist universal constants $p_0 > 0$ and $C > 0$ such that*

$$\|u\|_{L^{p_0}(Q_{1/4})} \leq C(\inf_{Q_{1/2}} u + \|f\|_{L^n(Q_1)}).$$

(2) Let $u \in \underline{S}(f)$ in Q_1, where f is continuous and bounded in Q_1. Then for any $p > 0$ there

exists a constant $C(p) > 0$ such that

$$\sup_{Q_{1/2}} u \leq C(p)(\|u^+\|_{L^p(Q_{3/4})} + \|f\|_{L^n(Q_1)}).$$

Proof. Omitted, see Theorem 4.8 in [**?**]. $\square$

2.1 Hölder Continuity

The purpose of this section is to apply the Harnack inequality to obtain C^α regularity estimates for viscosity solutions. We begin by showing a bound for the oscillation decay, which leads to a universal interior Hölder continuity estimate.

Theorem 2.4. *For any $u \in S^*(f)$ in Q_1 we have that*

$$\text{osc}_{Q_{1/2}} u \leq \gamma \ \text{osc}_{Q_1} u + 2\|f\|_{L^n(Q_1)},$$

where $\gamma < 1$ is a universal constant. Furthermore, $u \in C^\alpha(\overline{Q}_{1/2})$ with

$$\|u\|_{C^\alpha(\overline{Q}_{1/2})} \leq \widetilde{C}(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^n(Q_1)}),$$

with $\alpha \in (0, 1)$ $\widetilde{C} > 0$ both being universal.

Proof. The first part is rather simple: writing $m_r = \inf_{Q_r} u$ and $M_r = \sup_{Q_r} u$, we directly apply the Harnack inequality (Theorem 2.2) to the nonnegative functions $u - m_1, M_1 - u$ (observe that both of these functions are translations, and therefore remain in S^*), and get

$$M_{1/2} - m_1 \leq C(m_{1/2} - m_1 + \|f\|_{L^n(Q_1)}),$$

and

$$M_1 - m_{1/2} \leq C(M_1 - M_{1/2} + \|f\|_{L^n(Q_1)}).$$

Adding both inequalities, rearranging, and using that $2C/(C + 1) \leq 1$ yields

$$\text{osc}_{Q_{1/2}} u \leq \frac{C-1}{C+1} \text{osc}_{Q_1} u + 2\|f\|_{L^n(Q_1)},$$

which proves the first part, with $\gamma = C/(C + 1)$. Now, by rescaling the above inequality we get that

$$osc_{Q_{r/2}} u \leq \gamma \, \text{osc}_{Q_r} u + 2r^{(2n-1)/n}\|f\|_{L^n(Q_1)} \leq \text{osc}_{Q_r} u + 2r\|f\|_{L^n(Q_1)},$$

for all $r \leq 1$. By Lemma 8.23 in [4] we get that

$$\operatorname{osc}_{Q_r} u \leq \widetilde{C}(r^\alpha \operatorname{osc}_{Q_1} u + r^\mu \|f\|_{L^n(Q_1)}),$$

for all $r \leq 1$ and $\mu \in (0,1)$, where $\widetilde{C} > 0$ and α are universal constants, in the sense that they do not depend on r. Now let $x, y \in \overline{Q}_{1/2}$ be arbitrary with $r = \|x - y\|$ and write $\widetilde{\alpha} = \min(\alpha, \mu) < 1$. Picking any $\mu \in (0,1)$ the above inequality yields

$$\frac{u(x) - u(y)}{\|x - y\|^{\widetilde{\alpha}}} \leq \widetilde{C}(r^{\alpha - \widetilde{\alpha}} \operatorname{osc}_{Q_1} u + r^{\mu - \widetilde{\alpha}} \|f\|_{L^n(Q_1)}) \leq \widetilde{C}(\operatorname{osc}_{Q_1} u + \|f\|_{L^n(Q_1)}),$$

which was what we wanted. □

Using the maximum principle one can obtain a rough estimate for Hölder continuity at the boundary. It is then standard to combine this with the interior estimate above to obtain a global estimate up to the boundary (see Propositions 4.12 and 4.13 in [1]).

3 Evans-Krylov Theorem

In this section we present and prove the Evans-Krylov theorem. The importance of this theorem is that it will allow us to solve the Dirichlet problem for certain fully nonlinear elliptic equations by the method of continuity.

Theorem 3.1. *Assume that $F(D^2u) = 0$ where $u \in C^2(B_1)$ and F is uniformly elliptic and concave. Then $u \in C^{2,\alpha}(B_{1/2})$ for some universal $\alpha \in (0,1)$ and we have the estimate:*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{C^{1,1}(B_{3/4})},$$

where $C > 0$ is also universal.

Note that the theorem is stated in terms of classical solutions rather than in terms of viscosity solutions, however, when it comes to solving the Dirichlet problem this is actually no loss of generality: Using this estimate we will eventually be able to use the method of continuity, rendering classical solutions. By the uniqueness of viscosity solutions we therefore get the same result for viscosity solutions. While this is sort of trivial, it turns out to be useful in further regularity theory. Now, we have to prove an interior Hölder estimate on the second derivatives, and proceed similarly as in Theorem 2.4. The main lemma towards this is the following:

Lemma 3.2. *Under the hypothesis of Theorem 3.1, there exists a universal $\delta \in (0,1)$ such that*

$$\operatorname{diam} D^2u(B_\delta) \leq \operatorname{diam} D^2u(B_1)/2.$$

Theorem 3.1 follows easily from this lemma. Indeed, writing $w(r) = \operatorname{diam} D^2u(B_r)$ and rescaling the above inequality we see that $w(\delta r) \leq w(r)/2$ for all $r \leq 1$. By Lemma 8.23 in [4] we get that there exists $\alpha \in (0,1)$ such that $w(r) \leq Cr^\alpha w(3/4)$. So, proceeding as in the proof of Theorem 2.4, for $x, y \in \overline{B}_{1/2}$ arbitrary with $r = \|x-y\|$ we have that

$$\frac{|D^2u(x) - D^2u(y)|}{\|x-y\|^\alpha} \leq C \operatorname{diam} D^2u(B_{3/4}) \leq C\|D^2u\|_{L^\infty(B_{3/4})},$$

which proves the theorem. The radius $3/4 \leq 1$ is arbitrarily chosen such that the right-hand side is also an interior quantity. Now, to prove the lemma we use Proposition 1.3 as well as some auxilliary results:

Lemma 3.3. *Let $v \in \overline{S}(0)$ in B_1 satisfy $v \geq 0$ in B_1. Then there exists universal constants $C > 0$ and $\delta > 0$ such that*

$$\inf_{B_{1/2}} v \geq C|\{v \geq 1\} \cap B_{1/4}|^\delta$$

Proof. Rescaling the weak Harnack inequality in Theorem 2.3 for balls we have $\|v\|_{L^{p_0}(B_{1/4})} \leq C \inf_{B_{1/2}} v$, where p_0 and C are universal constants. Writing $\delta = 1/p_0$ we then get the desired result:

$$|\{v \geq 1\} \cap B_{1/4}|^\delta \leq \|v\|_{L^{p_0}(B_{1/4} \cap \{v \geq 1\})} \leq \|v\|_{L^{p_0}(B_{1/4})} \leq C \inf_{B_{1/2}} v.$$

□

Lemma 3.4. *If $F(M_1) = F(M_2) = 0$ then*

$$c_0 \|M_2 - M_1\| \leq \|(M_2 - M_1)^+\| = \sup_{e \in \partial B_1} (e^t (M_2 - M_1) e)^+,$$

where $c_0 = \lambda/(\lambda + \Lambda)$ only depends on the ellipticity constants of F.

The proof of this follows directly from the definition of uniform ellipticity of F, and diagonalization of $M_2 - M_1$ (see Lemma 6.4 in [1]).

Lemma 3.5. *Under the hypothesis of Theorem 3.1, suppose that* $\operatorname{diam} D^2u(B_1) \in (1, 2]$*, and $D^2u(B_1)$ is covered by balls $B^1, \ldots, B^N \subset \mathcal{S}$ of fixed radius (in the space $\mathcal{S}$ of symmetric matrices) $\varepsilon \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is a universal constant to be chosen later. Then we can remove one ball so that the remaining union still covers $D^2u(B_{1/2})$.*

Proof. The first step is to specialize the covering. Let c_0 be as in Lemma 3.4. For all $i = 1, \ldots, N$ there exists $x_i \in B_1$ such that $B^i \subseteq B_{2\varepsilon}(M_i)$, where $M_i = D^2u(x_i) \in \mathcal{S}$. So, with ε_0 such that $2\varepsilon_0 \leq c_0/16$ we have that

$$D^2u(B_1) \subseteq \bigcup_{i=1}^{N} B_{c_0/16}(M_i).$$

We can choose a subcollection $(M_i)_{i=1}^M$ of centers with $M \leq N$ universal such that

$$D^2u(B_1) \subseteq \bigcup_{i=1}^{M} B_{c_0/8}(M_i),$$

and after taking preimages and relabelling we get that $|(D^2u)^{-1}(B_{c_0/8}(M_1)) \cap B_{1/4}| \geq \eta$, where $\eta > 0$ is universal. Now, taking ε_0 such that $2\varepsilon_0 \leq 1/4$ and relabelling we get that $\|M_1 - M_2\| \geq 1/4$. Since $F(M_1) = F(M_2) = 0$ we have from Lemma 3.4 that there exists $e \in \partial B_1$ such that

$$0 < c_0/4 \leq e^t (M_2 - M_1) e = u_{ee}(x_2) - u_{ee}(x_1).$$

Writing $K = \sup_{B_1} u_{ee}$ and $v = K - u_{ee}$ we have that $v \in \overline{S}(\lambda/n, \Lambda, 0)$ by Lemma 1.3, and $v \geq 0$ in B_1. Furthermore we have $|\{v \geq c_0/8\} \cap B_{1/4}| \geq \eta$ by construction, so from Lemma 3.3 there exists a universal $C > 0$ such that

$$C \leq \inf_{B_{1/2}} (K - u_{ee}).$$

From the definition of K and the covering above, we get that there exists $1 \leq j \leq N$ such that $K - u_{ee}(x_j) < 3\varepsilon$. Taking ε_0 such that $5\varepsilon_0 \leq C$ and combining these observations we get that $D^2u(B_{1/2}) \cap B_{2\varepsilon}(M_j) = \emptyset$, which finishes the proof. $\square$

Proof of Lemma 3.2. We can assume $\operatorname{diam} D^2u(B_1) = 2$ without any loss of generality, since the transformation $w = u/(\operatorname{diam} D^2u(B_1))$ solves the equation $F^t(D^2u) := t^{-1}F(tD^2u) = 0$, and the operator F^t has the same ellipticity constants. From the previous lemma we can then cover $D^2u(B_1)$ with N balls of universal radius ε_0, and then apply the lemma such that $D^2u(B_{1/2})$ is covered by $N-1$ balls of radius ε_0. If $\operatorname{diam} D^2u(B_{1/2}) \leq 1$ then we are done. If not, defining $w(x) = 4u(x/2)$ on B_1 and applying the previous lemma to $D^2w(B_1) = D^2u(B_{1/2})$, we get that $D^2u(B_{1/4})$ is covered by $N-2$ balls of radius ε_0. Continuing this process finitely many times (since $N < \infty$) we conclude that there exists $k \leq N$ such that $\operatorname{diam} D^2u(B_{1/2^k}) \leq 1$. In particular this means that

$$\operatorname{diam} D^2u(B_{1/2^N}) \leq \operatorname{diam} D^2u(B_{1/2^k}) \leq 1,$$

where $\delta = 1/2^N$ is a universal constant. $\square$

4 The Dirichlet Problem for Concave Equations

4.1 Linearization and Standard Regularity Theory

The purpose of this section is to show that a $C^{2,\alpha}$ estimate is enough to apply standard regularity estimates to fully nonlinear elliptic equations, as long as $F \in C^\infty(\mathcal{S} \times \mathbb{R})$. This works inductively by using the classical Schauder estimates for linear operators. In this setting, we can also refine the estimate in Theorem 3.1 to only depend on the L^∞ norm of u. This is done using a clever trick (known as Bernstein's technique) involving a particular type of auxiliary function.

Proposition 4.1. *Let $\alpha \in (0,1)$ and suppose that $u \in C^{2,\alpha}$ is a solution of*

$$F(D^2u, x) = f(x)$$

for $x \in \Omega$. If $F \in C^\infty(\mathcal{S} \times \Omega)$ and $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

The proof is found in chapter 9 of [1], and is omitted here.

Lemma 4.2. *Suppose that $F \in C^\infty(\mathcal{S})$ is concave and satisfies $F(0) = 0$, and let $u \in C^4(\Omega)$ satisfy $F(D^2u) = 0$ in Ω. Consider the linearized operator $Lv = a_{ij}(x)v_{ij}$, where $a_{ij} = F_{ij}(D^2u(x))$. For any $e \in \partial B_1$, we have that $Lu \leq 0$, $Lu_e = 0$, and $Lu_{ee} \geq 0$ in Ω.*

The proof of Lemma 4.2 follows directly from differentiating the equation. Indeed, considering the function $\psi(t) = F((1-t)D^2u(x))$ we see that $\psi(0) = \psi(1) = 0$. Concavity of F then implies that $\psi(t) \geq 0$ for all $t \in [0,1]$. In particular this means that

$$0 \leq \psi'(0) = F_{ij}(D^2u(x))(-u_{ij}(x)) = -Lu.$$

Differentiating the equation $F(D^2u(x)) = 0$ with respect to e twice yields the rest of the lemma. Now, using a special case of Lemma 4.2, as well as a cleverly chosen auxiliary function, we can obtain $C^{1,1}$-estimates of u:

Proposition 4.3. *Suppose that $F \in C^\infty(\mathcal{S})$ is concave and satisfy $F(0) = 0$, and that $u \in C^4(\overline{B}_1)$ satisfy $F(D^2u) = 0$ in B_1. Then there exists $C > 0$ universal such that*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)},$$
$$\|D^2u\|_{L^\infty(B_{1/2})} \leq C\|\nabla u\|_{L^\infty(B_1)}.$$

Proof. Let $\varphi \in C^\infty(\mathbb{R})$ be a bump-function chosen such that $\varphi \equiv 1$ and in $B_{1/2}$, and $0 \leq \varphi \leq 1$ in B_1, and $\varphi|_{\partial B_1} \equiv 0$. In particular this means that $0 \leq \varphi + |\nabla\varphi| + \|D^2\varphi\| \leq C$ for some (universal) $C > 0$. Let $M = \sup_{\overline{B}_1} u$, and consider the auxiliary function

$$h = \delta(M-u)^2 + \varphi^2|\nabla u|^2 \in C^3(\overline{B}_1),$$

where $\delta > 0$ is a constant to be chosen later. Let L be the operator from Lemma 4.2. It follows from the product rule that

$$\begin{aligned}Lh = 2\delta(M-u)(-Lu) &+ 2\delta a_{ij}u_iu_j + |\nabla u|^2 L(\varphi^2) \\ &+ 2\varphi^2 u_k Lu_k + 2\varphi^2 a_{ij}u_{ki}u_{kj} + 8a_{ij}\varphi\varphi_i u_k u_{kj}.\end{aligned}$$

We have that $Lu \le 0$ and $Lu_k = 0$ by Lemma 4.2. The rightmost term can be expressed as $\langle 8\nabla u|\nabla\varphi|, \nabla u_j\rangle$, where the inner product is defined by the matrix $(a_{ij})_{i,j=1,\dots,n}$. By the Cauchy-Schwarz inequality we then get:

$$\begin{aligned}|8a_{ij}\varphi\varphi_i u_k u_{kj}| &= |8\varphi u_k(a_{ij}\varphi_i u_{kj})| \\ &= |8\varphi u_k||\langle\nabla\varphi, \nabla u_k\rangle| \\ &\le (8\varphi u_k\|\nabla\varphi\|)(\varphi\|\nabla u_k\|) \\ &\le C|\nabla u|^2 + 2\varphi^2 a_{ij}u_{ki}u_{kj},\end{aligned}$$

with the last bound being a consequence of $2xy \le x^2 + y^2$ and the boundedness of $|\nabla\varphi|$ assumed above. Putting all of this together with the ellipticity condition for L yields

$$Lh \ge 2\delta\lambda|\nabla u|^2 - C|\nabla u|^2,$$

which is nonnegative for $\delta \ge 0$ sufficiently (universally) large. This means that h is a supersolution (in the classical sense) of L, so by the maximum principle and the definition of h we get that

$$\sup_{B_{1/2}} |\nabla u|^2 \le \sup_{B_1} h = \sup_{\partial B_1} \delta(M-u)^2 \le \delta(\operatorname{osc}_{B_1} u)^2 \le \delta \sup_{B_1} |u|,$$

which was what we wanted. Now, for the second inequality we consider for any $e \in \partial B_1$ the auxiliary function

$$g = \delta(u_e)^2 + \varphi^2(u_{ee})^2 \in C^2(\overline{\Omega}),$$

where φ is as before, and $\Omega = \{u_{ee} > 0\} \subseteq B_1$. The same calculation as before yields that $Lg \ge 0$ in Ω for $\delta \ge 0$ sufficiently (universally) large. The maximum principle, and the fact that $\partial\Omega \subseteq \partial B_1 \cup \{u_{ee} = 0\}$ yields

$$\sup_{B_{1/2}}((u_{ee})^+)^2 = \sup_{\Omega\cap B_{1/2}} (u_{ee})^2 \le \sup_{\Omega} g = \sup_{\partial\Omega} g \le \delta \sup_{B_1}(u_e)^2 \le \delta\|\nabla u\|^2_{L^\infty(B_1)}.$$

The final inequality then follows from Lemma 3.4. $\square$

By Taylor's theorem we get that $\|u\|_{C^{1,1}(B_1)} \le C(\|u\|_{L^\infty(B_1)} + \|\nabla u\|_{L^\infty(B_1)} + \|D^2u\|_{L^\infty(B_1)})$ for some universal constant $C > 0$. Combining this with the results from this section we immediately obtain an interior $C^{2,\alpha}$ estimate in terms of the L^∞-norm for sufficiently nice equations. More precisely, consider a $F \in C^\infty(\mathcal{S})$ concave and $u \in C^2(B_1)$ with $F(D^2u) = 0$ in B_1. Suppose first that $F(0) = 0$. With this additional simplification, we now rescale the Evans-Krylov theorem and apply Propositions 4.1 and 4.3 to get the estimate

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/8})} \leq C\|u\|_{L^\infty(B_1)},$$

where $C > 0$ is universal. The standard rescaling and covering argument (for instance used in the proof of Theorem 2.3) yields that $u \in C^{2,\alpha}(B_1)$ with the estimate

$$\|u\|_{C^{2,\alpha}(\overline{B}_{r/2})} \leq C\|u\|_{L^\infty(B_r)},$$

with $C = C(n, r, \lambda, \Lambda) > 0$ now possibly a different universal constant. Finally from Proposition 4.1 we conclude that $u \in C^\infty(B_1)$. Now, the general case can easily be reclaimed through the following argument: Substituting $N = tI$ and $M = 0$ into the ellipticity condition for F and using the intermediate value theorem we get that there exists $|t| \leq |F(0)|/\lambda$ such that $F(tI) = 0$. Then for the paraboloid $P(x) = t|x|^2/2$ we have that

$$F(D^2u) = F(D^2(u - P) + tI) =: G(D^2(u - P)),$$

where G has the same ellipticity constants as F, and now satisfies $G(0) = 0$. We get all the properties as above for $u - P$, so $u = (u - P) + P$ also satisfy everything. By the triangle inequality we obtain the interior estimate:

$$\|u\|_{C^{2,\alpha}(\overline{B}_{r/2})} \leq C\|u - P\|_{C^{2,\alpha}(\overline{B}_{r/2})} \leq C(\|u - P\|_{L^\infty(B_r)}) \leq C(\|u\|_{L^\infty(B_r)} + |F(0)|).$$

Actually it is possible to obtain this estimate in a more general setting (in particular without needing F to be differentiable, see Theorem 6.6 in [1]). However this requires a very different style of argument than the one presented above.

4.2 The Case of $F(D^2u) = 0$ in B_1

So far we have only proved interior estimates to the equation $F(D^2u) = 0$. However, for the continuity method we need an estimate up to the boundary. The most difficult part, the estimate for $\|D^2u\|_{C^\alpha(\overline{B}_1)}$, is done by transforming it into a sufficiently regular boundary estimate which can then be resolved by the Harnack inequality. The details for this last part will be shown now. Let us for any $r > 0$ define the half-ball $B_r^+ := B_4 \cap \{x_n > 0\}$, and its "boundary" with respect to the last coordinate, $\Gamma_r := B_4 \cap \{x_n = 0\}$.

Theorem 4.4. *Let $u \in C^2(\overline{B}_4^+)$ be a solution of $Lu = f$ on $\overline{B}_4^+$ and $u = 0$ on Γ_4, where $L = a_{ij}\partial_{ij}$ is a uniformly elliptic operator with $a_{ij} \in L^\infty(\overline{B}_4^+) \cap L^1(\overline{B}_4^+)$ and ellipticity constants λ, Λ. Assume that there exists $K \geq 0$ such that*

$$\|w\|_{L^\infty(B_4^+)} + \|\nabla w\|_{L^\infty(B_4^+)} + \|f\|_{L^\infty(B_4^+)} \leq K.$$

Then there exists constants $\alpha \in (0,1)$ and $C > 0$, depending on n, λ, Λ and K, such that

$$\|\partial_n u\|_{C^\alpha(\overline{\Gamma}_1)} \leq C.$$

Proof. We proceed similarly to the proof of Proposition 2.4. In particular we have $u = 0$ on Γ_1 (since $\Gamma_1 \Gamma_4$), which means that

$$\partial_{x_n} u = \lim_{x_n \to 0} \frac{u(x)}{x_n} =: \lim_{x_n \to 0} v(x).$$

The proof will therefore be done if we can show a (uniform) Hölder-like bound on the function v, and then taking the limit $x_n \to 0$. For convenience we introduce notation for a "rectangle" $Q_r^{a,b} := \{|x'| \leq r,\, a \leq x_n \leq b\}$, where $(x', x_n) \in \mathbb{R}^n$ and $r \leq 1$. We also denote $m_r = \inf_{Q_r^{\delta r/2, \delta r}} v$ and $M_r = \sup_{Q_r^{\delta r/2, \delta r}} v$. In particular we have that $\sup v \leq 2/(\delta r) \sup u$ and $\inf u \leq 1/(\delta r) \inf v$ in $Q_r^{\delta r/2, \delta r}$, where $0 < \delta < 1/4$ is some universal constant. By the Harnack inequality, the standard rescaling and covering argument, and a simple technical Lemma (4.31 in [5]) on $u - m_{2r} x_n \geq 0$ in $Q_{2r}^{0,2r}$, we obtain

$$\begin{aligned}
\sup_{Q_r^{\delta r/2, \delta r}} (v - m_{2r}) &\leq C\big(\inf_{Q_r^{\delta r/2, \delta r}} (v - m_{2r}) + r\|f\|_{L^n(B_4^+)}\big) \\
&\leq C\big(\inf_{Q_r^{0, \delta r/2}} (v - m_{2r} + r\|f\|_{L^n(B_4^+)}\big) \\
&= C(m_{r/2} - m_{2r} + r\|f\|_{L^n(B_4^+)}),
\end{aligned}$$

where $C = C(n, \lambda, \Lambda, K) > 0$. Repeating the same calculation on $M_{2r} x_n - u \geq 0$ in $Q_{2r}^{0,2r}$ we obtain

$$\sup_{Q_r^{\delta r/2, \delta r}} (M_{2r} - v) \leq C(M_{2r} - M_{r/2} + r\|f\|_{L^n(B_4^+)}).$$

By adding the two inequalities we get

$$M_{2r} - m_{2r} \leq C(M_{2r} - m_{2r} - (M_{r/2} - m_{r/2}) + r\|f\|_{L^n(B_4^+)}).$$

Combining terms and writing $o(r)$ for the oscillation of u in $Q_r^{\delta r/2, \delta r}$ we see that $o(\tau r) \leq \gamma o(r) +$

$c_0 r/2$ for some constant c_0, with $\tau = 1/4$ and $\gamma < 1$. By Lemma 8.23 in [4] we therefore conclude that

$$o(r) \leq Cr^\alpha$$

for all $r \leq 1$, with $C > 0$ and $0 < \alpha < 1$ constant in r. This bound is uniform, so taking $x_n \to 0$ and interchanging limits yields

$$\operatorname{osc}_{Q_r^{\delta r/2, \delta r}} \partial_n u = \lim_{x_n \to \infty} o(r) \leq \lim_{n_x \to \infty} Cr^\alpha = Cr^\alpha,$$

from which we immediately get the desired Hölder continuity with exponent α. □

We now present a $C^{2,\alpha}$ estimate up to the boundary:

Theorem 4.5. *Let $F \in C^\infty(\mathcal{S})$ be concave and $g \in C^\infty(\overline{B}_1)$. Then there exists a universal $\alpha \in (0,1)$ such that for all solutions $u \in C^{2,\alpha}(\overline{B}_1)$ of*

$$\begin{cases} F(D^2 u) = 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases}$$

we have

$$\|u\|_{C^{2,\alpha}(\overline{B}_1)} \leq C_1(\|g\|_{C^3(\overline{B}_1)} + |F(0)|)$$

for some universal constant $C = C(n, \lambda, \Lambda) > 0$.

Proof. Similarly to before we can freely assume $F(0) = 0$. Furthermore, rescaling $v = u/\|g\|_{C^3(\overline{B}_1)}$ does not change the ellipticity constants for the equation, which means that we may assume $\|g\|_{C^3(\overline{B}_1)} \leq 1$. We therefore need to prove a universal bound $\|u\|_{C^{2,\alpha}(\overline{B}_1)} \leq C$; for the sake of length we only prove the final estimate for the Hölder seminorm of $D^2 u$. We wish to show a uniform local bound at the boundary, i.e. an inequality of the form

$$\|D^2 u(x_1) - D^2 u(x_0)\| \leq C\|x_1 - x_0\|^\beta \qquad \forall x_0 \in \partial B_1 \quad \forall x_1 \in \partial B_1 \cap B_r(x_0),$$

for some universal $C = C(\lambda, \Lambda, n) > 0$, $0 < \beta < 1$ and where $r > 0$ does not depend on x_0. By iterating this inequality finitely many times be obtain the estimate for arbitrary $x_1 \in \partial B_1$:

$$\begin{aligned}\|D^2u(x_1) - D^2u(x_0)\| &\leq \sum_{k=1}^{m(r)} \|D^2u(x_k) - D^2u(x_{k-1})\| \\ &\leq C \sum_{k=1}^{m(r)} \|x_k - x_{k-1}\| \\ &\leq C\|x_1 - x_0\|,\end{aligned}$$

where we relabelled $x_{m(r)} := x_1$. In particular, the final constant is again universal. Combining this with the Evans-Krylov theorem (which yields an interior estimate), we can then obtain an estimate up to the boundary in a similar fashion as in the previous section. What remains is therefore to show the inequality above. For this purpose, fix $x_0 \in \partial B_1$ and let A be a neighborhood of x_0 which contains a ball of universal radius. We flatten the boundary in this neighborhood, i.e. consider smooth diffeomorphisms $\varphi = \psi^{-1}$ such that $\varphi(x_0) = 0$, $\varphi(A \cap B_1) = B_4^+$, and $\varphi(A \cap \partial B_1) = \Gamma_4$. Consider the auxiliary function

$$v : \varphi(B_1) \to \mathbb{R}, \qquad\qquad v(y) = u(\psi(y)) - g(\psi(y)) \text{ for } y \in \Omega := \varphi(B_1)$$

We have that $v|_{\partial\Omega} = 0$. Furthermore, for all $x \in B_1$ we have by definition that $u(x) = v(\varphi(x)) - g(x)$, so applying the chain rule yields that v satisfies the equation

$$G(D^2v, Dv, y) := F\left(\left[\sum_{m,l=1}^{n} v_{mn}(y)\varphi_i^m(\psi(y))\varphi_j^n(\psi(y)) + \sum_{m=1}^{n} v_m\varphi_{ij}^m(\psi(y)) + g_{ij}(\psi(y))\right]_{ij}\right) = 0$$

on Ω. Now, the desired inequality will follow if we show $\|D^2v(y) - D^2v(0)\| \leq C\|y\|^\beta$ for all $y \in \Gamma_1$. Using the implicit function theorem on the pure second derivative v_{nn} in above equation, we get the bound:

$$|v_{nn}(y) - v_{nn}(0)| \leq C\left(\sup_{\substack{1\leq k\leq n-1 \\ 1\leq l\leq n}} |v_{kl}(y) - v_{kl}(0)| + |y|\right).$$

Since $v|_{\partial\Gamma_1} \equiv 0$ we get by the maximum principle that $v_{kl}|_{\partial\Gamma_1} \equiv 0$ for all $1 \leq k,\ l \geq n-1$. In particular, the above inequality can then be simplified to

$$|v_{nn}(y) - v_{nn}(0)| \leq C\left(\sup_{1\leq k\leq n-1} |v_{kn}(y) - v_{kn}(0)| + |y|\right).$$

Differentiating the equation $G(D^2v, Dv, y) = 0$ with respect to y_k we get that v_k satisfies some equation that satisfies all the assumptions in Theorem 4.4. We therefore conclude that v_{kn} satisfies a Hölder estimate, which proves the theorem.

□

We are now ready to solve the Dirichlet problem:

Theorem 4.6. *Let $F \in C^\infty(\mathcal{S})$ be concave and $g \in C^\infty(\overline{B}_1)$. Then the Dirichlet problem*

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases}$$

has a unique viscosity solution $u \in C(\overline{B}_1)$. Furthermore, u is smooth on $\overline{B}_1$ and there exist universal constants $\alpha \in (0,1)$, $C_1 > 0$ and $C_2 > 0$ such that

$$\|u\|_{C^{2,\alpha}(\overline{B}_1)} \leq C_1(\|g\|_{C^3(\overline{B}_1)} + |F(0)|)$$
$$\|u\|_{C^{2,\alpha}(\overline{B}_{r/2})} \leq C_2(\|u\|_{L^\infty(B_r)} + r^2|F(0)|),$$

for any (not necessarily centered) ball $B_r \subseteq B_1$.

Proof. All of the regularity properties of u are proved earlier in this section, so what remains is to use the method of continuity to show existence: For $t \in [0,1]$ we consider the family of Dirichlet problems:

$$\begin{cases} tF(D^2v + D^2g) + (1-t)(\Delta v + \Delta g) = 0 & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1. \end{cases}$$

For all $t \in [0,1]$ we have that the operator $F_t := tF(D^2u) + (1-t)\Delta u$ is C^∞, concave, and uniformly elliptic with ellipticity constants λ, Λ. To see this, observe that with $\lambda < 1$ (which can be freely assumed), and $M, N \in \mathcal{S}$ arbitrary symmetric matrices, an elementary computation shows

$$\begin{aligned} F_t(M+N) - F_t(M) &= tF(M+N) - tF(M) + (1-t)\sum_{i=1}^n N_{ii} \\ &\geq t\lambda\|N\| + (1-t)\lambda\|N\| \\ &= \lambda\|N\|. \end{aligned}$$

A similar computation shows for the upper bound holds whenever $\Lambda > 1$, which again can be assumed. Let $\phi(t,v) := tF(D^2v + D^2g) + (1-t)(\Delta v + \Delta g)$ and consider the set

$$A := \{t \in [0,1] \mid \exists v \in C_0^{2,\alpha}(\overline{B}_1) \text{ with } \phi(t,v) = 0 \text{ in } B_1\},$$

where $C_0^{2,\alpha}(\overline{B}_1)$ denotes the subspace of functions that vanish at the boundary. Note that for all $t \in A$ we have by Theorem 4.5 that the corresponding $v \in C_0^{2,\alpha}$ all satisfy a uniform bound $\|v\|_{C^{2,\alpha}(\overline{B}_1)} \leq C_1$, meaning that $C_1 > 0$ particular does not depend on t. By the theory of harmonic functions we get $0 \in A$, so A is nonempty. It remains to show that $1 \in A$, and proceed by showing that A is both open and closed with respect to the subspace topology on $[0,1]$ (which is a connected space). For openness, we wish to invoke the implicit function theorem, and define the unbounded (nonlinear) operator

$$\phi : C_0^{2,\alpha}(\overline{B}_1) \times [0,1] \to C^\alpha(\overline{B}_1), \qquad (v,t) \mapsto \phi(v,t),$$

which we first have to argue is well-defined: For v and t as in the domain, let $x, y \in B_1$ with $\|x-y\| = r$. By the ellipticity of F we get

$$\begin{aligned}
|\phi(v,t)(x) - \phi(v,t)(y)| &\leq t|F(D^2v(x) + D^2g(x)) - F(D^2v(y) + D^2g(y))| \\
&\qquad + (1-t)|\Delta v(x-y) + \Delta g(x-y)| \\
&\leq t\Lambda\|D^2v(x-y) + D^2g(x-y)\| + (1-t)\|D^2v(x-y) + D^2g(x-y)\| \\
&\leq Cr^\alpha,
\end{aligned}$$

where the last equality follows since $v, g \in C^{2,\alpha}$. Note that $C = C(n, \lambda, \Lambda, t)$ does not depend on x, y, which means that we have $\phi(v,t) \in C^\alpha(B_1)$. The Fréchet derivative with respect to t is constant, so clearly defines a continuous linear operator. The Fréchet derivative with respect to v defines the linear operator

$$D_v\phi(v,t)(w) : C_0^{2,\alpha}(\overline{B}_1) \to C^\alpha(\overline{B}_1), \qquad w \mapsto [tF_{ij}(D^2v + D^2g) + (1-)\delta_{ij}]w_{ij},$$

which again is well-defined, since the coefficients have ellipticity constants λ, Λ. From the standard uniqueness theorem for linear equations we therefore get that $D_v\varphi(v,t)$ is invertible. So, for all $t \in A$ we get from the implicit function theorem that there exists a neighborhood $U \subseteq A$ and $g : U \to C_0^{2\alpha}(\overline{B}_1)$ continuous, such that $\phi(t, g(t)) = 0$ for all $t \in U$. So A is open. For closedness, let $(t_n)_{n\in\mathbb{N}} \in A$ be given with $t_n \to t$, with the corresponding $(v_n)_{n\in\mathbb{N}} \in C_0^{2,\alpha}(\overline{B}_1)$. Recall by Theorem 4.5 that we have a uniform bound $\|v_n\|_{C^{2,\alpha}(\overline{B}_1)} \leq C_1$, so $v_n \to v$ uniformly with $v \in C_0^{2,\alpha}(\overline{B}_1)$. Taking limits yields $\phi(v,t) = 0$. We conclude that $A = [0,1]$, which in particular means that we have existence for the original problem. $\square$

By a standard approximation argument we can relax the smoothness assumption on F and f. What is omitted from this report is that it is also possible to show Hölder estimates for more general equations, such as with nonconstant coefficients $F(D^2u, x) = 0$, or even more general coefficients.

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